

OPTIMAL BOUNDARY REGULARITY FOR SOME SINGULAR MONGE-AMPÈRE EQUATIONS ON BOUNDED CONVEX DOMAINS

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ABSTRACT. By constructing explicit supersolutions, we obtain the optimal global Hölder regularity for several singular Monge-Ampère equations on general bounded open convex domains including those related to complete affine hyperbolic spheres, and proper affine hyperspheres. Our analysis reveals that certain singular-looking equations, such as $\det D^2 u = |u|^{-n-2-k}(x \cdot Du - u)^{-k}$ with zero boundary data, have unexpected degenerate nature.

1. Introduction and statements of the main results. This note is concerned with the optimal global Hölder regularity of the unique convex solution $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ to some singular Monge-Ampère equations whose prototype is

$$\begin{cases} \det D^2 u = |u|^{-p} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

where $p > 0$ and Ω is a general bounded open convex domain in \mathbb{R}^n ($n \geq 2$). Using suitable subsolutions, we show that $u \in C^{\frac{2}{n+p}}(\bar{\Omega})$. By constructing explicit supersolutions on some special bounded convex domains, we show that this global Hölder regularity is optimal when $p \geq 1$. The class of such bounded convex domains includes those containing parts of hyperplanes on their boundaries.

There is an extensive literature on (1) when Ω is a C^2 , bounded, strictly convex domain. Our focus here is on the general bounded convex domains which can contain parts of hypersurfaces on their boundaries. To the best of the author's knowledge, the only existence result available for (1) is the work of Cheng and Yau [6] for $p = n + 2$. In this case, if u is a solution to (1), then the Legendre transform of u is a complete affine hyperbolic sphere; see [4, 6, 7]. Also related to the exponent $p = n + 2$ is the work of Chen and Huang [5] who treated the following equation for all $k \geq 0$

$$\begin{cases} \det D^2 u = |u|^{-n-2-k}(x \cdot Du - u)^{-k} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2)$$

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which is related to proper affine hyperspheres. When $k > 0$, the domain Ω in (2) was required to contain the origin so that the expression $x \cdot Du - u$ is positive.

Other positive values of p in (1) are also of interest because of the close relation with the L_p -Minkowski problem (see, for example [17]) and the Minkowski problem in centro-affine geometry (see, for example [8, 13] and the references therein). The case $p = 1$ is closely related to the logarithmic Minkowski problem [1]. For $p > 0$, the existence of a unique convex solution to (1) is perhaps well known to experts. However, since we could not locate a reference and for the reader's convenience, we include its existence proof in part (i) of Theorem 1.1 below.

Our first main result states as follows.

Theorem 1.1. *Let Ω be a bounded, open convex domain in \mathbb{R}^n ($n \geq 2$). Let $p > 0$. Then,*

(i) *there exists a unique convex solution $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ to*

$$\begin{cases} \det D^2u = |u|^{-p} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (3)$$

(ii) *moreover, $u \in C^{\frac{2}{n+p}}(\bar{\Omega})$ and the following estimate holds:*

$$|u(x)| \leq C(n, p, \text{diam}(\Omega)) [\text{dist}(x, \partial\Omega)]^{\frac{2}{n+p}} \quad \text{for all } x \in \Omega.$$

(iii) *if $p \geq 1$, then the global Hölder regularity in (ii) is optimal in the following sense: there exist bounded convex domains $\Omega \subset \mathbb{R}^n$ such that $u \notin C^\beta(\bar{\Omega})$ for any $\beta > \frac{2}{n+p}$. In fact, the class of such bounded convex domains includes those containing parts of hyperplanes on their boundaries.*

Note that the global Hölder estimate in the case of $p = n + 2$ in Theorem 1.1 (ii) was treated by Lin and Wang [15, Section 4] and also by Jian and Li [11, Theorem 1.2]. Jian and Li observed that when $n = 2$ and $\Omega = (-1, 1) \times \mathbb{R}_+ \subset \mathbb{R}^2$, the $C^\infty(\Omega) \cap C^{\frac{1}{3}}(\bar{\Omega})$ convex function

$$u(x_1, x_2) = -\sqrt{3} \left(\frac{x_2}{2} \right)^{\frac{1}{3}} (1 - x_1^2)^{\frac{1}{3}} \quad (4)$$

is a solution of $\det D^2u = |u|^{-4}$ in Ω with $u = 0$ on $\partial\Omega$. This example, though for unbounded Ω , indicates that the global estimates in Theorem 1.1 (ii) should be optimal, at least when $p = n + 2$. The bounded convex domains Ω and the supersolutions constructed in Lemma 2.4 actually confirm the optimality of the global Hölder estimates in Theorem 1.1 (ii) for all $p \geq 1$. Our ansatz is quite different from the ansatz in (4); see also Remarks 3 and 4. It is an open question to investigate the remaining case $0 < p < 1$.

Remark 1. Since our method of proof of optimality relies only on explicit supersolutions but not on explicit solutions, it is quite flexible and can be extended to (2); see Theorem 1.2. When $k > 0$, we are not aware of any explicit solution to (2) on any convex domain Ω . During the course of the proof of Theorem 1.1 (iii), we find that when $p = n + 2$ or $p = 1$, (1) has an explicit solution when Ω is a circular cylinder; see Remarks 3 and 4.

The proof of Theorem 1.1 (ii) is based on a subsolution as in [14] which treated (1) in the case $p = -q < 0$; see also [5] for similar arguments. Interestingly, this argument gives almost the same Hölder exponent for $p > 0$ and $p < 0$.

When $p = -q < 0$, we have the following degenerate Monge-Ampère equation

$$\begin{cases} \det D^2 u = |u|^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (5)$$

For $0 < q \neq n$, the existence of a nonzero convex solution to (5) was established by Tso [18, Corollary 4.2 and Theorem E] when Ω is a bounded, smooth and uniformly convex domain, and by the author [14, Theorem 4.2] when Ω is a general bounded convex domain. When $0 \leq q < n$, the nonzero solution u is unique. The case $q = n$ corresponds to the Monge-Ampère eigenvalue problem [14, 16]. By [14, Proposition 2.8], if $u \in C(\bar{\Omega})$ is an Aleksandrov solution to (5), then $u \in C^\infty(\Omega)$.

If Ω is an open bounded convex domain in \mathbb{R}^n and $n - 2 \leq q \neq n$, the author established in [14, Proposition 5.4] the global almost Lipschitz property for solutions to (5), that is, $u \in C^{0,\beta}(\bar{\Omega})$ for all $\beta \in (0, 1)$. It is not known if this global almost Lipschitz result holds for the nonzero solution to (5) when $n \geq 3$, $q \in (0, n - 2)$ and Ω is a general open bounded convex domain in \mathbb{R}^n . The following shows that, in the remaining cases, we have the global almost $C^{\frac{2}{n-q}}$ property.

Proposition 1. *Let Ω be an open bounded convex domain in \mathbb{R}^n where $n \geq 3$ and $0 < q < n - 2$. Let $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ be the nonzero convex solution of*

$$\begin{cases} \det D^2 u = |u|^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, for all $\beta \in (0, \frac{2}{n-q})$, we have $u \in C^{0,\beta}(\bar{\Omega})$ and the estimate

$$|u(x)| \leq C(n, q, \beta, \text{diam } \Omega) [\text{dist}(x, \partial\Omega)]^\beta \text{ for all } x \in \Omega. \quad (6)$$

When $k > 0$ and Ω is a bounded, open convex domain in \mathbb{R}^n ($n \geq 2$) that contains the origin in its interior, Chen and Huang [5] proved the existence of a unique convex solution $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ to (2). Their proof (see [5, p. 871]) shows that $u \in C^{\frac{1}{n+k+1}}(\bar{\Omega})$. Exploring the fact that $\text{dist}(0, \partial\Omega) > 0$ together with suitable constructions of subsolutions and supersolutions, we show a higher global Hölder regularity $u \in C^{\frac{2+k}{2n+2k+2}}(\bar{\Omega})$ and this is in fact optimal.

Our final theorem states as follows.

Theorem 1.2. *Let Ω be a bounded, open convex domain in \mathbb{R}^n ($n \geq 2$) that contains the origin in its interior. Let $k > 0$. Let $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ be the unique convex solution to*

$$\begin{cases} \det D^2 u = |u|^{-n-2-k} (x \cdot Du - u)^{-k} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (7)$$

Then $u \in C^{\frac{2+k}{2n+2k+2}}(\bar{\Omega})$ with the estimate

$$|u(x)| \leq C(n, k, \text{diam}(\Omega), \text{dist}(0, \partial\Omega)) [\text{dist}(x, \partial\Omega)]^{\frac{2+k}{2n+2k+2}} \text{ for all } x \in \Omega, \quad (8)$$

and the exponent $\frac{2+k}{2n+2k+2}$ is optimal.

Remark 2. We now compare the results in Theorems 1.1 and 1.2.

- (i) If the right hand-side $|u|^{-p}$ of (3) is replaced by $f|u|^{-p}$ where f is a positive bounded function on $\bar{\Omega}$ then, from the proof of Theorem 1.1 (ii), the global $C^{\frac{2}{n+p}}(\bar{\Omega})$ estimate for u still holds.

- (ii) In the context of Theorem 1.2 where one can show that $x \cdot Du - u$ is bounded from below by a positive constant, if we just use the boundedness of $(x \cdot Du - u)^{-k}$, then the right hand-side of (7) is bounded from above by $C|u|^{-n-k-2}$ from which we can use (i) to deduce $u \in C^{\frac{1}{n+k+1}}(\bar{\Omega})$. This result is due to Chen and Huang as mentioned earlier.
- (iii) Our higher global Hölder regularity for u in Theorem 1.2 is based on the insight that Du is unbounded near the boundary $\partial\Omega$ so $(x \cdot Du - u)^{-k}$ tends to 0 near the boundary with certain rate. The estimate (28) in the proof of Lemma 4.1 suggests that the rate is at least $[\text{dist}(\cdot, \partial\Omega)]^{\frac{(2n+k)k}{2n+2k+2}}$.

Surprisingly, despite the singular-looking nature of the right-hand side of (7), we find that when $n > 4$ and k is large, right-hand side of (7) is not completely singular as it might tend to 0 near the boundary. Thus (7) can have a degeneracy nature as in (5). The following proposition makes this more precise.

Proposition 2. *Assume $k > 0$ and $0 < \gamma < 1$. Let*

$$\Omega = \{(x', x_n) : |x'| < 1, 0 < x_n + \gamma < 1 - |x'|^2\}.$$

Let $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ be the unique convex solution to (7). Let

$$f(x) := |u(x)|^{-n-2-k}(x \cdot Du(x) - u(x))^{-k}.$$

Then, for $x = (0, x_n) \in \Omega$ with $0 < x_n + \gamma$ small, we have

$$f(x) \leq C(n, k, \gamma)[\text{dist}(x, \partial\Omega)]^{\frac{(n-4)k-(2n+4)}{2n+2k+2}}. \tag{9}$$

In particular, if $n > 4$ and $k > (2n + 4)/(n - 4)$, then the right-hand side of (7) tends to 0 as x approaches the boundary $\partial\Omega \cap \{x_n = -\gamma\}$ along the x_n -axis.

In this note, we denote a point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ by (x', x_n) where $x' = (x_1, \dots, x_{n-1})$. In computations, we usually denote $r = |x'|$. The Lebesgue measure of Ω is denoted by $|\Omega|$.

The rest of this note is organized as follows: In Section 2, we construct subsolutions and supersolutions of singular Monge-Ampère equations (3). In Section 3, we prove Theorem 1.1 and Proposition 1. The proofs of Theorem 1.2 and Proposition 2 will be given in Section 4.

2. Subsolutions and supersolutions of singular Monge-Ampère equations.

In the proof of Theorem 1.1, we frequently use the following comparison principle whose short proof is included for completeness.

Lemma 2.1 (Comparison principle). *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain. Let $p > 0$.*

- (i) *Assume that $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$ are convex functions with*

$$\det D^2u \geq |u|^{-p}, \quad \det D^2v \leq |v|^{-p} \quad \text{in } \Omega$$

and $0 \geq v \geq u$ on $\partial\Omega$. Then $v \geq u$ in Ω .

- (ii) *Let $k > 0$. Assume that $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$ are convex functions with*

$$x \cdot Du - u > 0, \quad x \cdot Dv - v > 0 \quad \text{in } \Omega, \quad 0 \geq v \geq u \quad \text{on } \partial\Omega$$

and

$$\det D^2u \geq |u|^{-p}(x \cdot Du - u)^{-k}, \quad \det D^2v \leq |v|^{-p}(x \cdot Dv - v)^{-k} \quad \text{in } \Omega.$$

Then $v \geq u$ in Ω .

Proof. (i) If $v - u$ attains its minimum value on $\bar{\Omega}$ at $x_0 \in \Omega$ with $v(x_0) < u(x_0) < 0$, then $D^2v(x_0) \geq D^2u(x_0)$. It follows that

$$|v(x_0)|^{-p} \geq \det D^2v(x_0) \geq \det D^2u(x_0) \geq |u(x_0)|^{-p}.$$

Therefore, $|v(x_0)|^{-p} \geq |u(x_0)|^{-p}$ which contradicts $|v(x_0)| > |u(x_0)|$ and $p > 0$.

(ii) If $v - u$ attains its minimum value on $\bar{\Omega}$ at $x_0 \in \Omega$ with $v(x_0) < u(x_0) < 0$, then $Du(x_0) = Dv(x_0)$ and $D^2v(x_0) \geq D^2u(x_0)$. From $\det D^2v(x_0) \geq \det D^2u(x_0)$ and the assumptions on v and u , we deduce that

$$(x_0 \cdot Dv(x_0) - v(x_0))^{-k} |v(x_0)|^{-p} \geq |u(x_0)|^{-p} (x_0 \cdot Du(x_0) - u(x_0))^{-k}.$$

Since $k > 0$ and

$$x_0 \cdot Dv(x_0) - v(x_0) = x_0 \cdot Du(x_0) - v(x_0) > x_0 \cdot Du(x_0) - u(x_0),$$

we easily find $|v(x_0)|^{-p} > |u(x_0)|^{-p}$ which contradicts $|v(x_0)| > |u(x_0)|$ and $p > 0$. \square

The following lemma, motivated by [2, Lemma 1], constructs subsolutions to (3).

Lemma 2.2 (Subsolutions for (3)). *Let Ω be a bounded convex domain such that $0 \in \partial\Omega$ and $\Omega \subset \mathbb{R}_+^n = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$. For $\alpha \in (0, 1)$, we consider the following function on Ω*

$$v_\alpha(x) = x_n^\alpha (|x'|^2 - C_\alpha) \text{ where } C_\alpha = \frac{1 + 2[\text{diam}\Omega]^2}{\alpha(1 - \alpha)}. \tag{10}$$

Then

(i) v_α is convex in Ω with $v_\alpha \leq 0$ on $\partial\Omega$ and

$$\det D^2v_\alpha(x) \geq 2x_n^{n\alpha-2} \text{ in } \Omega.$$

(ii)

$$\det D^2v_{\frac{2}{n+p}} > |v_{\frac{2}{n+p}}|^{-p} \text{ if } p > 0.$$

Proof. (i) Note that

$$D_{ij}v_\alpha = 2x_n^\alpha \delta_{ij} \text{ for } i, j \leq n - 1, \quad D_{in}v_\alpha = 2\alpha x_i x_n^{\alpha-1},$$

and

$$D_{nn}v_\alpha = \alpha(\alpha - 1)(|x'|^2 - C_\alpha)x_n^{\alpha-2},$$

where δ_{ij} is the Kronecker symbol, that is, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

A short computation (see also the computation in the proof of Lemma 2.4) gives

$$\det D^2v_\alpha(x) = 2^{n-1} x_n^{n\alpha-2} [\alpha(1 - \alpha)C_\alpha - (\alpha^2 + \alpha)|x'|^2] \text{ for } x \in \Omega. \tag{11}$$

Therefore (i) easily follows, since, from the definition of C_α , we have

$$\alpha(1 - \alpha)C_\alpha - (\alpha^2 + \alpha)|x'|^2 \geq 1.$$

(ii) Now consider $p > 0$ and $\alpha := 2/(n + p)$. Using $\alpha(1 - \alpha) \leq \frac{1}{4}$, we deduce that $C_\alpha \geq 4 + 8[\text{diam}\Omega]^2$. Therefore, in view of (11) and $n \geq 2$, we find

$$\begin{aligned} |v_\alpha|^p \det D^2v_\alpha &> |v_\alpha|^p x_n^{n\alpha-2} [\alpha(1 - \alpha)C_\alpha - (\alpha^2 + \alpha)|x'|^2] \\ &= [\alpha(1 - \alpha)C_\alpha - (\alpha^2 + \alpha)|x'|^2] (C_\alpha - |x'|^2)^p \\ &\geq (C_\alpha - |x'|^2)^p > 1. \end{aligned}$$

\square

To establish a lower bound for the L^∞ norm of solution to (3), we use the following lemma.

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain in \mathbb{R}^n . Let $p, \varepsilon \geq 0$. Let $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ be the convex solution to*

$$\begin{cases} \det D^2u = (|u| + \varepsilon)^{-p} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If $\varepsilon < \varepsilon_0(n, p, |\Omega|)$, then

$$\|u\|_{L^\infty(\Omega)} \geq c(n, p)|\Omega|^{\frac{2}{n+p}}.$$

Proof. The proof is similar to that of [14, Lemma 3.1 (iii)]. Under the affine transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $\det T = 1$:

$$\Omega \rightarrow T(\Omega), \quad u(x) \rightarrow u(T^{-1}x),$$

the equation $\det D^2u = (|u| + \varepsilon)^{-p}$, the quantities $\|u\|_{L^\infty(\Omega)}$ and $|\Omega|$ are unchanged. Thus, by John’s lemma, we can assume that Ω is normalized, that is

$$B_R(0) \subset \Omega \subset B_{nR}(0) \text{ for some } R > 0.$$

Let $\alpha = \|u\|_{L^\infty(\Omega)} > 0$ and $v = u/\alpha$. Then, $v \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ with $v = 0$ on $\partial\Omega$, and $\|v\|_{L^\infty(\Omega)} = 1$. Furthermore, v satisfies $\alpha^n \det D^2v = (\alpha|v| + \varepsilon)^{-p}$ in Ω . It follows that

$$\frac{1}{(\alpha + \varepsilon)^p} \leq \alpha^n \det D^2v \quad \text{in } \Omega.$$

Integrating both sides over $B_{R/2}(0)$, we find

$$\frac{1}{(\alpha + \varepsilon)^p} |B_{R/2}(0)| \leq \alpha^n \int_{B_{R/2}(0)} \det D^2v \, dx.$$

Now we estimate $\int_{B_{R/2}(0)} \det D^2v \, dx$ from above. Since, $B_R(0) \subset \Omega \subset B_{nR}(0)$, the convexity of v and the fact that $v = 0$ on $\partial\Omega$ give for $x \in B_{R/2}(0)$

$$|Dv(x)| \leq \frac{|v(x)|}{\text{dist}(x, \Omega)} \leq \frac{\|v\|_{L^\infty(\Omega)}}{\text{dist}(x, \Omega)} \leq 2R^{-1}.$$

Hence

$$\int_{B_{R/2}(0)} \det D^2v \, dx = |Dv(B_{R/2}(0))| \leq |B_{2R^{-1}}(0)|.$$

Therefore

$$\frac{1}{\alpha^n(\alpha + \varepsilon)^p} \leq \frac{|B_{2R^{-1}}(0)|}{|B_{R/2}(0)|} = 4^n R^{-2n} \leq C(n)|\Omega|^{-2}.$$

By choosing $\varepsilon_0^n(2\varepsilon_0)^p C(n)|\Omega|^{-2} < 1/2$, we obtain the conclusion of the lemma. \square

Now, we construct supersolutions to (3) with optimal global Hölder regularity.

Lemma 2.4 (Supersolutions for (3)). *Assume $p \geq 1$. Let*

$$\Omega = \{(x', x_n) : |x'| < 1, 0 < x_n < 1 - |x'|^2\}.$$

Then there is a constant $C = C(n, p)$ such that the function

$$w = Cx_n - Cx_n^{\frac{2}{n+p}}(1 - |x'|^2)^{\frac{n+p-2}{n+p}}$$

is smooth, convex in Ω and satisfies

$$\det D^2w \leq |w|^{-p} \quad \text{in } \Omega, \quad \text{and } w = 0 \quad \text{on } \partial\Omega.$$

Proof. For $x = (x', x_n)$, we denote $r = |x'|$. Let

$$v(x) = -Cx_n^a(1 - r^2)^b$$

where $0 < a, b < 1$ and $C > 0$. Then

$$\begin{aligned} v_r &= 2Cb x_n^a(1 - r^2)^{b-1}r \\ v_{rr} &= 2Cb x_n^a(1 - r^2)^{b-2}[1 - (2b - 1)r^2] \\ v_{x_n} &= -Cax_n^{a-1}(1 - r^2)^b \\ v_{x_n x_n} &= Ca(1 - a)x_n^{a-2}(1 - r^2)^b \\ v_{x_n r} &= 2Cabx_n^{a-1}(1 - r^2)^{b-1}r. \end{aligned}$$

In suitable coordinate systems, such as cylindrical in x' , the Hessian of v has the following form

$$D^2v = \begin{pmatrix} \frac{v_r}{r} & 0 & \cdots & 0 & 0 \\ 0 & \frac{v_r}{r} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & v_{rr} & v_{rx_n} \\ 0 & 0 & \cdots & v_{rx_n} & v_{x_n x_n} \end{pmatrix}.$$

We have

$$\begin{aligned} \det D^2v &= \left(\frac{v_r}{r}\right)^{n-2}[v_{x_n x_n} v_{rr} - v_{x_n r}^2] \\ &= C^n(2b)^{n-1}x_n^{na-2}(1 - r^2)^{n(b-1)}a[1 - a + (1 - 2b - a)r^2]. \end{aligned}$$

It follows that v is convex in Ω provided that

$$0 < a, b < 1, \quad 1 - a + (1 - 2b - a)r^2 > 0 \text{ in } \Omega.$$

Since $r < 1$ in Ω , the last condition is equivalent to

$$0 \leq 1 - a + (1 - 2b - a) = 2(1 - a - b), \quad \text{or } a + b \leq 1.$$

We would like to have

$$\det D^2v \leq |v|^{-p} = C^{-p}x_n^{-ap}(1 - r^2)^{-bp}$$

which is equivalent to

$$C^{n+p}(2b)^{n-1}x_n^{(n+p)a-2}(1 - r^2)^{(n+p)b-n}a[(1 - a)(1 - r^2) + 2(1 - a - b)r^2] \leq 1 \quad \text{for all } r < 1. \quad (12)$$

Requiring v to be $C^{\frac{2}{n+p}}(\bar{\Omega})$, we choose

$$a = \frac{2}{n + p}.$$

Since $a + b \leq 1$, (12) then implies that

$$C^{n+p}(1 - r^2)^{(n+p)b-n+1}(2b)^{n-1}a(1 - a) \leq 1 \quad \text{for all } r < 1. \quad (13)$$

Note that

$$(n + p)b - n + 1 \leq (n + p)(1 - a) - n + 1 = p - 1$$

with equality if and only $b = 1 - a$.

Since we want (13) to hold for $p > 0$ as small as we want, a nature choice is to choose

$$b = 1 - a = \frac{n + p - 2}{n + p}.$$

Then (12) is exactly (13) and (13) holds for a suitable $C = C(n, p)$ for each $p \geq 1$.
 Set

$$w = Cx_n + v = C[x_n - x_n^a(1 - r^2)^b].$$

Then w is smooth, convex in Ω and $w = 0$ on $\partial\Omega$. Moreover, since $|w| = |v| - Cx_n$, we have

$$\det D^2w = \det D^2v \leq |v|^{-p} = |Cx_n + |w||^{-p} \leq |w|^{-p} \quad \text{in } \Omega.$$

□

Remark 3. When $p = 1$, the constant $C(n, 1)$ in Lemma 2.4 is given by

$$C(n, 1) = (n + 1)[2(n - 1)]^{-\frac{n}{n+1}}.$$

The proof of Lemma 2.4 (see (12)) shows that if

$$\Omega = \{(x', x_n) : |x'| < 1, x_n > 0\},$$

then

$$v(x) = -C(n, 1)x_n^{\frac{2}{n+1}}(1 - |x'|^2)^{\frac{n-1}{n+1}}$$

is a solution to the singular Monge-Ampère equation

$$\det D^2v = |v|^{-1} \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega.$$

Remark 4. In the proof of Lemma 2.4, if we choose

$$1 - 2b - a = 0, \quad \text{or } b = \tilde{b} := \frac{1 - a}{2} = \frac{n + p - 2}{2(n + p)},$$

then from

$$1 - a + (1 - 2b - a)r^2 = (1 - a)(1 - r^2) + 2(1 - a - b)r^2 = 1 - a,$$

we find that (12) is equivalent to

$$C^{n+p}(1 - r^2)^{(n+p)b-n}(2b)^{n-1}a(1 - a) \leq 1 \quad \text{for all } r < 1, \tag{14}$$

which requires

$$0 \leq (n + p)b - n = \frac{p - n - 2}{2}, \quad \text{or } p \geq n + 2.$$

Thus, for $p \geq n + 2$ and a suitable $C = C(n, p)$, the function

$$\tilde{v}(x', x_n) = -Cx_n^{\frac{2}{n+p}}(1 - |x'|^2)^{\frac{n+p-2}{2(n+p)}}$$

satisfies

$$\det D^2\tilde{v} \leq |\tilde{v}|^{-p} \quad \text{in } \{(x', x_n) : x_n > 0, |x'| < 1\}.$$

When $p = n + 2$ and $n = 2$, we recover the function (4) obtained in [11]; see also [12] for the case $n \geq 2$.

Modifying the last step in the proof of Lemma 2.4, we find that if $x_n^2 = 1 - |x'|^2$ on $\partial\Omega \setminus \{x_n = 0\}$, then

$$\tilde{v}(x', x_n) = -Cx_n^{\frac{2}{n+p}}(1 - |x'|^2)^{\frac{n+p-2}{2(n+p)}} = -Cx_n \quad \text{on } \partial\Omega \setminus \{x_n = 0\}.$$

We summarize these calculations in Lemma 2.5 below.

Lemma 2.5 (Supersolutions for (3)). *Assume $p \geq n + 2$. Let*

$$\Omega = \{(x', x_n) : |x'| < 1, 0 < x_n < \sqrt{1 - |x'|^2}\}.$$

Then there is a constant $C = C(n, p)$ such that the function

$$w = Cx_n - Cx_n^{\frac{2}{n+p}}(1 - |x'|^2)^{\frac{n+p-2}{2(n+p)}}$$

is smooth, convex in Ω and satisfies

$$\det D^2w \leq |w|^{-p} \text{ in } \Omega, \text{ and } w = 0 \text{ on } \partial\Omega.$$

We state the following rescaled version of Lemma 2.4.

Lemma 2.6. *Assume $p \geq 1$ and $t > 0$. Let*

$$\Omega_t = \{(x', x_n) : |x'| < t, 0 < x_n < t^2 - |x'|^2\}.$$

Let $C = C(n, p)$ be as in Lemma 2.4. Then, the function

$$w^t(x', x_n) = Ct^{\frac{2(1-p)}{n+p}} [x_n - x_n^{\frac{2}{n+p}}(t^2 - |x'|^2)^{\frac{n+p-2}{n+p}}]$$

is smooth, convex in Ω_t and satisfies

$$\det D^2w^t \leq |w^t|^{-p} \text{ in } \Omega_t, \text{ and } w^t = 0 \text{ on } \partial\Omega_t.$$

Proof. Let $a = \frac{2}{n+p}, b = \frac{n+p-2}{n+p}$, Ω and w be as in Lemma 2.4. Observe that $(x', x_n) \in \Omega_t \Leftrightarrow (\frac{x'}{t}, \frac{x_n}{t^2}) \in \Omega$. Note that

$$w^t(x', x_n) = Ct^{\frac{2(1+n)}{n+p}} \left[\frac{x_n}{t^2} - \left(\frac{x_n}{t^2}\right)^a \left[1 - \frac{|x'|^2}{t^2}\right]^b \right] = t^{\frac{2(1+n)}{n+p}} w\left(\frac{x'}{t}, \frac{x_n}{t^2}\right).$$

A direct calculation gives

$$\det D^2w^t(x', x_n) = t^{-\frac{2(1+n)p}{n+p}} (\det D^2w)\left(\frac{x'}{t}, \frac{x_n}{t^2}\right).$$

The properties of w^t now follow from those of w . □

3. Proofs of Theorem 1.1 and Proposition 1.

Proof of Theorem 1.1. (i) We divide the proof into two steps.

Step 1. Uniform global Hölder estimate for approximate solutions of (3). Let $\{\Omega_k\}_{k=1}^\infty$ be a sequence of open, bounded, smooth and uniformly convex domains in \mathbb{R}^n such that Ω_k converges to Ω in the Hausdorff distance. Let ε_0 be as in Lemma 2.3. For each k , consider the Monge-Ampère equation

$$\begin{cases} \det D^2u_k = (|u_k| + \varepsilon_k)^{-p} & \text{in } \Omega_k, \\ u_k = 0 & \text{on } \partial\Omega_k \end{cases}$$

where $\varepsilon_k = \frac{\varepsilon_0}{k}$. From [3, Theorem 7.1], there exists a unique convex solution $u_k \in C^\infty(\overline{\Omega_k})$ to the above equation. Since $|\Omega_k| \rightarrow |\Omega|$ when $k \rightarrow \infty$, we deduce from Lemma 2.3 the existence of a constant $c(n, p) > 0$ such that

$$\|u_k\|_{L^\infty(\Omega_k)} \geq c(n, p)|\Omega|^{\frac{2}{n+p}}. \tag{15}$$

Let

$$C_{\alpha,k} = \frac{1 + 2[\text{diam } \Omega_k]^2}{\alpha(1 - \alpha)}, \text{ where } \alpha := \frac{2}{n + p}.$$

For each $k \geq 1$ and $z \in \Omega_k$, we claim that

$$|u_k(z)| \leq C_{\alpha,k}[\text{dist}(z, \partial\Omega_k)]^\alpha. \tag{16}$$

Let z be an arbitrary point in Ω_k . By translation and rotation of coordinates, we can assume that: the origin 0 of \mathbb{R}^n lies on $\partial\Omega_k$, the x_n -axis points inward Ω_k , z lies on the x_n -axis, and the minimum distance to the boundary of Ω_k from z is achieved at the origin. Consider

$$v_\alpha = x_n^\alpha(|x'|^2 - C_{\alpha,k}).$$

By Lemma 2.2, $\det D^2 v_\alpha \geq |v_\alpha|^{-p}$. Thus, from $\det D^2 u_k \leq |u_k|^{-p}$, we can apply Lemma 2.1 to deduce that $u_k \geq v_\alpha$ in Ω_k . This implies (16).

Step 2. Convergence of a subsequence of $\{u_k\}$ to the solution of (3). From the convexity of u_k , we find that the functions u_k have uniformly bounded global $C^{0, \frac{2}{n+p}}$ norm on $\overline{\Omega_k}$. By the Arzela-Ascoli theorem, there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, that converges locally uniformly to a convex function u on Ω . The estimate (16) shows that $u = 0$ on $\partial\Omega$. Moreover, by (15), we have

$$\|u\|_{L^\infty(\Omega)} \geq c(n,p)|\Omega|^{\frac{2}{n+p}}.$$

In particular, $|u| > 0$ in Ω . The stability theorem of the Monge-Ampère equation (see [9, Proposition 2.6] and [10, Lemma 1.2.3]) then gives that the function u is actually an Aleksandrov solution of (3). Clearly, $u \in C^\infty(\Omega)$; see, for example [6, Theorem 5] or [14, Proposition 2.8]. The uniqueness of u follows from the comparison principle in Lemma 2.1.

(ii) The same argument as in (16) applied to Ω instead of Ω_k gives

$$|u(x)| \leq C(n,p, \text{diam}(\Omega))[\text{dist}(x, \partial\Omega)]^{\frac{2}{n+p}}. \tag{17}$$

By the convexity of u , we easily obtain $u \in C^{\frac{2}{n+p}}(\overline{\Omega})$.

(iii) Let Ω be a bounded convex domain in \mathbb{R}^n that contains parts of hyperplanes on its boundary. Let $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ be the solution to (3). We show that $u \notin C^\beta(\overline{\Omega})$ for any $\beta > \frac{2}{n+p}$. Indeed, by translating and rotating coordinates, we can assume that for some $t > 0$,

$$\Omega_t := \{(x', x_n) : |x'| < t, 0 < x_n < t^2 - |x'|^2\} \subset \Omega, \quad \text{and } \{(x', 0) : |x'| \leq t\} \subset \partial\Omega.$$

Let w^t be as in Lemma 2.6. Then,

$$\det D^2 w^t \leq |w^t|^{-p} \quad \text{in } \Omega_t.$$

From the convexity of u and $\Omega_t \subset \Omega$, we have $u \leq 0$ on $\partial\Omega_t$. Now, the comparison principle in Lemma 2.1 implies that $w^t \geq u$ in Ω_t so

$$|u(x)| \geq |w^t(x)| = Ct^{\frac{2(1-p)}{n+p}} \left[x_n^{\frac{2}{n+p}} (t^2 - r^2)^{\frac{n+p-2}{n+p}} - x_n \right].$$

For $x = (0, x_n)$, we have then

$$|u(0, x_n)| \geq Ct^{\frac{2(1-p)}{n+p}} \left[t^{\frac{2(n+p-2)}{n+p}} x_n^{\frac{2}{n+p}} - x_n \right] \geq \frac{Ct^{\frac{n-1}{n+p}}}{2} x_n^{\frac{2}{n+p}}$$

if $x_n > 0$ is small, depending only on n, p, t . This estimate show that the exponent $2/(n+p)$ in the upper bound for u in (17) is optimal. \square

Proof of Proposition 1. Let $K := \|u\|_{L^\infty(\Omega)}$. Then, by [14, Lemma 3.1 (iii)], we have the following uniform estimate

$$c(n,q)|\Omega|^{\frac{2}{n-q}} \leq K \leq C(n,q)|\Omega|^{\frac{2}{n-q}}.$$

By [14, Proposition 2.8], we have $u \in C^\infty(\Omega)$. By the convexity of u , the global regularity $u \in C^{0,\beta}(\bar{\Omega})$ for all $\beta \in (0, \frac{2}{n-q})$ follows from the boundary estimate (6). We will prove this estimate by an iterative argument in what follows. The proof is similar to that of [14, Proposition 5.3].

Let z be an arbitrary point in Ω . By translation and rotation of coordinates, we can assume that: the origin 0 of \mathbb{R}^n lies on $\partial\Omega$, the x_n -axis points inward Ω , z lies on the x_n -axis, and the minimum distance to the boundary of Ω from z is achieved at the origin.

By the convexity of u , in order to verify (6) at z , it suffices to prove that for all $x \in \Omega$, we have

$$|u(x)| \leq C(n, \alpha, \text{diam } \Omega)x_n^\beta \text{ for all } \beta \in (0, \frac{2}{n-q}). \tag{18}$$

Step 1. When $\beta = \frac{2}{n}$. In this step, we show

$$|u(x)| \leq C(K, n, \text{diam } \Omega)x_n^{\frac{2}{n}} \text{ for all } x \in \Omega. \tag{19}$$

As in Lemma 2.2, for $\alpha \in (0, 1)$, we consider

$$v_\alpha(x) = x_n^\alpha(|x'|^2 - C_\alpha) \text{ where } C_\alpha = \frac{1 + 2[\text{diam } \Omega]^2}{\alpha(1 - \alpha)}. \tag{20}$$

Then, v_α is convex in Ω with

$$\det D^2v_\alpha(x) \geq 2x_n^{n\alpha-2} \text{ in } \Omega \text{ and } v_\alpha \leq 0 \text{ on } \partial\Omega. \tag{21}$$

Let

$$v := K^{q/n}v_{\frac{2}{n}}.$$

Then, since $\det D^2v_{2/n} \geq 2$, we have

$$\det D^2v = K^q \det D^2v_{\frac{2}{n}} \geq 2K^q > \det D^2u.$$

Note that on $\partial\Omega$, $u = 0 \geq v$. Since u and v are C^2 in Ω , we can use a simple maximum principle argument to show that $u \geq v$ in Ω . Therefore $|u| \leq |v|$ which shows that

$$|u(x)| = |u(x', x_n)| \leq |v(x)| \leq K^{q/n}C_{\frac{2}{n}}x_n^{\frac{2}{n}} \text{ for all } x \in \Omega.$$

Step 2. Iterative argument: We show that, if for some $\beta_k \in (0, \frac{2}{n-q})$ we have

$$|u(x)| \leq C(n, q, \beta_k, \text{diam } \Omega)x_n^{\beta_k} \text{ for all } x \in \Omega, \tag{22}$$

then for all $x \in \Omega$,

$$|u(x)| \leq \bar{C}(n, q, \beta_k, \text{diam } \Omega)x_n^{\beta_{k+1}} \text{ where } \beta_{k+1} := \frac{\beta_k q + 2}{n}. \tag{23}$$

Note that if $\beta_k < \frac{2}{n-q}$ then $\beta_{k+1} < \frac{2}{n-q}$ and

$$\frac{2}{n-q} - \beta_{k+1} = (\frac{2}{n-q} - \beta_k)\frac{q}{n}. \tag{24}$$

Suppose we have (22). Then for $\hat{C} = \hat{C}(n, q, \beta_k, \text{diam } \Omega)$ large, we have

$$|u(x)|^{\frac{q}{n}} \leq [C(n, q, \beta_k, \text{diam } \Omega)]^{\frac{q}{n}} x_n^{\frac{q\beta_k}{n}} < \hat{C}x_n^{\frac{n\beta_{k+1}-2}{n}} \text{ in } \Omega. \tag{25}$$

Denote by $(U^{ij}) = (\det D^2u)(D^2u)^{-1}$ the cofactor matrix of the Hessian matrix $D^2u = (D_{ij}u)$. Then

$$\det U = (\det D^2u)^{n-1} \text{ and } U^{ij}D_{ij}u = n \det D^2u = n|u|^q.$$

Using (21), (25) and the matrix inequality

$$\text{trace}(AB) \geq n(\det A)^{1/n}(\det B)^{1/n} \text{ for } A, B \text{ symmetric } \geq 0,$$

we find that

$$\begin{aligned} U^{ij}D_{ij}(\hat{C}v_{\beta_{k+1}}) &\geq n\hat{C}(\det D^2u)^{\frac{n-1}{n}}(\det D^2v_{\beta_{k+1}})^{\frac{1}{n}} \\ &\geq n\hat{C}|u|^{\frac{q(n-1)}{n}}x_n^{\frac{n\beta_{k+1}-2}{n}} \\ &> n|u|^q = n \det D^2u = U^{ij}D_{ij}u \text{ in } \Omega. \end{aligned} \tag{26}$$

Now, the maximum principle for the operator $U^{ij}D_{ij}$ applied to u and $\hat{C}v_{\beta_{k+1}}$ gives $u \geq \hat{C}v_{\beta_{k+1}}$ in Ω . It follows that

$$|u(x)| = |u|(x', x_n) \leq -\hat{C}v_{\beta_{k+1}}(x', x_n) \leq \hat{C}C_{\beta_{k+1}}x_n^{\beta_{k+1}} \text{ for all } x \in \Omega.$$

This gives (23).

Step 3. Conclusion. From Step 1, we can choose $\beta_0 = \frac{2}{n}$ to initiate Step 2 and obtain a sequence β_k . From (24), we find

$$\frac{2}{n-q} - \beta_k = \left(\frac{2}{n-q} - \beta_0\right) \left(\frac{q}{n}\right)^k = \frac{2q}{n(n-q)} \left(\frac{q}{n}\right)^k.$$

Given $\beta \in (0, \frac{2}{n-q})$, we can find a positive integer k such that

$$\frac{2q}{n(n-q)} \left(\frac{q}{n}\right)^k < \frac{2}{n-q} - \beta.$$

With this k , we have $\beta < \beta_k < \frac{2}{n-q}$. The proposition follows by applying Step 2 k times. □

4. Proofs of Theorem 1.2 and Proposition 2. The outline of the proof of Theorem 1.2 is similar to that of Theorem 1.1. For the global Hölder regularity, we use the following construction of subsolutions which is similar to Lemma 2.2. The main difference here is to take into account the origin being in the interior of the convex domain to improve the Hölder exponent.

Lemma 4.1 (Subsolutions for (7)). *Let Ω be a bounded convex domain such that $x_0 = ((x_0)', -\gamma) \in \partial\Omega$ and $\Omega \subset \{x = (x', x_n) \in \mathbb{R}^n : x_n > -\gamma\}$ where $\gamma \geq \gamma_0 > 0$. Let $k \geq 0$. Then for $a = \frac{2+k}{2n+2k+2} \in (0, 1)$ and $C = C(n, k, \gamma_0, \text{diam}(\Omega))$ large, the following function*

$$v_a(x) = (x_n + \gamma)^a(|x'|^2 - C). \tag{27}$$

is smooth, convex in Ω and satisfies

$$(\det D^2v_a)(x \cdot Dv_a(x) - v_a(x))^k |v_a|^{n+k+2} \geq 1 \text{ on } \Omega, \text{ and } v_a \leq 0 \text{ on } \partial\Omega.$$

Proof. For $a \in (0, 1)$ to be chosen, let $v_a(x) = (x_n + \gamma)^a(r^2 - C)$ where $r = |x'|$ and $C > [\text{diam}(\Omega)]^2$. We calculate as in Lemmas 2.2 and 2.4 that v_a is convex and

$$\begin{aligned} \det D^2v_a &= \left(\frac{v_{a,r}}{r}\right)^{n-2} [v_{a,x_n x_n} v_{a,rr} - v_{a,x_n r}^2] \\ &= 2^{n-1} (x_n + \gamma)^{an-2} [(a - a^2)C - a(1 + a)r^2]. \end{aligned}$$

Moreover, since $\gamma \geq \gamma_0 > 0$,

$$\begin{aligned} x \cdot Dv_a - v_a &= x_n v_{a,x_n} + r v_{a,r} - v \\ &= (x_n + \gamma)^{a-1} [ax_n(r^2 - C) + 2r^2(x_n + \gamma) + (C - r^2)(x_n + \gamma)] \\ &= (x_n + \gamma)^{a-1} \{ (x_n + \gamma)[(1+a)r^2 + (1-a)C] + a\gamma(C - r^2) \} \\ &\geq (x_n + \gamma)^{a-1} a\gamma_0(C - r^2). \end{aligned} \tag{28}$$

Therefore

$$\begin{aligned} &(\det D^2 v_a)(x \cdot Dv_a(x) - v_a(x))^k |v_a|^{n+k+2} \\ &\geq 2^{n-1} (a\gamma_0)^k (x_n + \gamma)^{a(2n+2k+2)-(2+k)} [(a - a^2)C - a(1+a)r^2][C - r^2]^{n+2k+2} \\ &\geq 1 \end{aligned}$$

if $a = \frac{2+k}{2n+2k+2} \in (0, 1)$ and $C = C(n, k, \gamma_0, \text{diam}(\Omega))$ is large. □

For the optimality of the global Hölder exponent of solution to (7), we use the following construction of supersolutions which is similar to Lemma 2.4.

Lemma 4.2 (Supersolutions for (7)). *Assume $k \geq 0$ and $0 < \gamma < 1$. Let*

$$\Omega = \{ (x', x_n) : |x'| < 1, 0 < x_n + \gamma < 1 - |x'|^2 \}.$$

Then there is a positive constant $C_0(n, k, \gamma)$ such that the function

$$w = C_0(x_n + \gamma) - C_0(x_n + \gamma)^{\frac{2+k}{2n+2k+2}} (1 - |x'|^2)^{\frac{2n+k}{2n+2k+2}}$$

is smooth, convex in Ω and satisfies

$$\det D^2 w \leq |w|^{-n-2-k} (x \cdot Dw - w)^{-k} \quad \text{in } \Omega, \quad \text{and } w = 0 \quad \text{on } \partial\Omega.$$

Proof. For $0 < a, b < 1$ with $a + b \leq 1$, and $C > 0$ to be chosen, let

$$v = -C(x_n + \gamma)^a (1 - r^2)^b \quad \text{and } w = C(x_n + \gamma) + v, \quad \text{where } r = |x'|.$$

As in the proof of Lemma 2.4, we know that under these conditions on a and b , v and w are convex in Ω . Moreover, if $a + b = 1$ then $w = 0$ on $\partial\Omega$.

By the convexity of w , we have

$$x \cdot Dw(x) - w(x) \geq 0 \cdot Dw(x) - w(0) = C\gamma^a - C\gamma > 0$$

and

$$x \cdot Dw(x) - w(x) = x \cdot Dv(x) - v(x) - C\gamma \leq x \cdot Dv(x) - v(x).$$

Since $\det D^2 w = \det D^2 v$ and $|w| \leq |v|$ in Ω , in order to obtain the desired properties of w , it suffices to prove that for $a = \frac{2+k}{2n+2k+2}$, $b = 1 - a$, we have for suitable C

$$\det D^2 v \leq |v|^{-n-2-k} (x \cdot Dv - v)^{-k} \quad \text{in } \Omega. \tag{29}$$

We compute

$$\begin{aligned} \det D^2 v &= C^n (2b)^{n-1} (x_n + \gamma)^{na-2} (1 - r^2)^{n(b-1)} a [1 - a + (1 - 2b - a)r^2] \\ &= C^n (2b)^{n-1} (x_n + \gamma)^{na-2} (1 - r^2)^{n(b-1)+1} a(1 - a), \\ x \cdot Dv - v &= C(x_n + \gamma)^{a-1} (1 - r^2)^{b-1} [(x_n + \gamma - ax_n)(1 - r^2) + 2br^2(x_n + \gamma)] \\ &\leq 3C(x_n + \gamma)^{a-1} (1 - r^2)^{b-1}. \end{aligned}$$

It follows that

$$\begin{aligned} & (\det D^2v)|v|^{n+k+2}(x \cdot Dv - v)^k \\ & \leq 3^k(2b)^{n-1}a(1-a)C^{2n+2k+2}(x_n + \gamma)^{na-2+a(n+k+2)+(a-1)k} \\ & \quad \times (1-r^2)^{n(b-1)+1+b(n+k+2)+(b-1)k} \\ & = 3^k(2b)^{n-1}a(1-a)C^{2n+2k+2}(x_n + \gamma)^{a(2n+2k+2)-2-k}(1-r^2)^{b(2n+2k+2)-n-k+1}. \end{aligned}$$

Thus (29) holds for a suitable $C = C_0(n, k, \gamma)$ when $a = \frac{2+k}{2n+2k+2}$ and $b = 1 - a$. \square

Proof of Theorem 1.2. First, we prove (8) for the unique convex solution $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ to (7). Let

$$a = \frac{2+k}{2n+2k+2} \quad \text{and} \quad \gamma_0 = \text{dist}(0, \partial\Omega) > 0.$$

Let z be an arbitrary point in Ω . Let $x_0 \in \partial\Omega$ be such that $|z - x_0| = \text{dist}(z, \partial\Omega)$. Suppose that the supporting hyperplane l_{x_0} to $\partial\Omega$ at x_0 has equation $\vec{n} \cdot (x - x_0) = 0$ and

$$\Omega \subset \{x \in \mathbb{R}^n : \vec{n} \cdot (x - x_0) \geq 0\}.$$

Then $\gamma := -\vec{n} \cdot x_0 = \text{dist}(0, l_{x_0}) \geq \gamma_0$. From Lemma 4.1, we find that for a suitable $C = C(n, k, \text{diam}(\Omega), \gamma_0)$, the function

$$v(x) := [\vec{n} \cdot (x - x_0)]^a (|x - \vec{n} \cdot x|^2 - C)$$

is a subsolution to (7). To see this, we can use a rotation to assume that $\vec{n} = (0, \dots, 0, 1)$ and hence $x_0 = ((x_0)^\prime, -\gamma)$.

Using the comparison principle in Lemma 2.1, we find $u \geq v$ and hence

$$|u(z)| \leq |v(z)| \leq C|z - x_0|^a = C[\text{dist}(z, \partial\Omega)]^a. \tag{30}$$

This holds for all $z \in \Omega$ so (8) is proved. By the convexity of u , we easily obtain $u \in C^a(\bar{\Omega})$.

Finally, we note that the optimality of the exponent a follows from Lemma 4.2 and the comparison principle. Indeed, let Ω , w and C_0 be as in Lemma 4.2. Let $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ be the unique convex solution to (7). Since w is a supersolution to (7), by the comparison principle in Lemma 2.1, we find that $w \geq u$ in Ω . Hence for $x = (0, x_n) \in \Omega$ where $-\gamma < x_n < 1 - \gamma$, we have

$$\begin{aligned} |u(x)| \geq |w(x)| &= C_0[(x_n + \gamma)^a - (x_n + \gamma)] \\ &\geq C_0(x_n + \gamma)^a/2 = C_0[\text{dist}(x, \partial\Omega)]^a/2 \end{aligned} \tag{31}$$

if $x_n + \gamma > 0$ is small. Hence, the exponent a in $u \in C^a(\bar{\Omega})$ is optimal. \square

Proof of Proposition 2. Let $u \in C^\infty(\Omega) \cap C(\bar{\Omega})$ be the unique convex solution to (7) where

$$\Omega = \{(x', x_n) : |x'| < 1, 0 < x_n + \gamma < 1 - |x'|^2\}.$$

Let $C = C(n, k, \gamma, \text{diam}(\Omega))$ and $C_0 = C_0(n, k, \gamma)$ be as in Lemmas 4.1 and 4.2, respectively. Let

$$a = \frac{2+k}{2n+2k+2}.$$

The main technical point of the proof is to obtain a positive lower bound comparable to $[\text{dist}(x, \partial\Omega)]^{a-1}$ for $x \cdot Du - u$; see (34).

Consider $x = (0, x_n) \in \Omega$ with $0 < x_n + \gamma$ is small. Thus $x_n < 0$. As in (30), we have

$$-u(x) = -u(0, x_n) = |u(0, x_n)| \leq C(x_n + \gamma)^a. \tag{32}$$

For $m > 0$ large to be chosen, we find from (31) that as long as $(0, x_n + m(x_n + \gamma)) \in \Omega$

$$u(0, x_n + m(x_n + \gamma)) \leq C_0 \{(m+1)(x_n + \gamma) - [(m+1)(x_n + \gamma)]^a\}.$$

Hence, by the convexity of u , we have

$$\begin{aligned} u_{x_n}(0, x_n) &\leq \frac{u(0, x_n + m(x_n + \gamma)) - u(0, x_n)}{m(x_n + \gamma)} \\ &\leq C_0 \frac{m+1}{m} - (x_n + \gamma)^{a-1} \frac{C_0(m+1)^a - C}{m}. \end{aligned}$$

We first choose m large such that

$$C_0(m+1)^a - C > C_0/2.$$

Then

$$u_{x_n}(0, x_n) < C_0 \frac{m+1}{m} - (x_n + \gamma)^{a-1} \frac{1}{2m}.$$

Now, recalling (31), we choose $0 < x_n + \gamma \leq \bar{\gamma}$ small so that $-\gamma < x_n < -\gamma/2$ and

$$|u(x) \geq C_1(x_n + \gamma)^a, \quad u_{x_n}(0, x_n) < -C_1(x_n + \gamma)^{a-1}, \quad C_1 := \min\left\{\frac{C_0}{2}, \frac{1}{4m}\right\}. \quad (33)$$

With these choices of m and x_n , we have for $x = (0, x_n)$

$$\begin{aligned} x \cdot Du - u > x_n u_{x_n}(0, x_n) &> -C_1 x_n (x_n + \gamma)^{a-1} \\ &> C_2 (x_n + \gamma)^{a-1}, \quad \text{where } C_2 = \frac{C_1 \gamma}{2}. \end{aligned} \quad (34)$$

Using this estimate together with (33) and recalling the definition of a , we find

$$\begin{aligned} f(x) &= |u(x)|^{-n-2-k} (x \cdot Du(x) - u(x))^{-k} \\ &\leq C^{-n-2-k} C_2^{-k} (x_n + \gamma)^{-a(n+2+k)-k(a-1)} \\ &= C_3 (x_n + \gamma)^{\frac{(n-4)k-(2n+4)}{2n+2k+2}} \\ &= C_3 [\text{dist}(x, \partial\Omega)]^{\frac{(n-4)k-(2n+4)}{2n+2k+2}}. \end{aligned}$$

Therefore, we obtain (9), completing the proof of the proposition. \square

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REFERENCES

- [1] K. J. Böröczky, E. Lutwak, D. Yang and G. Zhang, [The logarithmic Minkowski problem](#), *J. Amer. Math. Soc.*, **26** (2013), 831–852.
- [2] L. A. Caffarelli, [A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity](#), *Ann. of Math.*, **131** (1990), 129–134.
- [3] L. Caffarelli, L. Nirenberg and J. Spruck, [The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation](#), *Comm. Pure Appl. Math.*, **37** (1984), 369–402.
- [4] E. Calabi, [Complete affine hyperspheres. I](#). *Symposia Mathematica*, Vol. X (Convegno di Geometria Differenziale, INDAM, Rome, 1971). Academic Press, London, 1972, 19–38.
- [5] H. Chen and G. Huang, [Existence and regularity of the solutions of some singular Monge-Ampère equations](#), *J. Differential Equations*, **267** (2019), 866–878.
- [6] S. Y. Cheng and S. T. Yau, [On the regularity of the Monge-Ampère equation \$\det\(\partial^2 u / \partial x^i \partial x^j\) = F\(x, u\)\$](#) , *Comm. Pure Appl. Math.*, **30** (1977), 41–68.
- [7] S. Y. Cheng and S.-T. Yau, [Complete affine hypersurfaces. I. The completeness of affine metrics](#), *Comm. Pure Appl. Math.*, **39** (1986), 839–866.
- [8] K.-S. Chou and X.-J. Wang, [The Lp-Minkowski problem and the Minkowski problem in centroaffine geometry](#), *Adv. Math.*, **205** (2006), 33–83.

- [9] A. Figalli, *The Monge-Ampère Equation and its Applications*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), 2017.
- [10] C. E. Gutiérrez, *The Monge-Ampère Equation*, Second edition. Birkhäuser, Boston, 2001.
- [11] H. Jian and Y. Li, [Optimal boundary regularity for a singular Monge-Ampère equation](#), *J. Differential Equations*, **264** (2018), 6873–6890.
- [12] H. Jian and Y. Li, [A singular Monge-Ampère equation on unbounded domains](#), *Sci. China Math.*, **61** (2018), 1473–1480.
- [13] H. Jian and X.-J. Wang, [Bernstein theorem and regularity for a class of Monge-Ampère equations](#), *J. Differential Geom.*, **93** (2013), 431–469.
- [14] N. Q. Le, [The eigenvalue problem for the Monge-Ampère operator on general bounded convex domains](#), *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **18** (2018), 1519–1559.
- [15] F. H. Lin and L. Wang, [A class of fully nonlinear elliptic equations with singularity at the boundary](#), *J. Geom. Anal.*, **8** (1998), 583–598.
- [16] P.-L. Lions, [Two remarks on Monge-Ampère equations](#), *Ann. Mat. Pura Appl.*, **142** (1985), 263–275.
- [17] E. Lutwak, [The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem](#), *J. Differential Geom.*, **38** (1993), 131–150.
- [18] K. Tso, [On a real Monge-Ampère functional](#), *Invent. Math.*, **101** (1990), 425–448.

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