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Hadamard-type inequalities for k-positive matrices

Nam Q. Le [1]

Department of Mathematics, Indiana University, 831 E 3rd St, Bloomington, IN 47405, USA

A R T I C L E I N F O

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Dedicated to Professor Duong Minh Duc on the occasion of his 70th birthday

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We establish Hadamard-type inequalities for a class of symmetric matrices called k-positive matrices for which the m-th elementary symmetric functions of their eigenvalues are positive for all $m \leq k$. These matrices arise naturally in the study of k-Hessian equations in Partial Differential Equations. For each k-positive matrix, we show that the sum of its principal minors of size k is not larger than the k-th elementary symmetric function of their diagonal entries. The case $k = n$ corresponds to the classical Hadamard inequality for positive definite matrices. Some consequences are also obtained.

1. Introduction

Let $n \geq 2$ and $1 \leq k \leq n$. We denote the k-th symmetric function of n variables $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ by

$$S_k(\lambda) := \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.$$

E-mail address: nqle@indiana.edu.

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It is convenient to set

$$S_0(\lambda) = 1.$$

Let $\Gamma_k(n)$ be an open symmetric convex cone in $\mathbb{R}^n$, with vertex at the origin, given by

$$\Gamma_k(n) = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \mid S_j(\lambda) > 0 \quad \forall j = 1, \cdots, k\}.$$

The convexity of $\Gamma_k(n)$ is a consequence of Gårding's theory of hyperbolic polynomials; see Example 5.2.

Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices with real entries. If $A = (a_{ij})_{1 \leq i,j \leq n} \in M_n(\mathbb{R})$ is an $n \times n$ symmetric matrix, we use $\lambda(A) = (\lambda_1, \cdots, \lambda_n)$ to denote its eigenvalues. For $A \in M_n(\mathbb{R})$, let $\mathrm{diag}(A)$ be its diagonal matrix:

$$\mathrm{diag}(A) = \mathrm{diag}(a_{11}, \cdots, a_{nn}).$$

Notation. We use the following notation:

$$S_k(A) = S_k(\lambda(A));$$
$$[n] = \{1, \cdots, n\}; \quad J^c = [n] \setminus J \text{ for } J \subset [n].$$

For $J \subset [n]$, we denote by $A[J]$ the principal submatrix of A of size $|J|$ obtained by deleting the ith row and column of A, for each $i \notin J$.

Let $E_k(A)$ be the sum of the principal minors of size k of $A \in M_n(\mathbb{R})$. Then, by [4, Theorem 1.2.16], we have

$$S_k(A) = E_k(A). \tag{1.1}$$

If $A = (a_{ij})_{1 \leq i,j \leq n} \in M_n(\mathbb{R})$ is positive definite, or equivalently, $\lambda(A) \in \Gamma_n(n)$, then Hadamard's determinant inequality (see, for example, [4, Theorem 7.8.1]) gives

$$S_n(\mathrm{diag}(A)) = a_{11} \cdots a_{nn} \geq \det A = S_n(A). \tag{1.2}$$

When $A \in M_n(\mathbb{R})$ is positive definite, and $1 \leq k \leq n$ is fixed, each principal submatrix of size k of A is also positive definite; thus, we can apply the Hadamard inequality to each of these principal submatrices of A and use (1.1) to conclude that

$$S_k(\mathrm{diag}(A)) \geq S_k(A). \tag{1.3}$$

In analogy with the classical Hadamard inequality (1.2), we call (1.3) a Hadamard-type inequality.

In this note, we show that (1.3) holds for a larger class of symmetric matrices, called k-positive.

Definition 1.1 *(k-positive matrices).* Let $1 \le k \le n$. A symmetric $n \times n$ matrix $A \in M_n(\mathbb{R})$ is call k-positive if $\lambda(A) \in \Gamma_k(n)$.

As will be seen in Example 5.3, the set of k-positive matrices is a convex cone. This is again a consequence of Gårding's theory of hyperbolic polynomials.

Note that the class of n-positive matrices is equal to the class of positive definite matrices. The class of k-positive matrices arises naturally in the study of k-Hessian equations

$$S_k(D^2u) = f$$

in Partial Differential Equations where D^2u denotes the Hessian matrix of u; see [5] for a survey.

Due to the following remark, we will focus on the case $k \ge 3$.

Remark 1.2. Let $A = (a_{ij})_{1 \le i,j \le n} \in M_n(\mathbb{R})$ be symmetric.

(i) If $k = 1$, then

$$S_1(A) = \sum_{i=1}^n \lambda_i(A) = \sum_{i=1}^n a_{ii} = S_1(\mathrm{diag}(A)).$$

(ii) If $k = 2$, then

$$\begin{aligned} S_2(A) = E_2(A) &= \sum_{1 \le i < j \le n} a_{ii}a_{jj} - \sum_{1 \le i < j \le n} a_{ij}^2 \\ &= S_2(\mathrm{diag}(A)) - \sum_{1 \le i < j \le n} a_{ij}^2 \le S_2(\mathrm{diag}(A)). \end{aligned}$$

Equality holds if and only if A is diagonal.

Our main result on Hadamard-type inequalities for k-positive matrices states as follows.

Theorem 1.3 *(Hadamard-type inequalities for k-positive matrices). Let $n \ge k \ge 3$. Let $A \in M_n(\mathbb{R})$ be k-positive. Then $S_k(diag(A)) \ge S_k(A)$. Moreover, equality holds if and only if A is diagonal.*

A simple corollary of Theorem 1.3 and Remark 1.2 is the following.

Corollary 1.4. *Let $n \ge k \ge 1$. Let $A = (a_{ij})_{1 \le i,j \le n} \in M_n(\mathbb{R})$ be k-positive. Then $diag(A)$ is k-positive. In other words, $(a_{11}, \cdots, a_{nn}) \in \Gamma_k(n)$. Moreover, $S_k(diag(A)) \ge S_k(A)$.*

For $p \in [n]$ and $\lambda = (\lambda_1, \cdots, \lambda_n) \equiv (\lambda_i)_{1\leq i\leq n} \in \mathbb{R}^n$, let us denote the following point in $\mathbb{R}^{\binom{n}{p}}$:

$$\lambda_{[p]} = \left(\lambda_{i_1} + \cdots + \lambda_{i_p}\right)_{1\leq i_1<\cdots<i_p\leq n}.$$

Note that $\lambda_{[1]} = \lambda$. We now state an interesting consequence of Corollary 1.4.

Theorem 1.5. *Let $A = (a_{ij})_{1\leq i,j\leq n} \in M_n(\mathbb{R})$ be symmetric. Let $p \in [n]$ and $1 \leq k \leq \binom{n}{p}$. If $\lambda(A)_{[p]} \in \Gamma_k(\binom{n}{p})$ then $(a_{11}, \cdots, a_{nn})_{[p]} \in \Gamma_k(\binom{n}{p})$ and $S_k((a_{11}, \cdots, a_{nn})_{[p]}) \geq S_k(\lambda(A)_{[p]})$.*

We deduce from Theorem 1.3 the following result.

Corollary 1.6. *Let $n \geq k \geq 2$. Let $A = (a_{ij})_{1\leq i,j\leq n}$, and $B = (b_{ij})_{1\leq i,j\leq n} \in M_n(\mathbb{R})$ be two k-positive matrices. Then*

$$\sum_{i=1}^{n} b_{ii} S_{k-1}(A[\{i\}^c]) \geq k[S_k(A)]^{\frac{k-1}{k}} [S_k(B)]^{\frac{1}{k}}.$$

The rest of this note is organized as follows. In Section 2, we prove Theorem 1.3. In Section 3, we prove Theorem 1.5. The proof of Corollary 1.6 will be given in Section 4. The final Section 5 relates the main results and concepts of this note with hyperbolic polynomials.

2. Proof of Theorem 1.3

In this section, the entries of $A \in M_n(\mathbb{R})$ will be denoted by a_{ij} so $A = (a_{ij})_{1\leq i,j\leq n}$. We start with the following useful expansion.

Lemma 2.1. *Let $A \in M_n(\mathbb{R})$ be symmetric. If $A[\{n\}^c]$ is diagonal, then for $k \geq 2$, we have*

$$\begin{aligned} S_k(A) &= S_k(diag(A)) - \sum_{i<n} a_{in}^2 \left(\sum_{i_1<\cdots<i_{k-2}\in\{i,n\}^c} a_{i_1 i_1} \cdots a_{i_{k-2} i_{k-2}} \right) \\ &\equiv S_k(diag(A)) - \sum_{i<n} a_{in}^2 S_{k-2}(diag(A[\{i,n\}^c])). \end{aligned}$$

Proof. Recall that $S_k(A)$ is the sum of the principle minors of size k of A. Using the definition of determinant of $k\times k$ matrices together with the fact that $A[\{n\}^c]$ is diagonal, we find

$$\begin{aligned} S_k(A) &= S_k(\mathrm{diag}(A)) \\ &\quad + \sum_{i<n} a_{in}^2 \left(\sum_{i_1<\cdots<i_{k-2}\in\{i,n\}^c} \mathrm{sign}\begin{pmatrix} i & n & i_1 & \cdots & i_{k-2} \\ n & i & i_1 & \cdots & i_{k-2}\end{pmatrix} a_{i_1i_1}\cdots a_{i_{k-2}i_{k-2}} \right) \\ &= S_k(\mathrm{diag}(A)) - \sum_{i<n} a_{in}^2 \left(\sum_{i_1<\cdots<i_{k-2}\in\{i,n\}^c} a_{i_1i_1}\cdots a_{i_{k-2}i_{k-2}} \right). \end{aligned}$$

Here

$$\mathrm{sign}\begin{pmatrix} i & n & i_1 & \cdots & i_{k-2} \\ n & i & i_1 & \cdots & i_{k-2}\end{pmatrix} = -1$$

is the sign of the permutation of k numbers $i, n, i_1, \cdots, i_{k-2}$. $\square$

Our key lemma in the proof of Theorem 1.3 is the following.

Lemma 2.2. *Let $n > k \geq 2$ and let $A \in M_n(\mathbb{R})$ be symmetric. Let $j \in [n]$. Assume that $S_{k-2}(diag(A[\{i,j\}^c])) > 0$ for all $i \neq j$. Then*

$$S_k(A) \leq S_k(A[\{j\}^c]) + a_{jj}S_{k-1}(A[\{j\}^c]).$$

Moreover, the equality holds if and only if $a_{ij} = 0$ for all $i \neq j$.

Proof. We can assume that $j = n$. Then, for all $i < n$, we have

$$S_{k-2}(\mathrm{diag}(A[\{i,j\}^c])) > 0.$$

Case 1. Consider the case $A[\{n\}^c] := (a_{ij})_{1\leq i,j\leq n-1}$ is diagonal. Then, from Lemma 2.1, we have

$$\begin{aligned} S_k(A[\{n\}^c]) + a_{nn}S_{k-1}(A[\{n\}^c]) - S_k(A) &= S_k(\mathrm{diag}(A)) - S_k(A) \\ &= \sum_{i<n} S_{k-2}(\mathrm{diag}(A[\{i,n\}^c]))a_{in}^2 \geq 0. \end{aligned}$$

Moreover, the equality holds if and only if $a_{in} = 0$ for all $i < n$.

Case 2. General case. We can find an orthogonal matrix $U \in O(n-1)$ such that $U^tA[\{n\}^c]U$ is diagonal. Let

$$W = U \bigoplus 1 := \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \in O(n)$$

and $B = (a_{in})_{1\leq i\leq n-1}$. Then

$$W^tAW = \begin{pmatrix} U^tA[\{n\}^c]U & U^tB \\ B^tU & a_{nn} \end{pmatrix}$$

has the form considered in **Case 1**. Note that $S_m(W^tAW) = S_m(A) > 0$ for $1 \leq m \leq k$. Therefore, from **Case 1**, we have

$$\begin{aligned} S_k(A) = S_k(W^tAW) &\leq S_k(U^tA[\{n\}^c]U) + a_{nn}S_{k-1}(U^tA[\{n\}^c]U) \\ &= S_k(A[\{n\}^c]) + a_{nn}S_{k-1}(A[\{n\}^c]). \end{aligned}$$

The equality occurs if and only if $U^tB = 0$, or equivalently, $a_{in} = 0$ for all $i < n$. $\square$

The key assumption in Lemma 2.2 can be deduced, in many cases, from the following result which is a consequence of Sylvestre's criterion established in [5, Theorem 2.1].

Theorem 2.3 *(Theorem 2.1 in [5]). Let $A \in M_n(\mathbb{R})$ be k-positive where $k \geq 2$. Then for all $i \in [n]$, we have that $A[\{i\}^c]$ is $(k-1)$-positive.*

For reader's convenience, we provide a different proof of Theorem 2.3 using Gårding's inequality in Section 4.

We begin the proof of Theorem 1.3 with the case $k = 3$.

Lemma 2.4. *Let $n \geq 4$. Let $A \in M_n(\mathbb{R})$ be 3-positive. Then $S_3(diag(A)) \geq S_3(A)$. Moreover, equality holds if and only if A is diagonal.*

Proof. Fix $j \in [n]$. Since A is 3-positive, we can apply Theorem 2.3 twice to find that if $i \neq j$, then $A[\{i,j\}^c]$ is 1-positive. Thus

$$\sum_{k\in\{i,j\}^c} a_{kk} = S_1(\mathrm{diag}(A[\{i,j\}^c])) > 0. \tag{2.1}$$

From Lemma 2.2, we have

$$S_3(A) \leq S_3(A[\{j\}^c]) + a_{jj}S_2(A[\{j\}^c]).$$

Adding these inequalities, and noting that

$$(n-3)S_3(A) = \sum_{j=1}^n S_3(A[\{j\}^c]),$$

we find

$$3S_3(A) \leq \sum_{i=1}^n a_{ii}S_2(A[\{i\}^c]) = 3S_3(\mathrm{diag}(A)) - \sum_{i=1}^n \left(a_{ii} \sum_{i\neq j\neq k\neq i} a_{jk}^2 \right)$$

$$= 3S_3(\mathrm{diag}(A)) - \sum_{i<j}\left(a_{ij}^2 \sum_{k\in\{i,j\}^c} a_{kk}\right)$$
$$\leq 3S_3(\mathrm{diag}(A))$$

where we used (2.1) in the last inequality. Clearly, equality occurs if and only if $a_{ij} = 0$ for all $i \neq j$ or if A is diagonal. $\square$

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. As remarked in the introduction, we have $S_2(\mathrm{diag}(A)) \geq S_2(A)$ for any symmetric matrix $A \in M_n(\mathbb{R})$ with equality holding if and only if A is diagonal. We only consider the case $k < n$ since the case $k = n$ is the classical Hadamard inequality.

The proof of the theorem is by induction on $k \geq 3$, the base case being Lemma 2.4. Suppose that the theorem is true up to $k \geq 3$. We prove it for $k + 1 < n$.

Assume $A \in M_n(\mathbb{R})$ is $(k+1)$-positive. Then, by Theorem 2.3, $A[\{i\}^c]$ is k-positive for all $i \in [n]$. If $j \neq i$, then again by Theorem 2.3, we have that $A[\{i, j\}^c]$ is $(k-1)$-positive. It follows from the induction hypothesis together with Remark 1.2 that for $1 \leq m \leq k$

$$S_{m-1}(\mathrm{diag}(A[\{i,j\}^c])) \geq S_{m-1}(A[\{i,j\}^c])) > 0. \tag{2.2}$$

For each $j \in [n]$, let $A^{j,0}$ be the matrix obtained from A by replacing all entries in the j-th row and column by 0, except a_{jj} being kept unchanged.

Step 1. We show that $A^{n,0}$ is $(k + 1)$-positive. Indeed, from (2.2), we find that the hypothesis of Lemma 2.2 is satisfied where k there being replaced by $(m + 1)$ here. We can then apply Lemma 2.2 to find that, for $1 \leq m \leq k$, we have

$$S_{m+1}(A^{n,0}) = S_{m+1}(A[\{n\}^c])) + a_{nn}S_m(A[\{n\}^c])) \geq S_{m+1}(A) > 0$$

with equality if and only if $a_{in} = 0$ for all $i < n$. This combined with $S_1(A^{n,0}) = S_1(A) > 0$ shows that $A^{n,0}$ is $(k + 1)$-positive.

Step 2. Next, for each $i \in [n-1]$, we replace the non-diagonal term in the i-th row and column of $A^{n,0}$ by 0, we obtain a new $(k + 1)$-positive matrix with no less S_{k+1} value. Repeating this process, we obtain the conclusion of the theorem for $k + 1$ with equality if and only if A is diagonal. $\square$

3. Proof of Theorem 1.5

In this section, we prove Theorem 1.5. The proof uses ideas from Harvey-Lawson [3] to interpret $\lambda(A)_{[p]}$ as eigenvalues of a suitable matrix associated with A. We recall this formalism.

Let $Sym^2(\mathbb{R}^n)$ be the space of symmetric endomorphisms of $\mathbb{R}^n$. Fix an orthonormal basis $(e_1, \cdots, e_n)$ of $\mathbb{R}^n$. For $p \in [n]$, let $\Lambda^p\mathbb{R}^n$ be the space of p-vectors $v_1 \wedge \cdots \wedge v_p$

where $v_i \in \mathbb{R}^n$ for $1 \leq i \leq p$. The inner product on $\mathbb{R}^n$ induces an inner product on $\Lambda^p\mathbb{R}^n$. Then, an induced orthonormal basis for $\Lambda^p\mathbb{R}^n$ is $\{e_{i_1} \wedge \cdots \wedge e_{i_p}\}$ where $(i_1, \cdots, i_p)$ runs over all increasing p-tuples which are ordered lexicographically.

For each symmetric matrix $A \in M_n(\mathbb{R})$, we can view its as a member of $Sym^2(\mathbb{R}^n)$. We define the linear derivation $\mathcal{D}_A$ of A on $\Lambda^p\mathbb{R}^n$ by assigning each p-vector $v_1 \wedge \cdots \wedge v_p \in \Lambda^p\mathbb{R}^n$ another p-vector

$$\mathcal{D}_A(v_1 \wedge \cdots \wedge v_p) = Av_1 \wedge \cdots \wedge v_p + v_1 \wedge Av_2 \wedge \cdots \wedge v_p + \cdots + v_1 \wedge \cdots \wedge Av_p \in \Lambda^p\mathbb{R}^n.$$

Clearly, $\mathcal{D}_A \in Sym^2(\Lambda^p\mathbb{R}^n)$, and $\mathcal{D}_A$ has a matrix representation with respect to the induced basis $\{e_{i_1} \wedge \cdots \wedge e_{i_p}\}$ with matrix entries being linear combinations of the entries of A. Moreover,

$$\operatorname{diag}(\mathcal{D}_A) = \mathcal{D}_{\operatorname{diag}(A)} = \operatorname{diag}\left(a_{i_1 i_1} + \cdots + a_{i_p i_p}\right)_{1 \leq i_1 < \cdots < i_p \leq n}. \tag{3.1}$$

In [3, Lemma 2.5], Harvey and Lawson showed that if A has eigenvalues $\lambda(A) = (\lambda_1, \cdots, \lambda_n)$ with corresponding eigenvectors $(v_1, \cdots v_n)$, then $\mathcal{D}_A$ has eigenvalues

$$\{\lambda_{i_1} + \cdots + \lambda_{i_p} : 1 \leq i_1 < \cdots < i_p \leq n\},$$

with corresponding eigenvectors

$$\{v_{i_1} \wedge \cdots \wedge v_{i_p} : 1 \leq i_1 < \cdots < i_p \leq n\}.$$

Thus, in our notation,

$$\lambda(\mathcal{D}_A) = \lambda(A)_{[p]} \quad \text{and } S_k(\lambda(A)_{[p]}) = S_k(\mathcal{D}_A). \tag{3.2}$$

Proof of Theorem 1.5. We use the above setup and notation. If $\lambda(A)_{[p]} \in \Gamma_k(\binom{n}{p})$, then $\lambda(\mathcal{D}_A) \in \Gamma_k\left(\binom{n}{p}\right)$. By Corollary 1.4, we then have $\operatorname{diag}(\mathcal{D}_A) \in \Gamma_k\left(\binom{n}{p}\right)$ and

$$S_k(\operatorname{diag}(\mathcal{D}_A)) \geq S_k(\mathcal{D}_A).$$

In view of (3.1) and (3.2), we obtain the conclusion of the theorem. □

4. Proofs of Theorem 2.3 and Corollary 1.6 via Gårding's inequality

In the proofs of Theorem 2.3 and Corollary 1.6, we will use the following form of Gårding's inequality [1].

Lemma 4.1 *(Gårding's inequality). Suppose that $A = (a_{ij})_{1 \leq i,j \leq n}$, and $D = (d_{ij})_{1 \leq i,j \leq n} \in M_n(\mathbb{R})$ are two k-positive matrices. Then*

$$\sum_{i,j=1}^{n} d_{ij} S_k^{ij}(A) \geq k[S_k(A)]^{\frac{k-1}{k}}[S_k(D)]^{\frac{1}{k}} \text{ where } S_k^{ij}(A) = \frac{\partial}{\partial a_{ij}} S_k(A). \tag{4.1}$$

Lemma 4.1 follows from the polarization inequality in [1, Theorem 5] for the polynomial $S_k(A)$; see also [6, inequality (3.2)] for a related version when A and D are Hessian matrices of two real-valued functions. Note that

$$S_k^{ii}(A) = S_{k-1}(A[\{i\}^c]). \tag{4.2}$$

Proof of Theorem 2.3 using Gårding's inequality. If $A \in M_n(\mathbb{R})$ is k-positive then A is m-positive for all $1 \leq m \leq k$. Thus, by an induction argument, it suffices to prove that if $A \in M_n(\mathbb{R})$ is k-positive then $S_{k-1}(A[\{i\}^c]) > 0$ for all $i \in [n]$.

Indeed, if $\delta_i > 0$, then $D = \operatorname{diag}(\delta_1, \cdots, \delta_n)$ is k-positive, and we deduce from (4.1)

$$\sum_{i=1}^{n} \delta_i S_{k-1}(A[\{i\}^c])) \geq k[S_k(A)]^{\frac{k-1}{k}}[S_k(D)]^{\frac{1}{k}}. \tag{4.3}$$

For a fixed $i \in [n]$, letting $\delta_i = 1$ and $\delta_j \to 0$ for $j \neq i$ in (4.3), we discover

$$S_{k-1}(A[\{i\}^c]) \geq 0.$$

It remains to prove that $S_{k-1}(A[\{i\}^c])) \neq 0$ for all $i \in [n]$. Assume that $S_{k-1}(A[\{1\}^c])) = 0$. In this case, consider $\delta_1 = 1$, $\delta_i = \varepsilon > 0$ for $i = 2, \cdots, k$ and $\delta_i = 0$, otherwise. Then D is still k-positive since $S_m(D) \geq \varepsilon^{m-1} > 0$ for all $1 \leq m \leq k$. Now, (4.3) and the assumption $S_{k-1}(A[\{1\}^c])) = 0$ give

$$\varepsilon \sum_{i=2}^{k} S_{k-1}(A[\{i\}^c]) \geq k[S_k(A)]^{\frac{k-1}{k}} \varepsilon^{\frac{k-1}{k}}. \tag{4.4}$$

Since $S_k(A) > 0$, by dividing both sides of (4.4) by ε and letting $\varepsilon \to 0^+$, we obtain

$$\sum_{i=2}^{k} S_{k-1}(A[\{i\}^c]) = \infty,$$

a contradiction. □

Proof of Corollary 1.6. Suppose $A, B \in M_n(\mathbb{R})$ are k-positive. By Corollary 1.4, $D := \operatorname{diag}(B)$ is k-positive. Applying (4.1) to A and $D = \operatorname{diag}(B)$, and recalling (4.2), we find

$$\sum_{i=1}^{n} b_{ii} S_{k-1}(A[\{i\}^c])) \geq k[S_k(A)]^{\frac{k-1}{k}}[S_k(\operatorname{diag}(B))]^{\frac{1}{k}} \geq k[S_k(A)]^{\frac{k-1}{k}}[S_k(B)]^{\frac{1}{k}}$$

where we used Theorem 1.3 in the last inequality. □

5. Hyperbolic polynomials and a conjectural inequality

In this section, we state a generalization of Theorem 1.3 for hyperbolic polynomials. Using the theory of hyperbolic polynomials, we prove the convexity of $\Gamma_k(n)$ and the convexity of the set of k-positive matrices.

First, we recall the concept of hyperbolic polynomials [1] (see also [2] for a self-contained account of Gårding's theory).

Let P be a homogeneous real polynomial of degree k on $\mathbb{R}^n$. Given $a \in \mathbb{R}^n$, we say that P is *a-hyperbolic* if $P(a) > 0$, and for each $x \in \mathbb{R}^n$, $P(ta + x)$ can be factored as

$$P(ta + x) = P(a) \prod_{i=1}^{k} (t + \lambda_i(P; a, x)) \quad \text{for all } t \in \mathbb{R}$$

where $\lambda_i(P; a, x)$'s $(i = 1, \cdots, k)$ are real numbers, called *a-eigenvalues of x*.

We recall the following fundamental theorem of hyperbolic polynomials; see [1, Theorem 2].

Theorem 5.1 *(Gårding). Let P be a homogeneous real polynomial of degree k on $\mathbb{R}^n$. Assume that P is a-hyperbolic. Denote the Gårding cone of P at a to be the set*

$$\Gamma_a(P) = \{x \in \mathbb{R}^n : \lambda_i(P; a, x) > 0 \textit{ for all } i = 1, \cdots, k\}.$$

Then the following hold:

(i) *If $b \in \Gamma_a(P)$, then P is b-hyperbolic and $\Gamma_a(P) = \Gamma_b(P)$.*
(ii) *$\Gamma_a(P)$ is convex.*

A self-contained proof of this theorem of Gårding can also be found in [2] which consists of Theorems 3.6 and 5.1 there.

Suppose now P is a-hyperbolic. By Gårding's theorem, we can define the Gårding cone of P to be

$$\Gamma(P) = \{x \in \mathbb{R}^n : \lambda_i(P; a, x) > 0 \text{ for all } i = 1, \cdots, k\},$$

and $\Gamma(P)$ is independent of a.

Example 5.2 *(Gårding cone and $\Gamma_k(n)$).* The k-th elementary symmetric function $S_k(\lambda)$ is a homogeneous real polynomial of degree k on $\mathbb{R}^n$ and it is λ-hyperbolic at any $\lambda \in \Gamma_k(n)$. Moreover,

$$\Gamma(S_k) = \Gamma_k(n).$$

From the convexity of $\Gamma(S_k)$ due to Gårding's theorem, we deduce the convexity of $\Gamma_k(n)$ from the above equality.

Proof of the statements in Example 5.2. By Example 2, p. 959 in [1], we know that S_k is a-hyperbolic where $a = (1, \cdots, 1) \in \mathbb{R}^n$. Thus, for any $x \in \mathbb{R}^n$, we have from the definition of a-hyperbolicity that the a-eigenvalues $\lambda_i(S_k; a, x)$ are real numbers, for all $i = 1, \cdots, k$.

Assume $x \in \Gamma_k(n)$. Then, from $\lambda_i(S_k; a, x) \in \mathbb{R}$,

$$S_k(ta + x) = \sum_{i=0}^{k} \binom{n-i}{k-i} t^{k-i} S_i(x) = S_k(a) \prod_{i=1}^{k} (t + \lambda_i(S_k; a, x))$$

and $S_i(x) > 0$ for all $i = 0, 1, \cdots, k$, we easily find that $\lambda_i(S_k; a, x) > 0$ for all $i = 1, \cdots, k$. Hence $x \in \Gamma(S_k)$ from which we deduce that $\Gamma_k(n) \subset \Gamma(S_k)$, and S_k is x-hyperbolic by Gårding's theorem. Recall that we use $\Gamma(S_k)$ to denote the Gårding cone of S_k at $a = (1, \cdots, 1)$.

Now, assume $x \in \Gamma(S_k)$. Then, by the definition of $\Gamma(S_k) = \Gamma_a(S_k)$, we have $\lambda_i(S_k; a, x) > 0$ for all $i = 1, \cdots, k$. Therefore, from the above expansion of $S_k(ta + x)$, we obtain $S_i(x) > 0$ for all $i = 1, \cdots, k$ which shows that $x \in \Gamma_k(n)$, or $\Gamma(S_k) \subset \Gamma_k(n)$.

Thus, we have $\Gamma(S_k) = \Gamma_k(n)$. $\square$

A different proof of the convexity of $\Gamma_k(n)$ can be found in Section 2 of [9].

Example 5.2 shows that k-positive matrices are those having eigenvalues lying in the Gårding cone of S_k.

Example 5.3 *(Gårding cone and the set of k-positive matrices).* Let $N = \frac{1}{2}n(n+1)$ and let $A = (a_{ij})_{1 \leq i,j \leq n} \in M_n(\mathbb{R})$ be symmetric. We can view A as a point in $\mathbb{R}^N$. Then $P(A) = \det A$ is A-hyperbolic for any positive definite matrix A. Let I_n be the identity $n \times n$ matrix. Define P_k by

$$\det(tI_n + A) = P(tI_n + A) = \sum_{k=0}^{n} t^{n-k} P_k(A) \quad \text{for all } t \in \mathbb{R}.$$

Then P_k is a homogeneous polynomial of degree k on $\mathbb{R}^N$; moreover, P_k is I_n-hyperbolic. This follows from Example 3 and the discussion at the end of p. 959 in [1].

Note that

$$P_k(A) = S_k(\lambda(A)).$$

Arguing as in the proof of statements in Example 5.2, we have

$$\Gamma(P_k) = \{A \in M_n(\mathbb{R}) : \lambda(A) \in \Gamma_k(n)\};$$

See also equation (2.10) in [6]. From the convexity of $\Gamma(P_k)$ due to Gårding's theorem, we deduce from the above equality the convexity of the set of k-positive matrices.

Example 5.4. In many geometric problems (see, for example [3,7,8]), the Hessian equation operators S_k are replaced by other hyperbolic polynomials P. One example is

$$\mathcal{P}_p(\lambda) = \prod_{1\leq i_1<\cdots<i_p\leq n} (\lambda_{i_1} + \cdots + \lambda_{i_p}), \text{for } \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n.$$

Note that $\mathcal{P}_1 = S_n$ while $\mathcal{P}_n = S_1$. Moreover, $\mathcal{P}_{n-1}(\lambda(A)) = \det(S_1(A)I_n - A)$.

We note that the statement of Theorem 1.5, without any appeal to hyperbolic polynomials, is modeled on the hyperbolic polynomial $\mathcal{P}_p$ in Example 5.4.

It is of interest to study matrices whose eigenvalues lying in the Gårding cone of a hyperbolic polynomial P other than S_k and $\mathcal{P}_p$. In this regard, we state the following generalization of Theorem 1.3.

Conjecture 5.5 *(Hadamard-type inequalities for hyperbolic polynomials). Let P be a homogeneous, real, symmetric, hyperbolic polynomial of degree k on $\mathbb{R}^n$. Let $A \in M_n(\mathbb{R})$. If $\lambda(A) \in \Gamma(P)$ then $(a_{11}, \cdots, a_{nn}) \in \Gamma(P)$ and*

$$P(a_{11}, \cdots, a_{nn}) \geq P(\lambda(A)).$$

Declaration of competing interest

There is no competing interest.

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