Linear Algebra and its Applications 645 (2022) 194–228

Contents lists available at ScienceDirect

Linear Algebra and its Applications

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Maximum determinant and permanent of sparse 0-1 matrices

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ARTICLE INFO

Article history:
Received 3 November 2020
Accepted 23 March 2022
Available online 29 March 2022
Submitted by R. Brualdi

MSC:
15A15
05B20

Keywords:
0,1 matrices
Sparse matrices
Determinant
Permanent

ABSTRACT

We prove that the maximum determinant of an $n \times n$ matrix, with entries in $\{0, 1\}$ and at most $n + k$ non-zero entries, is at most $2^{k/3}$, which is best possible when k is a multiple of 3. This result solves a conjecture of Bruhn and Rautenbach. We also obtain an upper bound on the number of perfect matchings in C_4-free bipartite graphs based on the number of edges, which, in the sparse case, improves on the classical Bregman's inequality for permanents. This bound is tight, as equality is achieved by the graph formed by vertex disjoint union of 6-vertex cycles.

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1 Research supported by NSF RTG Grant DMS-1937241, NSF Grant DMS-1764123, Arnold O. Beckman Research Award (UIUC Campus Research Board RB 18132), the Langan Scholar Fund (UIUC), and the Simons Fellowship.

2 Member of Illinois Combinatorics Lab for Undergraduate Experiences.

https://doi.org/10.1016/j.laa.2022.03.020
0024-3795/

1. Introduction

Many upper bounds on determinants have been given in the literature, in particular to matrices with all entries 0 or 1. Assuming the matrix $A \in \{0,1\}^{n\times n}$ has at most $2n$ non-zero entries, then the classical Hadamard's inequality [4] $\det(A) \le \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{i,j}^2\right)^{1/2}$, together with arithmetic and geometric mean inequality, imply $\det(A) \le 2^{n/2}$. This bound was improved to $\det(A) \le 2\left(2 - \frac{2}{n-1}\right)^{\frac{1}{2}(n-1)}$ by Ryser [5], who extended it as follows:

$$\det(A) \le k\left(k - \frac{k(k-1)}{n-1}\right)^{\frac{1}{2}(n-1)} \text{ when } A \in \{0,1\}^{n\times n} \text{ has } kn \text{ ones, for } 1 \le k \le \frac{n+1}{2}.$$

While both classical bounds above are best-possible in their general formulation, they are not best possible for sparse matrices. Ryser's inequality holds with equality only when we have at least $n\sqrt{n}$ non-zero entries (see [3] or [5] for a more detailed discussion on when equality holds for Ryser's inequality). Aiming to obtain better bounds for sparse combinatorial matrices, Bruhn and Rautenbach [3] proved the following.

Theorem 1 *(Bruhn, Rautenbach). If $A \in \{0,1\}^{n\times n}$ has at most $2n$ non-zero entries, then $|det(A)| \le 2^{n/6} \cdot 3^{n/6}$.*

They also conjectured that the determinant of A is at most $2^{n/3}$ in this case (see Conjecture 4 in [3]). Advancing towards this conjecture, Shitov [7] generalized its formulation and used induction to give a short and elegant proof of the following result.

Theorem 2 *(Shitov). If $A \in \{0,1\}^{n\times n}$ has at most $n+k$ non-zero entries, then $|det(A)| \le 3^{k/4}$.*

The conjectured maximum value comes from matrices of the form $A = \mathrm{diag}(C, \ldots, C)$, where $C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, which has determinant of $2^{n/3}$ with $n = k$. The main contribution of the present paper is a proof of the optimal result.

Theorem 3. *Every matrix $A \in \{0,1\}^{n\times n}$ containing at most $n+k$ non-zero entries has determinant at most α^k, where $\alpha = 2^{1/3}$.*

In particular, this bound is best possible only when k is a multiple of 3, and $k \le n$. We emphasize that Theorem 3 resolves the above conjecture of Bruhn and Rautenbach.

Corollary 4. *If $A \in \{0,1\}^{n\times n}$ has at most $2n$ non-zero entries, then $|det(A)| \le 2^{n/3}$.*

We can consider the same question for more 1's. Bruhn and Rautenbach [3] noted that the point-line incidence matrix of the Fano plane has determinant 24. It gives a lower bound of $24^{n/7} \approx 1.5746^n$ for the maximum determinant of matrices with at most $3n$ ones, the authors of [3] conjecture this to be the best possible. Scheinerman [6] showed $\det(A) \leq c(k)^n$ for some constant $c(k)$ depending only on the integer k for all matrices A with at most kn ones. For $k = 3$, Scheinerman bound is $\det(A) \leq 24^{n/6} \approx 1.6984^n$. Theorem 3 improves this bound.

Corollary 5. *If $A \in \{0,1\}^{n\times n}$ has at most $3n$ non-zero entries, then $|det(A)| \leq 2^{2n/3} \approx 1.5874^n$.*

Our proof of Theorem 3 extends some of the ideas of Shitov [7]. With a careful analysis, we push down the bound to the optimal value of $2^{k/3}$. We identify a matrix A with the graph whose bi-adjacency matrix is A. In other words, when G is a balanced bipartite graph, the determinant of G, denoted by $\det(G)$, is the absolute value of the determinant of the bi-adjacency matrix of G. We then aim to prove $\det(G) \leq 2^{k/3}$ for all balanced bipartite graphs G with $2n$ vertices and at most $n + k$ edges. The proof will be by induction on $n + k$: given a graph G, we will assume the inequality $\det(G') \leq \alpha^{e(G')-v(G')/2}$ for all proper balanced bipartite subgraphs G' of G, where $v(G')$ and $e(G')$ denote the number of vertices and edges of G', respectively. A bipartite graph is *balanced* if both parts have the same number of vertices.

For the sake of completeness of the induction argument, we highlight that the result holds for every $n > 0$ when $k = 0$. For bigger values of n and k, we break the proof into several cases.

Since in most cases we only make use of linearity and cofactor expansion along lines, we get as a byproduct an upper bound for permanents instead of only determinants. In what follows, $\mathrm{perm}(G)$ stands for the permanent of the bi-adjacency matrix of G or, equivalently, the number of perfect matchings in G.

We highlight that the first part of the next result might be of independent interest. Namely, that $\mathrm{perm}(G) \leq 2^{k/3}$ for all C_4-free bipartite graphs G with $2n$ vertices and $n+k$ edges, we state it in Theorem 6. We have additional stronger statements for connected graphs with a minimum degree of at most 2 that makes the proof of the induction step simpler. This stronger statement tells us that the inequality holds with an extra multiplicative factor whenever we avoid components inducing K_2 or C_6. To get an even stronger result and make our proof work, we rule out one further graph, namely the graph J (Fig. 1).

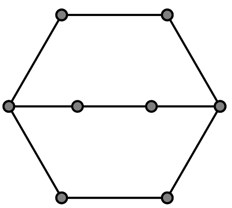


Fig. 1. The graph J.

From now on, we use J only to refer to that graph. For the sake of brevity, we write $\alpha = 2^{1/3}$.

Theorem 6.

(a) *Let G be a C_4-free balanced bipartite graph with $2n$ vertices, $n + k$ edges. Then* $\mathrm{perm}(G) \le \alpha^k$.
(b) *Let H be a connected C_4-free balanced bipartite graph with $2n$ vertices, $n+k$ edges, $\delta(H) \le 2$, and $\Delta(H) \le 3$. Assume further that H is not isomorphic to K_2, C_6, or J. Then* $\mathrm{perm}(H) \le c_1 \cdot \alpha^k$, *where $c_1 = \alpha^{-2} + \alpha^{-7} \le (3\alpha^{-4})^{-1}$ is a constant.*
(c) *Let H be a connected C_4-free balanced bipartite graph with $2n$ vertices, $n+k$ edges, and $\delta(H) \le 2$. Assume further that H is not isomorphic to K_2, C_6, or J. Then* $\mathrm{perm}(H) \le c_2 \cdot \alpha^k$, *where $c_2 = \alpha^{-3} + \alpha^{-4} \le (\alpha^{-1} + \alpha^{-5})^{-1}$ is a constant.*

Remark. We note that the values $c_1 = \alpha^{-2} + \alpha^{-7} \approx 0.828$ and $c_2 = \alpha^{-3} + \alpha^{-4} \approx 0.897$ that our proof gives in the above statement are not tight. Furthermore, parts (b) and (c) of Theorem 6 with $c_1 = (3\alpha^{-4})^{-1} \approx 0.840$ and $c_2 = (\alpha^{-1} + \alpha^{-5})^{-1} \approx 0.902$ would be sufficient to conclude part (a).

For sparse graphs, Theorem 6 improves on the classic Bregman's inequality [2] for permanents. For a C_4-free d-regular graph Theorem 6 gives an upper bound of $2^{(d-1)n/3}$, which is smaller than the bound $(d!)^{n/d}$ from Bregman's inequality when $d \le 5$.

We prove Theorem 6 in the Sections 2–4. In Section 2, we introduce notation and give a high-level overview for the proof of Theorem 6, explaining the organization of the proof. In Section 3, we show the induction step in order to prove items (b) and (c) of Theorem 6. In Section 4, we deal with the induction step for part (a) of Theorem 6. The proof of Theorem 3 is analogous, with the small difference that we have to take into consideration when the graph G has C_4 as a subgraph, where we use linearity or the fact the determinant is preserved after subtracting a line from another line instead of using cofactor expansion. As the majority of the proofs are similar, we only present a sketch proof for Theorem 3 in Section 5 with handling the additional cases where the proof of Theorem 6 has to be supplemented.

2. Proof of Theorem 6: overview

Throughout the proof we identify a matrix A with the graph whose bi-adjacency matrix is A. Therefore, we label the lines of a matrix by the vertex set of a graph. Let G be a balanced bipartite graph. The cofactor expansion along a vertex u adjacent to $v_1, \ldots, v_t$ for permanents implies

$$\mathrm{perm}(G) = \sum_{i=1}^{t} \mathrm{perm}(G - \{u, v_i\}),$$

where $G - \{u, v_i\}$ is the graph obtained after deleting the vertices u and v_i from G.

We introduce an auxiliary function

$$f(G) := \alpha^{-e(G)+\frac{1}{2}v(G)} \cdot \text{perm}(G),$$

where $e(G)$ and $v(G)$ are the number of the edges and vertices of G, respectively. In this notation, the cofactor expansion implies

$$f(G) = \sum_{i=1}^{t} \alpha^{2-d(u)-d(v_i)} \cdot f(G - \{u, v_i\}). \tag{1}$$

We can think of $f(G)$ as the normalized number of perfect matchings of G. Hence, we want to prove that $f(G) \leq 1$ for all C_4-free balanced bipartite graphs G, and $f(H) \leq c_1$ for connected graphs H with minimum degree at most 2 and maximum degree at most 3 that are not isomorphic to K_2, C_6 or J, and $f(H) \leq c_2$ for connected graphs H with minimum degree at most 2 that are not isomorphic to K_2, C_6 or J.

We will prove these statements with a simultaneous induction. In Section 3, we present the proofs of (b) and (c) for the graph H, assuming the results of (a), (b), and (c) hold for all proper balanced subgraphs of H. For the proof of (b), since H is connected, we do not have isolated vertices. If $H \neq K_2$ has vertex with degree 1, we expand the permanent along this vertex and conclude that $f(H) \leq \alpha^{-1} < c_1$. We thus assume $\delta(H) = 2$ and $\Delta(H) \leq 3$. If H has a path with three consecutive vertices of degree 2, then we either proceed as in Claim 8 to obtain $f(H) \leq \alpha^{-3} + \alpha^{-5} < c_1$ or H has to be a cycle, in which case we are done since $H \neq C_4, C_6$ and $f(C_{2n}) < c_1$ for $n \geq 4$.

If H has a vertex x of degree 2 adjacent to vertices y_1 and y_2 of degree 3, then we proceed as in Claim 9 to obtain

$$f(H) \leq \frac{1}{2} f(H - \{x, y_1\}) + \frac{1}{2} f(H - \{x, y_2\}). \tag{2}$$

Assuming $H - \{x, y_i\}$ is connected and not isomorphic to K_2, C_6, or J, the bound $f(H) \leq c_1$ follows from $f(H - \{x, y_i\}) \leq c_1$ for $i = 1, 2$. Otherwise, we will see, in Claim 11, that $H - \{x, y_i\}$ cannot be isomorphic to any of K_2, C_6, J; and, in Claim 12, that if $H - \{x, y_i\}$ is not connected, then $f(H) \leq c_1$.

If there is no path with three consecutive vertices of degree 2 in H and H has no vertex of degree 2 connected to two vertices of degree 3, then any vertex x of degree 2 in H is as in Fig. 3. We then proceed as in Claim 10 to obtain

$$f(H) \leq \alpha^{-2} f(H - \{x, y_1\}) + \alpha^{-5} f(H - \{x, x_1, y_1, y_2\}). \tag{3}$$

We will see, in Claims 13 through 16, that $H - \{x, y_1\}$ and $H - \{x, x_1, y_1, y_2\}$ cannot be isomorphic to K_2, C_6, or J; and that if $H - \{x, y_1\}$ or $H - \{x, x_1, y_1, y_2\}$ are not connected, then $f(H) \leq c_1$. Otherwise, we use the induction hypothesis to conclude $f(H) \leq c_1$ from $f(H - \{x, y_1\}) \leq c_1$ and $f(H - \{x, x_1, y_1, y_2\}) \leq c_1$.

Definition 7. We call $x \in V(H)$ a **Type I** vertex if x is of degree 2 and its neighbors are of degree 3 (Fig. 2). We call $x \in V(H)$ a **Type II** vertex if x is as in Fig. 3, i.e., x is adjacent to y_1 of degree 2 and y_2 of degree 3, while y_1 is also adjacent to a vertex x_1 of degree 3. Therefore, when we say "*Type I* deletion" or "*Type I* expansion" we mean expanding the permanent along a Type I vertex as in (2) above, and similarly for "Type II deletion" or "*Type II* expansion" as in (3). Further, whenever we fix a Type I vertex x, the variables y_1 and y_2 stand for the neighbors of x. For a Type II vertex x, the variables y_1 and y_2 stand for the neighbors of x, and $x_1 \neq x$ is the only other neighbor of y_1.

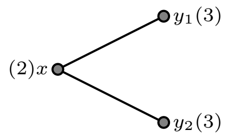


Fig. 2. *Type I* vertex.

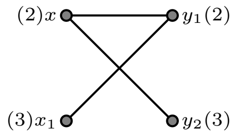


Fig. 3. *Type II* vertex.

We can think of Definition 7 as a way to classify degree 2 vertices in H. Each vertex of degree 2 (that is not contained in a path with three consecutive vertices of degree 2) in H is either *Type I* or *Type II*. Depending on whether there is a Type I vertex in H or all vertices of degree 2 are Type II vertices we will use the expansion (2) or (3), respectively. For the proof of (c), we proceed similarly to the one for (b), we deal with the proof of (c) in Section 3.3.

We prove (a) in Section 4. First, in Section 4.1, we use (b) and (c) to reduce the proof of (a) to fewer cases. Next, we deal with the cases when G is disconnected, or $\Delta(G) \geq 6$, or $\delta(G) \geq 4$ in Subsections 4.2, 4.3, and 4.4, by using (a) for some proper subgraphs of G.

After that, we can assume G is connected, $\delta(G) = 3$ and $\Delta(G) \leq 5$. We thus use (a) and (b) for proper subgraphs of G to deal with the case $\Delta(G) = 3$ in Section 4.5, and use (a) and (c) for proper subgraphs of G to deal with the cases $\Delta(G) = 4$ and $\Delta(G) = 5$ in Sections 4.6 and 4.7, respectively. In all cases, we use cofactor expansion or linearity to bound the permanent by a sum of permanents of subgraphs with minimum degree 2 in a way that the induction hypothesis implies $f(G) \leq 1$.

For example, when G is 3-regular, the cofactor expansion along a vertex u adjacent to v_1, v_2 and v_3 is equivalent to

$$f(G) = \alpha^{-4} \cdot \left(\sum_{i=1}^{3} f(G - \{u, v_i\}) \right).$$

Our goal is to use the induction hypothesis of part (b) for $H = G - \{u, v_i\}$ to get $f(H) \leq c_1$, therefore concluding

$$f(G) = \alpha^{-4} \cdot \left(\sum_{i=1}^{3} f(G - \{u, v_i\}) \right) \leq \alpha^{-4} \cdot \left(\sum_{i=1}^{3} \frac{\alpha^4}{3} \right) = 1.$$

However, we can use part (b) only when H is connected and $H \neq K_2, C_6, J$. We will see, in Claim 20, that $G - \{u, v_i\}$ cannot be isomorphic to K_2, C_6 or J. Finally, in Claim 21, that $f(G) \leq 1$ when $G - \{u, v_i\}$ is not connected follows from (a) for proper subgraphs of G.

3. Proofs of Theorem 6 (b) and (c)

The proofs are by induction on $n + k$. When $n \leq 1$ or $k = 0$, there is no connected bipartite graph $H \neq K_2$. For $n = 2$, the only connected C_4-free graph is a path with $n + k = 3$ edges, which has permanent $1 \leq c_1\alpha < c_2\alpha$. Then the induction hypothesis for both (b) and (c) is true when $n + k \leq 3$.

Further, as H is connected, it has no isolated vertices. If $H \neq K_2$ has a vertex v of degree 1, then the neighbor w of v has degree at least 2. Expanding on the line of vertex v, we get $\text{perm}(H) = \text{perm}(H - \{v, w\})$ and $f(H) \leq \alpha^{-1} \leq c_1$. If H is a cycle, we have $\text{perm}(C_{2n}) = 2$ and then $f(C_{2n}) = 2\alpha^{-n} \leq \alpha^{-1}$ since $H \neq C_4, C_6$ and $n \geq 4$. From now we can assume that the minimum degree of H is 2 and the maximum degree is at least 3.

If H is not a cycle and has a path with three consecutive vertices of degree 2, then it is sufficient to use part (a) of the induction hypothesis to obtain

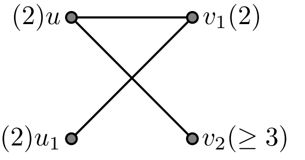
$$f(H) \leq \alpha^{-3} + \alpha^{-5} < 0.815 < 0.828 < c_1,$$

using the following Claim.

Claim 8. *Let u and v_1 be adjacent vertices with degree 2. Further, assume that u is adjacent to v_2, v_1 is adjacent to u_1, and v_2 is not adjacent to u_1. If $d(u_1) = 2$ and $d(v_2) \geq 3$, then* $\text{perm}(H) \leq \alpha^{k-3} + \alpha^{k-5} \leq c_1 \cdot \alpha^k$.

Proof. Assume u_1 is adjacent to a vertex v_3, and v_3 is adjacent to a vertex u_2 ($\neq u, u_1$). Then the bi-adjacency matrix of H is

$$A = \begin{array}{c} \\ u \\ u_1 \\ u_2 \\ \vdots \\ \\ \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & \dots & & \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & x & 1 & a_{34} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & a_{n3} & a_{n4} & \dots & a_{nn} \end{pmatrix} \end{array}$$



By expanding by the line of u and then u_1 and v_1, respectively we have

$$\begin{aligned} \text{perm}(H) &\leq \text{perm}(H - \{u, v_1\}) + \text{perm}(H - \{u, v_2\}) \\ &\leq \text{perm}(H - \{u, v_1, u_1, v_3\}) + \text{perm}(H - \{u, v_2, v_1, u_1\}). \end{aligned}$$

If $d(v_3) \geq 3$, then we conclude by part (a) of the induction hypothesis that $\text{perm}(H) \leq \alpha^{k-4} + \alpha^{k-4} < \alpha^{k-3} + \alpha^{k-5}$. Indeed, we delete four vertices and lose at least six edges in both cases by deleting $\{u, v_1, u_1, v_3\}$ or $\{u, v_2, v_1, u_1\}$. Assume, then, that $d(v_3) = 2$. We expand $H - \{u, v_2, v_1, u_1\}$ along the line of v_3. If $d(u_2) \geq 3$, or $d(u_2) = 2$ and u_2 is not adjacent to v_2, then

$$\begin{aligned} \text{perm}(H) &\leq \text{perm}(H - \{u, v_1, u_1, v_3\}) + \text{perm}(H - \{u, v_2, v_1, u_1\}) \\ &\leq \text{perm}(H - \{u, v_1, u_1, v_3\}) + \text{perm}(H - \{u, v_2, v_1, u_1, v_3, u_2\}) \\ &\leq \alpha^{k-3} + \alpha^{k-5}. \end{aligned}$$

Finally, if $d(u_2) = 2$ and u_2 is adjacent to v_2, then we expand $H - \{u, v_1, u_1, v_3\}$ along the line of u_2 to get

$$\begin{aligned} \text{perm}(H) &\leq \text{perm}(H - \{u, v_1, u_1, v_3\}) + \text{perm}(H - \{u, v_2, v_1, u_1\}) \\ &\leq \text{perm}(H - \{u, v_1, u_1, v_3, u_2, v_2\}) + \text{perm}(H - \{u, v_2, v_1, u_1, v_3, u_2\}) \\ &\leq \alpha^{k-4} + \alpha^{k-4}. \quad \square \end{aligned}$$

We thus assume that H has no path with three consecutive vertices of degree 2. This implies that every vertex of degree 2 has a neighbor of degree at least 3. Let y_1 and y_2 be the neighbors of a degree 2 vertex x. We analyze all possible degree combinations of y_1 and y_2. Since we use extensively the same proof method in what follows, we explain in detail how to obtain the bounds in Claims 9 and 10. We will expand the permanent along the vertices x of degree 2 that are either *Type I* or *Type II* vertices and use the induction hypothesis of parts (b) and (c). Hence, we check the hypothesis of being connected and not isomorphic to K_2, C_6, or J in Sections 3.1 and 3.2. This will be sufficient to conclude part (b). To conclude part (c), we note that x could be neither of Type I nor Type II, i.e. a neighbor of x can have a degree larger than 3, and we handle these remaining cases in Section 3.3. We first assume H has a Type I vertex x.

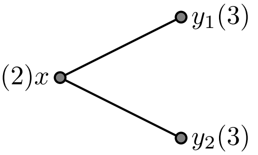
Claim 9. *Let x be a vertex with degree 2 and neighbors y_1 and y_2. If $d(y_1) = 3$ and $d(y_2) = 3$, then*

$$f(H) \leq \frac{1}{2} f(H - \{x, y_1\}) + \frac{1}{2} f(H - \{x, y_2\}).$$

Further, if both $H - \{x, y_1\}$ and $H - \{x, y_2\}$ are connected, not isomorphic to K_2, C_6, or J, then we have $f(H) \leq c_2$. Moreover, in addition to the previous conditions, if $\Delta(H) \leq 3$, then $f(H) \leq c_1$.

Proof. In this case, the bi-adjacency matrix of H is

$$A = \begin{array}{c} \\ x \\ \\ \vdots \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & \cdots & & \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \end{array}$$



The first inequality follows from the cofactor expansion along the line of x. If $H - \{x, y_i\}$ is connected, and not isomorphic to K_2, C_6, or J, then, as $H - \{x, y_i\}$ has minimum degree at most 2, we can use part (c) of the induction hypothesis to obtain that $f(H - \{x, y_i\}) \leq c_2$. We then conclude by (1) that

$$\begin{aligned} f(H) &= \alpha^{2-d(x)-d(y_1)} f(H - \{x, y_1\}) + \alpha^{2-d(x)-d(y_2)} f(H - \{x, y_2\}) \\ &\leq \alpha^{2-d(x)-d(y_1)} \cdot c_2 + \alpha^{2-d(x)-d(y_2)} \cdot c_2 = c_2. \end{aligned}$$

Similarly, if $\Delta(H) \leq 3$, then $f(H) \leq c_1$ by the induction hypothesis of part (b). □

It remains to check cases with at least one of y_1 or y_2 having degree 2. Without loss of generality, let $d(y_1) = 2$, then we can assume $d(y_2) \neq 2$ because we already dealt with the cases when H is a path and when H has a path with three vertices of degree 2. Besides x, let y_1 be adjacent to x_1. We can assume y_2 is not adjacent to x_1, otherwise x is contained in a C_4. As there is no path of three vertices each of degree 2, we have $d(x_1) = d(y_2) = 3$.

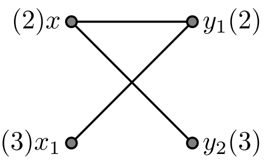
Claim 10. *Let x and y_1 be adjacent vertices with degree 2. Further, assume that x is adjacent to y_2, y_1 is adjacent to x_1, and y_2 is not adjacent to x_1. If $d(x_1) = d(y_2) = 3$, then*

$$f(H) \leq \alpha^{-2} f(H - \{x, y_1\}) + \alpha^{-5} f(H - \{x, x_1, y_1, y_2\}).$$

Further, if both $H - \{x, y_1\}$ and $H - \{x, x_1, y_1, y_2\}$ are connected, not isomorphic to K_2, C_6, or J, and have minimum degree at most 2, then we have $f(H) < c_2$. Moreover, in addition to the previous conditions, if $\Delta(H) \leq 3$, then $f(H) < c_1$.

Proof. In this case

$$A = \begin{array}{c} \\ x \\ x_1 \\ \\ \vdots \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} y_1 & y_2 & \cdots & & \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & a_{23} & \cdots & a_{2n} \\ 0 & a_{23} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \end{array}$$



By expanding through the row of x, $f(H) = \alpha^{2-d(x)-d(y_1)} f(H - \{x, y_1\}) + \alpha^{2-d(x)-d(y_2)} f(H - \{x, y_2\})$. Now assume $H - \{x, y_1\}$ is connected, not isomorphic to K_2, C_6, or J. By part (c) of the induction hypothesis, $f(H - \{x, y_1\}) \leq c_2$. For $H - \{x, y_2\}$, we can further expand along the column of y_1 to obtain

$$f(H - \{x, y_2\}) = \alpha^{1-d(x_1)} f(H - \{x, y_2, y_1, x_1\}) \leq \alpha^{1-d(x_1)} \cdot c_2,$$

when $H - \{x, x_1, y_1, y_2\}$ is connected, not isomorphic to K_2, C_6, or J, and have minimum degree at most 2. We conclude

$$f(H) \leq \alpha^{-2} \cdot c_2 + \alpha^{1-d(x_1)-d(y_2)} \cdot c_2 \leq \alpha^{-2} \cdot c_2 + \alpha^{-5} \cdot c_2 < 0.945 \cdot c_2.$$

Similarly, if $\Delta(H) \leq 3$, then $f(H) < c_1$ by the induction hypothesis of part (b). □

3.1. Expanding the permanent along a Type I vertex

We will see in Claim 11 that $H - \{x, y_i\}$ cannot be isomorphic to K_2, C_6 or J, and in Claim 12 we use part (a) of the induction hypothesis to obtain that if $H - \{x, y_i\}$ is disconnected then $f(H) \leq c_1$. Recall that H has the following properties: H is bipartite, C_4-free, and has maximum degree at least 3.

Claim 11. *If x is a* Type I *vertex, then $H - \{x, y_1\}$ is not isomorphic to K_2, C_6 or J.*

Proof. If $H - \{x, y_1\} = K_2$, then H has 4 vertices, which contradicts that H is bipartite and has a vertex of degree at least 3. If $H - \{x, y_1\} = C_6$, since $d(y_1) = 3$, then y_1 is adjacent to two vertices of the C_6. Since any two vertices in a C_6 have distance at most 3, H contains a C_3, C_4 or C_5. It contradicts either the graph being C_4-free or bipartite. Similarly, since any two vertices of J have distance at most 3, we cannot have $H - \{x, y_1\} = J$. □

Claim 12. *If x is a* Type I *vertex and $H - \{x, y_1\}$ is disconnected, then $f(H) \leq \alpha^{-1} \leq c_1$.*

Proof. By breaking into cases of when the edges that are going to be deleted are contained in the matching (when computing the permanent) or not, leads us to the following two cases.

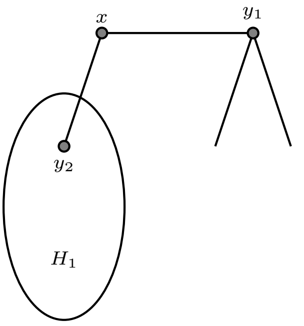


Fig. 4. Case 1 of Claim 12.

Case 1: There is a component containing y_2 and not containing any of the neighbors of y_1 (Fig. 4).

If $v(H_1)$ is even, then xy_2 cannot be in any perfect matching. Then H and $H - xy_2$ have the same number of perfect matchings, which means $\text{perm}(H) = \text{perm}(H - xy_2)$. Using the inductive hypothesis, we have that $\text{perm}(H - xy_2) \leq \alpha^{e(H-xy_2)-\frac{1}{2}v(H-xy_2)} = \alpha^{(e(H)-1)-\frac{1}{2}v(H)}$. Therefore, $\text{perm}(H) \leq \alpha^{e(H)-\frac{1}{2}v(H)-1}$ and $f(H) \leq \alpha^{-1}$.

If $v(H_1)$ is odd, then xy_2 must be in every perfect matching, then xy_1 cannot be in any of them. Then $\text{perm}(H) = \text{perm}(H - xy_1)$. Using the inductive hypothesis, we have that $\text{perm}(H - xy_1) \leq \alpha^{e(H-xy_1)-\frac{1}{2}v(H-xy_1)} = \alpha^{(e(H)-1)-\frac{1}{2}v(H)}$. Therefore, $\text{perm}(H) \leq \alpha^{e(H)-\frac{1}{2}v(H)-1}$ and $f(H) \leq \alpha^{-1}$.

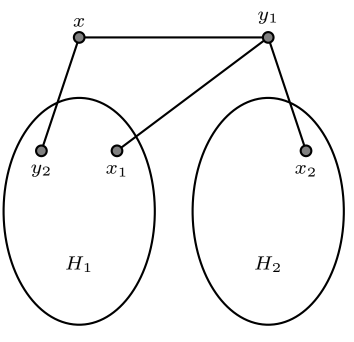


Fig. 5. Case 2 of Claim 12.

Case 2: The component containing y_2 also contains x_1, one of the neighbors of y_1 (Fig. 5).

Since $v(H_1) + v(H_2)$ is even, they can only both be either even or odd.

If $v(H_1)$ and $v(H_2)$ are both even, then y_1x_2 cannot be in any perfect matching. Thus $\text{perm}(H) \leq \alpha^{e(H)-\frac{1}{2}v(H)-1}$.

If $v(H_1)$ and $v(H_2)$ are both odd, then xy_1 cannot be in any perfect matching. Thus $\text{perm}(H) \leq \alpha^{e(H)-\frac{1}{2}v(H)-1}$.

In conclusion, in all cases $f(H) \leq \alpha^{-1} \leq c_1$. $\square$

3.2. Expanding the permanent along a Type II vertex

Now we assume H has no Type I vertex. Let x be a Type II vertex. We will see in Claims 13 and 14 that $H - \{x, y_1\}$ and $H - \{x, x_1, y_1, y_2\}$ cannot be isomorphic to K_2, C_6 or J; and in Claims 15 and 16 we use part (a) of the induction hypothesis to obtain that if $H - \{x, y_1\}$ or $H - \{x, x_1, y_1, y_2\}$ are disconnected then $f(H) \leq c_1$. Otherwise, if $H - \{x, y_1\}$ and $H - \{x, x_1, y_1, y_2\}$ are connected and non-isomorphic to K_2, C_6, or J; then by Claim 10 we conclude

$$f(H) \leq \alpha^{-2} f(H - \{x, y_1\}) + \alpha^{-5} f(H - \{x, x_1, y_1, y_2\}) < c_1,$$

when $\Delta(H) \leq 3$. Notice $H - \{x, y_1\}$ has minimum degree at most 2, but we need that $\Delta(H) \leq 3$ to guarantee $H - \{x, x_1, y_1, y_2\}$ has minimum degree at most 2. In general, $H - \{x, x_1, y_1, y_2\}$ has minimum degree at most 2 unless we have the structure of Claim 17, in which case we will see that $f(H) < c_2$.

Claim 13. *If x is a* Type II *vertex, then $H - \{x, x_1, y_1, y_2\}$ is not isomorphic to K_2, C_6 or J.*

Proof. If $H - \{x, x_1, y_1, y_2\} = K_2$, since x_1 have two neighbors in $H - \{x, x_1, y_1, y_2\}$, we have a copy of C_3 in H, a contradiction. As in the proof of Claim 11, if $H - \{x, x_1, y_1, y_2\} = C_6$, then x_1 is adjacent to two vertices of the C_6, creating a copy C_3, C_4 or C_5 in H. Notice that for the graph J, any two vertices are at distance at most 3. Similarly, assuming $H - \{x, x_1, y_1, y_2\} = J$, we conclude H has a copy of C_3, C_4 or C_5, a contradiction. □

Claim 14. *If $H \neq J$, H has no* Type I *vertex, and x is a* Type II *vertex, then $H - \{x, y_1\}$ is not isomorphic to K_2, C_6 or J.*

Proof. If $H - \{x, y_1\} = K_2$, then H has 4 vertices, contradicting x_1, y_2 have degree 3. If $H - \{x, y_1\} = C_6$, then x_1 and y_2 are vertices of this cycle and either we have a copy C_3, C_4 or C_5 in H, or H is isomorphic to J.

Notice that we obtain a similar contradiction when $H - \{x, y_1\} = J$, unless x_1 and y_2 are vertices with distance 3 in J. Without loss of generality, we can assume the graph H as Fig. 6.

Then z is a *Type I* vertex, a contradiction. □

Claim 15. *If x is a* Type II *vertex and $H - \{x, y_1\}$ is disconnected, then $f(H) \leq \alpha^{-1} \leq c_1$.*

Proof. When $H - \{x, y_1\}$ is disconnected, there is only one possible case. Namely, when y_2 and x_1 belong to different components H_1 and H_2, respectively, of $H - \{x, y_1\}$ (Fig.7).

Still, since $v(H_1) + v(H_2)$ is even, they can only both be either even or odd. If $v(H_1)$ and $v(H_2)$ are both even, then xy_2 and y_1x_1 cannot be in any perfect matching. If $v(H_1)$

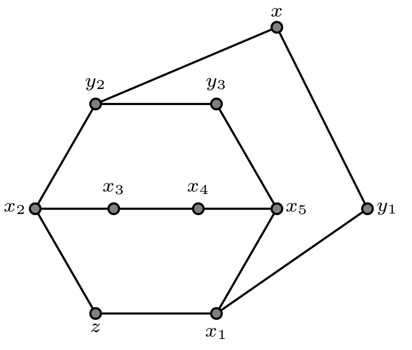


Fig. 6. The graph H in Claim 14.

and $v(H_2)$ are both odd, then xy_1 cannot be an edge in any perfect matching. Thus $\text{perm}(H) \le \alpha^{e(H)-\frac{1}{2}v(H)-1}$ and in both cases we get $f(H) \le \alpha^{-1}$. $\square$

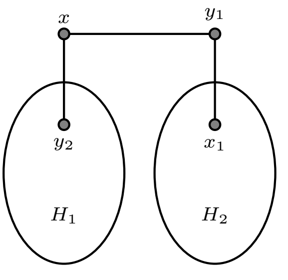


Fig. 7. The graph H in Claim 15.

Claim 16. *If H has no* Type I *vertex, x is a* Type II *vertex, and $H - \{x, x_1, y_1, y_2\}$ is disconnected, then $f(H) \le c_2$. Moreover, $f(H) \le c_1$ when $\Delta(H) \le 3$.*

Proof. Now we assume $H - \{x, x_1, y_1, y_2\}$ is disconnected and break the proof into cases depending on whether the edges that were going to be deleted are contained in the matching or not. We have the following three cases.

Case 1: There is a component of $H - \{x, x_1, y_1, y_2\}$ containing only one of the four neighbors of y_2 and x_1 (Fig. 8). Without loss of generality, we assume x_2 is the only such neighbor in the component H_1.

If $v(H_1)$ is odd, x_2y_2 must be in every perfect matchings, then xy_2 cannot be. Thus, $\text{perm}(H) \le \alpha^{e(H)-\frac{1}{2}v(H)-1}$.

If $v(H_1)$ is even, then x_2y_2 cannot be in any perfect matching, and $\text{perm}(H) \le \alpha^{e(H)-\frac{1}{2}v(H)-1}$. In both cases we get $f(H) \le \alpha^{-1}$.

Case 2: There are two components in $H - \{x, x_1, y_1, y_2\}$, each containing the two neighbors of y_2 or x_1 (Fig. 9). In this case, $H - \{x, y_1\}$ is disconnected and then $f(H) \le \alpha^{-1}$ by Claim 15.

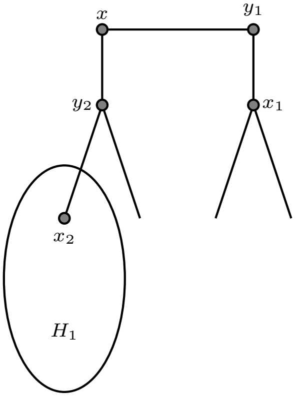


Fig. 8. Case 1 of Claim 16.

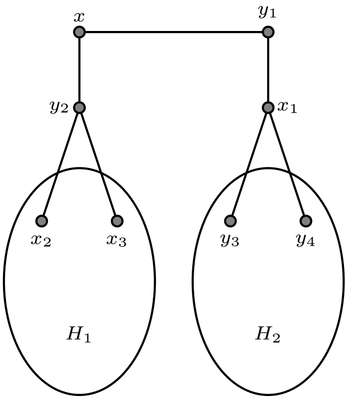


Fig. 9. Case 2 of Claim 16.

Case 3: There are two components in $H - \{x, x_1, y_1, y_2\}$, each containing one of the neighbors of both y_2 and x_1. Without loss of generality, we have the adjacencies shown in Fig. 10.

If $v(H_1)$ and $v(H_2)$ are both odd, then one of y_2x_2, x_1y_3 must be in a perfect matching, which means xy_1 has to be in the perfect matching and then xy_2 and y_1x_1 cannot be in any perfect matching. Thus, $\text{perm}(H) \leq \alpha^{e(H)-\frac{1}{2}v(H)-2}$.

If $v(H_1)$ and $v(H_2)$ are both even, we proceed by breaking into cases depending on whether the edges that were going to be deleted are contained in the matching or not.

When xy_1 is an edge in the perfect matching, then xy_2, y_1x_1 are not. Then either y_2x_2, x_1y_3 are edges in a perfect matching, or y_2x_3, x_1y_4 are edges in a perfect matching. In the first case, there are at most $\alpha^{e(H)-\frac{1}{2}v(H)-4-(d(x_2)-1)-(d(y_3)-1)+a}$ such perfect matchings; where $a = 1$ when x_2 is adjacent to y_3, and $a = 0$, otherwise. In the second case, there are at most $\alpha^{e(H)-\frac{1}{2}v(H)-4-(d(x_3)-1)-(d(y_4)-1)+b}$ such perfect matchings; where $b = 1$ when x_3 is adjacent to y_4, and $b = 0$, otherwise.

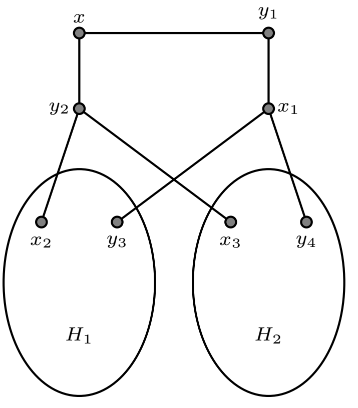


Fig. 10. Case 3 of Claim 16.

When a perfect matching does not contain xy_1, then the edges xy_2 and y_1x_1 must be in the perfect matching, which implies y_2x_2, y_2x_3, x_1y_3, x_1y_4 cannot be in the perfect matching. There are at most $\alpha^{e(H)-\frac{1}{2}v(H)-5}$ such perfect matchings.

Summing up the number of perfect matchings, we get

$$\operatorname{perm}(H) \leq (\alpha^{-(2+d(x_2)+d(y_3)-a)} + \alpha^{-(2+d(x_3)+d(y_4)-b)} + \alpha^{-5}) \cdot \alpha^{e(H)-\frac{1}{2}v(H)}.$$

If $a = b = 0$, we obtain

$$f(H) \leq \alpha^{-5} + 2\alpha^{-6} \leq c_1.$$

If $d(x_2) = 3$ and $a = 0$, or when $d(x_2) \geq 4$, we obtain

$$f(H) \leq 2\alpha^{-5} + \alpha^{-7} = c_1.$$

If $d(x_2) = 3$ and $a = 1$, then $d(y_3) \geq 3$, as if $d(y_3) = 2$, then y_3 is a Type I vertex. We get

$$f(H) \leq 2\alpha^{-5} + \alpha^{-7} = c_1.$$

By symmetry of x_2, y_3, x_3, and y_4, the only two cases remaining are shown in Fig. 11, when $d(x_2) = d(x_3) = d(y_3) = d(y_4) = 2$.

In the first case, H is isomorphic to J, a contradiction. In the second case, because x_2 is not a Type I vertex, hence it has to be adjacent to a vertex z_1 of degree $\neq 3$. If $d(z_1) \geq 4$, then expanding along the vertex x_2 yields $f(H) \leq \alpha^{-3}+\alpha^{-4} = c_2$. Otherwise, by symmetry, both x_2 and y_3 have to be adjacent to vertices with degree 2, say z_1 and z_2, which are not adjacent to each other (since, otherwise, there will be a path of vertices with degree 2 of length 3). If the other neighbor of z_1 or z_2 is a vertex of degree 2, then by Claim 8, we already have the desired bound on $f(H)$. Thus, both z_1 and z_2 have to be adjacent to vertices with degrees at least 3, say w_1 and w_2 (Fig. 12).

Fig. 11. Case 3: $a = b = 1$ and $a = 0$, $b = 1$.

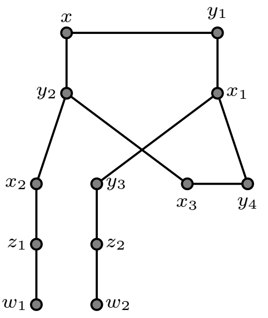


Fig. 12. Case 3: $a = 0$, $b = 1$.

If $d(w_1) \geq 4$, then

$$f(H) \leq \alpha^{-2} f(H - \{x_2, z_1\}) + \alpha^{-6} f(H - \{x_2, w_1, z_1, y_2\}) \leq \alpha^{-2} + \alpha^{-6} \leq c_2.$$

If $d(w_1) = 3$, then x_2 is a Type II vertex, with neighbors z_1 and y_2, while z_1 is also adjacent to w_1. Proceeding as in Claim 10, we get

$$f(H) \leq \alpha^{-2} f(H - \{x_2, z_1\}) + \alpha^{-5} f(H - \{x_2, w_1, z_1, y_2\}).$$

Notice that $H - \{x_2, z_1\}$ have a path of length 4 consisting of degree 2 vertices (namely $y_1 x y_2 x_3 y_4$) and, proceeding as in Claim 8, we have $f(H - \{x_2, z_1\}) \leq \alpha^{-3} + \alpha^{-5}$. Further, note that, in $H - \{x_2, w_1, z_1, y_2\}$, x has degree 1 and is adjacent to y_1 of degree 2. By expanding along the vertex x, we get $f(H - \{x_2, w_1, z_1, y_2\}) \leq \alpha^{-1}$.

We conclude

$$\begin{aligned} f(H) &\leq \alpha^{-2} f(H - \{x_2, z_1\}) + \alpha^{-5} f(H - \{x_2, w_1, z_1, y_2\}) \\ &\leq \alpha^{-2} \cdot (\alpha^{-3} + \alpha^{-5}) + \alpha^{-5} \cdot \alpha^{-1} = \alpha^{-5} + \alpha^{-6} + \alpha^{-7} \leq c_1. \quad \square \end{aligned}$$

This completes the proof of part (b), since we dealt with all possible cases for a connected C_4-free balanced bipartite graph H with $\Delta(H) \leq 3$ when $\delta(H) = 1$ or $\delta(H) = 2$. Indeed, for the latter, either

1. H is a path;
2. H has a path with three vertices of degree 2;
3. H has a Type I vertex;
4. H has a Type II vertex.

The first case follows from a direct computation, namely $f(C_{2n}) = 2\alpha^{-n}$. For the second case, we conclude $f(H) < c_1$ by Claim 8. For the third case, $f(H) \leq c_1$ follows from Claims 9, 11 and 12. Finally, for the last case, $f(H) \leq c_1$ follows from Claims 10, 13–16.

3.3. Proof of part (c)

When x is a Type II vertex, then $H - \{x, x_1, y_1, y_2\}$ has minimum degree at most 2 unless we have the following structure of Claim 17, in which case we will prove that $f(H) < c_2$.

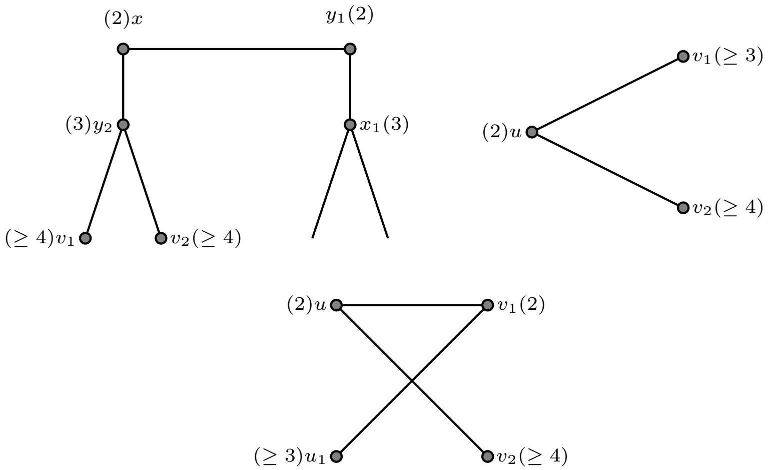


Fig. 13. Substructures in Claims 17, 18, and 19, respectively.

Claim 17. *If x is a* Type II *vertex, and $H - \{x, x_1, y_1, y_2\}$ has minimum degree at least 3, then $f(H) \leq \alpha^{-6} + \alpha^{-6} + \alpha^{-5} < c_2$.*

Proof. We use the notation in Fig. 13. If $H - \{x, x_1, y_1, y_2\}$ has minimum degree at least 3, then before the *Type II* deletion, we can expand on y_2 to get

$$\mathrm{perm}(H) = \mathrm{perm}(H - \{y_2, v_1\}) + \mathrm{perm}(H - \{y_2, v_2\}) + \mathrm{perm}(H - \{y_1, x\}).$$

Notice that $d(x) = 1$ in $H - \{y_1, v_i\}$, and $d(y_1) = 1$ in $H - \{x, y_2\}$. Thus, further expanding along x and y_1,

$$\mathrm{perm}(H) = \mathrm{perm}(H - \{x, y_1, y_2, v_1\}) + \mathrm{perm}(H - \{x, y_1, y_2, v_2\}) + \mathrm{perm}(H - \{x, x_1, y_1, y_2\}).$$

We conclude

$$f(H) \leq \alpha^{-6} + \alpha^{-6} + \alpha^{-5} < 0.8150 < 0.8968 < c_2. \quad \square$$

Now, Claims 18 and 19 will use part (a) of the induction hypothesis to deal with all other possible cases when $\Delta(H) > 3$. We have the following cases.

Claim 18. *Let u be a vertex with degree 2, and neighbors of v_1 and v_2. If $d(v_1) \geq 3$ and $d(v_2) \geq 4$, then*

$$f(H) \leq \alpha^{-3} + \alpha^{-4} = c_2.$$

Proof. Similarly to the proof of Claim 9, we have $\mathrm{perm}(H) = \mathrm{perm}(H - \{u, v_1\}) + \mathrm{perm}(H - \{u, v_2\}) \leq \alpha^{k-3} + \alpha^{k-4}$. That means $f(H) \leq \alpha^{-3} + \alpha^{-4} = c_2$. $\square$

Claim 19. *Let u and v_1 be adjacent vertices with degree 2. Further, assume that u is adjacent to v_2, v_1 is adjacent to u_1, and v_2 is not adjacent to u_1. If $d(u_1) \geq 3$ and $d(v_2) \geq 4$, then*

$$f(H) \leq \alpha^{-2} + \alpha^{-6} < c_2.$$

Proof. Similarly to the proof of Claim 10, $\mathrm{perm}(H) = \mathrm{perm}(H - \{u, v_1\}) + \mathrm{perm}(H - \{u, u_1, v_1, v_2\}) \leq \alpha^{k-2} + \alpha^{k-6}$. That means $f(H) \leq \alpha^{-2} + \alpha^{-6} < 0.88 < 0.8968 < c_2$. $\square$

This completes the proof of part (c), since we dealt with all possible cases for a connected C_4-free balanced bipartite graph H when $\delta(H) = 1$ or $\delta(H) = 2$. Indeed, for the latter, either

1. H is a path;
2. H has a path with three vertices of degree 2;
3. H has a Type I vertex;
4. H has a Type II vertex;
5. H has a vertex of degree 2 adjacent to vertices of degree at least 3 and 4, respectively;
6. H has a vertex of degree 2 adjacent to vertices of degree 2 and at least 3, respectively.

The first case follows from a direct computation, namely $f(C_{2n}) = 2\alpha^{-n}$. For the second case, we conclude $f(H) < c_2$ by Claim 8. For the third case, $f(H) \leq c_2$ follows from Claims 9, 11 and 12. For the fourth case, $f(H) \leq c_2$ follows from Claims 10, 13–17. For the fifth case, $f(H) \leq c_2$ follows from Claim 18. For the last case, $f(H) < c_2$ follows from Claim 19.

4. Proof of part (a)

As the proof is by induction on $n + k$, we begin with the base case when $n = 1$ or $k = 0$. When $n = 1$, every 1×1 matrix with entries in $\{0, 1\}$ has at most $1 = 1 + 0$ non-zero entry and permanent at most $1 = \alpha^0$. When $k = 0$, every $n \times n$ matrix with at most n non-zero entries has permanent at most 1, since the determinant is non-zero only when we have exactly one non-zero entry per row and column.

We first prove, in Sections 4.1–4.4, that the result follows unless G is connected, with $\delta(G) = 3$ and $\Delta(G) \leq 5$. We thus deal with the cases $\Delta(G) = 3$, 4, or 5 in Sections 4.5, 4.6, and 4.7, respectively.

4.1. G is connected and $\delta(G) \leq 2$

If G is connected and $\delta(G) \leq 2$, by (c), we have that $f(G) \leq c_2 < 1$ when $G \neq K_2, C_6, J$. As $f(K_2) = f(C_6) = 1$ and $f(J) = 3\alpha^{-5} < 1$, we conclude $f(G) \leq 1$ for all connected graphs with minimum degree at most 2.

4.2. G is disconnected

If A, the bi-adjacency matrix of G, has a block-diagonal form, say with square matrices D_1 and D_2 as diagonal blocks, then the result follows by induction, applied to each of the blocks, as $\mathrm{perm}(A) = \mathrm{perm}(D_1)\mathrm{perm}(D_2)$ and the order of the two matrices D_1 and D_2 add to the order of the matrix A.

4.3. $\Delta(G) \geq 6$

If A contains a line with at least 6 non-zero entries, then we can split $\mathrm{perm}(A)$ as the sum of two permanents with at least 3 ones missing in each. That is, reordering the rows and columns of A if needed, we can assume that

$$A = \begin{pmatrix} & & & & \cdots & & & & \\ & & & & \cdots & & & & \\ 1 & 1 & 1 & 1 & 1 & 1 & a_{i7} & \cdots & a_{in} \\ & & & & \cdots & & & & \\ & & & & \cdots & & & & \end{pmatrix},$$

then we have that $\mathrm{perm}(A) = \mathrm{perm}(B) + \mathrm{perm}(C)$, where

$$B = \begin{pmatrix} & & & & \cdots & & & & \\ & & & & \cdots & & & & \\ 1 & 1 & 1 & 0 & 0 & 0 & a_{i7} & \cdots & a_{in} \\ & & & & \cdots & & & & \\ & & & & \cdots & & & & \end{pmatrix} \quad \text{and}$$

$$C = \begin{pmatrix} & & & & \cdots & & & & \\ & & & & \cdots & & & & \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & \dots & 0 \\ & & & & \cdots & & & & \\ & & & & \cdots & & & & \end{pmatrix}.$$

As both B and C are $n \times n$ matrices with at most $n + k - 3$ non-zero entries, the induction hypothesis implies

$$\mathrm{perm}(A) \leq \alpha^{k-3} + \alpha^{k-3} = \alpha^k.$$

4.4. $\delta(G) = d \geq 4$

Let u be a vertex with minimum degree in G, and adjacent to $v_1, \dots, v_d$. If $d \geq 4$, then the induction hypothesis after expanding the permanent along the line of u gives

$$f(G) \leq \sum_{i=1}^{d} \alpha^{2-d(u)-d(v_i)} \leq d \cdot \alpha^{2-2d} \leq 1,$$

where the last inequality follows from the fact $d \cdot \alpha^{2-2d}$ is decreasing in d for $d \geq 4$, since $\alpha^2 > 1.25 \geq \frac{d+1}{d}$. For $d = 4$, we have $d \cdot \alpha^{2-2d} = 4\alpha^{-6} = 1$.

Remark. By the discussion above, the result follows unless G is connected, has minimum degree $\delta(G) = 3$ and maximum degree $\Delta(G) \leq 5$. We assume this is the case and we deal next with the cases $\Delta(G) = 3, 4, 5$.

4.5. *Connected 3-regular graphs*

Assume now G is a 3-regular C_4-free balanced bipartite graph. Let u be a vertex and v_1, v_2, v_3 its neighbors. Then expanding on the line of u gives

$$f(G) = \alpha^{-4} \cdot \left(\sum_{i=1}^{3} f(G - \{u, v_i\}) \right).$$

Note that $G - \{u, v_1\}$ has minimum degree 2, since v_3 has degree 2 in $G - \{u, v_1\}$. Similarly, $G - \{u, v_2\}$ and $G - \{u, v_3\}$ have minimum degree 2 as well. We will see in Claim 20 that $G - \{u, v_i\}$ cannot be isomorphic to K_2, C_6 or J. We are left to check what happens when $G - \{u, v_i\}$ is disconnected in Claim 21. Otherwise, we can use the induction hypothesis of part (b) for $H = G - \{u, v_i\}$ to conclude

$$f(G) = \alpha^{-4} \cdot \left(\sum_{i=1}^{3} f(G - \{u, v_i\}) \right) \leq \alpha^{-4} \cdot \left(\sum_{i=1}^{3} c_1 \right) = 3\alpha^{-4} \cdot c_1 < 1.$$

Claim 20. *$G - \{u, v_1\}$ is not isomorphic to K_2, C_6 or J.*

Proof. Note that the graph $G-\{u,v_1\}$ has minimum degree 2 hence $G-\{u,v_1\}$ cannot be a K_2. Indeed, the deletion of two adjacent vertices u and v_1 decreases the degree of four different vertices (neighbors of u and v_1) by 1 and maintains the degree of all other vertices. In particular, $G-\{u,v_1\}$ cannot be C_6 or J, since both graphs have six vertices of degree 2. □

Claim 21. *If $G-\{u,v_1\}$ is disconnected, then $f(G) \leq \alpha^{-1} < 1$.*

Proof. If $G-\{u,v_1\}$ is disconnected, then we have 3 cases, depending on how the neighbors of u and v_1 are distributed in different components.

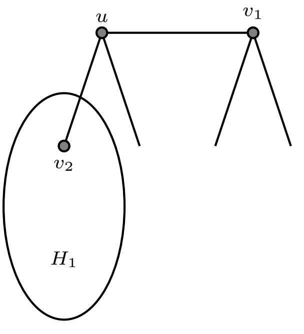


Fig. 14. Case 1 in Claim 21.

Case 1: There is a component containing only one of the four neighbors of u and v_1 (Fig. 14). Without loss of generality, we assume v_2 is the unique such neighbor in the component H_1.

If $v(H_1)$ is odd, uv_2 must be in every perfect matching of G, hence uv_3 cannot be. The number of perfect matchings of G does not change after deleting the edge uv_3. By the induction hypothesis, $\text{perm}(G) \leq \alpha^{k-1}$.

If $v(H_1)$ is even, then uv_2 cannot be in any perfect matching, and similarly we have $\text{perm}(G) \leq \alpha^{k-1}$.

We conclude $f(G) \leq \alpha^{-1} < 1$.

Case 2: There are two components, each containing the two neighbors of u or v_1 (Fig. 15). Let H_1 be the component containing v_2 and v_3, and H_2 containing the neighbors of v_1 (namely, u_1 and u_2).

Since $v(H_1)+v(H_2)$ is even, they can only both be either even or odd. If $v(H_1)$ and $v(H_2)$ are both even, then the edges uv_2, uv_3, v_1u_1 and v_1u_2 cannot be in any perfect matching of G, which implies $\text{perm}(G) \leq \alpha^{k-4}$. If $v(H_1)$ and $v(H_2)$ are both odd, then uv_1 cannot be in any perfect matching, implying $\text{perm}(G) \leq \alpha^{k-1}$.

We conclude $f(G) \leq \alpha^{-1} < 1$.

Case 3: There are two components, each containing one of the neighbors of both u and v_1 (Fig. 16). Without loss of generality, we have the following:

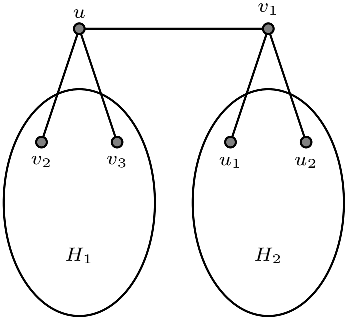


Fig. 15. Case 2 in Claim 21.

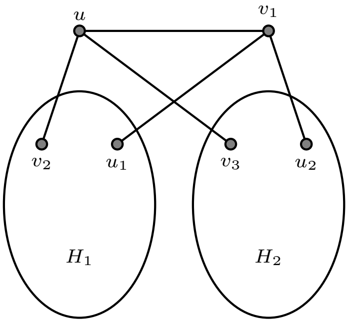


Fig. 16. Case 3 in Claim 21.

If $v(H_1)$ and $v(H_2)$ are both odd, then uv_1 cannot be in any perfect matching, implying $\mathrm{perm}(G) \leq \alpha^{k-1}$.

If $v(H_1)$ and $v(H_2)$ are both even, then we break the proof into cases according to which edge incident to u is contained in a perfect matching.

When uv_1 is an edge in a perfect matching, then uv_2, v_1u_1, v_1u_1 and v_1u_2 are not in a perfect matching. We have at most α^{k-4} such perfect matchings.

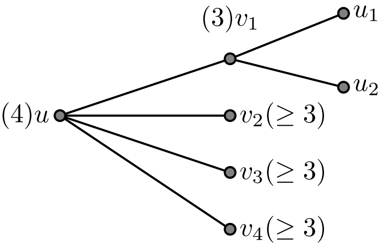
When uv_2 is an edge in a perfect matching, then v_1u_1 must be in the perfect matching and then uv_1, uv_3, v_1u_2 and further edges incident to v_2 and u_1 cannot be in the perfect matchings. There are at most α^{k-6} such perfect matchings. Similarly, there are at most α^{k-6} perfect matchings containing uv_3.

We conclude $\mathrm{perm}(G) \leq \alpha^{k-4} + 2\alpha^{k-6}$ and $f(G) \leq \alpha^{-4} + 2\alpha^{-6} < 0.8969 < 1$. $\square$

4.6. Connected graphs with $\delta(G) = 3$ *and* $\Delta(G) = 4$

Let u be a vertex of degree 4 and v_1, v_2, v_3, v_4 its neighbors. If $d(v_i) = 4$ for all $i \in \{1, 2, 3, 4\}$, then expanding on the line of u gives $f(G) \leq 4 \cdot \alpha^{-6} = 1$. Thus, we can assume that at least one of the neighbors of u is of degree 3. Without loss of generality, assume $d(v_1) = 3$. Let u_1 and u_2 be neighbors of v_1 as below.

$$G = \begin{array}{c} \\ u \\ u_1 \\ u_2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & \dots \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & \dots & 0 \\ 1 & a_{22} & \dots & & & & \\ 1 & a_{32} & \dots & & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \end{pmatrix} \end{array}$$



If $d(u_1) = d(u_2) = 4$, then expansion along the line of v_1 would give $f(G) \leq 3\cdot\alpha^{-5} < 1$. Thus, we assume $d(u_1) = 3$ and use linearity of the permanent to get $\operatorname{perm}(G) = \operatorname{perm}(G - uv_1) + \operatorname{perm}(G - \{u, v_1\})$, and then

$$f(G) = \alpha^{-1} \cdot f(G - uv_1) + \alpha^{-5} \cdot f(G - \{u, v_1\}).$$

We note that both $G - \{u, v_1\}$ and $G - uv_1$ have minimum degree 2, since u_1 has degree 2 in $G - \{u, v_1\}$ and v_1 has degree 2 in $G - uv_1$. We will see in Claims 22 and 23 that $G - uv_1$ and $G - \{u, v_1\}$ cannot be isomorphic to K_2, C_6 or J. We are left to check what happens when $G - uv_1$ is disconnected in Claim 24, and when $G - \{u, v_1\}$ is disconnected in Claim 25. Otherwise, we can use the induction hypothesis of part (c) for $H = G - uv_1$ and $H = G - \{u, v_1\}$ to obtain

$$f(G) = \alpha^{-1} \cdot f(G - uv_1) + \alpha^{-5} \cdot f(G - \{u, v_1\}) \leq c_2 \cdot (\alpha^{-1} + \alpha^{-5}) < 0.9944 < 1.$$

Claim 22. *$G - uv_1$ is not isomorphic to K_2, C_6, or J.*

Proof. Note that the graph $G - uv_1$ has minimum degree 2, hence $G - uv_1$ cannot be K_2. The deletion of the edge uv_1 only decreases the degree of u and v_1 by 1 and maintains the degree of all other vertices. In particular, $G - uv_1$ cannot be C_6 or J, since it would have six vertices of degree 2. □

Claim 23. *$G - \{u, v_i\}$ is not isomorphic to K_2, C_6, or J.*

Proof. Note that the graphs $G - \{u, v_1\}$ have minimum degree 2, hence $G - \{u, v_1\}$ cannot be K_2. Indeed, the deletion of two adjacent vertices u and v_1 decreases the degree of five different vertices (neighbors of u and v_1) by 1 and maintains the degree of all other vertices. In particular, $G - \{u, v_1\}$ cannot be C_6 or J, since it would have six vertices of degree 2. □

Claim 24. *If $G - uv_1$ is disconnected, then $f(G) \leq \alpha^{-1} < 1$ (Fig. 17).*

Proof. Assume $G - uv_1$ is disconnected. Let H_1 be the component containing v_1. Then u, v_2, v_3, v_4 are not in H_1.

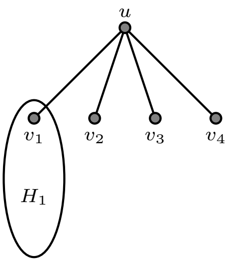


Fig. 17. The graph G in Claim 24.

If $v(H_1)$ is odd, then uv_1 must be in every perfect matching, implying that uv_2, uv_3, uv_4 are not. Thus, $\mathrm{perm}(G) \leq \alpha^{e(G)-\frac{1}{2}v(G)-3}$. If $v(H_1)$ is even, then uv_1 is not in any perfect matching, implying that $\mathrm{perm}(G) \leq \alpha^{e(G)-\frac{1}{2}v(G)-1}$. We conclude $f(G) \leq \alpha^{-1} < 1$. □

Claim 25. *If $G - \{u, v_1\}$ is disconnected, then $f(G) < 1$.*

Proof. If $G - \{u, v_1\}$ is disconnected, then we have 4 cases.

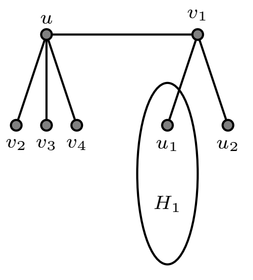


Fig. 18. Case 1 in Claim 25.

Case 1: There is a component containing only one of the five neighbors of u and v_1 (Fig. 18). We assume u_1 is the only such neighbor in the component H_1. The other cases can be dealt with similarly.

If $v(H_1)$ is odd, then v_1u_1 must be in every perfect matching of G, then uv_1 cannot be. The number of perfect matchings of G is the same after deleting the edge uv_1. By the induction hypothesis, $\mathrm{perm}(G) \leq \alpha^{e(G)-\frac{1}{2}v(G)-1}$. If $v(H_1)$ is even, v_1u_1 cannot be in any perfect matching, and similarly we have $\mathrm{perm}(G) \leq \alpha^{e(G)-\frac{1}{2}v(G)-1}$. We conclude $f(G) \leq \alpha^{-1}$. Similarly, we have $f(G) \leq \alpha^{-1}$ when v_2, v_3, v_4, or u_2 is the unique neighbor in a component of $G - \{u, v_1\}$.

Case 2: There are two components, each containing the neighborhood of u or v_1 (Fig. 19). Let H_1 be the component containing v_2, v_3 and v_4, and H_2 containing u_1 and u_2.

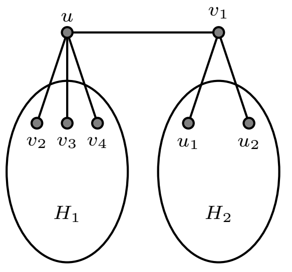


Fig. 19. Case 2 in Claim 25.

If $v(H_1)$ and $v(H_2)$ are odd, then uv_1 cannot be in any perfect matching of G, then $f(G) \le \alpha^{-1}$. If $v(H_1)$ and $v(H_2)$ are even, then uv_2, uv_3, uv_4, v_1u_1, and v_1u_2 cannot be in any perfect matching, hence $f(G) \le \alpha^{-5}$.

Case 3: There are two components, one contains two neighbors of u, the other one contains two neighbors of v_1 and one neighbor of u (Fig. 20). Assume H_1 is a component containing v_2, and v_3, and H_2 containing v_4, u_1 and u_2.

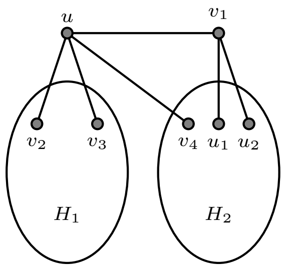


Fig. 20. Case 3 in Claim 25.

If $v(H_1)$ and $v(H_2)$ are odd, then uv_1 cannot be in any perfect matching, hence $f(G) \le \alpha^{-1}$. If $v(H_1)$ and $v(H_2)$ are even, then uv_2 and uv_3 cannot be in any perfect matching, hence $f(G) \le \alpha^{-2}$.

Case 4: There are two components, one contains two neighbors of u and one neighbor of v_1, the other component contains one neighbors of u and one neighbor of v_1 (Fig. 21). Assume H_1 is a component containing v_2, v_3, and u_1; and H_2, containing v_4 and u_2.

If $v(H_1)$ and $v(H_2)$ are odd, then uv_1 cannot be in any perfect matching, hence $f(G) \le \alpha^{-1}$.

If $v(H_1)$ and $v(H_2)$ are even, we break the proof into cases depending on which edge incident to u is contained in the matching. The number of perfect matchings containing uv_1 is at most $\alpha^{e(G)-\frac{1}{2}v(G)-5}$. When uv_2 or uv_3 is in a matching, then v_1u_1 also must be in the matching and there are at most $2 \cdot \alpha^{e(G)-\frac{1}{2}v(G)-7}$ such perfect matchings. When uv_4 is in a perfect matching, then v_1u_2 is also in and there are at most $\alpha^{e(G)-\frac{1}{2}v(G)-7}$ such perfect matchings. Summing up the bounds on the number of perfect matchings, we get $f(G) \le \alpha^{-5} + 3 \cdot \alpha^{-7} < 0.9103 < 1$. □

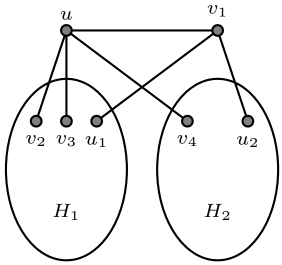
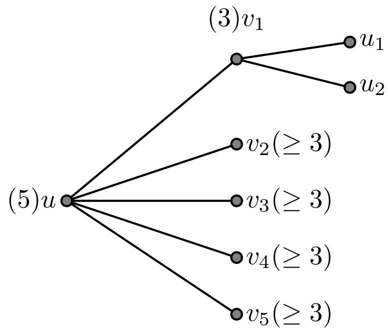


Fig. 21. Case 4 in Claim 25.

4.7. Connected graphs with $\delta(G) = 3$ and $\Delta(G) = 5$

Let u be a vertex of degree 5 and v_1, v_2, v_3, v_4, v_5 its neighbors. If $d(v_i) \geq 4$ for all $i \in \{1, 2, 3, 4, 5\}$, then expanding on the line of u gives $f(G) \leq 5 \cdot \alpha^{-7} < 1$. Thus, we can assume that at least one of the neighbors is of degree 3, without loss of generality, assume $d(v_1) = 3$. Let u_1 and u_2 be neighbors of v_1 as below.

$$G = \begin{array}{c} \\ u \\ u_1 \\ u_2 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & \dots & & \end{array} \\ \left(\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \\ 1 & a_{22} & \dots & & & & & \\ 1 & a_{32} & \dots & & & & & \\ \vdots & \vdots & & & & & & \end{array} \right) \end{array}$$



If $d(u_1), d(u_2) \geq 4$, then the expansion along the line of v_1 would give $f(G) \leq 2\alpha^{-5} + \alpha^{-6} < 1$. Thus, we assume $d(u_1) = 3$ and using the linearity of permanent we get $\operatorname{perm}(G) = \operatorname{perm}(G - uv_1) + \operatorname{perm}(G - \{u, v_1\})$, hence

$$f(G) = \alpha^{-1} \cdot f(G - uv_1) + \alpha^{-6} \cdot f(G - \{u, v_1\}).$$

We note that both $G - \{u, v_1\}$ and $G - uv_1$ have minimum degree 2. We will see in Claims 26 and 27 that $G - uv_1$ and $G - \{u, v_1\}$ cannot be isomorphic to K_2, C_6 or J. We are left to check what happens when $G - uv_1$ is disconnected in Claim 28 and $G - \{u, v_1\}$ is disconnected in Claim 29. Otherwise, we can use the induction hypothesis of part (c) for $H = G - uv_1$ and $H = G - \{u, v_1\}$ to obtain

$$f(G) = \alpha^{-1} \cdot f(G - uv_1) + \alpha^{-6} \cdot f(G - \{u, v_1\}) \leq c_2 \cdot (\alpha^{-1} + \alpha^{-6}) < 0.9361 < 1.$$

Claim 26. *$G - uv_1$ has minimum degree 2, and it is not isomorphic to K_2, C_6, or J.*

Proof. Note that the graph $G-uv_1$ has minimum degree 2, hence $G-uv_1$ cannot be K_2. The deletion of the edge uv_1 only decreases the degree of u and v_1 by 1 and maintains the degree of all other vertices. In particular, $G-uv_1$ cannot be C_6 or J, since it would have six vertices of degree 2. □

Claim 27. *$G-\{u,v_i\}$ has minimum degree 2, and it is not isomorphic to K_2, C_6, or J.*

Proof. Note that the graph $G-\{u,v_1\}$ has minimum degree 2, hence $G-\{u,v_1\}$ cannot be K_2. Since C_6 and J have six vertices of degree 2, if $G-\{u,v_1\}=C_6$ or $G-\{u,v_1\}=J$, then the neighbors of u and v_1 must be exactly the six vertices of degree 2, otherwise the minimum degree of G is less than 3. Thus, at least two of v_2, v_3, v_4, v_5 are adjacent, which contradicts to the graph being bipartite. □

Claim 28. *If $G-uv_1$ is disconnected, then $f(G)\leq\alpha^{-1}<1$ (Fig. 22).*

Proof. Assume that $G-uv_1$ is disconnected, let H_1 be the component containing v_1. Then u, v_2, v_3, v_4, and v_5 are not in H_1.

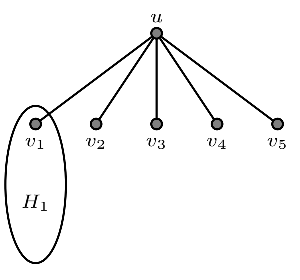


Fig. 22. The graph G in Claim 28.

If $v(H_1)$ is odd, then uv_1 must be in every perfect matching, implying that uv_2, uv_3, uv_4, uv_5 are not part of any. Thus, $\mathrm{perm}(G)\leq\alpha^{e(G)-\frac{1}{2}v(G)-4}$. If $v(H_1)$ is even, then uv_1 is not in any perfect matching, implying that $\mathrm{perm}(G)\leq\alpha^{e(G)-\frac{1}{2}v(G)-1}$. We conclude $f(G)\leq\alpha^{-1}<1$. □

Claim 29. *If $G-\{u,v_1\}$ is disconnected, then $f(G)<1$.*

Proof. If $G-\{u,v_1\}$ is disconnected, then we have three cases.

Case 1: There is a component containing neighbors of only one of u or v_1 (Fig. 23).

Assume that H_1 is such component, without loss of generality, $v_2\in H_1$, hence by assumption, H_1 contains no neighbor of v_1. If $v(H_1)$ is odd, then uv_1 cannot be in any perfect matching of G, hence $f(G)\leq\alpha^{-1}$. If $v(H_1)$ is even, then uv_2 cannot be in any perfect matching, hence $f(G)\leq\alpha^{-1}$.

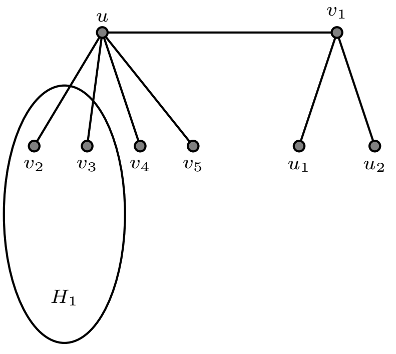


Fig. 23. Case 1 in Claim 29.

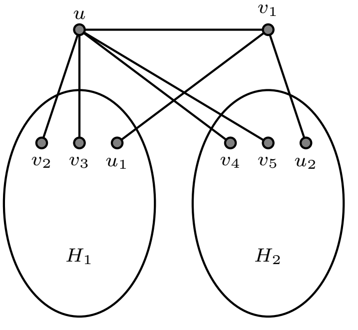


Fig. 24. Case 2 in Claim 29.

Case 2: There are two components, both contain two neighbors of u and one neighbor of v_1 (Fig. 24). We assume v_2, v_3, and u_1 are in the component H_1 and v_4, v_5, and u_2 are in H_2.

If $v(H_1)$ and $v(H_2)$ are odd, then uv_1 cannot be in any perfect matching, hence $f(G) \leq \alpha^{-1}$.

If $v(H_1)$ and $v(H_2)$ are even, we break into cases of which edge incident to u is contained in the matching. We have at most $\alpha^{e(G)-\frac{1}{2}v(G)-6}$ perfect matchings containing uv_1. When uv_2 or uv_3 is in a perfect matching, then v_1u_1 also must be in that matching and there are at most $2 \cdot \alpha^{e(G)-\frac{1}{2}v(G)-8}$ such perfect matchings. Similarly, when uv_4 or uv_5 is in a perfect matching, then v_1u_2 also is and there are at most $2 \cdot \alpha^{e(G)-\frac{1}{2}v(G)-8}$ such perfect matchings. Summing up the bounds on the number of perfect matchings, we get $f(G) \leq \alpha^{-6} + 4 \cdot \alpha^{-8} < 0.88 < 1$.

Case 3: There are two components, one contains three neighbors of u and one neighbor of v_1 (Fig. 25). We assume v_2, v_3, v_4 and u_1 are in the component H_1, and v_5 and u_2 are in H_2.

If $v(H_1)$ and $v(H_2)$ are odd, then uv_1 cannot be in any perfect matching, hence $f(G) \leq \alpha^{-1}$.

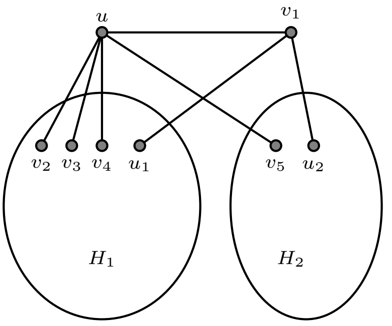


Fig. 25. Case 3 in Claim 29.

If $v(H_1)$ and $v(H_2)$ are even, we break into cases of which edge incident to u is contained in the matching. We have at most $\alpha^{e(G)-\frac{1}{2}v(G)-6}$ perfect matchings containing uv_1. When uv_2, uv_3, or uv_4 is in a perfect matching, then v_1u_1 also must be in and there are at most $3\cdot\alpha^{e(G)-\frac{1}{2}v(G)-8}$ such perfect matchings. When uv_5 is in the matching, v_1u_2 also is and there are at most $\alpha^{e(G)-\frac{1}{2}v(G)-8}$ such perfect matchings. Summing up the bounds on the number of perfect matchings, we get $f(G) \le \alpha^{-6}+4\cdot\alpha^{-8} < 0.88 < 1$. □

5. Determinant of graphs containing a C_4

We notice the cofactor expansion $\text{perm}(G) = \sum_{i=1}^{t} \text{perm}(G-\{u,v_i\})$ for the determinant is the following

$$\det(G) \le \sum_{i=1}^{t} \det(G-\{u,v_i\}).$$

Using the analogous auxiliary function $f'(G) = \alpha^{-e(G)+\frac{1}{2}v(G)}\cdot\det(G)$, instead of $f(G) = \alpha^{-e(G)+\frac{1}{2}v(G)}\cdot\text{perm}(G)$, we can mimic the proof of Theorem 6 to obtain Theorem 3. Note that the number of perfect matchings is an upper bound for the determinant. The only places where we used the C_4-free assumption in the proof above was in Claims 11, 13, and 14.

We get rid of the case when G has a vertex of degree 2 contained in a C_4 in Claim 30, and then discuss how to deal with the cases of Claims 11, 13, and 14 assuming the graph G can potentially have C_4 as a subgraph in Claims 31–36, but we shall bound the determinant rather than the permanent. We highlight that the assumption of C_4-free for the bound on permanents is needed, since $\text{perm}(C_4) = 2 > \alpha^2$.

For the next cases, it will be useful to first consider when there is a vertex u of degree 2 contained in a C_4.

Claim 30. *If G has a vertex u of degree 2 contained in a C_4, then $f'(G) \le \alpha^{-2} < c_1$.*

Proof. If $\{u, v_1, u_1, v_2\}$ induces a C_4 in G then

$$A = \begin{array}{c} \\ u \\ u_1 \\ \\ \vdots \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & \dots & & \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & a_{23} & \dots & a_{2n} \\ a_{13} & a_{23} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \end{array},$$

and we can subtract the line of u from the line of u_1, which does not change the determinant of the matrix. We obtain, using the induction hypothesis,

$$\det(G) = \det(G - \{u_1v_1, u_1v_2\}) \leq \alpha^{k-2} < c_1 \cdot \alpha^k. \quad \square$$

Since we used that the graph is C_4-free in Claims 11, 13, 14, and 27, so we get rid of those cases when there is a C_4, by explicitly computing the value of determinant in the following claims.

Claim 31. *If x is a Type I vertex and $H - \{x, y_1\} = C_6$, then $f'(H) \leq c_1$.*

Proof. If x is a Type I vertex and $H - \{x, y_1\} = C_6$, then y_2 is a vertex in the C_6, and y_1 is adjacent to another two vertices in the C_6. Given that H is bipartite, x is contained in a C_4. By Claim 30, $f'(H) \leq \alpha^{-2} < c_1$. $\square$

Claim 32. *If x is a Type I vertex and $H - \{x, y_1\} = J$, then $f'(H) \leq c_1$.*

Proof. If x is a Type I vertex and $H - \{x, y_1\} = J$, then y_2 is a vertex of degree 2 in J, and y_1 is adjacent to another two vertices in J. By Claim 30, we can assume that x is not in a C_4. Using the notation on Fig. 26, observe that y_1 can only be adjacent to the z_i's, otherwise H is not bipartite, and y_1 cannot be adjacent to z_1 or z_2, otherwise x is in a C_4. Therefore y_1 is adjacent to z_3 and z_4.

The determinant of the corresponding bi-adjacency matrix is 5. Notice that $k = 8$, hence

$$f'(H) = 5\alpha^{-8} < 0.7875 < 0.8283 < c_1. \quad \square$$

Claim 33. *If H has no Type I vertex, x is a Type II vertex and $H - \{x, y_1, x_1, y_2\} = C_6$, then $f'(H) \leq c_1$ (Fig. 27).*

Proof. If x is a Type II vertex and $H - \{x, y_1, x_1, y_2\} = C_6$, then both x_1 and y_2 are connected to two vertices in the C_6. First notice that x_1 and x_2 cannot have a common neighbor, since this would create a C_5. Further, the neighbors of x_1 (and of y_2) must have distance 2 to avoid C_3 or C_5.

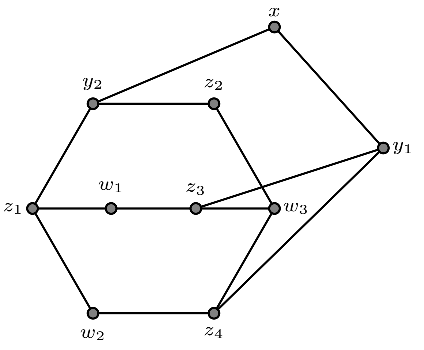


Fig. 26. The graph H in Claim 32.

Assume x_1 is adjacent to w_1 and w_3 as below. If y_2 is not adjacent to w_2, which is shown in the first graph below, then w_2 is a Type I vertex, a contradiction. If y_2 is adjacent to w_2, we will have the second graph below, up to isomorphism.

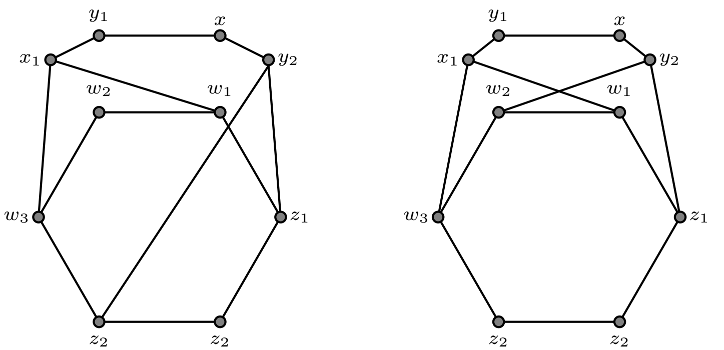


Fig. 27. Cases of Claim 33.

The determinant of the corresponding bi-adjacency matrix is 4. Notice that $k = 8$, hence

$$f'(H) = 4\alpha^{-8} < 0.63 < 0.8283 < c_1. \quad \square$$

Claim 34. *If H has no Type I vertex, x is a Type II vertex and $H - \{x, y_1, x_1, y_2\} = J$, then $f'(H) \le c_1$.*

Proof. If x is a Type II vertex and $H - \{x, y_1, x_1, y_2\} = J$, then both x_1 and y_2 are connected to two vertices in J. Similarly to the proof of the last claim, x_1 and y_2 cannot have a common neighbor and the neighbors of x_1 or y_2 must have distance 2.

We have two cases as in Fig. 28. In the first case, assume the neighbors of y_2 are vertices of degree 2 in J. Since H is bipartite, we can assume that the neighbors of y_2 are x_2 and x_3. Therefore y_3 and y_4 must be neighbors of x_1, otherwise, they would be

Type I vertices. The determinant of the corresponding bi-adjacency matrix is 5. Notice that $k = 10$, hence

$$f'(H) = 5\alpha^{-10} < 0.4961 < 0.8283 < c_1.$$

In the second case, assume y_2 has a neighbor with degree 3 in J, say x_2. Then the other neighbor of y_2 must be one of x_3, x_4, or x_5. Without loss of generality, let us assume it is x_3. Thus y_3 must be a neighbor of x_1, otherwise, it would be a Type I vertex. Consequently, the only choice for the other neighbor of x_1 is y_6, otherwise, x_4 or x_5 would be a Type I vertex. The determinant of the corresponding bi-adjacency matrix is 6. Notice that $k = 10$, hence

$$f'(H) = 6\alpha^{-10} < 0.5953 < 0.8283 < c_1. \quad \square$$

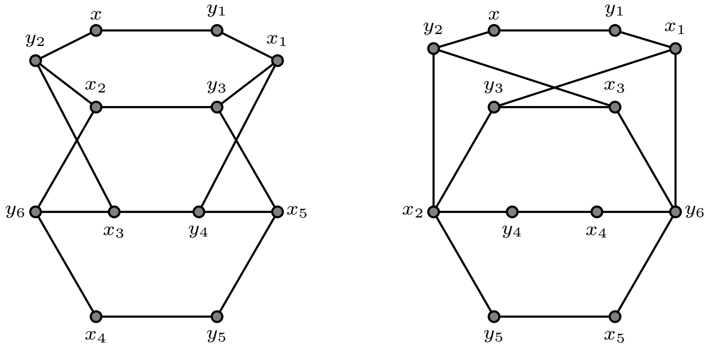


Fig. 28. Cases of Claim 34.

Claim 35. *If $H \neq J$, H has no Type I vertex, x is a Type II vertex and $H - \{x, y_1\} = C_6$, then $f'(H) \leq c_1$ (Fig. 29).*

Proof. If x is a Type II vertex and $H - \{x, y_1\} = C_6$, then both x_1 and y_2 are vertices of the C_6. We have 3 cases, up to isomorphism, shown below.

In the first graph, $d(x_1, y_2) = 1$, and x is in a C_4, as before we have $f'(H) \leq \alpha^{-2} < c_1$. In the second graph, x is in a C_5, a contradiction. For the third graph, $H = J$, a contradiction. $\square$

Claim 36. *If H has no Type I vertex, x is a Type II vertex, and $H - \{x, y_1\} = J$, then $f'(H) \leq c_1$.*

Proof. If x is a Type II vertex and $H - \{x, y_1\} = J$, then both x_1 and y_2 are vertices of degree 2 in J. As H is bipartite, we have two cases, up to isomorphism, shown below.

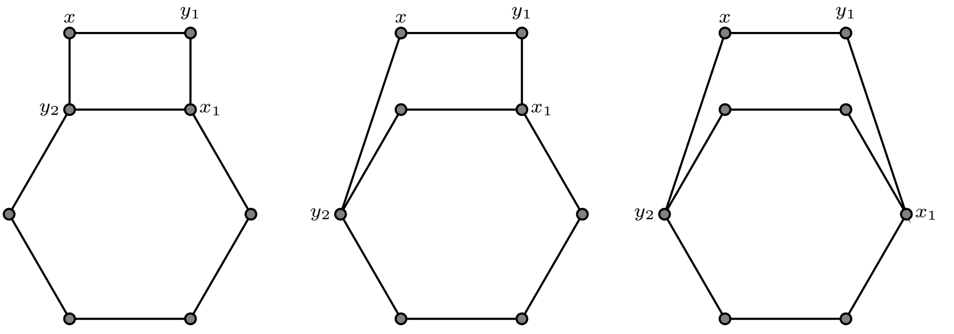


Fig. 29. Cases in Claim 35.

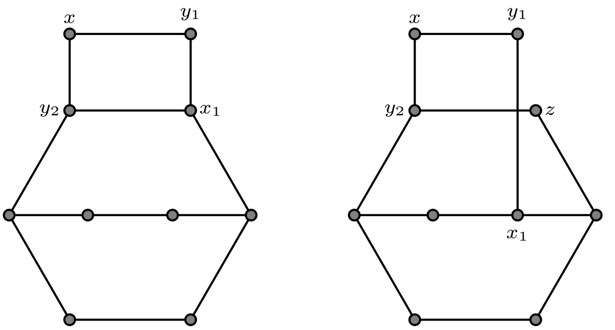


Fig. 30. Cases in Claim 36.

Either x is in a C_4, see the graph on the left side on Fig. 30, or there is a Type I vertex, namely z, see the graph on the right side on Fig. 30. We conclude both cases were handled before, hence $f'(H) \leq c_1$. $\square$

This concludes the proof of Theorem 3 since for a not necessarily C_4-free graph H we can use Claims 31 and 32 to obtain the statement similar to Claim 11 for determinants; Claims 33 and 34 to obtain Claim 13 for determinants; and Claims 35 and 36 to obtain Claim 14 for determinants. The only new case we have to deal while removing the C_4-freeness assumption is when a vertex of degree 2 is contained in a C_4, and this case is dealt with in Claim 30. In fact, we conclude more than Theorem 3, namely that

(a) $\det(G) \leq \alpha^k$ for any balanced bipartite graph G with $2n$ vertices and $n+k$ edges.
(b) When H is a connected balanced bipartite graph with $2n$ vertices, $n+k$ edges, $\delta(H) \leq 2$, $\Delta(H) \leq 3$, and H is not isomorphic to K_2, C_6, or J. Then $\det(H) \leq c_1 \cdot \alpha^k$.
(c) When H is a connected balanced bipartite graph with $2n$ vertices, $n+k$ edges, $\delta(H) \leq 2$, and H is not isomorphic to K_2, C_6, or J. Then $\det(H) \leq c_2 \cdot \alpha^k$.

6. Final remarks and open problems

We conclude that even that the initial aspiration was to prove Corollary 4, which is tight for every n multiple of 3, our generalized result of Theorem 3 is best possible only when $k \leq n$. In particular, Corollary 5 is not expected to be optimal. As mentioned before, Bruhn and Rautenbach [3] noted that the incidence matrix of the Fano plane has determinant 24. The graph formed by vertex disjoint copies of them gives a lower bound of $24^{n/7}$ for the maximum determinant of matrices with at most $3n$ ones.

Conjecture 1 *(Bruhn, Rautenbach). If* $A \in \{0,1\}^{n\times n}$ *has at most* $3n$ *non-zero entries, then* $\det(A) \leq 24^{n/7}$.

Similar questions for permanents can also be examined. Somewhat surprisingly, the permanent of the incidence matrix of the Fano plane is equal to its determinant. We conjecture that this is the maximum permanent among C_4-free bipartite graphs as well. Note that a variant of our method might solve this conjecture, however, we have not attempted to do so.

Conjecture 2. *If* $A \in \{0,1\}^{n\times n}$ *is* C_4*-free and has at most* $3n$ *non-zero entries, then* $\mathrm{perm}(A) \leq 24^{n/7}$.

Intuitively, to maximize the number of perfect matchings, all vertices should be in the largest possible number of short cycles. Therefore, the optimal regular graphs should be bipartite graphs with small girth and the least number of vertices. The existence of k-regular bipartite graphs with girth 6 is known for all $k \geq 2$ (see [1] for an example based on finite projective planes). Let $A_{k,6}$ denote the smallest k-regular bipartite graph with girth 6, and let $2v_k$ be the number of vertices in such graph. We conclude with the following more general conjecture.

Conjecture 3. *If* $A \in \{0,1\}^{n\times n}$ *is* C_4*-free and has at most* kn *non-zero entries, then*

$$\mathrm{perm}(A) \leq \mathrm{perm}(A_{k,6})^{n/v_k}.$$

Declaration of competing interest

There is no competing interest.

Acknowledgements

We thank the anonymous referee for their careful reading and useful comments that simplified the proof of some claims and improved the presentation of this manuscript.

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