



The Large Deviation Principle for Interacting Dynamical Systems on Random Graphs

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Abstract: Using the weak convergence approach to large deviations, we formulate and prove the large deviation principle (LDP) for W -random graphs in the cut-norm topology. This generalizes the LDP for Erdős–Rényi random graphs by Chatterjee and Varadhan. Furthermore, we translate the LDP for random graphs to a class of interacting dynamical systems on such graphs. To this end, we demonstrate that the solutions of the dynamical models depend continuously on the underlying graphs with respect to the cut-norm and apply the contraction principle.

1. Introduction

The problem of the macroscopic description of motion of interacting particles has a long history [11, 15]. When the number of particles is large, the analysis of individual trajectories becomes intractable and one is led to study statistical distribution of particles in the phase space. This is done using the Vlasov equation or other kinetic equations describing the state of the system in the continuum limit as the size of the system goes to infinity [3, 10, 25]. Modern applications ranging from neuronal networks to power grids feature models with spatially structured interactions. The derivation of the continuum limit for such models has to deal with the fact that in contrast to the classical setting used in [3, 10, 25], the particles are no longer identical, and it has also to take into account the limiting connectivity of the network assigned by the underlying graph sequences. This problem was addressed in [22, 23], where the ideas from the theory of graph limits [19] were used to formulate and to justify the continuum limit for interacting dynamical systems on certain convergent graph sequences. In particular, in [22] and in the followup paper [24], solutions of coupled dynamical systems on a sequence of W -random graphs were approximated by those of a deterministic nonlocal diffusion equation on a unit

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interval, representing a continuum of nodes in the spirit of the theory of graph limits. This result can be interpreted as a Law of Large Numbers for the solutions of the initial value problems (IVPs) of interacting dynamical systems on W-random graphs. In the present paper, we study the accuracy of the continuum limit for coupled dynamical systems on W-random graphs at the level of large deviations, i.e., we are interested in exponentially small probabilities of $O(1)$ deviations of the solutions of the discrete system from their typical behavior. This is the main goal of the work.

Motivated by applications, we consider the following model of n interacting particles

$$\dot{u}_i^n = f(u_i^n, \xi_i^n, t) + \frac{1}{n} \sum_{j=1}^n X_{ij}^n D(u_i^n, u_j^n), \quad (1.1)$$

$$u_i^n(0) = g_i^n, \quad i \in [n] := \{1, 2, \dots, n\}, \quad (1.2)$$

where $u_i^n : \mathbb{R}^+ \rightarrow \mathbb{X}$ stands for the state of particle i , f describes its intrinsic dynamics, and D models the pairwise interactions between the particles. The network connectivity is defined by the graph Γ^n with the adjacency matrix $\{X_{ij}^n\}$. The phase space $\mathbb{X}$ can be either $\mathbb{R}$, or $\mathbb{R}/\mathbb{Z}$, or $\mathbb{R}^d$ depending on the model at hand. Parameters $\xi_i^n \in \mathbb{R}^p$ and initial conditions g_i^n , $i \in [n]$, are random in general. Many network models in science and technology fit into the framework (1.1), (1.2). Examples include neuronal networks, power grid models, and various coupled oscillator systems to name a few. We present the Kuramoto model of coupled phase oscillators [17], as an illustrative example.

Example 1.1. In the Kuramoto model, $\mathbb{X} = \mathbb{R}/\mathbb{Z}$, $f(u, \xi, t) = \xi$, and $D(u, v) = \sin(2\pi(v - u))$. $\{\xi_i^n\}$ are independent and identically distributed (iid) random variables. This model was studied using Erdős–Rényi, small-world, and power law random graphs (see [24] and references therein).

In this paper, we use W-random graphs to define network connectivity in (1.1). This is a flexible framework for modeling random graphs [21], which fits seamlessly into the analysis of the continuum limit of interacting dynamical systems like (1.1) [24]. Specifically, given $W \in \mathcal{S} = \{U \in L^\infty([0, 1]^2) : 0 \leq U \leq 1\}$, which prescribes the asymptotic behavior of $\{\Gamma^n\}$, we define $\{X_{ij}^n, (i, j) \in [n]^2\}$ as independent random variables such that

$$\mathbb{P}(X_{ij}^n = 1) = W_{ij}^n \quad \text{and} \quad \mathbb{P}(X_{ij}^n = 0) = 1 - W_{ij}^n, \quad (1.3)$$

where

$$W_{ij}^n = n^2 \int_{Q_{ij}^n} W(x, y) dx dy, \quad Q_{ij}^n = Q_i^n \times Q_j^n, \quad Q_i^n = \left[\frac{i-1}{n}, \frac{i}{n} \right] \quad i, j \in [n]. \quad (1.4)$$

For large n the direct analysis of (1.1), (1.2) is not feasible and one is led to seek other ways. A common alternative to studying individual trajectories of (1.1), (1.2) is to consider a continuum limit as the size of the system tends to infinity. In this case, under the suitable assumptions on f , D , and W the discrete model (1.1), (1.2) can be approximated by the following continuum limit (cf. [24]):

$$\partial_t u(t, x) = f(u(t, x), t) + \int W(x, y) D(u(t, x), u(t, y)) dy, \quad (1.5)$$

$$u(0, x) = g(x). \quad (1.6)$$

Here and below, if the domain of integration is not specified, it is implicitly assumed to be $[0, 1]$. Also, we have dropped the dependence on ξ and let $\mathbb{X} = \mathbb{R}$ to simplify the presentation. At the end of the paper, we comment on how to extend the analysis to cover models depending on random parameters. Until then we will study the following discrete model:

$$\dot{u}_i^n = f(u_i^n, t) + n^{-1} \sum_{j=1}^n X_{ij}^n D(u_i^n, u_j^n), \quad (1.7)$$

$$u_i^n(0) = g_i^n, \quad i \in [n], \quad (1.8)$$

where $u_i^n : \mathbb{R}^+ \rightarrow \mathbb{R}$ and the rest is the same as in (1.1), (1.2).

Let $C([0, T], L^2([0, 1]))$ stand for the space of continuous vector-valued functions $[0, T] \ni t \mapsto u(t, \cdot) \in L^2([0, 1])$ equipped with the norm (cf. [18])

$$\|u\|_{C([0, T], L^2([0, 1]))} = \sup_{t \in [0, T]} \|u(t, \cdot)\|_{L^2([0, 1])}.$$

To compare solutions of the discrete and continuous models, we represent the former as an element of $C([0, T], L^2([0, 1]))$:

$$u^n(t, x) := \sum_{i=1}^n u_i^n(t) \mathbf{1}_{Q_i^n}(x), \quad (1.9)$$

where $\mathbf{1}_A$ stands for the indicator function of A . Then for the model (1.7) with a sequence of W-random graphs (1.3), (1.4) with deterministic initial conditions it was shown in [24] that

$$\lim_{n \rightarrow \infty} \|u^n - u\|_{C([0, T], L^2([0, 1]))} = 0 \quad \text{a.s.}$$

This statement can be interpreted as the Law of Large Numbers (LLN) for (1.7). Note that the continuum limit (1.5) is deterministic, while the discrete models (1.7) are posed on random graphs. Therefore, the solution of (1.5), (1.6) presents the typical behavior of the solutions of the discrete system (1.7), (1.8) on random graphs for large n . We are interested in the deviations of u^n from this typical behavior. Below we formulate and prove an LDP for solutions of the discrete model (1.7), (1.8). Before we address this problem, we first establish an LDP for a sequence of W-random graphs $\{\Gamma^n\}$. To this end, we represent them as elements of $\mathcal{S}$ through

$$H^n = \sum_{i,j=1}^n X_{ij}^n \mathbf{1}_{Q_{ij}^n}, \quad (1.10)$$

where $\{X_{ij}^n\}$ is the adjacency matrix of Γ^n . Then using the weak convergence method [4], one can show (see Theorem 4.1) that $\{H^n\}$, or to be more precise a sequence of equivalence classes for which each H^n provides a representative element, satisfies an LDP. This LDP for W-random graphs generalizes the LDP for Erdős–Rényi graphs in [8] and gives logarithmic asymptotics. The LDP is established using the same cut norm topology on $\mathcal{S}$ as in [8], which turns out to be suitable for our later application to dynamical models.

Remark 1.2. By construction, $\{\Gamma^n\}$ is a sequence of random directed graphs. The definition of Γ^n can be easily modified if graphs Γ^n are assumed to be undirected instead. To this end, the entries X_{ij}^n , $1 \leq i \leq j \leq n$, are defined as above and the rest are defined by symmetry: $X_{ij}^n = X_{ji}^n$, $1 \leq j < i < n$. Also, the scaling sequence and the rate function used in Theorem 4.1 need to be modified accordingly.

To translate the LDP for W-random graphs to the space of solutions of (1.5), we use the contraction principle [4]. To this end, we need to show that the solutions of the IVP for (1.5) depend continuously on $W \in \mathcal{S}$ in the appropriate topology. It would be natural and easier to establish an LDP for $\{\Gamma^n\}$ in the weak topology. However, the weak topology is not enough to construct the continuous mapping from $\mathcal{S}$ to $C([0, T], L^2([0, 1]))$, the space of solutions of (1.5), (1.6). Conversely, the strong topology, which would guarantee the continuous dependence, is too discriminate. Random graph sequences, like Erdős–Rényi graphs, do not converge in the L^2 -norm. This suggests that like in combinatorial problems involving random graphs (cf. [6]), the right topology for the model at hand is that generated by the cut-norm. On the one hand, it metrizes graph convergence [19], i.e., the random graph sequence used in (1.7) converges in the cut-norm. On the other hand, the cut-norm is strong enough to provide continuous dependence of solutions of (1.5), (1.6) on W . It is these considerations that motivate the use of the cut topology in the problem of large deviations for random graphs.

This work fits into two partially independent lines of research. On the one hand, there has been an interest in developing the theory of large deviations for large random graphs. This research is motivated by questions in combinatorics. Recently, building on the results of the theory of graph limits Chatterjee and Varadhan proved an LDP for Erdős–Rényi random graphs in cut-norm topology [8]. Our Theorem 4.1 generalizes the LDP of Chatterjee and Varadhan to W-random graphs, a large class of random graphs. We use the weak convergence techniques for large deviations [4], which afford a short proof of the LDP. On the other hand, there has been a search for rigorous methods for studying large networks of interacting dynamical systems. This research is motivated by problems in statistical physics, and it has been reinvigorated by widespread presence of networks in modern science. The continuum limit is one of the main tools for analyzing dynamics of large networks. The main result of this paper provides fine estimates of the accuracy of the continuum limit approximation developed in [24] for a large class of models on W-random graphs. Previous studies of large deviations for interacting dynamical systems on random graphs like (1.7) considered models forced by white noise. For such models in [9, 26] it was shown that if a spatially averaged model satisfies an LDP with respect to white noise forcing then so will the original model on the random graph. This does not address large deviations due to random connectivity. The rate function derived in this paper gives an explicit relation between the random connectivity of the network and the variability of the network dynamics, which is often sought in applications.

After this paper was submitted for publication, there was progress on large deviations for block and step graphon random graph models [2, 12]. Using our terminology, these papers present LDPs for W-random graph sequences, for which the graphon W is a step function. The rate functions and the scaling sequences derived in these papers are consistent with our results for bounded graphons. The proofs of the LDPs in [2, 12] are built upon the method of Chatterjee and Varadhan [6, 8]. We rely on the weak convergence techniques [4].

The outline of the paper is as follows. In the next section we formulate the assumptions on the model and impose random initial conditions. In Sect. 3, we review certain facts from the theory of graph limits [19], which will be used in the main part of the paper.

In Sect. 4, we formulate the LDPs for the combinatorial and dynamical problems. In Sect. 5, we prove the LDP for W-random graphs. In Sect. 6, we establish the contraction principle relating the LDPs for the combinatorial and dynamical models. In Sect. 7, the LDP is used to characterize the most likely realizations of the W-random graph model conditioned on large deviations from the expected value of the number of edges. In this case, we provide an explicit solution of the constrained optimization problem, which yields asymptotic edge distribution for the model at hand. Certain extensions of the main result are discussed in Sect. 8, and the lower semicontinuity of the rate function is proved in a concluding appendix.

2. The Model

In this section, we formulate our assumptions on the dynamical model (1.7), (1.8), except for assumptions on $\{X_{ij}^n\}$, which were given in (1.3), (1.4). Functions f and D describe the intrinsic dynamics of individual particles and interactions between two particles at the adjacent nodes of Γ^n respectively. We assume that $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is bounded, and uniformly Lipschitz continuous in u in that

$$|f(u, t) - f(v, t)| \leq L_f |u - v| \quad u, v \in \mathbb{R}, \quad t \in \mathbb{R}, \quad (2.1)$$

and continuous in t for each fixed u . D is a bounded and Lipschitz continuous function:

$$|D(u, v) - D(u', v')| \leq L_D (|u - u'| + |v - v'|). \quad (2.2)$$

By rescaling time in (1.7) if necessary, one can always achieve that f and D are bounded by 1. Thus, we assume

$$|f(u, t)| \leq 1 \quad \text{and} \quad |D(u, v)| \leq 1. \quad (2.3)$$

Finally, for the contraction principle in Sect. 6 we will need in addition to assume that $D \in H_{\text{loc}}^s(\mathbb{R}^2)$, $s > 1$, where H_{loc}^s stands for the Sobolev space of functions on $\mathbb{R}^2$ that are square integrable together with their generalized derivatives up to order s on any compact subset of $\mathbb{R}^2$.

We now turn to the initial condition. Let $\mathcal{B} = L^2([0, 1])$ with the usual norm and associated topology. Assume that $\{G^n\}$ is a sequence of $\mathcal{B}$ -valued random variables that are independent of $\{X_{ij}^n\}$ and that satisfy an LDP with function K and scaling sequence n^2 . To define an initial condition for the discrete system, we let

$$g_i^n = n \int_{Q_i^n} G^n(y) dy \quad \text{for } x \in Q_i^n.$$

Suppose that $\bar{G}^n$ is defined by $\bar{G}^n(x) = g_i^n$ for $x \in Q_i^n$.

Then it is not automatic that $\{G^n\}$ and $\{\bar{G}^n\}$ have the same large deviation asymptotics, and so we impose the following.

Assumption 2.1. $\{\bar{G}^n\}$ satisfies the LDP in $\mathcal{B}$ with the rate function K and scaling sequence n^2 .

Remark 2.2. As an alternative condition we could have simply assumed a large deviation property of $\{\bar{G}^n\}$. However, it seems easier to pose conditions on $\{G^n\}$ under which an LDP holds than to pose conditions on $\{g_i^n, i \in [n]\}$.

Example 2.3. If $G^n = g$ for some fixed deterministic $g \in B$ then Assumption 2.1 holds with K defined by $K(h) = 0$ if $h = g$ and $K(h) = \infty$ otherwise. Indeed, in this case we have

$$\begin{aligned} 0 \leq \|g - \bar{G}^n\|_{L^2([0,1])}^2 &= \sum_{i=1}^n \int_{Q_i^n} \left[g(x) - n \int_{Q_i^n} g(y) dy \right]^2 dx \\ &= \int_{[0,1]} g(x)^2 dx - \frac{1}{n} \sum_{i=1}^n \left[n \int_{Q_i^n} g(y) dy \right]^2. \end{aligned}$$

Letting $f_n(x) = n \int_{Q_i^n} g(y) dy$ for $x \in Q_i^n$ we find $f_n(x) \rightarrow g(x)$ a.e. with respect to Lebesgue measure, and thus by Fatou's lemma

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \left[n \int_{Q_i^n} g(y) dy \right]^2 \geq \int_{[0,1]} g(x)^2 dx.$$

Hence $\|g - \bar{G}^n\|_{L^2([0,1])} \rightarrow 0$.

Example 2.4. Suppose that there is $M < \infty$ such that G^n is Lipschitz continuous with constant M almost surely (a.s.). Then

$$\begin{aligned} \|G^n - \bar{G}^n\|_{L^2([0,1])}^2 &= \sum_{i=1}^n \int_{Q_i^n} \left[G^n(x) - n \int_{Q_i^n} G^n(y) dy \right]^2 dx \\ &\leq \sum_{i=1}^n \int_{Q_i^n} \left[M \frac{1}{n} \right]^2 dx \\ &= M^2 \frac{1}{n^2} \rightarrow 0. \end{aligned}$$

Since the convergence is uniform in ω $\{\bar{G}^n\}$ satisfies the same LDP as $\{G^n\}$, and therefore Assumption 2.1 holds. Note that the Lipschitz condition is stronger than needed. For instance, it can be relaxed to requiring that G^n belongs to a generalized Lipschitz space [16, Lemma 5.2].

Example 2.5. Suppose there is a probability distribution μ on $\mathbb{R}$ with bounded support (i.e., $M < \infty$ such that $\mu([-M, M]^c) = 0$) and that $\{h_i^n\}$ are iid μ for $n \in \mathbb{N}$ and $i \in [n^2]$. Define $F^n(x) = h_i^n$ for $x \in [i/n^2, (i+1)/n^2)$, and identify F^n with its periodic extension to $\mathbb{R}$. Let $\rho \geq 0$ be a smooth convolution kernel with compact support and define

$$G^n(x) = \int_{\mathbb{R}} \rho(x - y) F^n(y) dy.$$

Then Assumption 2.1 holds. Indeed, in this case one can show that $\{F^n\}$ satisfies the LDP on B with the weak topology on B and with the rate function

$$J(\ell) = \int_{[0,1]} L(\ell(x)) dx,$$

where $L(b) = \sup_{a \in \mathbb{R}} [ab - \log \int e^a \mu(da)]$.¹ If $\int_{\mathbb{R}} h f_n dx \rightarrow \int_{\mathbb{R}} h f dx$ for all $h \in \mathcal{B}$ then, in particular, $\int_{\mathbb{R}} \rho(x-y) f_n(y) dy \rightarrow \int_{\mathbb{R}} \rho(x-y) f(y) dy$ for each $x \in \mathbb{R}$. If in addition $|f_n(y)| \leq M$ for all $n \in \mathbb{N}$ and $y \in [0, 1]$ then $\{\int_{\mathbb{R}} \rho(x-y) f_n(y) dy\}$ are uniformly equicontinuous, and thus the convergence is uniform in x . Therefore

$$f \mapsto \int_{\mathbb{R}} \rho(x-y) f(y) dy$$

is a continuous mapping from $\mathcal{B} \cap \{f : |f(y)| \leq M \text{ for } y \in [0, 1]\}$ into itself, but with the weak topology on the domain and the strong topology on the range. By the contraction principle $\{G^n\}$ satisfies the LDP on $\mathcal{B}$ with rate function

$$K(h) = \inf \left\{ J(\ell) : h(x) = \int_{\mathbb{R}} \rho(x-y) \ell(y) dy \right\}.$$

The scaling used in Example 2.5 is needed so that the large deviation scaling sequence of the initial conditions matches that of the random graph. If another scaling is used that produces a different large deviation scaling sequence, e.g. $n^{2\alpha}$, then when $\alpha > 1$ the rate function for the initial conditions does not appear in the rate function for $\{u^n\}$, and from the perspective of large deviations the initial conditions are deterministic. If however $\alpha < 1$ then the scaling sequence for $\{u^n\}$ is necessarily $n^{2\alpha}$, and the LDP for $\{u^n\}$ will not reflect the randomness of $\{X_{ij}^n\}$.

3. The Space of Graphons

The key ingredient in the dynamical network models formulated in the previous section is a sequence of random adjacency matrices $\{X_{ij}^n\}$. The corresponding kernels H^n and their (averaged) limits W are called graphons in the language of the graph theory [19]. Before we can formulate the LDPs for dynamical models, we first need to understand large deviations for random graphons $\{H^n\}$. To this end, in this section, we review certain facts about graphons.

Recall the collection of random variables $\{X_{ij}^n, i, j \in [n]\}$ (cf. (1.3), (1.4)). Given such random variables, we define $H^n : [0, 1]^2 \rightarrow [0, 1]$ by

$$H^n = \sum_{i,j=1}^n X_{ij}^n \mathbf{1}_{Q_{ij}^n}. \quad (3.1)$$

We view $\{H^n\}$ as taking values in $\mathcal{S}$, the space of measurable functions from $[0, 1]^2$ to $[0, 1]$. $\mathcal{S}$ is equipped with the $\infty \rightarrow 1$ distance

$$d_{\infty \rightarrow 1}(f, g) = \sup_{-1 \leq a, b \leq 1} \left| \int_{[0,1]^2} a(t)b(s)[f(t,s) - g(t,s)] dt ds \right|, \quad (3.2)$$

¹ Several proofs are available, including one that extends Sanov's theorem. However, the most direct argument is to note that Mogulskii's theorem asserts that if $Y^n(x) = \int_0^x F^n(t) dt$, then $\{Y^n\}$ satisfies an LDP in $C([0, 1])$ with the rate function $I(\phi)$ equal to $\int_0^1 L(\dot{\phi}(t)) dt$ if ϕ is absolutely continuous with $\phi(0) = 0$ and ∞ otherwise. Using $F^n(x) = \dot{Y}^n(x)$ a.s. in x , we can find the result stated for the sequence $\{F^n\}$ using integration by parts.

where $a, b : [0, 1] \rightarrow [-1, 1]$ are measurable functions. The $\infty \rightarrow 1$ distance is derived from the $L^\infty \rightarrow L^1$ operator norm

$$\|W\|_{\infty \rightarrow 1} = \sup_{-1 \leq a, b \leq 1} \int_{[0,1]^2} a(x)b(y)W(x,y)dx dy, \quad (3.3)$$

which in turn is equivalent to the cut norm

$$\|W\|_{\square} := \sup_{S,T} \left| \int_{S \times T} W(x,y)dx dy \right| = \sup_{0 \leq a, b \leq 1} \left| \int_{[0,1]^2} a(x)b(y)W(x,y)dx dy \right|, \quad (3.4)$$

where the first supremum is taken over all measurable subsets of $[0, 1]$. In particular, we have (cf. [19, Lemma 8.11])

$$\|W\|_{\square} \leq \|W\|_{\infty \rightarrow 1} \leq 4\|W\|_{\square}. \quad (3.5)$$

$\mathcal{S}^n \subset \mathcal{S}$ stands for the set of piecewise constant functions with respect to the partition $\{Q_{ij}^n\}$. Specifically, $H^n \in \mathcal{S}^n$ is constant on each Q_{ij}^n . The $\infty \rightarrow 1$ distance on $\mathcal{S}^n$ is equivalent to

$$d_{\infty \rightarrow 1}^n(f, g) = \sup_{a^n, b^n} \frac{1}{n^2} \sum_{i,j=1}^n a_i^n b_j^n [f(i/n, j/n) - g(i/n, j/n)], \quad (3.6)$$

where $a^n = (a_1^n, a_2^n, \dots, a_n^n)$, $b^n = (b_1^n, b_2^n, \dots, b_n^n)$, and each $a_i^n, b_i^n \in [-1, 1]$, $i \in [n]$.

Let $\mathcal{P}$ be the set of all measure preserving bijections of $[0, 1]$. For every $\sigma \in \mathcal{P}$ and $f \in \mathcal{S}$ define

$$f_\sigma(t, s) = f(\sigma(t), \sigma(s)). \quad (3.7)$$

This defines an equivalence relation on $\mathcal{S}$. Two elements f and g of $\mathcal{S}$ are equivalent, $f \sim g$, if $g = f_\sigma$ for some $\sigma \in \mathcal{P}$. By identifying all elements in the same equivalence class, we obtain the quotient space $\hat{\mathcal{S}} = \mathcal{S}/\sim$. The distance on $\hat{\mathcal{S}}$ is defined as follows:

$$\delta_{\infty \rightarrow 1}(f, g) = \inf_{\sigma} d_{\infty \rightarrow 1}(f_\sigma, g) = \inf_{\sigma} d_{\infty \rightarrow 1}(f, g_\sigma)$$

By the Weak Regularity Lemma, $(\hat{\mathcal{S}}, \delta_{\infty \rightarrow 1})$ is a compact metric space [20].

4. The LDPs

For $\hat{V} \in \hat{\mathcal{S}}$ let

$$I(\hat{V}) = \inf_{V \in \hat{V}} \Upsilon(V, W), \quad (4.1)$$

where Υ is defined by

$$\begin{aligned} \Upsilon(V, W) &= \int_{[0,1]^2} R(\{V(y), 1 - V(y)\} \| \{W(y), 1 - W(y)\}) dy \\ &= \int_{[0,1]^2} \left\{ V(y) \log \left(\frac{V(y)}{W(y)} \right) + (1 - V(y)) \log \left(\frac{1 - V(y)}{1 - W(y)} \right) \right\} dy, \end{aligned} \quad (4.2)$$

and $R(\theta \parallel \mu)$ is the relative entropy of probability measures θ and μ , i.e.,

$$R(\theta \parallel \mu) = \int \left(\log \frac{d\theta}{d\mu} \right) d\theta \quad (4.3)$$

if $\theta \ll \mu$ and $R(\theta \parallel \mu) = \infty$ otherwise.

Theorem 4.1. *Let $\{H^n\}$ be defined by (3.1). Then $\{\hat{H}^n\}_{n \in \mathbb{N}}$ satisfies the LDP with scaling sequence n^2 and rate function (4.1): I has compact level sets on $\hat{\mathcal{S}}$,*

$$\liminf_{n \rightarrow \infty} \frac{1}{n^2} \log P\{\hat{H}^n \in O\} \geq - \inf_{\hat{V} \in O} I(\hat{V})$$

for open $O \subset \hat{\mathcal{S}}$, and

$$\limsup_{n \rightarrow \infty} \frac{1}{n^2} \log P\{\hat{H}^n \in F\} \leq - \inf_{\hat{V} \in F} I(\hat{V})$$

for closed $F \subset \hat{\mathcal{S}}$.

We now turn to the dynamical model (1.7), (1.8). Below, it will be convenient to rewrite (1.7), (1.8) as

$$\partial_t u^n(t, x) = f(u^n(t, x), t) + \int H^n(x, y) D(u^n(t, x), u^n(t, y)) dy, \quad (4.4)$$

$$u^n(0, x) = g^n(x), \quad (4.5)$$

where as before

$$\begin{aligned} H^n(x, y) &:= \sum_{i,j=1}^n X_{ij}^n \mathbf{1}_{Q_{ij}^n}(x, y), \quad u^n(t, x) := \sum_{i=1}^n u_i^n(t) \mathbf{1}_{Q_i^n}(x), \quad \text{and} \\ g^n(x) &:= \sum_{i=1}^n g_i^n \mathbf{1}_{Q_i^n}(x). \end{aligned} \quad (4.6)$$

Recall that $\mathcal{B}$ and $\mathcal{S}$ stand for the space of initial conditions and the space of graphons respectively. We use the L^2 -distance on $\mathcal{B}$ and the $\infty \rightarrow 1$ distance on $\mathcal{S}$. Let $\mathcal{X} := \mathcal{S} \times \mathcal{B}$ endowed with the product topology. On $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{X}$ we define the equivalence relations:

$$\begin{aligned} g &\sim g' \quad \text{if} \quad g' = g_\sigma, \\ W &\sim W' \quad \text{if} \quad W' = W_\sigma, \end{aligned}$$

and

$$(W, g) \sim (W', g') \quad \text{if} \quad W' = W_\sigma \text{ \& } g' = g_\sigma \quad \text{for some } \sigma \in \mathcal{P}.$$

Define the quotient spaces $\hat{\mathcal{B}} = \mathcal{B}/\sim$ and $\hat{\mathcal{X}} := \mathcal{X}/\sim$. The distance on $\hat{\mathcal{X}}$ is given by

$$d_{\hat{\mathcal{X}}}(\widehat{(U, g)}, \widehat{(V, h)}) = \inf_{\sigma} \left\{ \|U_\sigma - V\|_{\infty \rightarrow 1} + \|g_\sigma - h\|_{L^2([0,1])} \right\}, \quad (4.7)$$

where $(U, g) \in \widehat{(U, g)}$ and $(V, h) \in \widehat{(V, h)}$ are arbitrary representatives.

Likewise, let $\mathcal{Y} := C([0, T], B)$ and $\hat{\mathcal{Y}} := C([0, T], \hat{B})$. $\hat{\mathcal{Y}}$ is a quotient space under the following relation:

$$\mathcal{Y} \ni u \sim u' \quad \text{if} \quad u'(t, x) = u(t, \sigma(x)), \quad (t, x) \in [0, T] \times [0, 1]$$

for some $\sigma \in \mathcal{P}$. The distance on $\hat{\mathcal{Y}}$ is given by

$$d_{\hat{\mathcal{Y}}}(\hat{u}, \hat{v}) = \inf_{\sigma} \|u_{\sigma} - v\|_{C(0, T; L^2([0, 1]))}, \quad (4.8)$$

where $u \in \hat{u}$ and $v \in \hat{v}$ are arbitrary representatives.

Given $(W, g) \in \mathcal{X}$ let $u \in \mathcal{Y}$ stand for the corresponding solution of the IVP (1.5), (1.6). By uniqueness of solution of the IVP (1.5), (1.6)

$$F : \mathcal{X} \ni (W, g) \mapsto u \in \mathcal{Y}$$

is well-defined. Furthermore, it maps all members of a given equivalence class of $\mathcal{X}$ to the same equivalence class of $\mathcal{Y}$:

$$F(W_{\sigma}, g_{\sigma}) = u_{\sigma} \quad \forall \sigma \in \mathcal{P}.$$

Thus, F may be viewed as a map between $\hat{\mathcal{X}}$ and $\hat{\mathcal{Y}}$.

Lemma 4.2. $F : \hat{\mathcal{X}} \rightarrow \hat{\mathcal{Y}}$ is a continuous mapping.

Lemma 4.2 will be proved in Sect. 6. With Theorem 4.1 and Lemma 4.2 in place, we use the Contraction Principle to derive the LDP for solutions of the discrete model (4.4), (4.5). In addition, Lemma 4.2 justifies (1.5), (1.6) as a continuum limit for discrete models (1.7), (1.8) on any convergent sequence of dense graphs.

We remind the reader that initial conditions are assumed to be independent of the random graph.

Theorem 4.3. For $W \in \mathcal{S}$ let $\{(H^n, g^n)\}$ be a sequence of random graphons and random initial data (cf. (4.6)), and let Assumption 2.1 hold. Denote by $\{u^n\}$ the corresponding solutions of (4.4), (4.5). Then $\{\hat{u}^n\}$ satisfies an LDP on $\hat{\mathcal{X}}$ with scaling sequence n^2 and the rate function

$$J(\hat{u}) = \inf \{I(\hat{W}) + K(\hat{g}) : (\widehat{W, g}) = F^{-1}(\hat{u})\}.$$

5. The Proof of Theorem 4.1

5.1. The weak convergence approach. The proof of Theorem 4.1 is based on the weak convergence method of [4]. We use a “test function” characterization of large deviations (see [4, Theorem 1.8]). The proof that I has compact level sets appears in the appendix. To complete the proof of the LDP for $\{\hat{H}^n\}$, it is sufficient to show that for each bounded and continuous (with respect to $\delta_{\infty \rightarrow 1}$) $G : \hat{\mathcal{S}} \rightarrow \mathbb{R}$,

$$-\frac{1}{n^2} \log E e^{-n^2 G(\hat{H}^n)} \rightarrow \inf_{\hat{V} \in \hat{\mathcal{S}}} [I(\hat{V}) + G(\hat{V})] \quad \text{as } n \rightarrow \infty.$$

At the heart of the weak convergence approach lies the following representation for the Laplace integrals:

$$-\frac{1}{n^2} \log E e^{-n^2 G(\hat{F}^n)} = \inf E \left[\frac{1}{n^2} R(\theta^n \parallel \mu^n) + G(\hat{F}^n) \right], \quad (5.1)$$

where μ^n is the product measure corresponding to $\{X_{ij}^n\}$ on $\{0, 1\}^{n^2}$ and the infimum in (5.1) is taken over all probability measures θ^n on $\{0, 1\}^{n^2}$ (cf. [4, Proposition 2.2]). Here $\hat{F}^n$ is analogous to $\hat{F}$, in that

$$\bar{F}^n(y) = \bar{X}_{ij}^n \quad \text{for } y \in \mathcal{Q}_{ij}^n,$$

where $\{\bar{X}_{ij}^n\}$ has joint distribution θ^n , and $\hat{F}^n$ is the corresponding equivalence class.

By (5.1), the proof of Theorem 4.1 is reduced to showing the convergence of variational problems: for each bounded and continuous G

$$\inf_{\theta^n} E \left[\frac{1}{n^2} R(\theta^n \parallel \mu^n) + G(\hat{F}^n) \right] \rightarrow \inf_{\hat{V} \in \hat{\mathcal{S}}} [I(\hat{V}) + G(\hat{V})] \quad \text{as } n \rightarrow \infty. \quad (5.2)$$

5.2. A law of large numbers type result. Let $a_k, k = 1, \dots, n^2$ be some enumeration of the points in $\{1, \dots, n\}^2$, and let $k(i, j)$ be defined by $a_{k(i, j)} = (i, j)$. Let $\bar{\theta}_k^n$ be (a version of) the conditional distribution on variable $\bar{X}_{a_k}^n$, given $\bar{X}_{a_s}^n, s < k$. Thus for $m = 0, 1$,

$$\bar{\theta}_k^n(\{m\})(\omega) = P \left\{ \bar{X}_{a_k}^n = m \mid \bar{X}_{a_1}^n, \dots, \bar{X}_{a_{k-1}}^n \right\}(\omega).$$

We can decompose θ^n and μ^n into products of these conditional distributions, and then by the chain rule (see for example [4, Proposition 3.1]),

$$E \left[\frac{1}{n^2} R(\theta^n \parallel \mu^n) + G(\hat{H}^n) \right] = E \left[\frac{1}{n^2} \sum_{k=1}^{n^2} R(\bar{\theta}_k^n \parallel \mu_k^n) + G(\hat{H}^n) \right], \quad (5.3)$$

where $\mu_k^n(A) = P(X_{a_k}^n \in A)$. Note that $\bar{\theta}_k^n$ is random and measurable with respect to $\mathcal{F}_{k-1}^n$, where $\mathcal{F}_k^n = \sigma(\bar{X}_{a_s}^n, s \leq k)$, while μ_k^n is deterministic. For an analogous calculation but with more details see [4, Sect. 3.1].

We would like to relate the weak limits of $\{\hat{H}^n\}$ to a function that measures the “new” link probabilities under θ^n , as well as the cost to produce these new probabilities. The original probabilities are $\mu_k^n(\{1\})$, and the new ones are $\bar{\theta}_k^n(\{1\})$. Let

$$\bar{M}^n(y) = \bar{\theta}_k^n(\{1\}) \quad \text{if } y \in \mathcal{Q}_{ij}^n.$$

Note that $\{\bar{M}^n\}$ are random variables with values in $\mathcal{S}$, and and that since $\hat{\mathcal{S}}$ is compact $\{\hat{M}^n\}$ and $\{\hat{H}^n\}$ are automatically tight.

Letting

$$W^n(x, y) = \mu_k^n(\{1\}) \quad \text{if } (x, y) \in \mathcal{Q}_{ij}^n,$$

we can write

$$\frac{1}{n^2} \sum_{k=1}^{n^2} R(\bar{\theta}_k^n \parallel \mu_k^n) = \Upsilon(\bar{M}^n, W^n), \quad (5.4)$$

where $\Upsilon(\cdot, \cdot)$ is defined in (4.2). Note that while W^n is deterministic, $\bar{M}^n$ need not be. We will also want to note that trivially

$$\Upsilon(H, W) \geq \inf_{\sigma} \Upsilon(H_{\sigma}, W)$$

for any $H, W \in \mathcal{S}$.

We next state a LLN type result for the sequence of “controlled” random graphs $\{\bar{H}^n\}$.

Lemma 5.1. *For any $\delta > 0$*

$$P(d_{\infty \rightarrow 1}^n(\bar{H}^n, \bar{M}^n) \geq \delta) \rightarrow 0,$$

and therefore for any $\delta > 0$

$$P(\delta_{\infty \rightarrow 1}(\hat{\bar{H}}^n, \hat{\bar{M}}^n) \geq \delta) \rightarrow 0.$$

To prove Lemma 5.1, we use a new version of the Bernstein bound that allows dependence between the random variables.

Lemma 5.2. *Let $\{Z_i, i = 1, \dots, N\}$ be random variables with the following properties.*

1. $|Z_i| \leq c < \infty$ a.s.,
2. *There is a filtration $\{\mathcal{F}_i\}$ such that each Z_j for $1 \leq j < i$ is $\mathcal{F}_i$ -measurable. Let $m_i = E[Z_i | \mathcal{F}_i]$. Then for $\delta > 0$*

$$P\left(\frac{1}{N} \sum_{i=1}^N (Z_i - m_i) \geq \delta\right) \leq e^{-Nh(\delta/c)},$$

where $h(u) = (1+u) \log(1+u) - u > 0$ for $u > 0$.

Proof. Since $|Z_i| \leq c$, the conditional distribution of $|Z_i - m_i|$ given $\mathcal{F}_i$ is also bounded uniformly by c . By straightforward calculations using Taylor’s theorem,

$$E\left[e^{\alpha(Z_i - m_i)} | \mathcal{F}_i\right] \leq e^{e^{\alpha c} - 1 - \alpha c} \quad \text{a.s.}$$

(the same calculation is used in the proof of the Bernstein bound). For any $\alpha > 0$

$$\begin{aligned} P\left(\frac{1}{N} \sum_{i=1}^N (Z_i - m_i) \geq \delta\right) &= P\left(e^{\alpha \sum_{i=1}^N (Z_i - m_i)} \geq e^{N\alpha\delta}\right) \\ &\leq e^{-N\alpha\delta} E e^{\alpha \sum_{i=1}^N (Z_i - m_i)}. \end{aligned}$$

We then bound $E e^{\alpha \sum_{i=1}^N (Z_i - m_i)}$ by recurring backwards from $i = N$:

$$\begin{aligned} E e^{\alpha \sum_{i=1}^N (Z_i - m_i)} &= E \left[E \left[e^{\alpha \sum_{i=1}^N (Z_i - m_i)} \middle| \mathcal{F}_N \right] \right] \\ &= E \left[E \left[e^{\alpha(Z_N - m_N)} \middle| \mathcal{F}_N \right] e^{\alpha \sum_{i=1}^{N-1} (Z_i - m_i)} \right] \\ &\leq E e^{\alpha \sum_{i=1}^{N-1} (Z_i - m_i)} e^{e^{\alpha c} - 1 - \alpha c} \\ &\leq e^{N(e^{\alpha c} - 1 - \alpha c)}. \end{aligned}$$

Thus

$$P\left(\frac{1}{N} \sum_{i=1}^N (Z_i - m_i) \geq \delta\right) \leq e^{-N\alpha\delta} e^{N(e^{\alpha c} - 1 - \alpha c)}.$$

Now optimize on $\alpha > 0$. Calculus gives $\delta - ce^{\alpha c} + c = 0$, so

$$e^{\alpha c} = \frac{c + \delta}{c} \quad \text{or} \quad \alpha = \frac{1}{c} \log \left(\frac{c + \delta}{c} \right) > 0.$$

This choice gives the value

$$\begin{aligned} \delta \alpha - e^{\alpha c} + 1 + \alpha c &= \frac{\delta}{c} \log \left(\frac{c + \delta}{c} \right) - \left(\frac{c + \delta}{c} \right) + 1 + \log \left(\frac{c + \delta}{c} \right) \\ &= -\frac{\delta}{c} + \left(1 + \frac{\delta}{c} \right) \log \left(1 + \frac{\delta}{c} \right) \\ &= h \left(\frac{\delta}{c} \right). \end{aligned}$$

□

Proof of Lemma 5.1. We apply the previous lemma with Z_i replaced by $a_i^n b_j^n \bar{X}_{ij}^n$, m_i replaced by $a_i^n b_j^n \bar{\theta}_{k(i,j)}^n(\{1\})$, N replaced by n^2 , and $c = 1$ to get

$$P \left(\frac{1}{n^2} \sum_{i,j} a_i^n b_j^n [\bar{X}_{ij}^n - \bar{\theta}_{k(i,j)}^n(\{1\})] \geq \delta \right) \leq e^{-n^2 h(\delta)}.$$

Since $h(\delta) > 0$ for $\delta > 0$, we can proceed exactly as in a LLN argument for uncontrolled random graphs that appears in [13, Lemma 4.1]. Using (3.6) and that there are 2^n choices for a^n and b^n , the union bound gives

$$P(d_{\infty \rightarrow 1}^n(\bar{H}^n, \bar{M}^n) \geq \delta) \leq 2^{n+1} e^{-n^2 h(\delta)} = 2e^{n \log 2} e^{-n^2 h(\delta)} \rightarrow 0.$$

□

5.3. Completion of the proof of Theorem 4.1. By the discussion in Sect. 5.1, it remains to show

$$\lim_{n \rightarrow \infty} \inf_{\theta^n} E \left[\frac{1}{n^2} R(\theta^n \parallel \mu^n) + G(\hat{H}^n) \right] = \inf_{\hat{V} \in \hat{\mathcal{S}}} [I(\hat{V}) + G(\hat{V})]. \quad (5.5)$$

We first establish a lower bound. Let $\{\theta^n\}$ be any sequence for which θ^n is a probability measure on $\{0, 1\}^{n^2}$. Construct $\{\bar{H}^n\}$, $\{\bar{M}^n\}$, $\{\hat{H}^n\}$, $\{\hat{M}^n\}$ and $\{W^n\}$ as in Sects. 3 and 5.2, and note that $d_{\infty \rightarrow 1}(W^n, W) \rightarrow 0$. Since $(\hat{\mathcal{S}}, d_{\infty \rightarrow 1})$ is compact, $\{\hat{H}^n\}$ and $\{\hat{M}^n\}$ are automatically tight. Consider any subsequence along which $\{\hat{H}^n\}$ and $\{\hat{M}^n\}$ converge in distribution, and label the limits $\hat{H}$ and $\hat{M}$. By Lemma 5.1, $\hat{H} = \hat{M}$. We use Fatou's lemma, the Eqs. (5.3) and (5.4), and the lower semicontinuity of relative entropy (see the proof of the lower semicontinuity of I in the appendix) along this subsequence to obtain

$$\begin{aligned} &\liminf_{n \rightarrow \infty} E \left[\frac{1}{n^2} R(\theta^n \parallel \mu^n) + G(\hat{H}^n) \right] \\ &= \liminf_{n \rightarrow \infty} E \left[\Upsilon(\bar{M}^n, W^n) + G(\hat{H}^n) \right] \end{aligned}$$

$$\begin{aligned}
&\geq \liminf_{n \rightarrow \infty} E \left[\inf_{V \in \hat{M}^n} \Upsilon(V, W^n) + G(\hat{H}^n) \right] \\
&\geq E \left[\inf_{V \in \hat{M}} \Upsilon(V, W) + G(\hat{M}) \right] \\
&\geq \inf_{\hat{V} \in \hat{\mathcal{S}}} [I(\hat{V}) + G(\hat{V})].
\end{aligned}$$

Since $\{\theta^n\}$ is arbitrary, an argument by contradiction then gives

$$\liminf_{n \rightarrow \infty} \inf_{\theta^n} E \left[\frac{1}{n^2} R(\theta^n \parallel \mu^n) + G(\hat{H}^n) \right] \geq \inf_{\hat{V} \in \hat{\mathcal{S}}} [I(\hat{V}) + G(\hat{V})].$$

Next we consider the reverse bound. Let $\delta > 0$ and choose $V^* \in \mathcal{S}$ such that

$$[\Upsilon(V^*, W) + G(\hat{V}^*)] \leq \inf_{\hat{V} \in \hat{\mathcal{S}}} [I(\hat{V}) + G(\hat{V})] + \delta.$$

Letting $\theta^{*,n}$ correspond to V^* in exactly the same way that μ^n corresponds W , we can apply Lemma 5.1 (or the ordinary LLN) to establish that $\delta_{\infty \rightarrow 1}(\hat{H}^n, \hat{V}^*) \rightarrow 0$ in distribution. We also have by Jensen's inequality that

$$\begin{aligned}
\frac{1}{n^2} R(\theta^{*,n} \parallel \mu^n) &= \frac{1}{n^2} \sum_{k=1}^{n^2} \int_{Q_{a_k}} R(\{\bar{M}^n(y), 1 - \bar{M}^n(y)\} \parallel \{W^n(y), 1 - W^n(y)\}) dy \\
&\leq \int_{[0,1]^2} R(\{V^*(y), 1 - V^*(y)\} \parallel \{W(y), 1 - W(y)\}) dy \\
&= \Upsilon(V^*, W)
\end{aligned}$$

(the reverse bound also holds as $n \rightarrow \infty$ by lower semicontinuity). Since we have made a particular choice of θ^n , it follows from the dominated convergence theorem that

$$\begin{aligned}
&\limsup_{n \rightarrow \infty} \inf_{\theta^n} E \left[\frac{1}{n^2} R(\theta^n \parallel \mu^n) + G(\hat{H}^n) \right] \\
&\leq \limsup_{n \rightarrow \infty} E \left[\frac{1}{n^2} R(\theta^{*,n} \parallel \mu^n) + G(\hat{H}^n) \right] \\
&= [\Upsilon(V^*, W) + G(V^*)] \\
&\leq \inf_{\hat{V} \in \hat{\mathcal{S}}} [I(\hat{V}) + G(\hat{V})] + \delta.
\end{aligned}$$

Letting $\delta \rightarrow 0$ establishes the upper bound, and completes the proof.

6. Applying the Contraction Principle

In this section, we use Theorem 4.1 and the contraction principle to prove the LDP for dynamical model (1.7), (1.8). To this end, we need to establish continuous dependence of the solutions of the corresponding IVPs on a kernel W with respect to the cut norm and on initial data with respect to the topology of $L^2([0, 1])$.

6.1. Proof of Lemma 4.2. Let U and V be two measurable functions on $[0, 1]^2$ with values in $[0, 1]$ and consider the following IVPs

$$\partial_t u(t, x) = f(u(t, x), t) + \int U(x, y) D(u(t, x), u(t, y)) dy, \quad (6.1)$$

$$u(0, x) = g(x), \quad (6.2)$$

and

$$\partial_t v(t, x) = f(v(t, x), t) + \int V(x, y) D(v(t, x), v(t, y)) dy, \quad (6.3)$$

$$v(0, x) = h(x), \quad (6.4)$$

where $g, h \in L^\infty([0, 1])$ and $x \in [0, 1]$.

Lemma 6.1. *For a given $T > 0$, we have*

$$\|u - v\|_{C(0, T; L^2([0, 1]))} \leq C (\|U - V\|_{\infty \rightarrow 1} + \|g - h\|_{L^2([0, 1])}), \quad (6.5)$$

where C depends on T , but not on U, V or g, h .²

We will need the following finite-dimensional (Galerkin) approximation of (6.1), (6.2) and (6.3), (6.4), respectively:

$$\partial_t u^n(t, x) = f(u^n(t, x), t) + \int U^n(x, y) D(u^n(t, x), u^n(t, y)) dy, \quad (6.6)$$

$$u^n(0, x) = g^n(x), \quad (6.7)$$

and

$$\partial_t v^n(t, x) = f(v^n(t, x), t) + \int V^n(x, y) D(v^n(t, x), v^n(t, y)) dy, \quad (6.8)$$

$$v^n(0, x) = h^n(x), \quad (6.9)$$

where as in (4.6) U^n, V^n and g^n, h^n stand for the L^2 -projections of U, V and g, h onto finite-dimensional subspaces $\text{span}\{\mathbf{1}_{Q_{ij}^n} : (i, j) \in [n]^2\}$ and $\text{span}\{\mathbf{1}_{Q_i^n} : i \in [n]\}$ respectively:

$$\begin{aligned} w^n(x) &= \sum_{i=1}^n w_i^n \mathbf{1}_{Q_i^n}(x), \quad w_i^n = n \int_{Q_i^n} w(x) dx, \quad w \in \{g, h\}, \\ W^n(x, y) &= \sum_{i, j=1}^n W_{ij}^n \mathbf{1}_{Q_{ij}^n}(x, y), \quad W_{ij}^n = n^2 \int_{Q_{ij}^n} W(x, y) dx, \quad W \in \{U, V\}. \end{aligned} \quad (6.10)$$

For solutions of the finite-dimensional models, we will need the following lemma.

Lemma 6.2.

$$\|u^n - v^n\|_{C(0, T; L^2([0, 1]))} \leq C (\|U^n - V^n\|_{\infty \rightarrow 1} + \|g^n - h^n\|_{L^2([0, 1])}), \quad (6.11)$$

where C is independent of n .

² Here and below, C stands for a generic constant.

The proof of Lemma 6.2 will be presented after the proof of Lemma 6.1.

Proof of Lemma 6.1. 1. First, we show that

$$\|u - u^n\|_{C([0,T],L^2([0,1]))} \leq C \left(\|U - U^n\|_{L^2([0,1]^2)} + \|g - g^n\|_{L^2([0,1])} \right). \quad (6.12)$$

To this end, let $\xi := u - u^n$, subtract (6.6) from (6.1), multiply the resulting equation by ξ and integrate over $[0, 1]$ with respect to x :

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \xi(t, x)^2 dx \\ &= \int [f(u(t, x), t) - f(u^n(t, x))] \xi(t, x) dx + \int_{[0,1]^2} U(x, y) \{D(u(t, x), u(t, y)) \\ & - D(u^n(t, x), u^n(t, y))\} \xi(t, x) dx dy + \int_{[0,1]^2} (U(x, y) - U^n(x, y)) D(u^n(t, x), u^n(t, y)) \xi(t, x) dx dy. \end{aligned} \quad (6.13)$$

Using (2.1), (2.2), (2.3), and $|U| \leq 1$, from (6.13), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \xi(t, x)^2 dx \\ & \leq (L_f + 2L_D) \int \xi(t, x)^2 dx + \int_{[0,1]^2} |U(x, y) - U^n(x, y)| |\xi(t, x)| dx dy. \end{aligned}$$

By Young's inequality, we further have

$$\frac{d}{dt} \int \xi(t, x)^2 dx \leq 2(L_f + 2L_D + 1/2) \int \xi(t, x)^2 dx + \int_{[0,1]^2} |U(x, y) - U^n(x, y)|^2 dx dy. \quad (6.14)$$

We obtain (6.12) from (6.14) via Gronwall's inequality. Similarly, we have

$$\|v - v^n\|_{C([0,T],L^2([0,1]))} \leq C \left(\|V - V^n\|_{L^2([0,1]^2)} + \|h - h^n\|_{L^2([0,1])} \right). \quad (6.15)$$

2. Using contractivity of the L^2 -projection operator with respect to the cut norm (cf. [19]), $\|U^n\|_{\square} \leq \|U\|_{\square}$, $\|V^n\|_{\square} \leq \|V\|_{\square}$, and (3.5), we have

$$\|U^n - V^n\|_{\infty \rightarrow 1} \leq 4\|U - V\|_{\infty \rightarrow 1}. \quad (6.16)$$

This and Lemma 6.2 imply

$$\|u^n - v^n\|_{C([0,T],L^2([0,1]))} \leq C \left(\|U - V\|_{\infty \rightarrow 1} + \|g^n - h^n\|_{L^2([0,1])} \right). \quad (6.17)$$

3. From (6.12), (6.15), and (6.17), by the triangle inequality, we have

$$\begin{aligned} \|u - v\|_{C([0,T],L^2([0,1]))} & \leq C \left(\|U - V\|_{\infty \rightarrow 1} + \|U - U^n\|_{L^2([0,1]^2)} + \|V - V^n\|_{L^2([0,1]^2)} \right. \\ & \quad \left. + \|g - g^n\|_{L^2([0,1])} + \|h - h^n\|_{L^2([0,1])} + \|g^n - h^n\|_{L^2([0,1])} \right). \end{aligned}$$

We obtain (6.5) after sending $n \rightarrow \infty$. $\square$

It remains to prove Lemma 6.2. We follow the lines of the proof of Proposition 2 in [26].

Proof of Lemma 6.2. 1. Using the bounds on U, V, f, D ,

$$|U| \leq 1, |V| \leq 1, |f| \leq 1, |D| \leq 1,$$

and the initial data

$$\max\{\|u^n(0, \cdot)\|_{L^\infty([0,1])}, \|v^n(0, \cdot)\|_{L^\infty([0,1])}\} \leq \max\{\|g\|_{L^\infty([0,1])}, \|h\|_{L^\infty([0,1])}\},$$

it follows from (6.6)–(6.9) that

$$\max_{(t,x) \in [0,T] \times [0,1]} |w^n(t, x)| \leq M, \quad w^n \in \{u^n, v^n\} \quad (6.18)$$

for some $M \in (0, \infty)$ independent of n .

2. Since D is a Lipschitz continuous bounded function and $D \in H_{\text{loc}}^s(\mathbb{R}^2)$, $s > 1$, there is a Lipschitz continuous bounded function $D_M \in H^s(\mathbb{R}^2)$, $s > 1$, which coincides with D on the ball of radius $\sqrt{2}M$ centered at the origin, $B(0, \sqrt{2}M)$. Indeed, as D_M one can take $D_M(\mathbf{x}) = \xi_M(\mathbf{x})D(\mathbf{x})$, where ξ_M is an infinitely differentiable bump function equal to 1 on $B(0, \sqrt{2}M)$ and equal to 0 outside of $B(0, 2\sqrt{2}M)$.

In view of (6.18), replacing D with D_M is not going to affect the solutions of the IVPs (6.6), (6.7) and (6.8), (6.9) on $[0, T]$. Thus, without loss of generality for the remainder of the proof we assume that $D \in H^s(\mathbb{R}^2)$, $s > 1$. In this case, letting ϕ be the Fourier transform of D , we have $\phi \in L^1(\mathbb{R}^2)$ and D can be written as

$$D(u) = \int_{\mathbb{R}^2} e^{2\pi i u \cdot z} \phi(z) dz, \quad u = (u_1, u_2), \quad z := (z_1, z_2), \quad u \cdot z = u_1 z_1 + u_2 z_2. \quad (6.19)$$

3. Recall that U^n and V^n are step functions (cf. (6.10)). Likewise, the solutions of the finite-dimensional IVPs (6.6), (6.7) can be written as

$$w^n(t, x) = \sum_{i=1}^n w_i^n(t) \mathbf{1}_{Q_i^n}(x), \quad w \in \{u, v\}. \quad (6.20)$$

Denote

$$\delta_i^n(t) := u_i^n(t) - v_i^n(t), \quad i \in [n]. \quad (6.21)$$

By subtracting (6.8) from (6.6), we have

$$\begin{aligned} \delta_i^n(s) = \delta_i^n(0) + \int_0^s \left\{ n^{-1} \sum_{j=1}^n U_{ij}^n \left(D(u_i^n(\tau), u_j^n(\tau)) - D(v_i^n(\tau), v_j^n(\tau)) \right) \right. \\ \left. + [f(u_i^n(\tau), \tau) - f(v_i^n(\tau), \tau)] + n^{-1} \sum_{j=1}^n (U_{ij}^n - V_{ij}^n) D(v_i^n(\tau), v_j^n(\tau)) \right\} d\tau, \end{aligned} \quad (6.22)$$

where U^n and V^n were defined in (6.10).

By continuity, there are $0 \leq t_i \leq T$ and $\sigma_i \in \{1, -1\}$ such that

$$\sup_{s \in [0, T]} |\delta_i^n(s)| = \sigma_i \delta_i^n(t_i), \quad i \in [n]. \quad (6.23)$$

Thus,

$$\begin{aligned}
\Delta(T) &:= \int \sup_{s \in [0, T]} |u^n(s, x) - v^n(s, x)| dx = n^{-1} \sum_{i=1}^n \sigma_i \delta_i^n(t_i) \\
&= n^{-1} \sum_{i=1}^n \sigma_i \delta_i^n(0) \\
&\quad + \int_0^T n^{-2} \sum_{i,j=1}^n \sigma_i U_{ij}^n \left(D(u_i^n(\tau), u_j^n(\tau)) - D(v_i^n(\tau), v_j^n(\tau)) \right) \mathbf{1}_{[0, t_i]}(\tau) d\tau \\
&\quad + \int_0^T n^{-1} \sum_{i=1}^n \sigma_i [f(u_i^n(\tau), \tau) - f(v_i^n(\tau), \tau)] \mathbf{1}_{[0, t_i]}(\tau) d\tau \\
&\quad + \int_0^T n^{-2} \sum_{i,j=1}^n \sigma_i (U_{ij}^n - V_{ij}^n) D(v_i^n(\tau), v_j^n(\tau)) \mathbf{1}_{[0, t_i]}(\tau) d\tau \\
&= n^{-1} \sum_{i=1}^n \sigma_i \delta_i^n(0) + I_1 + I_2 + I_3. \tag{6.24}
\end{aligned}$$

Using Lipschitz continuity of D and f (cf. (2.2) and (2.1)) and the fact that $|U_{ij}^n| \leq 1$, we have

$$I_1 + I_2 \leq \int_0^T (2L_D + L_f) \Delta(\tau) d\tau. \tag{6.25}$$

On the other hand, using (6.19), we estimate

$$I_3 \leq n^{-2} \int_0^T \int_{\mathbb{R}^2} \left| \sum_{i,j}^n (V_{ij}^n - U_{ij}^n) e^{2\pi i u_i^n(\tau) z_1} e^{2\pi i v_j^n(\tau) z_2} \right| |\phi(z)| dz d\tau \tag{6.26}$$

Decomposing $e^{2\pi i u_i^n(\tau) z_1}$ and $e^{2\pi i v_j^n(\tau) z_2}$ into sums of real and imaginary parts, each not exceeding 1 in absolute value, we have

$$I_3 \leq 4T \|U^n - V^n\|_{\infty \rightarrow 1} \|\phi\|_{L^1(\mathbb{R}^2)}. \tag{6.27}$$

Combining (6.24), (6.25), and (6.27), and using Gronwall's inequality and the definition of $\delta_i^n(0)$, we obtain

$$\begin{aligned}
&\int \sup_{s \in [0, T]} |u^n(s, x) - v^n(s, x)| dx \\
&\leq e^{(2L_D + L_f)T} (4T \|\phi\|_{L^1(\mathbb{R}^2)} \|U^n - V^n\|_{\infty \rightarrow 1} + \|g^n - h^n\|_{L^2([0, 1])}). \tag{6.28}
\end{aligned}$$

4. Using (6.28) and (6.18), we have

$$\begin{aligned}
&\sup_{t \in [0, T]} \int (u^n(t, x) - v^n(t, x))^2 dx \\
&\leq 2M \sup_{t \in [0, T]} \int |u^n(t, x) - v^n(t, x)| dx \leq 2M \int \sup_{t \in [0, T]} |u^n(t, x) - v^n(t, x)| dx \\
&\leq 2M e^{(2L_D + L_f)T} (4T \|\phi\|_{L^1(\mathbb{R}^2)} \|U^n - V^n\|_{\infty \rightarrow 1} + \|g_n - h_n\|_{L^2([0, 1])}).
\end{aligned}$$

□

Finally, given $(\widehat{U}, g)$ and $(\widehat{V}, h)$, fix two representatives $(U, g) \in (\widehat{U}, g)$ and $(V, h) \in (\widehat{V}, h)$. Denote the corresponding solutions of the IVP and their equivalence classes by u, v and $\hat{u}, \hat{v}$ respectively. Using Lemma 6.1, we have

$$\begin{aligned} d_{\hat{\mathcal{Y}}}(\hat{u}, \hat{v}) &= \inf_{\sigma} \|u_{\sigma} - v\|_{C(0,T;L^2([0,1]))} \\ &\leq C \inf_{\sigma} \{ \|U_{\sigma} - V\|_{\infty \rightarrow 1} + \|g_{\sigma} - h\|_{L^2([0,1])} \} \\ &= C d_{\hat{\mathcal{X}}}((\widehat{U}, g), (\widehat{V}, h)). \end{aligned}$$

This shows the continuity of $F : \hat{\mathcal{X}} \rightarrow \hat{\mathcal{Y}}$ needed for the application of the contraction principle to the dynamical model at hand.

7. Application to Atypical Edge Counts

Throughout this section, $W \in \mathcal{S}$ is arbitrary but fixed. Let $\mathbb{G}(n, W)$ denote a W -random graph model defined by (3.1). Specifically, $H^n \in \mathbb{G}(n, W)$ means that

$$H^n = \sum_{i,j} X_{ij}^n \mathbf{1}_{Q_{ij}^n},$$

where X_{ij}^n are independent random variables defined by (1.3), (1.4). We will refer to H^n as an element of $\mathcal{S}$ or as an element of $\mathbb{G}(n, W)$ interchangeably, i.e., H^n may be interpreted as a function or as a graph depending on the context.

The number of directed edges (counting self-edges) in $H^n \in \mathbb{G}(n, W)$ is given by

$$|E(H^n)| = n^2 \mathcal{L}(H^n), \quad (7.1)$$

where

$$\mathcal{L}(V) := \int_{[0,1]^2} V(x) dx, \quad V \in \mathcal{S}.$$

Note that $\mathcal{L}(f_{\sigma}) = \mathcal{L}(f)$ for every $f \in \mathcal{S}$ and $\sigma \in \mathcal{P}$ [cf. (3.7)]. Thus, $\mathcal{L}$ is a well-defined functional on $\hat{\mathcal{S}}$ as well as on $\mathcal{S}$. By the SLLN,

$$n^{-2}|E(H^n)| \rightarrow w := \mathcal{L}(W) \quad \text{as } n \rightarrow \infty \quad \text{a.s..}$$

The following theorem characterizes realizations of $\mathbb{G}(n, W)$ conditioned on large deviations of $n^{-2}|E(H^n)|$ from its expected value w .

Theorem 7.1. *Let $0 < \delta < 1 - w$. Then for every $\epsilon > 0$ and $n \in \mathbb{N}$*

$$\mathbb{P} \left(\delta_{\infty \rightarrow 1} \left(\hat{H}^n, \hat{V}_{\delta}^+ \right) \geq \epsilon \mid n^{-2}|E(H^n)| \geq w + \delta \right) \leq e^{-C^+(\epsilon, \delta)n^2}, \quad (7.2)$$

where the positive constant $C^+(\epsilon, \delta)$ depends on ϵ and δ but not on n . $V_{\delta}^+ \in \mathcal{S}$ is the unique minimizer of I [cf. (4.1)] restricted to

$$F_{\delta}^+ = \{V \in \mathcal{S} : \mathcal{L}(V) = w + \delta\},$$

i.e.,

$$I(V_\delta^+) = \inf_{V \in F_\delta^+} I(V).$$

Likewise, given $0 < \delta < w$ for every $\epsilon > 0$ and $n \in \mathbb{N}$, we have

$$\mathbb{P} \left(\delta_{\infty \rightarrow 1} \left(\hat{H}^n, \hat{V}_\delta^- \right) \geq \epsilon \mid n^{-2} |E(H^n)| \leq w - \delta \right) \leq e^{-C^-(\epsilon, \delta)n^2}, \quad (7.3)$$

where $C^-(\epsilon, \delta) > 0$ is independent of n and $V_\delta^- \in \mathcal{S}$ is the unique minimizer of I restricted to

$$F_\delta^- = \{V \in \mathcal{S} : \mathcal{L}(V) = w - \delta\}.$$

The proof of Theorem 7.1 relies on the following lemma, which also identifies the minimizers $V_\delta^\pm$ in the theorem.

Lemma 7.2. *Recall that $W \in \mathcal{S}$ is given and $w = \mathcal{L}(W)$. Consider the constrained optimization problem*

$$\inf \left\{ \Upsilon(V, W) : \int_{[0,1]^2} V(x, y) dx dy \geq m \right\} \quad (7.4)$$

with $m \in (w, 1)$. Then the unique (up to sets of Lebesgue measure zero) minimizer of (7.4) is given by

$$V^*(x, y) = \frac{W(x, y)e^{-\lambda^*}}{1 - W(x, y) + W(x, y)e^{-\lambda^*}}, \quad (7.5)$$

where λ^* is the unique solution to

$$\int_{[0,1]^2} \frac{W(x, y)e^{-\lambda^*}}{1 - W(x, y) + W(x, y)e^{-\lambda^*}} dx dy = m.$$

The analogous statement holds for the case of $m \in (0, w)$.

Remark 7.3. Suppose W is measurable with respect to a certain σ -algebra of subsets of $[0, 1]^2$. Then the minimizer is measurable with respect to the same σ -algebra. In particular, if W is a step function with respect to a given partition of $[0, 1]^2$ then so is V^* .

The proof of the lemma will be given after the proof of Theorem 7.1.

Proof of Theorem 7.1. Consider

$$F_{\geq \delta}^+ = \{V \in \mathcal{S} : \mathcal{L}(V) \geq w + \delta\}.$$

We continue to use $\hat{F}_{\geq \delta}^+$ to denote the corresponding subset of $\hat{\mathcal{S}}$:

$$\hat{F}_{\geq \delta}^+ = \left\{ \hat{V} \in \hat{\mathcal{S}} : V \in F_{\geq \delta}^+ \right\}.$$

$\hat{F}_{\geq \delta}^+$ is a closed subset of the metric space $(\hat{\mathcal{S}}, \delta_{\infty \rightarrow 1})$. Moreover,

$$\inf_{\text{int}(\hat{F}_{\geq \delta}^+)} I = \inf_{\hat{F}_{\geq \delta}^+} I,$$

since the right hand side is a finite convex function of δ on $(0, 1 - w)$. Here, $\text{int}(A)$ stands for the interior of A . Lemma 7.2 implies existence of a unique minimizer of $\Upsilon(\cdot, W)$ on $F_{\geq \delta}^+$. Denote this minimizer by V_{δ}^+ . By Lemma 7.2, $V_{\delta}^+ \in F_{\geq \delta}^+$.

Recall the definition of the rate function I on $\hat{S}$ [cf. (4.1)]. Clearly, $\hat{V}_{\delta}^+ \in \hat{F}_{\delta}^+$ is the unique minimizer of I on $\hat{F}_{\delta}^+$. Estimate (7.2) is standard in the large deviations theory (cf. [4, Theorem 1.4]). Estimate (7.3) is proved similarly. $\square$

Proof of Lemma 7.2. Note that $V \rightarrow \Upsilon(V, W)$ is convex and non-negative with its minimum of zero uniquely attained at $V = W$. Since the constraint $\mathcal{L}(V) = b$ is linear in V , the solution to (7.4) with $\geq m$ replaced by $= b$ is convex and non-negative with its minimum of zero uniquely attained at $b = w$. Hence the solution to (7.4) is the same as that obtained when $\geq m$ replaced by $= m$.

We first consider the case where

$$W(x, y) = w_k \quad \text{for } (x, y) \in D_k, \quad k = 1, \dots, K,$$

and with each D_k measurable and $[0, 1]^2$ is the disjoint union of D_k , $k = 1, \dots, K$. Let

$$R(p \| q) = p \log \left(\frac{p}{q} \right) + (1 - p) \log \left(\frac{1 - p}{1 - q} \right).$$

It is easy to argue that a minimizer exists, which we call V^* . With $m(\cdot)$ denoting Lebesgue measure, Jensen's inequality gives

$$\begin{aligned} \Upsilon(V^*, W) &= \sum_{k=1}^K \int_{D_k} R(V^*(x, y) \| W(x, y)) dx dy \\ &= \sum_{k=1}^K \int_{D_k} R(V^*(x, y) \| w_k) dx dy \\ &\geq \sum_{k=1}^K m(D_k) R \left(\frac{1}{m(D_k)} \int_{D_k} V^*(x, y) dx dy \| w_k \right), \end{aligned}$$

which by strict convexity of $p \rightarrow R(p \| q)$ shows that V^* must be constant on each D_k . Let the values be v_k . Then the minimization problem becomes

$$\inf \left\{ \sum_{k=1}^K m(D_k) R(v_k \| w_k) : \sum_{k=1}^K m(D_k) v_k = m \right\},$$

where $0 \leq v_k \leq 1$ for $k = 1, \dots, K$.

Using the convexity, the unique solution can be easily found using Lagrange multipliers. Note that

$$\frac{d}{dp} R(p \| q) = \log \left(\frac{p}{q} \right) - \log \left(\frac{1 - p}{1 - q} \right).$$

Thus we obtain the equations

$$\log \left(\frac{v_k}{w_k} \right) - \log \left(\frac{1 - v_k}{1 - w_k} \right) + \lambda = 0,$$

for $k = 1, \dots, K$, which have the unique solution

$$v_k = \frac{w_k e^{-\lambda}}{1 - w_k + w_k e^{-\lambda}}.$$

Since we have limits of zero and one for the right hand side as $\lambda \rightarrow \infty$ and $\lambda \rightarrow -\infty$ whenever $w_k \notin \{0, 1\}$ as well as monotonicity, it is easy to check that there is the unique solution λ^* to

$$\sum_{k=1}^K m(D_k) \frac{w_k e^{-\lambda}}{1 - w_k + w_k e^{-\lambda}} = m,$$

which identifies the minimizer.

Next we consider the general case. We can find graphons $\{W^\delta, \delta \in (0, 1)\}$ such that the following properties hold:

1. W^δ converges to W uniformly,
2. $\{(x, y) : W(x, y) = 0\} = \{(x, y) : W^\delta(x, y) = 0\}$ and $\{(x, y) : W(x, y) = 1\} = \{(x, y) : W^\delta(x, y) = 1\}$,
3. $W^\delta(x, y) \downarrow W(x, y)$ if $W(x, y) \leq 1/3$ and $W^\delta(x, y) \uparrow W(x, y)$ if $W(x, y) \geq 2/3$,
4. W^δ takes on a finite collection of values.

Let $V^{*,\delta}$ and $\lambda^{*,\delta}$ correspond to W^δ according to the first part of the proof. Since

$$\int_{[0,1]^2} \frac{W^\delta(x, y) e^{-\lambda^{*,\delta}}}{1 - W^\delta(x, y) + W^\delta(x, y) e^{-\lambda^{*,\delta}}} dx dy = m$$

and W^δ converges to W uniformly, it follows that $\lambda^{*,\delta} \rightarrow \lambda^*$. Let $V^* = W e^{-\lambda^*} / (1 - W + W e^{-\lambda^*})$, and suppose that $\bar{V}$ is the minimizer for $\delta = 0$, which will satisfy $\Upsilon(\bar{V}, W) < \infty$. Later in the proof we will show that

$$\Upsilon(\bar{V}, W^\delta) \rightarrow \Upsilon(\bar{V}, W). \quad (7.6)$$

Temporarily assuming this, we also know that

$$\Upsilon(V^{*,\delta}, W^\delta) \leq \Upsilon(\bar{V}, W^\delta),$$

since $V^{*,\delta}$ is the optimizer for W^δ . It follows from lower semicontinuity that

$$\liminf_{\delta \rightarrow 0} \Upsilon(V^{*,\delta}, W^\delta) \geq \Upsilon(V^*, W),$$

where V^* defined above is the uniform limit of $V^{*,\delta}$. Combining the last three displays gives $\Upsilon(V^*, W) \leq \Upsilon(\bar{V}, W)$. Since by assumption $\bar{V}$ is the minimizer there must be equality, and again using the strict convexity of $p \rightarrow R(p \| q)$ except possibly on a set of Lebesgue measure zero $V^* = \bar{V}$.

Thus we need to prove (7.6). Let $\ell(z) = z \log z - z + 1$. We first claim that

$$\int_{[0,1]^2} W(x, y) \ell\left(\frac{\bar{V}(x, y)}{W(x, y)}\right) dx dy + \int_{[0,1]^2} [1 - W(x, y)] \ell\left(\frac{1 - \bar{V}(x, y)}{1 - W(x, y)}\right) dx dy < \infty.$$

To see this we rewrite as

$$\begin{aligned} & \int_{[0,1]^2} \bar{V}(x, y) \log \left(\frac{\bar{V}(x, y)}{W(x, y)} \right) dx dy - \int_{[0,1]^2} \bar{V}(x, y) dx dy + \int_{[0,1]^2} W(x, y) dx dy \\ & + \int_{[0,1]^2} [1 - \bar{V}(x, y)] \log \left(\frac{1 - \bar{V}(x, y)}{1 - W(x, y)} \right) dx dy - \int_{[0,1]^2} [1 - \bar{V}(x, y)] dx dy \\ & + \int_{[0,1]^2} [1 - W(x, y)] dx dy. \end{aligned}$$

Since the integrals without log are bounded, the claim follows from $\Upsilon(\bar{V}, W) < \infty$. To show (7.6) it suffices to prove

$$\int_{[0,1]^2} \bar{V}(x, y) \log \left(\frac{\bar{V}(x, y)}{W^\delta(x, y)} \right) dx dy \rightarrow \int_{[0,1]^2} \bar{V}(x, y) \log \left(\frac{\bar{V}(x, y)}{W(x, y)} \right) dx dy$$

and the analogous result for

$$\int_{[0,1]^2} [1 - \bar{V}(x, y)] \log \left(\frac{1 - \bar{V}(x, y)}{1 - W^\delta(x, y)} \right) dx dy.$$

Choose a number $a \in (0, 1/3)$ such that $\{W = a\}$ has zero Lebesgue measure. Then

$$\int_{\{W \geq a\}} \bar{V}(x, y) \log \left(\frac{\bar{V}(x, y)}{W^\delta(x, y)} \right) dx dy \rightarrow \int_{\{W \geq a\}} \bar{V}(x, y) \log \left(\frac{\bar{V}(x, y)}{W(x, y)} \right) dx dy$$

follows from the dominated convergence theorem. For $\{W \leq a\}$ we use

$$\begin{aligned} \int_{\{W \leq a\}} \bar{V}(x, y) \log \left(\frac{\bar{V}(x, y)}{W^\delta(x, y)} \right) dx dy &= \int_{\{W \leq a\}} W^\delta(x, y) \ell \left(\frac{\bar{V}(x, y)}{W^\delta(x, y)} \right) dx dy \\ &+ \int_{\{W \leq a\}} \bar{V}(x, y) dx dy - \int_{\{W \leq a\}} W^\delta(x, y) dx dy. \end{aligned}$$

Since the last term converges to $\int_{\{W \leq a\}} W(x, y) dx dy$, it is enough to show

$$\int_{\{W \leq a\}} W^\delta(x, y) \ell \left(\frac{\bar{V}(x, y)}{W^\delta(x, y)} \right) dx dy \rightarrow \int_{\{W \leq a\}} W(x, y) \ell \left(\frac{\bar{V}(x, y)}{W(x, y)} \right) dx dy.$$

To do this we construct a dominating function for

$$W^\delta(x, y) \ell \left(\frac{\bar{V}(x, y)}{W^\delta(x, y)} \right).$$

Note that $\bar{V} \leq 1$. If $\bar{V} = 0$ the quantity in the last display is bounded by 1. If $\bar{V} > 0$ then we compute $\sup_{1 \geq a \geq W} a \ell(\bar{V}/a)$ to obtain the dominating function. Since by calculus the unique extreme point is $a = \bar{V}$ at which it is a minimum, the maximum must be on the boundary, and we find

$$\sup_{1 \geq a \geq W} a \ell \left(\frac{\bar{V}}{a} \right) \leq W \ell \left(\frac{\bar{V}}{W} \right) \vee \ell(\bar{V}).$$

Since W^δ decreases to W in $\{W \leq a\}$ this is indeed a dominating function, and so (7.6) holds. $\square$

8. Generalization and an Open Problem

In this section, we first describe a useful generalization of the analysis in the main part of the paper, which is to extend the LDP to cover the original model (1.7) with random parameters. We also discuss the case of the dynamical model on a sequence of sparse graphs, which is an interesting open problem. Questions involving large deviation estimates for sparse random graphs arise in probabilistic combinatorics [5]. Starting with the work of Chatterjee and Dembo [7], there has been a big development in this area recently. For the current state of research, we refer an interested reader to the preprint by Harel et al. [14] and references therein. The analysis in the previous sections of the present paper suggests a natural extension of the LDP derived for the dense networks to their sparse counterparts. To explain this extension, we formulate the dynamical model on a convergent sequence of sparse W-random graphs [1]. We then conjecture, based on the similarity between a rescaled version of the sparse random graph and an appropriate Poisson random measure, that the LDP holds for sparse W-random graphs in the space of nonnegative finite measures with the vague topology and with the same rate function as for the sequence of Poisson random measures. We further conjecture that this LDP can be upgraded to the LDP with the same rate function in the space of graphons with the cut norm topology, which would afford further application to the dynamical problem. We support this conjecture by demonstrating the key estimate needed for the proof of the lower large deviations bound and outlining the steps needed for the proof of the upper bound. The latter however leads to new technical difficulties, which will be addressed elsewhere.

8.1. Random parameters. We now revisit (1.1), (1.2) to address the dependence of f on random parameters. To this end, we rewrite (1.1), (1.2) as follows

$$\begin{aligned} \dot{u}_i^n &= f(u_i^n, \eta_i^n, t) + \frac{1}{n} \sum_{j=1}^n X_{ij}^n D(u_i^n, u_j^n), \quad \dot{\eta}_i^n = 0, \\ u_i^n(0) &= g_i^n, \quad \eta_i^n(0) = \xi_i^n, \quad i \in [n], \end{aligned} \tag{8.1}$$

where $\xi_i^n \in \mathbb{R}^d$ is a random array. Thus, the random parameters can be treated in the same way as the initial data.

We formulate the assumptions on $\{\xi_i^n\}$ in analogy to how this was done for $\{g_i^n\}$ in Sect. 2. Specifically, let $\{J^n\}$ be a sequence of iid $\mathcal{B}^d$ -valued random variables independent from $\{X_{ij}^n\}$ and $\{G^n\}$. Then

$$\xi_i^n = n \int_{Q_i^n} J^n(y) dy, \quad i \in [n]$$

and

$$\bar{J}^n(x) = \xi_i^n \quad \text{for } x \in Q_i^n.$$

In analogy to Assumption 2.1, we impose the following.

Assumption 8.1. $\{\bar{J}^n\}$ satisfies the LDP in $\mathcal{B}^d$ with the rate function L and scaling sequence n^2 .

All other assumptions on the data in (8.1) remain the same with one exception: the Lipschitz condition (2.1) is replaced by the condition

$$|f(u, \xi, t) - f(u', \xi', t)| \leq L_f (|u - u'| + |\xi - \xi'|) \quad u, u' \in \mathbb{R}, \xi, \xi' \in \mathbb{R}^d, t \geq 0.$$

The continuum limit for (8.1) is given by

$$\begin{aligned} \partial_t u(t, x) &= f(u, j(x), t) + \int W(x, y) D(u(t, x), u(t, y)) dy, \\ u(0, x) &= g(x), \end{aligned} \quad (8.2)$$

where $g \in \mathcal{B}$ and $j \in \mathcal{B}^d$. The initial value problem (8.2) has a unique solution $u \in \mathcal{Y}$. (Recall that $\mathcal{Y}$ stands for $C([0, T], L^2([0, 1]))$.) Furthermore, we can formulate a counterpart of Lemma 4.2 for the model at hand. Specifically, let

$$(W, g, j) \sim (W', g', j') \quad \text{if } W' = W_\sigma, g' = g_\sigma, \& j' = j_\sigma \quad \text{for some } \sigma \in \mathcal{P}.$$

and redefine $\mathcal{X} := \mathcal{S} \times \mathcal{B} \times \mathcal{B}^d$ and $\hat{\mathcal{X}} = \mathcal{X}/\sim$. Let

$$F : \mathcal{X} \ni (W, g, j) \mapsto u \in \mathcal{Y}$$

denote the map between the data and the solution of the initial value problem (8.1). As before,

$$F(W_\sigma, g_\sigma, j_\sigma) = u_\sigma \quad \forall \sigma \in \mathcal{P}.$$

Thus, $F : \hat{\mathcal{X}} \rightarrow \hat{\mathcal{Y}}$ is well defined. The analysis of Sect. 6 with straightforward modifications then implies that that F is a continuous mapping. Here, the metric in $\hat{\mathcal{Y}}$ is unchanged and

$$d_{\hat{\mathcal{X}}} \left((\widehat{U, g, j}), (\widehat{U', g', j'}) \right) = \inf_{\sigma} \left\{ \|U_\sigma - U'\|_{\infty \rightarrow 1} + \|g_\sigma - g'\|_{\mathcal{B}} + \|j_\sigma - j'\|_{\mathcal{B}^d} \right\}, \quad (8.3)$$

where $(U, g, j) \in (\widehat{U, g, j})$ and $(U', g', j') \in (\widehat{U', g', j'})$ are arbitrary representatives.

This leads to the following.

Theorem 8.2. *For $W \in \mathcal{S}$ let (H^n, g^n, J^n) be a sequence of random graphons and random initial data and parameters and let Assumptions 2.1 and 8.1 hold. Denote by u^n the corresponding solutions of (8.1). Then $\{\hat{u}^n\}$ satisfies an LDP on $\hat{\mathcal{X}}$ with scaling sequence n^2 and the rate function*

$$\mathbf{J}(\hat{u}) = \inf \{ I(\hat{W}) + K(\hat{g}) + L(\hat{j}) : (\widehat{W, g, j}) = F^{-1}(\hat{u}) \}.$$

8.2. Sparsity. Let $W : [0, 1]^2 \rightarrow [0, 1]$ as before and let $0 < \alpha_n \leq 1$, $n \in \mathbb{N}$, be a nonincreasing sequence. If $\alpha_n \rightarrow 0$, we in addition assume that

$$\alpha_n^2 n \rightarrow \infty. \quad (8.4)$$

Define $\{\Gamma^n\}$ by

$$\mathbb{P}(X_{ij}^n = 1) = \alpha_n W_{ij}^n. \quad (8.5)$$

If $\alpha_n \rightarrow \alpha_\infty > 0$, $\{\Gamma^n\}$ is a sequence of dense graphs as before. If $\alpha_n \searrow 0$ then graphs $\{\Gamma^n\}$ are sparse.

Example 8.3. Sparse Erdős–Rényi graphs: $W \equiv 1$, $\alpha_n \searrow 0$. Dynamical models on sparse Erdős–Rényi graphs were studied in [9, 26].

On sparse graphs $\{\Gamma^n\}$, the dynamical model takes the form

$$\dot{u}_i^n = f(u_i^n, t) + (\alpha_n n)^{-1} \sum_{j=1}^n X_{ij}^n D(u_i^n, u_j^n). \quad (8.6)$$

The new scaling of the interaction term, $(\alpha_n n)^{-1}$, is used to account for sparsity.

To apply the analysis of the large deviations in the main part of the paper to the model at hand, note that

$$(\alpha_n n)^{-1} \sum_{j=1}^n X_{ij}^n D(u_i^n, u_j^n) = n^{-1} \sum_{j=1}^n \mathbf{X}_{ij}^n D(u_i^n, u_j^n),$$

where

$$\mathbf{X}_{ij}^n = \alpha_n^{-1} X_{ij}^n, \quad \mathbb{E} \mathbf{X}_{ij}^n = W_{ij}^n.$$

This suggests that in the sparse case one needs to study rescaled random variables

$$\mathbf{H}^n = \alpha_n^{-1} H^n.$$

By considering $\mathbf{H}^n(y)dy$ we can view $\mathbf{H}^n$ as taking values in the set of non-negative finite measures on $[0, 1]^2$ with the vague topology. This topology can be metrized so that the space is a Polish space [4, Section A.4.1]. Let $\ell(z) = z \log z - z + 1$ for $z \geq 0$. To identify a candidate for the rate function, consider the relative entropy for the graphons corresponding to the rescaled Bernoulli random variables $\{X_{ij}^n\}$ [cf. (8.5)]:

$$\begin{aligned} R(\{\alpha_n V, 1 - \alpha_n V\} \parallel \{\alpha_n W, 1 - \alpha_n W\}) &= \alpha_n V \log \left(\frac{V}{W} \right) + (1 - \alpha_n V) \log \left(\frac{1 - \alpha_n V}{1 - \alpha_n W} \right) \\ &= \alpha_n \left(V \log \left(\frac{V}{W} \right) + W - V + O(\alpha_n) \right) \\ &= \alpha_n \left(W \ell \left(\frac{V}{W} \right) + O(\alpha_n) \right), \end{aligned}$$

This suggests that $\{\mathbf{H}^n\}_{n \in \mathbb{N}}$ as defined above satisfies the LDP with rate function

$$\int_{[0,1]^2} W(y) \ell \left(\frac{V(y)}{W(y)} \right) dy \quad (8.7)$$

and the scaling sequence $n^2 \alpha_n$.

However, we would like to strengthen this to the cut-norm topology. One direction is straightforward, in that we can still use Bernstein's bound for the rescaled array, and thereby establish the large deviation lower bound in the stronger topology. For example, if we want to compare $\mathbf{H}^n$ and W^n as would be needed to establish the LLN in the stronger topology, we find

$$\begin{aligned}
 \mathbb{P}(d_{\infty \rightarrow 1}(\mathbf{H}^n, W^n) \geq \delta) &\leq \sup_{a^n, b^n} \mathbb{P}\left(\frac{1}{n^2} \sum_{i,j=1}^n a_i^n b_j^n \left[\frac{a_{ij}}{\alpha_n} - W_{ij}^n\right] \geq \delta\right) \\
 &= \sup_{a^n, b^n} \mathbb{P}\left(\frac{1}{n^2} \sum_{i,j=1}^n a_i^n b_j^n [a_{ij} - \alpha_n W_{ij}^n] \geq \alpha_n \delta\right) \\
 &\leq e^{n \log 2} e^{-n^2 h(\alpha_n \delta)} \rightarrow 0
 \end{aligned}$$

owing to our assumption (8.4) and $h(\alpha_n \delta) \approx \alpha_n^2 \delta^2 / 2$. The analogous estimate as needed to establish the large deviation estimate in the cut norm also holds.

For the large deviation upper bound it was essential to work with the equivalence class, and there it was crucial that the set $\hat{\mathcal{S}}$ was compact. We conjecture that an analogous compactness holds here as well. Specifically, let

$$I(\hat{V}) = \inf_{V \in \hat{V}} \int_{[0,1]^2} W(y) \ell\left(\frac{V(y)}{W(y)}\right) dy.$$

Then we conjecture that under reasonable conditions on W the superlinear growth of ℓ implies level sets of $I(\hat{V})$ are compact in the natural generalization of $\hat{\mathcal{S}}$, and that this together with the Bernstein's bound suffices to establish the upper bound.

9. Appendix: Proof of Lower Semicontinuity of I

We want to prove that

$$\liminf_{n \rightarrow \infty} I(\hat{V}^n) \geq I(\hat{V})$$

when $\hat{V}^n \rightarrow \hat{V}$. The latter means $V^n \rightarrow V$ in $d_{\infty \rightarrow 1}$. Then we have to show that

$$\liminf_{n \rightarrow \infty} \inf_{V \in \hat{V}^n} \Upsilon(V, W) \geq \inf_{V \in \hat{V}} \Upsilon(V, W),$$

where

$$\Upsilon(V, W) = \int_{[0,1]^2} G(V(x), W(x)) dx, \quad G(v, w) = v \log\left(\frac{v}{w}\right) + (1-v) \log\left(\frac{1-v}{1-w}\right).$$

To simplify the proof we assume that W is continuous and bounded away from 0 and 1. If W is measurable and bounded away from 0 and 1 the assumption of continuity can be justified by Lusin's Theorem, since for a general W we can find W_ε with the same bounds such that W_ε is continuous and the set where $W_\varepsilon \neq W$ has Lebesgue measure less than ε .

If W is not bounded away from 0 and 1 then we can replace G by $G^h \geq 0$ that is continuous and satisfies

$$G^h(v, w) \leq G(v, w) \quad \text{and} \quad G^h(v, w) \uparrow G(v, w).$$

We can then apply the argument below with G^h replacing G to show lower semicontinuity of the analogue I^h of I for arbitrary W , and then need only show

$$\liminf_{h \rightarrow 0} I^h(\hat{V}) \geq I(\hat{V}).$$

This can also be done by a minor adaptation of the argument used below.

The proof will be based on weak convergence and a construction analogous to the “chattering lemma” of control theory. Let $\bar{V}^n$ come within $1/n$ of the infimum in $\inf_{V \in \hat{V}^n} \Upsilon(V, W)$. Then there is $\sigma^n \in \mathcal{P}$ such that

$$\Upsilon(\bar{V}^n, W) = \Upsilon(V^n, W \circ \sigma^n),$$

and it is enough to show the following. Let any subsequence of n be given and consider a further subsequence (say $\bar{n}$). Then given $\varepsilon > 0$ we can find $\sigma \in \mathcal{P}$ such that

$$\liminf_{\bar{n} \rightarrow \infty} \Upsilon(V^{\bar{n}}, W \circ \sigma^{\bar{n}}) \geq \Upsilon(V, W \circ \sigma) - \varepsilon.$$

To simplify notation we write n rather than $\bar{n}$.

We define probability measures $\{\mu^n\}$ on $[0, 1]^5$ by

$$\mu^n(A_1 \times A_2 \times \cdots \times A_5) = \int_{A_2 \times A_3} 1_{A_1}(V^n(x_1, x_2)) 1_{A_4}(\sigma^n(x_1)) dx_1 1_{A_5}(\sigma^n(x_2)) dx_2.$$

By compactness we can assume that for the subsubsequence these converge weakly with limit μ . Also, we can write

$$\begin{aligned} \Upsilon(V^n, W \circ \sigma^n) &= \int_{[0,1]^2} G(V(x_1, x_2), W(\sigma^n(x_1), \sigma^n(x_2))) dx_1 dx_2 \\ &= \int_{[0,1]^5} G(v, W(y_1, y_2)) \mu^n(dv \times dx_1 \times dx_2 \times dy_1 \times dy_2), \end{aligned}$$

and using the properties of G (bounded and lsc) and W (bounded and continuous)

$$\liminf_{n \rightarrow \infty} \Upsilon(V^n, W \circ \sigma^n) \geq \int_{[0,1]^5} G(v, W(y_1, y_2)) \mu(dv \times dx_1 \times dx_2 \times dy_1 \times dy_2).$$

Given $\varepsilon > 0$ we need to construct $\sigma \in \mathcal{P}$ such that

$$\int_{[0,1]^5} G(v, W(y_1, y_2)) \mu(dv \times dx_1 \times dx_2 \times dy_1 \times dy_2) \geq \Upsilon(V, W \circ \sigma) - \varepsilon.$$

With subscripts denoting marginal distributions, it is clear that

$$\mu_{2,4}(dx_1 \times dy_1) = \mu_{3,5}(dx_2 \times dy_2)$$

and, since each σ^n is a measure preserving bijection, that

$$\mu_2(dx_1) = dx_1, \mu_3(dx_2) = dx_2, \mu_4(dy_1) = dy_1, \mu_5(dy_2) = dy_2.$$

Thus both marginals of $\mu_{2,4}$ (and $\mu_{3,5}$) are Lebesgue measure, but we do not know that $\mu_{2,4}$ is the measure induced by a measure preserving bijection σ . We will approximate $\mu_{2,4}$ to construct σ , and in doing so incur a small error in the integral which will be smaller than ε .

Let $\nu(dx \times dy) = \mu_{2,4}(dx \times dy)$. Then it suffices to find a sequence $\theta_k \in \mathcal{P}$ such that if $\nu_k(A_1 \times A_2) = \int_{A_1} 1_{A_2}(\theta_k(x)) dx$, then ν_k converges to ν in the weak topology. The construction is as follows. Let $\delta = 1/k$. Then we partition $[0, 1]^2$ according to

$$T_{i,j}^k = [(i-1)\delta, i\delta) \times [(j-1)\delta, j\delta), \quad 1 \leq i, j \leq k$$

and define $m_{i,j}^k = \nu(T_{i,j}^k)$. Let

$$S(\mathbf{x}, a) = \{\mathbf{x} + t(1, 1) : 0 \leq t < a\}.$$

The graph $G \subset [0, 1]^2$ of θ_k is constructed recursively as follows.

Let $j = 1$, and set $\mathbf{x}_{1,1} = (0, 0)$. Then set $G_{1,1} = S(\mathbf{x}_{1,1}, m_{1,1})$, and define $\mathbf{x}_{2,1} = (\delta, m_{1,1})$. We then iterate, setting

$$G_{i+1,1} = G_{i,1} \cup S(\mathbf{x}_{i,1}, m_{i,1}) \quad \text{and} \quad \mathbf{x}_{i+1,1} = \left(i\delta, \sum_{r=1}^i m_{r,1} \right)$$

until $i = k - 1$. This assigns all the mass of $\nu([0, 1) \times [0, \delta)) = \delta$ to nearby points consistent with a piecewise continuous measure preserving bijection. Specifically, the projection of $G_{k,1}$ onto the y-axis gives the set $[0, \delta)$.

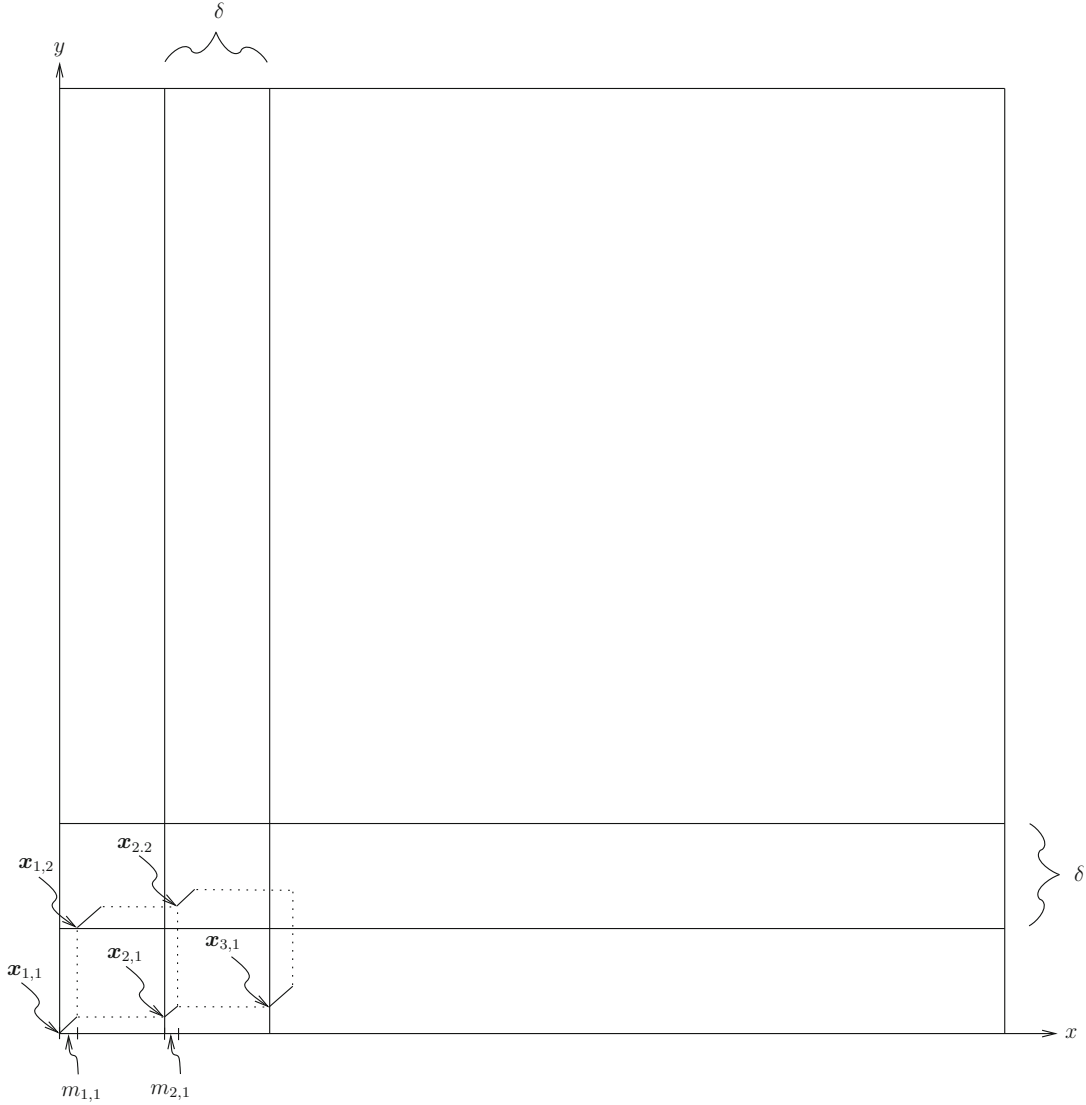


Fig. 1. Construction of σ

Next consider $j = 2$. To maintain that the graph generate a measure preserving bijection, we now start with $\mathbf{x}_{1,2} = (m_{1,1}, \delta)$. The iteration is now

$$G_{i+1,2} = G_{i,2} \cup S(\mathbf{x}_{i,2}, m_{i,2}) \quad \text{and} \quad \mathbf{x}_{i+1,1} = \left(i\delta + m_{i,1}, \sum_{r=1}^i m_{r,2} \right).$$

For $1 < j \leq k$ the definitions are $\mathbf{x}_{1,j} = (\sum_{l=1}^j m_{1,l}, j\delta)$ and

$$G_{i+1,j} = G_{i,j} \cup S(\mathbf{x}_{i,j}, m_{i,j}) \quad \text{and} \quad \mathbf{x}_{i+1,j} = \left(i\delta + \sum_{l=1}^j m_{i,l}, \sum_{r=1}^i m_{r,j} \right).$$

See Fig. 1. Finally we set $G = G_{k,k}$. This graph defines an element θ_k of $\mathcal{P}$. If $\nu_k(A_1 \times A_2) = \int_{A_1} 1_{A_2}(\theta_k(x)) dx$, then all mass inside $T_{i,j}^k$ under ν has stayed inside $T_{i,j}^k$. As a consequence ν_k converges weakly to ν , and the proof is complete.

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