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Electromagnetic-gravitational perturbations of Kerr–Newman spacetime: The Teukolsky and Regge–Wheeler equations

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Abstract. We derive the equations governing the linear stability of Kerr–Newman spacetime to coupled electromagnetic-gravitational perturbations. The equations generalize the celebrated Teukolsky equation for curvature perturbations of Kerr, and the Regge–Wheeler equation for metric perturbations of Reissner–Nordström. Because of the "apparent indissolubility of the coupling between the spin-1 and spin-2 fields", as put by Chandrasekhar, the stability of Kerr–Newman spacetime cannot be obtained through standard decomposition in modes. Due to the impossibility to decouple the modes of the gravitational and electromagnetic fields, the equations governing the linear stability of Kerr–Newman have not been previously derived. Using a tensorial approach that was applied to Kerr, we produce a set of generalized Regge–Wheeler equations for perturbations of Kerr–Newman, which are suitable for the study of linearized stability by physical space methods. The physical space analysis overcomes the issue of coupling of spin-1 and spin-2 fields and represents the first step towards an analytical proof of the stability of the Kerr–Newman black hole.

Keywords: Kerr–Newman black hole; Teukolsky equation; Regge–Wheeler equation.

Mathematics Subject Classification: 83C05, 83C57

1. Introduction

One of the fundamental problems in General Relativity is to understand the final state of evolution of initial data for the Einstein equation. Through gravitational collapse and dispersion of gravitational waves, the geometry to which solutions to the Einstein equation are expected to relax outside the event horizon of a black hole is the one given by the known stationary and axisymmetric explicit solutions: the Kerr and the Kerr–Newman black hole.

According to General Relativity, the interaction between gravitational and electromagnetic fields in a spacetime is governed by the *Einstein–Maxwell equation*:

$$\mathrm{Ric}_{\mu\nu}(g) = 2F_{\mu\lambda}F^{\lambda}{}_{\nu} - \frac{1}{2}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}, \quad D_{[\mu}F_{\nu\lambda]} = 0, \quad D^{\mu}F_{\mu\nu} = 0, \tag{1.1}$$

where Ric denotes the Ricci curvature of the metric, D its covariant derivative and F is an antisymmetric 2-tensor, called the electromagnetic tensor, which satisfies the Maxwell equations.

The *Kerr–Newman metric* [30] is the most general known explicit black hole solution to the Einstein–Maxwell equation (1.1), and it is a 3-parameter family which describes the gravitational field around an isolated rotating charged black hole of mass M, angular momentum Ma and electric charge Q, within the subextremal range $\sqrt{a^2+Q^2} < M$. Its expression in Boyer–Lindquist coordinates (t, r, θ, φ) is given by

$$\mathbf{g}_{M,a,Q} = -\frac{\Delta}{|q|^2}(dt - a\sin^2\theta d\varphi)^2 + \frac{|q|^2}{\Delta}dr^2 + |q|^2 d\theta^2 + \frac{\sin^2\theta}{|q|^2}(adt - (r^2+a^2)d\varphi)^2,$$

where

$$\Delta = r^2 - 2Mr + a^2 + Q^2, \quad |q|^2 = r^2 + a^2\cos^2\theta.$$

The Kerr–Newman metric generalizes the Reissner–Nordström ̈solution (for $a = 0$), and also the Kerr (for $Q = 0$) and Schwarzschild metric (for $Q = a = 0$), which are solutions to the Einstein vacuum equation. As such, the Kerr–Newman spacetime plays a fundamental role in describing the final state of evolution in General Relativity.

As part of the resolution of the description of the final state, we focus on the issue of stability of the Kerr–Newman black hole, which consists in showing that solutions to the Einstein equation which are given as small perturbations of the initial data of such a black hole asymptotically converge in time to a member of the Kerr–Newman family. The stability of the Kerr–Newman family can be analyzed at different levels:

(1) the *linear stability* consists in the analysis of the linearized Einstein–Maxwell equation around the background metric $\mathbf{g}_{M,a,Q}$. It can be further divided into (a) mode stability and (b) full linear stability.
(2) the *nonlinear stability* consists in the analysis of the full Einstein–Maxwell equation for a perturbation of a member of the Kerr–Newman family.

The mode analysis (a) of the Einstein equation consists in analyzing only special solutions, the so-called mode solutions. In the simplified case of the linear wave equation

$$\square_{\mathbf{g}_{M,a,Q}}\psi = 0, \tag{1.2}$$

where $\Box_{\mathbf{g}_{M,a,Q}}$ is the D'Alembertian associated to the Kerr–Newman metric, mode solutions are solutions of the separated form

$$\psi(r,t,\theta,\phi) = e^{-i\omega t} e^{im\phi} R(r) S(\theta), \tag{1.3}$$

where $\omega \in \mathbb{C}$ is the time frequency and m is the azimuthal mode. Because of the integrability of the geodesic flow in the Kerr–Newman metric, functions of the form (1.3) are solutions to the wave equation (1.2), as long as $R(r)$ satisfies a radial ODE and $S(\theta)$ satisfies an angular ODE (which defines spheroidal harmonics $S_{\omega m \ell}$). The mode stability consists in proving that solutions of the form (1.3) with finite initial energy do not have imaginary part of ω which is positive, i.e. do not exponentially grow in time. The mode stability of Schwarzschild, Reissner–Nordström and Kerr black hole was obtained as a combination of many results in black hole perturbation theory by the physics community in the 1970s and 1980s, see [3, 4, 32, 37–39].

Particularly relevant are the case of axial metric perturbations of Schwarzschild, which are governed by the so-called *Regge–Wheeler equation* [32], of the form

$$\Box_{\mathbf{g}_M} \psi = \frac{4}{r^2}\left(1 - \frac{2M}{r}\right)\psi. \tag{1.4}$$

Observe that the potential on the right-hand side of (1.4) is positive in the exterior of the black hole. Inspired by (1.4), we denote by *Regge–Wheeler equation* any equation of the form $\Box_g \psi - V\psi = 0$ for a positive real potential V.

In the case of gravitational perturbations of Kerr, in order to obtain an equation decoupled from any other component, one needs to consider perturbations at the level of curvature. The extreme null Weyl scalars then satisfy the *Teukolsky equation* of spin $s = \pm 2$ and $s = \pm 1$ for gravitational and electromagnetic perturbations of Kerr, respectively [37], of the form

$$\begin{aligned} \mathcal{T}^{[s]}(\psi) := \Box_{\mathbf{g}_{M,a}} \psi^{[s]} + \frac{2s}{|q|^2}(r-M)\partial_r \psi^{[s]} + \frac{2s}{|q|^2}\left(\frac{a(r-M)}{\Delta} + i\frac{\cos\theta}{\sin^2\theta}\right)\partial_\varphi \psi^{[s]} \\ + \frac{2s}{|q|^2}\left(\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right)\partial_t \psi^{[s]} - \frac{s}{|q|^2}(s\cot^2\theta - 1)\psi^{[s]} = 0, \end{aligned} \tag{1.5}$$

which is also a separable equation. In [32, 38] (see also [33, 36]), the mode analysis of the Regge–Wheeler and the Teukolsky equation was proved, and a transformation theory [4] (now known as *Chandrasekhar transformation*) was discovered to connect the metric perturbations approach to the curvature perturbations one. The results in the mode analysis are collected in the monumental book by Chandrasekhar [4].

Nevertheless, mode stability for Eq. (1.2) is still consistent with the unboundedness of (finite initial energy) solutions as the time increases, i.e. it does not exclude the possibility that

$$\limsup_{t\to\infty} \psi(r,t,\theta,\phi) = \infty. \tag{1.6}$$

This is because statements at the level of the single mode solutions do not imply boundedness statements for the infinite superposition of those modes. The full linear stability (b) consists in proving a uniform bound for a general solution ψ of (1.2), and therefore excluding (1.6).

Extensive progress has been obtained in the last 15 years which allowed to go beyond the mode analysis in Kerr spacetime, tackling the full linear stability (b) for the linear wave equation. A robust geometric interpretation of the redshift effect [8], a physical space analysis of the trapping region and the superradiance [11], a hierarchy of r-weighted decay [9] all contributed to a complete understanding of the boundedness of solutions to the linear wave equation, finally proving that (1.6) indeed does not happen. The complete resolution of boundedness and decay statements for the linear wave equation (1.2) was obtained for slowly rotating Kerr solutions [10] and then for the full subextremal range $|a| < M$ [12] (see also [2, 35]). Similarly, proofs of boundedness and decay statements for the Teukolsky equation have been obtained in Schwarzschild [6] (see also [24–27]) in Reissner–Nordström [14, 16] and in Kerr, for slowly rotating [7, 29] and very recently in the full subextremal range [34]. These results have been used to obtain proof of the full linear stability of Schwarzschild [6], Reissner–Nordström, for small charge [17] and then in the full subextremal range [15], and for slowly rotating Kerr [1, 21]. Concerning the full nonlinear stability of black hole solutions to the Einstein equation, the only known result is the proof of nonlinear stability of Schwarzschild under the class of symmetry of axially symmetric polarized perturbations [28]. In the presence of a positive cosmological constant, the Kerr–de Sitter and the Kerr–Newman–de Sitter family with small angular momentum have also been proved to be nonlinearly stable [22, 23].

Quite strikingly, the Kerr–Newman solution stands up as genuinely different from the similar cases of Kerr or Reissner–Nordström, even in the simplest possible form of stability, i.e. the mode stability as studied by the black hole perturbation theory community. As stated by Chandrasekhar in [4, Sec. 111], "the methods that have proved to be so successful in treating the gravitational perturbations of the Kerr spacetime do not seem to be applicable (nor susceptible to easy generalizations) for treating the coupled electromagnetic-gravitational perturbations of the Kerr–Newman spacetime." The techniques applied in those early works, which relied on decomposition in frequency modes of perturbations of the solutions, failed to be extended to the case of Kerr–Newman spacetime, despite the manifest similarity of the metric to the Kerr case. Again as pointed out by Chandrasekhar in [4, Sec. 111], "the principal obstacle is in finding separated equation" and in the "apparent indissolubility of the coupling between the spin-1 and spin-2 fields in the perturbed spacetime". Following the same procedure as in the case of Kerr or Reissner–Nordström, one reaches a point where the equations cannot be decoupled or separated any further. In [4, p. 583], Chandrasekhar gives an explanation of "why the system of equations proves intractable in contrast to apparently similar system of equations encountered in the treatment of the perturbations of the

Reissner–Nordström and Kerr spacetimes". The reason has to do with the interaction of the spin-1 and spin-2 fields in a non-spherically symmetric background. We summarize his argument here and describe how we intend to overcome such difficulties towards an analytical proof of the stability of the Kerr–Newman black hole.

1.1. *Why the analytical proof of mode stability for Kerr–Newman fails*

Being the most general explicit black hole solution of the Einstein equation coupled with matter, the Kerr–Newman spacetime has been at the center of analytical and numerical research for decades. Numerical works strongly support the mode stability of Kerr–Newman spacetime [13, 31], and the Kerr–Newman metric is expected to be stable as a solution to the fully nonlinear Einstein–Maxwell equation. Nevertheless, an analytical proof of even its mode stability is missing, and the state of the art on this problem is pretty much the same as described by Chandrasekhar [4] in 1983. We now explain what are the main issues.

The Einstein–Maxwell equation (1.1) governs the interaction between the gravitational radiation, encoded in the left-hand side of the equation (i.e. the curvature), and the electromagnetic radiation, encoded in the right-hand side (i.e. the electromagnetic tensor). From the study of perturbations of Kerr, we know that the gravitational and the electromagnetic radiation are transported by a spin-2 field $\psi^{[2]}$ and a spin-1 $\psi^{[1]}$, respectively. This is more precisely related to the fact that the extreme null component of the Weyl curvature is a 2-tensor on the sphere (θ, φ), while the extreme null component of the electromagnetic tensor is a 1-tensor on the sphere.

When taken independently, the gravitational and electromagnetic perturbations of Kerr satisfy the Teukolsky equation (1.5) for spin $s = \pm 2$ or $s = \pm 1$, respectively. On the other hand, when considering coupled electromagnetic-gravitational perturbations of Kerr–Newman, one should expect a *system of coupled Teukolsky equations*, as in the case of Reissner–Nordström [14, 16], of the schematic form:

$$\begin{aligned} \mathcal{T}^{[1]}(\psi^{[1]}) &= \mathrm{div}(\psi^{[2]}), \\ \mathcal{T}^{[2]}(\psi^{[2]}) &= \nabla\widehat{\otimes}(\psi^{[1]}), \end{aligned} \tag{1.7}$$

where the angular operators on the right-hand side relate 1-tensors and 2-tensors. More precisely, if $\psi^{[1]}$ is a 1-tensor and $\psi^{[2]}$ is a symmetric traceless 2-tensor, then $\mathrm{div}(\psi^{[2]})_a := \nabla^b \psi^{[2]}_{ab}$ is a 1-tensor and $2\nabla\widehat{\otimes}(\psi^{[1]})_{ab} = \nabla_a \psi^{[1]}_b + \nabla_b \psi^{[1]}_a - \delta_{ab}\mathrm{div}\psi^{[1]}$ is a symmetric traceless 2-tensor on the sphere.

The issue in the analysis of a coupled system like (1.7) comes from the decomposition in modes. The mode decomposition of the Teukolsky variables

$$\psi^{[s[}(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} R^{[s]}(r) S^{[s]}_{m\ell}(a\omega, \cos\theta)$$

involves the *spin s-weighted spheroidal harmonics* $S^{[s]}_{m\ell}(a\omega, \cos\theta)$ which are eigenfunctions of the spin s-weighted Laplacian

$$\Delta^{[s]} = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) - \frac{m^2 + 2ms\cos\theta + s^2}{\sin^2\theta} + a^2\omega^2\cos^2\theta - 2a\omega s\cos\theta.$$

For $a = 0$, they reduce to the spherical harmonics $S^{[s]}_{m\ell}(0, \cos\theta) = Y^{[s]}_{m\ell}(\cos\theta)$. *Spin-weighted spherical harmonics* of different spins are simply related through the angular operators div and $\nabla\widehat{\otimes}$, and have the same eigenvalues. Schematically:

$$\nabla\widehat{\otimes}(Y^{[1]}_{m\ell}) = -\lambda Y^{[2]}_{m\ell},$$
$$\text{div}(Y^{[2]}_{m\ell}) = \lambda Y^{[1]}_{m\ell}.$$

On the other hand, in the general axisymmetric case, as in Kerr or Kerr–Newman, the spin-weighted spheroidal harmonics of different spins are not simply related through those angular operators.

We are now ready to explain the "apparent indissolubility of the coupling between the spin-1 and spin-2 fields" [4] for electromagnetic-gravitational perturbations of Kerr–Newman, in contrast with Reissner–Nordström or Kerr. In a spherically symmetric background, as in Reissner–Nordström, the fact that the spherical harmonics of different spins are simply related through the angular derivatives implies that the decomposition in modes of the system of Teukolsky equations (1.7) passes through. When considering the separated versions of the equations, one obtains

$$\mathcal{T}^{[1]}(Y^{[1]}_{m\ell}) = \text{div}(Y^{[2]}_{m\ell}) = \lambda Y^{[1]}_{m\ell},$$
$$\mathcal{T}^{[2]}(Y^{[2]}_{m\ell}) = \nabla\widehat{\otimes}(Y^{[1]}_{m\ell}) = -\lambda Y^{[2]}_{m\ell},$$

giving two decoupled equations for the spin-1 and the spin-2 fields. For gravitational perturbations of Kerr one only uses the spin-2 decomposition for $\mathcal{T}^{[2]}(Y^{[2]}_{m\ell}) = 0$, so the problem of the coupling does not arise.

In electromagnetic-gravitational perturbations of the axially symmetric Kerr–Newman, the interaction between the spin-2 and spin-1 prevents the separability in modes. In particular, when trying to derive equations for $S^{[1]}_{m\ell}$ and for $S^{[2]}_{m\ell}$, one cannot separate them:

$$\mathcal{T}^{[1]}(S^{[1]}_{m\ell}) = \text{div}(S^{[2]}_{m\ell}),$$
$$\mathcal{T}^{[2]}(S^{[2]}_{m\ell}) = \nabla\widehat{\otimes}(S^{[1]}_{m\ell}),$$

as the right-hand side of the first equation cannot be written in terms of $S^{[1]}_{m\ell}$ and the right-hand side of the second equation cannot be written in terms of $S^{[2]}_{m\ell}$.

These are the main obstacles to separability of the equations in the case of electromagnetic-gravitational perturbations of Kerr–Newman spacetime. As Chandrasekhar ends at [4, p. 583], "one might be inclined to conclude that a decoupling of the system of equations and a separation of the variables will be possible, if at all, only by contemplating equations of order 4 or higher".

1.2. *Towards the full linear stability of Kerr–Newman*

In treating the coupled electromagnetic-gravitational perturbations of Kerr–Newman spacetime, the decomposition in modes of the equations, which had the objective of simplifying the analysis of the perturbations, actually makes them unsolvable as consequence of the discussion in the previous section. Observe that such failure is explicitly related to the fact that the equations as analyzed in [4] required the decomposition in spheroidal harmonics, which yields the problem of non-separability of the decomposition. There is no reason to believe that if one does not decompose in modes nor separate the equations using the spheroidal harmonics such problems could not be circumvented.

Our approach to solve this issue is to abandon the decomposition in modes, and perform a *physical space analysis* of the equations. Following the road map that mathematicians have taken in the last few years in interpreting in physical space the mode analysis done by the physics community, the Kerr–Newman solution may be the case where a physical space approach could succeed where the mode analysis in physics failed. Observe that our proof of boundedness of a general solution through a physical space analysis will in particular imply the absence of exponentially growing modes, therefore proving mode stability.

We summarize here the four main ingredients in the analysis: the formalism to study perturbations of the Kerr–Newman spacetime, the identification of the gauge-invariant quantities in the linear perturbations, the derivation of the system of coupled Teukolsky equations, and finally the derivation, through the Chandrasekhar transformation, of a system of generalized Regge–Wheeler equations.

1.2.1. *The GKS formalism*

As a first step, we present the formalism which we use to treat perturbations of axially symmetric Petrov Type D spacetimes, like Kerr or Kerr–Newman . One way to analyze the perturbations is to use the Newman–Penrose (NP) formalism, which consists in decomposing all the components in null frames, obtaining complex scalars. We instead make use of a more geometric formalism, more commonly used in the mathematical community, and first developed in the proof of nonlinear stability of Minkowski space [5]. Such formalism was extended in [19] for general Petrov Type D spacetime in the context of the nonlinear stability of Kerr.

We recall that a Petrov Type D spacetime's Weyl curvature $\mathbf{W}_{\mu\nu\alpha\beta}$ is diagonalizable by two linearly independent eigenvectors, the so-called principal null-directions. We call the outgoing null direction e_4 and the ingoing one e_3. The tangent space orthogonal to them is spanned by two orthonormal vectors, e_a, for $a = 1, 2$. Observe that in Kerr, the orthogonal structure determined by the principal null frames e_3, e_4 is not integrable, i.e. e_1 and e_2 do not span the tangent space of a 2-surface, like in Schwarzschild. This can be seen in the non-symmetry of the 2-tensors χ and $\underline{\chi}$ defined by

$$\chi(e_a, e_b) = g(D_a e_4, e_b), \quad \underline{\chi}(e_a, e_b) = g(D_a e_3, e_b), \quad a, b = 1, 2,$$

which in the case of an integrable horizontal structure would be the null second fundamental forms of the embedding of the sphere in the spacetime, therefore being symmetric. In the case of Kerr or Kerr–Newman, the 2-tensor χ and $\underline{\chi}$ are not symmetric in a and b.

In the GKS formalism (from the authors in [19]), the non-integrability of the horizontal structure is allowed, and all components are decomposed in null frames, obtaining a range of complex 2-tensors, 1-tensors and scalars. See Sec. 2 for the description of the formalism, with particular attention to the comparison with the NP formalism in Sec. 2.5. We extend it here to the case of the Einstein–Maxwell equation, by deriving the equations in their full generality in Sec. 3.

We then apply this general formalism to the case of Kerr–Newman and its linear perturbations. More precisely, the GKS quantities which vanish in Kerr–Newman, are considered to be $O(\epsilon)$, where ϵ is a smallness parameter, in linear perturbations of Kerr–Newman, see Sec. 4. The next step is to identify the $O(\epsilon)$-quantities which govern the linear perturbations.

1.2.2. *The identification of the gauge-invariant quantities*

The first issue to specifically treat electromagnetic-gravitational perturbations of Kerr–Newman spacetime is to identify what are the Teukolsky variables which represent the electromagnetic and gravitational radiations, respectively. Since those variables have a physical meaning, they should be independent of the choice of coordinates to a certain extent, or more precisely being (quadratically) invariant under infinitesimal tetrad transformations. For example, the spin-2 complex Teukolsky variable given by

$$A_{ab} = \mathbf{W}(e_4, e_a, e_4, e_b) + i\ {}^*\mathbf{W}(e_4, e_a, e_4, e_b),$$

where * denotes the Hodge dual, is a symmetric traceless 2-tensor on the horizontal structure (which corresponds to Ψ_0 in NP formalism) and is known to be invariant under infinitesimal rotations of the frame. More precisely, if a rotation is applied to the frame (e_3, e_4, e_1, e_2) into a new frame (e_3', e_4', e_1', e_2') which is ϵ-close to the previous one, the variable A' computed with respect to the primed frame is ϵ^2-close to the original one, i.e. $A' = A + O(\epsilon^2)$. The quantity A is precisely the Teukolsky variable representing gravitational perturbations of Kerr, and satisfies the Teukolsky equation of spin 2 [19, 37].

For electromagnetic-gravitational perturbations of Kerr–Newman, the Teukolsky variable A is not sufficient to describe the full perturbation. In particular, one would need a quantity satisfying a spin 1 Teukolsky equation as the electromagnetic contribution. The Teukolsky variable of spin 1 in electromagnetic perturbations of Kerr, i.e.

$$^{(\mathbf{F})}B_a = \mathbf{F}(e_4, e_a) + i\ {}^*\mathbf{F}(e_4, e_a),$$

where $\mathbf{F}$ is the electromagnetic tensor, is not invariant under infinitesimal rotations of the frame, and therefore cannot represent electromagnetic radiation. This problem appears already in the perturbations of Reissner–Nordström spacetime, where $^{(\mathbf{F})}B$ also fails to be gauge-invariant. In the case of Reissner–Nordström, two additional quantities $\mathfrak{f}$ and $\tilde{\beta}$, a 2-tensor and a 1-tensor, respectively, were identified to be gauge-invariant and satisfy a coupled system of Teukolsky equation [14, 16].

Inspired by the quantities in Reissner–Nordström, in Sec. 5, we define the symmetric traceless 2-tensor $\mathfrak{F}$ and the 1-tensor $\mathfrak{B}$ which are quadratically invariant upon infinitesimal rotation of the frame (see (5.2) and (5.3) for the precise definition). In addition to those, we have the gauge-invariant 1-tensor $\mathfrak{X}$ which is auxiliary in the derivation.

1.2.3. *The system of coupled Teukolsky equations*

As described above, we have defined four gauge-invariant quantities for linear electromagnetic-gravitational perturbations of Kerr–Newman, given by

$$A, \quad \mathfrak{F}, \quad \mathfrak{B}, \quad \mathfrak{X},$$

where A and $\mathfrak{F}$ are symmetric traceless 2-tensors, and therefore good candidates to represent gravitational radiation and $\mathfrak{B}$ and $\mathfrak{X}$ are 1-tensors, to represent electromagnetic radiation. As in the case of Reissner–Nordström it turns out that $\mathfrak{F}$ and $\mathfrak{B}$ are the most significant quantities, while A and $\mathfrak{X}$ can be thought of as auxiliary quantities.

Observe that under a rotation of the frame given by a conformal rescaling of the null vectors e_3 and e_4, i.e. if $e_3' = \lambda e_3$ and $e_4' = \lambda^{-1} e_4$ for a real scalar λ, the quantities A and $\mathfrak{X}$ change as $A' = \lambda^2 A$, $\mathfrak{X}' = \lambda^2 \mathfrak{X}$, while $\mathfrak{F}$ and $\mathfrak{B}$ change as $\mathfrak{F}' = \lambda \mathfrak{F}$, $\mathfrak{B}' = \lambda \mathfrak{B}$. We say that $\mathfrak{F}$ and $\mathfrak{B}$ are of conformal type 1 and A and $\mathfrak{X}$ of conformal type 2.

In Sec. 6, we derive the wave-like equations satisfied by A, $\mathfrak{F}$ and $\mathfrak{B}$ as a consequence of the Einstein–Maxwell equations. We obtain the following system of coupled Teukolsky equations, see Theorem 6.1:

$$\mathcal{T}_1(\mathfrak{B}) = \mathbf{M}_1[\mathfrak{F}, \mathfrak{X}],$$

$$\mathcal{T}_2(\mathfrak{F}) = \mathbf{M}_2[A, \mathfrak{X}, \mathfrak{B}],$$

$$\mathcal{T}_3(A) = \mathbf{M}_3[\mathfrak{F}, \mathfrak{X}],$$

where the $\mathcal{T}_s$ are Teukolsky operators and $\mathbf{M}_s$ denotes the dependence on the right-hand side of each equation.

The projection to the first component of the Teukolsky operators gives the Teukolsky equation for complex scalars. It turns out that in Kerr–Newman, not only the spin of the variable is not the only parameter appearing in the Teukolsky equation, but so does its conformal type. We define for a complex scalar ψ of spin s and conformal type c the Teukolsky operator

$$\begin{aligned}\mathcal{T}^{[s,c]}(\psi) := \square_{\mathbf{g}_{M,a,Q}}\psi + \frac{2c}{|q|^2}(r-M)\partial_r\psi + \frac{2}{|q|^2}\left(c\frac{a(r-M)}{\Delta} + si\frac{\cos\theta}{\sin^2\theta}\right)\partial_\varphi\psi \\ + \frac{2}{|q|^2}\left(c\left(\frac{M(r^2-a^2)-Q^2r}{\Delta} - r\right) - sia\cos\theta\right)\partial_t\psi \\ + \frac{1}{|q|^2}(s - s^2\cot^2\theta)\psi.\end{aligned}$$

By comparing it to (1.5), one can see that for $c = s$, the Teukolsky operator $\mathcal{T}^{[s,s]}(\psi)$ reduces to the standard Teukolsky operator in Kerr. In Kerr–Newman, it is important to keep the distinction since $\mathfrak{F}$ is of spin 2 and conformal type 1.

Due to the non-separability in modes, our goal is to analyze the Teukolsky equations in physical space. Unfortunately, this is not possible, even in the case of Schwarzschild or Kerr. Recall that to prove boundedness of the energy for a solution of the wave equation $\square\psi = 0$, one multiplies it by $\partial_t\psi$ and integrate it by parts. For example, in the case of Minkowski:

$$\begin{aligned}0 = \square\psi\cdot\partial_t\psi &= \big(-\partial_t^2\psi + \partial_x^2\psi\big)\cdot\partial_t\psi \\ &= -\partial_t^2\psi\cdot\partial_t\psi - \partial_t\partial_x\psi\cdot\partial_x\psi + \partial_x(\partial_x\psi\cdot\partial_t\psi) \\ &= -\frac{1}{2}\partial_t(|\partial_t\psi|^2 + |\partial_x\psi|^2) + \text{boundary terms.}\end{aligned}$$

Upon integration on a causal domain, one can neglect the boundary terms obtaining conservation of the energy. Similarly for a Regge–Wheeler equation, the term with the potential can be written as a boundary term:

$$\begin{aligned}0 &= \big(\square\psi - V\psi\big)\cdot\partial_t\psi \\ &= -\frac{1}{2}\partial_t(|\partial_t\psi|^2 + |\partial_x\psi|^2) - \frac{1}{2}V\partial_t(|\psi|^2) + \text{boundary terms} \\ &= -\frac{1}{2}\partial_t(|\partial_t\psi|^2 + |\partial_x\psi|^2 + V|\psi|^2) + \text{boundary terms.}\end{aligned}$$

If the potential V is positive, one obtains the conservation of a positive definite energy. The Teukolsky equation is instead of the form $\square\psi - V\psi = c_1\partial_r\psi + c_2\partial_\varphi\psi + c_3\partial_t\psi$, and so clearly one cannot obtain boundedness of the energy directly in this way, because of the presence of the first-order terms. This motivates our search for a more amenable system of equations, of the Regge–Wheeler type.

1.2.4. *The system of generalized Regge-Wheeler equations*

One would like to transform the system of Teukolsky equations, which are intractable to physical space energy estimates, to a system of Regge–Wheeler-type equations. Such transformation is related to the passage from curvature perturbations to metric perturbations, and was referred to as "transformation theory" in [4]. Chandrasekhar describes such transformation in the mode analysis as consisting in taking derivatives along the null direction of the Teukolsky variables, in order to obtain solutions to the Regge–Wheeler equation. Dafermos–Holzegel–Rodnianski crucially extended the Chandrasekhar transformation to a physical space one, first in Schwarzschild [6] and then in Kerr [7], see also [29]. In [19, 28], physical space nonlinear analogue of the Chandrasekhar transformation have also been introduced.

Following this idea, in Sec. 7, we define the Chandrasekhar-transformed of the quantities $\mathfrak{B}$ and $\mathfrak{F}$ in the case of linear perturbations of Kerr–Newman, and obtain the main result of the paper, see Theorem 7.3 for the precise statement.

Theorem 1.1. *Consider a linear electromagnetic-gravitational perturbation of Kerr–Newman spacetime* $\mathbf{g}_{M,a,Q}$, *with associated gauge-invariant quantities* $\mathfrak{B}$ *and* $\mathfrak{F}$. *Then there exist a* 1*-tensor* $\mathfrak{p}$ *and a symmetric traceless* 2*-tensor* $\mathfrak{q}^{\mathbf{F}}$, *obtained as Chandrasekhar-transformed of* $\mathfrak{B}$ *and* $\mathfrak{F}$, *respectively, such that* $\mathfrak{p}$ *and* $\mathfrak{q}^{\mathbf{F}}$ *satisfy the following coupled system of generalized Regge–Wheeler equations*:

$$\Box_{\mathbf{g}_{M,a,Q}}\mathfrak{p} - i\frac{2a\cos\theta}{|q|^2}\nabla_t\mathfrak{p} - V_1\mathfrak{p} = 4Q^2\frac{(r-ia\cos\theta)^3}{|q|^5}\mathrm{div}\mathfrak{q}^{\mathbf{F}} + \ l.o.t. \tag{1.8}$$

$$\Box_{\mathbf{g}_{M,a,Q}}\mathfrak{q}^{\mathbf{F}} - i\frac{4a\cos\theta}{|q|^2}\nabla_t\mathfrak{q}^{\mathbf{F}} - V_2\mathfrak{q}^{\mathbf{F}} = -\frac{1}{2}\frac{(r+ia\cos\theta)^3}{|q|^5}\nabla\widehat{\otimes}\mathfrak{p} + \ l.o.t. \tag{1.9}$$

with real positive potentials V_1 *and* V_2 *and l.o.t. denotes lower order terms with respect to* $\mathfrak{p}$ *and* $\mathfrak{q}^{\mathbf{F}}$.

These equations represent the main system of equations governing electromagnetic-gravitational perturbations of the Kerr–Newman spacetime. In particular observe that, in applying the Chandrasekhar transformation, the dependence on the auxiliary quantities A and $\mathfrak{X}$ disappears. The above equations have the same structure as the generalized Regge–Wheeler equation in Kerr obtained in [19]. Also, for $a=0$ the above system of equations reduces to the Regge–Wheeler system of Reissner–Nordström in [15], for which boundedness of the energy was obtained in the full subextremal range.

The above theorem is proved through a careful (and lengthy) computation which consists in applying an ingoing null derivative to both Teukolsky equations for $\mathfrak{B}$ and $\mathfrak{F}$, and then choose a precise complex rescaling of the transformed quantities. Such rescaling is applied in order to obtain the above structure of the equations, for which boundedness of the energy can be obtained in physical space. Observe that the above equations are not precisely of the Regge–Wheeler form, but have additional terms, like the first-order term ∇_t and the coupling terms on the right-hand side. Nevertheless, in Theorem 1.1, we are careful to obtain precisely a structure which allows for the proof of boundedness of energy in physical space. More precisely, upon multiplying Eq. (1.8) by $\nabla_t \overline{\mathfrak{p}}$ and Eq. (1.9) by $\nabla_t \overline{\mathfrak{q}^{\mathbf{F}}}$ and taking their real part, we obtain the following simplifications:

(1) The Regge–Wheeler pieces of the equations, i.e. $\big(\square_{\mathbf{g}_{M,a,Q}}\mathfrak{p} - V_1\mathfrak{p}\big)\nabla_t\overline{\mathfrak{p}}$ and $\big(\square_{\mathbf{g}_{M,a,Q}}\mathfrak{q}^{\mathbf{F}} - V_2\mathfrak{q}^{\mathbf{F}}\big)\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}$, for real positive potentials, can be written as boundary term,

(2) The first-order terms, being of the form $i\nabla_t$, get cancelled in the energy estimates:

$$i\frac{2a\cos\theta}{|q|^2}\nabla_t\mathfrak{p}\nabla_t\overline{\mathfrak{p}} + \overline{i\frac{2a\cos\theta}{|q|^2}}\nabla_t\overline{\mathfrak{p}}\nabla_t\mathfrak{p} = 0,$$

$$i\frac{4a\cos\theta}{|q|^2}\nabla_t\mathfrak{q}^{\mathbf{F}}\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}} + \overline{i\frac{4a\cos\theta}{|q|^2}}\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}\nabla_t\mathfrak{q}^{\mathbf{F}} = 0.$$

(3) The coupling terms, given by adjoint operators div and $\nabla\widehat{\otimes}$ multiplied by complex conjugate functions, get simplified upon summing the estimates for the two equations:

$$\frac{(r - ia\cos\theta)^3}{|q|^5}\mathrm{div}\mathfrak{q}^{\mathbf{F}}\nabla_t\overline{\mathfrak{p}} - \frac{(r + ia\cos\theta)^3}{|q|^5}\nabla\widehat{\otimes}\mathfrak{p}\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}} = \text{ l.o.t.}$$

(4) The lower order terms have a favorable structure in using transport equations to be estimated.

As a consequence of the above theorem, the good properties of the equations obtained in Reissner–Nordström and in Kerr can be generalized to the case of Kerr–Newman. This strikingly compares with the equations in separated modes as described at the beginning of this introduction, which could not be generalized from the Kerr and Reissner–Nordström case. By avoiding the decomposition in modes, and maintaining the above equations for the 1-tensor $\mathfrak{p}$ and the 2-tensor $\mathfrak{q}^{\mathbf{F}}$, the issue of non-commutativity of the decomposition is not present and a physical space analysis of the above system is possible. In Sec. 7.4, we sketch how to prove that solutions to the generalized Regge–Wheeler equations as obtained in Theorem 1.1 have bounded energy. Such proof has to be combined with spacetime Morawetz estimates to obtain the complete statement of boundedness and decay for the Teukolsky system of equations. More precisely, such analysis would have to avoid decomposition in modes for the solutions, for example in the spirit of [2] for small angular momentum. Nevertheless, the Morawetz estimates, which will be obtained in a future work, are much less sensitive to the structure of the equation, and the procedure described in Sec. 7.4 to obtain bounded energy is crucial to justify the precise structure of the equations here derived.

This paper is organized as follows. In Sec. 2, we recall the general formalism introduced in [19] and in Sec. 3, we derive the Einstein–Maxwell equations in their full generality. In Sec. 4, we introduce the Kerr–Newman spacetime and its linear perturbations. In Sec. 5, we define the main gauge-invariant quantities in electromagnetic-gravitational perturbations of Kerr–Newman spacetime. In Sec. 6, we derive the system of Teukolsky equations satisfied by the gauge-invariant quantities. Finally, in Sec. 7, we define the Chandrasekhar transformation in Kerr–Newman and derive the Regge–Wheeler-type equations for the perturbations, proving the main theorem of the paper.

To facilitate the reading of the paper, we diverted most of the proofs (involving lengthy computations) to the appendix. In Appendix A, we collect the explicit computations needed in the first five sections of the paper. In Appendix B, we derive the system of Teukolsky equations and in Appendix C, we derive the system of generalized Regge–Wheeler equations.

2. The GKS Formalism

In this section, we collect the main definitions and preliminaries to the formalism introduced in [19]. From the authors of [19] we refer to this formalism as GKS formalism. We refer to [19, Sec. 2] for more details.

2.1. *Null pairs and horizontal structures*

Let $(\mathcal{M}, \mathbf{g})$ be a Lorentzian 4-dimensional manifold. Consider an arbitrary null pair $e_3 = \underline{L}$ and $e_4 = L$, i.e.

$$\mathbf{g}(e_3, e_3) = \mathbf{g}(e_4, e_4) = 0, \quad \mathbf{g}(e_3, e_4) = -2.$$

We say that a vectorfield X is horizontal if

$$\mathbf{g}(L, X) = \mathbf{g}(\underline{L}, X) = 0.$$

On the set of horizontal vectors, given a fixed orientation we define the induced volume form by $\in(X, Y) := \frac{1}{2} \in (X, Y, \underline{L}, L)$.

Observe that the commutator $[X, Y]$ of two horizontal vectorfields may fail to be horizontal. We say that the pair $(L, \underline{L})$ is integrable if the set of horizontal vectorfields forms an integrable distribution, i.e. X, Y horizontal implies that $[X, Y]$ is horizontal.

Given an arbitrary vectorfield X we denote by ${}^{(h)}X$ its horizontal projection,

$${}^{(h)}X = X + \frac{1}{2}\mathbf{g}(X, \underline{L})L + \frac{1}{2}\mathbf{g}(X, L)\,\underline{L}.$$

A k-covariant tensor-field U is said to be horizontal if for any $X_1, \ldots, X_k$, we have

$$U(X_1, \ldots, X_k) = U({}^{(h)}X_1, \ldots, {}^{(h)}X_k).$$

Definition 2.1. *For any horizontal X, Y we define*[a]

$$\gamma(X, Y) = \mathbf{g}(X, Y) \tag{2.1}$$

[a]In the particular case where the horizontal structure is integrable, γ is the induced metric and χ and $\underline{\chi}$ are the null second fundamental forms.

and

$$\chi(X,Y) = \mathbf{g}(\mathbf{D}_X L, Y), \quad \underline{\chi}(X,Y) = \mathbf{g}(\mathbf{D}_X \underline{L}, Y),$$

where $\mathbf{D}$ *is the covariant derivative of* $\mathbf{g}$.

Observe that χ and $\underline{\chi}$ are symmetric if and only if the horizontal structure is integrable, as follows from

$$\chi(X,Y) - \chi(Y,X) = \mathbf{g}(\mathbf{D}_X L, Y) - \mathbf{g}(\mathbf{D}_Y L, X) = -\mathbf{g}(L, [X,Y]).$$

Given X, Y horizontal vectors, the covariant derivative $\mathbf{D}_X Y$ fails in general to be horizontal. We thus define,[b]

$$\nabla_X Y :=^{(h)} (\mathbf{D}_X Y) = \mathbf{D}_X Y - \frac{1}{2}\underline{\chi}(X,Y)L - \frac{1}{2}\chi(X,Y)\,\underline{L}.$$

Given a general covariant, horizontal tensor-field U we define its horizontal covariant derivative according to the formula

$$\begin{aligned}\nabla_Z U(X_1,\dots,X_k) = Z(U(X_1,\dots,X_k)) - U(\nabla_Z X_1,\dots,X_k)\\ - \cdots - U(X_1,\dots,\nabla_Z X_k).\end{aligned}$$

Given X horizontal, $\mathbf{D}_L X$ and $\mathbf{D}_{\underline{L}} X$ are in general not horizontal. We thus define

$$\begin{aligned}\nabla_4 X &:= {}^{(h)}(\mathbf{D}_L X) = \mathbf{D}_L X - \frac{1}{2}\mathbf{g}(X, \mathbf{D}_L \underline{L})L - \frac{1}{2}\mathbf{g}(X, \mathbf{D}_L L)\,\underline{L},\\ \nabla_3 X &:= {}^{(h)}(\mathbf{D}_{\underline{L}} X) = \mathbf{D}_{\underline{L}} X - \frac{1}{2}\mathbf{g}(X, \mathbf{D}_{\underline{L}} \underline{L})L - \frac{1}{2}\mathbf{g}(X, \mathbf{D}_{\underline{L}} L)\,\underline{L}.\end{aligned}$$

We can extend the operators ∇_4 and ∇_3 to arbitrary k-covariant, horizontal tensor-fields U as above. Therefore, with the above definitions ∇, ∇_4 and ∇_3 take horizontal tensor-fields into horizontal tensor-fields. We can then extend the definition of horizontal covariant derivative to any X in the tangent space of M and Y horizontal as

$$\dot{\mathbf{D}}_X Y :=^{(h)} (\mathbf{D}_X Y) \tag{2.2}$$

and we can extend it to horizontal tensor-fields as above.

Given a horizontal structure defined by $e_3 = \underline{L}$, $e_4 = L$, we associate a null frame by choosing orthonormal horizontal vectorfields e_1, e_2 such that $\gamma(e_a, e_b) = \delta_{ab}$ for $a, b = 1, 2$. For an arbitrary orthonormal horizontal frame $(e_a)_{a=1,2}$, we denote $\nabla_a Y = \nabla_{e_a} Y$. We write ∇Y to denote the 2-tensor whose contraction with e_a results in $\nabla_a Y$, i.e. $\nabla Y(e_a) = \nabla_a Y$.

In what follows, we fix a null pair $e_3 = \underline{L}$ and $e_4 = L$ and an orientation on the horizontal tensors.

[b]In the integrable case, ∇ coincides with the Levi-Civita connection of the metric induced on the integral surfaces of the horizontal distribution.

Definition 2.2. *Given a 2-covariant horizontal tensor-field U and an arbitrary orthonormal horizontal frame $(e_a)_{a=1,2}$ we define the trace of U as*

$$\mathrm{tr}(U) := \delta^{ab}U_{ab} = \delta^{ab}\,{}^{(s)}U_{ab},$$

where ${}^{(s)}U_{ab} = \frac{1}{2}(U_{ab} + U_{ba})$. We define the anti-trace of U by

$${}^{(a)}tr(U) := \in^{ab} U_{ab} = \in^{ab}\ {}^{(a)}U_{ab},$$

where ${}^{(a)}U_{ab} = \frac{1}{2}(U_{ab} - U_{ba})$.

A general horizontal, 2-tensor U can be decomposed according to

$$U_{ab} = {}^{(s)}U_{ab} + {}^{(a)}U_{ab} = \widehat{U}_{ab} + \frac{1}{2}\delta_{ab}\,\mathrm{tr}(U) + \frac{1}{2}\in_{ab}\ {}^{(a)}\mathrm{tr}(U).$$

Definition 2.3. *We denote by $\mathfrak{s}_0 = \mathfrak{s}_0(\mathcal{M})$ the set of pairs of real scalar functions on $\mathcal{M}$, $\mathfrak{s}_1 = \mathfrak{s}_1(\mathcal{M})$ the set of real horizontal 1-forms on $\mathcal{M}$ and by $\mathfrak{s}_2 = \mathfrak{s}_2(\mathcal{M})$ the set of real symmetric traceless horizontal 2-tensors on $\mathcal{M}$.*

We define the following operators on horizontal tensors.

Definition 2.4. *We define the duals of $f \in \mathfrak{s}_1$ and $u \in \mathfrak{s}_2$ by*

$${}^*f_a = \in_{ab} f_b, \quad ({}^*u)_{ab} = \in_{ac} u_{cb}.$$

Given $\xi, \eta \in \mathfrak{s}_1$ we denote

$$\begin{aligned}
\xi \cdot \eta &:= \delta^{ab}\xi_a\eta_b,\\
\xi \wedge \eta &:= \in^{ab} \xi_a\eta_b = \xi \cdot {}^*\eta,\\
(\xi\widehat{\otimes}\eta)_{ab} &:= \frac{1}{2}\big(\xi_a\eta_b + \xi_b\eta_a - \delta_{ab}\xi\cdot\eta\big).
\end{aligned}$$

Given $\xi \in \mathfrak{s}_1$, $u \in \mathfrak{s}_2$ we denote

$$(\xi \cdot u)_a := \delta^{bc}\xi_b u_{ac}.$$

Given $u, v \in \mathfrak{s}_2$ we denote

$$(u \wedge v)_{ab} := \in^{ab} u_{ac}v_{cb}.$$

For $f \in \mathfrak{s}_1$ and $u \in \mathfrak{s}_2$ we define the frame-dependent operators

$$\begin{aligned}
\mathrm{div} f &= \delta^{ab}\nabla_b f_a, \quad \mathrm{curl} f = \in^{ab} \nabla_a f_b,\\
(\nabla\widehat{\otimes} f)_{ba} &= \frac{1}{2}\big(\nabla_b f_a + \nabla_a f_b - \delta_{ab}(\mathrm{div} f)\big),\\
(\mathrm{div} u)_a &= \delta^{bc}\nabla_b u_{ca}.
\end{aligned}$$

2.2. *Ricci, electromagnetic and curvature components*

In what follows, we define Ricci coefficients, electromagnetic and curvature components of a general spacetime $(\mathcal{M}, \mathbf{g})$.

Definition 2.5. *We define the horizontal* 1*-forms*

$$
\begin{aligned}
\underline{\eta}(X) &:= \frac{1}{2}\mathbf{g}(X, \mathbf{D}_L \underline{L}), \quad \eta(X) := \frac{1}{2}\mathbf{g}(X, \mathbf{D}_{\underline{L}} L),\\
\underline{\xi}(X) &:= \frac{1}{2}\mathbf{g}(X, \mathbf{D}_{\underline{L}} \underline{L}), \quad \xi(X) := \frac{1}{2}\mathbf{g}(X, \mathbf{D}_L L),\\
\zeta(X) &= \frac{1}{2}\mathbf{g}(\mathbf{D}_X L, \underline{L}).
\end{aligned}
$$

and the scalars

$$
\underline{\omega} := \frac{1}{4}\mathbf{g}(\mathbf{D}_{\underline{L}} \underline{L}, L), \quad \omega := \frac{1}{4}\mathbf{g}(\mathbf{D}_L L, \underline{L}).
$$

Observe that the quantities underlined are obtained from the non-underlined ones by interchanging the null vectors $e_3 = \underline{L}$ and $e_4 = L$.

Definition 2.6. *The horizontal tensor-fields* $\chi, \underline{\chi}, \eta, \underline{\eta}, \zeta, \xi, \underline{\xi}, \omega, \underline{\omega}$ *are called the connection coefficients of the null pair* $(L, \underline{L})$*. Given an arbitrary basis of horizontal vectorfields* e_1, e_2*, we write using the short hand notation* $\mathbf{D}_a = \mathbf{D}_{e_a}, a = 1, 2,$

$$
\begin{aligned}
\underline{\chi}_{ab} &= \mathbf{g}(\mathbf{D}_a \underline{L}, e_b), \quad \chi_{ab} = \mathbf{g}(\mathbf{D}_a L, e_b),\\
\underline{\xi}_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} \underline{L}, e_a), \quad \xi_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_L L, e_a),\\
\underline{\omega} &= \frac{1}{4}\mathbf{g}(\mathbf{D}_{\underline{L}} \underline{L}, L), \quad \omega = \frac{1}{4}\mathbf{g}(\mathbf{D}_L L, \underline{L}),\\
\underline{\eta}_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_L \underline{L}, e_a), \quad \eta_a = \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}} L, e_a),\\
\zeta_a &= \frac{1}{2}\mathbf{g}(\mathbf{D}_a L, \underline{L}).
\end{aligned}
$$

We easily derive the Ricci formulae

$$
\begin{aligned}
\mathbf{D}_a e_b &= \nabla_a e_b + \frac{1}{2}\chi_{ab} e_3 + \frac{1}{2}\underline{\chi}_{ab} e_4,\\
\mathbf{D}_a e_4 &= \chi_{ab} e_b - \zeta_a e_4,\\
\mathbf{D}_a e_3 &= \underline{\chi}_{ab} e_b + \zeta_a e_3,\\
\mathbf{D}_3 e_a &= \nabla_3 e_a + \eta_a e_3 + \underline{\xi}_a e_4,\\
\mathbf{D}_3 e_3 &= -2\underline{\omega} e_3 + 2\underline{\xi}_b e_b,\\
\mathbf{D}_3 e_4 &= 2\underline{\omega} e_4 + 2\eta_b e_b,\\
\mathbf{D}_4 e_a &= \nabla_4 e_a + \underline{\eta}_a e_4 + \xi_a e_3,
\end{aligned}
$$

$$\begin{aligned}\mathbf{D}_4 e_4 &= -2\omega e_4 + 2\xi_b e_b,\\ \mathbf{D}_4 e_3 &= 2\omega e_3 + 2\underline{\eta}_b e_b.\end{aligned} \tag{2.3}$$

Definition 2.7. *We introduce the notation*

$$tr\,\chi := \mathrm{tr}(\chi), \quad {}^{(a)}tr\chi := {}^{(a)}tr(\chi), \quad \mathrm{tr}\,\underline{\chi} := \mathrm{tr}(\underline{\chi}), \quad {}^{(a)}tr\underline{\chi} := {}^{(a)}tr(\underline{\chi}).$$

The symmetric traceless part of χ and $\underline{\chi}$, denoted $\hat{\chi}$ and $\hat{\underline{\chi}}$, are called the (outgoing and ingoing, respectively) shear of the horizontal distribution, while the scalars $tr\,\chi$ and $\mathrm{tr}\,\underline{\chi}$ are the (outgoing and ingoing respectively) expansion of the distribution. The scalars ${}^{(a)}tr\chi$ and ${}^{(a)}tr\underline{\chi}$ measure the integrability defects of the distribution.

In particular we can write

$$\chi_{ab} = \hat{\chi}_{ab} + \frac{1}{2}\delta_{ab}\,tr\,\chi + \frac{1}{2}\in_{ab}\ {}^{(a)}tr\chi. \tag{2.4}$$

Let $\mathbf{F}$ be an antisymmetric 2-tensor on $(\mathcal{M}, \mathbf{g})$. We define the null components of $\mathbf{F}$ as the horizontal vectors $\beta(\mathbf{F}), \underline{\beta}(\mathbf{F})$ by the formulas

$$\begin{aligned}{}^{(\mathbf{F})}\beta(X) &= \beta(\mathbf{F})(X) = \mathbf{F}(X, L),\\ {}^{(\mathbf{F})}\underline{\beta}(X) &= \underline{\beta}(\mathbf{F})(X) = \mathbf{F}(X, \underline{L}),\\ \varrho(\mathbf{F})(X, Y) &= \mathbf{F}(X, Y).\end{aligned}$$

For a 2-form $\mathbf{F}$, the dual ${}^*\mathbf{F}$ denotes the Hodge dual on $(\mathcal{M}, \mathbf{g})$ of $\mathbf{F}$, defined by ${}^*\mathbf{F}_{\alpha\beta} = \frac{1}{2}\in_{\mu\nu\alpha\beta}\mathbf{F}^{\mu\nu}$.

It is convent to express in terms of the following two scalar quantities:

$${}^{(\mathbf{F})}\rho = \rho(\mathbf{F}) = \frac{1}{2}\mathbf{F}(\underline{L}, L), \qquad {}^{*\,(\mathbf{F})}\rho = {}^*\rho(\mathbf{F}) = \frac{1}{2}\,{}^*\mathbf{F}(\underline{L}, L). \tag{2.5}$$

Thus, $\varrho(\mathbf{F})(X, Y) = -\ {}^*\rho(\mathbf{F}) \in (X, Y)$ for horizontal vectors X, Y. i.e. $\mathbf{F}_{ab} = -\in_{ab}\ {}^*\rho$.

Let $\mathbf{W}$ be a Weyl field on $(\mathcal{M}, \mathbf{g})$. We define the null components of the Weyl field $\mathbf{W}$, horizontal 2-tensors $\alpha(\mathbf{W}), \underline{\alpha}(\mathbf{W}), \varrho(\mathbf{W})$ and horizontal 1-tensors $\beta(\mathbf{W}), \underline{\beta}(\mathbf{W})$ by the formulas

$$\begin{aligned}\alpha(\mathbf{W})(X, Y) &= \mathbf{W}(L, X, L, Y),\\ \underline{\alpha}(\mathbf{W})(X, Y) &= \mathbf{W}(\underline{L}, X, \underline{L}, Y),\\ \beta(\mathbf{W})(X) &= \frac{1}{2}\mathbf{W}(X, L, \underline{L}, L),\\ \underline{\beta}(\mathbf{W})(X) &= \frac{1}{2}\mathbf{W}(X, \underline{L}, \underline{L}, L),\\ \varrho(\mathbf{W})(X, Y) &= \mathbf{W}(X, \underline{L}, Y, L).\end{aligned}$$

Recall that if $\mathbf{W}$ is a Weyl field its Hodge dual ${}^*\mathbf{W}$, defined by ${}^*\mathbf{W}_{\alpha\beta\mu\nu} = \frac{1}{2}\in_{\mu\nu}{}^{\rho\sigma}\mathbf{W}_{\alpha\beta\rho\sigma}$, is also a Weyl field. It is convenient to express it in terms of the following two scalar quantities:

$$\rho(\mathbf{W}) = \frac{1}{4}\mathbf{W}(L, \underline{L}, L, \underline{L}), \qquad {}^*\rho(\mathbf{W}) = \frac{1}{4}\,{}^*\mathbf{W}(L, \underline{L}, L, \underline{L}). \tag{2.6}$$

Thus, $\varrho(X,Y) = -\rho\,\gamma(X,Y) + \ {}^*\rho \in (X,Y)$ for horizontal vectors X, Y. We have

$$\begin{aligned}
\mathbf{W}_{a3b4} &= \varrho_{ab} = (-\rho\delta_{ab} + \ {}^*\rho \in_{ab}),\\
\mathbf{W}_{ab34} &= 2 \in_{ab} \ {}^*\rho,\\
\mathbf{W}_{abcd} &= - \in_{ab}\in_{cd} \rho,\\
\mathbf{W}_{abc3} &= \in_{ab} \ {}^*\underline{\beta}_c,\\
\mathbf{W}_{abc4} &= - \in_{ab} \ {}^*\beta_c.
\end{aligned}$$

2.3. *Complex notations*

Recall Definition 2.3 of the set of horizontal tensors $\mathfrak{s}_k$ on $\mathcal{M}$. By Definition 2.4, the duals of real horizontal tensors are real horizontal tensors of the same type. We define the complexified version of horizontal tensors on $\mathcal{M}$.

Definition 2.8. *We denote by $\mathfrak{s}_k(\mathbb{C})$ the set of complex anti-self-dual k-tensors on $\mathcal{M}$. More precisely, $a + ib \in \mathfrak{s}_0(\mathbb{C})$ is a complex scalar function on $\mathcal{M}$ with $(a,b) \in \mathfrak{s}_0$, $F = f + i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$ is a complex anti-self-dual 1-tensor on $\mathcal{M}$ with $f \in \mathfrak{s}_1$ and $U = u + i\,{}^*u \in \mathfrak{s}_2(\mathbb{C})$ is a complex anti-self-dual symmetric traceless 2-tensor on $\mathcal{M}$ with $u \in \mathfrak{s}_2$.*

Observe that $F \in \mathfrak{s}_1(\mathbb{C})$ and $U \in \mathfrak{s}_2(\mathbb{C})$ are indeed anti-self-dual tensors, i.e.

$${}^*F = -iF, \qquad {}^*U = -iU.$$

In particular, $F_2 = -iF_1$ and $U_{12} = -iU_{11}$, $U_{22} = -U_{11}$, where we denote $F_1 := F(e_1)$ and $U_{11} := U(e_1, e_1)$ the contraction of the tensors with the horizontal frame.

We extend the definitions for the Ricci, electromagnetic and curvature components given in Sec. 2.2 to the complex case by using the anti-self-dual tensors.

Definition 2.9. *Let $(\mathcal{M}, \mathbf{g})$ be a Lorentzian 4-dimensional manifold. We define the following complexified versions of the Ricci components*:

$$\begin{aligned}
X &= \chi + i\,{}^*\chi, & \underline{X} &= \underline{\chi} + i\,{}^*\underline{\chi},\\
H &= \eta + i\,{}^*\eta, & \underline{H} &= \underline{\eta} + i\,{}^*\underline{\eta},\\
\Xi &= \xi + i\,{}^*\xi, & \underline{\Xi} &= \underline{\xi} + i\,{}^*\underline{\xi},\\
Z &= \zeta + i\,{}^*\zeta.
\end{aligned}$$

In particular, note that

$$\mathrm{tr}X = tr\,\chi - i\,{}^{(a)}tr\chi, \quad \widehat{X} = \hat{\chi} + i\,{}^*\hat{\chi}, \quad \mathrm{tr}\underline{X} = \mathrm{tr}\,\underline{\chi} - i\,{}^{(a)}tr\underline{\chi}, \quad \widehat{\underline{X}} = \underline{\hat{\chi}} + i\,{}^*\underline{\hat{\chi}}.$$

We define the following complexified versions of the electromagnetic components:

$$ {}^{(\mathbf{F})}B = {}^{(\mathbf{F})}\beta + i\,{}^*{}^{(\mathbf{F})}\beta, \quad {}^{(\mathbf{F})}\underline{B} = {}^{(\mathbf{F})}\underline{\beta} + i\,{}^*{}^{(\mathbf{F})}\underline{\beta} $$
$$ {}^{(\mathbf{F})}P = {}^{(\mathbf{F})}\rho + i\,{}^*{}^{(\mathbf{F})}\rho, $$

and of the curvature components

$$ A = \alpha + i\,{}^*\alpha, \quad \underline{A} = \underline{\alpha} + i\,{}^*\underline{\alpha}, $$
$$ B = \beta + i\,{}^*\beta, \quad \underline{B} = \underline{\beta} + i\,{}^*\underline{\beta}, $$
$$ P = \rho + i\,{}^*\rho. $$

With the above definition, the complex scalars, one-forms and symmetric traceless 2-tensors are, respectively, given by

$$ \mathrm{tr}X, \mathrm{tr}\underline{X}, P, {}^{(\mathbf{F})}P \in \mathfrak{s}_0(\mathbb{C}), $$
$$ H, \underline{H}, Z, \Xi, \underline{\Xi}, {}^{(\mathbf{F})}B, {}^{(\mathbf{F})}\underline{B}, B, \underline{B} \in \mathfrak{s}_1(\mathbb{C}), $$
$$ \widehat{X}, \widehat{\underline{X}}, A, \underline{A} \in \mathfrak{s}_2(\mathbb{C}). $$

Definition 2.10. *We define the complexified version of the ∇ horizontal derivative as*

$$ \mathcal{D} = \nabla + i\,{}^*\nabla, \quad \overline{\mathcal{D}} = \nabla - i\,{}^*\nabla. $$

More precisely,

- *For $a + ib \in \mathfrak{s}_0(\mathbb{C})$*

$$ \mathcal{D}(a+ib) := (\nabla + i\,{}^*\nabla)(a+ib), $$
$$ \overline{\mathcal{D}}(a+ib) := (\nabla - i\,{}^*\nabla)(a+ib). $$

- *For $F = f + i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$*

$$ \overline{\mathcal{D}} \cdot (f + i\,{}^*f) := (\nabla - i\,{}^*\nabla)\cdot(f + i\,{}^*f) = 2\big(\mathrm{div} f + i \mathrm{curl} f\big), $$
$$ \mathcal{D}\widehat{\otimes}(f + i\,{}^*f) := (\nabla + i\,{}^*\nabla)\widehat{\otimes}(f + i\,{}^*f) = 2\big(\nabla\widehat{\otimes} f + i\,{}^*(\nabla\widehat{\otimes} f)\big). $$

- *For $U = u + i\,{}^*u \in \mathfrak{s}_2(\mathbb{C})$,*

$$ \overline{\mathcal{D}}(u + i\,{}^*u) := (\nabla - i\,{}^*\nabla)(u + i\,{}^*u) = 2\big(\mathrm{div} u + i\,{}^*(\mathrm{div} u)\big). $$

For $F \in \mathfrak{s}_1(\mathbb{C})$, the operator $-\mathcal{D}\widehat{\otimes}$ is formally adjoint to the operator $\overline{\mathcal{D}} \cdot U$ applied to $U \in \mathfrak{s}_2(\mathbb{C})$, as shown in the following lemma.

Lemma 2.11. *For $F = f + i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$ and $U = u + i\,{}^*u \in \mathfrak{s}_2(\mathbb{C})$, we have*

$$(\mathcal{D}\widehat{\otimes}F)\cdot\overline{U} = -F\cdot(\mathcal{D}\cdot\overline{U}) - ((H+\underline{H})\widehat{\otimes}F)\cdot\overline{U} + \mathbf{D}_\alpha(F\cdot\overline{U}). \tag{2.7}$$

Proof. We have

$$\begin{aligned}(\nabla\widehat{\otimes}f)\cdot u &= \frac{1}{2}\big(\nabla_a f_b + \nabla_b f_a - \delta_{ab}\text{div}f\big)u_{ab} = (\nabla_a f_b)u_{ab}\\ &= \nabla_a(u_{ab}f_b) - (\text{div}u)\cdot f\end{aligned}$$

Let $\xi \in \mathfrak{s}_1$. Then the difference between the spacetime and the horizontal divergence is given by

$$\mathbf{D}_\alpha\xi^\alpha - \nabla_a\xi^a = -\frac{1}{2}\big(\mathbf{D}_3\xi_4 + \mathbf{D}_4\xi_3\big) = (\eta+\underline{\eta})\cdot\xi,$$

which applied to $\xi = u\cdot f$ gives

$$\begin{aligned}(\nabla\widehat{\otimes}f)\cdot u &= \mathbf{D}_\alpha(u_{\alpha b}f_b) - (\eta+\underline{\eta})\cdot(u\cdot f) - (\text{div}u)\cdot f\\ &= -(\text{div}u)\cdot f - ((\eta+\underline{\eta})\widehat{\otimes}f)\cdot u + \mathbf{D}_\alpha(u_{\alpha b}f_b).\end{aligned}$$

By complexifying the above, we obtain the stated identity. □

2.4. *Frame transformations and conformal invariance*

A general frame transformation of the null frame basis vectors $\{e_3, e_4, e_a\}$ into a transformed null frame $\{e_3', e_4', e_a'\}$ can be decomposed into the following three elementary types:

- *rotations of class* I, which leave the vector e_4 unchanged

$$e_4' = e_4, \quad e_3' = e_3 + \underline{\mu}_a e_a + \frac{1}{4}|\underline{\mu}|^2 e_4, \quad e_a' = e_a + \frac{1}{2}\underline{\mu}_a e_4 \tag{2.8}$$

- *rotations of class* II, which leave the vector e_3 unchanged

$$e_3' = e_3, \quad e_4' = e_4 + \mu_a e_a + \frac{1}{4}|\mu|^2 e_3 \quad e_a' = e_a + \frac{1}{2}\mu_a e_3, \tag{2.9}$$

- *rotations of class* III, which leave the directions of e_3 and e_4 unchanged and rotate e_a

$$e_3' = \lambda^{-1}e_3, \quad e_4' = \lambda e_4, \quad e_a' = O_{ab}e^b, \tag{2.10}$$

where μ and $\underline{\mu}$ are real one-forms, λ is a real function, O_{ab} is a orthogonal matrix, and the repeated indices indicate the sum on those.

Definition 2.12. *We say that a frame transformation is conformal if it is a rotation of class III with $O_{ab} = I_{ab}$ the identity matrix, i.e. such that*

$$e_3' = \lambda^{-1}e_3, \quad e_4' = \lambda e_4, \quad e_a' = e_a \tag{2.11}$$

for a real function λ.

Note that under a conformal frame transformation, the Ricci, electromagnetic, curvature components get modified in the following way:

$$\mathrm{tr}\,\underline{\chi}' = \lambda^{-1}\mathrm{tr}\,\underline{\chi}, \quad {}^{(a)}\mathrm{tr}\underline{\chi}' = \lambda^{-1}\,{}^{(a)}\mathrm{tr}\underline{\chi}, \quad tr\,\chi' = \lambda tr\,\chi, \quad {}^{(a)}\mathrm{tr}\chi' = \lambda\,{}^{(a)}\mathrm{tr}\chi$$

$$\xi' = \lambda^2\xi, \quad \eta' = \eta, \quad \underline{\eta}' = \underline{\eta}, \quad \underline{\xi}' = \lambda^{-2}\underline{\xi}$$

$${}^{(\mathbf{F})}\beta' = \lambda\,{}^{(\mathbf{F})}\beta, \quad {}^{(\mathbf{F})}\rho' = {}^{(\mathbf{F})}\rho, \quad {}^{*\,(\mathbf{F})}\rho' = {}^{*\,(\mathbf{F})}\rho, \quad {}^{(\mathbf{F})}\underline{\beta}' = \lambda^{-1}\,{}^{(\mathbf{F})}\underline{\beta}$$

$$\alpha' = \lambda^2\alpha, \quad \beta' = \lambda\beta, \quad \rho' = \rho, \quad {}^*\rho' = {}^*\rho, \quad \underline{\beta}' = \lambda^{-1}\underline{\beta}, \quad \underline{\alpha}' = \lambda^{-2}\underline{\alpha}$$

and similarly for their complex counterparts, and

$$\underline{\omega}' = \lambda^{-1}\left(\underline{\omega} + \frac{1}{2}e_3(\log\lambda)\right), \quad \omega' = \lambda\left(\omega - \frac{1}{2}e_4(\log\lambda)\right), \quad \zeta' = \zeta - \nabla(\log\lambda).$$

Definition 2.13. *We say that a horizontal tensor f is conformal invariant of type s if, under the conformal frame transformation (2.11), it changes as $f' = \lambda^s f$.*

Remark 2.14. Note that s is precisely what in [5] is called the signature of the tensor.[c]

Observe that if f is conformal invariant of type s, then $\nabla_3 f, \nabla_4 f, \nabla_a f$ are not conformal invariant. In GKS formalism, we correct the lacking of being conformal invariant by making the following definition.

Lemma 2.15. *If f is conformal invariant of type s, then*

(1) ${}^{(c)}\nabla_3 f := \nabla_3 f - 2s\underline{\omega} f$ *is conformal invariant of type* $(s-1)$.
(2) ${}^{(c)}\nabla_4 f := \nabla_4 f + 2s\omega f$ *is conformal invariant of type* $(s+1)$.
(3) ${}^{(c)}\nabla_a f := \nabla_a f + s\zeta_a f$ *is conformal invariant of type* s.

Also, ${}^{(c)}\mathcal{D}f := {}^{(c)}\nabla f + i\,{}^{*\,(c)}\nabla f = \mathcal{D}f + sZf$ *is conformal invariant of type* s.

2.5. *Comparison with the Newman–Penrose formalism*

The GKS formalism here recalled is strongly connected with the more familiar NP formalism. In NP formalism, one chooses a basis of null vectors $(n, l, m, \overline{m})$ with n and l real and m complex, scaled such that $\mathbf{g}(n,l) = -1$. They are related to the null frame (e_3, e_4, e_1, e_2) here presented for example by

$$n = \frac{1}{2}e_3, \quad l = e_4, \quad m = \frac{1}{\sqrt{2}}(e_1 + ie_2), \quad \overline{m} = \frac{1}{\sqrt{2}}(e_1 - ie_2).$$

In NP formalism, the connection coefficients, electromagnetic and curvature components are all complex scalar functions obtained by contracting the tensors with

[c] For a horizontal tensor f defined in terms of the null frames e_3 and e_4, its signature is the number of $e_4 = L$ used in its definition minus the number of $e_3 = \underline{L}$. For example, the signature of $\alpha = \mathbf{W}(L, e_a, L, e_b)$ is $2 - 0 = 2$, while the signature of $\eta = \frac{1}{2}\mathbf{g}(\mathbf{D}_{\underline{L}}L, e_a)$ is $1 - 1 = 0$.

the null frame. For example, the extreme null curvature component Ψ_0 is the spin-2 complex scalar defined as

$$\Psi_0 = -\mathbf{W}_{\mu\nu\gamma\sigma} l^\mu m^\nu l^\gamma m^\sigma$$
$$= -\mathbf{W}(l, m, l, m).$$

In GKS formalism, the extreme null curvature component is a complex horizontal 2-tensor, as in

$$A_{ab} = \mathbf{W}(e_4, e_a, e_4, e_b) + i \, {}^*\mathbf{W}(e_4, e_a, e_4, e_b)$$
$$= \alpha_{ab} + i \, {}^*\alpha_{ab}.$$

The relation between Ψ_0 and A_{ab} is the following: the projection of the horizontal 2-tensor A into its first component gives, up to a scalar, precisely the complex scalar Ψ_0, i.e.

$$A(e_1, e_1) = \alpha(e_1, e_1) + i \, {}^*\alpha(e_1, e_1) = -\Psi_0.$$

Such relation also explains why the complex scalar Ψ_0 is of spin-2, as it can be realized as a projection of a 2-tensor. Similarly, in NP formalism the extreme electromagnetic component ϕ_0 is the spin-1 complex scalar defined as

$$\phi_0 = -\mathbf{F}_{\mu\nu} l^\mu m^\nu = -\mathbf{F}(l, m).$$

In GKS formalism, the extreme electromagnetic component is a complex horizontal 1-tensor, as in

$${}^{(\mathbf{F})}B_a = \mathbf{F}(L, e_a) + i \, {}^*\mathbf{F}(L, e_a) = {}^{(\mathbf{F})}\beta_a + i \, {}^* \, {}^{(\mathbf{F})}\beta_a$$

and the projection of the horizontal 1-tensor ${}^{(\mathbf{F})}B$ into its first component gives, up to a scalar, precisely the complex scalar ϕ_0, i.e.

$${}^{(\mathbf{F})}B(e_1) = {}^{(\mathbf{F})}\beta(e_1) + i \, {}^* \, {}^{(\mathbf{F})}\beta(e_1) = -\phi_0.$$

The information about the spin of the complex scalars in NP formalism is encoded in the tensors in GKS formalism: a spin 2-scalar is substituted by a horizontal 2-tensor, and a spin 1-scalar by a horizontal 1-tensor.

For future reference, we collect here a table of conversion from NP and GKS formalism, where it is understood that the correspondence between the curvature, electromagnetic and Ricci components holds up to projection on the first component.

NP formalism	GKS formalism
D, Δ	∇_4, ∇_3
$\delta, \overline{\delta}$	$\mathcal{D}\widehat{\otimes}, \overline{\mathcal{D}}\cdot$
Ψ_0, Ψ_4	$A, \underline{A}$
Ψ_1, Ψ_3	$B, \underline{B}$
Ψ_2	P
$\phi_0,\ \phi_2$	${}^{(\mathbf{F})}B,\ {}^{(\mathbf{F})}\underline{B}$
ϕ_1	${}^{(\mathbf{F})}P$
σ, λ	$\widehat{X}, \underline{\widehat{X}}$
τ, π	$H,\ \underline{H}$
κ, ν	$\Xi, \underline{\Xi}$
α, β	Z
ρ, μ	$\mathrm{tr}X, \mathrm{tr}\underline{X}$
ϵ, γ	$\omega, \underline{\omega}$

The conformal derivatives in GKS formalism are the equivalent of the spin and boost weight operators defined in GHP formalism. Just as in GHP formalism, the derivatives absorb in their definitions the Ricci coefficients ϵ, γ, α and β, similarly our ${}^{(c)}\nabla_3$, ${}^{(c)}\nabla_4$ and ${}^{(c)}\nabla_a$ absorb ω, $\underline{\omega}$ and Z.

3. The Einstein–Maxwell Equations in Full Generality

The Einstein–Maxwell equations are given by

$$\mathbf{R}_{\mu\nu} = 2\mathbf{F}_{\mu\lambda}\mathbf{F}_{\nu}{}^{\lambda} - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathbf{F}^{\alpha\beta}\mathbf{F}_{\alpha\beta}, \tag{3.1}$$

$$\mathbf{D}_{[\mu}\mathbf{F}_{\nu\lambda]} = 0, \quad \mathbf{D}^{\mu}\mathbf{F}_{\mu\nu} = 0. \tag{3.2}$$

where $\mathbf{R}_{\mu\nu}$ denotes the Ricci curvature of $(\mathcal{M}, \mathbf{g})$ and $\mathbf{F}_{\mu\nu}$ is an antisymmetric 2-tensor. In this section, we derive the null Einstein–Maxwell equations in full generality, for a spacetime with a non-integrable null frame, therefore paying particular attention to the symmetric and antisymmetric part of χ and $\underline{\chi}$.

3.1. *The Maxwell equations*

The equation $\mathbf{D}_{[\mu}\mathbf{F}_{\nu\lambda]} = 0$ in (3.2) gives three independent equations. The first one is obtained in the following way, using (2.3):

$$\begin{aligned} 0 &= \mathbf{D}_a\mathbf{F}_{34} + \mathbf{D}_3\mathbf{F}_{4a} + \mathbf{D}_4\mathbf{F}_{a3} \\ &= \nabla_a(\mathbf{F}_{34}) - \mathbf{F}(\underline{\chi}_{ab}e_b + \zeta_a e_3, e_4) - \mathbf{F}(e_3, -\zeta_a e_4 + \chi_{ab}e_b) \\ &\quad + \nabla_3(\mathbf{F}_{4a}) - \mathbf{F}(2\underline{\omega}e_4 + 2\eta_b e_b, e_a) - \mathbf{F}(e_4, \eta_a e_3 + \underline{\xi}_a e_4) \end{aligned}$$

$$+\nabla_4(\mathbf{F}_{a3}) - \mathbf{F}(\underline{\eta}_a e_4 + \xi_a e_3, e_3) - \mathbf{F}(e_a, 2\omega e_3 + 2\underline{\eta}_b e_b)$$
$$= 2\nabla_a({}^{(\mathbf{F})}\rho) - \underline{\chi}_{ab}\,{}^{(\mathbf{F})}\beta_b + \chi_{ab}\,{}^{(\mathbf{F})}\underline{\beta}_b - \nabla_3\,{}^{(\mathbf{F})}\beta_a + 2\underline{\omega}\,{}^{(\mathbf{F})}\beta_a$$
$$-2\in_{ab}\eta_b\,{}^*\,{}^{(\mathbf{F})}\rho + 2\eta_a\,{}^{(\mathbf{F})}\rho + \nabla_4\,{}^{(\mathbf{F})}\underline{\beta}_a + 2\underline{\eta}_a\,{}^{(\mathbf{F})}\rho - 2\omega\,{}^{(\mathbf{F})}\underline{\beta}_a + 2\in_{ab}\underline{\eta}_b\,{}^*\,{}^{(\mathbf{F})}\rho.$$

By writing

$$\chi_{ab}\,{}^{(\mathbf{F})}\underline{\beta}_b = \left(\hat{\chi}_{ab} + \frac{1}{2}\delta_{ab}\,\mathrm{tr}\,\chi + \frac{1}{2}\in_{ab}\,{}^{(a)}\mathrm{tr}\chi\right){}^{(\mathbf{F})}\underline{\beta}_b$$
$$= \hat{\chi}_{ab}\,{}^{(\mathbf{F})}\underline{\beta}_b + \frac{1}{2}\mathrm{tr}\,\chi\,{}^{(\mathbf{F})}\underline{\beta}_a + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^*\,{}^{(\mathbf{F})}\underline{\beta}_a$$

and using the definition of Hodge duals, we obtain

$$\nabla_3\,{}^{(\mathbf{F})}\beta_a - \nabla_4\,{}^{(\mathbf{F})}\underline{\beta}_a = -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\beta_a + 2\underline{\omega}\,{}^{(\mathbf{F})}\beta_a - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\,{}^{(\mathbf{F})}\beta_a$$
$$+\frac{1}{2}\mathrm{tr}\,\chi\,{}^{(\mathbf{F})}\underline{\beta}_a - 2\omega\,{}^{(\mathbf{F})}\underline{\beta}_a + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^*\,{}^{(\mathbf{F})}\underline{\beta}_a$$
$$+2\nabla_a({}^{(\mathbf{F})}\rho) + 2\left(\eta_a + \underline{\eta}_a\right){}^{(\mathbf{F})}\rho + 2\left({}^*\underline{\eta}_a - {}^*\eta_a\right){}^*\,{}^{(\mathbf{F})}\rho$$
$$-\hat{\underline{\chi}}_{ab}\,{}^{(\mathbf{F})}\beta_b + \hat{\chi}_{ab}\,{}^{(\mathbf{F})}\underline{\beta}_b. \tag{3.3}$$

The second equation is obtained in the following way:

$$0 = \mathbf{D}_a\mathbf{F}_{b3} + \mathbf{D}_b\mathbf{F}_{3a} + \mathbf{D}_3\mathbf{F}_{ab}$$
$$= \nabla_a(\mathbf{F}_{b3}) - \mathbf{F}(\frac{1}{2}\chi_{ab}e_3 + \frac{1}{2}\underline{\chi}_{ab}e_4, e_3) - \mathbf{F}(e_b, \underline{\chi}_{ac}e_c + \zeta_a e_3)$$
$$+\nabla_b(\mathbf{F}_{3a}) - \mathbf{F}(\underline{\chi}_{bc}e_c + \zeta_b e_3, e_a) - \mathbf{F}(e_3, \frac{1}{2}\chi_{ba}e_3 + \frac{1}{2}\underline{\chi}_{ba}e_4)$$
$$+\nabla_3(\mathbf{F}_{ab}) - \mathbf{F}(\eta_a e_3 + \underline{\xi}_a e_4, e_b) - \mathbf{F}(e_a, \eta_b e_3 + \underline{\xi}_b e_4)$$
$$= \nabla_a\,{}^{(\mathbf{F})}\underline{\beta}_b - \nabla_b\,{}^{(\mathbf{F})}\underline{\beta}_a + (\underline{\chi}_{ab} - \underline{\chi}_{ba})\,{}^{(\mathbf{F})}\rho + \underline{\chi}_{ac}\in_{bc}\,{}^*\,{}^{(\mathbf{F})}\rho - \zeta_a\,{}^{(\mathbf{F})}\underline{\beta}_b$$
$$+\underline{\chi}_{bc}\in_{ca}\,{}^*\,{}^{(\mathbf{F})}\rho + \zeta_b\,{}^{(\mathbf{F})}\underline{\beta}_a - \in_{ab}\nabla_3\,{}^*\,{}^{(\mathbf{F})}\rho + \eta_a\,{}^{(\mathbf{F})}\underline{\beta}_b + \underline{\xi}_a\,{}^{(\mathbf{F})}\underline{\beta}_b$$
$$-\eta_b\,{}^{(\mathbf{F})}\underline{\beta}_a - \underline{\xi}_b\,{}^{(\mathbf{F})}\beta_a.$$

Contracting the above with $\in^{ab}$ and recalling that $\mathrm{curl}\,{}^{(\mathbf{F})}\underline{\beta} = \in^{ab}\nabla_a\,{}^{(\mathbf{F})}\underline{\beta}_b$, we have

$$\nabla_3\,{}^*\,{}^{(\mathbf{F})}\rho - \mathrm{curl}\,{}^{(\mathbf{F})}\underline{\beta} = -\left(\mathrm{tr}\,\underline{\chi}\,{}^*\,{}^{(\mathbf{F})}\rho - {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^{(\mathbf{F})}\rho\right) + \left(\eta - \zeta\right)\cdot{}^*\,{}^{(\mathbf{F})}\underline{\beta} + \underline{\xi}\cdot{}^*\,{}^{(\mathbf{F})}\beta.$$

The third equation is obtained from symmetrization of the above:

$$\nabla_4\,{}^*\,{}^{(\mathbf{F})}\rho - \mathrm{curl}\,{}^{(\mathbf{F})}\beta = -\left(\mathrm{tr}\,\chi\,{}^*\,{}^{(\mathbf{F})}\rho + {}^{(a)}\mathrm{tr}\chi\,{}^{(\mathbf{F})}\rho\right) + \left(\underline{\eta} + \zeta\right)\cdot{}^*\,{}^{(\mathbf{F})}\beta + \xi\cdot{}^*\,{}^{(\mathbf{F})}\underline{\beta}.$$

The equation $\mathbf{D}^\mu \mathbf{F}_{\mu\nu} = \delta_{bc}\mathbf{D}_b\mathbf{F}_{c\nu} - \frac{1}{2}\mathbf{D}_4\mathbf{F}_{3\nu} - \frac{1}{2}\mathbf{D}_3\mathbf{F}_{4\nu} = 0$ gives three additional independent equations. The first one is obtained in the following way:

$$
\begin{aligned}
0 &= \delta_{bc}\mathbf{D}_b\mathbf{F}_{ca} - \frac{1}{2}\mathbf{D}_4\mathbf{F}_{3a} - \frac{1}{2}\mathbf{D}_3\mathbf{F}_{4a} \\
&= \delta_{bc}\left(\in_{ac}\nabla_b\, {}^*{}^{(\mathbf{F})}\rho - \mathbf{F}\left(\frac{1}{2}\underline{\chi}_{bc}e_4 + \frac{1}{2}\chi_{bc}e_3, e_a\right) - \mathbf{F}\left(e_c, \frac{1}{2}\underline{\chi}_{ba}e_4 + \frac{1}{2}\chi_{ba}e_3\right)\right) \\
&\quad + \frac{1}{2}\nabla_4\, {}^{(\mathbf{F})}\underline{\beta}_a + \frac{1}{2}\mathbf{F}(2\omega e_3 + 2\underline{\eta}_c e_c, e_a) + \frac{1}{2}\mathbf{F}(e_3, \underline{\eta}_a e_4 + \xi_a e_3) \\
&\quad + \frac{1}{2}\nabla_3\, {}^{(\mathbf{F})}\beta_a + \frac{1}{2}\mathbf{F}(2\underline{\omega} e_4 + 2\eta_c e_c, e_a) + \frac{1}{2}\mathbf{F}(e_4, \eta_a e_3 + \underline{\xi}_a e_4) \\
&= \in_{ac}\nabla_c\, {}^*{}^{(\mathbf{F})}\rho + \frac{1}{2}\mathrm{tr}\,\underline{\chi}\, {}^{(\mathbf{F})}\beta_a + \frac{1}{2}\mathrm{tr}\,\chi\, {}^{(\mathbf{F})}\underline{\beta}_a - \frac{1}{2}\underline{\chi}_{ca}\, {}^{(\mathbf{F})}\beta_c - \frac{1}{2}\chi_{ca}\, {}^{(\mathbf{F})}\underline{\beta}_c \\
&\quad + \frac{1}{2}\nabla_4\, {}^{(\mathbf{F})}\underline{\beta}_a - \omega\, {}^{(\mathbf{F})}\underline{\beta}_a - \underline{\eta}_c\, {}^*{}^{(\mathbf{F})}\rho \in_{ca} + \underline{\eta}_a\, {}^{(\mathbf{F})}\rho + \frac{1}{2}\nabla_3\, {}^{(\mathbf{F})}\beta_a \\
&\quad - \underline{\omega}\, {}^{(\mathbf{F})}\beta_a - \eta_c\, {}^*{}^{(\mathbf{F})}\rho \in_{ca} - \eta_a\, {}^{(\mathbf{F})}\rho.
\end{aligned}
$$

By writing

$$
\begin{aligned}
\chi_{ca}\, {}^{(\mathbf{F})}\underline{\beta}_c &= \left(\hat{\chi}_{ca} + \frac{1}{2}\delta_{ca}\,\mathrm{tr}\,\chi + \frac{1}{2}\in_{ca}\, {}^{(a)}\mathrm{tr}\chi\right) {}^{(\mathbf{F})}\underline{\beta}_c \\
&= \hat{\chi}_{ca}\, {}^{(\mathbf{F})}\underline{\beta}_c + \frac{1}{2}\mathrm{tr}\,\chi\, {}^{(\mathbf{F})}\underline{\beta}_a - \frac{1}{2}\, {}^{(a)}\mathrm{tr}\chi\, {}^*{}^{(\mathbf{F})}\underline{\beta}_a
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\nabla_4\, {}^{(\mathbf{F})}\underline{\beta}_a + \nabla_3\, {}^{(\mathbf{F})}\beta_a &= -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\, {}^{(\mathbf{F})}\beta_a - \frac{1}{2}\, {}^{(a)}\mathrm{tr}\underline{\chi}\, {}^*{}^{(\mathbf{F})}\beta_a + 2\underline{\omega}\, {}^{(\mathbf{F})}\beta_a \\
&\quad - \frac{1}{2}\mathrm{tr}\,\chi\, {}^{(\mathbf{F})}\underline{\beta}_a - \frac{1}{2}\, {}^{(a)}\mathrm{tr}\chi\, {}^*{}^{(\mathbf{F})}\underline{\beta}_a + 2\omega\, {}^{(\mathbf{F})}\underline{\beta}_a \\
&\quad - 2\in_{ac}\nabla_c\, {}^*{}^{(\mathbf{F})}\rho - 2\big({}^*\eta_a + {}^*\underline{\eta}_a\big)\, {}^*{}^{(\mathbf{F})}\rho + 2\big(\eta_a - \underline{\eta}_a\big)\, {}^{(\mathbf{F})}\rho \\
&\quad + \hat{\underline{\chi}}_{ca}\, {}^{(\mathbf{F})}\beta_c + \hat{\chi}_{ca}\, {}^{(\mathbf{F})}\underline{\beta}_c.
\end{aligned} \tag{3.4}
$$

Summing and subtracting (3.3) and (3.4), we obtain

$$
\begin{aligned}
\nabla_3\, {}^{(\mathbf{F})}\beta - \nabla({}^{(\mathbf{F})}\rho) + {}^*\nabla\, {}^*{}^{(\mathbf{F})}\rho &= -\frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}\, {}^{(\mathbf{F})}\beta + {}^{(a)}\mathrm{tr}\underline{\chi}\, {}^*{}^{(\mathbf{F})}\beta\big) + 2\underline{\omega}\, {}^{(\mathbf{F})}\beta \\
&\quad + 2\big(\eta\, {}^{(\mathbf{F})}\rho - {}^*\eta\, {}^*{}^{(\mathbf{F})}\rho\big) + \hat{\chi}\cdot {}^{(\mathbf{F})}\underline{\beta}
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_4\, {}^{(\mathbf{F})}\underline{\beta} + \nabla({}^{(\mathbf{F})}\rho) + {}^*\nabla\, {}^*{}^{(\mathbf{F})}\rho &= -\frac{1}{2}\big(\mathrm{tr}\,\chi\, {}^{(\mathbf{F})}\underline{\beta} + {}^{(a)}\mathrm{tr}\chi\, {}^*{}^{(\mathbf{F})}\underline{\beta}\big) + 2\omega\, {}^{(\mathbf{F})}\underline{\beta} \\
&\quad - 2\big(\underline{\eta}\, {}^{(\mathbf{F})}\rho + {}^*\underline{\eta}\, {}^*{}^{(\mathbf{F})}\rho\big) + \hat{\underline{\chi}}\cdot {}^{(\mathbf{F})}\beta.
\end{aligned}
$$

The last equation is obtained by

$$
\begin{aligned}
0 &= \delta_{bc}\mathbf{D}_b\mathbf{F}_{c4} - \frac{1}{2}\mathbf{D}_4\mathbf{F}_{34}\\
&= \delta_{bc}\left(\nabla_b\, {}^{(\mathbf{F})}\beta_c - \mathbf{F}\left(\frac{1}{2}\underline{\chi}_{bc}e_4 + \frac{1}{2}\chi_{bc}e_3, e_4\right) - \mathbf{F}(e_c, -\zeta_b e_4 + \chi_{ba}e_a)\right)\\
&\quad -\frac{1}{2}\left(2\nabla_4\, {}^{(\mathbf{F})}\rho - \mathbf{F}(2\omega e_3 + 2\underline{\eta}_a e_a, e_4) - \mathbf{F}(e_3, -2\omega e_4 + 2\xi_a e_a)\right)\\
&= \mathrm{div}\, {}^{(\mathbf{F})}\beta - \mathrm{tr}\,\chi\, {}^{(\mathbf{F})}\rho + \zeta\cdot {}^{(\mathbf{F})}\beta + {}^{(a)}\mathrm{tr}\chi\, {}^{*}{}^{(\mathbf{F})}\rho - \nabla_4\, {}^{(\mathbf{F})}\rho + \underline{\eta}\cdot {}^{(\mathbf{F})}\beta - \xi\cdot {}^{(\mathbf{F})}\underline{\beta},
\end{aligned}
$$

which gives

$$
\begin{aligned}
\nabla_4\, {}^{(\mathbf{F})}\rho - \mathrm{div}\, {}^{(\mathbf{F})}\beta &= -\left(\mathrm{tr}\,\chi\, {}^{(\mathbf{F})}\rho - {}^{(a)}\mathrm{tr}\chi\, {}^{*}{}^{(\mathbf{F})}\rho\right) + (\zeta+\underline{\eta})\cdot {}^{(\mathbf{F})}\beta - \xi\cdot {}^{(\mathbf{F})}\underline{\beta},\\
\nabla_3\, {}^{(\mathbf{F})}\rho + \mathrm{div}\, {}^{(\mathbf{F})}\underline{\beta} &= -\left(\mathrm{tr}\,\underline{\chi}\, {}^{(\mathbf{F})}\rho + {}^{(a)}\mathrm{tr}\underline{\chi}\, {}^{*}{}^{(\mathbf{F})}\rho\right) + (\zeta-\eta)\cdot {}^{(\mathbf{F})}\underline{\beta} + \underline{\xi}\cdot {}^{(\mathbf{F})}\beta.
\end{aligned}
\tag{3.5}
$$

We summarize the Maxwell equations in the following proposition.

Proposition 3.1. *We have*

$$
\begin{aligned}
&\nabla_3\, {}^{(\mathbf{F})}\beta - \nabla({}^{(\mathbf{F})}\rho) + {}^{*}\nabla\, {}^{*}{}^{(\mathbf{F})}\rho\\
&\qquad = -\frac{1}{2}\left(\mathrm{tr}\,\underline{\chi}\, {}^{(\mathbf{F})}\beta + {}^{(a)}tr\underline{\chi}\, {}^{*}{}^{(\mathbf{F})}\beta\right) + 2\underline{\omega}\, {}^{(\mathbf{F})}\beta + 2\left(\eta\, {}^{(\mathbf{F})}\rho - {}^{*}\eta\, {}^{*}{}^{(\mathbf{F})}\rho\right) + \hat{\chi}\cdot {}^{(\mathbf{F})}\underline{\beta},\\
&\nabla_4\, {}^{(\mathbf{F})}\underline{\beta} + \nabla({}^{(\mathbf{F})}\rho) + {}^{*}\nabla\, {}^{*}{}^{(\mathbf{F})}\rho\\
&\qquad = -\frac{1}{2}\left(tr\,\chi\, {}^{(\mathbf{F})}\underline{\beta} + {}^{(a)}tr\chi\, {}^{*}{}^{(\mathbf{F})}\underline{\beta}\right) + 2\omega\, {}^{(\mathbf{F})}\underline{\beta} + 2\left(-\underline{\eta}\, {}^{(\mathbf{F})}\rho - {}^{*}\underline{\eta}\, {}^{*}{}^{(\mathbf{F})}\rho\right) + \hat{\underline{\chi}}\cdot {}^{(\mathbf{F})}\beta,\\
&\nabla_4\, {}^{(\mathbf{F})}\rho - \mathrm{div}\, {}^{(\mathbf{F})}\beta = -\left(tr\,\chi\, {}^{(\mathbf{F})}\rho - {}^{(a)}tr\chi\, {}^{*}{}^{(\mathbf{F})}\rho\right) + (\zeta+\underline{\eta})\cdot {}^{(\mathbf{F})}\beta - \xi\cdot {}^{(\mathbf{F})}\underline{\beta},\\
&\nabla_3\, {}^{(\mathbf{F})}\rho + \mathrm{div}\, {}^{(\mathbf{F})}\underline{\beta} = -\left(\mathrm{tr}\,\underline{\chi}\, {}^{(\mathbf{F})}\rho + {}^{(a)}tr\underline{\chi}\, {}^{*}{}^{(\mathbf{F})}\rho\right) + (\zeta-\eta)\cdot {}^{(\mathbf{F})}\underline{\beta} + \underline{\xi}\cdot {}^{(\mathbf{F})}\beta,\\
&\nabla_4\, {}^{*}{}^{(\mathbf{F})}\rho - \mathrm{curl}\, {}^{(\mathbf{F})}\beta = -\left(tr\,\chi\, {}^{*}{}^{(\mathbf{F})}\rho + {}^{(a)}tr\chi\, {}^{(\mathbf{F})}\rho\right) + (\underline{\eta}+\zeta)\cdot {}^{*}{}^{(\mathbf{F})}\beta + \xi\cdot {}^{*}{}^{(\mathbf{F})}\underline{\beta},\\
&\nabla_3\, {}^{*}{}^{(\mathbf{F})}\rho - \mathrm{curl}\, {}^{(\mathbf{F})}\underline{\beta} = -\left(\mathrm{tr}\,\underline{\chi}\, {}^{*}{}^{(\mathbf{F})}\rho - {}^{(a)}tr\underline{\chi}\, {}^{(\mathbf{F})}\rho\right) + (\eta-\zeta)\cdot {}^{*}{}^{(\mathbf{F})}\underline{\beta} + \underline{\xi}\cdot {}^{*}{}^{(\mathbf{F})}\beta.
\end{aligned}
$$

In complex notations and using conformal derivatives, we have

$$
\begin{aligned}
{}^{(c)}\nabla_3\, {}^{(\mathbf{F})}B - {}^{(c)}\mathcal{D}\, {}^{(\mathbf{F})}P &= -\frac{1}{2}\mathrm{tr}\underline{X}\, {}^{(\mathbf{F})}B + 2\, {}^{(\mathbf{F})}PH + \frac{1}{2}\widehat{X}\cdot \overline{{}^{(\mathbf{F})}\underline{B}},\\
{}^{(c)}\nabla_4\, {}^{(\mathbf{F})}\underline{B} + {}^{(c)}\mathcal{D}\,\overline{{}^{(\mathbf{F})}P} &= -\frac{1}{2}\mathrm{tr}X\, {}^{(\mathbf{F})}\underline{B} - 2\,\overline{{}^{(\mathbf{F})}P}\,\underline{H} + \frac{1}{2}\widehat{\underline{X}}\cdot \overline{{}^{(\mathbf{F})}B},\\
{}^{(c)}\nabla_4\, {}^{(\mathbf{F})}P - \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot {}^{(\mathbf{F})}B &= -\overline{\mathrm{tr}X}\, {}^{(\mathbf{F})}P + \frac{1}{2}\overline{\underline{H}}\cdot {}^{(\mathbf{F})}B - \frac{1}{2}\Xi\cdot \overline{{}^{(\mathbf{F})}\underline{B}},\\
{}^{(c)}\nabla_3\, {}^{(\mathbf{F})}P + \frac{1}{2}\, {}^{(c)}\mathcal{D}\cdot \overline{{}^{(\mathbf{F})}\underline{B}} &= -\mathrm{tr}\underline{X}\, {}^{(\mathbf{F})}P - \frac{1}{2}H\cdot \overline{{}^{(\mathbf{F})}\underline{B}} + \frac{1}{2}\underline{\Xi}\cdot {}^{(\mathbf{F})}B.
\end{aligned}
$$

Proof. We derive the equation for ${}^{(\mathbf{F})}B$. From the above equation for ${}^{(\mathbf{F})}\beta$ and its dual, we have

$$\begin{aligned}\nabla_3\,{}^{(\mathbf{F})}B &= \nabla_3\big({}^{(\mathbf{F})}\beta + i\,{}^*\,{}^{(\mathbf{F})}\beta\big)\\ &= \nabla\,{}^{(\mathbf{F})}\rho - {}^*\nabla\,{}^*\,{}^{(\mathbf{F})}\rho + i\big({}^*\nabla\,{}^{(\mathbf{F})}\rho + \nabla\,{}^*\,{}^{(\mathbf{F})}\rho\big)\\ &\quad -\frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\beta + {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\,{}^{(\mathbf{F})}\beta\big) - \frac{1}{2}i\big(\mathrm{tr}\,\underline{\chi}\,{}^*\,{}^{(\mathbf{F})}\beta - {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^{(\mathbf{F})}\beta\big)\\ &\quad + 2\underline{\omega}\big({}^{(\mathbf{F})}\beta + i\,{}^*\,{}^{(\mathbf{F})}\beta\big) + 2\big(\eta\,{}^{(\mathbf{F})}\rho - {}^*\eta\,{}^*\,{}^{(\mathbf{F})}\rho\big) + 2i\big({}^*\eta\,{}^{(\mathbf{F})}\rho + \eta\,{}^*\,{}^{(\mathbf{F})}\rho\big)\\ &\quad + \hat{\chi}\cdot{}^{(\mathbf{F})}\underline{\beta} + i\,{}^*(\hat{\chi}\cdot{}^{(\mathbf{F})}\underline{\beta}),\end{aligned}$$

which gives

$$\nabla_3\,{}^{(\mathbf{F})}B - \mathcal{D}\,{}^{(\mathbf{F})}P = -\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}B + 2\underline{\omega}\,{}^{(\mathbf{F})}B + 2\,{}^{(\mathbf{F})}PH + \frac{1}{2}\widehat{X}\cdot\overline{{}^{(\mathbf{F})}\underline{B}}.$$

From the equations for ${}^{(\mathbf{F})}\rho$ and ${}^*\,{}^{(\mathbf{F})}\rho$, we obtain

$$\begin{aligned}\nabla_4\,{}^{(\mathbf{F})}P &= \nabla_4\big({}^{(\mathbf{F})}\rho + i\,{}^*\,{}^{(\mathbf{F})}\rho\big)\\ &= \mathrm{div}\,{}^{(\mathbf{F})}\beta + i\mathrm{curl}\,{}^{(\mathbf{F})}\beta - \big(tr\,\chi\,{}^{(\mathbf{F})}\rho - {}^{(a)}\mathrm{tr}\chi\,{}^*\,{}^{(\mathbf{F})}\rho\big)\\ &\quad - i\big(\mathrm{tr}\,\chi\,{}^*\,{}^{(\mathbf{F})}\rho + {}^{(a)}\mathrm{tr}\chi\,{}^{(\mathbf{F})}\rho\big) + \big(\zeta + \underline{\eta}\big)\cdot{}^{(\mathbf{F})}\beta + i\big(\underline{\eta} + \zeta\big)\cdot{}^*\,{}^{(\mathbf{F})}\beta\\ &\quad - \xi\cdot{}^{(\mathbf{F})}\underline{\beta} + i\xi\cdot{}^*\,{}^{(\mathbf{F})}\underline{\beta},\end{aligned}$$

which gives

$$\nabla_4\,{}^{(\mathbf{F})}P - \frac{1}{2}\overline{\mathcal{D}}\cdot{}^{(\mathbf{F})}B = -\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}P + \frac{1}{2}\big(\overline{Z + \underline{H}}\big)\cdot{}^{(\mathbf{F})}B - \frac{1}{2}\Xi\cdot\overline{{}^{(\mathbf{F})}\underline{B}}$$

as desired. The other equations are obtained by symmetrization. Using the fact that ${}^{(\mathbf{F})}B$ is conformal invariant of type 1, ${}^{(\mathbf{F})}\underline{B}$ is conformal of type -1 and ${}^{(\mathbf{F})}P$ is conformal of type 0, we easily deduce the equations with conformal derivatives. □

3.2. *The Ricci identities*

We now compute the Ricci curvature $\mathbf{R}_{\mu\nu}$ of $(\mathcal{M},\mathbf{g})$ in terms of the decomposition in frames according to the Einstein equation (3.1):

$$\mathbf{R}_{a3} = 2\mathbf{F}_{a\lambda}\mathbf{F}_3{}^{\lambda} = 2\delta_{bc}\mathbf{F}_{ab}\mathbf{F}_{3c} - \mathbf{F}_{a3}\mathbf{F}_{34} = 2\,{}^*\,{}^{(\mathbf{F})}\rho\,{}^*\,{}^{(\mathbf{F})}\underline{\beta}_a - 2\,{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\underline{\beta}_a,$$

$$\mathbf{R}_{a4} = 2\,{}^*\,{}^{(\mathbf{F})}\rho\,{}^*\,{}^{(\mathbf{F})}\beta_a + 2\,{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta_a,$$

$$\mathbf{R}_{33} = 2g^{\lambda\mu}\mathbf{F}_{3\lambda}\mathbf{F}_{3\mu} = 2\delta_{ab}\mathbf{F}_{3a}\mathbf{F}_{3b} = 2\,{}^{(\mathbf{F})}\underline{\beta}\cdot{}^{(\mathbf{F})}\underline{\beta},$$

$$\mathbf{R}_{44} = 2\,{}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\beta,$$

$$\begin{aligned}\mathbf{R}_{34} &= (\mathbf{F}_{34})^2 + 2\,{}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\underline{\beta} + \big(-2\,{}^{(\mathbf{F})}\rho^2 + 2\,{}^*\,{}^{(\mathbf{F})}\rho^2 - 2\,{}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\underline{\beta}\big)\\ &= 2\,{}^{(\mathbf{F})}\rho^2 + 2\,{}^*\,{}^{(\mathbf{F})}\rho^2,\end{aligned}$$

$$\begin{aligned}\mathbf{R}_{ab} &= -\mathbf{F}_{a3}\mathbf{F}_{b4} - \mathbf{F}_{a4}\mathbf{F}_{b3} + 2\delta_{cd}\mathbf{F}_{ac}\mathbf{F}_{bd} - \frac{1}{2}\delta_{ab}\big(-2\,{}^{(\mathbf{F})}\rho^2 + 2\,{}^{*\,(\mathbf{F})}\rho^2 - 2\,{}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\underline{\beta}\big)\\ &= -\,{}^{(\mathbf{F})}\underline{\beta}_a\,{}^{(\mathbf{F})}\beta_b - {}^{(\mathbf{F})}\beta_a\,{}^{(\mathbf{F})}\underline{\beta}_b + 2\,{}^{*\,(\mathbf{F})}\rho^2\,\in_{ac}\in_{bc}\\ &\quad - \frac{1}{2}\delta_{ab}\big(-2\,{}^{(\mathbf{F})}\rho^2 + 2\,{}^{*\,(\mathbf{F})}\rho^2 - 2\,{}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\underline{\beta}\big)\\ &= -2\big({}^{(\mathbf{F})}\underline{\beta}\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{\beta}\big)_{ab} + \big({}^{(\mathbf{F})}\rho^2 + {}^{*\,(\mathbf{F})}\rho^2\big)\delta_{ab}.\end{aligned}$$

Using the decomposition of the Riemann curvature in Weyl curvature and Ricci tensor:

$$\mathbf{R}_{\alpha\beta\gamma\delta} = \mathbf{W}_{\alpha\beta\gamma\delta} + \frac{1}{2}(g_{\beta\delta}\mathbf{R}_{\alpha\gamma} + g_{\alpha\gamma}\mathbf{R}_{\beta\delta} - g_{\beta\gamma}\mathbf{R}_{\alpha\delta} - g_{\alpha\delta}\mathbf{R}_{\beta\gamma}), \tag{3.6}$$

we compute the components of the Riemann tensor:

$$\begin{aligned}\mathbf{R}_{a33b} &= \mathbf{W}_{a33b} - \frac{1}{2}\delta_{ab}\mathbf{R}_{33} = -\underline{\alpha}_{ab} - \big({}^{(\mathbf{F})}\underline{\beta}\cdot{}^{(\mathbf{F})}\underline{\beta}\big)\delta_{ab},\\ \mathbf{R}_{a34b} &= \mathbf{W}_{a34b} + \mathbf{R}_{ab} - \frac{1}{2}\delta_{ab}\mathbf{R}_{34} = \rho\delta_{ab} - {}^*\rho\in_{ab} - 2\big({}^{(\mathbf{F})}\beta\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{\beta}\big)_{ab},\\ \mathbf{R}_{a334} &= \mathbf{W}_{a334} - \mathbf{R}_{a3} = 2\underline{\beta}_a - 2\,{}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\underline{\beta}_a + 2\,{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\underline{\beta}_a,\\ \mathbf{R}_{3434} &= \mathbf{W}_{3434} + 2\mathbf{R}_{34} = 4\rho + 4\,{}^{(\mathbf{F})}\rho^2 + 4\,{}^{*\,(\mathbf{F})}\rho^2,\\ \mathbf{R}_{a3cb} &= \mathbf{W}_{a3cb} + \frac{1}{2}(\delta_{ac}\mathbf{R}_{3b} - \delta_{ab}\mathbf{R}_{3c})\\ &= \in_{cb}\,{}^*\underline{\beta}_a + \delta_{ac}\big({}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\underline{\beta}_b - {}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\underline{\beta}_b\big)\\ &\quad - \delta_{ab}\big({}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\underline{\beta}_c - {}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\underline{\beta}_c\big).\end{aligned}$$

The Ricci identities are obtained from the definition of Riemann curvature and are given by, see [19]:

$$\begin{aligned}\nabla_3\underline{\chi}_{ba} &= 2\nabla_b\underline{\xi}_a - 2\underline{\omega}\,\underline{\chi}_{ba} - \underline{\chi}_{bc}\underline{\chi}_{ca} + 2\big(-2\zeta_b\underline{\xi}_a + \eta_b\underline{\xi}_a + \underline{\eta}_a\underline{\xi}_b\big) + \mathbf{R}_{b33a},\\ \nabla_3\chi_{ba} &= 2\nabla_b\eta_a + 2\underline{\omega}\chi_{ba} - \underline{\chi}_{bc}\chi_{ca} + 2(\underline{\xi}_b\xi_a + \eta_a\,\eta_b) + \mathbf{R}_{a43b},\\ \nabla_4\underline{\chi}_{ba} &= 2\nabla_b\underline{\eta}_a + 2\omega\underline{\chi}_{ba} - \chi_{bc}\underline{\chi}_{ca} + 2(\xi_b\underline{\xi}_a + \underline{\eta}_a\,\underline{\eta}_b) + \mathbf{R}_{a34b},\\ \nabla_4\chi_{ba} &= 2\nabla_b\xi_a - 2\omega\,\chi_{ba} - \chi_{bc}\chi_{ca} + 2\big(2\zeta_b\xi_a + \underline{\eta}_b\xi_a + \eta_a\xi_b\big) + \mathbf{R}_{b44a}.\end{aligned}$$

$$\nabla_3\zeta_a + 2\nabla_a\underline{\omega} = -\underline{\chi}_{ab}(\zeta_b + \eta_b) + 2\underline{\omega}(\zeta-\eta)_a + \chi_{ab}\underline{\xi}_b + 2\omega\underline{\xi}_a - \frac{1}{2}\mathbf{R}_{a334}.$$

$$\nabla_4\zeta_a - 2\nabla_a\omega = \chi_{ab}(-\zeta_b + \underline{\eta}_b) + 2\omega(\zeta+\underline{\eta})_a - \underline{\chi}_{ab}\xi_b - 2\underline{\omega}\xi_a + \frac{1}{2}\mathbf{R}_{a443},$$

$$\nabla_3\underline{\eta}_a - \nabla_4\underline{\xi}_a = -\underline{\chi}_{ba}(\underline{\eta}-\eta)_b - 4\omega\underline{\xi}_a + \frac{1}{2}\mathbf{R}_{a334},$$

$$\nabla_4\eta_a - \nabla_3\xi_a = -\chi_{ba}(\eta-\underline{\eta})_b - 4\underline{\omega}\xi_a + \frac{1}{2}\mathbf{R}_{a443},$$

$$\nabla_3\omega + \nabla_4\underline{\omega} = 4\omega\underline{\omega} + \xi\cdot\underline{\xi} + (\eta-\underline{\eta})\cdot\zeta - \eta\cdot\underline{\eta} + \frac{1}{4}\mathbf{R}_{3434},$$

$$\nabla_a\chi_{bc} + \zeta_a\chi_{bc} = \nabla_b\chi_{ac} + \zeta_b\chi_{ac} + (\chi_{ab}-\chi_{ba})\eta_c + (\underline{\chi}_{ab}-\underline{\chi}_{ba})\xi_c + \mathbf{R}_{b4ac},$$

$$\nabla_a\underline{\chi}_{bc} - \zeta_a\underline{\chi}_{bc} = \nabla_b\underline{\chi}_{ac} - \zeta_b\underline{\chi}_{ac} + (\underline{\chi}_{ab}-\underline{\chi}_{ba})\underline{\eta}_c + (\chi_{ab}-\chi_{ba})\underline{\xi}_c + \mathbf{R}_{b3ac}.$$

We summarize the result of their complexification, with the above values of the Riemann curvature in the following.

Proposition 3.2. *In complex notations and using the conformal derivatives we have the following Ricci identities*:

$$^{(c)}\nabla_3\mathrm{tr}\underline{X} + \frac{1}{2}(\mathrm{tr}\underline{X})^2 = {}^{(c)}\mathcal{D}\cdot\overline{\underline{\Xi}} + \underline{\Xi}\cdot\overline{\underline{H}} + \overline{\underline{\Xi}}\cdot H - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{\underline{X}}} - {}^{(\mathbf{F})}\underline{B}\cdot\overline{{}^{(\mathbf{F})}\underline{B}},$$

$$^{(c)}\nabla_4\mathrm{tr}X + \frac{1}{2}(\mathrm{tr}X)^2 = {}^{(c)}\mathcal{D}\cdot\overline{\Xi} + \Xi\cdot\overline{H} + \overline{\Xi}\cdot H - \frac{1}{2}\widehat{X}\cdot\overline{\widehat{X}} - {}^{(\mathbf{F})}B\cdot\overline{{}^{(\mathbf{F})}B},$$

$$^{(c)}\nabla_3\mathrm{tr}X + \frac{1}{2}\mathrm{tr}\underline{X}\mathrm{tr}X = {}^{(c)}\mathcal{D}\cdot\overline{H} + H\cdot\overline{H} + 2P + \underline{\Xi}\cdot\overline{\Xi} - \frac{1}{2}\widehat{\underline{X}}\cdot\overline{\widehat{X}},$$

$$^{(c)}\nabla_4\mathrm{tr}\underline{X} + \frac{1}{2}\mathrm{tr}X\mathrm{tr}\underline{X} = {}^{(c)}\mathcal{D}\cdot\overline{\underline{H}} + \underline{H}\cdot\overline{\underline{H}} + 2\overline{P} + \Xi\cdot\overline{\underline{\Xi}} - \frac{1}{2}\widehat{X}\cdot\overline{\widehat{\underline{X}}},$$

$$^{(c)}\nabla_3\widehat{\underline{X}} + \Re(\mathrm{tr}\underline{X})\widehat{\underline{X}} = {}^{(c)}\mathcal{D}\widehat{\otimes}\underline{\Xi} + \underline{\Xi}\widehat{\otimes}(H+\underline{H}) - \underline{A},$$

$$^{(c)}\nabla_4\widehat{X} + \Re(\mathrm{tr}X)\widehat{X} = {}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \Xi\widehat{\otimes}(\underline{H}+H) - A,$$

$$^{(c)}\nabla_3\widehat{X} + \frac{1}{2}\mathrm{tr}\underline{X}\,\widehat{X} = {}^{(c)}\mathcal{D}\widehat{\otimes}H + H\widehat{\otimes}H - \frac{1}{2}\overline{\mathrm{tr}X}\widehat{\underline{X}} + \frac{1}{2}\underline{\Xi}\widehat{\otimes}\Xi - \frac{1}{2}\,{}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B},$$

$$^{(c)}\nabla_4\widehat{\underline{X}} + \frac{1}{2}\mathrm{tr}X\,\widehat{\underline{X}} = {}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H} + \underline{H}\widehat{\otimes}\,\underline{H} - \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\widehat{X} + \frac{1}{2}\Xi\widehat{\otimes}\underline{\Xi} - \frac{1}{2}\,{}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B},$$

$$^{(c)}\nabla_3\,\underline{H} - {}^{(c)}\nabla_4\underline{\Xi} = -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(\,\underline{H}-H) - \frac{1}{2}\widehat{\underline{X}}\cdot(\overline{\underline{H}}-\overline{H}) + \underline{B} + {}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B},$$

$$^{(c)}\nabla_4 H - {}^{(c)}\nabla_3\Xi = -\frac{1}{2}\overline{\mathrm{tr}X}(H-\underline{H}) - \frac{1}{2}\widehat{X}\cdot(\overline{H}-\overline{\underline{H}}) - B - \overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B,$$

$$\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{X} = \frac{1}{2}\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr}X} - i\Im(\mathrm{tr}X)H - i\Im(\mathrm{tr}\underline{X})\Xi - B + \overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B,$$

$$\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{\underline{X}} = \frac{1}{2}\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}} - i\Im(\mathrm{tr}\underline{X})\,\underline{H} - i\Im(\mathrm{tr}X)\underline{\Xi} + \underline{B} - {}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B}.$$

Also, for the non-conformal Z, ω *and* $\underline{\omega}$:

$$\nabla_3 Z + \frac{1}{2}\mathrm{tr}\underline{X}(Z+H) - 2\underline{\omega}(Z-H)$$
$$= -2\mathcal{D}\underline{\omega} - \frac{1}{2}\widehat{\underline{X}}\cdot(\overline{Z}+\overline{H}) + \frac{1}{2}\mathrm{tr}X\underline{\Xi} + 2\omega\underline{\Xi} - \underline{B} - {}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B} + \frac{1}{2}\overline{\underline{\Xi}}\cdot\widehat{X},$$

$$\nabla_4 Z + \frac{1}{2}\mathrm{tr}X(Z - \underline{H}) - 2\omega(Z + \underline{H})$$
$$= 2\mathcal{D}\omega + \frac{1}{2}\widehat{X}\cdot(-\overline{Z} + \overline{\underline{H}}) - \frac{1}{2}\mathrm{tr}\underline{X}\Xi - 2\underline{\omega}\Xi - B - \overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B - \frac{1}{2}\overline{\Xi}\cdot\widehat{\underline{X}},$$
$$\nabla_3\omega + \nabla_4\underline{\omega} - 4\omega\underline{\omega} - \xi\cdot\underline{\xi} - (\eta - \underline{\eta})\cdot\zeta + \eta\cdot\underline{\eta} = \rho + {}^{(\mathbf{F})}\rho^2 + {}^{*\,(\mathbf{F})}\rho^2.$$

Note that we are missing the traditional Gauss equation which, in the integrable case, connects the Gauss curvature of a sphere to a Riemann curvature component. We now state the non-integrable analogue of the Gauss equation, see [19].

Proposition 3.3. *The following identities hold true for $f \in \mathfrak{s}_1$ and $u \in \mathfrak{s}_2$:*

$$\big(\nabla_a\nabla_b - \nabla_b\nabla_a\big)f_c = \frac{1}{2}\in_{ab}\,\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)f_c - \frac{1}{2}E_{cdab}f^d + \mathbf{R}_{cdab}f^d$$
$$\big(\nabla_a\nabla_b - \nabla_b\nabla_a\big)u_{st} = \frac{1}{2}\in_{ab}\,\big({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\big)u_{st} - \frac{1}{2}E_{sdab}u_{dt} - \frac{1}{2}E_{tdab}u_{sd}$$
$$+\mathbf{R}_{sdab}u_{dt} + \mathbf{R}_{tdab}u_{sd},$$

where

$$E_{cdab} := \chi_{ac}\underline{\chi}_{bd} + \underline{\chi}_{ac}\chi_{bd} - \chi_{bc}\underline{\chi}_{ad} - \underline{\chi}_{bc}\chi_{ad}. \tag{3.7}$$

Proof. See [19, Proposition 2.34]. □

3.3. *The Bianchi identities*

The Bianchi identities for the Weyl curvature are given by

$$\mathbf{D}^\alpha\mathbf{W}_{\alpha\beta\gamma\delta} = \frac{1}{2}(\mathbf{D}_\gamma\mathbf{R}_{\beta\delta} - \mathbf{D}_\delta\mathbf{R}_{\beta\gamma}) =: J_{\beta\gamma\delta}. \tag{3.8}$$

The null Bianchi identities are given by, see [19],

$$\nabla_3\alpha - 2\nabla\widehat{\otimes}\beta$$
$$= -\frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}\alpha + {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*\alpha\big) + 4\underline{\omega}\alpha + 2(\zeta + 4\eta)\widehat{\otimes}\beta - 3(\rho\hat{\chi} + {}^*\rho\,{}^*\hat{\chi}) + \mathfrak{a},$$
$$\nabla_4\underline{\alpha} + 2\nabla\widehat{\otimes}\underline{\beta}$$
$$= -\frac{1}{2}\big(\mathrm{tr}\,\chi\underline{\alpha} - {}^{(a)}\mathrm{tr}\chi\,{}^*\underline{\alpha}\big) + 4\omega\underline{\alpha} + 2(\zeta - 4\underline{\eta})\widehat{\otimes}\underline{\beta} - 3(\rho\underline{\hat{\chi}} - {}^*\rho\,{}^*\underline{\hat{\chi}}) + \underline{\mathfrak{a}},$$

where

$$\mathfrak{a}_{ab} = J_{ba4} + J_{ab4} - \frac{1}{2}\delta_{ab}J_{434}, \quad \underline{\mathfrak{a}}_{ab} = J_{ba3} + J_{ab3} - \frac{1}{2}\delta_{ab}J_{343}.$$

We also have

$$\begin{aligned}
&\nabla_4\beta - \mathrm{div}\alpha \\
&\quad = -2(\mathrm{tr}\,\chi\beta - {}^{(a)}\mathrm{tr}\chi\ {}^*\beta) - 2\omega\beta + \alpha\cdot(2\zeta+\underline{\eta}) + 3(\xi\rho + {}^*\xi\ {}^*\rho) - J_{4a4},\\
&\nabla_3\underline{\beta} + \mathrm{div}\underline{\alpha} \\
&\quad = -2(\mathrm{tr}\,\underline{\chi}\,\underline{\beta} - {}^{(a)}\mathrm{tr}\underline{\chi}\ {}^*\underline{\beta}) - 2\underline{\omega}\underline{\beta} - \underline{\alpha}\cdot(-2\zeta+\eta) - 3(\underline{\xi}\rho - {}^*\underline{\xi}\ {}^*\rho) + J_{3a3},\\
&\nabla_3\beta + \mathrm{div}\varrho \\
&\quad = -(\mathrm{tr}\,\underline{\chi}\beta + {}^{(a)}\mathrm{tr}\underline{\chi}\ {}^*\beta) + 2\underline{\omega}\,\beta + 2\underline{\beta}\cdot\hat{\chi} + 3(\rho\eta + {}^*\rho\ {}^*\eta) + \alpha\cdot\underline{\xi} + J_{3a4},\\
&\nabla_4\underline{\beta} - \mathrm{div}\check{\varrho} \\
&\quad = -(\mathrm{tr}\,\chi\underline{\beta} + {}^{(a)}\mathrm{tr}\chi\ {}^*\underline{\beta}) + 2\omega\,\underline{\beta} + 2\beta\cdot\underline{\hat{\chi}} - 3(\rho\underline{\eta} - {}^*\rho\ {}^*\underline{\eta}) - \underline{\alpha}\cdot\xi - J_{4a3},
\end{aligned}$$

where

$$\mathrm{div}\varrho = -(\nabla\rho + {}^*\nabla\ {}^*\rho), \quad \mathrm{div}\check{\varrho} = -(\nabla\rho - {}^*\nabla\ {}^*\rho).$$

Finally,

$$\begin{aligned}
\nabla_4\rho - \mathrm{div}\beta &= -\frac{3}{2}(\mathrm{tr}\,\chi\rho + {}^{(a)}\mathrm{tr}\chi\ {}^*\rho) + (2\underline{\eta}+\zeta)\cdot\beta - 2\xi\cdot\underline{\beta} - \frac{1}{2}\underline{\hat{\chi}}\cdot\alpha - \frac{1}{2}J_{434},\\
\nabla_4\ {}^*\rho + \mathrm{curl}\beta &= -\frac{3}{2}(\mathrm{tr}\,\chi\ {}^*\rho - {}^{(a)}\mathrm{tr}\chi\rho) - (2\underline{\eta}+\zeta)\cdot\ {}^*\beta - 2\xi\cdot\ {}^*\underline{\beta}\\
&\quad + \frac{1}{2}\underline{\hat{\chi}}\cdot\ {}^*\alpha - \frac{1}{2}\ {}^*J_{434},\\
\nabla_3\rho + \mathrm{div}\underline{\beta} &= -\frac{3}{2}(\mathrm{tr}\,\underline{\chi}\rho - {}^{(a)}\mathrm{tr}\underline{\chi}\ {}^*\rho) - (2\eta-\zeta)\cdot\underline{\beta} + 2\underline{\xi}\cdot\beta - \frac{1}{2}\hat{\chi}\cdot\underline{\alpha} - \frac{1}{2}J_{343},\\
\nabla_3\ {}^*\rho + \mathrm{curl}\underline{\beta} &= -\frac{3}{2}(\mathrm{tr}\,\underline{\chi}\ {}^*\rho + {}^{(a)}\mathrm{tr}\underline{\chi}\rho) - (2\eta-\zeta)\cdot\ {}^*\underline{\beta} - 2\underline{\xi}\cdot\ {}^*\beta\\
&\quad - \frac{1}{2}\hat{\chi}\cdot\ {}^*\underline{\alpha} + \frac{1}{2}\ {}^*J_{343}.
\end{aligned}$$

We compute the Js terms through the Ricci curvature, and then use the Maxwell equations to simplify them. We summarize the final Bianchi identities in the following, and defer the proof to the appendix.

Proposition 3.4. *In complex notations and using the conformal derivatives we have the following Bianchi identities*:

$$\begin{aligned}
&{}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}A\\
&\quad = {}^{(c)}\mathcal{D}\widehat{\otimes}B + 4H\widehat{\otimes}B - 3\overline{P}\widehat{X} - 2\,\overline{{}^{(\mathbf{F})}P}\left(-\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B + {}^{(\mathbf{F})}P\widehat{X}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\nabla_4({}^{(\mathbf{F})}B\widehat{\otimes}{}^{(\mathbf{F})}\underline{B})+\frac{1}{2}{}^{(c)}\nabla_3({}^{(\mathbf{F})}B\widehat{\otimes}{}^{(\mathbf{F})}B)\\
&+\left(-\frac{1}{2}\mathrm{tr}X\,{}^{(\mathbf{F})}\underline{B}+\frac{1}{2}\widehat{\underline{X}}\cdot\overline{{}^{(\mathbf{F})}B}+\frac{1}{2}\widehat{X}\cdot{}^{(\mathbf{F})}\underline{B}+{}^{(\mathbf{F})}P\Xi\right)\widehat{\otimes}{}^{(\mathbf{F})}B,
\end{aligned}
$$

$$
\begin{aligned}
&{}^{(c)}\nabla_4\underline{A}+\frac{1}{2}\mathrm{tr}X\underline{A}\\
&=-{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B}-4\,\underline{H}\widehat{\otimes}\underline{B}-3P\widehat{\underline{X}}+2\,{}^{(\mathbf{F})}P\left(-\frac{1}{2}{}^{(c)}\mathcal{D}\widehat{\otimes}{}^{(\mathbf{F})}\underline{B}-\overline{{}^{(\mathbf{F})}P}\widehat{\underline{X}}\right)\\
&+\frac{1}{2}\nabla_3({}^{(\mathbf{F})}\underline{B}\widehat{\otimes}{}^{(\mathbf{F})}B)+\frac{1}{2}{}^{(c)}\nabla_4({}^{(\mathbf{F})}\underline{B}\widehat{\otimes}{}^{(\mathbf{F})}\underline{B})\\
&+\left(-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}B+\frac{1}{2}\widehat{X}\cdot\overline{{}^{(\mathbf{F})}\underline{B}}+\frac{1}{2}\widehat{\underline{X}}\cdot{}^{(\mathbf{F})}B-{}^{(\mathbf{F})}P\underline{\Xi}\right)\widehat{\otimes}{}^{(\mathbf{F})}\underline{B}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
&{}^{(c)}\nabla_4B-\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot A\\
&=-2\overline{\mathrm{tr}X}B+\frac{1}{2}A\cdot\overline{\underline{H}}+\left(3\overline{P}-2\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}P\right)\Xi+\overline{{}^{(\mathbf{F})}P}\,{}^{(c)}\nabla_4({}^{(\mathbf{F})}B)\\
&+\frac{1}{2}{}^{(c)}\mathcal{D}({}^{(\mathbf{F})}B\cdot\overline{{}^{(\mathbf{F})}B}),\\
&{}^{(c)}\nabla_3\underline{B}+\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\mathring{A}\\
&=-2\overline{\mathrm{tr}\underline{X}}\,\underline{B}-\frac{1}{2}\underline{A}\cdot\overline{H}-\left(3P-2\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}P\right)\underline{\Xi}+{}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3({}^{(\mathbf{F})}\underline{B})\\
&+\frac{1}{2}{}^{(c)}\mathcal{D}({}^{(\mathbf{F})}\underline{B}\cdot\overline{{}^{(\mathbf{F})}\underline{B}})
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{(c)}\nabla_3B-{}^{(c)}\mathcal{D}\overline{P}&=-\mathrm{tr}\underline{X}B+\overline{\underline{B}}\cdot\widehat{X}+3\overline{P}H+\frac{1}{2}A\cdot\overline{\underline{\Xi}}\\
&+\overline{{}^{(\mathbf{F})}P}\,{}^{(c)}\mathcal{D}({}^{(\mathbf{F})}P)-\frac{1}{2}\mathrm{tr}\underline{X}\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B-\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B}\\
&+\frac{1}{2}(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(\mathbf{F})}B)\,{}^{(\mathbf{F})}\underline{B}-{}^{(\mathbf{F})}P\widehat{X}\cdot{}^{(\mathbf{F})}\underline{B}-\frac{1}{2}\overline{{}^{(\mathbf{F})}P}\widehat{\underline{X}}\cdot{}^{(\mathbf{F})}B,\\
{}^{(c)}\nabla_4\underline{B}+{}^{(c)}\mathcal{D}P&=-\mathrm{tr}X\underline{B}+\overline{B}\cdot\widehat{\underline{X}}-3P\,\underline{H}-\frac{1}{2}\underline{A}\cdot\overline{\Xi}\\
&-{}^{(\mathbf{F})}P\,{}^{(c)}\mathcal{D}(\overline{{}^{(\mathbf{F})}P})-\frac{1}{2}\mathrm{tr}X\,{}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B}-\overline{\mathrm{tr}\underline{X}}\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B\\
&+\frac{1}{2}(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(\mathbf{F})}\underline{B})\,{}^{(\mathbf{F})}B+{}^{(\mathbf{F})}P\widehat{\underline{X}}\cdot{}^{(\mathbf{F})}B+\frac{1}{2}\overline{{}^{(\mathbf{F})}P}\widehat{X}\cdot{}^{(\mathbf{F})}\underline{B}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
{}^{(c)}\nabla_4 P - \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{B} &= -\frac{3}{2}\mathrm{tr}X P - \mathrm{tr}X\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} + \underline{H}\cdot\overline{B} - \Xi\cdot\underline{B} - \frac{1}{4}\underline{\widehat{X}}\cdot\overline{A}\\
&\quad + \frac{1}{2}\,{}^{(\mathbf{F})}P\,{}^{(c)}\mathcal{D}\cdot\overline{{}^{(\mathbf{F})}B} + \overline{H}\cdot\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B\\
&\quad + {}^{(c)}\nabla_3({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}B) + \left(-\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}B - \frac{1}{2}\widehat{X}\cdot{}^{(\mathbf{F})}\underline{B}\right)\widehat{\otimes}\,{}^{(\mathbf{F})}B,\\
{}^{(c)}\nabla_3 P + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{B} &= -\frac{3}{2}\overline{\mathrm{tr}\underline{X}}P - \overline{\mathrm{tr}\underline{X}}\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} - \overline{H}\cdot\underline{B} + \underline{\Xi}\cdot\overline{B} - \frac{1}{4}\overline{\widehat{X}}\cdot\underline{A}\\
&\quad - \frac{1}{2}\,{}^{(\mathbf{F})}P\,\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(\mathbf{F})}B - \underline{H}\cdot\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}\underline{B}\\
&\quad + {}^{(c)}\nabla_4({}^{(\mathbf{F})}\underline{B}\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B}) + \left(-\mathrm{tr}X\,{}^{(\mathbf{F})}\underline{B} - \frac{1}{2}\underline{\widehat{X}}\cdot{}^{(\mathbf{F})}B\right)\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B}.
\end{aligned}
$$

Proof. See Appendix A.1. □

4. The Kerr–Newman Spacetime and its Linear Perturbations

In this section, we introduce the Kerr–Newman spacetime and its representation within the formalism above introduced. For a more complete description of the Kerr–Newman spacetime see [20].

4.1. *The Kerr–Newman metric*

The Kerr–Newman black hole $\mathbf{g}_{M,a,Q}$ represents the most general explicit solution of a stationary, rotating (with spin a) and charged (with charge Q) black hole of mass M. We consider the Kerr–Newman metric in standard Boyer–Lindquist coordinates (t, r, θ, φ):

$$
\mathbf{g}_{M,a,Q} = -\frac{\Delta}{|q|^2}(dt - a\sin^2\theta d\varphi)^2 + \frac{|q|^2}{\Delta}dr^2 + |q|^2 d\theta^2 + \frac{\sin^2\theta}{|q|^2}(adt - (r^2+a^2)d\varphi)^2,
$$

where

$$
q = r + ia\cos\theta, \quad \overline{q} = r - ia\cos\theta \tag{4.1}
$$

and

$$
\begin{aligned}
\Delta &= r^2 - 2Mr + a^2 + Q^2,\\
|q|^2 &= r^2 + a^2(\cos\theta)^2.
\end{aligned}
$$

The metric $\mathbf{g}_{M,a,Q}$ is a solution to the Einstein–Maxwell equations (3.1) and (3.2), with electromagnetic tensor $\mathbf{F} = d\mathbf{A}$, and vector potential $\mathbf{A}$ given by

$$
\mathbf{A} = -\frac{Qr}{|q|^2}(dt - a\sin^2\theta d\phi).
$$

We note that ∂_t and ∂_φ are Killing vectorfields of the metric.

The Kerr–Newman metric is of Petrov Type D, i.e. its Weyl curvature can be diagonalized with two linearly independent eigenvectors, the so-called principal null directions. The principal null frame is given[d] by

$$e_4 = \frac{r^2+a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\varphi, \quad e_3 = \frac{r^2+a^2}{|q|^2}\partial_t - \frac{\Delta}{|q|^2}\partial_r + \frac{a}{|q|^2}\partial_\varphi,$$

$$e_1 = \frac{1}{\sqrt{|q|^2}}\partial_\theta, \quad e_2 = \frac{a\sin\theta}{\sqrt{|q|^2}}\partial_t + \frac{1}{\sqrt{|q|^2}\sin\theta}\partial_\varphi.$$

With respect to the principal null frame, we have

$$\hat{\chi} = \underline{\hat{\chi}} = \xi = \underline{\xi} = 0, \quad {}^{(\mathbf{F})}\beta = {}^{(\mathbf{F})}\underline{\beta} = 0, \quad \alpha = \beta = \underline{\beta} = \underline{\alpha} = 0$$

or their complexified versions,[e]

$$\widehat{X} = \underline{\widehat{X}} = \Xi = \underline{\Xi} = 0, \quad {}^{(\mathbf{F})}B = {}^{(\mathbf{F})}\underline{B} = 0, \quad A = B = \underline{B} = \underline{A} = 0. \tag{4.2}$$

With the above choice of principal null frame, the Ricci coefficients are given by

$$\operatorname{tr}\chi = \frac{2r}{|q|^2}, \quad {}^{(a)}\operatorname{tr}\chi = \frac{2a\cos\theta}{|q|^2}, \quad \operatorname{tr}\underline{\chi} = -\frac{2r\Delta}{|q|^4}, \quad {}^{(a)}\operatorname{tr}\underline{\chi} = \frac{2a\Delta\cos\theta}{|q|^4}$$

$$\underline{\omega} = \frac{a^2\cos^2\theta(r-M) + Mr^2 - a^2 r - Q^2 r}{|q|^4}, \quad \omega = 0, \quad \underline{\eta} = -\zeta.$$

Also, we have

$$\eta_1 = -\frac{a^2\sin\theta\cos\theta}{|q|^3}, \quad \eta_2 = \frac{a\sin\theta r}{|q|^3},$$

$${}^*\eta_1 = \frac{a\sin\theta r}{|q|^3}, \quad {}^*\eta_2 = \frac{a^2\sin\theta\cos\theta}{|q|^3},$$

$$\underline{\eta}_1 = -\frac{a^2\sin\theta\cos\theta}{|q|^3}, \quad \underline{\eta}_2 = -\frac{a\sin\theta r}{|q|^3},$$

$${}^*\underline{\eta}_1 = -\frac{a\sin\theta r}{|q|^3}, \quad {}^*\underline{\eta}_2 = \frac{a^2\sin\theta\cos\theta}{|q|^3}.$$

Their complexified values are given by

$$\operatorname{tr}X = \frac{2}{q}, \quad \operatorname{tr}\underline{X} = -\frac{2\Delta}{q\overline{q}^2}, \quad \underline{H} = -Z,$$

$$H_1 = \frac{ai\sin\theta\, q}{|q|^3}, \quad H_2 = \frac{a\sin\theta\, q}{|q|^3},$$

$$Z_1 = \frac{ai\sin\theta\,\overline{q}}{|q|^3}, \quad Z_2 = \frac{a\sin\theta\,\overline{q}}{|q|^3}.$$

[d]There is an indeterminacy in the principal null frame as one may replace the pair (e_3, e_4) with $(\lambda^{-1}e_3, \lambda e_4)$ for any $\lambda > 0$. The formulas in this section correspond to the choice of λ such that e_4 is geodesic.

[e]In NP formalism, this corresponds to the vanishing of $\sigma = \lambda = \kappa = \nu = 0$, $\Phi_0 = \Phi_1 = \Phi_3 = \Phi_4 = 0$ and $\phi_0 = \phi_2 = 0$.

The non-vanishing electromagnetic components are given by

$$ {}^{(\mathbf{F})}\rho = \frac{Q(r^2 - a^2\cos^2\theta)}{|q|^4}, \qquad {}^{*\,(\mathbf{F})}\rho = \frac{2aQr\cos\theta}{|q|^4} $$

with complexified value

$$ {}^{(\mathbf{F})}P = \frac{Q}{\overline{q}^2}. $$

The non-vanishing curvature components are given by

$$ \rho = \frac{1}{|q|^6}(-2Mr^3 + 2Q^2r^2 + 6Ma^2\cos^2\theta r - 2Q^2a^2\cos^2\theta), $$

$$ {}^*\rho = \frac{a\cos\theta}{|q|^6}(6Mr^2 - 4Q^2r - 2Ma^2\cos^2\theta), $$

with complexified value

$$ P = -\frac{2M}{q^3} + \frac{2Q^2}{q^3\overline{q}}. $$

4.2. *The Einstein–Maxwell equations in Kerr–Newman*

Using the vanishing of the Ricci, curvature and electromagnetic components given by (4.2), one can see that many of the Einstein–Maxwell equations obtained in Sec. 3 become trivial in Kerr–Newman. We denote those which are not trivially satisfied as reduced equations, and we collect them in the following proposition.

Proposition 4.1. *The reduced Maxwell equations in Kerr–Newman are*

$$ {}^{(c)}\nabla_4\,{}^{(\mathbf{F})}P = -\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}P, \qquad {}^{(c)}\nabla_3\,{}^{(\mathbf{F})}P = -\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}P, \tag{4.3} $$

$$ {}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}P = -2\,{}^{(\mathbf{F})}PH, \qquad {}^{(c)}\mathcal{D}\,\overline{{}^{(\mathbf{F})}P} = -2\,\overline{{}^{(\mathbf{F})}P}\,\underline{H}. \tag{4.4} $$

The reduced Ricci identities in Kerr–Newman are

$$ {}^{(c)}\nabla_3\mathrm{tr}\underline{X} + \frac{1}{2}(\mathrm{tr}\underline{X})^2 = 0, \qquad {}^{(c)}\nabla_4\mathrm{tr}X + \frac{1}{2}(\mathrm{tr}X)^2 = 0, \tag{4.5} $$

$$ {}^{(c)}\nabla_3\mathrm{tr}X + \frac{1}{2}\mathrm{tr}\underline{X}\mathrm{tr}X = {}^{(c)}\mathcal{D}\cdot\overline{H} + H\cdot\overline{H} + 2P, \tag{4.6} $$

$$ {}^{(c)}\nabla_4\mathrm{tr}\underline{X} + \frac{1}{2}\mathrm{tr}X\mathrm{tr}\underline{X} = {}^{(c)}\mathcal{D}\cdot\overline{\underline{H}} + \underline{H}\cdot\overline{\underline{H}} + 2\overline{P}, \tag{4.7} $$

$$ {}^{(c)}\mathcal{D}\widehat{\otimes}H + H\widehat{\otimes}H = 0, \qquad {}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H} + \underline{H}\widehat{\otimes}\,\underline{H} = 0, \tag{4.8} $$

$$ {}^{(c)}\nabla_3\,\underline{H} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}(\,\underline{H} - H) = 0, \qquad {}^{(c)}\nabla_4 H + \frac{1}{2}\overline{\mathrm{tr}X}(H - \,\underline{H}) = 0, \tag{4.9} $$

$$ {}^{(c)}\mathcal{D}\overline{\mathrm{tr}X} - (\mathrm{tr}X - \overline{\mathrm{tr}X})H = 0, \qquad {}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}} - (\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}})\,\underline{H} = 0. \tag{4.10} $$

The reduced Bianchi identities in Kerr–Newman are

$$ {}^{(c)}\mathcal{D}\overline{P} = -\big(3\overline{P} - 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)H, \qquad {}^{(c)}\mathcal{D}P = -\big(3P - 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\,\underline{H}, \tag{4.11} $$

$$ {}^{(c)}\nabla_4 P = -\frac{3}{2}\mathrm{tr}X P - \mathrm{tr}X\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}, \qquad {}^{(c)}\nabla_3 P = -\frac{3}{2}\overline{\mathrm{tr}\underline{X}}P - \overline{\mathrm{tr}\underline{X}}\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} \tag{4.12} $$

From the above we deduce (see also [19]) the following identities for $q = r + ia\cos\theta$:

$$ \nabla_3 q = \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\, q, \quad \nabla_4 q = \frac{1}{2}\mathrm{tr}X q \quad \overline{\mathcal{D}}q = q\overline{H}, \quad \mathcal{D}q = q\,\underline{H}. \tag{4.13} $$

4.3. *The wave operators in Kerr–Newman spacetime*

In what follows, we will need to express the equations governing electromagnetic-gravitational perturbations of Kerr–Newman in terms of wave operators applied to k-horizontal tensor fields. In this section, we collect useful formulas for those operators.

Consider the wave operator for $\Psi \in \mathfrak{s}_k(\mathbb{C})$ defined as

$$ \dot{\square}_k \Psi_{ab} := \mathbf{g}^{\mu\nu}\dot{\mathbf{D}}_\mu \dot{\mathbf{D}}_\nu \Psi_{ab}, \tag{4.14} $$

where $\dot{\mathbf{D}}$ is the horizontal covariant derivative as defined in (2.2).

Lemma 4.2. *The wave operator for $\Psi \in \mathfrak{s}_k(\mathbb{C})$ is given by*

$$ \begin{aligned} \dot{\square}_k \Psi = &-\frac{1}{2}\big(\nabla_3\nabla_4\Psi + \nabla_4\nabla_3\Psi\big) + \Delta_k\Psi + \left(\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4\Psi \\ &+ \left(\omega - \frac{1}{2}tr\,\chi\right)\nabla_3\Psi + (\eta + \underline{\eta})\cdot\nabla\Psi, \end{aligned} \tag{4.15} $$

where $\Delta_k = \delta^{ab}\nabla_a\nabla_b$ is the Laplacian operator for horizontal k-tensors. More precisely, for $F \in \mathfrak{s}_1(\mathbb{C})$ we have

$$ \begin{aligned} \dot{\square}_1 F = &-\nabla_3\nabla_4 F + \Delta_1 F + \left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4 F - \frac{1}{2}tr\,\chi\nabla_3 F + 2\eta\cdot\nabla F \\ &+ i\left(-\ {}^*\rho + \eta\wedge\underline{\eta}\right)F \\ = &-\nabla_3\nabla_4 F + \frac{1}{2}\overline{\mathcal{D}}\cdot(\mathcal{D}\widehat{\otimes}F) + \left(2\underline{\omega} - \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right)\nabla_4 F \\ &- \frac{1}{2}\overline{\mathrm{tr}X}\nabla_3 F + (H + \overline{H})\cdot\nabla F \\ &+ \left(\frac{1}{4}tr\,\chi\mathrm{tr}\,\underline{\chi} + \frac{1}{4}{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + \rho - {}^{(\mathbf{F})}\rho^2 - {}^{*\,(\mathbf{F})}\rho^2 + i\left(-\ {}^*\rho + \eta\wedge\underline{\eta}\right)\right)F \end{aligned} $$

and for $U \in \mathfrak{s}_2(\mathbb{C})$ we have

$$\dot{\square}_2 U = -\nabla_3\nabla_4 U + \Delta_2 U + \left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4 U - \frac{1}{2}tr\,\chi\nabla_3 U + 2\eta\cdot\nabla U$$
$$+ i\left(-2\ {}^*\rho + 2\eta\wedge\underline{\eta}\right)U$$
$$= -\nabla_3\nabla_4 U + \frac{1}{2}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot U) + \left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\underline{X}\right)\nabla_4 U$$
$$-\frac{1}{2}\mathrm{tr}X\nabla_3 U + (H+\overline{H})\cdot\nabla U + \left(-\frac{1}{2}tr\,\chi\mathrm{tr}\,\underline{\chi} - \frac{1}{2}\,{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} - 2\rho\right.$$
$$\left. + 2\,{}^{(\mathbf{F})}\rho^2 + 2\,{}^{*\,(\mathbf{F})}\rho^2 + i(-2\ {}^*\rho + 2\eta\wedge\underline{\eta})\right)U.$$

Proof. For the proof of (4.15) see [19, Lemma 5.4].

For $F \in \mathfrak{s}_1(\mathbb{C})$ and $U \in \mathfrak{s}_2(\mathbb{C})$, using the commutators, see [19, Lemma 5.2]:

$$[\nabla_3,\nabla_4]F = -2\omega\nabla_3 F + 2\underline{\omega}\nabla_4 F + 2(\eta-\underline{\eta})\cdot\nabla F + 2i\left(-\ {}^*\rho + \eta\wedge\underline{\eta}\right)F,$$
$$[\nabla_3,\nabla_4]U = -2\omega\nabla_3 U + 2\underline{\omega}\nabla_4 U + 2(\eta-\underline{\eta})\cdot\nabla U + 4i\left(-\ {}^*\rho + \eta\wedge\underline{\eta}\right)U,$$

we obtain

$$\dot{\square}_1 F = -\nabla_3\nabla_4 F + \Delta_1 F + \left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4 F - \frac{1}{2}\mathrm{t}r\,\chi\nabla_3 F$$
$$+ 2\eta\cdot\nabla F + i\left(-\ {}^*\rho + \eta\wedge\underline{\eta}\right)F,$$
$$\dot{\square}_2 U = -\nabla_3\nabla_4 U + \Delta_2 U + \left(2\underline{\omega} - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\right)\nabla_4 U - \frac{1}{2}\mathrm{t}r\,\chi\nabla_3 U$$
$$+ 2\eta\cdot\nabla U + i\left(-2\ {}^*\rho + 2\eta\wedge\underline{\eta}\right)U.$$

Using the following Gauss relations, see Lemma A.7:

$$\overline{\mathcal{D}}\cdot(\mathcal{D}\widehat{\otimes}F) = 2\Delta_1 F + i(\,{}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)F$$
$$-\left(\frac{1}{2}\mathrm{t}r\,\chi\mathrm{tr}\,\underline{\chi} + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + 2\rho - 2\,{}^{(\mathbf{F})}\rho^2 - 2\,{}^{*\,(\mathbf{F})}\rho^2\right)F,$$
$$\left(\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot U)\right) = 2\Delta_2 U - i(\,{}^{(a)}\mathrm{tr}\chi\nabla_3 + {}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4)U$$
$$+\left(\mathrm{t}r\,\chi\mathrm{tr}\,\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + 4\rho - 4\,{}^{(\mathbf{F})}\rho^2 - 4\,{}^{*\,(\mathbf{F})}\rho^2\right)U$$

we obtain

$$\dot{\square}_1 F = -\nabla_3\nabla_4 F + \frac{1}{2}\overline{\mathcal{D}}\cdot(\mathcal{D}\widehat{\otimes}F) + \left(2\underline{\omega} - \frac{1}{2}(\mathrm{tr}\,\underline{\chi} + i\,{}^{(a)}\mathrm{tr}\underline{\chi})\right)\nabla_4 F$$
$$-\frac{1}{2}(\mathrm{t}r\,\chi + i\,{}^{(a)}\mathrm{tr}\chi)\nabla_3 F + 2\eta\cdot\nabla F + \left(\frac{1}{4}\mathrm{t}r\,\chi\mathrm{tr}\,\underline{\chi} + \frac{1}{4}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}\right.$$
$$\left. + \rho - {}^{(\mathbf{F})}\rho^2 - {}^{*\,(\mathbf{F})}\rho^2 + i(-\ {}^*\rho + \eta\wedge\underline{\eta})\right)F$$

and

$$\dot{\square}_2 U = -\nabla_3\nabla_4 U + \frac{1}{2}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot U) + \left(2\underline{\omega} - \frac{1}{2}(\mathrm{tr}\,\underline{\chi} - i\,{}^{(a)}\mathrm{tr}\underline{\chi})\right)\nabla_4 U$$
$$- \frac{1}{2}(\mathrm{tr}\,\chi - i\,{}^{(a)}\mathrm{tr}\chi)\nabla_3 U 2\eta\cdot\nabla U + \left(-\frac{1}{2}tr\,\chi\mathrm{tr}\,\underline{\chi} - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}\right.$$
$$\left. - 2\rho + 2\,{}^{(\mathbf{F})}\rho^2 + 2\,{}^{*\,(\mathbf{F})}\rho^2 + i(-2\,{}^{*}\rho + 2\eta\wedge\underline{\eta})\right)U.$$

By writing $\mathrm{tr}\,\underline{\chi} + i\,{}^{(a)}\mathrm{tr}\underline{\chi} = \overline{\mathrm{tr}\underline{X}}$, $\mathrm{tr}\,\chi + i\,{}^{(a)}\mathrm{tr}\chi = \overline{\mathrm{tr}X}$ and $2\eta = H + \overline{H}$, we obtain the stated expressions. □

4.4. *Linear perturbations of Kerr–Newman*

In this section, we define the linear electromagnetic-gravitational perturbations of the Kerr–Newman spacetime. Recall that as Kerr–Newman is of Petrov Type D, the following coefficients:

$$\widehat{X}, \widehat{\underline{X}}, \Xi, \underline{\Xi}, \quad A, B, \underline{B}, \underline{A}, \quad {}^{(\mathbf{F})}B,\ {}^{(\mathbf{F})}\underline{B}$$

vanish in the background. For this reason, our definition of linear perturbations of Kerr–Newman consists in solutions to the Einstein–Maxwell equations where quadratic expressions in the above terms are neglected.

Definition 4.3. *A linear electromagnetic-gravitational perturbation of the Kerr–Newman spacetime*[f] *is a solution to the Einstein–Maxwell equations of Sec. 3, where quadratic expressions of terms which vanish in the background (i.e.* $\widehat{X}, \widehat{\underline{X}}, \Xi, \underline{\Xi}, A, B, \underline{B}, \underline{A}, {}^{(\mathbf{F})}B, {}^{(\mathbf{F})}\underline{B}$*) are neglected.*

For example, consider the Maxwell equation:

$${}^{(c)}\nabla_3\,{}^{(\mathbf{F})}B - {}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}P = -\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}B + 2\,{}^{(\mathbf{F})}PH + \frac{1}{2}\widehat{X}\cdot\overline{{}^{(\mathbf{F})}\underline{B}}.$$

The last term, $\frac{1}{2}\widehat{X}\cdot\overline{{}^{(\mathbf{F})}\underline{B}}$, is quadratic in $\widehat{X}$ and ${}^{(\mathbf{F})}\underline{B}$, and therefore it is neglected in linear electromagnetic-gravitational perturbations of Kerr–Newman. The linearized version of the above Maxwell equation then reduces to

$${}^{(c)}\nabla_3\,{}^{(\mathbf{F})}B - {}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}P = -\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}B + 2\,{}^{(\mathbf{F})}PH.$$

A similar procedure can be applied to all the Einstein–Maxwell equations of Sec. 3. We collect them in the following proposition.

[f] This same definition can also be used to define linear electromagnetic-gravitational perturbations of any Petrov Type D spacetime.

Proposition 4.4. *A linear electromagnetic-gravitational perturbation of the Kerr–Newman spacetime consists in a set of complex horizontal scalars, one-forms, 2-tensors*

$$\begin{aligned} \mathrm{tr}X, \mathrm{tr}\underline{X}, P, {}^{(\mathbf{F})}P &\in \mathfrak{s}_0(\mathbb{C}), \\ H, \underline{H}, Z, \Xi, \underline{\Xi}, {}^{(\mathbf{F})}B, {}^{(\mathbf{F})}\underline{B}, B, \underline{B} &\in \mathfrak{s}_1(\mathbb{C}), \\ \widehat{X}, \widehat{\underline{X}}, A, \underline{A} &\in \mathfrak{s}_2(\mathbb{C}), \end{aligned}$$

which satisfy the following linearized Einstein–Maxwell equations, comprised of the linearized Maxwell equations:

$$ {}^{(c)}\nabla_3 {}^{(\mathbf{F})}B - {}^{(c)}\mathcal{D} {}^{(\mathbf{F})}P = -\frac{1}{2}\mathrm{tr}\underline{X} {}^{(\mathbf{F})}B + 2 {}^{(\mathbf{F})}PH, \tag{4.16}$$

$$ {}^{(c)}\nabla_4 {}^{(\mathbf{F})}\underline{B} + {}^{(c)}\mathcal{D} \overline{{}^{(\mathbf{F})}P} = -\frac{1}{2}\mathrm{tr}X {}^{(\mathbf{F})}\underline{B} - 2\overline{{}^{(\mathbf{F})}P}\,\underline{H}, \tag{4.17}$$

$$ {}^{(c)}\nabla_4 {}^{(\mathbf{F})}P - \frac{1}{2}\overline{{}^{(c)}\mathcal{D}} \cdot {}^{(\mathbf{F})}B = -\overline{\mathrm{tr}X} {}^{(\mathbf{F})}P + \frac{1}{2}\overline{\underline{H}} \cdot {}^{(\mathbf{F})}B, \tag{4.18}$$

$$ {}^{(c)}\nabla_3 {}^{(\mathbf{F})}P + \frac{1}{2} {}^{(c)}\mathcal{D} \cdot \overline{{}^{(\mathbf{F})}\underline{B}} = -\mathrm{tr}\underline{X} {}^{(\mathbf{F})}P - \frac{1}{2}H \cdot \overline{{}^{(\mathbf{F})}\underline{B}}, \tag{4.19}$$

the linearized Ricci identities:

$$ {}^{(c)}\nabla_3\mathrm{tr}\underline{X} + \frac{1}{2}(\mathrm{tr}\underline{X})^2 = {}^{(c)}\mathcal{D} \cdot \overline{\underline{\Xi}} + \underline{\Xi} \cdot \overline{\underline{H}} + \overline{\underline{\Xi}} \cdot H, \tag{4.20}$$

$$ {}^{(c)}\nabla_4\mathrm{tr}X + \frac{1}{2}(\mathrm{tr}X)^2 = {}^{(c)}\mathcal{D} \cdot \overline{\Xi} + \Xi \cdot \overline{H} + \overline{\Xi} \cdot H, \tag{4.21}$$

$$ {}^{(c)}\nabla_3\mathrm{tr}X + \frac{1}{2}\mathrm{tr}\underline{X}\mathrm{tr}X = {}^{(c)}\mathcal{D} \cdot \overline{H} + H \cdot \overline{H} + 2P, \tag{4.22}$$

$$ {}^{(c)}\nabla_4\mathrm{tr}\underline{X} + \frac{1}{2}\mathrm{tr}X\mathrm{tr}\underline{X} = {}^{(c)}\mathcal{D} \cdot \overline{\underline{H}} + \underline{H} \cdot \overline{\underline{H}} + 2\overline{P}, \tag{4.23}$$

$$ {}^{(c)}\nabla_3\widehat{\underline{X}} + \Re(\mathrm{tr}\underline{X})\widehat{\underline{X}} = {}^{(c)}\mathcal{D}\widehat{\otimes}\underline{\Xi} + \underline{\Xi}\widehat{\otimes}(H + \underline{H}) - \underline{A}, \tag{4.24}$$

$$ {}^{(c)}\nabla_4\widehat{X} + \Re(\mathrm{tr}X)\widehat{X} = {}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \Xi\widehat{\otimes}(\underline{H} + H) - A, \tag{4.25}$$

$$ {}^{(c)}\nabla_3\widehat{X} + \frac{1}{2}\mathrm{tr}\underline{X}\,\widehat{X} = {}^{(c)}\mathcal{D}\widehat{\otimes}H + H\widehat{\otimes}H - \frac{1}{2}\overline{\mathrm{tr}X}\widehat{\underline{X}}, \tag{4.26}$$

$$ {}^{(c)}\nabla_4\widehat{\underline{X}} + \frac{1}{2}\mathrm{tr}X\,\widehat{\underline{X}} = {}^{(c)}\mathcal{D}\widehat{\otimes}\,\underline{H} + \underline{H}\widehat{\otimes}\,\underline{H} - \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\widehat{X}, \tag{4.27}$$

$$ {}^{(c)}\nabla_3\,\underline{H} - {}^{(c)}\nabla_4\underline{\Xi} = -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}(\,\underline{H} - H) - \frac{1}{2}\widehat{\underline{X}} \cdot (\overline{\underline{H}} - \overline{H}) + \underline{B} + {}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B}, \tag{4.28}$$

$$
{}^{(c)}\nabla_4 H - {}^{(c)}\nabla_3 \Xi = -\frac{1}{2}\overline{\mathrm{tr} X}(H - \underline{H}) - \frac{1}{2}\widehat{X}\cdot(\overline{H} - \overline{\underline{H}}) - B - \overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B, \tag{4.29}
$$

$$
\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{X} = \frac{1}{2}\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr} X} - i\Im(\mathrm{tr} X)H - i\Im(\mathrm{tr}\underline{X})\Xi - B + \overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B, \tag{4.30}
$$

$$
\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{\underline{X}} = \frac{1}{2}\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}} - i\Im(\mathrm{tr}\underline{X})\,\underline{H} - i\Im(\mathrm{tr} X)\underline{\Xi} + \underline{B} - {}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B}, \tag{4.31}
$$

$$
\nabla_3\omega + \nabla_4\underline{\omega} - 4\omega\underline{\omega} - (\eta - \underline{\eta})\cdot\zeta + \eta\cdot\underline{\eta} = \rho + {}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2 \tag{4.32}
$$

and the linearized Bianchi identities:

$$
\begin{aligned}
&{}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}A \\
&\quad = {}^{(c)}\mathcal{D}\widehat{\otimes}B + 4H\widehat{\otimes}B - 3\overline{P}\widehat{X} - 2\overline{{}^{(\mathbf{F})}P}\left(-\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B + {}^{(\mathbf{F})}P\widehat{X}\right)
\end{aligned} \tag{4.33}
$$

$$
\begin{aligned}
&{}^{(c)}\nabla_4 \underline{A} + \frac{1}{2}\mathrm{tr} X\underline{A} \\
&\quad = -\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} - 4\,\underline{H}\widehat{\otimes}\underline{B} - 3P\widehat{\underline{X}} + 2\,{}^{(\mathbf{F})}P\left(-\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B} - \overline{{}^{(\mathbf{F})}P}\widehat{\underline{X}}\right)
\end{aligned} \tag{4.34}
$$

$$
\begin{aligned}
&{}^{(c)}\nabla_4 B - \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot A \\
&\quad = -2\overline{\mathrm{tr} X}B + \frac{1}{2}A\cdot\overline{\underline{H}} + \left(3\overline{P} - 2\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}P\right)\Xi + \overline{{}^{(\mathbf{F})}P}\,{}^{(c)}\nabla_4({}^{(\mathbf{F})}B),
\end{aligned} \tag{4.35}
$$

$$
\begin{aligned}
&{}^{(c)}\nabla_3 \underline{B} + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\underline{A} \\
&\quad = -2\overline{\mathrm{tr}\underline{X}}\,\underline{B} - \frac{1}{2}\underline{A}\cdot\overline{H} - \left(3P - 2\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}P\right)\underline{\Xi} + {}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3({}^{(\mathbf{F})}\underline{B})
\end{aligned} \tag{4.36}
$$

$$
\begin{aligned}
&{}^{(c)}\nabla_3 B - {}^{(c)}\mathcal{D}\overline{P} \\
&\quad = -\mathrm{tr}\underline{X}B + 3\overline{P}H + \overline{{}^{(\mathbf{F})}P}\,{}^{(c)}\mathcal{D}({}^{(\mathbf{F})}P) - \frac{1}{2}\mathrm{tr}\underline{X}\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B - \overline{\mathrm{tr} X}\,{}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B},
\end{aligned} \tag{4.37}
$$

$$
\begin{aligned}
&{}^{(c)}\nabla_4 \underline{B} + {}^{(c)}\mathcal{D}P \\
&\quad = -\mathrm{tr} X\underline{B} - 3P\,\underline{H} - {}^{(\mathbf{F})}P\,{}^{(c)}\mathcal{D}(\overline{{}^{(\mathbf{F})}P}) - \frac{1}{2}\mathrm{tr} X\,{}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B} - \overline{\mathrm{tr}\underline{X}}\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B
\end{aligned} \tag{4.38}
$$

$$
\begin{aligned}
&{}^{(c)}\nabla_4 P - \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{B} \\
&\quad = -\frac{3}{2}\mathrm{tr} XP - \mathrm{tr} X\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} + \underline{H}\cdot\overline{B} + \frac{1}{2}\,{}^{(\mathbf{F})}P\,{}^{(c)}\mathcal{D}\cdot\overline{{}^{(\mathbf{F})}B} + \overline{H}\cdot\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B,
\end{aligned} \tag{4.39}
$$

$$
\begin{aligned}
&{}^{(c)}\nabla_3 P + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}} \cdot \underline{B} \\
&\quad = -\frac{3}{2}\overline{\mathrm{tr}\underline{X}} P - \overline{\mathrm{tr}\underline{X}}\, {}^{(\mathbf{F})}P\, \overline{{}^{(\mathbf{F})}P} - \overline{H} \cdot \underline{B} - \frac{1}{2}\, {}^{(\mathbf{F})}P\, \overline{{}^{(c)}\mathcal{D}} \cdot {}^{(\mathbf{F})}B - \underline{H} \cdot \overline{{}^{(\mathbf{F})}P}\, {}^{(\mathbf{F})}\underline{B}.
\end{aligned} \tag{4.40}
$$

Remark 4.5. Observe that the above definition of linear perturbations does not rely on a linear expansion of the metric of the form

$$
\mathbf{g} = \mathbf{g}_{m,a,Q} + \epsilon\ \mathbf{g}^{(1)} + \epsilon^2\ \mathbf{g}^{(2)} + \cdots
$$

for a smallness parameter ϵ. If one performs the above decomposition, applied to the Ricci, curvature and electromagnetic components, and selects the ϵ-expansion of the Einstein–Maxwell equations, then a choice of gauge would be needed in order to evaluate the values of the background metric $\mathbf{g}_{m,a,Q}$. Definition 4.3 instead is more general, and has the only effect of discarding the ϵ^2 terms, without choosing a gauge in doing so.

5. Gauge-Invariant Quantities in Perturbations of Kerr–Newman

In this section, we identify the gauge-invariant quantities in linear gravitational-electromagnetic perturbations of Kerr–Newman spacetime. Those quantities play a fundamental role in the resolution of the stability of Kerr–Newman, as, being affected only quadratically by a change of coordinates, they are good candidate to represent gravitational and electromagnetic radiation.

5.1. *Linear frame transformations*

Recall the rotations of class I, class II and class III which transform the basis vectors $\{e_3, e_4, e_a\}$ into $\{e'_3, e'_4, e'_a\}$ as introduced in Sec. 2.4. Those rotations depend on the one-forms μ and $\underline{\mu}$, on the scalar function λ and on the orthogonal matrix O_{ab}. Observe that the dependence of the equations and the coefficients on λ has already been taken into account in the definitions of conformal derivatives, and the dependence on the matrix O_{ab} is accounted for in the use of horizontal tensors as opposed to scalars. In this way, all the equations we have are already invariant to a rotation of class III. We now consider the change in the coefficients caused by rotations of class I and class II.

In the context of linear perturbations of a spacetime, we consider linear frame transformations, i.e. those where quadratic expressions in the μ and $\underline{\mu}$ are neglected. Combining the transformations given by (2.8) and (2.9), and neglecting the quadratic terms $|\mu|^2$ and $|\underline{\mu}|^2$, we obtain the general linear frame transformations as defined here.

Definition 5.1. *A linear frame transformation of the basis vectors* $\{e_3, e_4, e_a\}$ *into* $\{e_3', e_4', e_a'\}$ *is a transformation of the form*

$$\begin{aligned} e_4' &= e_4 + \mu_a e_a, \\ e_3' &= e_3 + \underline{\mu}_a e_a, \\ e_a' &= e_a + \frac{1}{2}\underline{\mu}_a e_4 + \frac{1}{2}\mu_a e_3, \end{aligned} \tag{5.1}$$

where μ *and* $\underline{\mu}$ *are real one-forms.*

When a linear frame transformation is applied to the frame, the Ricci, curvature and electromagnetic components change accordingly. For example, the electromagnetic component ${}^{(\mathbf{F})}\beta$ is modified in the following way:

$$\begin{aligned} {}^{(\mathbf{F})}\beta_a' &= \mathbf{F}(e_a', e_4') = \mathbf{F}(e_a + \frac{1}{2}\underline{\mu}_a e_4 + \frac{1}{2}\mu_a e_3, e_4 + \mu_b e_b) \\ &= \mathbf{F}(e_a, e_4) + \frac{1}{2}\mu_a \mathbf{F}(e_3, e_4) + \mu_b \mathbf{F}(e_a, e_b) + \frac{1}{2}\underline{\mu}_a \mu_b \mathbf{F}(e_4, e_b) \\ &\quad + \frac{1}{2}\mu_a \mu_b \mathbf{F}(e_3, e_b). \end{aligned}$$

By neglecting the quadratic terms in μ and $\underline{\mu}$, we then obtain

$${}^{(\mathbf{F})}\beta_a' = {}^{(\mathbf{F})}\beta_a + \mu_a \, {}^{(\mathbf{F})}\rho - \in_{ab} \mu_b \, {}^{*\,(\mathbf{F})}\rho = {}^{(\mathbf{F})}\beta_a + \mu_a \, {}^{(\mathbf{F})}\rho - {}^*\mu_a \, {}^{*\,(\mathbf{F})}\rho.$$

By considering the complexification of the ${}^{(\mathbf{F})}\beta$, i.e. ${}^{(\mathbf{F})}B$, and by defining $M := \mu + i\,{}^*\mu$, we deduce

$$\begin{aligned} {}^{(\mathbf{F})}B' &= {}^{(\mathbf{F})}\beta' + i\,{}^{*\,(\mathbf{F})}\beta' = {}^{(\mathbf{F})}\beta + \mu\,{}^{(\mathbf{F})}\rho - {}^*\mu\,{}^{*\,(\mathbf{F})}\rho \\ &\quad + i\,{}^*({}^{(\mathbf{F})}\beta + \mu\,{}^{(\mathbf{F})}\rho - {}^*\mu\,{}^{*\,(\mathbf{F})}\rho) \\ &= {}^{(\mathbf{F})}\beta + i\,{}^{*\,(\mathbf{F})}\beta + \mu\,{}^{(\mathbf{F})}\rho - {}^*\mu\,{}^{*\,(\mathbf{F})}\rho + i\,{}^*\mu\,{}^{(\mathbf{F})}\rho + i\mu\,{}^{*\,(\mathbf{F})}\rho \\ &= {}^{(\mathbf{F})}\beta + i\,{}^{*\,(\mathbf{F})}\beta + ({}^{(\mathbf{F})}\rho + i\,{}^{*\,(\mathbf{F})}\rho)(\mu + i\,{}^*\mu) = {}^{(\mathbf{F})}B + {}^{(\mathbf{F})}PM. \end{aligned}$$

In the same way, we can compute how all the Ricci, curvature and electromagnetic components get transformed by a linear frame transformation of the form (5.1). We collect those transformations in the following lemma.

Lemma 5.2. *The linear frame transformation* (5.1) *modifies the Ricci, curvature and electromagnetic components in the following way*:

$$\mathrm{tr}X' = \mathrm{tr}X + \frac{1}{2}{}^{(c)}\mathcal{D}\cdot\overline{M} + \frac{1}{2}\overline{H}\cdot M, \quad \mathrm{tr}\underline{X}' = \mathrm{tr}\underline{X} + \frac{1}{2}{}^{(c)}\mathcal{D}\cdot\overline{\underline{M}} + \frac{1}{2}\overline{\underline{H}}\cdot\underline{M},$$

$$\widehat{X}' = \widehat{X} + \frac{1}{2}{}^{(c)}\mathcal{D}\widehat{\otimes}M + \frac{1}{2}H\widehat{\otimes}M, \quad \underline{\widehat{X}}' = \underline{\widehat{X}} + \frac{1}{2}{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{M} + \frac{1}{2}\underline{H}\widehat{\otimes}\underline{M}.$$

$$H' = H + \frac{1}{2}{}^{(c)}\nabla_3 M + \frac{1}{4}\overline{\mathrm{tr}X}\underline{M}, \quad \underline{H}' = \underline{H} + \frac{1}{2}{}^{(c)}\nabla_4\underline{M} + \frac{1}{4}\overline{\mathrm{tr}\underline{X}}M.$$

$$\Xi' = \Xi + \frac{1}{2}{}^{(c)}\nabla_4 M + \frac{1}{4}\overline{\mathrm{tr}X}M, \quad \underline{\Xi}' = \underline{\Xi} + \frac{1}{2}{}^{(c)}\nabla_3\underline{M} + \frac{1}{4}\overline{\mathrm{tr}\underline{X}}\underline{M}$$

and

$$
{}^{(\mathbf{F})}B' = {}^{(\mathbf{F})}B + {}^{(\mathbf{F})}PM, \quad {}^{(\mathbf{F})}P' = {}^{(\mathbf{F})}P, \quad {}^{(\mathbf{F})}\underline{B}' = {}^{(\mathbf{F})}\underline{B} - \overline{{}^{(\mathbf{F})}P}\underline{M},
$$
$$
A' = A, \quad B' = B + \frac{3}{2}\overline{P}M, \quad P' = P, \quad \underline{B}' = \underline{B} - \frac{3}{2}P\underline{M}, \quad \underline{A}' = \underline{A},
$$

where $M =:= \mu + i\,{}^*\mu$ *and* $\underline{M} := \underline{\mu} + i\,{}^*\underline{\mu}$.

Proof. See [28]. □

5.2. *Gauge-invariant quantities and their relations*

The linear frame transformations of the form (5.1) can be used to pick a gauge in the perturbations. In particular, the quantities which are not modified by such a linear frame transformation may have a physical meaning, since they do not depend at the linear level on the choice of coordinates. We call such quantities gauge-invariant, and they are good candidates to represent electromagnetic or gravitational radiation.

Definition 5.3. *A horizontal tensor* $\Psi \in \mathfrak{s}(\mathbb{C})$ *is said to be gauge-invariant if it is not modified by a linear frame transformations of the form* (5.1)*, i.e. if* $\Psi' = \Psi$.

One of the main steps in analyzing electromagnetic-gravitational perturbations of Kerr–Newman is then to identify the gauge-invariant quantities. We identify in the following lemma four gauge-invariant quantities and their symmetric version.

Lemma 5.4. *For a linear electromagnetic-gravitational perturbation of the Kerr–Newman spacetime, the following symmetric traceless* 2*-tensors*:

$$
A, \quad \underline{A}
$$

and

$$
\begin{aligned}
\mathfrak{F} &= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{3}{2}H\widehat{\otimes}\,{}^{(\mathbf{F})}B + {}^{(\mathbf{F})}P\widehat{X},\\
\underline{\mathfrak{F}} &= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B} - \frac{3}{2}\underline{H}\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B} - \overline{{}^{(\mathbf{F})}P}\widehat{\underline{X}}
\end{aligned}
\tag{5.2}
$$

and the following 1*-tensors*:

$$
\mathfrak{B} = 2\,{}^{(\mathbf{F})}PB - 3\overline{P}\,{}^{(\mathbf{F})}B, \quad \underline{\mathfrak{B}} = 2\,\overline{{}^{(\mathbf{F})}P}\underline{B} - 3P\,{}^{(\mathbf{F})}\underline{B}
\tag{5.3}
$$

and

$$
\mathfrak{X} = {}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B + \frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}B - 2\,{}^{(\mathbf{F})}P\Xi, \quad \underline{\mathfrak{X}} = {}^{(c)}\nabla_3\,{}^{(\mathbf{F})}\underline{B} + \frac{3}{2}\overline{\mathrm{tr}\underline{X}}\,{}^{(\mathbf{F})}\underline{B} + 2\,\overline{{}^{(\mathbf{F})}P}\underline{\Xi}
\tag{5.4}
$$

are gauge-invariant.

Proof. The invariance of A and $\underline{A}$ is straightforward from Lemma 5.2. We check the invariance of $\mathfrak{F}$:

$$\begin{aligned}
\mathfrak{F}' &= -\frac{1}{2}\,{}^{(c)}\mathcal{D}'\widehat{\otimes}\,{}^{(\mathbf{F})}B' - \frac{3}{2}H'\widehat{\otimes}\,{}^{(\mathbf{F})}B' + {}^{(\mathbf{F})}P'\widehat{X}' \\
&= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\left({}^{(\mathbf{F})}B + {}^{(\mathbf{F})}PM\right) - \frac{3}{2}H\widehat{\otimes}\left({}^{(\mathbf{F})}B + {}^{(\mathbf{F})}PM\right) \\
&\quad + {}^{(\mathbf{F})}P\left(\widehat{X} + \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}M + \frac{1}{2}H\widehat{\otimes}M\right) \\
&= \mathfrak{F} - \frac{1}{2}\,{}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}P\widehat{\otimes}M - \frac{1}{2}\,{}^{(\mathbf{F})}P\,{}^{(c)}\mathcal{D}\widehat{\otimes}M - \frac{3}{2}\,{}^{(\mathbf{F})}PH\widehat{\otimes}M \\
&\quad + {}^{(\mathbf{F})}P\left(\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}M + \frac{1}{2}H\widehat{\otimes}M\right) \\
&= \mathfrak{F} - \frac{1}{2}(-2\,{}^{(\mathbf{F})}PH)\widehat{\otimes}M - {}^{(\mathbf{F})}PH\widehat{\otimes}M = \mathfrak{F},
\end{aligned}$$

where we used[g] (4.4). Similarly for $\underline{\mathfrak{F}}$. We check the invariance of $\mathfrak{B}$:

$$\begin{aligned}
\mathfrak{B}' &= 2\,{}^{(\mathbf{F})}P'B' - 3\overline{P}'\,{}^{(\mathbf{F})}B' = 2\,{}^{(\mathbf{F})}P\left(B + \frac{3}{2}\overline{P}F\right) - 3\overline{P}\left({}^{(\mathbf{F})}B + {}^{(\mathbf{F})}PF\right) \\
&= \mathfrak{B} + 2\,{}^{(\mathbf{F})}P\left(\frac{3}{2}\overline{P}F\right) - 3\overline{P}\left({}^{(\mathbf{F})}PF\right) = \mathfrak{B}
\end{aligned}$$

and similarly for $\underline{\mathfrak{B}}$. Finally, we check the invariance of $\mathfrak{X}$:

$$\begin{aligned}
\mathfrak{X}' &= {}^{(c)}\nabla_4({}^{(\mathbf{F})}B + {}^{(\mathbf{F})}PF) + \frac{3}{2}\overline{\mathrm{tr}X}({}^{(\mathbf{F})}B + {}^{(\mathbf{F})}PF) \\
&\quad - 2\,{}^{(\mathbf{F})}P\left(\Xi + \frac{1}{2}\,{}^{(c)}\nabla_4 F + \frac{1}{4}\overline{\mathrm{tr}X}F\right) \\
&= {}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B + \frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}B - 2\,{}^{(\mathbf{F})}P\Xi \\
&\quad - \overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}PF + {}^{(\mathbf{F})}P\,{}^{(c)}\nabla_4(F) + \frac{3}{2}\overline{\mathrm{tr}X}({}^{(\mathbf{F})}PF) - 2\,{}^{(\mathbf{F})}P\left(\frac{1}{2}\,{}^{(c)}\nabla_4 F + \frac{1}{4}\overline{\mathrm{tr}X}F\right) \\
&= {}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B + \frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}B - 2\,{}^{(\mathbf{F})}P\Xi = \mathfrak{X}
\end{aligned}$$

and similarly for $\underline{\mathfrak{X}}$. □

[g]Observe that by using the linearized Maxwell equation (4.16), one obtains

$$\begin{aligned}
({}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}P)\widehat{\otimes}M &= \left({}^{(c)}\nabla_3\,{}^{(\mathbf{F})}B + \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}B - 2\,{}^{(\mathbf{F})}PH\right)\widehat{\otimes}M \\
&= -2\,{}^{(\mathbf{F})}PH\widehat{\otimes}M + \text{ quadratic terms}
\end{aligned}$$

since the terms in ${}^{(\mathbf{F})}B$ and M are quadratic for linear perturbations of Kerr–Newman, and are therefore neglected. From now on, when multiplied by a quantity which vanishes on the background, we can then use the reduced equations of Proposition 4.1.

Remark 5.5. The identification of the above gauge-invariant quantities for linear perturbations of Kerr–Newman is a crucial new part of this work. In order to compare them in NP formalism, we collect here their equivalent, where the correspondence has to be understood through the projection to the first component, as explained in Sec. 2.5.

NP formalism	GKS formalism
Ψ_0	A
$\mathfrak{f} := -\delta\phi_0 + (2\beta + 3\tau)\phi_0 - 2\sigma\phi_1$	$\mathfrak{F}$
$\mathfrak{b} := 3\phi_0\Psi_2 - 2\phi_1\Psi_1$	$\mathfrak{B}$
$\mathfrak{x} := D\phi_0 - (3\rho + 2\epsilon)\phi_0 + 2\kappa\phi_1$	$\mathfrak{X}$

We point out that in [4, p. 240], Eq. (213), in the context of perturbations of Reissner–Nordström, Chandrasekhar notes "parenthetically, that while a gauge, in which Ψ_1 and ϕ_1 vanish simultaneously, cannot be chosen, the combination $2\Psi_1\phi_1 - 3\phi_0\Psi_2$ is invariant to first order for infinitesimal rotations".

The complex scalar $\mathfrak{b}$ identified in [4] corresponds precisely to the projection to the first component of our gauge-invariant quantity $\mathfrak{B}$. Nevertheless, such quantity was not used in the subsequent analysis in [4]. Indeed, it was used to show that a gauge where Ψ_1 and ϕ_1 vanish identically cannot be chosen, while a gauge where $\phi_0 = \phi_2 = 0$, the so-called phantom gauge, can be chosen. In [4], the equations governing the perturbations in the NP formalism were written in the phantom gauge, and all the analysis was performed in such a gauge. In particular, by choosing the phantom gauge, the above quantity was being reduced to a rescaled version of the curvature component Ψ_1.

No previous mention of $\mathfrak{f}$ nor $\mathfrak{x}$ in the context of perturbations of Kerr–Newman is known to the author.

Remark 5.6. In the case of linear electromagnetic-gravitational perturbations of Reissner–Nordström, the gauge-invariant quantities A, $\mathfrak{F}$, $\mathfrak{B}$ and $\mathfrak{X}$, respectively reduce to the quantities α, $\mathfrak{f}$, $\tilde{\beta}$ and $\mathfrak{x}$, first appearing in [14, 16]. More precisely, in Reissner–Nordström the Ricci coefficients H, $\underline{H}$, ${}^{(a)}\mathrm{tr}\chi$, ${}^{(a)}\mathrm{tr}\underline{\chi}$, ${}^{*\,(\mathbf{F})}\rho$, ${}^{*}\rho$ vanish in the background, and therefore the terms $H\widehat{\otimes}\,{}^{(\mathbf{F})}B$, $\underline{H}\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B}$, ${}^{*\,(\mathbf{F})}\rho\widehat{X}$, ${}^{*\,(\mathbf{F})}\rho\underline{\widehat{X}}$ in the definition of $\mathfrak{F}$, $\mathfrak{B}$ and $\mathfrak{X}$ become quadratic for linear perturbations of Reissner–Nordström. The real parts of A, $\mathfrak{F}$, $\mathfrak{B}$ and $\mathfrak{X}$ reduce to

$$\Re(\mathfrak{F}) = -\nabla\widehat{\otimes}\,{}^{(\mathbf{F})}\beta + {}^{(\mathbf{F})}\rho\hat{\chi} + \text{ quadratic terms} = \mathfrak{f},$$

$$\Re(\mathfrak{B}) = 2\,{}^{(\mathbf{F})}\rho\beta - 3\rho\,{}^{(\mathbf{F})}\beta + \text{ quadratic terms} = \tilde{\beta},$$

$$\Re(\mathfrak{X}) = \nabla_4\,{}^{(\mathbf{F})}\beta + \frac{3}{2}\mathrm{tr}\,\chi\,{}^{(\mathbf{F})}\beta - 2\,{}^{(\mathbf{F})}\rho\xi + \text{ quadratic terms} = \mathfrak{x},$$

with $\mathfrak{f}$, $\tilde{\beta}$ and $\mathfrak{x}$, as defined in [14, 16].

In the case of gravitational perturbations of Kerr, the only gauge-invariant quantity which has relevance is Ψ_0, Ψ_4 or A, $\underline{A}$, the well-known Teukolsky variables. The quantities $\mathfrak{F}$, $\mathfrak{B}$ and $\mathfrak{X}$, since contain electromagnetic components, only make sense for solutions to the Einstein–Maxwell equations.

Observe that by adding and subtracting $3\,\overline{{}^{(\mathbf{F})}P}H\widehat{\otimes}\,{}^{(\mathbf{F})}B$ and $3\,{}^{(\mathbf{F})}P\,\underline{H}\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B}$ to the linearized Bianchi identities (4.33) and (4.34), respectively, using the definition of $\mathfrak{F}$ and $\underline{\mathfrak{F}}$ (5.2), those Bianchi identity become

$$\begin{aligned} {}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}A &= {}^{(c)}\mathcal{D}\widehat{\otimes}B + H\widehat{\otimes}\big(4B - 3\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B\big) \\ &\quad - 3\overline{P}\widehat{X} - 2\,\overline{{}^{(\mathbf{F})}P}\mathfrak{F}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} {}^{(c)}\nabla_4 \underline{A} + \frac{1}{2}\mathrm{tr}X\underline{A} &= -\,{}^{(c)}\mathcal{D}\widehat{\otimes}\underline{B} - \underline{H}\widehat{\otimes}\big(4\underline{B} - 3\,{}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B}\big) \\ &\quad - 3P\widehat{\underline{X}} + 2\,{}^{(\mathbf{F})}P\underline{\mathfrak{F}} \end{aligned} \tag{5.6}$$

We summarize here some fundamental relations between the above gauge invariant quantities A, $\mathfrak{F}$, $\mathfrak{B}$ and $\mathfrak{X}$ obtained as consequence of the linearized Einstein–Maxwell equation. The relations between $\underline{A}$, $\underline{\mathfrak{F}}$, $\underline{\mathfrak{B}}$ and $\underline{\mathfrak{X}}$ can be obtained by symmetrization.

Proposition 5.7. *In a linear electromagnetic-gravitational perturbation of the Kerr–Newman spacetime, the following relations among the gauge invariant quantities* A, $\mathfrak{F}$, $\mathfrak{B}$ *and* $\mathfrak{X}$ *hold true*[h]:

- *The following relation between the* ${}^{(c)}\nabla_3$ *derivative of* A, *the* ${}^{(c)}\mathcal{D}$ *derivative of* $\mathfrak{B}$ *and* $\mathfrak{F}$:

$$ {}^{(\mathbf{F})}P\big({}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}A\big) = \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} + 3H\widehat{\otimes}\mathfrak{B} - \big(3\overline{P} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\mathfrak{F} \tag{5.7}$$

- *The following relation between the* ${}^{(c)}\nabla_4$ *derivative of* $\mathfrak{F}$, *the* ${}^{(c)}\mathcal{D}$ *derivative of* $\mathfrak{X}$ *and* A:

$$ {}^{(c)}\nabla_4\mathfrak{F} + \left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right)\mathfrak{F} = -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} - \frac{1}{2}(3H + \underline{H})\widehat{\otimes}\mathfrak{X} - {}^{(\mathbf{F})}PA \tag{5.8}$$

[h]In NP formalism, the above relations have the following form:

$$\begin{aligned} 2\phi_1\big(-\Delta\Psi_0 + (4\gamma-\mu)\Psi_0\big) &= \delta\mathfrak{b} - 2(\beta+3\tau)\mathfrak{b} + \big(3\Psi_2 + 2\phi_1\overline{\phi_1}\big)\mathfrak{f} \\ D\mathfrak{f} - \big(3\rho + \overline{\rho} + 3\epsilon - \overline{\epsilon}\big)\mathfrak{f} &= -\delta\mathfrak{x} + \big(3\beta + 3\tau - \overline{\pi} + \overline{\alpha}\big)\mathfrak{x} - 2\phi_1\Psi_0 \\ D\mathfrak{b} - 2\big(\epsilon + 3\rho\big)\mathfrak{b} &= -2\phi_1\big(\overline{\delta}\Psi_0 - 4\alpha\Psi_0 + \pi\Psi_0\big) + \big(3\Psi_2 - 2\phi_1\overline{\phi_1}\big)\mathfrak{x} \\ \Delta\mathfrak{x} + \big(\overline{\mu} - 3\gamma - \overline{\gamma}\big)\mathfrak{x} &= -\overline{\delta}\mathfrak{f} + \big(\overline{\tau} + 3\alpha - \overline{\beta}\big)\mathfrak{f} + 2\mathfrak{b}, [4pt] \end{aligned}$$

where $\mathfrak{f}$, $\mathfrak{b}$, $\mathfrak{x}$ are defined as in Remark 5.5.

- *The following relation between the* ${}^{(c)}\nabla_4$ *derivative of* $\mathfrak{B}$, *the* $\overline{{}^{(c)}\mathcal{D}}$ *derivative of* A *and* $\mathfrak{X}$:

$$ {}^{(c)}\nabla_4\mathfrak{B} + 3\overline{\mathrm{tr}X}\mathfrak{B} = {}^{(\mathbf{F})}P\big(\overline{{}^{(c)}\mathcal{D}} \cdot A + \overline{\underline{H}} \cdot A\big) - \big(3\overline{P} - 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\mathfrak{X}. \tag{5.9} $$

- *The following relation between the* ${}^{(c)}\nabla_3$ *derivative of* $\mathfrak{X}$, *the* $\overline{{}^{(c)}\mathcal{D}}$ *derivative of* $\mathfrak{F}$ *and* $\mathfrak{B}$:

$$ {}^{(c)}\nabla_3\mathfrak{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\mathfrak{X} = -\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F} - \overline{H} \cdot \mathfrak{F} - 2\mathfrak{B}. \tag{5.10} $$

Proof. See Appendix A.2. □

6. The System of Teukolsky Equations

In this section, we state the first theorem of the paper, which contains the coupled system of Teukolsky equations for the gauge-invariant quantities A, $\mathfrak{F}$ and $\mathfrak{B}$. These equations govern the linear electromagnetic-gravitational perturbations of Kerr–Newman spacetime, and generalize the Teukolsky equation for A in the case of Kerr. The system of Teukolsky equations for $\underline{\mathfrak{B}}$, $\underline{\mathfrak{F}}$ and $\underline{A}$ can be obtained by symmetry.

Theorem 6.1. *Consider a linear electromagnetic-gravitational perturbation of Kerr–Newman spacetime* $\mathbf{g}_{M,a,Q}$ *as in Definition 4.3. Then its associated complex tensors and gauge-invariant quantities* A, $\mathfrak{F}$, $\mathfrak{B}$ *and* $\mathfrak{X}$, *satisfy the following coupled system of Teukolsky equations*:

$$ \mathcal{T}_1(\mathfrak{B}) = \mathbf{M}_1[\mathfrak{F}, \mathfrak{X}], \tag{6.1} $$

$$ \mathcal{T}_2(\mathfrak{F}) = \mathbf{M}_2[A, \mathfrak{X}, \mathfrak{B}], \tag{6.2} $$

$$ \mathcal{T}_3(A) = \mathbf{M}_3[\mathfrak{F}, \mathfrak{X}], \tag{6.3} $$

where

- *on the left-hand side of the equations,* $\mathcal{T}$ *denotes the Teukolsky differential operators, respectively, given by*

$$ \begin{aligned} \mathcal{T}_1(\mathfrak{B}) :=& -{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\mathfrak{B} + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}} \cdot \big({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\big) - 3\overline{\mathrm{tr}X}\,{}^{(c)}\nabla_3\mathfrak{B} \\ & - \left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right){}^{(c)}\nabla_4\mathfrak{B} + \big(6H + \overline{H} + 3\overline{\underline{H}}\big) \cdot {}^{(c)}\nabla\mathfrak{B} \\ & + \left(-\frac{9}{2}\mathrm{tr}\underline{X}\,\overline{\mathrm{tr}X} - 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} + 9\,\overline{\underline{H}} \cdot H\right)\mathfrak{B}, \end{aligned} \tag{6.4} $$

$$ \begin{aligned} \mathcal{T}_2(\mathfrak{F}) :=& -{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\mathfrak{F} + \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F}\big) - \left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right){}^{(c)}\nabla_3\mathfrak{F} \\ & - \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right){}^{(c)}\nabla_4\mathfrak{F} + \big(4H + \overline{H} + \underline{H}\big) \cdot {}^{(c)}\nabla\mathfrak{F} \end{aligned} $$

$$+\left(-\frac{3}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X}-\frac{1}{4}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}X+3\overline{P}-P+4\,{}^{(\mathbf{F})}\!P\,\overline{{}^{(\mathbf{F})}\!P}-\frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H\right)\mathfrak{F}$$
$$+\frac{1}{2}\underline{H}\widehat{\otimes}(\overline{H}\cdot\mathfrak{F}), \tag{6.5}$$

$$\mathcal{T}_3(A):=-{}^{(c)}\nabla_4{}^{(c)}\nabla_3A+\frac{1}{2}{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot A)-\left(\frac{1}{2}\mathrm{tr}X+2\overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3A$$
$$-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4A+\left(4H+\underline{H}+\overline{\underline{H}}\right)\cdot{}^{(c)}\nabla A+\big(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X}$$
$$+2\overline{P}-2\,{}^{(\mathbf{F})}\!P\,\overline{{}^{(\mathbf{F})}\!P}\big)A+2H\widehat{\otimes}\big(\overline{\underline{H}}\cdot A\big) \tag{6.6}$$

- *on the right-hand side of the equations,* $\mathbf{M}$ *denotes the coupling terms, where the terms in squared parenthesis indicate the quantities involved in the expressions, respectively, given by*

$$\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]:=2\,{}^{(\mathbf{F})}\!P\,\overline{{}^{(\mathbf{F})}\!P}\left(2\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}+4\overline{\underline{H}}\cdot\mathfrak{F}-\left(2\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\right)\,\mathfrak{X}\right), \tag{6.7}$$

$$\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]:=-{}^{(\mathbf{F})}\!P\left({}^{(c)}\nabla_3A+\frac{1}{2}\left(3\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\right)A\right)+\left(\frac{3}{2}{}^{(c)}\nabla_3H\right)\widehat{\otimes}\mathfrak{X}$$
$$+\,(2H-\underline{H})\,\widehat{\otimes}\mathfrak{B}, \tag{6.8}$$

$$\mathbf{M}_3[\mathfrak{F},\mathfrak{X}]:=2\,\overline{{}^{(\mathbf{F})}\!P}\big(2\,{}^{(c)}\nabla_4\mathfrak{F}+2\overline{\mathrm{tr}X}\mathfrak{F}+(\underline{H}+H)\widehat{\otimes}\mathfrak{X}\big). \tag{6.9}$$

Proof. The derivation of the above Teukolsky equations relies on Proposition 5.7, and is obtained in Appendix B. □

We collect here few remarks about Theorem 6.1.

(1) The Teukolsky operators $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are wave-like operators, as it can be seen by comparing the expressions for $\dot{\square}_1$ and $\dot{\square}_2$ given in Lemma 4.2. More precisely, their highest order terms are given by a wave operator, with the additional presence of first-order terms.
(2) Observe that the system of Eqs. (6.1)–(6.3) for $\mathfrak{B}$, $\mathfrak{F}$ and A also involve the gauge-invariant quantity $\mathfrak{X}$. Nevertheless, $\mathfrak{X}$ is considered here an auxiliary quantity which only appears on the right-hand side $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ at the first order. More precisely, the system of Teukolsky equations (6.1)–(6.3), combined with the transport equation (5.10) for $\mathfrak{X}$, gives a complete system of equations.
(3) In the case of linear gravitational perturbations of Kerr spacetimes (which corresponds to Kerr–Newman for ${}^{(\mathbf{F})}\!P=0$), Eq. (6.3) for A reduces to $\mathcal{T}_3(A)=0$, i.e. the Teukolsky equation of spin +2 in Kerr, as obtained in [19].
(4) In the case of linear electromagnetic-gravitational perturbations of Reissner–Nordström (which corresponds to Kerr–Newman for $H=\underline{H}={}^{(a)}\mathrm{tr}\chi={}^{(a)}\mathrm{tr}\underline{\chi}=0$), the real parts of the Teukolsky system (6.1)–(6.3) reduces to the

following Teukolsky equations for $\tilde{\beta}$, $\mathfrak{f}$ and α

$$\mathcal{T}_1(\tilde{\beta}) = 2\,{}^{(\mathbf{F})}\rho^2\big(4\mathrm{div}\mathfrak{f} - \mathrm{tr}\,\underline{\chi}\,\mathfrak{x}\big),$$
$$\mathcal{T}_2(\mathfrak{f}) = -\,{}^{(\mathbf{F})}\rho\big({}^{(c)}\nabla_3\alpha + \mathrm{tr}\,\underline{\chi}\alpha\big),$$
$$\mathcal{T}_3(\alpha) = 4\,{}^{(\mathbf{F})}\rho\big({}^{(c)}\nabla_4\mathfrak{f} + \mathrm{tr}\,\chi\mathfrak{f}\big),$$

where

$$\begin{aligned}
\mathcal{T}_1(\tilde{\beta}) :=& -{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\tilde{\beta} - 2\mathrm{div}\,\mathcal{D}_2^*\tilde{\beta} - 3\mathrm{tr}\,\chi\,{}^{(c)}\nabla_3\tilde{\beta} - 2\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_4\tilde{\beta}\\
&+\left(-\frac{9}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - 4\,{}^{(\mathbf{F})}\rho^2\right)\tilde{\beta},\\
\mathcal{T}_2(\mathfrak{f}) :=& -{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\mathfrak{f} - 2\,\mathcal{D}_2^*\mathrm{div}\mathfrak{f} - \mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_4\mathfrak{f}\\
&- 2\mathrm{tr}\,\chi\,{}^{(c)}\nabla_3\mathfrak{f} + (-\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + 2\rho + 4\,{}^{(\mathbf{F})}\rho^2)\mathfrak{f},\\
\mathcal{T}_3(\alpha) :=& -{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\alpha - 2\,\mathcal{D}_2^*\mathrm{div}\alpha - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_4\alpha - \frac{5}{2}\mathrm{tr}\,\chi\,{}^{(c)}\nabla_3\alpha\\
&+\left(-\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + 2\rho - 2\,{}^{(\mathbf{F})}\rho^2\right)\alpha.
\end{aligned}$$

The quantities $\tilde{\beta}$, $\mathfrak{f}$ and α (as recalled in Remark 5.6) and the above quantities were obtained in [14, 16].

Finally, we relate the above system of Teukolsky equations for the horizontal tensors $\mathfrak{B}$, $\mathfrak{F}$ and A to the equations verified by their projection to the first component, as one would have obtained using the NP formalism.

Using that for $F \in \mathfrak{s}_1(\mathbb{C})$ and $U \in \mathfrak{s}_2(\mathbb{C})$ with $f = F_1 = F(e_1)$ and $u = u_{11} = u(e_1, e_1)$ their scalar projections, we have, see [19, Appendix D]

$$(\dot{\square}_2 F)_{11} = \square_{\mathbf{g}} f + i\frac{2}{|q|^2}\frac{\cos\theta}{\sin^2\theta}\partial_\varphi f + \left(-2\frac{(r^2+a^2)^2}{|q|^6}\cot^2\theta + \frac{2a^2\cos^2\theta\Delta}{|q|^6}\right)f,$$
$$(\dot{\square}_2 U)_{11} = \square_{\mathbf{g}} u + i\frac{4}{|q|^2}\frac{\cos\theta}{\sin^2\theta}\partial_\varphi u + \left(-4\frac{(r^2+a^2)^2}{|q|^6}\cot^2\theta + \frac{4a^2\cos^2\theta\Delta}{|q|^6}\right)u,$$

we can deduce the scalar Teukolsky equations satisfied by the projections of $\mathfrak{B}$, $\mathfrak{F}$ and A.

An interesting aspect of the system of Teukolsky equations in Kerr–Newman is that we have to differentiate between the spin and the conformal type of the quantities $\mathfrak{B}$, $\mathfrak{F}$ and A. As $\mathfrak{B}$ is a horizontal 1-tensor and $\mathfrak{F}$ and A are horizontal 2-tensors, their projections $\mathfrak{B}_1$ and $\mathfrak{F}_{11}$, A_{11} will respectively be scalars of spin 1 and spin 2. On the other hand, $\mathfrak{B}$ and $\mathfrak{F}$ are of conformal type 1, while A is of conformal type 2. We define the relevant rescaling of those projections up to some functions of q and $\overline{q}$ so that we can relate them to the standard Teukolsky equation in the literature [37].

We define the following rescaled projected quantities:

$$\mathfrak{b} = \frac{\overline{q}^{7/2}}{q^{1/2}}\mathfrak{B}_1, \quad \mathfrak{f} = \overline{q}\mathfrak{F}_{11}, \quad \alpha = \frac{\overline{q}}{q}A_{11}$$

and we collect in the following table their respective spin and conformal type:

	Spin type s	Conformal type c
$\mathfrak{b}$	1	1
$\mathfrak{f}$	2	1
α	2	2

We define the following Teukolsky operator of spin type s and conformal type c in Kerr–Newman, applied to a scalar ψ of spin type s and conformal type c to be given by

$$\begin{aligned}\mathcal{T}^{[s,c]}(\psi) := {}& \square_{\mathbf{g}_{M,a,Q}}\psi + \frac{2c}{|q|^2}(r-M)\partial_r\psi + \frac{2}{|q|^2}\left(c\frac{a(r-M)}{\Delta} + si\frac{\cos\theta}{\sin^2\theta}\right)\partial_\varphi\psi \\ & + \frac{2}{|q|^2}\left(c\left(\frac{M(r^2-a^2)-Q^2r}{\Delta} - r\right) - sia\cos\theta\right)\partial_t\psi \\ & + \frac{1}{|q|^2}(s - s^2\cot^2\theta)\psi. \end{aligned} \tag{6.10}$$

In particular notice that the conformal type c is relevant in the real parts of the coefficients of the first derivative, while the spin type s is relevant in the imaginary parts. Observe that the above Teukolsky operator reduces to the standard one [37] in Kerr for spin s applied to Ψ_0, Ψ_4, ϕ_0, ϕ_2 by using $c = s$, since these quantities have the same spin and conformal type.

One can then show that the projections to the first components of the Teukolsky differential operators $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ given by (6.4)–(6.4) can be written in terms of the scalar Teukolsky operator (6.10). More precisely:

$$\frac{\overline{q}^{7/2}}{q^{1/2}}(\mathcal{T}_1(\mathfrak{B}))_1 = \mathcal{T}^{[1,1]}(\mathfrak{b}), \quad \overline{q}(\mathcal{T}_2(\mathfrak{F}))_{11} = \mathcal{T}^{[2,1]}(\mathfrak{f}), \quad \frac{\overline{q}}{q}(\mathcal{T}_3(A))_{11} = \mathcal{T}^{[2,2]}(\alpha).$$

Just like in Schwarzschild, Kerr and Reissner–Nordström, boundedness and decay for solutions to the Teukolsky equations cannot be obtained directly. Since in Kerr–Newman is crucial to avoid the decomposition in modes as recalled in the introduction, we proceed in deriving a system of generalized Regge–Wheeler equations from the Teukolsky ones.

7. The System of Generalized Regge–Wheeler Equations

In this section, we derive the Regge–Wheeler system of equations governing the electromagnetic-gravitational perturbations of Kerr–Newman spacetime, therefore proving the main result of the paper.

7.1. *The invariant quantities* $\mathfrak{P}$, $\mathfrak{Q}$ *and* $\mathfrak{p}$, $\mathfrak{q}$

We introduce here the crucial invariant quantities satisfying the Regge–Wheeler equations. Those quantities are derived from the gauge-invariant quantities $\mathfrak{B}$ and $\mathfrak{F}$, through the following conformal operator.

Definition 7.1. *Let* $\Psi \in \mathfrak{s}_k(\mathbb{C})$ *be a gauge-invariant quantity of conformal type* s. *We define the Chandrasekhar operator* $\mathcal{P}_C : \mathfrak{s}_k(\mathbb{C}) \to \mathfrak{s}_k(\mathbb{C})$ *to be*

$$\mathcal{P}_C(\Psi) := {}^{(c)}\nabla_3\Psi + C\Psi \in \mathfrak{s}_k(\mathbb{C}) \tag{7.1}$$

for a scalar function C *of conformal type* -1.

We immediately observe that $\mathcal{P}_C(\Psi) \in \mathfrak{s}_k(\mathbb{C})$ is gauge-invariant of conformal type $s-1$.

We define the invariant quantities $\mathfrak{P}$ and $\mathfrak{Q}$ as the Chandrasekhar-transformed of the gauge-invariant quantities $\mathfrak{B}$ and $\mathfrak{F}$, respectively. In addition we allow for a rescaling of those quantities.

Definition 7.2. *We define the invariant quantities* $\mathfrak{P} \in \mathfrak{s}_1(\mathbb{C})$ *and* $\mathfrak{Q} \in \mathfrak{s}_2(\mathbb{C})$ *as*

$$\mathfrak{P} := \mathcal{P}_{C_1}(\mathfrak{B}) = {}^{(c)}\nabla_3\mathfrak{B} + C_1\mathfrak{B} \in \mathfrak{s}_1(\mathbb{C}), \tag{7.2}$$

$$\mathfrak{Q} := \mathcal{P}_{C_2}(\mathfrak{F}) = {}^{(c)}\nabla_3\mathfrak{F} + C_2\mathfrak{F} \in \mathfrak{s}_2(\mathbb{C}) \tag{7.3}$$

for scalar functions C_1 *and* C_2 *to be determined. We also define their rescaled version* $\mathfrak{p} \in \mathfrak{s}_1(\mathbb{C})$ *and* $\mathfrak{q}^{\mathbf{F}} \in \mathfrak{s}_2(\mathbb{C})$ *as*

$$\mathfrak{p} := f_1(q,\overline{q})\mathfrak{P} = f_1(q,\overline{q})\big({}^{(c)}\nabla_3\mathfrak{B} + C_1\mathfrak{B}\big) \in \mathfrak{s}_1(\mathbb{C}), \tag{7.4}$$

$$\mathfrak{q}^{\mathbf{F}} := f_2(q,\overline{q})\mathfrak{Q} = f_2(q,\overline{q})\big({}^{(c)}\nabla_3\mathfrak{F} + C_2\mathfrak{F}\big) \in \mathfrak{s}_2(\mathbb{C}), \tag{7.5}$$

where f_1 *and* f_2 *are functions of* $q = r + ia\cos\theta$ *and* $\overline{q} = r - ia\cos\theta$ *to be determined.*

The quantities $\mathfrak{p}$ and $\mathfrak{q}$ can be seen as first-order differential operators applied to the gauge-invariant quantities $\mathfrak{B}$ and $\mathfrak{F}$, which satisfy the Teukolsky system of equations. Observe that $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{p}$, $\mathfrak{q}^{\mathbf{F}}$ are all of conformal type 0.

7.2. *Statement of the main theorem and remarks*

We now state the main result regarding the wave equations satisfied by $\mathfrak{p}$ and $\mathfrak{q}^{\mathbf{F}}$.

Theorem 7.3. *Consider a linear electromagnetic-gravitational perturbation of Kerr–Newman spacetime* $\mathbf{g}_{M,a,Q}$ *as in Definition 4.3, with associated gauge-invariant quantities* $\mathfrak{B}$ *and* $\mathfrak{F}$.

Then there exist choices of complex scalar functions C_1, C_2 f_1, f_2, *in the definitions of* $\mathfrak{p}$ *and* $\mathfrak{q}^{\mathbf{F}}$, *explicitly*:

$$\mathfrak{p} = q^{\frac{1}{2}}\overline{q}^{\frac{9}{2}}\left({}^{(c)}\nabla_3\mathfrak{B} + \left(2\mathrm{tr}\,\underline{\chi} - \frac{5}{2}i\,{}^{(a)}tr\underline{\chi}\right)\mathfrak{B}\right) \in \mathfrak{s}_1(\mathbb{C}),$$

$$\mathfrak{q}^{\mathbf{F}} = q\overline{q}^2\big({}^{(c)}\nabla_3\mathfrak{F} + \big(\mathrm{tr}\,\underline{\chi} - 3i\,{}^{(a)}tr\underline{\chi}\big)\mathfrak{F}\big) \in \mathfrak{s}_2(\mathbb{C}),$$

such that the invariant 1*-tensor* $\mathfrak{p} \in \mathfrak{s}_1(\mathbb{C})$ *and the symmetric traceless* 2*-tensor* $\mathfrak{q}^{\mathbf{F}} \in \mathfrak{s}_2(\mathbb{C})$ *satisfy the following coupled system of wave equations*:

$$\dot{\square}_1\mathfrak{p} - i\frac{2a\cos\theta}{|q|^2}\nabla_t\mathfrak{p} - V_1\mathfrak{p} = 4Q^2\frac{\overline{q}^3}{|q|^5}\left(\overline{\mathcal{D}}\cdot\mathfrak{q}^{\mathbf{F}}\right) + L_{\mathfrak{p}}[\mathfrak{B},\mathfrak{F}], \tag{7.6}$$

$$\dot{\square}_2\mathfrak{q}^{\mathbf{F}} - i\frac{4a\cos\theta}{|q|^2}\nabla_t\mathfrak{q}^{\mathbf{F}} - V_2\mathfrak{q}^{\mathbf{F}} = -\frac{1}{2}\frac{q^3}{|q|^5}\left(\mathcal{D}\widehat{\otimes}\mathfrak{p} - \frac{3}{2}\left(H - \underline{H}\right)\widehat{\otimes}\mathfrak{p}\right) + L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B},\mathfrak{F}] \tag{7.7}$$

where

- $\dot{\square}_1$ *and* $\dot{\square}_2$ *denote the wave operators for horizontal* 1*-tensors and* 2*-tensors, respectively, as defined in* (4.14),
- *the potentials* V_1 *and* V_2 *are* real *positive scalar functions* (*whose precise expression is given by* (7.39)), *which for* $a = 0$ *coincide with the potentials of the Regge–Wheeler system of equations in Reissner–Nordström* [15], *i.e.*

$$V_1 = -\frac{1}{4}tr\,\chi\mathrm{tr}\,\underline{\chi} + 5\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^4}\right), \quad V_2 = -tr\,\chi\mathrm{tr}\,\underline{\chi} + 2\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^4}\right).$$

- $L_{\mathfrak{p}}[\mathfrak{B},\mathfrak{F}]$ *and* $L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B},\mathfrak{F}]$ *are linear first-order operators in* $\mathfrak{B}$ *and* $\mathfrak{F}$, *respectively, given by*

$$\begin{aligned}L_{\mathfrak{p}}[\mathfrak{B},\mathfrak{F}] = q^{1/2}\overline{q}^{9/2}\big[&- Z_a^{\mathfrak{B}}\cdot{}^{(c)}\nabla\mathfrak{B} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\;Y_a^{\mathfrak{F}}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}\big)\\ &+ \big(2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\;Y_0^{\mathfrak{B}} - Z_0^{\mathfrak{B}}\big)\mathfrak{B} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big(Y_0^{\mathfrak{F}}\cdot\mathfrak{F} + Y_0^{\mathfrak{X}}\;\mathfrak{X}\big)\big]\end{aligned}$$

and

$$\begin{aligned}L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B},\mathfrak{F}] = q\overline{q}^2\big[&W_4^{\mathfrak{F}}\,{}^{(c)}\nabla_4\mathfrak{F} + \big(W_a^{\mathfrak{F}} - Z_a^{\mathfrak{F}}\big)\cdot{}^{(c)}\nabla\mathfrak{F} + W_a^{\mathfrak{X}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} + W_a^{\mathfrak{B}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\\ &+ \big(W_0^{\mathfrak{F}} - Z_0^{\mathfrak{F}}\big)\mathfrak{F} + W_0^{\mathfrak{B}}\widehat{\otimes}\mathfrak{B} + W_0^{\mathfrak{X}}\widehat{\otimes}\mathfrak{X}\big],\end{aligned}$$

where

— $W_4^{\mathfrak{F}}$ *and* $W_a^{\mathfrak{X}}$ *are* real *functions*,
— $Z_a^{\mathfrak{B}}$ *and* $(W_a^{\mathfrak{F}} - Z_a^{\mathfrak{F}})$ *are* real *one-forms*,
— $W_a^{\mathfrak{B}}$ *and* $Y_a^{\mathfrak{F}}$ *are* imaginary *functions given by* $W_a^{\mathfrak{B}} = \frac{3}{4}i\,{}^{(a)}tr\underline{\chi}$ *and* $Y_a^{\mathfrak{F}} = -3i\,{}^{(a)}tr\underline{\chi}$

and $Y_0^{\mathfrak{B}}$, $Z_0^{\mathfrak{B}}$, $Y_0^{\mathfrak{X}}$, $(W_0^{\mathfrak{F}} - Z_0^{\mathfrak{F}})$ *are complex functions, and* $Y_0^{\mathfrak{F}}$, $W_0^{\mathfrak{B}}$, $W_0^{\mathfrak{X}}$ *are complex one-forms, all of which vanish for zero angular momentum.*

We call the system of Eqs. (7.6)–(7.7) *a system of* generalized Regge–Wheeler equations*.*

We now remark what are the crucial structures of the system of generalized Regge–Wheeler equations (7.6)–(7.7) which make them analyzable in physical space.

(1) The only first-order terms present in both equations is of the form $i\nabla_t$, as in the generalized Regge–Wheeler equation obtained in Kerr [7, 19, 29]. Such first-order term has good divergence properties in the derivation of the energy estimates. Schematically, when multiplying Eq. (7.6) by $\nabla_t \overline{\mathfrak{p}}$ and taking the real part, one obtains a cancellation from the first-order term:

$$i\nabla_t \mathfrak{p} \cdot \nabla_t \overline{\mathfrak{p}} + \overline{i\nabla_t \mathfrak{p}} \cdot \nabla_t \mathfrak{p} = i\nabla_t \mathfrak{p} \cdot \nabla_t \overline{\mathfrak{p}} - i\nabla_t \overline{\mathfrak{p}} \cdot \nabla_t \mathfrak{p} = 0.$$

This allows to derive the energy estimates without loss of derivatives. Similarly for Eq. (7.7).

(2) The reality of the potentials V_1 and V_2 is also crucial in the derivation of the estimates. When deriving the energy estimates and multiplying Eq. (7.6) by $\nabla_t \overline{\mathfrak{p}}$ and taking the real part, one obtains

$$\begin{aligned} V_1 \mathfrak{p} \cdot \nabla_t \overline{\mathfrak{p}} + \overline{V_1 \mathfrak{p}} \cdot \nabla_t \mathfrak{p} &= \Re(V_1)(\mathfrak{p} \cdot \nabla_t \overline{\mathfrak{p}} + \overline{\mathfrak{p}} \cdot \nabla_t \mathfrak{p}) + i\Im(V_1)(\mathfrak{p} \cdot \nabla_t \overline{\mathfrak{p}} - i\overline{\mathfrak{p}} \cdot \nabla_t \mathfrak{p}) \\ &= \frac{1}{2}\Re(V_1)\nabla_t(|\mathfrak{p}|^2) + i\Im(V_1)(\mathfrak{p} \cdot \nabla_t \overline{\mathfrak{p}} - i\overline{\mathfrak{p}} \cdot \nabla_t \mathfrak{p}). \end{aligned}$$

If the imaginary part of the potential is not zero, then the last term cannot be written as a boundary term, and the energy estimates cannot be closed. In addition, the positivity of the real part of the potentials give positive contribution to the energy in the boundary terms. Similarly for Eq. (7.7).

(3) The highest order coupling terms on the right-hand side of the equations are of the form $\frac{\overline{q}^3}{|q|^5}(\overline{\mathcal{D}} \cdot \mathfrak{q}^{\mathbf{F}})$ and $-\frac{q^3}{|q|^5}(\mathcal{D}\widehat{\otimes}\mathfrak{p})$, up to a multiplication by the positive constant $8Q^2$. In particular observe that the functions multiplying the operators $\overline{\mathcal{D}}\cdot$ and $\mathcal{D}\widehat{\otimes}$ are complex conjugate. Such structure is crucial in the cancellation of those coupling terms once the estimates for the two equations are summed, since $\overline{\mathcal{D}}\cdot$ and $\mathcal{D}\widehat{\otimes}$ are adjoint operators up to lower order terms, as obtained in Lemma 2.11.

Such lower order terms, together with the derivatives falling to the functions $\frac{\overline{q}^3}{|q|^5}$ and $\frac{q^3}{|q|^5}$, are crucial in treating the coupling terms $\frac{\overline{q}^3}{|q|^5}(\overline{\mathcal{D}} \cdot \mathfrak{q}^{\mathbf{F}})$ and $-\frac{q^3}{|q|^5}(\mathcal{D}\widehat{\otimes}\mathfrak{p} - \frac{3}{2}(H - \underline{H})\widehat{\otimes}\mathfrak{p})$, precisely to cancel the term $-\frac{3}{2}(H - \underline{H})\widehat{\otimes}\mathfrak{p}$ in the estimates.

(4) The first-order operators $L_{\mathfrak{p}}[\mathfrak{B}, \mathfrak{F}]$ and $L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B}, \mathfrak{F}]$ contain terms which are \ order in differentiability with respect to $\mathfrak{p}$ and $\mathfrak{q}^{\mathbf{F}}$. In particular, in the derivation of the energy estimates it is crucial that the highest order terms relative to the corresponding equations have real coefficients. More precisely, in the equation for $\mathfrak{p}$, the operator $L_{\mathfrak{p}}[\mathfrak{B}, \mathfrak{F}]$ should have real coefficients for the quantities which are lower order with respect to $\mathfrak{p}$, i.e. ${}^{(c)}\nabla\mathfrak{B}$, while in the equation for $\mathfrak{q}^{\mathbf{F}}$, the

operator $L^{\mathbf{F}}_{\mathfrak{q}}[\mathfrak{B}, \mathfrak{F}]$ should have real coefficients for the quantities which are lower order with respect to $\mathfrak{q}^{\mathbf{F}}$, i.e. ${}^{(c)}\nabla_4\mathfrak{F}$ and ${}^{(c)}\nabla\mathfrak{F}$.

On the other hand, the lower order terms which are coupled should cancel, similarly to the coupling terms above. It is therefore crucial to obtain a cancellation in the terms $W^{\mathfrak{B}}_a = \frac{3}{4}i\,{}^{(a)}\mathrm{tr}\underline{\chi}$ and $Y^{\mathfrak{F}}_a = -3i\,{}^{(a)}\mathrm{tr}\underline{\chi}$.

It is remarkable that a choice of complex functions C_1, C_2, f_1 and f_2 which realizes all the above exists and can be found. In particular, the freedom in the choice of these functions is not enough to impose each one of the above conditions. We instead will prove the theorem by imposing condition 1 (i.e. the only first-order term is of the form $i\nabla_t$) and condition 2 (i.e. the potentials are real), and this will uniquely determine the functions C_1, C_2, f_1 and f_2. We then show that with those choices, conditions 3 and 4 are also satisfied.

We summarize here the main steps of the proof.

(a) We compute the commutator between the Chandrasekhar operator $\mathcal{P}_C$ and the Teukolsky operators $\mathcal{T}_1$ and $\mathcal{T}_2$. In order to cancel the lower order terms in the commutator, we impose conditions on the real part of the functions C_1 and C_2, and obtain

$$\Re(C_1) = 2\mathrm{tr}\,\underline{\chi}, \quad \Re(C_2) = \mathrm{tr}\,\underline{\chi}.$$

This is done in Sec. 7.3.1. With such choice we can compute the wave equations for the Chandrasekhar-transformed $\mathfrak{P}$ and $\mathfrak{Q}$ quantities, in Sec. 7.3.2.

(b) We compute the effect on the wave equations of the rescaling of $\mathfrak{P}$ and $\mathfrak{Q}$ through functions f_1 and f_2. In order to get only first-order terms of the form $i\nabla_t$ (condition 1), we impose conditions on functions f_1 and f_2, and obtain

$$f_1 = (q)^{1/2}(\overline{q})^{9/2}, \quad f_2 = q\overline{q}^2.$$

This is done in Sec. 7.3.3.

(c) We compute the right-hand side of the respective equations, and show that the above choice of f_1 and f_2 implies the structure of the higher coupling terms (condition 3). This is done in Sec. 7.3.4.

(d) We compute the potentials of the two equations, and impose the vanishing of their imaginary parts. This uniquely determines the imaginary parts of the functions C_1 and C_2, giving

$$\Im(C_1) = -\frac{5}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}, \quad \Im(C_2) = -3i\,{}^{(a)}\mathrm{tr}\underline{\chi}.$$

This is done in Sec. 7.3.5.

(e) Finally, we compute the lower order terms in $L_{\mathfrak{p}}[\mathfrak{B}, \mathfrak{F}]$ and $L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B}, \mathfrak{F}]$ and show that the above choices for C_1 and C_2 imply the reality or the cancellation of the relevant coefficients. This is done in Sec. 7.3.6.

7.3. *Proof of Theorem 7.3*

In this section, we derive the proof of Theorem 7.3 while relying on the computations in the appendix. Recall the Teukolsky equations (6.1) and (6.2) in Theorem 6.1, i.e.

$$\mathcal{T}_1(\mathfrak{B}) = \mathbf{M}_1[\mathfrak{F}, \mathfrak{X}],$$
$$\mathcal{T}_2(\mathfrak{F}) = \mathbf{M}_2[A, \mathfrak{X}, \mathfrak{B}].$$

We apply the Chandrasekhar operators $\mathcal{P}_{C_1}$ and $\mathcal{P}_{C_2}$, for C_1 and C_2 to be determined, to the above equations, respectively. Recalling that $\mathfrak{P} := \mathcal{P}_{C_1}(\mathfrak{B})$ and $\mathfrak{Q} := \mathcal{P}_{C_2}(\mathfrak{F})$, we obtain

$$\mathcal{T}_1(\mathfrak{P}) + [\mathcal{P}_{C_1}, \mathcal{T}_1](\mathfrak{B}) = \mathcal{P}_{C_1}\big(\mathbf{M}_1[\mathfrak{F}, \mathfrak{X}]\big), \tag{7.8}$$
$$\mathcal{T}_2(\mathfrak{Q}) + [\mathcal{P}_{C_2}, \mathcal{T}_2](\mathfrak{F}) = \mathcal{P}_{C_2}\big(\mathbf{M}_2[A, \mathfrak{X}, \mathfrak{B}]\big). \tag{7.9}$$

7.3.1. *The commutators* $[\mathcal{P}_C, \mathcal{T}]$

We compute the commutators between the Teukolsky operators $\mathcal{T}_1$ and $\mathcal{T}_2$ and the first-order differential operator $\mathcal{P}_C$ as defined in (7.1) for any scalar function C. In order to eliminate the highest order terms which cannot be expressed in terms of $\mathfrak{P}$ or $\mathfrak{Q}$ (i.e. ${}^{(c)}\nabla_4\mathfrak{B}$ and ${}^{(c)}\nabla_4\mathfrak{F}$), we need to impose conditions on the real part of the functions C_1 and C_2. We obtain the following proposition.

Proposition 7.4. *Let* $\mathfrak{P} = \mathcal{P}_{C_1}(\mathfrak{B}) = {}^{(c)}\nabla_3\mathfrak{B} + C_1\mathfrak{B}$ *and* $\mathfrak{Q} = \mathcal{P}_{C_2}(\mathfrak{F}) = {}^{(c)}\nabla_3\mathfrak{F} + C_2\mathfrak{F}$, *such that* C_1 *and* C_2 *satisfy, respectively,*

$${}^{(c)}\nabla_3 C_1 + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right) C_1 - \mathrm{tr}\underline{X}\,\overline{\mathrm{tr}\underline{X}} = 0, \tag{7.10}$$
$${}^{(c)}\nabla_3 C_2 + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right) C_2 - \frac{1}{2}\mathrm{tr}\underline{X}\,\overline{\mathrm{tr}\underline{X}} = 0. \tag{7.11}$$

Then the commutators between the Chandrasekhar operators $\mathcal{P}_{C_1}$ *and* $\mathcal{P}_{C_2}$ *and the Teukolsky operators* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are, respectively, given by*

$$\begin{aligned}[\mathcal{P}_{C_1}, \mathcal{T}_1](\mathfrak{B}) &= 2\underline{\eta}\cdot\nabla\mathfrak{P} - \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\nabla_4\mathfrak{P} + \hat{V}_1\mathfrak{P}\\ &\quad - \frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\mathbf{M}_1[\mathfrak{F}, \mathfrak{X}] - L_{\mathfrak{P}}[\mathfrak{B}, \mathfrak{F}],\\ [\mathcal{P}_{C_2}, \mathcal{T}_2](\mathfrak{F}) &= 2\underline{\eta}\cdot\nabla\mathfrak{Q} - \frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\nabla_4\mathfrak{Q} + \hat{V}_2\mathfrak{Q} - \frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\\ &\quad \times \mathbf{M}_2[A, \mathfrak{X}, \mathfrak{B}] - L_{\mathfrak{Q}}[\mathfrak{B}, \mathfrak{F}],\end{aligned}$$

where

- *the potentials* $\hat{V}_1$ *and* $\hat{V}_2$ *are given by*

$$\hat{V}_1 = I_3^{\mathfrak{B}} + J_3^{\mathfrak{B}} + K_3^{\mathfrak{B}} + M_3^{\mathfrak{B}} = -\frac{5}{2} tr\, \chi \mathrm{tr}\, \underline{\chi} - 4\rho - 2\, {}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^3}\right), \tag{7.12}$$

$$\hat{V}_2 = I_3^{\mathfrak{F}} + J_3^{\mathfrak{F}} + K_3^{\mathfrak{F}} + M_3^{\mathfrak{F}} = -\frac{3}{2} tr\, \chi \mathrm{tr}\, \underline{\chi} - 4\rho - 2\, {}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^3}\right), \tag{7.13}$$

where the precise coefficients are given in Appendix C.

- $L_{\mathfrak{P}}[\mathfrak{B}, \mathfrak{F}]$ *and* $L_{\mathfrak{Q}}[\mathfrak{B}, \mathfrak{F}]$ *are linear first-order operators in* $\mathfrak{B}$ *and* $\mathfrak{F}$*, given by*

$$L_{\mathfrak{P}}[\mathfrak{B}, \mathfrak{F}] := -Z_a^{\mathfrak{B}} \cdot {}^{(c)}\nabla \mathfrak{B} - \left(4\mathrm{tr}\, \underline{\chi}\, {}^{(\mathbf{F})}\rho^2 + Z_0^{\mathfrak{B}}\right)\mathfrak{B}, \tag{7.14}$$

$$L_{\mathfrak{Q}}[\mathfrak{B}, \mathfrak{F}] := -Z_a^{\mathfrak{F}} \cdot {}^{(c)}\nabla \mathfrak{F} + \left(4\mathrm{tr}\, \underline{\chi}\, {}^{(\mathbf{F})}\rho^2 - Z_0^{\mathfrak{F}}\right)\mathfrak{F}, \tag{7.15}$$

where $Z_a^{\mathfrak{B}}$ *and* $Z_a^{\mathfrak{F}}$ *are complex one-forms and* $Z_0^{\mathfrak{B}}$ *and* $Z_0^{\mathfrak{B}}$ *are complex functions of* (r, θ)*, all of which vanish for zero angular momentum, having the following fall-off in* r*:*

$$Z_a^{\mathfrak{B}}, Z_a^{\mathfrak{F}} = O\left(\frac{|a|}{r^3}\right), \qquad Z_0^{\mathfrak{B}}, Z_0^{\mathfrak{F}} = O\left(\frac{|a|}{r^4}\right).$$

Proof. See Appendix C.1. □

Observe that the transport equations (7.10) and (7.11) only impose conditions on the real parts of C_1 and C_2. Indeed, for any real constants p_1, p_2, the scalar functions C_1 and C_2 given by

$$C_1 = 2\mathrm{tr}\, \underline{\chi} + ip_1\, {}^{(a)}\mathrm{tr}\underline{\chi}, \qquad C_2 = \mathrm{tr}\, \underline{\chi} + ip_2\, {}^{(a)}\mathrm{tr}\underline{\chi} \tag{7.16}$$

are of conformal type -1 and satisfy (7.10) and (7.11), respectively.

7.3.2. *The wave equations for* $\mathfrak{P}$ *and* $\mathfrak{Q}$

Recall the Teukolsky operators $\mathcal{T}_1$ and $\mathcal{T}_2$, see (6.4) and (6.5). We then obtain for $\mathfrak{P}$ and $\mathfrak{Q}$, respectively,

$$\begin{aligned}\mathcal{T}_1(\mathfrak{P}) &= -\nabla_3\nabla_4\mathfrak{P} + \frac{1}{2}\overline{\mathcal{D}} \cdot (\mathcal{D}\widehat{\otimes}\mathfrak{P}) - 3\overline{\mathrm{tr}X}\nabla_3\mathfrak{P} - \left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}} - 2\underline{\omega}\right)\nabla_4\mathfrak{P} \\ &\quad + (6H + \overline{H} + 3\overline{\underline{H}}) \cdot \nabla\mathfrak{P} + \left(-\frac{9}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} - 4\, {}^{(\mathbf{F})}P\, \overline{{}^{(\mathbf{F})}P} + 9\overline{\underline{H}} \cdot H\right)\mathfrak{P},\end{aligned}$$

$$\mathcal{T}_2(\mathfrak{Q}) = -\nabla_3\nabla_4\mathfrak{Q} + \frac{1}{2}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}} \cdot \mathfrak{Q}) - \left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right)\nabla_3\mathfrak{Q}$$

$$
\begin{aligned}
&-\frac{1}{2}\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}-4\underline{\omega}\right)\nabla_4\mathfrak{Q}+\left(4H+\overline{H}+\underline{H}\right)\cdot\nabla\mathfrak{Q}\\
&+\Big(-\frac{3}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X}-\frac{1}{4}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}X+3\overline{P}-P+4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\\
&-\frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H+\eta\cdot\underline{\eta}+i\eta\wedge\underline{\eta}\Big)\mathfrak{Q},
\end{aligned}
$$

where recall that $\mathfrak{P}$ and $\mathfrak{Q}$ are both of conformal type 0.

From the commuted equations (7.8) and (7.9) and using the formulas for the commutators given by Proposition 7.4, we obtain, respectively, by writing $2\underline{\eta}=\underline{H}+\overline{\underline{H}}$:

$$
\begin{aligned}
&-\nabla_3\nabla_4\mathfrak{P}+\frac{1}{2}\overline{\mathcal{D}}\cdot(\mathcal{D}\widehat{\otimes}\mathfrak{P})-3\overline{\mathrm{tr}X}\nabla_3\mathfrak{P}-\left(2\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}-2\underline{\omega}\right)\nabla_4\mathfrak{P}\\
&\qquad+\left(6H+\overline{H}+\underline{H}+4\overline{\underline{H}}\right)\cdot\nabla\mathfrak{P}\\
&\qquad+\left(-\frac{9}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X}-4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}+9\,\overline{\underline{H}}\cdot H+\hat{V}_1\right)\mathfrak{P}\\
&\quad=\mathcal{P}_{C_1}\left(\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]\right)+\frac{1}{2}\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]+L_{\mathfrak{P}}[\mathfrak{B},\mathfrak{F}]
\end{aligned}
$$

and

$$
\begin{aligned}
&-\nabla_3\nabla_4\mathfrak{Q}+\frac{1}{2}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\mathfrak{Q})-\left(\frac{3}{2}\overline{\mathrm{tr}X}+\frac{1}{2}\mathrm{tr}X\right)\nabla_3\mathfrak{Q}\\
&-(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}-2\underline{\omega})\nabla_4\mathfrak{Q}+(4H+\overline{H}+2\,\underline{H}+\overline{\underline{H}})\cdot\nabla\mathfrak{Q}\\
&+\Big(-\frac{3}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X}-\frac{1}{4}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}X+3\overline{P}-P+4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\\
&-\frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H+\eta\cdot\underline{\eta}+i\eta\wedge\underline{\eta}+\hat{V}_2\Big)\mathfrak{Q}\\
&=\mathcal{P}_{C_2}\left(\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\right)+\frac{1}{2}\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]+L_{\mathfrak{Q}}[\mathfrak{B},\mathfrak{F}].
\end{aligned}
$$

Using the formulas for the wave operator according to Lemma 4.2

$$
\begin{aligned}
\dot{\square}_1\mathfrak{P}&=-\nabla_3\nabla_4\mathfrak{P}+\frac{1}{2}\overline{\mathcal{D}}\cdot(\mathcal{D}\widehat{\otimes}\mathfrak{P})+\left(2\underline{\omega}-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right)\nabla_4\mathfrak{P}\\
&\quad-\frac{1}{2}\overline{\mathrm{tr}X}\nabla_3\mathfrak{P}+(H+\overline{H})\cdot\nabla\mathfrak{P}\\
&\quad+\left(\frac{1}{4}tr\,\chi\mathrm{tr}\,\underline{\chi}+\frac{1}{4}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}+\rho-{}^{(\mathbf{F})}\rho^2-{}^{*\,(\mathbf{F})}\rho^2+i\left(-{}^{*}\rho+\eta\wedge\underline{\eta}\right)\right)\mathfrak{P},\\
\dot{\square}_2\mathfrak{Q}&=-\nabla_3\nabla_4\mathfrak{Q}+\frac{1}{2}\mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}}\cdot\mathfrak{Q})+\left(2\underline{\omega}-\frac{1}{2}\mathrm{tr}\underline{X}\right)\nabla_4\mathfrak{Q}
\end{aligned}
$$

$$
-\frac{1}{2}\mathrm{tr}X\nabla_3\mathfrak{Q}+(H+\overline{H})\cdot\nabla\mathfrak{Q}+\Bigg(-\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}-\frac{1}{2}{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}-2\rho
$$

$$
+2\,{}^{(\mathbf{F})}\rho^2+2\,{}^*\,{}^{(\mathbf{F})}\rho^2+i(-2\ {}^*\rho+2\eta\wedge\underline{\eta})\Bigg)\mathfrak{Q},
$$

we can rewrite the above as

$$
\dot{\square}_1\mathfrak{P}=\tfrac{5}{2}\overline{\mathrm{tr}X}\nabla_3\mathfrak{P}+\left(2\mathrm{tr}\underline{X}+\tfrac{1}{2}\overline{\mathrm{tr}\underline{X}}\right)\nabla_4\mathfrak{P}-\left(5H+\underline{H}+4\overline{\underline{H}}\right)\cdot\nabla\mathfrak{P}+\tilde{V}_1\mathfrak{P}
$$

$$
\mathcal{P}_{C_1}\left(\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]\right)+\tfrac{1}{2}\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]+L_{\mathfrak{P}}[\mathfrak{B},\mathfrak{F}], \tag{7.17}
$$

where

$$
\tilde{V}_1=\frac{9}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X}+4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}-9\overline{\underline{H}}\cdot H-\hat{V}_1+\frac{1}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}
$$

$$
+\frac{1}{4}{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}+\rho-\ {}^{(\mathbf{F})}\rho^2-\ {}^*\,{}^{(\mathbf{F})}\rho^2+i\left(-\ {}^*\rho+\eta\wedge\underline{\eta}\right) \tag{7.18}
$$

and

$$
\dot{\square}_2\mathfrak{Q}=\frac{3}{2}\overline{\mathrm{tr}X}\nabla_3\mathfrak{Q}+\left(\frac{1}{2}\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\nabla_4\mathfrak{Q}-\left(3H+2\,\underline{H}+\overline{\underline{H}}\right)\cdot\nabla\mathfrak{Q}+\tilde{V}_2\mathfrak{Q}
$$

$$
+\mathcal{P}_{C_2}\left(\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\right)+\frac{1}{2}\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]+L_{\mathfrak{Q}}[\mathfrak{B},\mathfrak{F}], \tag{7.19}
$$

where

$$
\tilde{V}_2=\frac{3}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X}+\frac{1}{4}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}X-3\overline{P}+P-4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}+\frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H-\eta\cdot\underline{\eta}
$$

$$
-i\eta\wedge\underline{\eta}-\hat{V}_2-\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}-\frac{1}{2}{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}-2\rho+2\,{}^{(\mathbf{F})}\rho^2+2\,{}^*\,{}^{(\mathbf{F})}\rho^2
$$

$$
+i(-2\ {}^*\rho+2\eta\wedge\underline{\eta}). \tag{7.20}
$$

7.3.3. *The rescaling from $\mathfrak{P}$ to $\mathfrak{p}$ and from $\mathfrak{Q}$ to $\mathfrak{q}^{\mathbf{F}}$*

Observe that the wave equations (7.17) and (7.19) satisfied by $\mathfrak{P}$ and $\mathfrak{Q}$ present first-order derivatives ∇_3, ∇_4 and ∇ on their right-hand side. In order to have only a first-order term of the form $i\nabla_t$, we need to define rescaled versions of $\mathfrak{P}$ and $\mathfrak{Q}$. The rescaling is obtained through functions of $q=r+ia\cos\theta$ and $\overline{q}=r-ia\cos\theta$, i.e.

$$
\mathfrak{p}=f_1(q,\overline{q})\mathfrak{P}\in\mathfrak{s}_1(\mathbb{C}),\qquad \mathfrak{q}^{\mathbf{F}}=f_2(q,\overline{q})\mathfrak{Q}\in\mathfrak{s}_2(\mathbb{C}).
$$

Proposition 7.5. *Let f_1 and f_2 be of the respective forms*

$$
f_1=(q)^{n_1}(\overline{q})^{5-n_1},\quad \textit{for any real } n_1,
$$

$$
f_2=(q)^{n_2}(\overline{q})^{3-n_2},\quad \textit{for any real } n_2,
$$

Then

$$\dot{\square}_1\mathfrak{p} = if_1\left[\frac{2a\cos\theta}{|q|^2}\nabla_t\mathfrak{P} + (1-2n_1)\left(\frac{2a\Delta\cos\theta}{|q|^4}\nabla_r\mathfrak{P} + \frac{2a\sin\theta r}{|q|^4}\nabla_\theta\mathfrak{P}\right)\right]$$
$$+\big(\tilde{V}_1 + f_1^{-1}\square(f_1)\big)\mathfrak{p}$$
$$+ f_1\left[\mathcal{P}_{C_1}\big(\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]\big) + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\mathbf{M}_1[\mathfrak{F},\mathfrak{X}] + L_{\mathfrak{P}}[\mathfrak{B},\mathfrak{F}]\right]$$

and

$$\dot{\square}_2\mathfrak{q}^{\mathbf{F}} = if_2\left[\frac{4a\cos\theta}{|q|^2}\nabla_t\mathfrak{Q} + (1-n_2)\left(\frac{4a\Delta\cos\theta}{|q|^4}\nabla_r\mathfrak{Q} + \frac{4a\sin\theta r}{|q|^4}\nabla_\theta\mathfrak{Q}\right)\right]$$
$$+\big(\tilde{V}_2 + f_2^{-1}\square(f_2)\big)\mathfrak{q}^{\mathbf{F}}$$
$$+ f_2\left[\mathcal{P}_{C_2}\big(\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\big) + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}] + L_{\mathfrak{Q}}[\mathfrak{B},\mathfrak{F}]\right].$$

In particular, observe that for $n_1 = \frac{1}{2}$ *and* $n_2 = 1$*, i.e.*

$$f_1 = (q)^{1/2}(\overline{q})^{9/2}, \quad f_2 = q\overline{q}^2, \tag{7.21}$$

the dependence on the ∇_r *and* ∇_θ *derivatives cancels out and we obtain*

$$\dot{\square}_1\mathfrak{p} = i\frac{2a\cos\theta}{|q|^2}\nabla_t\mathfrak{p} + \left(\tilde{V}_1 + f_1^{-1}\square(f_1)\right)\mathfrak{p}$$
$$+ f_1\left[\mathcal{P}_{C_1}\big(\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]\big) + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\mathbf{M}_1[\mathfrak{F},\mathfrak{X}] + L_{\mathfrak{P}}[\mathfrak{B},\mathfrak{F}]\right] \tag{7.22}$$

and

$$\dot{\square}_2\mathfrak{q}^{\mathbf{F}} = i\frac{4a\cos\theta}{|q|^2}\nabla_t\mathfrak{q}^{\mathbf{F}} + \left(\tilde{V}_2 + f_2^{-1}\square(f_2)\right)\mathfrak{q}^{\mathbf{F}}$$
$$+ f_2\left[\mathcal{P}_{C_2}\big(\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\big) + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}] + L_{\mathfrak{Q}}[\mathfrak{B},\mathfrak{F}]\right]. \tag{7.23}$$

Proof. See Appendix C.2. □

7.3.4. *The right-hand side of the equations*

Proposition 7.6. *Let* $\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]$ *and* $\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]$ *be the right-hand sides of the Teukolsky equations, as defined in* (6.7) *and* (6.8)*. Let* $\mathcal{P}_{C_1}$ *and* $\mathcal{P}_{C_2}$ *be the operators defined in* (7.1) *with* C_1 *and* C_2 *given by* (7.16)*. Then the following relations hold true:*

$$\mathcal{P}_{C_1}\big(\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]\big) + \frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]$$
$$= 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big(\overline{\mathcal{D}}\cdot\mathfrak{Q} + (2\underline{H} + \overline{H})\cdot\mathfrak{Q}\big) + L_{\mathbf{M}_1}[\mathfrak{B},\mathfrak{F},\mathfrak{X}],$$

$$\mathcal{P}_{C_2}\big(\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\big)+\frac{1}{2}\big(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\big)\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]$$
$$=\big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\mathfrak{Q}-\frac{1}{2}\big(\mathcal{D}\widehat{\otimes}\mathfrak{P}+(3H+2\,\underline{H})\widehat{\otimes}\mathfrak{P}\big)+L_{\mathbf{M}_2}[\mathfrak{B},\mathfrak{F},\mathfrak{X}],$$

where $L_{\mathbf{M}_1}[\mathfrak{F},\mathfrak{X}]$ *and* $L_{\mathbf{M}_2}[\mathfrak{B},\mathfrak{F},\mathfrak{X}]$ *are linear first-order operator in* $\mathfrak{B}$*,* $\mathfrak{F}$ *and* $\mathfrak{X}$*, given by*

$$L_{\mathbf{M}_1}[\mathfrak{B},\mathfrak{F},\mathfrak{X}]:=(4\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2)\ \mathfrak{B}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big(Y_a^{\mathfrak{F}}\,\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}+Y_0^{\mathfrak{F}}\cdot\mathfrak{F}$$
$$+Y_0^{\mathfrak{B}}\mathfrak{B}+Y_0^{\mathfrak{X}}\ \mathfrak{X}\big) \tag{7.24}$$

and

$$L_{\mathbf{M}_2}[\mathfrak{B},\mathfrak{F},\mathfrak{X}]:=-(4\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2)\ \mathfrak{F}+W_4^{\mathfrak{F}}\,{}^{(c)}\nabla_4\mathfrak{F}+W_a^{\mathfrak{F}}\cdot{}^{(c)}\nabla\mathfrak{F}+W_0^{\mathfrak{F}}\mathfrak{F}$$
$$+W_a^{\mathfrak{B}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}+W_0^{\mathfrak{B}}\widehat{\otimes}\mathfrak{B}+W_a^{\mathfrak{X}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X}+W_0^{\mathfrak{X}}\widehat{\otimes}\mathfrak{X}, \tag{7.25}$$

where $Y_a^{\mathfrak{F}}$*,* $Y_0^{\mathfrak{B}}$ *and* $Y_0^{\mathfrak{X}}$ *are complex functions of* (r,θ) *and* $Y_0^{\mathfrak{F}}$ *is a complex one-form, all of which vanish for zero angular momentum, having the following fall-off in* r*:*

$$Y_a^{\mathfrak{F}},Y_0^{\mathfrak{B}}=O\left(\frac{|a|}{r^2}\right),\quad Y_0^{\mathfrak{F}},Y_0^{\mathfrak{X}}=O\left(\frac{|a|}{r^3}\right)$$

and $W_4^{\mathfrak{F}}$*,* $W_0^{\mathfrak{F}}$*,* $W_a^{\mathfrak{B}}$ *and* $W_a^{\mathfrak{X}}$ *are complex functions of* (r,θ) *and* $W_a^{\mathfrak{F}}$*,* $W_0^{\mathfrak{B}}$ *and* $W_0^{\mathfrak{X}}$ *are complex one-forms, all of which vanish for zero angular momentum, having the following fall-off in* r*:*

$$W_a^{\mathfrak{B}}=O\left(\frac{|a|}{r^2}\right),\quad W_4^{\mathfrak{F}},W_a^{\mathfrak{F}},W_0^{\mathfrak{B}},W_a^{\mathfrak{X}}=O\left(\frac{|a|}{r^3}\right),\quad W_0^{\mathfrak{F}},W_0^{\mathfrak{X}}=O\left(\frac{|a|}{r^4}\right).$$

Proof. See Appendix C.3. ◻

Using (7.22) and (7.23) and the above proposition, we deduce

$$\dot{\square}_1\mathfrak{p}=i\frac{2a\cos\theta}{|q|^2}\nabla_t\mathfrak{p}+\big(\tilde{V}_1+f_1^{-1}\square(f_1)\big)\mathfrak{p}$$
$$+f_1\big[4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big(\overline{\mathcal{D}}\cdot\mathfrak{Q}+\big(2\overline{\underline{H}}+\overline{H}\big)\cdot\mathfrak{Q}\big)+L_{\mathbf{M}_1}[\mathfrak{B},\mathfrak{F},\mathfrak{X}]+L_{\mathfrak{P}}[\mathfrak{B},\mathfrak{F}]\big] \tag{7.26}$$

and

$$\dot{\square}_2\mathfrak{q}^{\mathbf{F}}=i\frac{4a\cos\theta}{|q|^2}\nabla_t\mathfrak{q}^{\mathbf{F}}+\left(\tilde{V}_2+f_2^{-1}\square(f_2)+3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\mathfrak{q}^{\mathbf{F}}$$
$$+f_2\left[-\frac{1}{2}\left(\mathcal{D}\widehat{\otimes}\mathfrak{P}+(3H+2\,\underline{H})\widehat{\otimes}\mathfrak{P}\right)+L_{\mathbf{M}_2}[\mathfrak{B},\mathfrak{F},\mathfrak{X}]+L_{\mathfrak{Q}}[\mathfrak{B},\mathfrak{F}]\right]. \tag{7.27}$$

We are now left to express the right-hand side in terms of $\mathfrak{p} = f_1\mathfrak{P}$ and $\mathfrak{q}^{\mathbf{F}} = f_2\mathfrak{Q}$. We write

$$\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{Q} = f_2^{-1}(\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{q}^{\mathbf{F}}) + \overline{{}^{(c)}\mathcal{D}}(f_2^{-1}) \cdot \mathfrak{q}^{\mathbf{F}} = f_2^{-1}(\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{q}^{\mathbf{F}}) - f_2^{-2}\,\overline{{}^{(c)}\mathcal{D}}(f_2) \cdot \mathfrak{q}^{\mathbf{F}},$$
$${}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{P} = f_1^{-1}({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{p}) + {}^{(c)}\mathcal{D}(f_1^{-1})\widehat{\otimes}\mathfrak{p} = f_1^{-1}({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{p}) - f_1^{-2}\,{}^{(c)}\mathcal{D}(f_1)\widehat{\otimes}\mathfrak{p}.$$

This implies

$$\dot{\square}_1\mathfrak{p} = i\frac{2a\cos\theta}{|q|^2}\nabla_t\mathfrak{p} + V_1\mathfrak{p} + 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}(f_1f_2^{-1})$$
$$\times\left(\overline{\mathcal{D}} \cdot \mathfrak{q}^{\mathbf{F}} + \left(2\,\overline{\underline{H}} + \overline{H} - f_2^{-1}\,\overline{{}^{(c)}\mathcal{D}}(f_2)\right) \cdot \mathfrak{q}^{\mathbf{F}}\right) + L_{\mathfrak{p}}[\mathfrak{B}, \mathfrak{F}] \tag{7.28}$$

and

$$\dot{\square}_2\mathfrak{q}^{\mathbf{F}} = i\frac{4a\cos\theta}{|q|^2}\nabla_t\mathfrak{q}^{\mathbf{F}} + V_2\mathfrak{q}^{\mathbf{F}} - \frac{1}{2}(f_2f_1^{-1})\big(\mathcal{D}\widehat{\otimes}\mathfrak{p}$$
$$+ \left(3H + 2\,\underline{H} - f_1^{-1}\,{}^{(c)}\mathcal{D}(f_1)\right)\widehat{\otimes}\mathfrak{p}\big) + L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B}, \mathfrak{F}], \tag{7.29}$$

where we define

$$V_1 := \tilde{V}_1 + f_1^{-1}\square(f_1), \tag{7.30}$$
$$V_2 := \tilde{V}_2 + f_2^{-1}\square(f_2) + 3\overline{P} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} \tag{7.31}$$

and

$$L_{\mathfrak{p}}[\mathfrak{B}, \mathfrak{F}] := f_1\left[L_{\mathbf{M}_1}[\mathfrak{B}, \mathfrak{F}, \mathfrak{X}] + L_{\mathfrak{P}}[\mathfrak{B}, \mathfrak{F}]\right], \tag{7.32}$$
$$L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B}, \mathfrak{F}] := f_2\left[L_{\mathbf{M}_2}[\mathfrak{B}, \mathfrak{F}, \mathfrak{X}] + L_{\mathfrak{Q}}[\mathfrak{B}, \mathfrak{F}]\right]. \tag{7.33}$$

We now simplify the coupling terms on the right-hand sides of the above. From (4.13), we deduce

$$\begin{aligned}\mathcal{D}(q^n\overline{q}^m) &= nq^{n-1}(\mathcal{D}q)\overline{q}^m + mq^n\overline{q}^{m-1}(\mathcal{D}\overline{q}) = (n\,\underline{H} + mH)q^n\overline{q}^m,\\ \overline{\mathcal{D}}(q^n\overline{q}^m) &= (m\,\overline{\underline{H}} + n\overline{H})q^n\overline{q}^m.\end{aligned} \tag{7.34}$$

We therefore have, for $f_1 = (q)^{1/2}(\overline{q})^{9/2}$ and $f_2 = q\overline{q}^2$:

$$2\,\overline{\underline{H}} + \overline{H} - f_2^{-1}\,\overline{{}^{(c)}\mathcal{D}}(f_2) = 2\,\overline{\underline{H}} + \overline{H} - (2\,\overline{\underline{H}} + \overline{H}) = 0,$$
$$3H + 2\,\underline{H} - f_1^{-1}\,{}^{(c)}\mathcal{D}(f_1) = 3H + 2\,\underline{H} - \left(\frac{1}{2}\,\underline{H} + \frac{9}{2}H\right) = -\frac{3}{2}H + \frac{3}{2}\,\underline{H}.$$

We also write

$$4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}(f_1f_2^{-1}) = 4\frac{Q}{\overline{q}^2}\frac{Q}{q^2}(q)^{1/2}(\overline{q})^{9/2}(q)^{-1}(\overline{q})^{-2} = 4Q^2\frac{\overline{q}^{1/2}}{q^{5/2}} = \frac{4Q^2}{|q|^5}\overline{q}^3,$$
$$(f_2f_1^{-1}) = q\overline{q}^2(q)^{-1/2}(\overline{q})^{-9/2} = \frac{q^{1/2}}{\overline{q}^{5/2}} = \frac{q^3}{|q|^5}.$$

We therefore finally obtain

$$\dot{\square}_1\mathfrak{p} - i\frac{2a\cos\theta}{|q|^2}\nabla_t\mathfrak{p} - V_1\mathfrak{p} = 4Q^2\frac{\overline{q}^3}{|q|^5}\left(\overline{\mathcal{D}}\cdot\mathfrak{q}^{\mathbf{F}}\right) + L_{\mathfrak{p}}[\mathfrak{B},\mathfrak{F}], \tag{7.35}$$

$$\dot{\square}_2\mathfrak{q}^{\mathbf{F}} - i\frac{4a\cos\theta}{|q|^2}\nabla_t\mathfrak{q}^{\mathbf{F}} - V_2\mathfrak{q}^{\mathbf{F}} = -\frac{1}{2}\frac{q^3}{|q|^5}\left(\mathcal{D}\widehat{\otimes}\mathfrak{p} - \frac{3}{2}\left(H - \underline{H}\right)\widehat{\otimes}\mathfrak{p}\right) + L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B},\mathfrak{F}]. \tag{7.36}$$

7.3.5. *The potentials of the equations*

In this section, we compute the potentials V_1 and V_2 as obtained in (7.30) and (7.31). We determine the imaginary parts of the complex functions $C_1 = 2\mathrm{tr}\,\underline{\chi} + ip_1\,{}^{(a)}\mathrm{tr}\underline{\chi}$ and $C_2 = \mathrm{tr}\,\underline{\chi} + ip_2\,{}^{(a)}\mathrm{tr}\underline{\chi}$ given in (7.16) such that the imaginary part of the potentials vanish.

Proposition 7.7. *Choosing $p_1 = -\frac{5}{2}$ and $p_2 = -3$ in the definition of C_1 and C_2* (7.16), *i.e. for*

$$C_1 = 2\mathrm{tr}\,\underline{\chi} - \frac{5}{2}i\,{}^{(a)}tr\underline{\chi}, \tag{7.37}$$

$$C_2 = \mathrm{tr}\,\underline{\chi} - 3i\,{}^{(a)}tr\underline{\chi} \tag{7.38}$$

the potentials V_1 and V_2 in Eqs. (7.35) *and* (7.36) *are real, i.e.* $\Im(V_1) = \Im(V_2) = 0$, *and are given by*

$$V_1 = -\frac{1}{4}tr\,\chi\mathrm{tr}\,\underline{\chi} + \mathrm{div}\underline{\eta} + \frac{1}{2}\eta\cdot\underline{\eta} + \frac{1}{2}|\underline{\eta}|^2 + 5\,{}^{(\mathbf{F})}\rho^2 + 5\,{}^{*\,(\mathbf{F})}\rho^2,$$

$$V_2 = -tr\,\chi\mathrm{tr}\,\underline{\chi} + 4\mathrm{div}\underline{\eta} + 2\eta\cdot\underline{\eta} + 2|\underline{\eta}|^2 + 2\,{}^{(\mathbf{F})}\rho^2 + 2\,{}^{*\,(\mathbf{F})}\rho^2.$$

In particular, modulo $O(|a|)$ terms, we have

$$V_1 = -\frac{1}{4}tr\,\chi\mathrm{tr}\,\underline{\chi} + 5\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^3}\right),$$

$$V_2 = -tr\,\chi\mathrm{tr}\,\underline{\chi} + 2\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^3}\right).$$

Proof. See Appendix C.4. ◻

Using the values in Kerr–Newman given in Sec. 4.1, we obtain

$$-\frac{1}{4}tr\,\chi\mathrm{tr}\,\underline{\chi} + \mathrm{div}\underline{\eta} + \frac{1}{2}\eta\cdot\underline{\eta} + \frac{1}{2}|\underline{\eta}|^2 = \frac{r^4 - 2Mr^3 + (2 - 3\cos^2\theta)a^2r^2 + Q^2r^2 - 2a^4\cos^2\theta}{|q|^6}$$

and therefore, explicitly,

$$
\begin{aligned}
V_1 &= \frac{r^4 - 2Mr^3 + (2-3\cos^2\theta)a^2r^2 + Q^2r^2 - 2a^4\cos^2\theta}{|q|^6} + \frac{5Q^2}{|q|^4},\\
V_2 &= 4\frac{r^4 - 2Mr^3 + (2-3\cos^2\theta)a^2r^2 + Q^2r^2 - 2a^4\cos^2\theta}{|q|^6} + \frac{2Q^2}{|q|^4}.
\end{aligned} \tag{7.39}
$$

7.3.6. *The lower order terms*

We finally simplify the lower order terms $L_{\mathfrak{p}}[\mathfrak{B},\mathfrak{F}]$ and $L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B},\mathfrak{F}]$ as defined in (7.32) and (7.33).

Using (7.14) and (7.24), we obtain

$$
\begin{aligned}
L_{\mathfrak{p}}[\mathfrak{B},\mathfrak{F}] &= f_1\big[L_{\mathbf{M}_1}[\mathfrak{B},\mathfrak{F},\mathfrak{X}] + L_{\mathfrak{P}}[\mathfrak{B},\mathfrak{F}]\big]\\
&= q^{1/2}\overline{q}^{9/2}\big[(4\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2)\mathfrak{B} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\\
&\quad\times\big(Y_a^{\mathfrak{F}}\,\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + Y_0^{\mathfrak{F}}\cdot\mathfrak{F} + Y_0^{\mathfrak{B}}\mathfrak{B} + Y_0^{\mathfrak{X}}\,\mathfrak{X}\big)\\
&\quad - Z_a^{\mathfrak{B}}\cdot{}^{(c)}\nabla\mathfrak{B} - \big(4\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 + Z_0^{\mathfrak{B}}\big)\mathfrak{B}\big]\\
&= q^{1/2}\overline{q}^{9/2}\big[-Z_a^{\mathfrak{B}}\cdot{}^{(c)}\nabla\mathfrak{B} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\;Y_a^{\mathfrak{F}}(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F})\\
&\quad + (2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\;Y_0^{\mathfrak{B}} - Z_0^{\mathfrak{B}})\mathfrak{B} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big(Y_0^{\mathfrak{F}}\cdot\mathfrak{F} + Y_0^{\mathfrak{X}}\,\mathfrak{X}\big)\big].
\end{aligned}
$$

Using (7.15) and (7.25), we obtain

$$
\begin{aligned}
L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B},\mathfrak{F}] &= f_2\big[L_{\mathbf{M}_2}[\mathfrak{B},\mathfrak{F},\mathfrak{X}] + L_{\mathfrak{Q}}[\mathfrak{B},\mathfrak{F}]\big]\\
&= q\overline{q}^2\big[-(4\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2)\mathfrak{F} + W_4^{\mathfrak{F}}\,{}^{(c)}\nabla_4\mathfrak{F} + W_a^{\mathfrak{F}}\cdot{}^{(c)}\nabla\mathfrak{F} + W_0^{\mathfrak{F}}\mathfrak{F}\\
&\quad + W_a^{\mathfrak{B}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} + W_0^{\mathfrak{B}}\widehat{\otimes}\mathfrak{B} + W_a^{\mathfrak{X}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} + W_0^{\mathfrak{X}}\widehat{\otimes}\mathfrak{X} - Z_a^{\mathfrak{F}}\cdot{}^{(c)}\nabla\mathfrak{F}\\
&\quad + \big(4\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 - Z_0^{\mathfrak{F}}\big)\mathfrak{F}\big]\\
&= q\overline{q}^2\big[W_4^{\mathfrak{F}}\,{}^{(c)}\nabla_4\mathfrak{F} + \big(W_a^{\mathfrak{F}} - Z_a^{\mathfrak{F}}\big)\cdot{}^{(c)}\nabla\mathfrak{F} + W_a^{\mathfrak{X}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} + W_a^{\mathfrak{B}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\\
&\quad + (W_0^{\mathfrak{F}} - Z_0^{\mathfrak{F}})\mathfrak{F} + W_0^{\mathfrak{B}}\widehat{\otimes}\mathfrak{B} + W_0^{\mathfrak{X}}\widehat{\otimes}\mathfrak{X}\big].
\end{aligned}
$$

We now summarize in the following the structure of the above terms.

Lemma 7.8. *With the above choices of C_1 and C_2, we have that the highest order terms in $L_{\mathfrak{p}}[\mathfrak{B},\mathfrak{F}]$ and $L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B},\mathfrak{F}]$ satisfies the following*:

- $W_4^{\mathfrak{F}}$ *and* $W_a^{\mathfrak{X}}$ *are real functions*,
- $Z_a^{\mathfrak{B}}$ *and* $W_a^{\mathfrak{F}} - Z_a^{\mathfrak{F}}$ *are real one-forms*,
- $W_a^{\mathfrak{B}} = \frac{3}{4}i\,{}^{(a)}\mathrm{tr}\underline{\chi}$ *and* $Y_a^{\mathfrak{F}} = -3i\,{}^{(a)}\mathrm{tr}\underline{\chi}$.

Proof. See Appendix C.5. □

7.4. *Sketch of boundedness of the energy*

We here sketch how to prove boundedness of the energy for the system of equations (7.6) and (7.7) as obtained in Theorem 7.3:

$$\dot{\square}_1\mathfrak{p} - i\frac{2a\cos\theta}{|q|^2}\nabla_t\mathfrak{p} - V_1\mathfrak{p} = 4Q^2\frac{\overline{q}^3}{|q|^5}\left(\overline{\mathcal{D}}\cdot\mathfrak{q}^{\mathbf{F}}\right) + L_{\mathfrak{p}}[\mathfrak{B},\mathfrak{F}],$$

$$\dot{\square}_2\mathfrak{q}^{\mathbf{F}} - i\frac{4a\cos\theta}{|q|^2}\nabla_t\mathfrak{q}^{\mathbf{F}} - V_2\mathfrak{q}^{\mathbf{F}} = -\frac{1}{2}\frac{q^3}{|q|^5}\left(\mathcal{D}\widehat{\otimes}\mathfrak{p} - \frac{3}{2}\left(H - \underline{H}\right)\widehat{\otimes}\mathfrak{p}\right) + L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B},\mathfrak{F}].$$

To fully close energy estimates, we need to combine them with spacetime local integrated Morawetz estimates, which will be done in a future work [18] by making use of the hidden symmetry in Kerr–Newman to avoid decomposition in modes. Nevertheless, in this section, we show that all the crucial structures obtained in Theorem 7.3 are precisely what one needs to perform energy estimates, once the Morawetz estimate, which is less sensitive to the structure of the lower order terms, are achieved.

As a general rule,[i] in order to obtain energy estimates for the wave equation $\square\psi = 0$, we multiply the equation by $\nabla_t\psi$, and integrate by parts. Since we are dealing with complex tensors, we then multiply the equation $\dot{\square}_1\mathfrak{p}$ by $\nabla_t\overline{\mathfrak{p}}$ and the equation $\dot{\square}_2\mathfrak{q}^{\mathbf{F}}$ by $\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}$, respectively, and then add the conjugate of each one to take the real part.

Doing so, we obtain from each one of the above equations the following:

$$\begin{aligned}
&\dot{\square}_1\mathfrak{p}\cdot\nabla_t\overline{\mathfrak{p}} + \dot{\square}_1\overline{\mathfrak{p}}\cdot\nabla_t\mathfrak{p}\\
&\quad = i\frac{2a\cos\theta}{|q|^2}\nabla_t\mathfrak{p}\cdot\nabla_t\overline{\mathfrak{p}} + \overline{i\frac{2a\cos\theta}{|q|^2}}\nabla_t\mathfrak{p}\cdot\nabla_t\overline{\mathfrak{p}} + V_1\mathfrak{p}\cdot\nabla_t\overline{\mathfrak{p}} + \overline{V_1}\overline{\mathfrak{p}}\cdot\nabla_t\mathfrak{p}\\
&\qquad + 4Q^2\frac{\overline{q}^3}{|q|^5}\left(\overline{\mathcal{D}}\cdot\mathfrak{q}^{\mathbf{F}}\right)\cdot\nabla_t\overline{\mathfrak{p}} + 4Q^2\frac{q^3}{|q|^5}\left(\mathcal{D}\cdot\overline{\mathfrak{q}^{\mathbf{F}}}\right)\cdot\nabla_t\mathfrak{p}\\
&\qquad + L_{\mathfrak{p}}[\mathfrak{B},\mathfrak{F}]\cdot\nabla_t\overline{\mathfrak{p}} + \overline{L_{\mathfrak{p}}[\mathfrak{B},\mathfrak{F}]}\cdot\nabla_t\mathfrak{p}
\end{aligned}\tag{7.40}$$

and

$$\begin{aligned}
&\dot{\square}_2\mathfrak{q}^{\mathbf{F}}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}} + \dot{\square}_2\overline{\mathfrak{q}^{\mathbf{F}}}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}\\
&\quad = i\frac{4a\cos\theta}{|q|^2}\nabla_t\mathfrak{q}^{\mathbf{F}}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}} + \overline{i\frac{4a\cos\theta}{|q|^2}}\nabla_t\mathfrak{q}^{\mathbf{F}}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}} + V_2\mathfrak{q}^{\mathbf{F}}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}} + \overline{V_2}\overline{\mathfrak{q}^{\mathbf{F}}}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}\\
&\quad - \frac{1}{2}\frac{q^3}{|q|^5}\left(\mathcal{D}\widehat{\otimes}\mathfrak{p}\right)\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}} - \frac{1}{2}\frac{\overline{q}^3}{|q|^5}\left(\overline{\mathcal{D}\widehat{\otimes}\mathfrak{p}}\right)\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}
\end{aligned}$$

[i]In the case of Kerr–Newman, in the ergoregion we need to multiply by the timelike $\partial_t + \frac{a}{r^2+a^2}\partial_\varphi$. The analysis is identical since the term involving the ∂_φ can be absorbed for small a by the non-degenerate Morawetz estimates away from the trapping region.

$$+\frac{3}{4}\frac{q^3}{|q|^5}\left((H-\underline{H})\widehat{\otimes}\mathfrak{p}\right)\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}+\frac{3}{4}\frac{\overline{q}^3}{|q|^5}\left((\overline{H}-\overline{\underline{H}})\widehat{\otimes}\overline{\mathfrak{p}}\right)\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}$$
$$+L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B},\mathfrak{F}]\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}+\overline{L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B},\mathfrak{F}]}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}. \tag{7.41}$$

We now analyze each term on the left-hand side.

(1) The structure of the first-order terms ∇_t in the equations of the form $if(r,\theta)\nabla_t$, for a real function $f(r,\theta)$ is crucial for the cancellation of these terms. Indeed,

$$i\frac{2a\cos\theta}{|q|^2}\nabla_t\mathfrak{p}\cdot\nabla_t\overline{\mathfrak{p}}+\overline{i\frac{2a\cos\theta}{|q|^2}}\nabla_t\mathfrak{p}\cdot\nabla_t\overline{\mathfrak{p}}$$
$$=i\frac{2a\cos\theta}{|q|^2}\nabla_t\mathfrak{p}\cdot\nabla_t\overline{\mathfrak{p}}-i\frac{2a\cos\theta}{|q|^2}\nabla_t\mathfrak{p}\cdot\nabla_t\overline{\mathfrak{p}}=0,$$
$$i\frac{4a\cos\theta}{|q|^2}\nabla_t\mathfrak{q}^{\mathbf{F}}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}+\overline{i\frac{4a\cos\theta}{|q|^2}}\nabla_t\mathfrak{q}^{\mathbf{F}}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}$$
$$=i\frac{4a\cos\theta}{|q|^2}\nabla_t\mathfrak{q}^{\mathbf{F}}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}-i\frac{4a\cos\theta}{|q|^2}\nabla_t\mathfrak{q}^{\mathbf{F}}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}=0.$$

(2) The reality of the potentials V_1 and V_2 allows to write the terms involving the potential as boundary terms in the usual way:

$$V_1\mathfrak{p}\cdot\nabla_t\overline{\mathfrak{p}}+\overline{V_1\mathfrak{p}}\cdot\nabla_t\mathfrak{p}=V_1\left(\mathfrak{p}\cdot\nabla_t\overline{\mathfrak{p}}+\overline{\mathfrak{p}}\cdot\nabla_t\mathfrak{p}\right)=V_1\partial_t(|\mathfrak{p}|^2)=\partial_t(V_1|\mathfrak{p}|^2),$$
$$V_2\mathfrak{q}^{\mathbf{F}}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}+\overline{V_2\mathfrak{q}^{\mathbf{F}}}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}=V_2\left(\mathfrak{q}^{\mathbf{F}}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}+\overline{\mathfrak{q}^{\mathbf{F}}}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}\right)$$
$$=V_2\partial_t(|\mathfrak{q}^{\mathbf{F}}|^2)=\partial_t(V_2|\mathfrak{q}^{\mathbf{F}}|^2).$$

Being V_1 and V_2 positive for $|a|/M\ll 1$, they give a coercive contribution to the energies.

(3) In order to obtain cancellation for the terms involving coupling, we need to sum the estimates for the two equations. Observe that the complex functions which multiply the coupling terms, i.e. $\frac{\overline{q}^3}{|q|^5}$ and $\frac{q^3}{|q|^5}$, are conjugate complex functions, and such structure is crucial for the cancellation. Since the coupling terms differ by a constant factor $8Q^2$, we multiply the second identity (7.41) by $8Q^2$ and sum to (7.40) and obtain

$$\dot{\square}_1\mathfrak{p}\cdot\nabla_t\overline{\mathfrak{p}}+\dot{\square}_1\overline{\mathfrak{p}}\cdot\nabla_t\mathfrak{p}-\partial_t(V_1|\mathfrak{p}|^2)$$
$$+8Q^2\left(\dot{\square}_2\mathfrak{q}^{\mathbf{F}}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}+\dot{\square}_2\overline{\mathfrak{q}^{\mathbf{F}}}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}-\partial_t(V_2|\mathfrak{q}^{\mathbf{F}}|^2)\right)$$
$$=4Q^2\frac{\overline{q}^3}{|q|^5}\left(\overline{\mathcal{D}}\cdot\mathfrak{q}^{\mathbf{F}}\right)\cdot\nabla_t\overline{\mathfrak{p}}+4Q^2\frac{q^3}{|q|^5}\left(\mathcal{D}\cdot\overline{\mathfrak{q}^{\mathbf{F}}}\right)\cdot\nabla_t\mathfrak{p}$$
$$-4Q^2\frac{q^3}{|q|^5}\left(\mathcal{D}\widehat{\otimes}\mathfrak{p}\right)\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}-4Q^2\frac{\overline{q}^3}{|q|^5}\left(\overline{\mathcal{D}\widehat{\otimes}\mathfrak{p}}\right)\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}$$

$$+ 6Q^2 \frac{q^3}{|q|^5} \left((H - \underline{H}) \widehat{\otimes} \mathfrak{p} \right) \cdot \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}} + 6Q^2 \frac{\overline{q}^3}{|q|^5} \left((\overline{H} - \overline{\underline{H}}) \widehat{\otimes} \overline{\mathfrak{p}} \right) \cdot \nabla_t \mathfrak{q}^{\mathbf{F}}$$
$$+ L_{\mathfrak{p}}[\mathfrak{B}, \mathfrak{F}] \cdot \nabla_t \overline{\mathfrak{p}} + \overline{L_{\mathfrak{p}}[\mathfrak{B}, \mathfrak{F}]} \cdot \nabla_t \mathfrak{p}$$
$$+ 8Q^2 (L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B}, \mathfrak{F}] \cdot \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}} + \overline{L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B}, \mathfrak{F}]} \cdot \nabla_t \mathfrak{q}^{\mathbf{F}}).$$

We now consider the first two lines on the right-hand side of the above. We put together the terms which are multiplied by the function $\frac{\overline{q}^3}{|q|^5}$ and those multiplied by $\frac{q^3}{|q|^5}$. We first integrate by parts in t in the first term, and obtain

$$2Q^2 \left(2 \frac{\overline{q}^3}{|q|^5} \left(\overline{\mathcal{D}} \cdot \mathfrak{q}^{\mathbf{F}} \right) \cdot \nabla_t \overline{\mathfrak{p}} - 2 \frac{\overline{q}^3}{|q|^5} \left(\overline{\mathcal{D} \widehat{\otimes} \mathfrak{p}} \right) \cdot \nabla_t \mathfrak{q}^{\mathbf{F}} + 3 \frac{\overline{q}^3}{|q|^5} \left((\overline{H} - \overline{\underline{H}}) \widehat{\otimes} \overline{\mathfrak{p}} \right) \cdot \nabla_t \mathfrak{q}^{\mathbf{F}} \right)$$
$$+ 2Q^2 \left(2 \frac{q^3}{|q|^5} \left(\mathcal{D} \cdot \overline{\mathfrak{q}^{\mathbf{F}}} \right) \cdot \nabla_t \mathfrak{p} - 2 \frac{q^3}{|q|^5} \left(\mathcal{D} \widehat{\otimes} \mathfrak{p} \right) \cdot \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}} \right.$$
$$\left. + 3 \frac{q^3}{|q|^5} \left((H - \underline{H}) \widehat{\otimes} \mathfrak{p} \right) \cdot \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}} \right)$$
$$= 2Q^2 \left(-2 \frac{\overline{q}^3}{|q|^5} \left(\overline{\mathcal{D}} \cdot \nabla_t \mathfrak{q}^{\mathbf{F}} \right) \cdot \overline{\mathfrak{p}} - 2 \frac{\overline{q}^3}{|q|^5} \left(\overline{\mathcal{D} \widehat{\otimes} \mathfrak{p}} \right) \cdot \nabla_t \mathfrak{q}^{\mathbf{F}} \right.$$
$$\left. + 3 \frac{\overline{q}^3}{|q|^5} \left((\overline{H} - \overline{\underline{H}}) \widehat{\otimes} \overline{\mathfrak{p}} \right) \cdot \nabla_t \mathfrak{q}^{\mathbf{F}} \right)$$
$$+ 2Q^2 \left(-2 \frac{q^3}{|q|^5} \left(\mathcal{D} \cdot \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}} \right) \cdot \mathfrak{p} - 2 \frac{q^3}{|q|^5} \left(\mathcal{D} \widehat{\otimes} \mathfrak{p} \right) \cdot \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}} \right.$$
$$\left. + 3 \frac{q^3}{|q|^5} \left((H - \underline{H}) \widehat{\otimes} \mathfrak{p} \right) \cdot \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}} \right).$$

Recall Lemma 2.11 that relates the operator $\mathcal{D}\widehat{\otimes}$ and $\mathcal{D}\cdot$. Applying it to $F = \mathfrak{p}$, $U = \nabla_t \mathfrak{q}^{\mathbf{F}}$, we obtain

$$(\mathcal{D} \widehat{\otimes} \mathfrak{p}) \cdot \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}} = -\mathfrak{p} \cdot (\mathcal{D} \cdot \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}}) - ((H + \underline{H}) \widehat{\otimes} \mathfrak{p}) \cdot \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}} + \mathbf{D}_\alpha (\mathfrak{p} \cdot \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}}).$$

Using the above we write, modulo spacetime divergence terms:

$$-2 \frac{q^3}{|q|^5} \left(\mathcal{D} \cdot \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}} \right) \cdot \mathfrak{p}$$
$$= 2\mathcal{D} \left(\frac{q^3}{|q|^5} \right) \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}} \cdot \mathfrak{p} + 2 \frac{q^3}{|q|^5} \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}} \cdot (\mathcal{D} \widehat{\otimes} \mathfrak{p}) + 2 \frac{q^3}{|q|^5} ((H + \underline{H}) \widehat{\otimes} \mathfrak{p}) \cdot \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}}$$
$$= 2\mathcal{D}(q^{1/2} (\overline{q})^{-5/2}) \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}} \cdot \mathfrak{p} + 2 \frac{q^3}{|q|^5} \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}} \cdot (\mathcal{D} \widehat{\otimes} \mathfrak{p})$$
$$+ 2 \frac{q^3}{|q|^5} ((H + \underline{H}) \widehat{\otimes} \mathfrak{p}) \cdot \nabla_t \overline{\mathfrak{q}^{\mathbf{F}}}$$

$$
\begin{aligned}
&= (\,\underline{H} - 5H)\frac{q^3}{|q|^5}\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}\cdot\mathfrak{p} + 2\frac{q^3}{|q|^5}\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}\cdot(\mathcal{D}\widehat{\otimes}\mathfrak{p})\\
&\quad + 2\frac{q^3}{|q|^5}((H+\,\underline{H})\widehat{\otimes}\mathfrak{p})\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}\\
&= (3\,\underline{H} - 3H)\frac{q^3}{|q|^5}\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}\cdot\mathfrak{p} + 2\frac{q^3}{|q|^5}\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}\cdot(\mathcal{D}\widehat{\otimes}\mathfrak{p})
\end{aligned}
$$

since from (7.34) we have $\mathcal{D}(q^{1/2}\overline{q}^{-5/2}) = (\frac{1}{2}\,\underline{H} - \frac{5}{2}H)\frac{q^3}{|q|^5}$. Similarly,

$$
-2\frac{\overline{q}^3}{|q|^5}\left(\overline{\mathcal{D}}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}\right)\cdot\overline{\mathfrak{p}} = (3\,\overline{\underline{H}} - 3\overline{H})\frac{\overline{q}^3}{|q|^5}\nabla_t\mathfrak{q}^{\mathbf{F}}\cdot\overline{\mathfrak{p}} + 2\frac{\overline{q}^3}{|q|^5}\nabla_t\mathfrak{q}^{\mathbf{F}}\cdot(\overline{\mathcal{D}\widehat{\otimes}\mathfrak{p}}).
$$

We finally obtain

$$
\begin{aligned}
&2Q^2\left(-2\frac{\overline{q}^3}{|q|^5}\left(\overline{\mathcal{D}}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}\right)\cdot\overline{\mathfrak{p}} - 2\frac{\overline{q}^3}{|q|^5}\left(\overline{\mathcal{D}\widehat{\otimes}\mathfrak{p}}\right)\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}\right.\\
&\qquad\left. + 3\frac{\overline{q}^3}{|q|^5}\left(\left(\overline{H} - \overline{\underline{H}}\right)\widehat{\otimes}\overline{\mathfrak{p}}\right)\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}\right) + 2Q^2\left(-2\frac{q^3}{|q|^5}\left(\mathcal{D}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}\right)\cdot\mathfrak{p}\right.\\
&\qquad\left. - 2\frac{q^3}{|q|^5}\left(\mathcal{D}\widehat{\otimes}\mathfrak{p}\right)\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}} + 3\frac{q^3}{|q|^5}\left((H - \,\underline{H})\widehat{\otimes}\mathfrak{p}\right)\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}\right)\\
&\quad = 2Q^2\frac{\overline{q}^3}{|q|^5}\Big((3\,\overline{\underline{H}} - 3\overline{H})\nabla_t\mathfrak{q}^{\mathbf{F}}\cdot\overline{\mathfrak{p}} + 2\nabla_t\mathfrak{q}^{\mathbf{F}}\cdot(\overline{\mathcal{D}\widehat{\otimes}\mathfrak{p}}) - 2\big(\overline{\mathcal{D}\widehat{\otimes}\mathfrak{p}}\big)\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}\\
&\qquad + 3\big(\big(\overline{H} - \overline{\underline{H}}\big)\widehat{\otimes}\overline{\mathfrak{p}}\big)\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}\Big) + 2Q^2\frac{q^3}{|q|^5}\Big((3\,\underline{H} - 3H)\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}\cdot\mathfrak{p}\\
&\qquad + 2\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}\cdot(\mathcal{D}\widehat{\otimes}\mathfrak{p}) - 2\big(\mathcal{D}\widehat{\otimes}\mathfrak{p}\big)\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}} + 3\big((H - \,\underline{H})\widehat{\otimes}\mathfrak{p}\big)\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}\Big)\\
&\quad = 0.
\end{aligned}
$$

Observe that upon a spacetime integration, the coupling terms cancel out, and therefore they only give contributions to boundary terms. Even though those terms do not have a definite sign, the modified energy terms are positive in the case of Reissner–Nordström for $|Q| < M$, as proved in [15]. In particular, for small angular momentum $|a| \ll M$ they remain positive in Kerr–Newman.

By putting the above together we have

$$
\begin{aligned}
&\dot{\square}_1\mathfrak{p}\cdot\nabla_t\overline{\mathfrak{p}} + \dot{\square}_1\overline{\mathfrak{p}}\cdot\nabla_t\mathfrak{p} - \partial_t(V_1|\mathfrak{p}|^2) + 8Q^2\big(\dot{\square}_2\mathfrak{q}^{\mathbf{F}}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}} + \dot{\square}_2\overline{\mathfrak{q}^{\mathbf{F}}}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}\\
&\qquad - \partial_t(V_2|\mathfrak{q}^{\mathbf{F}}|^2)\big) - \partial_t\left(\frac{4Q^2}{|q|^5}\overline{q}^3\left(\overline{\mathcal{D}}\cdot\mathfrak{q}^{\mathbf{F}}\right)\cdot\overline{\mathfrak{p}} + \frac{4Q^2}{|q|^5}q^3\left(\mathcal{D}\cdot\overline{\mathfrak{q}^{\mathbf{F}}}\right)\cdot\mathfrak{p}\right)\\
&\quad = L_{\mathfrak{p}}[\mathfrak{B},\mathfrak{F}]\cdot\nabla_t\overline{\mathfrak{p}} + \overline{L_{\mathfrak{p}}[\mathfrak{B},\mathfrak{F}]}\cdot\nabla_t\mathfrak{p} + 8Q^2(L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B},\mathfrak{F}]\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}\\
&\qquad + \overline{L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B},\mathfrak{F}]}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}).
\end{aligned}
$$

(4) In order to absorb the lower order terms on the right of the above estimates, one needs to combine the above energy estimates with boundedness of trapped spacetime energies, as given by Morawetz estimates. Moreover, through transport estimates one can show to bound all first derivatives of $\mathfrak{B}$, $\mathfrak{F}$ and $\mathfrak{X}$ by a degenerate Morawetz bulk for $\mathfrak{p}$ and $\mathfrak{q}^{\mathbf{F}}$.

Assuming such estimates, we briefly explain how to absorb the lower order terms above. Recall that

$$\begin{aligned} L_{\mathfrak{p}}[\mathfrak{B},\mathfrak{F}] &= q^{1/2}\overline{q}^{9/2}\big[-Z_a^{\mathfrak{B}}\cdot {}^{(c)}\nabla\mathfrak{B} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\;Y_a^{\mathfrak{F}}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}\big) \\ &\quad + \big(2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\;Y_0^{\mathfrak{B}} - Z_0^{\mathfrak{B}}\big)\mathfrak{B} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big(Y_0^{\mathfrak{F}}\cdot\mathfrak{F} + Y_0^{\mathfrak{X}}\;\mathfrak{X}\big)\big] \end{aligned}$$

and

$$\begin{aligned} L_{\mathfrak{q}^{\mathbf{F}}}[\mathfrak{B},\mathfrak{F}] &= q\overline{q}^2\big[W_4^{\mathfrak{F}}\,{}^{(c)}\nabla_4\mathfrak{F} + \big(W_a^{\mathfrak{F}} - Z_a^{\mathfrak{F}}\big)\cdot{}^{(c)}\nabla\mathfrak{F} + W_a^{\mathfrak{X}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} \\ &\quad + W_a^{\mathfrak{B}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} + \big(W_0^{\mathfrak{F}} - Z_0^{\mathfrak{F}}\big)\mathfrak{F} + W_0^{\mathfrak{B}}\widehat{\otimes}\mathfrak{B} + W_0^{\mathfrak{X}}\widehat{\otimes}\mathfrak{X}\big] \end{aligned}$$

The terms on the second line of the above expressions (i.e. the lowest order terms) can be absorbed for small $|a| \ll M$, by integration by parts in t and then bounding by Cauchy–Schwarz. For example,

$$\begin{aligned} &q^{1/2}\overline{q}^{9/2}\big(2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\;Y_0^{\mathfrak{B}} - Z_0^{\mathfrak{B}}\big)\mathfrak{B}\cdot\nabla_t\overline{\mathfrak{p}} \\ &\quad = -q^{1/2}\overline{q}^{9/2}\big(2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\;Y_0^{\mathfrak{B}} - Z_0^{\mathfrak{B}}\big)\nabla_t\mathfrak{B}\cdot\overline{\mathfrak{p}} \\ &\quad \leq O(ar)\big(|\nabla_t\mathfrak{B}|^2 + |\mathfrak{p}|^2\big). \end{aligned}$$

Both terms on the right-hand side appear without degeneracy at the trapping region in the Morawetz bulks, and therefore they can be absorbed by that for small $|a| \ll M$. The same will be true for the other terms of lower order, which contains only one derivative of $\mathfrak{B}$, $\mathfrak{F}$ or $\mathfrak{X}$.

In what follows, we therefore only look at the terms which highest number of derivatives, since the lower order terms can be treated as above. We now consider

$$q\overline{q}^2 W_4^{\mathfrak{F}}\,{}^{(c)}\nabla_4\mathfrak{F}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}} + \overline{q\overline{q}^2 W_4^{\mathfrak{F}}\,{}^{(c)}\nabla_4\mathfrak{F}}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}.$$

Since $W_4^{\mathfrak{F}}$ is real, we have

$$\begin{aligned} &= W_4^{\mathfrak{F}}\big(q\overline{q}^2\,{}^{(c)}\nabla_4\mathfrak{F}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}} + q^2\overline{q}\,{}^{(c)}\nabla_4\overline{\mathfrak{F}}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}\big) \\ &= W_4^{\mathfrak{F}}\big(q\overline{q}^2\nabla_t\mathfrak{F}\cdot\nabla_4\overline{\mathfrak{q}^{\mathbf{F}}} + q^2\overline{q}\nabla_t\overline{\mathfrak{F}}\cdot\nabla_4\mathfrak{q}^{\mathbf{F}}\big) \\ &= W_4^{\mathfrak{F}}\big(q\overline{q}^2\nabla_3\mathfrak{F}\cdot\nabla_4\overline{\mathfrak{q}^{\mathbf{F}}} + q^2\overline{q}\nabla_3\overline{\mathfrak{F}}\cdot\nabla_4\mathfrak{q}^{\mathbf{F}}\big) + \cdots. \end{aligned}$$

Writing $q\overline{q}^2\nabla_3\mathfrak{F} = \mathfrak{q}^{\mathbf{F}} +$ l.o.t., we obtain

$$= W_4^{\mathfrak{F}}\big(\mathfrak{q}^{\mathbf{F}}\cdot\nabla_4\overline{\mathfrak{q}^{\mathbf{F}}} + \overline{\mathfrak{q}^{\mathbf{F}}}\cdot\nabla_4\mathfrak{q}^{\mathbf{F}}\big) + \cdots = W_4^{\mathfrak{F}}\nabla_4(|\mathfrak{q}^{\mathbf{F}}|^2),$$

which gives a boundary term. The same happens for the terms $W_a^{\mathfrak{X}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X}$, $Z_a^{\mathfrak{B}}\cdot{}^{(c)}\nabla\mathfrak{B}$ and $\big(W_a^{\mathfrak{F}} - Z_a^{\mathfrak{F}}\big)\cdot{}^{(c)}\nabla\mathfrak{F}$, which because of the reality of the coefficients, can be written as boundary terms.

We now look at the coupling terms in the lower order terms, i.e.

$$(q^{1/2}\overline{q}^{9/2}2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\;Y_a^{\mathfrak{F}}(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}))\cdot\nabla_t\overline{\mathfrak{p}}+\overline{(q^{1/2}\overline{q}^{9/2}2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\;Y_a^{\mathfrak{F}}(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}))}$$
$$\cdot\nabla_t\mathfrak{p}+8Q^2(q\overline{q}^2W_a^{\mathfrak{B}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}+\overline{q\overline{q}^2W_a^{\mathfrak{B}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}).$$

Writing that ${}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}=\frac{Q^2}{|q|^4}$ and $W_a^{\mathfrak{B}}=\frac{3}{4}i\,{}^{(a)}\mathrm{tr}\underline{\chi}$ and $Y_a^{\mathfrak{F}}=-3i\,{}^{(a)}\mathrm{tr}\underline{\chi}$, we have

$$=-6Q^2\frac{q^{1/2}\overline{q}^{9/2}}{|q|^4}\;i\,{}^{(a)}\mathrm{tr}\underline{\chi}(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F})\cdot\nabla_t\overline{\mathfrak{p}}+6Q^2\frac{q^{9/2}\overline{q}^{1/2}}{|q|^4}\;i\,{}^{(a)}\mathrm{tr}\underline{\chi}({}^{(c)}\mathcal{D}\cdot\overline{\mathfrak{F}})\cdot\nabla_t\mathfrak{p}$$
$$+6Q^2q\overline{q}^2i\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}-6Q^2q^2\overline{q}i\,{}^{(a)}\mathrm{tr}\underline{\chi}\,\overline{{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}$$
$$=-6Q^2i\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big[\frac{q^{1/2}\overline{q}^{9/2}}{|q|^4}\;(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F})\cdot\nabla_t\overline{\mathfrak{p}}+q^2\overline{q}\,\overline{{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}\Big]$$
$$+6Q^2i\,{}^{(a)}\mathrm{tr}\underline{\chi}\Big[\frac{q^{9/2}\overline{q}^{1/2}}{|q|^4}\;({}^{(c)}\mathcal{D}\cdot\overline{\mathfrak{F}})\cdot\nabla_t\mathfrak{p}+q\overline{q}^2\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\cdot\nabla_t\overline{\mathfrak{q}^{\mathbf{F}}}\Big].$$

Now by recalling that $q\overline{q}^2\nabla_3\mathfrak{F}=\mathfrak{q}^{\mathbf{F}}+$ l.o.t. and $q^{1/2}\overline{q}^{9/2}\nabla_3\mathfrak{B}=\mathfrak{p}+$ l.o.t., we obtain, only looking at the highest order terms:

$$\frac{q^{1/2}\overline{q}^{9/2}}{|q|^4}(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F})\cdot\nabla_t\overline{\mathfrak{p}}+q^2\overline{q}\,\overline{{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}}\cdot\nabla_t\mathfrak{q}^{\mathbf{F}}$$
$$=-\frac{q^{1/2}\overline{q}^{9/2}}{|q|^4}\;(\overline{{}^{(c)}\mathcal{D}}\cdot\nabla_t\mathfrak{F})\cdot\overline{\mathfrak{p}}-q^2\overline{q}\,\overline{{}^{(c)}\mathcal{D}\widehat{\otimes}\nabla_t\mathfrak{B}}\cdot\mathfrak{q}^{\mathbf{F}}$$
$$=-\frac{q^{1/2}\overline{q}^{9/2}}{|q|^4}\;(\overline{{}^{(c)}\mathcal{D}}\cdot\nabla_3\mathfrak{F})\cdot\overline{\mathfrak{p}}-q^2\overline{q}\,\overline{{}^{(c)}\mathcal{D}\widehat{\otimes}\nabla_3\mathfrak{B}}\cdot\mathfrak{q}^{\mathbf{F}}$$
$$=-\frac{q^{1/2}\overline{q}^{9/2}}{|q|^4}\;\frac{1}{q\overline{q}^2}(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{q}^{\mathbf{F}})\cdot\overline{\mathfrak{p}}-q^2\overline{q}\frac{1}{\overline{q}^{1/2}q^{9/2}}\overline{{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{p}}\cdot\mathfrak{q}^{\mathbf{F}}$$
$$=-\frac{\overline{q}^{1/2}}{q^{5/2}}\big[(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{q}^{\mathbf{F}})\cdot\overline{\mathfrak{p}}+\overline{{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{p}}\cdot\mathfrak{q}^{\mathbf{F}}\big]=0.$$

The remaining terms are therefore only of lower order, and can be absorbed as shown before for small $|a|\ll M$ by Cauchy–Schwarz.

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Appendix A. Explicit Computations

We collect here some explicit computations needed in Secs. 2–5.

A.1. *Derivation of the Bianchi identities*

Lemma A.1. *We have the following for the decomposition in frames of $J_{\beta\gamma\delta}$ in* (3.8):

$$\begin{aligned} J_{434} &= -\nabla_4({}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2) + 2(\underline{\eta} - 2\eta) \cdot \big({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\beta + {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\beta\big) \\ &\quad + 2\xi \cdot \big({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\underline{\beta} - {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\underline{\beta}\big) + \nabla_3({}^{(\mathbf{F})}\beta \cdot {}^{(\mathbf{F})}\beta) - 4\underline{\omega}({}^{(\mathbf{F})}\beta \cdot {}^{(\mathbf{F})}\beta), \end{aligned} \tag{A.1}$$

$$\begin{aligned} J_{ab4} &= \nabla_b({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\beta_a + {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\beta_a) + (\zeta_b + \underline{\eta}_b)({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\beta_a + {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\beta_a) \\ &\quad + \underline{\eta}_a({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\beta_b + {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\beta_b) - \frac{1}{2}\nabla_4({}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2)\delta_{ab} \\ &\quad - \chi_{ba}({}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2) + \nabla_4({}^{(\mathbf{F})}\beta \widehat{\otimes} {}^{(\mathbf{F})}\underline{\beta})_{ab} + \frac{1}{2}\chi_{bc}({}^{(\mathbf{F})}\beta \widehat{\otimes} {}^{(\mathbf{F})}\underline{\beta})_{ca} \\ &\quad - \frac{1}{2}\underline{\chi}_{ba} \, {}^{(\mathbf{F})}\beta \cdot {}^{(\mathbf{F})}\beta + \xi_a({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\underline{\beta}_b - {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\underline{\beta}_b) \\ &\quad + \xi_b({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\underline{\beta}_a - {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\underline{\beta}_a), \end{aligned} \tag{A.2}$$

$$\begin{aligned} J_{4a4} &= - {}^{(c)}\nabla_4({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\beta_a + {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\beta_a) - tr\,\chi({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\beta_a + {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\beta_a) \\ &\quad + {}^{(a)}tr\chi({}^{*(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\beta_a - {}^{(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\beta_a) + 2\big({}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2\big)\xi_a \\ &\quad + \nabla_a({}^{(\mathbf{F})}\beta \cdot {}^{(\mathbf{F})}\beta) + (2\zeta_a + \underline{\eta}_a) \, {}^{(\mathbf{F})}\beta \cdot {}^{(\mathbf{F})}\beta - 2\xi_b({}^{(\mathbf{F})}\underline{\beta} \widehat{\otimes} {}^{(\mathbf{F})}\underline{\beta})_{ab}, \end{aligned} \tag{A.3}$$

$$\begin{aligned} J_{3a4} &= \nabla_a({}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2) - \frac{1}{2}tr\,\chi({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\underline{\beta}_a - {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\underline{\beta}_a) \\ &\quad - \frac{1}{2}\mathrm{tr}\,\underline{\chi}({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\beta_a + {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\beta_a) - \frac{1}{2} {}^{(a)}tr\chi \, {}^{*}({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\underline{\beta}_a - {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\underline{\beta}_a) \\ &\quad - \frac{1}{2} {}^{(a)}tr\underline{\chi} \, {}^{*}({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\beta_a + {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\beta_a) - {}^{(c)}\nabla_4({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\underline{\beta}_a - {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\underline{\beta}_a) \\ &\quad + 2\left({}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2\right)\underline{\eta}_a + \xi_a \, {}^{(\mathbf{F})}\underline{\beta} \cdot {}^{(\mathbf{F})}\underline{\beta} - \hat{\chi}_{ab}({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\underline{\beta}_b - {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\underline{\beta}_b) \\ &\quad - \hat{\underline{\chi}}({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\beta_b + {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\beta_b) - 2\underline{\eta}_b({}^{(\mathbf{F})}\underline{\beta} \widehat{\otimes} {}^{(\mathbf{F})}\underline{\beta})_{ab}, \end{aligned} \tag{A.4}$$

$$\begin{aligned} {}^{*}J_{434} &= 2\mathrm{curl}({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\beta + {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\beta) + 2\zeta \cdot {}^{*}({}^{*(\mathbf{F})}\rho \, {}^{*(\mathbf{F})}\beta + {}^{(\mathbf{F})}\rho \, {}^{(\mathbf{F})}\beta) \\ &\quad - 2 \, {}^{(a)}tr\chi({}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2) + {}^{(a)}tr\underline{\chi} \, {}^{(\mathbf{F})}\beta \cdot {}^{(\mathbf{F})}\beta. \end{aligned} \tag{A.5}$$

The other quantities are obtained by symmetrization, where in interchanging the 3 *with the* 4, *one interchanges* ${}^{(\mathbf{F})}\beta \leftrightarrow {}^{(\mathbf{F})}\underline{\beta}$, ${}^{(\mathbf{F})}\rho \leftrightarrow -{}^{(\mathbf{F})}\rho$, ${}^{*(\mathbf{F})}\rho \leftrightarrow {}^{*(\mathbf{F})}\rho$, $\zeta \leftrightarrow -\zeta$, $\eta \leftrightarrow \underline{\eta}$, $\hat{\chi} \leftrightarrow \underline{\chi}$ *and* $tr\,\chi \leftrightarrow \mathrm{tr}\,\underline{\chi}$, $\omega \leftrightarrow \underline{\omega}$.

Proof. We compute J_{434}:

$$\begin{aligned} 2J_{434} &= \mathbf{D}_3\mathbf{R}_{44} - \mathbf{D}_4\mathbf{R}_{43} \\ &= \nabla_3(\mathbf{R}_{44}) - 2\mathbf{R}(\mathbf{D}_3 e_4, e_4) - \nabla_4(\mathbf{R}_{34}) + \mathbf{R}(\mathbf{D}_4 e_4, e_3) + \mathbf{R}(e_4, \mathbf{D}_4 e_3) \end{aligned}$$

$$
\begin{aligned}
&= 2\nabla_3({}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\beta) - \nabla_4(2\,{}^{(\mathbf{F})}\rho^2 + 2\,{}^{*(\mathbf{F})}\rho^2) - 2\mathbf{R}(2\underline{\omega}e_4 + 2\eta_a e_a, e_4)\\
&\quad + \mathbf{R}(-2\omega e_4 + 2\xi_a e_a, e_3) + \mathbf{R}(e_4, 2\omega e_3 + 2\underline{\eta}_a e_a)\\
&= -2\nabla_4({}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2) - 4\underline{\omega}\mathbf{R}_{44} + 2(\underline{\eta}_a - 2\eta_a)\mathbf{R}_{a4}\\
&\quad + 2\xi_a\mathbf{R}_{a3} + 2\nabla_3({}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\beta)\\
&= -2\nabla_4({}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2) + 4\,{}^{*(\mathbf{F})}\rho\left(\underline{\eta} - 2\eta\right)\cdot{}^{*(\mathbf{F})}\beta + 4\,{}^{(\mathbf{F})}\rho\left(\underline{\eta} - 2\eta\right)\cdot{}^{(\mathbf{F})}\beta\\
&\quad + 4\,{}^{*(\mathbf{F})}\rho\xi\cdot{}^{*(\mathbf{F})}\underline{\beta} - 4\,{}^{(\mathbf{F})}\rho\xi\cdot{}^{(\mathbf{F})}\underline{\beta} + 2\nabla_3({}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\beta) - 8\underline{\omega}({}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\beta).
\end{aligned}
$$

This proves (A.1). We compute J_{ab4}:

$$
\begin{aligned}
2J_{ab4} &= \mathbf{D}_b\mathbf{R}_{4a} - \mathbf{D}_4\mathbf{R}_{ab}\\
&= \nabla_b(\mathbf{R}_{4a}) - \mathbf{R}(\mathbf{D}_b e_4, e_a) - \mathbf{R}(e_4, \mathbf{D}_b e_a) - \nabla_4(\mathbf{R}_{ab})\\
&\quad + \mathbf{R}(\mathbf{D}_4 e_a, e_b) + \mathbf{R}(e_a, \mathbf{D}_4 e_b)\\
&= \nabla_b(2\,{}^{*(\mathbf{F})}\rho\,{}^{*(\mathbf{F})}\beta_a + 2\,{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta_a) - \nabla_4(-2({}^{(\mathbf{F})}\beta\widehat{\otimes}{}^{(\mathbf{F})}\underline{\beta})_{ab}\\
&\quad + ({}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2)\delta_{ab}) - \mathbf{R}(-\zeta_b e_4 + \chi_{bc}e_c, e_a) - \mathbf{R}\left(e_4, \frac{1}{2}\underline{\chi}_{ba}e_4 + \frac{1}{2}\chi_{ba}e_3\right)\\
&\quad + \mathbf{R}(\underline{\eta}_a e_4 + \xi_a e_3, e_b) + \mathbf{R}(e_a, \underline{\eta}_b e_4 + \xi_b e_3)\\
&= \nabla_b(2\,{}^{*(\mathbf{F})}\rho\,{}^{*(\mathbf{F})}\beta_a + 2\,{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta_a) - \nabla_4(-2({}^{(\mathbf{F})}\beta\widehat{\otimes}{}^{(\mathbf{F})}\underline{\beta})_{ab}\\
&\quad + ({}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2)\delta_{ab}) + \zeta_b\mathbf{R}_{a4} - \chi_{bc}\mathbf{R}_{ca} - \frac{1}{2}\underline{\chi}_{ba}\mathbf{R}_{44}\\
&\quad - \frac{1}{2}\chi_{ba}\mathbf{R}_{43} + \underline{\eta}_a\mathbf{R}_{4b} + \xi_a\mathbf{R}_{3b} + \underline{\eta}_b\mathbf{R}_{a4} + \xi_b\mathbf{R}_{a3}.
\end{aligned}
$$

Using the Ricci decomposition, this proves (A.2). We compute J_{4a4}:

$$
\begin{aligned}
2J_{4a4} &= \mathbf{D}_a\mathbf{R}_{44} - \mathbf{D}_4\mathbf{R}_{4a}\\
&= \nabla_a(\mathbf{R}_{44}) - 2\mathbf{R}(\mathbf{D}_a e_4, e_4) - \nabla_4(\mathbf{R}_{4a}) + \mathbf{R}(\mathbf{D}_4 e_4, e_a) + \mathbf{R}(e_4, \mathbf{D}_4 e_a)\\
&= 2\nabla_a({}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\beta) - 2\nabla_4({}^{*(\mathbf{F})}\rho\,{}^{*(\mathbf{F})}\beta_a + {}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta_a) - 2\mathbf{R}(-\zeta_a e_4 + \chi_{ab}e_b, e_4)\\
&\quad + \mathbf{R}(-2\omega e_4 + 2\xi_b e_b, e_a) + \mathbf{R}(e_4, \underline{\eta}_a e_4 + \xi_a e_3)\\
&= -2\nabla_4({}^{*(\mathbf{F})}\rho\,{}^{*(\mathbf{F})}\beta_a + {}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta_a) - 2\chi_{ab}(2\,{}^{*(\mathbf{F})}\rho\,{}^{*(\mathbf{F})}\beta_b + 2\,{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta_b)\\
&\quad - 2\omega(2\,{}^{*(\mathbf{F})}\rho\,{}^{*(\mathbf{F})}\beta_a + 2\,{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta_a) + 4\big({}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2\big)\xi_a\\
&\quad + 2\nabla_a({}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\beta) + (4\zeta_a + 2\underline{\eta}_a)\,{}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\beta - 4\xi_b({}^{(\mathbf{F})}\underline{\beta}\widehat{\otimes}{}^{(\mathbf{F})}\underline{\beta})_{ab},
\end{aligned}
$$

which proves (A.3). We compute J_{3a4}:

$$
2J_{3a4} = \mathbf{D}_a\mathbf{R}_{43} - \mathbf{D}_4\mathbf{R}_{3a}
$$

$$
\begin{aligned}
&= \nabla_a(\mathbf{R}_{34}) - \mathbf{R}(\mathbf{D}_a e_4, e_3) - \mathbf{R}(e_4, \mathbf{D}_a e_3) - \nabla_4(\mathbf{R}_{3a}) + \mathbf{R}(\mathbf{D}_4 e_3, e_a)\\
&\quad + \mathbf{R}(e_3, \mathbf{D}_4 e_a)\\
&= \nabla_a(2\,{}^{(\mathbf{F})}\rho^2 + 2\,{}^{*\,(\mathbf{F})}\rho^2) - \mathbf{R}(\chi_{ab}e_b - \zeta_a e_4, e_3) - \mathbf{R}(e_4, \underline{\chi}_{ab}e_b + \zeta_a e_3)\\
&\quad - \nabla_4(2\,{}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\underline{\beta}_a - 2\,{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\underline{\beta}_a) + \mathbf{R}(2\omega e_3 + 2\underline{\eta}_b e_b, e_a)\\
&\quad + \mathbf{R}(e_3, \underline{\eta}_a e_4 + \xi_a e_3)\\
&= 2\nabla_a({}^{(\mathbf{F})}\rho^2 + {}^{*\,(\mathbf{F})}\rho^2) - \chi_{ab}(2\,{}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\underline{\beta}_b - 2\,{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\underline{\beta}_b)\\
&\quad - \underline{\chi}_{ab}(2\,{}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\beta_b + 2\,{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta_b)\\
&\quad - 2\nabla_4({}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\underline{\beta}_a - {}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\underline{\beta}_a) + 2\omega(2\,{}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\underline{\beta}_a - 2\,{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\underline{\beta}_a)\\
&\quad + 2\underline{\eta}_b(-2({}^{(\mathbf{F})}\underline{\beta}\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{\beta})_{ab} + \big({}^{(\mathbf{F})}\rho^2 + {}^{*\,(\mathbf{F})}\rho^2\big)\delta_{ab}) + \underline{\eta}_a(2\,{}^{(\mathbf{F})}\rho^2 + 2\,{}^{*\,(\mathbf{F})}\rho^2)\\
&\quad + 2\xi_a\,{}^{(\mathbf{F})}\underline{\beta}\cdot{}^{(\mathbf{F})}\underline{\beta},
\end{aligned}
$$

which proves (A.4). We compute ${}^*J_{434}$ using (A.2):

$$
\begin{aligned}
{}^*J_{434} &= -2J_{ab4}\in_{ab}\\
&= 2\mathrm{curl}({}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\beta + {}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta) + 2\zeta\cdot{}^*({}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\beta + {}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta)\\
&\quad -2\,{}^{(a)}\mathrm{tr}\chi({}^{(\mathbf{F})}\rho^2 + {}^{*\,(\mathbf{F})}\rho^2) + {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\beta,
\end{aligned}
$$

which proves (A.5). □

Lemma A.2. *Using Maxwell equations as in Proposition 3.1, we obtain*

$$
\begin{aligned}
J_{434} &= -2\mathrm{div}({}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\beta + {}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta) - 2\big(\zeta + 4\eta\big)\cdot\big({}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta + {}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\beta\big)\\
&\quad + 2tr\,\chi({}^{(\mathbf{F})}\rho^2 + {}^{*\,(\mathbf{F})}\rho^2) + 2\left(2\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}\beta + \frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\beta - \hat{\chi}\cdot{}^{(\mathbf{F})}\underline{\beta}\right)\cdot{}^{(\mathbf{F})}\beta.
\end{aligned}
\tag{A.6}
$$

Proof. We compute

$$
\begin{aligned}
&\nabla_4({}^{(\mathbf{F})}\rho^2 + {}^{*\,(\mathbf{F})}\rho^2)\\
&\quad = 2\,{}^{(\mathbf{F})}\rho\nabla_4\,{}^{(\mathbf{F})}\rho + 2\,{}^{*\,(\mathbf{F})}\rho\nabla_4\,{}^{*\,(\mathbf{F})}\rho\\
&\quad = 2\,{}^{(\mathbf{F})}\rho\big(\mathrm{div}\,{}^{(\mathbf{F})}\beta - \big(\mathrm{tr}\,\chi\,{}^{(\mathbf{F})}\rho - {}^{(a)}\mathrm{tr}\chi\,{}^{*\,(\mathbf{F})}\rho\big) + \big(\zeta + \underline{\eta}\big)\cdot{}^{(\mathbf{F})}\beta - \xi\cdot{}^{(\mathbf{F})}\underline{\beta}\big)\\
&\qquad + 2\,{}^{*\,(\mathbf{F})}\rho\big(\mathrm{curl}\,{}^{(\mathbf{F})}\beta - \big(\mathrm{tr}\,\chi\,{}^{*\,(\mathbf{F})}\rho + {}^{(a)}\mathrm{tr}\chi\,{}^{(\mathbf{F})}\rho\big) + \big(\underline{\eta} + \zeta\big)\cdot{}^{*\,(\mathbf{F})}\beta\\
&\qquad + \xi\cdot{}^{*\,(\mathbf{F})}\underline{\beta}\big)\\
&\quad = 2\,{}^{(\mathbf{F})}\rho\mathrm{div}\,{}^{(\mathbf{F})}\beta + 2\,{}^{*\,(\mathbf{F})}\rho\mathrm{curl}\,{}^{(\mathbf{F})}\beta - 2tr\,\chi({}^{(\mathbf{F})}\rho^2 + {}^{*\,(\mathbf{F})}\rho^2)\\
&\qquad + 2(\zeta + \underline{\eta})\cdot\big({}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta + {}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\beta\big) + 2\xi\cdot\big({}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\underline{\beta} - {}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\underline{\beta}\big).
\end{aligned}
$$

We now compute $\mathrm{div}({}^*{}^{(\mathbf{F})}\rho\, {}^*{}^{(\mathbf{F})}\beta + {}^{(\mathbf{F})}\rho\, {}^{(\mathbf{F})}\beta)$. It is given by

$$\begin{aligned}
&\mathrm{div}({}^*{}^{(\mathbf{F})}\rho\, {}^*{}^{(\mathbf{F})}\beta + {}^{(\mathbf{F})}\rho\, {}^{(\mathbf{F})}\beta)\\
&\quad = \nabla\, {}^*{}^{(\mathbf{F})}\rho \cdot {}^*{}^{(\mathbf{F})}\beta + {}^*{}^{(\mathbf{F})}\rho \mathrm{div}\, {}^*{}^{(\mathbf{F})}\beta + \nabla\, {}^{(\mathbf{F})}\rho \cdot {}^{(\mathbf{F})}\beta + {}^{(\mathbf{F})}\rho \mathrm{div}\, {}^{(\mathbf{F})}\beta\\
&\quad = {}^{(\mathbf{F})}\rho \mathrm{div}\, {}^{(\mathbf{F})}\beta + {}^*{}^{(\mathbf{F})}\rho \mathrm{curl}\, {}^{(\mathbf{F})}\beta + \big(\nabla\, {}^{(\mathbf{F})}\rho - {}^*\nabla\, {}^*{}^{(\mathbf{F})}\rho\big) \cdot {}^{(\mathbf{F})}\beta.
\end{aligned}$$

Using the Maxwell equation

$$\begin{aligned}
\nabla({}^{(\mathbf{F})}\rho) - {}^*\nabla\, {}^*{}^{(\mathbf{F})}\rho = \nabla_3\, {}^{(\mathbf{F})}\beta + \frac{1}{2}\big(\mathrm{tr}\,\underline{\chi}\, {}^{(\mathbf{F})}\beta + {}^{(a)}\mathrm{tr}\underline{\chi}\, {}^*{}^{(\mathbf{F})}\beta\big) - 2\underline{\omega}\, {}^{(\mathbf{F})}\beta\\
- 2\big(\eta\, {}^{(\mathbf{F})}\rho - {}^*\eta\, {}^*{}^{(\mathbf{F})}\rho\big) - \hat{\chi} \cdot {}^{(\mathbf{F})}\underline{\beta}
\end{aligned}$$

we obtain

$$\begin{aligned}
&\mathrm{div}({}^*{}^{(\mathbf{F})}\rho\, {}^*{}^{(\mathbf{F})}\beta + {}^{(\mathbf{F})}\rho\, {}^{(\mathbf{F})}\beta)\\
&\quad = {}^{(\mathbf{F})}\rho \mathrm{div}\, {}^{(\mathbf{F})}\beta + {}^*{}^{(\mathbf{F})}\rho \mathrm{curl}\, {}^{(\mathbf{F})}\beta - 2\eta \cdot \big({}^{(\mathbf{F})}\rho\, {}^{(\mathbf{F})}\beta + {}^*{}^{(\mathbf{F})}\rho\, {}^*{}^{(\mathbf{F})}\beta\big)\\
&\qquad + \left({}^{(c)}\nabla_3\, {}^{(\mathbf{F})}\beta + \frac{1}{2}\mathrm{tr}\,\underline{\chi}\, {}^{(\mathbf{F})}\beta - \hat{\chi} \cdot {}^{(\mathbf{F})}\underline{\beta}\right) \cdot {}^{(\mathbf{F})}\beta. \qquad (A.7)
\end{aligned}$$

This therefore gives

$$\begin{aligned}
&\nabla_4({}^{(\mathbf{F})}\rho^2 + {}^*{}^{(\mathbf{F})}\rho^2)\\
&\quad = 2\mathrm{div}({}^*{}^{(\mathbf{F})}\rho\, {}^*{}^{(\mathbf{F})}\beta + {}^{(\mathbf{F})}\rho\, {}^{(\mathbf{F})}\beta) - 2\mathrm{tr}\,\chi({}^{(\mathbf{F})}\rho^2 + {}^*{}^{(\mathbf{F})}\rho^2)\\
&\qquad + 2\big(\zeta + \underline{\eta} + 2\eta\big) \cdot \big({}^{(\mathbf{F})}\rho\, {}^{(\mathbf{F})}\beta + {}^*{}^{(\mathbf{F})}\rho\, {}^*{}^{(\mathbf{F})}\beta\big)\\
&\qquad + 2\xi \cdot \big({}^*{}^{(\mathbf{F})}\rho\, {}^*{}^{(\mathbf{F})}\underline{\beta} - {}^{(\mathbf{F})}\rho\, {}^{(\mathbf{F})}\underline{\beta}\big)\\
&\qquad - 2\left({}^{(c)}\nabla_3\, {}^{(\mathbf{F})}\beta + \frac{1}{2}\mathrm{tr}\,\underline{\chi}\, {}^{(\mathbf{F})}\beta - \hat{\chi} \cdot {}^{(\mathbf{F})}\underline{\beta}\right) \cdot {}^{(\mathbf{F})}\beta.
\end{aligned}$$

Using (A.1) and the above, we deduce (A.6). □

The complexified Bianchi identity for A is given by

$$\nabla_3 A - \mathcal{D}\widehat{\otimes} B = -\frac{1}{2}\mathrm{tr}\underline{X} A + 4\underline{\omega} A + (Z + 4H)\widehat{\otimes} B - 3\overline{P}\widehat{X} + \mathfrak{a} + i\, {}^*\mathfrak{a}. \qquad (A.8)$$

We compute $\mathfrak{a}$, using (A.1) and (A.2):

$$\begin{aligned}
\mathfrak{a} &= 2\nabla\widehat{\otimes}({}^*{}^{(\mathbf{F})}\rho\, {}^*{}^{(\mathbf{F})}\beta + {}^{(\mathbf{F})}\rho\, {}^{(\mathbf{F})}\beta) + 2(\zeta + 2\underline{\eta})\widehat{\otimes}({}^*{}^{(\mathbf{F})}\rho\, {}^*{}^{(\mathbf{F})}\beta + {}^{(\mathbf{F})}\rho\, {}^{(\mathbf{F})}\beta_b)\\
&\quad - 2({}^{(\mathbf{F})}\rho^2 + {}^*{}^{(\mathbf{F})}\rho^2)\hat{\chi} + 2\nabla_4({}^{(\mathbf{F})}\beta\widehat{\otimes}\, {}^{(\mathbf{F})}\underline{\beta}) + {}^{(c)}\nabla_3({}^{(\mathbf{F})}\beta\widehat{\otimes}\, {}^{(\mathbf{F})}\beta) + \chi \cdot ({}^{(\mathbf{F})}\beta\widehat{\otimes}\, {}^{(\mathbf{F})}\underline{\beta})\\
&\quad + 2\xi\widehat{\otimes}({}^*{}^{(\mathbf{F})}\rho\, {}^*{}^{(\mathbf{F})}\underline{\beta} - 2\, {}^{(\mathbf{F})}\rho\, {}^{(\mathbf{F})}\underline{\beta}). \qquad (A.9)
\end{aligned}$$

Using (A.9), we have

$$\begin{aligned}\mathfrak{a}+i\,{}^*\mathfrak{a} &= \mathcal{D}\widehat{\otimes}\big(\big({}^*\,{}^{(\mathbf{F})}\rho\,{}^*\,{}^{(\mathbf{F})}\beta+{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta\big)+i\,{}^*\big({}^*\,{}^{(\mathbf{F})}\rho\,{}^*\,{}^{(\mathbf{F})}\beta+{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta\big)\big)\\
&\quad+((\zeta+2\underline{\eta})+i\,{}^*(\zeta+2\underline{\eta}))\widehat{\otimes}(({}^*\,{}^{(\mathbf{F})}\rho\,{}^*\,{}^{(\mathbf{F})}\beta+{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta)\\
&\quad+i\,{}^*({}^*\,{}^{(\mathbf{F})}\rho\,{}^*\,{}^{(\mathbf{F})}\beta+{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta))-2({}^{(\mathbf{F})}\rho^2+{}^*\,{}^{(\mathbf{F})}\rho^2)(\hat{\chi}+i\,{}^*\hat{\chi})\\
&\quad+\nabla_4({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B})+\frac{1}{2}\,{}^{(c)}\nabla_3({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}B)\\
&\quad+\frac{1}{2}\widehat{X}\cdot({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B})+{}^{(\mathbf{F})}P(\Xi\widehat{\otimes}\,{}^{(\mathbf{F})}B).\end{aligned}$$

Observe that

$$\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B=\big({}^*\,{}^{(\mathbf{F})}\rho\,{}^*\,{}^{(\mathbf{F})}\beta+{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta\big)+i\,{}^*\big({}^*\,{}^{(\mathbf{F})}\rho\,{}^*\,{}^{(\mathbf{F})}\beta+{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta\big)$$

and ${}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}={}^{(\mathbf{F})}\rho^2+{}^*\,{}^{(\mathbf{F})}\rho^2$. We can therefore write

$$\begin{aligned}\mathfrak{a}+i\,{}^*\mathfrak{a} &= \mathcal{D}\widehat{\otimes}(\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B)+(Z+2\,\underline{H})\widehat{\otimes}(\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B)-2\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}P\widehat{X}\\
&\quad+\nabla_4({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B})+\frac{1}{2}\,{}^{(c)}\nabla_3({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}B)\\
&\quad+\frac{1}{2}\widehat{X}\cdot({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B})+{}^{(\mathbf{F})}P(\Xi\widehat{\otimes}\,{}^{(\mathbf{F})}B)\\
&= \overline{{}^{(\mathbf{F})}P}\mathcal{D}\widehat{\otimes}({}^{(\mathbf{F})}B)+\mathcal{D}\,\overline{{}^{(\mathbf{F})}P}\widehat{\otimes}\,{}^{(\mathbf{F})}B+(Z+2\,\underline{H})\widehat{\otimes}(\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B)-2\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}P\widehat{X}\\
&\quad+\nabla_4({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B})+\frac{1}{2}\,{}^{(c)}\nabla_3({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}B)\\
&\quad+\frac{1}{2}\widehat{X}\cdot({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B})+{}^{(\mathbf{F})}P(\Xi\widehat{\otimes}\,{}^{(\mathbf{F})}B).\end{aligned}$$

Using the Maxwell equation for ${}^{(c)}\mathcal{D}\,\overline{{}^{(\mathbf{F})}P}$ we write

$$\begin{aligned}\mathfrak{a}+i\,{}^*\mathfrak{a} &= \overline{{}^{(\mathbf{F})}P}\mathcal{D}\widehat{\otimes}({}^{(\mathbf{F})}B)+(-2\,\overline{{}^{(\mathbf{F})}P}\,\underline{H})\widehat{\otimes}\,{}^{(\mathbf{F})}B+(Z+2\,\underline{H})\widehat{\otimes}(\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B)\\
&\quad-2\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}P\widehat{X}+\frac{1}{2}\nabla_4({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B})+\frac{1}{2}\,{}^{(c)}\nabla_3({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}B)\\
&\quad+\left(-\frac{1}{2}\mathrm{tr}X\,{}^{(\mathbf{F})}\underline{B}+\frac{1}{2}\underline{\widehat{X}}\cdot\overline{{}^{(\mathbf{F})}B}+\frac{1}{2}\widehat{X}\cdot{}^{(\mathbf{F})}\underline{B}+{}^{(\mathbf{F})}P\Xi\right)\widehat{\otimes}\,{}^{(\mathbf{F})}B\\
&= -2\,\overline{{}^{(\mathbf{F})}P}\left(-\frac{1}{2}(\mathcal{D}+Z)\widehat{\otimes}\,{}^{(\mathbf{F})}B+{}^{(\mathbf{F})}P\widehat{X}\right)+\frac{1}{2}\nabla_4({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B})\\
&\quad+\frac{1}{2}\,{}^{(c)}\nabla_3({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}B)\\
&\quad+\left(-\frac{1}{2}\mathrm{tr}X\,{}^{(\mathbf{F})}\underline{B}+\frac{1}{2}\underline{\widehat{X}}\cdot\overline{{}^{(\mathbf{F})}B}+\frac{1}{2}\widehat{X}\cdot{}^{(\mathbf{F})}\underline{B}+{}^{(\mathbf{F})}P\Xi\right)\widehat{\otimes}\,{}^{(\mathbf{F})}B.\end{aligned}$$

From (A.8), we obtain

$$\begin{aligned}\nabla_3 A - \mathcal{D}\widehat{\otimes} B = &-\frac{1}{2}\mathrm{tr}\underline{X}A + 4\underline{\omega}A + (Z+4H)\widehat{\otimes}B - 3\overline{P}\widehat{X} - 2\overline{{}^{(\mathbf{F})}P}\\ &\times\left(-\frac{1}{2}(\mathcal{D}+Z)\widehat{\otimes}\,{}^{(\mathbf{F})}B + {}^{(\mathbf{F})}P\widehat{X}\right)\\ &+\frac{1}{2}\nabla_4({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B}) + \frac{1}{2}\,{}^{(c)}\nabla_3({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}B)\\ &+\left(-\frac{1}{2}\mathrm{tr}X\,{}^{(\mathbf{F})}\underline{B} + \frac{1}{2}\widehat{\underline{X}}\cdot\overline{{}^{(\mathbf{F})}B} + \frac{1}{2}\widehat{X}\cdot{}^{(\mathbf{F})}\underline{B} + {}^{(\mathbf{F})}P\Xi\right)\widehat{\otimes}\,{}^{(\mathbf{F})}B,\end{aligned}$$

which proves the first equation.

The complexified Bianchi identity for B is given by

$$\nabla_4 B - \frac{1}{2}\overline{\mathcal{D}}\cdot A = -2\overline{\mathrm{tr}X}B - 2\omega B + \frac{1}{2}A\cdot(\overline{2Z+\underline{H}}) + 3\overline{P}\,\Xi - (J_{4a4} + i\,{}^*J_{4a4}).$$

Using (A.3), we have

$$\begin{aligned}{}^*J_{4a4} = &-\nabla_4\,{}^*(\,{}^*\,{}^{(\mathbf{F})}\rho\,{}^*\,{}^{(\mathbf{F})}\beta_a + {}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta_a) - \mathrm{tr}\,\chi(-\,{}^*\,{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta_a + {}^{(\mathbf{F})}\rho\,{}^*\,{}^{(\mathbf{F})}\beta_a)\\ &+{}^{(a)}\mathrm{tr}\chi(\,{}^*\,{}^{(\mathbf{F})}\rho\,{}^*\,{}^{(\mathbf{F})}\beta_a + {}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta_a) - 2\omega\,{}^*(\,{}^*\,{}^{(\mathbf{F})}\rho\,{}^*\,{}^{(\mathbf{F})}\beta_a + {}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta_a)\\ &+2\big({}^{(\mathbf{F})}\rho^2 + {}^*\,{}^{(\mathbf{F})}\rho^2\big)\,{}^*\xi_a + {}^*\nabla_a({}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\beta) + {}^*(2\zeta_a + \underline{\eta}_a)\,{}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\beta\\ &-2\,{}^*\xi_b({}^{(\mathbf{F})}\underline{\beta}\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{\beta})_{ab}\end{aligned}$$

and therefore

$$\begin{aligned}J_{4a4} + i\,{}^*J_{4a4} = &-\nabla_4(\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B_a) - \mathrm{tr}X\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B_a - 2\omega\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B_a + 2\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}P\Xi_a\\ &+\frac{1}{4}\mathcal{D}({}^{(\mathbf{F})}B\cdot\overline{{}^{(\mathbf{F})}B}) + \frac{1}{2}(2Z+\underline{H})\,{}^{(\mathbf{F})}B\cdot\overline{{}^{(\mathbf{F})}B} - \frac{1}{2}\overline{\Xi}\cdot({}^{(\mathbf{F})}\underline{B}\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B}).\end{aligned}$$

Using the Maxwell equation for ${}^{(c)}\nabla_4\,{}^{(\mathbf{F})}P$ we obtain

$$\begin{aligned}J_{4a4} + i\,{}^*J_{4a4} = &-\overline{{}^{(\mathbf{F})}P}\nabla_4({}^{(\mathbf{F})}B_a) - 2\omega\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B_a + 2\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}P\Xi_a\\ &+\frac{1}{2}\mathcal{D}({}^{(\mathbf{F})}B\cdot\overline{{}^{(\mathbf{F})}B}) + Z\,{}^{(\mathbf{F})}B\cdot\overline{{}^{(\mathbf{F})}B},\end{aligned}$$

which proves the second equation. The other complexified Bianchi identity for β is given by

$$\nabla_3 B - \mathcal{D}\overline{P} = -\mathrm{tr}\underline{X}B + 2\underline{\omega}B + \overline{\underline{B}}\cdot\widehat{X} + 3\overline{P}H + \frac{1}{2}A\cdot\overline{\underline{\Xi}} + \big(J_{3a4} + i\,{}^*J_{3a4}\big).$$

Using (A.4), we have

$$\begin{aligned}J_{3a4} + i\,{}^*J_{3a4} = &\,\mathcal{D}({}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}) + \frac{1}{2}\mathrm{tr}X\,{}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B} - 2\omega\,{}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B} - \frac{1}{2}\mathrm{tr}\underline{X}\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B\\ &+\nabla_4({}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B}) + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\,\underline{H} + \frac{1}{2}\Xi({}^{(\mathbf{F})}\underline{B}\cdot\overline{{}^{(\mathbf{F})}\underline{B}}) - {}^{(\mathbf{F})}P\widehat{X}\cdot{}^{(\mathbf{F})}\underline{B}\\ &-\overline{{}^{(\mathbf{F})}P}\widehat{\underline{X}}\cdot{}^{(\mathbf{F})}B - \overline{\underline{H}}\cdot({}^{(\mathbf{F})}\underline{B}\widehat{\otimes}\,{}^{(\mathbf{F})}\underline{B}).\end{aligned}$$

Using the Maxwell equations, we obtain

$$\begin{aligned} J_{3a4}+i\,{}^*J_{3a4} &= \mathcal{D}({}^{(\mathbf{F})}P)\overline{{}^{(\mathbf{F})}P}+{}^{(\mathbf{F})}P\mathcal{D}(\overline{{}^{(\mathbf{F})}P})+\frac{1}{2}\mathrm{tr}X\,{}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B}-2\omega\,{}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B} \\ &\quad -\frac{1}{2}\mathrm{tr}\underline{X}\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B-\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B}+{}^{(\mathbf{F})}P(-\mathcal{D}\overline{{}^{(\mathbf{F})}P}-\frac{1}{2}\mathrm{tr}X\,{}^{(\mathbf{F})}\underline{B} \\ &\quad +2\omega\,{}^{(\mathbf{F})}\underline{B}-2\overline{{}^{(\mathbf{F})}P}\,\underline{H})+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\,\underline{H}+\frac{1}{2}(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(\mathbf{F})}B)\,{}^{(\mathbf{F})}\underline{B} \\ &\quad -{}^{(\mathbf{F})}P\widehat{X}\cdot{}^{(\mathbf{F})}\underline{B}-\frac{1}{2}\overline{{}^{(\mathbf{F})}P}\underline{\widehat{X}}\cdot{}^{(\mathbf{F})}B \\ &= \overline{{}^{(\mathbf{F})}P}\mathcal{D}({}^{(\mathbf{F})}P)-\frac{1}{2}\mathrm{tr}\underline{X}\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B-\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}\underline{B} \\ &\quad +\frac{1}{2}(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(\mathbf{F})}B)\,{}^{(\mathbf{F})}\underline{B}-{}^{(\mathbf{F})}P\widehat{X}\cdot{}^{(\mathbf{F})}\underline{B}-\frac{1}{2}\overline{{}^{(\mathbf{F})}P}\underline{\widehat{X}}\cdot{}^{(\mathbf{F})}B, \end{aligned}$$

which gives the desired formula. The complexified Bianchi identity for P is given by

$$\begin{aligned} \nabla_4P-\frac{1}{2}\mathcal{D}\cdot\overline{B} &= -\frac{3}{2}\mathrm{tr}XP+\frac{1}{2}(2\,\underline{H}+Z)\cdot\overline{B}-\overline{\Xi}\cdot\underline{B}-\frac{1}{4}\underline{\widehat{X}}\cdot\overline{A} \\ &\quad -\frac{1}{2}\big(J_{434}+i\,{}^*J_{434}\big). \end{aligned}$$

We compute using (A.6) and (A.5):

$$\begin{aligned} J_{434}+i\,{}^*J_{434} &= -2\mathrm{div}({}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\beta+{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta)+2i\mathrm{curl}({}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\beta+{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta) \\ &\quad -2\big(\zeta+i\,{}^*\zeta+4\eta\big)\cdot\big({}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta+{}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\beta\big) \\ &\quad +2(\mathrm{tr}\,\chi-i\,{}^{(a)}\mathrm{tr}\chi)({}^{(\mathbf{F})}\rho^2+{}^{*\,(\mathbf{F})}\rho^2) \\ &\quad +2\left(2\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}\beta+\frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\beta-\hat{\chi}\cdot{}^{(\mathbf{F})}\underline{\beta}\right)\cdot{}^{(\mathbf{F})}\beta+i\,{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^{(\mathbf{F})}\beta\cdot{}^{(\mathbf{F})}\beta. \end{aligned}$$

Observe that

$${}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}B}=\big({}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\beta+{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta\big)-i\,{}^*\big({}^{*\,(\mathbf{F})}\rho\,{}^{*\,(\mathbf{F})}\beta+{}^{(\mathbf{F})}\rho\,{}^{(\mathbf{F})}\beta\big)$$

and therefore

$$\begin{aligned} J_{434}+i\,{}^*J_{434} &= -\mathcal{D}\cdot({}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}B})-Z\cdot{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}B}+2\mathrm{tr}X\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} \\ &\quad -2\big(H+\overline{H}\big)\cdot\big({}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}B}+\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B\big)+2\,{}^{(c)}\nabla_3({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}B) \\ &\quad +\left(-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}B-\widehat{X}\cdot{}^{(\mathbf{F})}\underline{B}\right)\widehat{\otimes}\,{}^{(\mathbf{F})}B. \end{aligned}$$

Using the Maxwell equations for ${}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}P$ we obtain

$$\begin{aligned} J_{434}+i\,{}^*J_{434} &= 2\mathrm{tr}X\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}-{}^{(\mathbf{F})}P\mathcal{D}\cdot\overline{{}^{(\mathbf{F})}B}-Z\cdot{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}B}-2\overline{H}\cdot\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B \\ &\quad +{}^{(c)}\nabla_3({}^{(\mathbf{F})}B\widehat{\otimes}\,{}^{(\mathbf{F})}B)+\left(-\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}B-\frac{1}{2}\widehat{X}\cdot{}^{(\mathbf{F})}\underline{B}\right)\widehat{\otimes}\,{}^{(\mathbf{F})}B, \end{aligned}$$

which gives the desired formula.

A.2. *Proof of Proposition 5.7*

Here we derive (5.7)–(5.10).

A.2.1. *Derivation of* (5.7)

Multiply the Bianchi identity (5.5) by ${}^{(\mathbf{F})}P$:

$$
\begin{aligned}
&{}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}PA\\
&\quad= {}^{(\mathbf{F})}P\,{}^{(c)}\mathcal{D}\widehat{\otimes}B + H\widehat{\otimes}\big(4\,{}^{(\mathbf{F})}PB - 3\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}B\big) - 3\overline{P}\,{}^{(\mathbf{F})}P\widehat{X} - 2\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}P\mathfrak{F}.
\end{aligned}
$$

Multiply the definition of $\mathfrak{F}$ (5.2) by $3\overline{P}$:

$$
3\overline{P}\mathfrak{F} = -\frac{3}{2}\overline{P}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{9}{2}H\widehat{\otimes}\overline{P}\,{}^{(\mathbf{F})}B + 3\overline{P}\,{}^{(\mathbf{F})}P\widehat{X}.
$$

Summing the above we obtain the cancellation of $3\overline{P}\,{}^{(\mathbf{F})}P\widehat{X}$:

$$
\begin{aligned}
&\big(3\overline{P} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\mathfrak{F} + {}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3 A + \frac{1}{2}\,{}^{(\mathbf{F})}P\mathrm{tr}\underline{X}A\\
&\quad= \frac{1}{2}\big(2\,{}^{(\mathbf{F})}P\,{}^{(c)}\mathcal{D}\widehat{\otimes}B - 3\overline{P}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B\big)\\
&\qquad + H\widehat{\otimes}\left(4\,{}^{(\mathbf{F})}PB - 3\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}B - \frac{9}{2}\overline{P}\,{}^{(\mathbf{F})}B\right).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} &= {}^{(c)}\mathcal{D}\widehat{\otimes}\big(2\,{}^{(\mathbf{F})}PB - 3\overline{P}\,{}^{(\mathbf{F})}B\big)\\
&= \big(2\,{}^{(\mathbf{F})}P\,{}^{(c)}\mathcal{D}\widehat{\otimes}B - 3\overline{P}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B\big) + 2\,{}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}P\widehat{\otimes}B - 3\,{}^{(c)}\mathcal{D}\overline{P}\widehat{\otimes}\,{}^{(\mathbf{F})}B\\
&= \big(2\,{}^{(\mathbf{F})}P\,{}^{(c)}\mathcal{D}\widehat{\otimes}B - 3\overline{P}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B\big) - 4H\widehat{\otimes}\,{}^{(\mathbf{F})}PB\\
&\quad + \big(9\overline{P} - 6\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)H\widehat{\otimes}\,{}^{(\mathbf{F})}B.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\big(3\overline{P} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\mathfrak{F} + {}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3 A + \frac{1}{2}\,{}^{(\mathbf{F})}P\mathrm{tr}\underline{X}A\\
&\quad= \frac{1}{2}\big({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} + 4H\widehat{\otimes}\,{}^{(\mathbf{F})}PB - \big(9\overline{P} - 6\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)H\widehat{\otimes}\,{}^{(\mathbf{F})}B\big)\\
&\qquad + H\widehat{\otimes}\big(4\,{}^{(\mathbf{F})}PB - 3\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}P\,{}^{(\mathbf{F})}B - \frac{9}{2}\overline{P}\,{}^{(\mathbf{F})}B\big) = \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} + 3H\widehat{\otimes}\mathfrak{B},
\end{aligned}
$$

which proves (5.7).

A.2.2. *Derivation of* (5.8)

Using the definition of $\mathfrak{F}$ (5.2), we compute

$$
{}^{(c)}\nabla_4\mathfrak{F} = {}^{(c)}\nabla_4\left(-\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{3}{2}H\widehat{\otimes}\,{}^{(\mathbf{F})}B + {}^{(\mathbf{F})}P\widehat{X}\right)
$$

$$= -\frac{1}{2}\,{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{3}{2}\,{}^{(c)}\nabla_4 H\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{3}{2}H\widehat{\otimes}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B$$
$$+\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}P\widehat{X} + {}^{(\mathbf{F})}P\,{}^{(c)}\nabla_4\widehat{X}.$$

Recall the following commutator formula, for $F = f + i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$ of conformal type s, see [19, Lemma 5.3]:

$$[\,{}^{(c)}\nabla_4,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]F = -\frac{1}{2}\mathrm{tr}X(\,{}^{(c)}\mathcal{D}\widehat{\otimes}F + (1-s)\,\underline{H}\widehat{\otimes}F) + \underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4 F. \tag{A.10}$$

Applying (A.10) to $F = {}^{(\mathbf{F})}B$ and $s=1$, using (4.9), (4.3) and (4.25), we obtain

$$\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{F} &= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B - \frac{1}{2}\left(-\frac{1}{2}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B + \underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B\right)\\
&\quad - \frac{3}{2}\left(-\frac{1}{2}\overline{\mathrm{tr}X}(H-\underline{H})\right)\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{3}{2}H\widehat{\otimes}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B - \overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}P\widehat{X}\\
&\quad + {}^{(\mathbf{F})}P\left(-\frac{1}{2}(\mathrm{tr}X+\overline{\mathrm{tr}X})\widehat{X} + {}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + (H+\underline{H})\widehat{\otimes}\Xi - A\right)\\
&= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B + \frac{1}{4}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{1}{2}(3H+\underline{H})\widehat{\otimes}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B\\
&\quad + \frac{3}{4}\overline{\mathrm{tr}X}(H-\underline{H})\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}P\widehat{X} - \frac{1}{2}\mathrm{tr}X\,{}^{(\mathbf{F})}P\widehat{X}\\
&\quad + {}^{(\mathbf{F})}P\big(\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + (H+\underline{H})\widehat{\otimes}\Xi - A\big).
\end{aligned}$$

On the other hand, using the definition of $\mathfrak{X}$ (5.4) we compute using (4.10) and (4.4):

$$\begin{aligned}
{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} &= {}^{(c)}\mathcal{D}\widehat{\otimes}\left(\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B + \frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}B - 2\,{}^{(\mathbf{F})}P\Xi\right)\\
&= {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B + \frac{3}{2}\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr}X}\widehat{\otimes}\,{}^{(\mathbf{F})}B + \frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B\\
&\quad - 2\,{}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}P\widehat{\otimes}\Xi - 2\,{}^{(\mathbf{F})}P\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi\\
&= {}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B + \frac{3}{2}(\mathrm{tr}X-\overline{\mathrm{tr}X})H\widehat{\otimes}\,{}^{(\mathbf{F})}B + \frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B\\
&\quad + 4\,{}^{(\mathbf{F})}PH\widehat{\otimes}\Xi - 2\,{}^{(\mathbf{F})}P\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi.
\end{aligned}$$

This implies

$$\begin{aligned}
-\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B &= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} + \frac{3}{4}(\mathrm{tr}X-\overline{\mathrm{tr}X})H\widehat{\otimes}\,{}^{(\mathbf{F})}B + \frac{3}{4}\overline{\mathrm{tr}X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B\\
&\quad + 2\,{}^{(\mathbf{F})}PH\widehat{\otimes}\Xi - {}^{(\mathbf{F})}P\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi.
\end{aligned}$$

By plugging in to the above expression for ${}^{(c)}\nabla_4\mathfrak{F}$, we obtain

$$
\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{F} &= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} + \frac{3}{4}(\mathrm{tr}X - \overline{\mathrm{tr}X})H\widehat{\otimes}\,{}^{(\mathbf{F})}B + \frac{3}{4}\overline{\mathrm{tr}X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B + 2\,{}^{(\mathbf{F})}PH\widehat{\otimes}\Xi \\
&\quad - {}^{(\mathbf{F})}P\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \frac{1}{4}\mathrm{tr}X\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{1}{2}\,(3H + \underline{H})\,\widehat{\otimes}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B \\
&\quad + \frac{3}{4}\overline{\mathrm{tr}X}(H - \underline{H})\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}P\widehat{X} - \frac{1}{2}\mathrm{tr}X\,{}^{(\mathbf{F})}P\widehat{X} \\
&\quad + {}^{(\mathbf{F})}P\big({}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + (H + \underline{H})\widehat{\otimes}\Xi - A\big) \\
&= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} + \left(\frac{3}{4}\overline{\mathrm{tr}X} + \frac{1}{4}\mathrm{tr}X\right){}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B - \left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right){}^{(\mathbf{F})}P\widehat{X} \\
&\quad + \frac{3}{4}\mathrm{tr}X H\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{3}{4}\overline{\mathrm{tr}X}\,\underline{H}\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{1}{2}\big(3H + \underline{H}\big)\widehat{\otimes}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B \\
&\quad + {}^{(\mathbf{F})}P((3H + \underline{H})\widehat{\otimes}\Xi - A).
\end{aligned}
$$

Using again the definition of $\mathfrak{F}$ (5.2) to write $-\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B + {}^{(\mathbf{F})}P\widehat{X} = \mathfrak{F} + \frac{3}{2}H\widehat{\otimes}\,{}^{(\mathbf{F})}B$, we finally obtain

$$
\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{F} &= -\left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right)\mathfrak{F} - \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} - \left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right)\frac{3}{2}H\widehat{\otimes}\,{}^{(\mathbf{F})}B \\
&\quad + \frac{3}{4}\mathrm{tr}X H\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{3}{4}\overline{\mathrm{tr}X}\,\underline{H}\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{1}{2}\big(3H + \underline{H}\big)\widehat{\otimes}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B \\
&\quad + {}^{(\mathbf{F})}P((3H + \underline{H})\widehat{\otimes}\Xi - A) \\
&= -\left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right)\mathfrak{F} - \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} - \frac{1}{2}(3H + \underline{H}) \\
&\quad \widehat{\otimes}\left({}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B + \frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}B - 2\,{}^{(\mathbf{F})}P\Xi\right) - {}^{(\mathbf{F})}PA.
\end{aligned}
$$

Using again the definition of $\mathfrak{X}$ (5.4), this proves (5.8).

A.2.3. *Derivation of* (5.9)

Using the definition of $\mathfrak{B}$ (5.3), we compute

$$
\begin{aligned}
{}^{(c)}\nabla_4\mathfrak{B} + 3\overline{\mathrm{tr}X}\mathfrak{B} &= 2\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}PB + 2\,{}^{(\mathbf{F})}P\,{}^{(c)}\nabla_4 B - 3\,{}^{(c)}\nabla_4\overline{P}\,{}^{(\mathbf{F})}B - 3\overline{P}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B \\
&\quad + 3\overline{\mathrm{tr}X}\big(2\,{}^{(\mathbf{F})}PB - 3\overline{P}\,{}^{(\mathbf{F})}B\big).
\end{aligned}
$$

Using (4.35), (4.3), (4.12), we obtain

$$
\begin{aligned}
&{}^{(c)}\nabla_4\mathfrak{B} + 3\overline{\mathrm{tr}X}\mathfrak{B} \\
&\quad = -2\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}PB + 2\,{}^{(\mathbf{F})}P\left(\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot A - 2\overline{\mathrm{tr}X}B + \frac{1}{2}A\cdot\overline{\underline{H}} + 3\overline{P}\,\Xi\right.
\end{aligned}
$$

$$+\overline{{}^{(\mathbf{F})}P}\big({}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B-2\,{}^{(\mathbf{F})}P\Xi\big)\Big)-3\left(-\frac{3}{2}\overline{\mathrm{tr}X}\ \overline{P}-\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right){}^{(\mathbf{F})}B$$

$$-3\overline{P}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B+3\overline{\mathrm{tr}X}\big(2\,{}^{(\mathbf{F})}PB-3\overline{P}\,{}^{(\mathbf{F})}B\big),$$

which gives

$$\begin{aligned}{}^{(c)}\nabla_4\mathfrak{B}+3\overline{\mathrm{tr}X}\mathfrak{B}= {}&{}^{(\mathbf{F})}P\big(\overline{{}^{(c)}\mathcal{D}}\cdot A+\overline{\underline{H}}\cdot A\big)-\big(3\overline{P}-2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\\ &\times\left({}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B+\frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}B-2\,{}^{(\mathbf{F})}P\Xi\right),\end{aligned}$$

which, using the definition of $\mathfrak{X}$ (5.4), proves (5.9).

A.2.4. *Derivation of* (5.10)

Using the definition of $\mathfrak{X}$ (5.4), we compute

$$\begin{aligned}{}^{(c)}\nabla_3\mathfrak{X}&={}^{(c)}\nabla_3\left({}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B+\frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}B-2\,{}^{(\mathbf{F})}P\Xi\right)\\ &={}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}B+[\,{}^{(c)}\nabla_3,\,{}^{(c)}\nabla_4]\,{}^{(\mathbf{F})}B+\frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}B\\ &\quad+\frac{3}{2}\,{}^{(c)}\nabla_3\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}B-2\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}P\Xi-2\,{}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3\Xi.\end{aligned}$$

We compute each term. Using (4.16) and (4.7), we obtain

$$\begin{aligned}{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}B&={}^{(c)}\nabla_4\left(-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}B+{}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}P+2\,{}^{(\mathbf{F})}PH\right)\\ &=-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B-\frac{1}{2}\,{}^{(c)}\nabla_4\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}B+{}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}P\\ &\quad+2\,{}^{(\mathbf{F})}P\,{}^{(c)}\nabla_4H+2\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}PH\\ &=-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B+\left(\frac{1}{4}\mathrm{tr}X\mathrm{tr}\underline{X}-\frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{H}}-\frac{1}{2}\underline{H}\cdot\overline{\underline{H}}-\overline{P}\right)\\ &\quad\times{}^{(\mathbf{F})}B+{}^{(c)}\mathcal{D}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}P+[\,{}^{(c)}\nabla_4,\,{}^{(c)}\mathcal{D}]\,{}^{(\mathbf{F})}P+2\,{}^{(\mathbf{F})}P\,{}^{(c)}\nabla_4H\\ &\quad+2\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}PH\\ &=L_1+L_2+L_3+L_4+L_5.\end{aligned}$$

We simplify L_1 by making use of the definition of $\mathfrak{X}$ and writing ${}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B=\mathfrak{X}-\frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}B+2\,{}^{(\mathbf{F})}P\Xi$. We obtain

$$L_1=-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B+\left(\frac{1}{4}\mathrm{tr}X\mathrm{tr}\underline{X}-\frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{H}}-\frac{1}{2}\underline{H}\cdot\overline{\underline{H}}-\overline{P}\right){}^{(\mathbf{F})}B$$

$$
\begin{aligned}
&= -\frac{1}{2}\mathrm{tr}\underline{X}\left(\mathfrak{X} - \frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}B + 2\,{}^{(\mathbf{F})}P\Xi\right)\\
&\quad + \left(\frac{1}{4}\mathrm{tr}X\mathrm{tr}\underline{X} - \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{H}} - \frac{1}{2}\,\underline{H}\cdot\overline{\underline{H}} - \overline{P}\right){}^{(\mathbf{F})}B,
\end{aligned}
$$

which gives

$$
\begin{aligned}
L_1 &= -\frac{1}{2}\mathrm{tr}\underline{X}\ \mathfrak{X} + \left(\frac{1}{4}\mathrm{tr}X\mathrm{tr}\underline{X} + \frac{3}{4}\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} - \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{H}} - \frac{1}{2}\,\underline{H}\cdot\overline{\underline{H}} - \overline{P}\right){}^{(\mathbf{F})}B\\
&\quad + {}^{(\mathbf{F})}P(-\mathrm{tr}\underline{X}\Xi).
\end{aligned}
\tag{A.11}
$$

We compute L_2 using (4.18):

$$
\begin{aligned}
L_2 &= {}^{(c)}\mathcal{D}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}P = {}^{(c)}\mathcal{D}\left(-\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}P + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(\mathbf{F})}B + \frac{1}{2}\overline{\underline{H}}\cdot{}^{(\mathbf{F})}B\right)\\
&= -\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}P - \overline{\mathrm{tr}X}\,{}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}P + \frac{1}{2}\,{}^{(c)}\mathcal{D}(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(\mathbf{F})}B) + \frac{1}{2}\,{}^{(c)}\mathcal{D}(\overline{\underline{H}}\cdot{}^{(\mathbf{F})}B).
\end{aligned}
$$

Using (4.30) to write

$$
{}^{(c)}\mathcal{D}\overline{\mathrm{tr}X} = \overline{{}^{(c)}\mathcal{D}}\cdot\widehat{X} + (\mathrm{tr}X - \overline{\mathrm{tr}X})H + (\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}})\Xi + 2B - 2\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B
$$

and using the Leibniz rules to write ${}^{(c)}\mathcal{D}(\overline{\underline{H}}\cdot{}^{(\mathbf{F})}B) = ({}^{(c)}\mathcal{D}\cdot\overline{\underline{H}})\,{}^{(\mathbf{F})}B + \overline{\underline{H}}\cdot{}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}B$, we obtain

$$
\begin{aligned}
L_2 &= -\overline{\mathrm{tr}X}\,{}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}P - (\mathrm{tr}X - \overline{\mathrm{tr}X})H\,{}^{(\mathbf{F})}P + \frac{1}{2}\,{}^{(c)}\mathcal{D}(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(\mathbf{F})}B) + \frac{1}{2}\overline{\underline{H}}\cdot{}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}B\\
&\quad - 2\,{}^{(\mathbf{F})}PB + \left(2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} + \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{H}}\right){}^{(\mathbf{F})}B\\
&\quad + {}^{(\mathbf{F})}P\big(-\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{X} - (\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}})\Xi\big).
\end{aligned}
$$

Using the definition of $\mathfrak{B}$ to write $2\,{}^{(\mathbf{F})}PB = \mathfrak{B} + 3\overline{P}\,{}^{(\mathbf{F})}B$, we finally obtain

$$
\begin{aligned}
L_2 &= -\mathfrak{B} - \overline{\mathrm{tr}X}\,{}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}P - (\mathrm{tr}X - \overline{\mathrm{tr}X})H\,{}^{(\mathbf{F})}P + \frac{1}{2}\,{}^{(c)}\mathcal{D}(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(\mathbf{F})}B)\\
&\quad + \frac{1}{2}\overline{\underline{H}}\cdot{}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}B + \left(-3\overline{P} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} + \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{H}}\right){}^{(\mathbf{F})}B\\
&\quad + {}^{(\mathbf{F})}P\big(-\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{X} - (\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}})\Xi\big).
\end{aligned}
\tag{A.12}
$$

To compute L_3 we first have the following lemma.

Lemma A.3. *Let $G = g_1 + ig_2 \in \mathfrak{s}_0(\mathbb{C})$ be a 0-conformal invariant scalar function. Then*

$$[\,^{(c)}\nabla_4, \,^{(c)}\mathcal{D}]G = -\frac{1}{2}\mathrm{tr}X \,^{(c)}\mathcal{D}G + \underline{H} \,^{(c)}\nabla_4 G + \Xi \,^{(c)}\nabla_3 G - \frac{1}{2}\widehat{X} \cdot \overline{\,^{(c)}\mathcal{D}}G. \tag{A.13}$$

Proof. From the commutators, see [19, Lemma 2.39]

$$[\nabla_4, \nabla_a]G = -\frac{1}{2}\left(\mathrm{tr}\,\chi\nabla_a G + \,^{(a)}\mathrm{tr}\chi \,^*\nabla G\right) + (\underline{\eta}_a + \zeta_a)\nabla_4 G - \hat{\chi}_{ab}\nabla_b G + \xi_a \nabla_3 G,$$

$$\begin{aligned}[\nabla_4, \,^*\nabla_a]G = &-\frac{1}{2}\left(\mathrm{tr}\,\chi \,^*\nabla_a G - \,^{(a)}\mathrm{tr}\chi\nabla G\right) + \,^*(\underline{\eta}_a + \zeta_a)\nabla_4 G\\ &- \,^*\hat{\chi}_{ab}\nabla_b G + \,^*\xi_a\nabla_3 G,\end{aligned}$$

we obtain

$$\begin{aligned}[\nabla_4, \mathcal{D}]G = &-\frac{1}{2}\big(\mathrm{tr}\,\chi\nabla_a G + \,^{(a)}\mathrm{tr}\chi \,^*\nabla G\big) + (\underline{H} + Z)\nabla_4 G - \hat{\chi}_{ab}\nabla_b G + \Xi\nabla_3 G\\ &-\frac{1}{2}i\big(\mathrm{tr}\,\chi \,^*\nabla_a G - \,^{(a)}\mathrm{tr}\chi\nabla G\big) - i \,^*\hat{\chi}_{ab}\nabla_b G.\end{aligned}$$

Writing $\nabla = \frac{1}{2}\mathcal{D} + \frac{1}{2}\overline{\mathcal{D}}$, we obtain the desired formula with non-conformal derivatives. Using conformal derivatives we have

$$[\,^{(c)}\nabla_4, \,^{(c)}\mathcal{D}]G = \,^{(c)}\nabla_4 \,^{(c)}\mathcal{D}G - \,^{(c)}\mathcal{D} \,^{(c)}\nabla_4 G = \nabla_4\mathcal{D}G - \mathcal{D}\nabla_4 G - Z\nabla_4 G,$$

which gives the desired formula. □

We compute L_3 by applying (A.13) to $^{(\mathbf{F})}P$ (which is of conformal type 0) and using (4.18), (4.3) and (4.4), we obtain

$$\begin{aligned}L_3 &= [\,^{(c)}\nabla_4, \,^{(c)}\mathcal{D}] \,^{(\mathbf{F})}P\\ &= -\frac{1}{2}\mathrm{tr}X \,^{(c)}\mathcal{D} \,^{(\mathbf{F})}P + \underline{H} \,^{(c)}\nabla_4 \,^{(\mathbf{F})}P + \Xi \,^{(c)}\nabla_3 \,^{(\mathbf{F})}P - \frac{1}{2}\widehat{X} \cdot \overline{\,^{(c)}\mathcal{D}} \,^{(\mathbf{F})}P\\ &= -\frac{1}{2}\mathrm{tr}X \,^{(c)}\mathcal{D} \,^{(\mathbf{F})}P + \underline{H}\left(-\overline{trX} \,^{(\mathbf{F})}P + \frac{1}{2}\overline{\,^{(c)}\mathcal{D}} \cdot \,^{(\mathbf{F})}B + \frac{1}{2}\overline{\underline{H}} \cdot \,^{(\mathbf{F})}B\right)\\ &\quad - \mathrm{tr}\underline{X} \,^{(\mathbf{F})}P\Xi - \frac{1}{2}\widehat{X} \cdot (-2\overline{\underline{H}} \,^{(\mathbf{F})}P),\end{aligned}$$

which gives

$$\begin{aligned}L_3 = &-\frac{1}{2}\mathrm{tr}X \,^{(c)}\mathcal{D} \,^{(\mathbf{F})}P - \overline{\mathrm{tr}X} \,^{(\mathbf{F})}P\,\underline{H} + \frac{1}{2}\underline{H} \cdot \overline{\,^{(c)}\mathcal{D}} \,^{(\mathbf{F})}B + \frac{1}{2}(\underline{H} \cdot \overline{\underline{H}}) \,^{(\mathbf{F})}B\\ &+ \,^{(\mathbf{F})}P\big(\widehat{X} \cdot \overline{\underline{H}} - \mathrm{tr}\underline{X}\Xi\big).\end{aligned} \tag{A.14}$$

We compute L_4 using (4.29)

$$\begin{aligned}L_4 &= 2 \,^{(\mathbf{F})}P \,^{(c)}\nabla_4 H\\ &= 2 \,^{(\mathbf{F})}P\left(-\frac{1}{2}\overline{\mathrm{tr}X}(H - \underline{H}) + \,^{(c)}\nabla_3\Xi - \frac{1}{2}\widehat{X} \cdot (\overline{H} - \overline{\underline{H}}) - B - \overline{\,^{(\mathbf{F})}P} \,^{(\mathbf{F})}B\right),\end{aligned}$$

which can be written as

$$L_4 = -\mathfrak{B} - {}^{(\mathbf{F})}\!P\overline{\mathrm{tr}X}(H - \underline{H}) + \big(-3\overline{P} - 2\overline{{}^{(\mathbf{F})}\!P}\,{}^{(\mathbf{F})}\!P\big)\,{}^{(\mathbf{F})}\!B + {}^{(\mathbf{F})}\!P\big(2\,{}^{(c)}\nabla_3\Xi - \widehat{X}\cdot(\overline{H} - \overline{\underline{H}})\big). \tag{A.15}$$

We compute L_5 using (4.18)

$$L_5 = 2\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}\!PH = 2\left(-\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}\!P + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(\mathbf{F})}\!B + \frac{1}{2}\overline{\underline{H}}\cdot{}^{(\mathbf{F})}\!B\right)H,$$

which can be written as

$$L_5 = -2\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}\!PH + H\cdot\overline{{}^{(c)}\mathcal{D}}\,{}^{(\mathbf{F})}\!B + \big(H\cdot\overline{\underline{H}}\big)\,{}^{(\mathbf{F})}\!B. \tag{A.16}$$

Putting together (A.11), (A.12), (A.14)–(A.16), we obtain

$$\begin{aligned}
&{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}\!B\\
&\quad= -\frac{1}{2}\mathrm{tr}\underline{X}\;\mathfrak{X} - 2\mathfrak{B} - \left(\frac{1}{2}\mathrm{tr}X + \overline{\mathrm{tr}X}\right)\big({}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}\!P + 2H\,{}^{(\mathbf{F})}\!P\big)\\
&\qquad+ \frac{1}{2}\,{}^{(c)}\mathcal{D}\big(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(\mathbf{F})}\!B\big) + \frac{1}{2}\overline{\underline{H}}\cdot{}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}\!B + \left(H + \frac{1}{2}\underline{H}\right)\cdot\overline{{}^{(c)}\mathcal{D}}\,{}^{(\mathbf{F})}\!B\\
&\qquad+ \left(\frac{1}{4}\mathrm{tr}X\mathrm{tr}\underline{X} + \frac{3}{4}\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} - 7\overline{P} + H\cdot\overline{\underline{H}}\right){}^{(\mathbf{F})}\!B\\
&\qquad+ {}^{(\mathbf{F})}\!P\big(- \overline{{}^{(c)}\mathcal{D}}\cdot\widehat{X} + \widehat{X}\cdot(-\overline{H} + 2\overline{\underline{H}}) + 2\,{}^{(c)}\nabla_3\Xi - (3\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}})\Xi\big).
\end{aligned}$$

Using (4.16) to substitute ${}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}\!P + 2\,{}^{(\mathbf{F})}\!PH = {}^{(c)}\nabla_3\,{}^{(\mathbf{F})}\!B + \frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}\!B$ we obtain

$$\begin{aligned}
&{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}\!B\\
&\quad= -\frac{1}{2}\mathrm{tr}\underline{X}\;\mathfrak{X} - 2\mathfrak{B} - \left(\frac{1}{2}\mathrm{tr}X + \overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3\,{}^{(\mathbf{F})}\!B + \frac{1}{2}\,{}^{(c)}\mathcal{D}\big(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(\mathbf{F})}\!B\big)\\
&\qquad+ \frac{1}{2}\overline{\underline{H}}\cdot{}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}\!B + \left(H + \frac{1}{2}\underline{H}\right)\cdot\overline{{}^{(c)}\mathcal{D}}\,{}^{(\mathbf{F})}\!B\\
&\qquad+ \left(\frac{1}{4}\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} - 7\overline{P} + H\cdot\overline{\underline{H}}\right){}^{(\mathbf{F})}\!B + {}^{(\mathbf{F})}\!P\big(- \overline{{}^{(c)}\mathcal{D}}\cdot\widehat{X} + \widehat{X}\cdot(-\overline{H}\\
&\qquad+ 2\overline{\underline{H}}) + 2\,{}^{(c)}\nabla_3\Xi - (3\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}})\Xi\big).
\end{aligned}\tag{A.17}$$

In order to compute $[\,{}^{(c)}\nabla_3, {}^{(c)}\nabla_4]\,{}^{(\mathbf{F})}\!B$, we first have the following lemma.

Lemma A.4. *Let $F = f + i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$ be of conformal type s. Then*

$$\begin{aligned}
&[\,{}^{(c)}\nabla_3,\,{}^{(c)}\nabla_4]F\\
&\quad= \frac{1}{2}(H - \underline{H})\cdot \overline{{}^{(c)}\mathcal{D}}F + \frac{1}{2}(\overline{H} - \overline{\underline{H}})\cdot {}^{(c)}\mathcal{D}F\\
&\qquad + \left((s-1)P + (s+1)\overline{P} + 2s\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} - \frac{s+1}{2}(H\cdot\overline{\underline{H}}) + \frac{1-s}{2}(\overline{H}\cdot\underline{H})\right)F.
\end{aligned}\tag{A.18}$$

Proof. From, see [19, Lemma 2.39]

$$\begin{aligned}
[\nabla_4,\nabla_3]f_a &= 2\omega\nabla_3 f_a - 2\underline{\omega}\nabla_4 f_a + 2(\underline{\eta}_b - \eta_b)\nabla_b f_a + 2(\underline{\eta}\cdot f)\eta_a\\
&\quad - 2(\eta\cdot f)\underline{\eta}_a - 2\,{}^*\rho\,{}^*f_a\\
[\nabla_4,\nabla_3]\,{}^*f_a &= 2\omega\nabla_3\,{}^*f_a - 2\underline{\omega}\nabla_4\,{}^*f_a + 2(\underline{\eta}_b - \eta_b)\nabla_b\,{}^*f_a + 2(\underline{\eta}\cdot\,{}^*f)\eta_a\\
&\quad - 2(\eta\cdot\,{}^*f)\underline{\eta}_a + 2\,{}^*\rho f_a
\end{aligned}$$

we derive for $F = f + i\,{}^*f$,

$$\begin{aligned}
[\nabla_4,\nabla_3]F_a &= 2\omega\nabla_3 F_a - 2\underline{\omega}\nabla_4 F_a + 2(\underline{\eta}_b - \eta_b)\nabla_b F_a\\
&\quad + (\overline{\underline{H}}\cdot F)\eta_a - (\overline{H}\cdot F)\underline{\eta}_a + (P - \overline{P})F_a\\
&= 2\omega\nabla_3 F_a - 2\underline{\omega}\nabla_4 F_a + 2(\underline{\eta}_b - \eta_b)\nabla_b F_a + \frac{1}{2}(\overline{\underline{H}}\cdot H)F_a\\
&\quad - \frac{1}{2}(\overline{H}\cdot\underline{H})F_a + (P - \overline{P})F_a.
\end{aligned}$$

We have

$$[\,{}^{(c)}\nabla_3,\,{}^{(c)}\nabla_4]F = [\nabla_3,\nabla_4]F - 2\underline{\omega}\nabla_4 F + 2\omega\nabla_3 F + 2s(\nabla_3\omega + \nabla_4\underline{\omega} - 4\omega\underline{\omega})F$$

and using (4.32), we obtain

$$\begin{aligned}
[\,{}^{(c)}\nabla_3,\,{}^{(c)}\nabla_4]F &= -2\omega\nabla_3 F + 2\underline{\omega}\nabla_4 F - 2(\underline{\eta} - \eta)\cdot\nabla F - \frac{1}{2}(\overline{\underline{H}}\cdot H)F_a\\
&\quad + \frac{1}{2}(\overline{H}\cdot\underline{H})F_a - (P - \overline{P})F - 2\underline{\omega}\nabla_4 F + 2\omega\nabla_3 F\\
&\quad + 2s\left(\frac{1}{2}(P + \overline{P}) + {}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} + (\eta - \underline{\eta})\cdot\zeta - \eta\cdot\underline{\eta}\right)F\\
&= 2(\eta - \underline{\eta})\cdot\,{}^{(c)}\nabla F + \left((s-1)P + (s+1)\overline{P} + 2s\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)F\\
&\quad - \frac{1}{2}(\overline{\underline{H}}\cdot H)F_a + \frac{1}{2}(\overline{H}\cdot\underline{H})F_a - \frac{s}{2}(H\cdot\overline{\underline{H}} + \overline{H}\cdot\underline{H})F.
\end{aligned}$$

Finally observe that $2(\eta - \underline{\eta})\cdot\,{}^{(c)}\nabla F = \frac{1}{2}(H - \underline{H})\cdot\overline{{}^{(c)}\mathcal{D}}F + \frac{1}{2}(\overline{H} - \overline{\underline{H}})\cdot\,{}^{(c)}\mathcal{D}F$. ∎

Specializing (A.18) to $F = {}^{(\mathbf{F})}B$ and $s = 1$, we compute

$$[\,{}^{(c)}\nabla_3,\ {}^{(c)}\nabla_4]\,{}^{(\mathbf{F})}B = \frac{1}{2}(H - \underline{H}) \cdot \overline{{}^{(c)}\mathcal{D}}\,{}^{(\mathbf{F})}B + \frac{1}{2}(\overline{H} - \overline{\underline{H}}) \cdot {}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}B + \left(2\overline{P} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} - H \cdot \overline{\underline{H}}\right){}^{(\mathbf{F})}B. \tag{A.19}$$

We also compute using (4.22)

$$\frac{3}{2}\,{}^{(c)}\nabla_3\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}B = \left(-\frac{3}{4}\overline{\mathrm{tr}\underline{X}}\ \overline{\mathrm{tr}X} + \frac{3}{2}\overline{{}^{(c)}\mathcal{D}} \cdot H + \frac{3}{2}H \cdot \overline{H} + 3\overline{P}\right){}^{(\mathbf{F})}B \tag{A.20}$$

and using (4.3)

$$-2\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}P\Xi = 2\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}P\Xi. \tag{A.21}$$

Using (A.17), (A.19)–(A.21), we obtain

$$\begin{aligned}
&{}^{(c)}\nabla_3\mathfrak{X}\\
&= {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}B + [\,{}^{(c)}\nabla_3,\ {}^{(c)}\nabla_4]\,{}^{(\mathbf{F})}B + \frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}B + \frac{3}{2}\,{}^{(c)}\nabla_3\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}B\\
&\quad - 2\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}P\Xi - 2\,{}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3\Xi\\
&= -\frac{1}{2}\mathrm{tr}\underline{X}\ \mathfrak{X} - 2\mathfrak{B} - \left(\frac{1}{2}\mathrm{tr}X + \overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3\,{}^{(\mathbf{F})}B + \frac{1}{2}\,{}^{(c)}\mathcal{D}(\overline{{}^{(c)}\mathcal{D}} \cdot {}^{(\mathbf{F})}B) + \frac{1}{2}\overline{\underline{H}}\\
&\quad \cdot {}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}B + \left(H + \frac{1}{2}\underline{H}\right) \cdot \overline{{}^{(c)}\mathcal{D}}\,{}^{(\mathbf{F})}B + \left(\frac{1}{4}\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} - 7\overline{P} + H \cdot \overline{\underline{H}}\right){}^{(\mathbf{F})}B\\
&\quad + {}^{(\mathbf{F})}P\left(-\overline{{}^{(c)}\mathcal{D}} \cdot \widehat{X} + \widehat{X} \cdot (-\overline{H} + 2\overline{\underline{H}}) + 2\,{}^{(c)}\nabla_3\Xi - (3\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}})\Xi\right)\\
&\quad + \frac{1}{2}(H - \underline{H}) \cdot \overline{{}^{(c)}\mathcal{D}}\,{}^{(\mathbf{F})}B + \frac{1}{2}(\overline{H} - \overline{\underline{H}}) \cdot {}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}B\\
&\quad + \left(2\overline{P} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} - H \cdot \overline{\underline{H}}\right){}^{(\mathbf{F})}B + \frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}B\\
&\quad + \left(-\frac{3}{4}\overline{\mathrm{tr}\underline{X}}\ \overline{\mathrm{tr}X} + \frac{3}{2}\overline{{}^{(c)}\mathcal{D}} \cdot H + \frac{3}{2}H \cdot \overline{H} + 3\overline{P}\right){}^{(\mathbf{F})}B\\
&\quad + 2\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}P\Xi - 2\,{}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3\Xi,
\end{aligned}$$

which gives

$$\begin{aligned}
&{}^{(c)}\nabla_3\mathfrak{X} + \frac{1}{2}\mathrm{tr}\underline{X}\ \mathfrak{X} + 2\mathfrak{B}\\
&= \frac{1}{2}\,{}^{(c)}\mathcal{D}(\overline{{}^{(c)}\mathcal{D}} \cdot {}^{(\mathbf{F})}B) + \frac{1}{2}\overline{H} \cdot {}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}B + \frac{3}{2}H \cdot \overline{{}^{(c)}\mathcal{D}}\,{}^{(\mathbf{F})}B\\
&\quad - \frac{1}{2}\left(\mathrm{tr}X - \overline{\mathrm{tr}X}\right){}^{(c)}\nabla_3\,{}^{(\mathbf{F})}B
\end{aligned}$$

$$+\left(\frac{1}{4}\overline{\mathrm{tr}X}\mathrm{tr}\underline{X}-\frac{3}{4}\overline{\mathrm{tr}X}\ \overline{\mathrm{tr}\underline{X}}-2\overline{P}+2\,{}^{(\mathbf{F})}\!P\,\overline{{}^{(\mathbf{F})}\!P}+\frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H+\frac{3}{2}H\cdot\overline{H}\right){}^{(\mathbf{F})}\!B$$
$$+\,{}^{(\mathbf{F})}\!P\big(-\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{X}+\widehat{X}\cdot(-\overline{H}+2\underline{\overline{H}})-(\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}})\Xi\big). \tag{A.22}$$

We now write the right-hand side of the equation in terms of $\mathfrak{F}$. We have the following lemma.

Lemma A.5. *The following formula for the divergence of* $\mathfrak{F}$ *holds*:

$$\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}+\overline{H}\cdot\mathfrak{F}$$
$$=-\frac{1}{2}(\overline{\mathrm{tr}\underline{X}}-\mathrm{tr}\underline{X})\mathfrak{X}-\frac{1}{2}\,{}^{(c)}\mathcal{D}(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(\mathbf{F})}\!B)-\frac{3}{2}H\cdot\overline{{}^{(c)}\mathcal{D}}\,{}^{(\mathbf{F})}\!B$$
$$-\frac{1}{2}\overline{H}\cdot{}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}\!B-\frac{1}{2}(\overline{\mathrm{tr}X}-\mathrm{tr}X)\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}\!B$$
$$+\left[\frac{1}{4}\mathrm{tr}X\overline{\mathrm{tr}\underline{X}}+\frac{3}{4}\overline{\mathrm{tr}\underline{X}\mathrm{tr}X}-\frac{1}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X}-\underline{\omega}(\overline{\mathrm{tr}X}-\mathrm{tr}X)+\frac{1}{2}\mathcal{D}\cdot\overline{\underline{Z}}-\frac{1}{2}\overline{\mathcal{D}}\cdot Z\right.$$
$$\left.+P+\overline{P}-2\,{}^{(\mathbf{F})}\!P\,\overline{{}^{(\mathbf{F})}\!P}-\frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H-\frac{3}{2}\overline{H}\cdot H\right]{}^{(\mathbf{F})}\!B$$
$$+\,{}^{(\mathbf{F})}\!P\big(\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{X}+\big(\overline{H}-2\underline{\overline{H}}\big)\cdot\widehat{X}-(\overline{\mathrm{tr}\underline{X}}-\mathrm{tr}\underline{X})\,{}^{(\mathbf{F})}\!P\Xi\big).$$

Proof. Using the definition of $\mathfrak{F}$ (5.2) we compute

$$\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}+\overline{H}\cdot\mathfrak{F}=\overline{{}^{(c)}\mathcal{D}}\cdot\left(-\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}\!B-\frac{3}{2}H\widehat{\otimes}\,{}^{(\mathbf{F})}\!B+{}^{(\mathbf{F})}\!P\widehat{X}\right)$$
$$+\overline{H}\cdot\left(-\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}\!B-\frac{3}{2}H\widehat{\otimes}\,{}^{(\mathbf{F})}\!B+{}^{(\mathbf{F})}\!P\widehat{X}\right)$$
$$=-\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot(\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}\!B)-\frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot(H\widehat{\otimes}\,{}^{(\mathbf{F})}\!B)+{}^{(\mathbf{F})}\!P\,\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{X}$$
$$+\overline{{}^{(c)}\mathcal{D}}\,{}^{(\mathbf{F})}\!P\cdot\widehat{X}-\frac{1}{2}\overline{H}\cdot{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}\!B-\frac{3}{2}\overline{H}\cdot\big(H\widehat{\otimes}\,{}^{(\mathbf{F})}\!B\big)$$
$$+\,{}^{(\mathbf{F})}\!P\widehat{X}\cdot\overline{H}.$$

Using Leibniz rules, the above simplifies to

$$\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}+\overline{H}\cdot\mathfrak{F}=-\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot(\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}\!B)-\frac{3}{2}H\cdot\overline{{}^{(c)}\mathcal{D}}\,{}^{(\mathbf{F})}\!B-\frac{1}{2}\overline{H}\cdot{}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}\!B$$
$$+\left(-\frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H-\frac{3}{2}\overline{H}\cdot H\right){}^{(\mathbf{F})}\!B+{}^{(\mathbf{F})}\!P\big(\overline{{}^{(c)}\mathcal{D}}\cdot\widehat{X}$$
$$+\big(\overline{H}-2\underline{\overline{H}}\big)\cdot\widehat{X}\big).$$

Since ${}^{(\mathbf{F})}B$ is of conformal type 1, we have

$$
\begin{aligned}
\overline{{}^{(c)}\mathcal{D}} \cdot ({}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B) &= \overline{\mathcal{D}} \cdot (\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B) + Z \cdot \overline{\mathcal{D}}\,{}^{(\mathbf{F})}B + \overline{Z} \cdot \mathcal{D}\,{}^{(\mathbf{F})}B \\
&\quad + (\overline{\mathcal{D}} \cdot Z + \overline{Z} \cdot Z)\,{}^{(\mathbf{F})}B, \\
{}^{(c)}\mathcal{D}(\overline{{}^{(c)}\mathcal{D}} \cdot {}^{(\mathbf{F})}B) &= \mathcal{D}(\overline{\mathcal{D}} \cdot {}^{(\mathbf{F})}B) + Z \cdot \overline{\mathcal{D}}\,{}^{(\mathbf{F})}B + \overline{Z} \cdot \mathcal{D}\,{}^{(\mathbf{F})}B + (\mathcal{D} \cdot \overline{Z} + \overline{Z} \cdot Z)\,{}^{(\mathbf{F})}B.
\end{aligned}
$$

Using the relation (A.30) of Lemma A.7, which we prove below, applied to ${}^{(\mathbf{F})}B$:

$$
\begin{aligned}
\overline{\mathcal{D}} \cdot (\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B) &= \mathcal{D}(\overline{\mathcal{D}} \cdot {}^{(\mathbf{F})}B) + (\overline{\mathrm{tr}X} - \mathrm{tr}X)\nabla_3\,{}^{(\mathbf{F})}B + (\overline{\mathrm{tr}\underline{X}} - \mathrm{tr}\underline{X})\nabla_4\,{}^{(\mathbf{F})}B \\
&\quad - \left(\frac{1}{2}\mathrm{tr}X\overline{\mathrm{tr}\underline{X}} + \frac{1}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + 2P + 2\overline{P} - 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right){}^{(\mathbf{F})}B
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\overline{{}^{(c)}\mathcal{D}} \cdot ({}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B) \\
&\quad = \mathcal{D}(\overline{\mathcal{D}} \cdot {}^{(\mathbf{F})}B) + Z \cdot \overline{\mathcal{D}}\,{}^{(\mathbf{F})}B + \overline{Z} \cdot \mathcal{D}\,{}^{(\mathbf{F})}B + (\overline{\mathcal{D}} \cdot Z + \overline{Z} \cdot Z)\,{}^{(\mathbf{F})}B \\
&\qquad + (\overline{\mathrm{tr}X} - \mathrm{tr}X)\nabla_3\,{}^{(\mathbf{F})}B + (\overline{\mathrm{tr}\underline{X}} - \mathrm{tr}\underline{X})\nabla_4\,{}^{(\mathbf{F})}B \\
&\qquad - \left(\frac{1}{2}\mathrm{tr}X\overline{\mathrm{tr}\underline{X}} + \frac{1}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + 2P + 2\overline{P} - 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right) \\
&\quad = {}^{(c)}\mathcal{D}(\overline{{}^{(c)}\mathcal{D}} \cdot {}^{(\mathbf{F})}B) + (\overline{\mathrm{tr}X} - \mathrm{tr}X)\big({}^{(c)}\nabla_3\,{}^{(\mathbf{F})}B + 2\underline{\omega}\,{}^{(\mathbf{F})}B\big) \\
&\qquad + (\overline{\mathrm{tr}\underline{X}} - \mathrm{tr}\underline{X})\nabla_4\,{}^{(\mathbf{F})}B \\
&\qquad - \left(\frac{1}{2}\mathrm{tr}X\overline{\mathrm{tr}\underline{X}} + \frac{1}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + 2P + 2\overline{P} - 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} + \mathcal{D} \cdot \overline{Z} - \overline{\mathcal{D}} \cdot Z\right).
\end{aligned}
$$

This finally gives

$$
\begin{aligned}
\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F} + \overline{H} \cdot \mathfrak{F} &= -\frac{1}{2}\,{}^{(c)}\mathcal{D}(\overline{{}^{(c)}\mathcal{D}} \cdot {}^{(\mathbf{F})}B) - \frac{3}{2}H \cdot \overline{{}^{(c)}\mathcal{D}}\,{}^{(\mathbf{F})}B - \frac{1}{2}\overline{H} \cdot {}^{(c)}\mathcal{D}\,{}^{(\mathbf{F})}B \\
&\quad - \frac{1}{2}(\overline{\mathrm{tr}X} - \mathrm{tr}X)\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}B - \frac{1}{2}(\overline{\mathrm{tr}\underline{X}} - \mathrm{tr}\underline{X})\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B \\
&\quad + \Bigg[\frac{1}{4}\mathrm{tr}X\overline{\mathrm{tr}\underline{X}} + \frac{1}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} - \underline{\omega}(\overline{\mathrm{tr}X} - \mathrm{tr}X) + P + \overline{P} - 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} \\
&\quad - \frac{3}{2}\overline{{}^{(c)}\mathcal{D}} \cdot H - \frac{3}{2}\overline{H} \cdot H + \frac{1}{2}\mathcal{D} \cdot \overline{Z} - \frac{1}{2}\overline{\mathcal{D}} \cdot Z\Bigg]\,{}^{(\mathbf{F})}B \\
&\quad + {}^{(\mathbf{F})}P\big(\overline{{}^{(c)}\mathcal{D}} \cdot \widehat{X} + \big(\overline{H} - 2\underline{H}\big) \cdot \widehat{X}\big).
\end{aligned}
$$

Using the definition of $\mathfrak{X}$ to write ${}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B = \mathfrak{X} - \frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}B + 2\,{}^{(\mathbf{F})}P\Xi$ we obtain the final expression. $\square$

We can therefore write the right-hand side of (A.22) as

$$
\begin{aligned}
{}^{(c)}\nabla_3\mathfrak{X} + \frac{1}{2}\mathrm{tr}\underline{X}\ \mathfrak{X} + 2\mathfrak{B} = & -\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} - \overline{H}\cdot\mathfrak{F} - \frac{1}{2}(\overline{\mathrm{tr}\underline{X}} - \mathrm{tr}\underline{X})\mathfrak{X}\\
& + \left(\frac{1}{4}\mathrm{tr}X\overline{\mathrm{tr}\underline{X}} - \frac{1}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} - \underline{\omega}(\overline{\mathrm{tr}X} - \mathrm{tr}X) + \frac{1}{2}\mathcal{D}\cdot\overline{Z}\right.\\
& \left. - \frac{1}{2}\overline{\mathcal{D}}\cdot Z + P - \overline{P}\right)\ {}^{(\mathbf{F})}B.
\end{aligned}
$$

Observe that the coefficient of ${}^{(\mathbf{F})}B$ vanishes. We therefore obtain (5.10).

We are now left to prove Lemma A.7. We first recall the following Gauss equation.

Proposition A.6. *We have*

(1) *For* $\psi \in \mathfrak{s}_1(\mathbb{C})$:

$$
\begin{aligned}
\big(\nabla_1\nabla_2 - \nabla_2\nabla_1\big)\psi = & \frac{1}{2}\left({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\right)\psi\\
& + i\left(\frac{1}{4}tr\,\chi\mathrm{tr}\,\underline{\chi} + \frac{1}{4}\,{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + \rho - {}^{(\mathbf{F})}\rho^2 - {}^{*\,(\mathbf{F})}\rho^2\right)\psi
\end{aligned}
\tag{A.23}
$$

or also:

$$
\begin{aligned}
\big(\nabla_1\nabla_2 - \nabla_2\nabla_1\big)\psi = & \frac{1}{2}\left({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\right)\psi\\
& + \frac{1}{2}i\left(\frac{1}{4}\mathrm{tr}X\overline{\mathrm{tr}\underline{X}} + \frac{1}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + P + \overline{P} - 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\psi.
\end{aligned}
\tag{A.24}
$$

(2) *For* $\Psi \in \mathfrak{s}_2(\mathbb{C})$:

$$
\begin{aligned}
\big(\nabla_1\nabla_2 - \nabla_2\nabla_1\big)\Psi = & \frac{1}{2}\left({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\right)\Psi\\
& + 2i\left(\frac{1}{4}tr\,\chi\mathrm{tr}\,\underline{\chi} + \frac{1}{4}\,{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi} + \rho - {}^{(\mathbf{F})}\rho^2 - {}^{*\,(\mathbf{F})}\rho^2\right)\Psi
\end{aligned}
\tag{A.25}
$$

or also

$$
\begin{aligned}
\big(\nabla_1\nabla_2 - \nabla_2\nabla_1\big)\Psi = & \frac{1}{2}\left({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4\right)\Psi\\
& + i\left(\frac{1}{4}\mathrm{tr}X\overline{\mathrm{tr}\underline{X}} + \frac{1}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + P + \overline{P} - 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\Psi.
\end{aligned}
\tag{A.26}
$$

Proof. See [19, Proposition 5.5]. □

We are now left to prove the following lemma.

Lemma A.7. *Let* $\psi \in \mathfrak{s}_1(\mathbb{C})$ *and* $\Psi \in \mathfrak{s}_2(\mathbb{C})$. *The following relations hold true*:

$$\begin{aligned}\mathcal{D}(\overline{\mathcal{D}} \cdot \psi) &= 2\Delta_1\psi - i({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4)\psi \\ &\quad + \left(\frac{1}{4}\mathrm{tr}X\overline{\mathrm{tr}\underline{X}} + \frac{1}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + P + \overline{P} - 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\psi, \qquad (A.27)\\ \overline{\mathcal{D}} \cdot (\mathcal{D}\widehat{\otimes}\psi) &= 2\Delta_1\psi + i({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4)\psi \\ &\quad - \left(\frac{1}{4}\mathrm{tr}X\overline{\mathrm{tr}\underline{X}} + \frac{1}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + P + \overline{P} - 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\psi, \qquad (A.28)\\ \mathcal{D}\widehat{\otimes}(\overline{\mathcal{D}} \cdot \Psi) &= 2\Delta_2\Psi - i({}^{(a)}tr\chi\nabla_3 + {}^{(a)}tr\underline{\chi}\nabla_4)\Psi \\ &\quad + \left(\frac{1}{2}\mathrm{tr}X\overline{\mathrm{tr}\underline{X}} + \frac{1}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + 2P + 2\overline{P} - 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\Psi. \qquad (A.29)\end{aligned}$$

In particular,

$$\begin{aligned}\overline{\mathcal{D}} \cdot (\mathcal{D}\widehat{\otimes}\psi) &= \mathcal{D}(\overline{\mathcal{D}} \cdot \psi) + (\overline{\mathrm{tr}X} - \mathrm{tr}X)\nabla_3\psi + (\overline{\mathrm{tr}\underline{X}} - \mathrm{tr}\underline{X})\nabla_4\psi \\ &\quad - \left(\frac{1}{2}\mathrm{tr}X\overline{\mathrm{tr}\underline{X}} + \frac{1}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + 2P + 2\overline{P} - 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\psi. \qquad (A.30)\end{aligned}$$

Proof. Define $Z_a := (\mathcal{D}(\overline{\mathcal{D}} \cdot \psi))_a = \mathcal{D}_a\overline{\mathcal{D}}^b\psi_b$ and evaluate it in the frame. We have

$$\begin{aligned}Z_1 &= \mathcal{D}_1\overline{\mathcal{D}}_1\psi_1 + \mathcal{D}_1\overline{\mathcal{D}}_2\psi_2\\ &= \big(\nabla_1 + i\,{}^*\nabla_1\big)\big(\nabla_1 - i\,{}^*\nabla_1\big)\psi_1 + \big(\nabla_1 + i\,{}^*\nabla_1\big)\big(\nabla_2 - i\,{}^*\nabla_2\big)\psi_2\\ &= \big(\nabla_1 + i\nabla_2\big)\big(\nabla_1 - i\nabla_2\big)\psi_1 + \big(\nabla_1 + i\nabla_2\big)\big(\nabla_2 + i\nabla_1\big)\psi_2\\ &= \big(\nabla_1\nabla_1 + \nabla_2\nabla_2 - i(\nabla_1\nabla_2 - \nabla_2\nabla_1)\big)\psi_1\\ &\quad + \big(\nabla_1\nabla_2 - \nabla_2\nabla_1 + i(\nabla_1\nabla_1 + \nabla_2\nabla_2)\big)\psi_2.\end{aligned}$$

Using that $\psi_2 = -i\psi_1$, we obtain $Z_1 = 2\Delta_1\psi_1 - 2i(\nabla_1\nabla_2 - \nabla_2\nabla_1)\psi_1$. Also,

$$\begin{aligned}Z_2 &= \mathcal{D}_2\overline{\mathcal{D}}_1\psi_1 + \mathcal{D}_2\overline{\mathcal{D}}_2\psi_2\\ &= \big(\nabla_2 + i\,{}^*\nabla_2\big)\big(\nabla_1 - i\,{}^*\nabla_1\big)\psi_1 + \big(\nabla_2 + i\,{}^*\nabla_2\big)\big(\nabla_2 - i\,{}^*\nabla_2\big)\psi_2\\ &= \big(\nabla_2 - i\nabla_1\big)\big(\nabla_1 - i\nabla_2\big)\psi_1 + \big(\nabla_2 - i\nabla_1\big)\big(\nabla_2 + i\nabla_1\big)\psi_2\\ &= \big(\nabla_2\nabla_1 - \nabla_1\nabla_2 - i(\nabla_1\nabla_1 + \nabla_2\nabla_2)\big)\psi_1\\ &\quad + \big(\nabla_1\nabla_1 + \nabla_2\nabla_2 + i(\nabla_2\nabla_1 - \nabla_1\nabla_2)\big)\psi_2\\ &= \big(2\nabla_1\nabla_1 + 2\nabla_2\nabla_2 + 2i(\nabla_2\nabla_1 - \nabla_1\nabla_2)\big)\psi_2,\end{aligned}$$

which gives

$$(\mathcal{D}(\overline{\mathcal{D}}\cdot\psi))_a = 2\Delta_1\psi_a - 2i(\nabla_1\nabla_2 - \nabla_2\nabla_1)\psi_a.$$

Using the Gauss equation (A.24) we obtain (A.27). Define $Y_a := 2\big(\overline{\mathcal{D}}\cdot(\mathcal{D}\widehat{\otimes}\psi)\big)_a$, and evaluate it in coordinates, i.e. $Y_a = \overline{\mathcal{D}}^b\mathcal{D}_a\psi_b + \overline{\mathcal{D}}^b\mathcal{D}_b\psi_a - \delta_{ab}\overline{\mathcal{D}}^b\mathcal{D}^d\psi_d$. We have

$$\begin{aligned}
Y_1 &= \overline{\mathcal{D}}^b\mathcal{D}_1\psi_b + \overline{\mathcal{D}}^b\mathcal{D}_b\psi_1 - \delta_{1b}\overline{\mathcal{D}}^b\mathcal{D}^d\psi_d\\
&= \big(\overline{\mathcal{D}}_1\mathcal{D}_1 + \overline{\mathcal{D}}_2\mathcal{D}_2\big)\psi_1 + \big(\overline{\mathcal{D}}_2\mathcal{D}_1 - \overline{\mathcal{D}}_1\mathcal{D}_2\big)\psi_2\\
&= \big(\big(\nabla_1 - i\,{}^*\nabla_1\big)\big(\nabla_1 + i\,{}^*\nabla_1\big) + \big(\nabla_2 - i\,{}^*\nabla_2\big)\big(\nabla_2 + i\,{}^*\nabla_2\big)\big)\psi_1\\
&\quad + \big(\big(\nabla_2 - i\,{}^*\nabla_2\big)\big(\nabla_1 + i\,{}^*\nabla_1\big) - \big(\nabla_1 - i\,{}^*\nabla_1\big)\big(\nabla_2 + i\,{}^*\nabla_2\big)\big)\psi_2\\
&= \big(\big(\nabla_1 - i\nabla_2\big)\big(\nabla_1 + i\nabla_2\big) + \big(\nabla_2 + i\nabla_1\big)\big(\nabla_2 - i\nabla_1\big)\big)\psi_1\\
&\quad + \big(\big(\nabla_2 + i\nabla_1\big)\big(\nabla_1 + i\nabla_2\big) - \big(\nabla_1 - i\nabla_2\big)\big(\nabla_2 - i\nabla_1\big)\big)\psi_2\\
&= \big(2\nabla_1\nabla_1 + 2\nabla_2\nabla_2 + 2i(\nabla_1\nabla_2 - \nabla_2\nabla_1)\big)\psi_1\\
&\quad + \big(2\nabla_2\nabla_1 - 2\nabla_1\nabla_2 + 2i(\nabla_1\nabla_1 + \nabla_2\nabla_2)\big)\psi_2.
\end{aligned}$$

Using that $\psi_2 = -i\psi_1$, we obtain $Y_1 = 4\Delta_1\psi_1 + 4i(\nabla_1\nabla_2 - \nabla_2\nabla_1)\psi_1$. Also, $Y_2 = \overline{\mathcal{D}}^b\mathcal{D}_2\psi_b + \overline{\mathcal{D}}^b\mathcal{D}_b\psi_2 - \delta_{2b}\overline{\mathcal{D}}^b\mathcal{D}^d\psi_d = \big(\overline{\mathcal{D}}_1\mathcal{D}_2 - \overline{\mathcal{D}}_2\mathcal{D}_1\big)\psi_1 + \big(\overline{\mathcal{D}}_1\mathcal{D}_1 + \overline{\mathcal{D}}_2\mathcal{D}_2\big)\psi_2$, which gives

$$2\big(\overline{\mathcal{D}}\cdot(\mathcal{D}\widehat{\otimes}\psi)\big)_a = 4\Delta_1\psi_a + 4i(\nabla_1\nabla_2 - \nabla_2\nabla_1)\psi_a.$$

Using the Gauss equation (A.24) we obtain (A.28). Finally, (A.29) is proved in [19, Proposition 5.6]. □

Appendix B. Derivation of the Teukolsky Equations

In this section, we derive the system of Teukolsky equations for $\mathfrak{B}$, $\mathfrak{F}$, A.

B.1. *The Teukolsky equation for* $\mathfrak{B}$

Recall the relation (5.9):

$${}^{(c)}\nabla_4\mathfrak{B} + 3\overline{\mathrm{tr}X}\mathfrak{B} = {}^{(\mathbf{F})}P\big(\overline{{}^{(c)}\mathcal{D}}\cdot A + \underline{\overline{H}}\cdot A\big) + \big(2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} - 3\overline{P}\big)\mathfrak{X}.$$

We apply ${}^{(c)}\nabla_3$ to the above, and using (4.22) we obtain

$$\begin{aligned}
{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\mathfrak{B} &= -3\overline{\mathrm{tr}X}\,{}^{(c)}\nabla_3\mathfrak{B} + \left(\frac{3}{2}\overline{\mathrm{tr}\underline{X}}\ \overline{\mathrm{tr}X} - 3\,\overline{{}^{(c)}\mathcal{D}}\cdot H - 3H\cdot\overline{\underline{H}} - 6\overline{P}\right)\mathfrak{B}\\
&\quad + I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= {}^{(c)}\nabla_3\big({}^{(\mathbf{F})}P\big(\overline{{}^{(c)}\mathcal{D}}\cdot A + \underline{\overline{H}}\cdot A\big)\big), \quad I_2 = {}^{(c)}\nabla_3\big(2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} - 3\overline{P}\big)\mathfrak{X},\\
I_3 &= \big(2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} - 3\overline{P}\big)\,{}^{(c)}\nabla_3\mathfrak{X}.
\end{aligned}$$

We compute I_1. Using (4.3), we obtain

$$I_1 = {}^{(c)}\nabla_3\,{}^{(\mathbf{F})}P\big(\overline{{}^{(c)}\mathcal{D}}\cdot A+\overline{\underline{H}}\cdot A\big)+{}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3\big(\overline{{}^{(c)}\mathcal{D}}\cdot A+\overline{\underline{H}}\cdot A\big)$$
$$= -\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}P\big(\overline{{}^{(c)}\mathcal{D}}\cdot A+\overline{\underline{H}}\cdot A\big)+{}^{(\mathbf{F})}P\big(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(c)}\nabla_3 A+[{}^{(c)}\nabla_3,\overline{{}^{(c)}\mathcal{D}}\cdot]A$$
$$+\overline{\underline{H}}\cdot{}^{(c)}\nabla_3 A+{}^{(c)}\nabla_3\overline{\underline{H}}\cdot A\big).$$

Recall the following commutator formula, for $U = u + i\,{}^*u \in \mathfrak{s}_2(\mathbb{C})$, see [19, Lemma 5.3]:

$$[{}^{(c)}\nabla_3,\overline{{}^{(c)}\mathcal{D}}\cdot]U = -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,(\overline{{}^{(c)}\mathcal{D}}\cdot U+(s-2)\overline{H}\cdot U)+\overline{H}\cdot\nabla_3 U. \tag{B.1}$$

Applying (B.1) to A with $s=2$ we obtain

$$I_1 = -\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}P\big(\overline{{}^{(c)}\mathcal{D}}\cdot A+\overline{\underline{H}}\cdot A\big)-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,{}^{(\mathbf{F})}P\,\overline{{}^{(c)}\mathcal{D}}\cdot A+{}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3\overline{\underline{H}}\cdot A$$
$$+{}^{(\mathbf{F})}P\big(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(c)}\nabla_3 A+\big(\overline{H}+\overline{\underline{H}}\big)\cdot{}^{(c)}\nabla_3 A\big).$$

We write the last term of the above using (4.4) as

$${}^{(\mathbf{F})}P\big(\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(c)}\nabla_3 A+\big(\overline{H}+\overline{\underline{H}}\big)\cdot{}^{(c)}\nabla_3 A\big)$$
$$=\overline{{}^{(c)}\mathcal{D}}\cdot({}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3 A)-\overline{{}^{(c)}\mathcal{D}}\,{}^{(\mathbf{F})}P\cdot{}^{(c)}\nabla_3 A+\big(\overline{H}+\overline{\underline{H}}\big)\cdot{}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3 A$$
$$=\overline{{}^{(c)}\mathcal{D}}\cdot({}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3 A)+\big(\overline{H}+3\overline{\underline{H}}\big)\cdot{}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3 A.$$

Using (5.7) to write

$${}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3 A=-\frac{1}{2}\,{}^{(\mathbf{F})}P\mathrm{tr}\underline{X}A+\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}+3H\widehat{\otimes}\mathfrak{B}-\big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\mathfrak{F}$$

we compute

$$\overline{{}^{(c)}\mathcal{D}}\cdot({}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3 A)$$
$$=-\frac{1}{2}\,{}^{(\mathbf{F})}P\mathrm{tr}\underline{X}\,\overline{{}^{(c)}\mathcal{D}}\cdot A-\frac{1}{2}\mathrm{tr}\underline{X}\,\overline{{}^{(c)}\mathcal{D}}\,{}^{(\mathbf{F})}P\cdot A-\frac{1}{2}\,{}^{(\mathbf{F})}P\,\overline{{}^{(c)}\mathcal{D}}\mathrm{tr}\underline{X}\cdot A$$
$$+\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}+3H\,\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{B}+3\,\overline{{}^{(c)}\mathcal{D}}\cdot H\mathfrak{B}$$
$$-\big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}-\overline{{}^{(c)}\mathcal{D}}\big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\cdot\mathfrak{F}.$$

Using (4.4), (4.10), (4.11), the above becomes

$$\overline{{}^{(c)}\mathcal{D}}\cdot({}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3 A)$$
$$=-\frac{1}{2}\,{}^{(\mathbf{F})}P\mathrm{tr}\underline{X}\,\overline{{}^{(c)}\mathcal{D}}\cdot A+{}^{(\mathbf{F})}P\left(\frac{3}{2}\mathrm{tr}\underline{X}-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right)\overline{\underline{H}}\cdot A+\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}$$
$$+3H\,\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{B}+3(\overline{{}^{(c)}\mathcal{D}}\cdot H)\mathfrak{B}-\big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}$$
$$+\big(\big(9\overline{P}-2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\overline{\underline{H}}+4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\overline{H}\big)\cdot\mathfrak{F}.$$

We therefore obtain

$$I_1 = -\left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right) {}^{(\mathbf{F})}P\left(\overline{{}^{(c)}\mathcal{D}}\cdot A\right) + {}^{(\mathbf{F})}P\left(\frac{1}{2}\mathrm{tr}\underline{X}\overline{H} - \frac{1}{2}\overline{\mathrm{tr}\underline{X}\,\underline{H}}\right)\cdot A$$
$$+\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot {}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} + 3H\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{B} + 3(\overline{{}^{(c)}\mathcal{D}}\cdot H)\mathfrak{B}$$
$$-\left(3\overline{P} + 2\,{}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P}\right)\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + \left(\left(9\overline{P} - 2\,{}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P}\right)\underline{H} + 4\,{}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P}\overline{H}\right)\cdot\mathfrak{F}$$
$$+\left(\overline{H} + 3\underline{H}\right)\cdot\left(-\frac{1}{2}\,{}^{(\mathbf{F})}P\mathrm{tr}\underline{X}A + \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} + 3H\widehat{\otimes}\mathfrak{B} - \left(3\overline{P} + 2\,{}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P}\right)\mathfrak{F}\right),$$

which finally gives

$$I_1 = -\left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right) {}^{(\mathbf{F})}P\left(\overline{{}^{(c)}\mathcal{D}}\cdot A + \underline{H}\cdot A\right) + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot {}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}$$
$$+3H\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{B} + \left(\overline{H} + 3\underline{H}\right)\cdot\left(\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\right) + 3\left(\overline{{}^{(c)}\mathcal{D}}\cdot H + \overline{H}\cdot H\right)\mathfrak{B}$$
$$+9\underline{H}\cdot\left(H\widehat{\otimes}\mathfrak{B}\right) - \left(3\overline{P} + 2\,{}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P}\right)\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}$$
$$+\left(\left(-8\,{}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P}\right)\underline{H} + \left(-3\overline{P} + 2\,{}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P}\right)\overline{H}\right)\cdot\mathfrak{F}.$$

We compute I_2. Using (4.3) and (4.12) we have

$$I_2 = \left(2\,{}^{(c)}\nabla_3\,{}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P} + 2\,{}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3\overline{{}^{(\mathbf{F})}P} - 3\,{}^{(c)}\nabla_3\overline{P}\right)\mathfrak{X}$$
$$= \left(\frac{9}{2}\mathrm{tr}\underline{X}\overline{P} + \left(\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}}\right){}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P}\right)\mathfrak{X}$$
$$= \left(\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}}\right){}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P}\;\mathfrak{X} + \left(\frac{3}{2}\mathrm{tr}\underline{X}\right)\left(3\overline{P}\mathfrak{X}\right).$$

We use (5.9) again to write

$$3\overline{P}\mathfrak{X} = -\,{}^{(c)}\nabla_4\mathfrak{B} - 3\overline{\mathrm{tr}X}\mathfrak{B} + {}^{(\mathbf{F})}P\left(\overline{{}^{(c)}\mathcal{D}}\cdot A + \underline{H}\cdot A\right) + 2\,{}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P}\mathfrak{X}$$

and substituting in the above we obtain

$$I_2 = -\frac{3}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4\mathfrak{B} - \frac{9}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X}\mathfrak{B} + \frac{3}{2}\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}P\left(\overline{{}^{(c)}\mathcal{D}}\cdot A + \underline{H}\cdot A\right)$$
$$+\left(4\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}}\right){}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P}\;\mathfrak{X}.$$

We compute I_3. Using (5.10), we obtain

$$I_3 = \left(2\,{}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P} - 3\overline{P}\right){}^{(c)}\nabla_3\mathfrak{X}$$
$$= -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\left(2\,{}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P} - 3\overline{P}\right)\mathfrak{X} - \left(2\,{}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P} - 3\overline{P}\right)\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}$$
$$-\left(2\,{}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P} - 3\overline{P}\right)\overline{H}\cdot\mathfrak{F} + \left(-4\,{}^{(\mathbf{F})}P\overline{{}^{(\mathbf{F})}P} + 6\overline{P}\right)\mathfrak{B}.$$

Putting the above together we obtain

$$
\begin{aligned}
{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\mathfrak{B} = {}& -3\overline{\mathrm{tr}X}\,{}^{(c)}\nabla_3\mathfrak{B} - \frac{3}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4\mathfrak{B}\\
&+\left(-\frac{9}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + \frac{3}{2}\overline{\mathrm{tr}\underline{X}}\;\overline{\mathrm{tr}X} - 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\mathfrak{B}\\
&+\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} + 3H\,\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{B} + \left(\overline{H}+3\underline{\overline{H}}\right)\cdot\left(\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\right)\\
&+9\underline{\overline{H}}\cdot\left(H\widehat{\otimes}\mathfrak{B}\right) - \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,{}^{(\mathbf{F})}P\left(\overline{{}^{(c)}\mathcal{D}}\cdot A + \underline{\overline{H}}\cdot A\right)\\
&-\left(4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + \left(\left(-8\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\underline{\overline{H}}\right)\cdot\mathfrak{F}\\
&+\left(4\mathrm{tr}\underline{X} - 3\overline{\mathrm{tr}\underline{X}}\right)\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\;\mathfrak{X} + \frac{3}{2}\overline{\mathrm{tr}\underline{X}}\;\overline{P}\mathfrak{X}.
\end{aligned}
$$

Finally using (5.9) to write

$$
{}^{(\mathbf{F})}P\left(\overline{{}^{(c)}\mathcal{D}}\cdot A + \underline{\overline{H}}\cdot A\right) = {}^{(c)}\nabla_4\mathfrak{B} + 3\overline{\mathrm{tr}X}\mathfrak{B} - \left(2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} - 3\overline{P}\right)\mathfrak{X}
$$

we obtain

$$
\begin{aligned}
{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\mathfrak{B} = {}& -3\overline{\mathrm{tr}X}\,{}^{(c)}\nabla_3\mathfrak{B} - \left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right)\,{}^{(c)}\nabla_4\mathfrak{B}\\
&+\left(-\frac{9}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} - 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\mathfrak{B} + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\\
&+3H\,\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{B} + \left(\overline{H}+3\underline{\overline{H}}\right)\cdot\left(\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\right) + 9\underline{\overline{H}}\cdot\left(H\widehat{\otimes}\mathfrak{B}\right)\\
&-\left(4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + \left(\left(-8\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\underline{\overline{H}}\right)\cdot\mathfrak{F}\\
&+\left(4\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}}\right)\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\;\mathfrak{X},
\end{aligned}
$$

which gives the operator $\mathcal{T}_1$ as in (6.4). This completes the derivation of the Teukolsky equation for $\mathfrak{B}$.

B.2. *The Teukolsky equation for* $\mathfrak{F}$

Recall the relation (5.8):

$$
{}^{(c)}\nabla_4\mathfrak{F} + \left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right)\mathfrak{F} = -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} - \left(\frac{3}{2}H + \frac{1}{2}\underline{H}\right)\widehat{\otimes}\mathfrak{X} - {}^{(\mathbf{F})}PA.
$$

We apply ${}^{(c)}\nabla_3$ to the above, and using (4.22) we obtain

$$
\begin{aligned}
{}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\mathfrak{F} = {}& -\left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right)\,{}^{(c)}\nabla_3\mathfrak{F} + \left(\frac{1}{4}\mathrm{tr}\underline{X}\mathrm{tr}X + \frac{3}{4}\overline{\mathrm{tr}\underline{X}}\;\overline{\mathrm{tr}X} - 3\overline{P} - P\right.\\
&\left. -\frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H - \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{H} - 2H\cdot\overline{H}\right)\mathfrak{F} + K_1 + K_2 + K_3,
\end{aligned}
$$

where

$$K_1 = -\frac{1}{2}\,{}^{(c)}\nabla_3({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X}), \quad K_2 = -\,{}^{(c)}\nabla_3\left(\left(\frac{3}{2}H + \frac{1}{2}\,\underline{H}\right)\widehat{\otimes}\mathfrak{X}\right),$$
$$K_3 = -\,{}^{(c)}\nabla_3({}^{(\mathbf{F})}PA).$$

We compute K_1. Recall the following commutator formula, for $F = f + i\,{}^*f \in \mathfrak{s}_1(\mathbb{C})$, see [19, Lemma 5.3]:

$$[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]F = -\frac{1}{2}\mathrm{tr}\underline{X}\big({}^{(c)}\mathcal{D}\widehat{\otimes}F + (1+s)H\widehat{\otimes}F\big) + H\widehat{\otimes}\,{}^{(c)}\nabla_3 F. \tag{B.2}$$

Using (5.10) and (B.2) for $s=2$, we have

$$\begin{aligned}
K_1 &= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}({}^{(c)}\nabla_3\mathfrak{X}) - \frac{1}{2}[\,{}^{(c)}\nabla_3,\,{}^{(c)}\mathcal{D}\widehat{\otimes}]\mathfrak{X}\\
&= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}({}^{(c)}\nabla_3\mathfrak{X}) + \frac{1}{4}\mathrm{tr}\underline{X}\left({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} + 3H\widehat{\otimes}\mathfrak{X}\right) - \frac{1}{2}H\widehat{\otimes}\,{}^{(c)}\nabla_3\mathfrak{X}\\
&= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\left(-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\mathfrak{X} - \overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} - \overline{H}\cdot\mathfrak{F} - 2\mathfrak{B}\right) + \frac{1}{4}\mathrm{tr}\underline{X}\big({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} + 3H\widehat{\otimes}\mathfrak{X}\big)\\
&\quad - \frac{1}{2}H\widehat{\otimes}\left(-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\mathfrak{X} - \overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} - \overline{H}\cdot\mathfrak{F} - 2\mathfrak{B}\right),
\end{aligned}$$

which gives using (4.10)

$$\begin{aligned}
K_1 &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + \overline{H}\cdot\mathfrak{F}) + \frac{1}{2}H\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + \overline{H}\cdot\mathfrak{F})\\
&\quad + {}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} + H\widehat{\otimes}\mathfrak{B} + \frac{1}{4}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X}\\
&\quad + \left(\frac{1}{4}(\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}})\,\underline{H} + \frac{3}{4}\mathrm{tr}\underline{X}H + \frac{1}{4}\overline{\mathrm{tr}\underline{X}}H\right)\widehat{\otimes}\mathfrak{X}.
\end{aligned}$$

Using (5.8) to write

$${}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} = -2\,{}^{(c)}\nabla_4\mathfrak{F} - \big(3\overline{\mathrm{tr}X} + \mathrm{tr}X\big)\mathfrak{F} - \big(3H + \underline{H}\big)\widehat{\otimes}\mathfrak{X} - 2\,{}^{(\mathbf{F})}PA$$

we obtain

$$\begin{aligned}
K_1 &= \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + \overline{H}\cdot\mathfrak{F}) + \frac{1}{2}H\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + \overline{H}\cdot\mathfrak{F}) + {}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} + H\widehat{\otimes}\mathfrak{B}\\
&\quad + \frac{1}{4}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\big(-2\,{}^{(c)}\nabla_4\mathfrak{F} - \big(3\overline{\mathrm{tr}X} + \mathrm{tr}X\big)\mathfrak{F} - \big(3H + \underline{H}\big)\widehat{\otimes}\mathfrak{X} - 2\,{}^{(\mathbf{F})}PA\big)\\
&\quad + \left(\frac{1}{4}(\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}})\,\underline{H} + \frac{3}{4}\mathrm{tr}\underline{X}H + \frac{1}{4}\overline{\mathrm{tr}\underline{X}}H\right)\widehat{\otimes}\mathfrak{X}
\end{aligned}$$

$$= -\frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\, {}^{(c)}\nabla_4 \mathfrak{F} - \frac{1}{4}\big(\mathrm{tr}\underline{X}\mathrm{tr}X + 3\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + 3\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\mathrm{tr}X\big)\mathfrak{F}$$
$$+ \frac{1}{2}\, {}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + \overline{H}\cdot\mathfrak{F}) + \frac{1}{2}H\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + \overline{H}\cdot\mathfrak{F}) + {}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} + H\widehat{\otimes}\mathfrak{B}$$
$$- \frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\, {}^{(\mathbf{F})}PA - \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\big(\underline{H} + H\big)\widehat{\otimes}\mathfrak{X}.$$

Using (5.7) to write

$$ {}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} = 2\, {}^{(\mathbf{F})}P\, {}^{(c)}\nabla_3 A + {}^{(\mathbf{F})}P\mathrm{tr}\underline{X}A - 6H\widehat{\otimes}\mathfrak{B} + 2\big(3\overline{P} + 2\, {}^{(\mathbf{F})}P\, \overline{{}^{(\mathbf{F})}P}\big)\mathfrak{F}$$

we obtain

$$K_1 = -\frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\, {}^{(c)}\nabla_4 \mathfrak{F} - \frac{1}{4}\big(\mathrm{tr}\underline{X}\mathrm{tr}X + 3\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + 3\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\mathrm{tr}X\big)\mathfrak{F}$$
$$+ 2\big(3\overline{P} + 2\, {}^{(\mathbf{F})}P\, \overline{{}^{(\mathbf{F})}P}\big)\mathfrak{F}\frac{1}{2}\, {}^{(c)}\mathcal{D}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + \overline{H}\cdot\mathfrak{F}\big) + \frac{1}{2}H\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + \overline{H}\cdot\mathfrak{F}\big)$$
$$+ 2\, {}^{(\mathbf{F})}P\, {}^{(c)}\nabla_3 A + \frac{1}{2}\big(\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\big)\, {}^{(\mathbf{F})}PA - 5H\widehat{\otimes}\mathfrak{B} - \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\big(\underline{H} + H\big)\widehat{\otimes}\mathfrak{X}.$$

We compute K_2. Using (4.9) and (5.10)

$$K_2 = -\, {}^{(c)}\nabla_3\left(\frac{3}{2}H + \frac{1}{2}\underline{H}\right)\widehat{\otimes}\mathfrak{X} - \left(\frac{3}{2}H + \frac{1}{2}\underline{H}\right)\widehat{\otimes}\, {}^{(c)}\nabla_3\mathfrak{X}$$
$$= \left(-\frac{3}{2}\, {}^{(c)}\nabla_3 H + \frac{1}{4}\overline{\mathrm{tr}\underline{X}}(\underline{H} - H)\right)\widehat{\otimes}\mathfrak{X} - \left(\frac{3}{2}H + \frac{1}{2}\underline{H}\right)$$
$$\widehat{\otimes}\left(-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\mathfrak{X} - \overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} - \overline{H}\cdot\mathfrak{F} - 2\mathfrak{B}\right)$$
$$= \left(\frac{3}{2}H + \frac{1}{2}\underline{H}\right)\widehat{\otimes}\,\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + \left(\frac{3}{2}H + \frac{1}{2}\underline{H}\right)\widehat{\otimes}\left(\overline{H}\cdot\mathfrak{F}\right)$$
$$+ \left(-\frac{3}{2}\, {}^{(c)}\nabla_3 H + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}(\underline{H} + H)\right)\widehat{\otimes}\mathfrak{X} + \big(3H + \underline{H}\big)\widehat{\otimes}\mathfrak{B}.$$

We compute K_3. Using (4.3), we obtain

$$K_3 = -\, {}^{(c)}\nabla_3({}^{(\mathbf{F})}P)A - {}^{(\mathbf{F})}P\, {}^{(c)}\nabla_3 A = -\, {}^{(\mathbf{F})}P\, {}^{(c)}\nabla_3 A + \mathrm{tr}\underline{X}\, {}^{(\mathbf{F})}PA.$$

We therefore obtain

$$ {}^{(c)}\nabla_3\, {}^{(c)}\nabla_4\mathfrak{F} = -\left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right)\, {}^{(c)}\nabla_3\mathfrak{F} - \frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\, {}^{(c)}\nabla_4\mathfrak{F}$$

$$
\begin{aligned}
&+\left(-\frac{3}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X}-\frac{1}{4}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}X+3\overline{P}-P+4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right.\\
&\left.-\frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H-\frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{H}-2H\cdot\overline{H}\right)\mathfrak{F}\\
&+\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}+\overline{H}\cdot\mathfrak{F})+\left(2H+\frac{1}{2}\underline{H}\right)\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}+\overline{H}\cdot\mathfrak{F})\\
&+{}^{(\mathbf{F})}P\,{}^{(c)}\nabla_3A+\frac{1}{2}\big(3\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\big)\,{}^{(\mathbf{F})}PA+\left(-\frac{3}{2}\,{}^{(c)}\nabla_3H\right)\widehat{\otimes}\mathfrak{X}\\
&+(-2H+\underline{H})\widehat{\otimes}\mathfrak{B},
\end{aligned}
$$

which gives the operator $\mathcal{T}_2$ as in (6.5) and this ends the derivation of the Teukolsky equation for $\mathfrak{F}$.

B.3. *The Teukolsky equation for A*

This derivation is similar to the one obtained in [19]. Recall (5.5):

$$
{}^{(c)}\nabla_3A+\frac{1}{2}\mathrm{tr}\underline{X}A={}^{(c)}\mathcal{D}\widehat{\otimes}B+H\widehat{\otimes}\big(4B-3\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B\big)-3\overline{P}\widehat{X}-2\,{}^{(\mathbf{F})}P\mathfrak{F}.
$$

We apply ${}^{(c)}\nabla_4$ to the above, and using (4.23) we obtain

$$
\begin{aligned}
{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3A=&-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4A+\left(\frac{1}{4}\mathrm{tr}X\mathrm{tr}\underline{X}-\frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{H}}-\frac{1}{2}\underline{H}\cdot\overline{\underline{H}}-\overline{P}\right)A\\
&+J_1+J_2+J_3+J_4,
\end{aligned}
$$

where

$$
\begin{aligned}
&J_1={}^{(c)}\nabla_4\,{}^{(c)}\mathcal{D}\widehat{\otimes}B,\quad J_2={}^{(c)}\nabla_4\left(H\widehat{\otimes}\left(4B-3\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B\right)\right),\\
&J_3=-3\,{}^{(c)}\nabla_4(\overline{P}\widehat{X}),\quad J_4=-2\,{}^{(c)}\nabla_4({}^{(\mathbf{F})}P\mathfrak{F}).
\end{aligned}
$$

We compute J_1. Using (A.10) for $s=1$ we obtain

$$
\begin{aligned}
J_1&={}^{(c)}\mathcal{D}\widehat{\otimes}({}^{(c)}\nabla_4B)+[{}^{(c)}\nabla_4,{}^{(c)}\mathcal{D}\widehat{\otimes}]B\\
&={}^{(c)}\mathcal{D}\widehat{\otimes}({}^{(c)}\nabla_4B)-\frac{1}{2}\mathrm{tr}X({}^{(c)}\mathcal{D}\widehat{\otimes}B)+\underline{H}\widehat{\otimes}\,{}^{(c)}\nabla_4B.
\end{aligned}
$$

Observe that the Bianchi identity (4.35) can be written in terms of $\mathfrak{X}$ as

$$
{}^{(c)}\nabla_4B+2\overline{\mathrm{tr}X}B+\frac{3}{2}\overline{\mathrm{tr}X}\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B-3\overline{P}\,\Xi=\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot A+\frac{1}{2}A\cdot\overline{\underline{H}}+\overline{{}^{(\mathbf{F})}P}\mathfrak{X}.
\tag{B.3}
$$

This gives

$$
\begin{aligned}
{}^{(c)}\mathcal{D}\widehat{\otimes}({}^{(c)}\nabla_4 B) = & -2\overline{\mathrm{tr}X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}B - 2\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr}X}\widehat{\otimes}B - \frac{3}{2}\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr}X}\widehat{\otimes}\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B \\
& - \frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(c)}\mathcal{D}\,\overline{{}^{(\mathbf{F})}P}\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{3}{2}\overline{\mathrm{tr}X}\,\overline{{}^{(\mathbf{F})}P}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B + 3\overline{P}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi \\
& + 3\,{}^{(c)}\mathcal{D}\overline{P}\widehat{\otimes}\Xi + \overline{{}^{(\mathbf{F})}P}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} + {}^{(c)}\mathcal{D}\,\overline{{}^{(\mathbf{F})}P}\widehat{\otimes}\mathfrak{X} \\
& + \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,\overline{{}^{(c)}\mathcal{D}}\cdot A + \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(A\cdot\overline{\underline{H}}).
\end{aligned}
$$

We therefore obtain using again (B.3):

$$
\begin{aligned}
J_1 = & -\left(\frac{1}{2}\mathrm{tr}X + 2\overline{\mathrm{tr}X}\right){}^{(c)}\mathcal{D}\widehat{\otimes}B + \big(-2\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr}X} - 2\overline{\mathrm{tr}X}\,\underline{H}\big)\widehat{\otimes}B \\
& - \frac{3}{2}\overline{\mathrm{tr}X}\,\overline{{}^{(\mathbf{F})}P}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B - \frac{3}{2}\big(\overline{{}^{(\mathbf{F})}P}\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr}X} + \overline{\mathrm{tr}X}\,{}^{(c)}\mathcal{D}\,\overline{{}^{(\mathbf{F})}P} + \overline{\mathrm{tr}X}\,\overline{{}^{(\mathbf{F})}P}\,\underline{H}\big) \\
& \widehat{\otimes}\,{}^{(\mathbf{F})}B + 3\overline{P}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + 3\big({}^{(c)}\mathcal{D}\overline{P} + \underline{H}\overline{P}\big)\widehat{\otimes}\Xi + \overline{{}^{(\mathbf{F})}P}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} \\
& + \big({}^{(c)}\mathcal{D}\,\overline{{}^{(\mathbf{F})}P} + \underline{H}\,\overline{{}^{(\mathbf{F})}P}\big)\widehat{\otimes}\mathfrak{X} + \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,\overline{{}^{(c)}\mathcal{D}}\cdot A \\
& + \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(A\cdot\overline{\underline{H}}) + \frac{1}{2}\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot A\big) + \frac{1}{2}\underline{H}\widehat{\otimes}\big(A\cdot\overline{\underline{H}}\big).
\end{aligned}
$$

We compute J_2. Using (4.9), (B.3) and (4.3), we obtain

$$
\begin{aligned}
J_2 = & \ {}^{(c)}\nabla_4 H\widehat{\otimes}\big(4B - 3\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B\big) + H\widehat{\otimes}\big(4\,{}^{(c)}\nabla_4 B - 3\,\overline{{}^{(\mathbf{F})}P}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B \\
& - 3\,{}^{(c)}\nabla_4\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B\big) \\
= & -\frac{1}{2}\overline{\mathrm{tr}X}(H - \underline{H})\widehat{\otimes}\big(4B - 3\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B\big) \\
& + H\widehat{\otimes}\Bigg[4(-2\overline{\mathrm{tr}X}B - \frac{3}{2}\overline{\mathrm{tr}X}\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B + 3\overline{P}\,\Xi) \\
& + 4\left(\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot A + \frac{1}{2}A\cdot\overline{\underline{H}} + \overline{{}^{(\mathbf{F})}P}\mathfrak{X}\right) - 3\,\overline{{}^{(\mathbf{F})}P}\,{}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B - 3(-\mathrm{tr}X\,\overline{{}^{(\mathbf{F})}P})\,{}^{(\mathbf{F})}B\Bigg].
\end{aligned}
$$

Writing ${}^{(c)}\nabla_4\,{}^{(\mathbf{F})}B = \mathfrak{X} - \frac{3}{2}\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}B + 2\,{}^{(\mathbf{F})}P\Xi$, the above becomes

$$
\begin{aligned}
J_2 = & \big(-2\overline{\mathrm{tr}X}(5H - \underline{H})\big)\widehat{\otimes}B + \frac{3}{2}\big(-\overline{\mathrm{tr}X}\ \overline{{}^{(\mathbf{F})}P}\,\underline{H} + 2\mathrm{tr}X\,\overline{{}^{(\mathbf{F})}P}H\big)\widehat{\otimes}\,{}^{(\mathbf{F})}B \\
& + \big(12\overline{P} - 6\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)H\widehat{\otimes}\Xi + \overline{{}^{(\mathbf{F})}P}H\widehat{\otimes}\mathfrak{X} + 2H\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot A\big) + 2H\widehat{\otimes}\big(A\cdot\overline{\underline{H}}\big).
\end{aligned}
$$

We compute J_3. Using (4.12) and (4.25), we obtain

$$
\begin{aligned}
J_3 &= -3\,{}^{(c)}\nabla_4(\overline{P})\widehat{X} - 3\overline{P}\,{}^{(c)}\nabla_4(\widehat{X})\\
&= -3\left(-\frac{3}{2}\overline{\mathrm{tr}X}\ \overline{P} - \overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\widehat{X}\\
&\quad - 3\overline{P}\left(-\frac{1}{2}(\mathrm{tr}X + \overline{\mathrm{tr}X})\widehat{X} + {}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \Xi\widehat{\otimes}(\,\underline{H} + H) - A\right)\\
&= 3\overline{P}A + \left(\left(\frac{3}{2}\mathrm{tr}X + 6\overline{\mathrm{tr}X}\right)\overline{P} + 3\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\widehat{X}\\
&\quad - 3\overline{P}\left({}^{(c)}\mathcal{D}\widehat{\otimes}\Xi + \Xi\widehat{\otimes}(\,\underline{H} + H)\right).
\end{aligned}
$$

We compute J_4. Using (4.3) we obtain

$$
J_4 = -2\,{}^{(c)}\nabla_4(\overline{{}^{(\mathbf{F})}P})\mathfrak{F} - 2\,\overline{{}^{(\mathbf{F})}P}\,{}^{(c)}\nabla_4\mathfrak{F} = 2\,\overline{{}^{(\mathbf{F})}P}\big(-\,{}^{(c)}\nabla_4\mathfrak{F} + \mathrm{tr}X\mathfrak{F}\big).
$$

Summing the expressions obtained for J_1, J_2, J_3 and J_4 we obtain

$$
\begin{aligned}
&{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3 A\\
&= -\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_4 A + \left(\frac{1}{4}\mathrm{tr}X\mathrm{tr}\underline{X} - \frac{1}{2}\,{}^{(c)}\mathcal{D}\cdot\overline{\underline{H}} - \frac{1}{2}\underline{H}\cdot\overline{\underline{H}} + 2\overline{P}\right)A\\
&\quad + \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A + \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(A\cdot\overline{\underline{H}}) + \frac{1}{2}\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot A\big) + \frac{1}{2}\underline{H}\widehat{\otimes}\big(A\cdot\overline{\underline{H}}\big)\\
&\quad + 2H\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot A\big) + 2H\widehat{\otimes}\big(A\cdot\overline{\underline{H}}\big) - \left(\frac{1}{2}\mathrm{tr}X + 2\overline{\mathrm{tr}X}\right){}^{(c)}\mathcal{D}\widehat{\otimes}B\\
&\quad + \big(-2\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr}X} - 10\overline{\mathrm{tr}X}H\big)\widehat{\otimes}B - \frac{3}{2}\overline{\mathrm{tr}X}\,\overline{{}^{(\mathbf{F})}P}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(\mathbf{F})}B\\
&\quad - \frac{3}{2}\big(\overline{{}^{(\mathbf{F})}P}\,{}^{(c)}\mathcal{D}\overline{\mathrm{tr}X} + \overline{\mathrm{tr}X}\,{}^{(c)}\mathcal{D}\,\overline{{}^{(\mathbf{F})}P} + 2\overline{\mathrm{tr}X}\,\overline{{}^{(\mathbf{F})}P}\,\underline{H} - 2\mathrm{tr}X\,\overline{{}^{(\mathbf{F})}P}H\big)\widehat{\otimes}\,{}^{(\mathbf{F})}B\\
&\quad + 3\big({}^{(c)}\mathcal{D}\overline{P} + \big(3\overline{P} - 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)H\big)\widehat{\otimes}\Xi + \overline{{}^{(\mathbf{F})}P}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X}\\
&\quad + \big({}^{(c)}\mathcal{D}\,\overline{{}^{(\mathbf{F})}P} + \underline{H}\,\overline{{}^{(\mathbf{F})}P} + \overline{{}^{(\mathbf{F})}P}H\big)\widehat{\otimes}\mathfrak{X}\\
&\quad + \left(\left(\frac{3}{2}\mathrm{tr}X + 6\overline{\mathrm{tr}X}\right)\overline{P} + 3\overline{\mathrm{tr}X}\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\widehat{X} + 2\,\overline{{}^{(\mathbf{F})}P}\left(-\,{}^{(c)}\nabla_4\mathfrak{F} + \mathrm{tr}X\mathfrak{F}\right).
\end{aligned}
$$

Using (5.5) to write

$$
{}^{(c)}\mathcal{D}\widehat{\otimes}B = {}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}A - H\widehat{\otimes}\big(4B - 3\,\overline{{}^{(\mathbf{F})}P}\,{}^{(\mathbf{F})}B\big) + 3\overline{P}\widehat{X} + 2\,\overline{{}^{(\mathbf{F})}P}\mathfrak{F}
$$

the above becomes

$$\begin{aligned}
{}^{(c)}\nabla_4 {}^{(c)}\nabla_3 A = &-\left(\frac{1}{2}\mathrm{tr}X + 2\overline{\mathrm{tr}X}\right) {}^{(c)}\nabla_3 A - \frac{1}{2}\mathrm{tr}\underline{X}\, {}^{(c)}\nabla_4 A \\
&+\left(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} - \frac{1}{2}\, {}^{(c)}\mathcal{D}\cdot\overline{\underline{H}} - \frac{1}{2}\,\underline{H}\cdot\overline{\underline{H}} + 2\overline{P}\right) A \\
&+\frac{1}{2}\, {}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A + \frac{1}{2}\, {}^{(c)}\mathcal{D}\widehat{\otimes}(A\cdot\overline{\underline{H}}) + \frac{1}{2}\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot A\big) \\
&+\frac{1}{2}\underline{H}\widehat{\otimes}\big(A\cdot\overline{\underline{H}}\big) + 2H\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot A\big) + 2H\widehat{\otimes}\big(A\cdot\overline{\underline{H}}\big) + \big(-2\, {}^{(c)}\mathcal{D}\overline{\mathrm{tr}X} \\
&+2\mathrm{tr}XH - 2\overline{\mathrm{tr}X}H\big)\widehat{\otimes}B - \frac{3}{2}\overline{\mathrm{tr}X}\,\overline{{}^{(\mathbf{F})}P}\, {}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(\mathbf{F})}B - \frac{3}{2}\big(\overline{{}^{(\mathbf{F})}P}\, {}^{(c)}\mathcal{D}\overline{\mathrm{tr}X} \\
&+\overline{\mathrm{tr}X}\, {}^{(c)}\mathcal{D}\,\overline{{}^{(\mathbf{F})}P} + 2\overline{\mathrm{tr}X}\,\overline{{}^{(\mathbf{F})}P}\,\underline{H} + 4\overline{\mathrm{tr}X}H - \mathrm{tr}X\,\overline{{}^{(\mathbf{F})}P}H\big)\widehat{\otimes}\, {}^{(\mathbf{F})}B \\
&+3\big({}^{(c)}\mathcal{D}\overline{P} + \big(3\overline{P} - 2\, {}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)H\big)\widehat{\otimes}\Xi + \overline{{}^{(\mathbf{F})}P}\, {}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} \\
&+\big({}^{(c)}\mathcal{D}\,\overline{{}^{(\mathbf{F})}P} + \underline{H}\,\overline{{}^{(\mathbf{F})}P} + \overline{{}^{(\mathbf{F})}P}H\big)\widehat{\otimes}\mathfrak{X} + \big(3\overline{\mathrm{tr}X}\, {}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\widehat{X} \\
&+2\,\overline{{}^{(\mathbf{F})}P}\left(-\, {}^{(c)}\nabla_4\mathfrak{F} + \left(\frac{1}{2}\mathrm{tr}X - 2\overline{\mathrm{tr}X}\right)\mathfrak{F}\right).
\end{aligned}$$

Observe that using (4.10) and (4.11), the coefficients of B and Ξ vanish. Finally writing ${}^{(c)}\mathcal{D}\widehat{\otimes}\, {}^{(\mathbf{F})}B = -2\mathfrak{F} - 3H\widehat{\otimes}\, {}^{(\mathbf{F})}B + 2\, {}^{(\mathbf{F})}P\widehat{X}$, we obtain

$$\begin{aligned}
{}^{(c)}\nabla_4 {}^{(c)}\nabla_3 A = &-\left(\frac{1}{2}\mathrm{tr}X + 2\overline{\mathrm{tr}X}\right) {}^{(c)}\nabla_3 A - \frac{1}{2}\mathrm{tr}\underline{X}\, {}^{(c)}\nabla_4 A \\
&+\left(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} - \frac{1}{2}\, {}^{(c)}\mathcal{D}\cdot\overline{\underline{H}} - \frac{1}{2}\,\underline{H}\cdot\overline{\underline{H}} + 2\overline{P}\right) A \\
&+\frac{1}{2}\, {}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A + \frac{1}{2}\, {}^{(c)}\mathcal{D}\widehat{\otimes}(A\cdot\overline{\underline{H}}) + \frac{1}{2}\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot A\big) \\
&+\frac{1}{2}\underline{H}\widehat{\otimes}\big(A\cdot\overline{\underline{H}}\big) + 2H\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot A\big) + 2H\widehat{\otimes}\big(A\cdot\overline{\underline{H}}\big) \\
&-\frac{3}{2}\big(\overline{{}^{(\mathbf{F})}P}\, {}^{(c)}\mathcal{D}\overline{\mathrm{tr}X} + \overline{\mathrm{tr}X}\, {}^{(c)}\mathcal{D}\,\overline{{}^{(\mathbf{F})}P} + 2\overline{\mathrm{tr}X}\,\overline{{}^{(\mathbf{F})}P}\,\underline{H} + \overline{\mathrm{tr}X}\,\overline{{}^{(\mathbf{F})}P}H \\
&-\mathrm{tr}X\,\overline{{}^{(\mathbf{F})}P}H\big)\widehat{\otimes}\, {}^{(\mathbf{F})}B + \overline{{}^{(\mathbf{F})}P}\, {}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} + \big({}^{(c)}\mathcal{D}\,\overline{{}^{(\mathbf{F})}P} + \underline{H}\,\overline{{}^{(\mathbf{F})}P} \\
&+\overline{{}^{(\mathbf{F})}P}H\big)\widehat{\otimes}\mathfrak{X} + 2\,\overline{{}^{(\mathbf{F})}P}\left(-\, {}^{(c)}\nabla_4\mathfrak{F} + \left(\frac{1}{2}\mathrm{tr}X - \frac{1}{2}\overline{\mathrm{tr}X}\right)\mathfrak{F}\right).
\end{aligned}$$

Observe that using (4.10) and (4.4), the coefficient of ${}^{(\mathbf{F})}B$ vanishes. Finally, we can make use of (5.8) to write

$$ {}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X} = -2\, {}^{(c)}\nabla_4\mathfrak{F} - \big(3\overline{\mathrm{tr}X} + \mathrm{tr}X\big)\mathfrak{F} - \big(3H + \underline{H}\big)\widehat{\otimes}\mathfrak{X} - 2\, {}^{(\mathbf{F})}PA $$

and obtain

$$
\begin{aligned}
&{}^{(c)}\nabla_4\, {}^{(c)}\nabla_3 A \\
&\quad = -\left(\frac{1}{2}\mathrm{tr}X + 2\overline{\mathrm{tr}X}\right) {}^{(c)}\nabla_3 A - \frac{1}{2}\mathrm{tr}\underline{X}\, {}^{(c)}\nabla_4 A \\
&\qquad + \left(-\overline{\mathrm{tr}X}\mathrm{tr}\underline{X} - \frac{1}{2}\, {}^{(c)}\mathcal{D}\cdot\overline{\underline{H}} - \frac{1}{2}\underline{H}\cdot\overline{\underline{H}} + 2\overline{P} - 2\, {}^{(\mathbf{F})}P\, \overline{{}^{(\mathbf{F})}P}\right) A \\
&\qquad + \frac{1}{2}\, {}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot A + \frac{1}{2}\, {}^{(c)}\mathcal{D}\widehat{\otimes}(A\cdot\overline{\underline{H}}) + \frac{1}{2}\underline{H}\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot A\big) + \frac{1}{2}\underline{H}\widehat{\otimes}(A\cdot\overline{\underline{H}}) \\
&\qquad + 2H\widehat{\otimes}\big(\overline{{}^{(c)}\mathcal{D}}\cdot A\big) + 2H\widehat{\otimes}\big(A\cdot\overline{\underline{H}}\big) - 2\,\overline{{}^{(\mathbf{F})}P}\big(2\, {}^{(c)}\nabla_4\mathfrak{F} + 2\overline{\mathrm{tr}X}\mathfrak{F} \\
&\qquad + \big(\underline{H}+H\big)\widehat{\otimes}\mathfrak{X}\big),
\end{aligned}
$$

which gives the operator $\mathcal{T}_3(A)$, as in [19]. This completes the derivation of the Teukolsky equation for A.

Appendix C. Derivation of the Generalized Regge–Wheeler System

C.1. *Proof of Proposition 7.4*

C.1.1. *The commutators for $\mathcal{P}_C$*

Lemma C.1. *Let $\Psi \in \mathfrak{s}_k(\mathbb{C})$ of conformal type s. Recall the definition of $\mathcal{P}_C(\Psi)$, see (7.1),*

$$\mathcal{P}_C(\Psi) = {}^{(c)}\nabla_3\Psi + C\Psi \in \mathfrak{s}_k(\mathbb{C}).$$

Let $F \in \mathfrak{s}_1(\mathbb{C})$ of conformal type s. Then the following commutators hold:

$$
\begin{aligned}
[\mathcal{P}_C, {}^{(c)}\nabla_3]F &= -({}^{(c)}\nabla_3 C)F, \\
[\mathcal{P}_C, {}^{(c)}\nabla_4]F &= 2(\eta-\underline{\eta})\cdot {}^{(c)}\nabla F + \big(2s\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^{*\,(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\big) \\
&\quad + 2i\big(-\ {}^*\rho + \eta\wedge\underline{\eta}\big) - {}^{(c)}\nabla_4 C\big)F, \\
[\mathcal{P}_C, {}^{(c)}\nabla_a]F &= -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\, {}^{(c)}\nabla_a F - \frac{1}{2}\, {}^{(a)}tr\underline{\chi}\ {}^{*\,(c)}\nabla_a F + \eta_a\, {}^{(c)}\nabla_3 F \\
&\quad + (\mathcal{V}^s_{[3,a]} - {}^{(c)}\nabla_a C)F, \\
[\mathcal{P}_C, {}^{(c)}\mathcal{D}\widehat{\otimes}]F &= -\frac{1}{2}\mathrm{tr}\underline{X}\, {}^{(c)}\mathcal{D}\widehat{\otimes}F + H\widehat{\otimes}\, {}^{(c)}\nabla_3 F \\
&\quad + \left(-\,{}^{(c)}\mathcal{D}C - \frac{1}{2}(s+1)\mathrm{tr}\underline{X}H\right)\widehat{\otimes}F.
\end{aligned}
$$

Let $U \in \mathfrak{s}_2(\mathbb{C})$ of conformal type s. The following commutators hold:

$$
\begin{aligned}
[\mathcal{P}_C, {}^{(c)}\nabla_3]U &= -({}^{(c)}\nabla_3 C)U,\\
[\mathcal{P}_C, {}^{(c)}\nabla_4]U &= 2(\eta-\underline{\eta})\cdot {}^{(c)}\nabla U + \big(2s\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\big)\\
&\quad + 4i\big(-{}^*\rho + \eta\wedge\underline{\eta}\big) - {}^{(c)}\nabla_4 C\big)U,\\
[\mathcal{P}_C, {}^{(c)}\nabla_a]U &= -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_a U - \frac{1}{2}{}^{(a)}tr\underline{\chi}\,{}^{*(c)}\nabla_a U + \eta_a\,{}^{(c)}\nabla_3 U\\
&\quad + (\mathcal{V}^s_{[3,a]} - {}^{(c)}\nabla_a C)U,\\
[\mathcal{P}_C, \overline{{}^{(c)}\mathcal{D}}\cdot]U &= -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\overline{{}^{(c)}\mathcal{D}}\cdot U + \overline{H}\cdot{}^{(c)}\nabla_3 U + \left(-\overline{{}^{(c)}\mathcal{D}}C - \frac{1}{2}(s-2)\overline{\mathrm{tr}\underline{X}}\,\overline{H}\right)\cdot U.
\end{aligned}
$$

Proof. We compute

$$[\mathcal{P}_C, {}^{(c)}\nabla_3]F = ({}^{(c)}\nabla_3 + C)({}^{(c)}\nabla_3 F) - {}^{(c)}\nabla_3({}^{(c)}\nabla_3 F + CF) = -({}^{(c)}\nabla_3 C)F$$

and similarly for U. We compute

$$
\begin{aligned}
[\mathcal{P}_C, {}^{(c)}\nabla_4]F &= ({}^{(c)}\nabla_3 + C)({}^{(c)}\nabla_4 F) - {}^{(c)}\nabla_4({}^{(c)}\nabla_3 F + C\ F)\\
&= [{}^{(c)}\nabla_3, {}^{(c)}\nabla_4]F - ({}^{(c)}\nabla_4 C)F.
\end{aligned}
$$

Recall from (A.18) and from [19]:

$$
\begin{aligned}
[{}^{(c)}\nabla_3, {}^{(c)}\nabla_4]F &= 2(\eta-\underline{\eta})\cdot{}^{(c)}\nabla F + \big(2s\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\big)\\
&\quad + 2i\big(-{}^*\rho + \eta\wedge\underline{\eta}\big)\big), F \qquad (C.1)\\
[{}^{(c)}\nabla_3, {}^{(c)}\nabla_4]U &= 2(\eta-\underline{\eta})\cdot{}^{(c)}\nabla U + \big(2s\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\big)\\
&\quad + 4i\big(-{}^*\rho + \eta\wedge\underline{\eta}\big)\big).U \qquad (C.2)
\end{aligned}
$$

Using (C.1) and (C.2), we obtain the stated expressions.

We compute

$$
\begin{aligned}
[\mathcal{P}_C, {}^{(c)}\nabla_a]F &= ({}^{(c)}\nabla_3 + C)({}^{(c)}\nabla_a F) - {}^{(c)}\nabla_a({}^{(c)}\nabla_3 F + C\ F)\\
&= [{}^{(c)}\nabla_3, {}^{(c)}\nabla_a]F - ({}^{(c)}\nabla_a C)F.
\end{aligned}
$$

Using that, see [19, Lemma 5.3]

$$
\begin{aligned}
[{}^{(c)}\nabla_3, {}^{(c)}\nabla_a]F_b &= -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_a F_b - \frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^{*(c)}\nabla_a F_b + \eta_a\,{}^{(c)}\nabla_3 F_b\\
&\quad + \mathcal{V}^s_{[3,a]}(F), \qquad (C.3)\\
[{}^{(c)}\nabla_3, {}^{(c)}\nabla_a]U_{bc} &= -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(c)}\nabla_a U_{bc} - \frac{1}{2}{}^{(a)}\mathrm{tr}\underline{\chi}\,{}^{*(c)}\nabla_a U_{bc} + \eta_a\,{}^{(c)}\nabla_3 U_{bc}\\
&\quad + \mathcal{V}^s_{[3,a]}(U), \qquad (C.4)
\end{aligned}
$$

where

$$\mathcal{V}^s_{[3,a]}(F) = -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\big(s\eta_a U_b + \eta_b U_a - \delta_{ab}\eta\cdot U\big)$$
$$- \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\big(s\,{}^*\eta_a U_b + \eta_b\,{}^*U_a - \in_{ab}\eta\cdot U\big),$$
$$\mathcal{V}^s_{[3,a]}(U) = -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\big(s(\eta_a)U_{bc} + \eta_b U_{ac} + \eta_c U_{ab} - \delta_{ab}(\eta\cdot U)_c - \delta_{ac}(\eta\cdot U)_b\big)$$
$$-\frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\big(s({}^*\eta_a)U_{bc} + \eta_b\,{}^*U_{ac} + \eta_c\,{}^*U_{ab} - \in_{ab}(\eta\cdot U)_c - \in_{ac}(\eta\cdot U)_b\big),$$

we obtain the stated expressions.

We compute

$$[\mathcal{P}_C, {}^{(c)}\mathcal{D}\widehat{\otimes}]F = ({}^{(c)}\nabla_3 + C)({}^{(c)}\mathcal{D}\widehat{\otimes}F) - {}^{(c)}\mathcal{D}\widehat{\otimes}({}^{(c)}\nabla_3 F + CF)$$
$$= [{}^{(c)}\nabla_3, {}^{(c)}\mathcal{D}\widehat{\otimes}]F - ({}^{(c)}\mathcal{D}C)\widehat{\otimes}F.$$

Using (B.2), we obtain the stated expression. We compute

$$[\mathcal{P}_C, \overline{{}^{(c)}\mathcal{D}}\cdot]U = ({}^{(c)}\nabla_3 + C)(\overline{{}^{(c)}\mathcal{D}}\cdot U) - \overline{{}^{(c)}\mathcal{D}}\cdot({}^{(c)}\nabla_3 U + CU)$$
$$= [{}^{(c)}\nabla_3, \overline{{}^{(c)}\mathcal{D}}\cdot]U - (\overline{{}^{(c)}\mathcal{D}}C)\cdot U.$$

Using (B.1), we obtain the stated expression. □

C.1.2. *The commutators for* $[\mathcal{P}_{C_1}, \mathcal{T}_1]$ *and* $[\mathcal{P}_{C_2}, \mathcal{T}_2]$

Using (6.4) and (6.5), we separate the computations of $[\mathcal{P}_{C_1}, \mathcal{T}_1]$ and $[\mathcal{P}_{C_2}, \mathcal{T}_2]$ into the following terms:

$$[\mathcal{P}_{C_1}, \mathcal{T}_1] = I^{\mathfrak{B}} + J^{\mathfrak{B}} + K^{\mathfrak{B}} + L^{\mathfrak{B}} + M^{\mathfrak{B}} + N^{\mathfrak{B}}, \tag{C.5}$$
$$[\mathcal{P}_{C_2}, \mathcal{T}_2] = I^{\mathfrak{F}} + J^{\mathfrak{F}} + K^{\mathfrak{F}} + L^{\mathfrak{F}} + M^{\mathfrak{F}} + N^{\mathfrak{F}}, \tag{C.6}$$

where

$$I^{\mathfrak{B}} = -[\mathcal{P}_{C_1}, {}^{(c)}\nabla_3\,{}^{(c)}\nabla_4]\mathfrak{B}, \quad I^{\mathfrak{F}} = -[\mathcal{P}_{C_2}, {}^{(c)}\nabla_3\,{}^{(c)}\nabla_4]\mathfrak{F},$$
$$J^{\mathfrak{B}} = \frac{1}{2}[\mathcal{P}_{C_1}, \overline{{}^{(c)}\mathcal{D}}\cdot{}^{(c)}\mathcal{D}\widehat{\otimes}]\mathfrak{B}, \quad J^{\mathfrak{F}} = \frac{1}{2}[\mathcal{P}_{C_2}, {}^{(c)}\mathcal{D}\widehat{\otimes}\overline{{}^{(c)}\mathcal{D}}\cdot]\mathfrak{F},$$
$$K^{\mathfrak{B}} = [\mathcal{P}_{C_1}, -3\overline{\mathrm{tr}X}\,{}^{(c)}\nabla_3]\mathfrak{B}, \quad K^{\mathfrak{F}} = [\mathcal{P}_{C_2}, -\left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right){}^{(c)}\nabla_3]\mathfrak{F},$$
$$L^{\mathfrak{B}} = [\mathcal{P}_{C_1}, -\left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right){}^{(c)}\nabla_4]\mathfrak{B}, \quad L^{\mathfrak{F}} = [\mathcal{P}_{C_2}, -\frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right){}^{(c)}\nabla_4]\mathfrak{F},$$
$$M^{\mathfrak{B}} = [\mathcal{P}_{C_1}, \big(6H + \overline{H} + 3\overline{\underline{H}}\big)\cdot{}^{(c)}\nabla]\mathfrak{B}, \quad M^{\mathfrak{F}} = [\mathcal{P}_{C_2}, \big(4H + \overline{H} + \underline{H}\big)\cdot{}^{(c)}\nabla]\mathfrak{F},$$

$$N^{\mathfrak{B}} = \left[\mathcal{P}_{C_1}, \left(-\frac{9}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} - 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} + 9\,\overline{\underline{H}}\cdot H\right)\right]\mathfrak{B},$$

$$N^{\mathfrak{F}} = \left[\mathcal{P}_{C_2}, + \left(-\frac{3}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} - \frac{1}{4}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}X + 3\overline{P} - P + 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} - \frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H\right)\right]\mathfrak{F}$$
$$+\frac{1}{2}[\mathcal{P}_{C_2}, \underline{H}\widehat{\otimes}\overline{H}\cdot]\mathfrak{F}.$$

C.1.3. *Expressions for $I^{\mathfrak{B}}$ and $I^{\mathfrak{F}}$*

We have

$$I^{\mathfrak{B}} = -[\mathcal{P}_{C_1}, {}^{(c)}\nabla_3\,{}^{(c)}\nabla_4]\mathfrak{B} = -[\mathcal{P}_{C_1}, {}^{(c)}\nabla_3]\,{}^{(c)}\nabla_4\mathfrak{B} - {}^{(c)}\nabla_3([\mathcal{P}_{C_1}, {}^{(c)}\nabla_4]\mathfrak{B})$$
$$= ({}^{(c)}\nabla_3 C_1)\,{}^{(c)}\nabla_4\mathfrak{B} - {}^{(c)}\nabla_3([\mathcal{P}_{C_1}, {}^{(c)}\nabla_4]\mathfrak{B}).$$

Using Lemma C.1 applied to $F=\mathfrak{B}$ of conformal type $s=1$,

$$[\mathcal{P}_{C_1}, {}^{(c)}\nabla_4]\mathfrak{B} = 2(\eta-\underline{\eta})\cdot{}^{(c)}\nabla\mathfrak{B} + \big(2\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^*{}^{(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\big)$$
$$+2i\big(-{}^*\rho + \eta\wedge\underline{\eta}\big) - {}^{(c)}\nabla_4 C_1\big)\mathfrak{B} \tag{C.7}$$

we deduce

$${}^{(c)}\nabla_3([\mathcal{P}_{C_1}, {}^{(c)}\nabla_4]\mathfrak{B})$$
$$= 2(\eta-\underline{\eta})\cdot{}^{(c)}\nabla_3\,{}^{(c)}\nabla\mathfrak{B} + 2\,{}^{(c)}\nabla_3(\eta-\underline{\eta})\cdot{}^{(c)}\nabla\mathfrak{B}$$
$$+\big(2\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^*{}^{(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\big) + 2i\big(-{}^*\rho + \eta\wedge\underline{\eta}\big) - {}^{(c)}\nabla_4 C_1\big)\,{}^{(c)}\nabla_3\mathfrak{B}$$
$$+{}^{(c)}\nabla_3\big(2\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^*{}^{(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\big) + 2i\big(-{}^*\rho + \eta\wedge\underline{\eta}\big) - {}^{(c)}\nabla_4 C_1\big)\mathfrak{B}.$$

Using (C.3) we write the above as

$${}^{(c)}\nabla_3([\mathcal{P}_{C_1}, {}^{(c)}\nabla_4]\mathfrak{B})$$
$$= 2(\eta-\underline{\eta})\cdot{}^{(c)}\nabla\,{}^{(c)}\nabla_3\mathfrak{B} + \big(2\,{}^{(c)}\nabla_3(\eta-\underline{\eta}) - \mathrm{tr}\,\underline{\chi}(\eta-\underline{\eta}) + {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*(\eta-\underline{\eta})\big)$$
$$\cdot{}^{(c)}\nabla\mathfrak{B} + \big(2\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^*{}^{(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta} + \eta\cdot(\eta-\underline{\eta})\big)$$
$$+2i\big(-{}^*\rho + \eta\wedge\underline{\eta}\big) - {}^{(c)}\nabla_4 C_1\big)\,{}^{(c)}\nabla_3\mathfrak{B} + \big[{}^{(c)}\nabla_3\big(2\big(\rho + {}^{(\mathbf{F})}\rho^2$$
$$+{}^*{}^{(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\big) + 2i\big(-{}^*\rho + \eta\wedge\underline{\eta}\big) - {}^{(c)}\nabla_4 C_1\big) + 2(\eta-\underline{\eta})\cdot\mathcal{V}^{s=1}_{[3,a]}\big]\mathfrak{B}.$$

We therefore obtain

$$I^{\mathfrak{B}} = -2(\eta-\underline{\eta})\cdot{}^{(c)}\nabla\,{}^{(c)}\nabla_3\mathfrak{B} + I^{\mathfrak{B}}_4\,{}^{(c)}\nabla_4\mathfrak{B} + I^{\mathfrak{B}}_3\,{}^{(c)}\nabla_3\mathfrak{B} + I^{\mathfrak{B}}_a\cdot{}^{(c)}\nabla_a\mathfrak{B} + I^{\mathfrak{B}}_0\mathfrak{B}, \tag{C.8}$$

where

$$\begin{aligned}
I_4^{\mathfrak{B}} &= {}^{(c)}\nabla_3 C_1,\\
I_3^{\mathfrak{B}} &= -2\rho - 2\,{}^{(\mathbf{F})}\rho^2 - 2\,{}^{*\,(\mathbf{F})}\rho^2 - 2\eta\cdot(\eta - 2\underline{\eta}) + i\big(2\ {}^*\rho - 2\eta\wedge\underline{\eta}\big) + {}^{(c)}\nabla_4 C_1,\\
I_a^{\mathfrak{B}} &= -2\,{}^{(c)}\nabla_3(\eta-\underline{\eta}) + \operatorname{tr}\underline{\chi}(\eta-\underline{\eta}) - {}^{(a)}\operatorname{tr}\underline{\chi}\ {}^*(\eta-\underline{\eta}),\\
I_0^{\mathfrak{B}} &= {}^{(c)}\nabla_3\big[-2\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^{*\,(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\big)\\
&\quad + 2i\big({}^*\rho - \eta\wedge\underline{\eta}\big) + {}^{(c)}\nabla_4 C_1\big] - 2(\eta-\underline{\eta})\cdot\mathcal{V}^{s=1}_{[3,a]}.
\end{aligned}\tag{C.9}$$

We have

$$I^{\mathfrak{F}} = \big({}^{(c)}\nabla_3 C_2\big)\,{}^{(c)}\nabla_4\mathfrak{F} - {}^{(c)}\nabla_3\big([\mathcal{P}_{C_2}, {}^{(c)}\nabla_4]\mathfrak{F}\big).$$

Using Lemma C.1 applied to $U = \mathfrak{F}$ of conformal type $s = 1$,

$$\begin{aligned}
[\mathcal{P}_{C_2}, {}^{(c)}\nabla_4]\mathfrak{F} &= 2(\eta-\underline{\eta})\cdot{}^{(c)}\nabla\mathfrak{F} + \big(2\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^{*\,(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\big)\\
&\quad + 4i\big(-\ {}^*\rho + \eta\wedge\underline{\eta}\big) - {}^{(c)}\nabla_4 C_2\big)\mathfrak{F}.
\end{aligned}\tag{C.10}$$

We similarly obtain

$$I^{\mathfrak{F}} = -2(\eta-\underline{\eta})\cdot{}^{(c)}\nabla\,{}^{(c)}\nabla_3\mathfrak{F} + I_4^{\mathfrak{F}}\,{}^{(c)}\nabla_4\mathfrak{F} + I_3^{\mathfrak{F}}\,{}^{(c)}\nabla_3\mathfrak{F} + I_a^{\mathfrak{F}}\cdot{}^{(c)}\nabla_a\mathfrak{F} + I_0^{\mathfrak{F}}\mathfrak{F},\tag{C.11}$$

where

$$\begin{aligned}
I_4^{\mathfrak{F}} &= {}^{(c)}\nabla_3 C_2,\\
I_3^{\mathfrak{F}} &= -2\rho - 2\,{}^{(\mathbf{F})}\rho^2 - 2\,{}^{*\,(\mathbf{F})}\rho^2 - 2\eta\cdot(\eta - 2\underline{\eta}) + i\big(4\ {}^*\rho - 4\eta\wedge\underline{\eta}\big) + {}^{(c)}\nabla_4 C_2,\\
I_a^{\mathfrak{F}} &= -2\,{}^{(c)}\nabla_3(\eta-\underline{\eta}) + \operatorname{tr}\underline{\chi}(\eta-\underline{\eta}) - {}^{(a)}\operatorname{tr}\underline{\chi}\ {}^*(\eta-\underline{\eta}),\\
I_0^{\mathfrak{F}} &= {}^{(c)}\nabla_3\big[-2\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^{*\,(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\big) + 4i\big({}^*\rho - \eta\wedge\underline{\eta}\big) + {}^{(c)}\nabla_4 C_2\big]\\
&\quad - 2(\eta-\underline{\eta})\cdot\mathcal{V}^{s=1}_{[3,a]}.
\end{aligned}\tag{C.12}$$

C.1.4. *Expressions for $J^{\mathfrak{B}}$ and $J^{\mathfrak{F}}$*

We have

$$J^{\mathfrak{B}} = \frac{1}{2}[\mathcal{P}_{C_1}, \overline{{}^{(c)}\mathcal{D}}\cdot]\big({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\big) + \frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\big([\mathcal{P}_{C_1}, {}^{(c)}\mathcal{D}\widehat{\otimes}]\mathfrak{B}\big).$$

Using Lemma C.1 applied to $U = {}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}$ of conformal type $s = 1$, we have

$$\begin{aligned}
&[\mathcal{P}_{C_1}, \overline{{}^{(c)}\mathcal{D}}\cdot]\big({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\big)\\
&\quad = -\frac{1}{2}\overline{\operatorname{tr}\underline{X}}\,\overline{{}^{(c)}\mathcal{D}}\cdot\big({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\big) + \overline{H}\cdot\nabla_3\big({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\big)
\end{aligned}$$

$$
\begin{aligned}
&+\left(-\overline{{}^{(c)}\mathcal{D}}C_1+\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\overline{H}\right)\cdot\left({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\right)\\
=&-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\overline{{}^{(c)}\mathcal{D}}\cdot\left({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\right)+\overline{H}\cdot{}^{(c)}\mathcal{D}\widehat{\otimes}\nabla_3\mathfrak{B}\\
&+\overline{H}\cdot\left(-\frac{1}{2}\mathrm{tr}\underline{X}\big({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}+2H\widehat{\otimes}\mathfrak{B}\big)+H\widehat{\otimes}\,{}^{(c)}\nabla_3\mathfrak{B}\right)\\
&+\left(-\overline{{}^{(c)}\mathcal{D}}C_1+\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\overline{H}\right)\cdot\left({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\right)\\
=&-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\overline{{}^{(c)}\mathcal{D}}\cdot\left({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\right)+2\overline{H}\cdot{}^{(c)}\nabla\,{}^{(c)}\nabla_3\mathfrak{B}\\
&+\big(-2\,\overline{{}^{(c)}\mathcal{D}}C_1+(\overline{\mathrm{tr}\underline{X}}-\mathrm{tr}\underline{X})\,\overline{H}\big)\cdot{}^{(c)}\nabla\mathfrak{B}+(H\cdot\overline{H})\,{}^{(c)}\nabla_3\mathfrak{B}\\
&-\mathrm{tr}\underline{X}(\overline{H}\cdot H)\mathfrak{B}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
&\overline{{}^{(c)}\mathcal{D}}\cdot\left([\mathcal{P}_{C_1},{}^{(c)}\mathcal{D}\widehat{\otimes}]\mathfrak{B}\right)\\
&=\overline{{}^{(c)}\mathcal{D}}\cdot\Big(-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}+H\widehat{\otimes}\,{}^{(c)}\nabla_3\mathfrak{B}+\big(-{}^{(c)}\mathcal{D}C_1-\mathrm{tr}\underline{X}H\big)\widehat{\otimes}\mathfrak{B}\Big)\\
&=-\frac{1}{2}\mathrm{tr}\underline{X}\,\overline{{}^{(c)}\mathcal{D}}\cdot\left({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\right)+2H\cdot{}^{(c)}\nabla\,{}^{(c)}\nabla_3\mathfrak{B}\\
&\quad+\big(-2\,{}^{(c)}\mathcal{D}C_1-2\mathrm{tr}\underline{X}H-(\overline{\mathrm{tr}\underline{X}}-\mathrm{tr}\underline{X})\,\overline{\underline{H}}\big)\cdot{}^{(c)}\nabla\mathfrak{B}\\
&\quad+\big(\overline{{}^{(c)}\mathcal{D}}\cdot H\big)\,{}^{(c)}\nabla_3\mathfrak{B}+\overline{{}^{(c)}\mathcal{D}}\cdot\big(-{}^{(c)}\mathcal{D}C_1-\mathrm{tr}\underline{X}H\big)\mathfrak{B},
\end{aligned}
$$

where we used ${}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}}-(\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}})\,\underline{H}=0$.

Putting the above together we obtain

$$
\begin{aligned}
J^{\mathfrak{B}}=&-\frac{1}{2}(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}})\left(\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\left({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\right)\right)+2\eta\cdot{}^{(c)}\nabla\,{}^{(c)}\nabla_3\mathfrak{B}+\tilde{J}^{\mathfrak{B}}_3\,{}^{(c)}\nabla_3\mathfrak{B}\\
&+\tilde{J}^{\mathfrak{B}}_a\cdot{}^{(c)}\nabla\mathfrak{B}+\tilde{J}^{\mathfrak{B}}_0\mathfrak{B},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{J}^{\mathfrak{B}}_3&=\frac{1}{2}\big(\overline{{}^{(c)}\mathcal{D}}\cdot H\big)+\frac{1}{2}(H\cdot\overline{H}),\\
\tilde{J}^{\mathfrak{B}}_a&=-2\,{}^{(c)}\nabla C_1-\mathrm{tr}\underline{X}H-\frac{1}{2}(\overline{\mathrm{tr}\underline{X}}-\mathrm{tr}\underline{X})\,\overline{\underline{H}}+\frac{1}{2}(\overline{\mathrm{tr}\underline{X}}-\mathrm{tr}\underline{X})\,\overline{H},\\
\tilde{J}^{\mathfrak{B}}_0&=\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot\big(-{}^{(c)}\mathcal{D}C_1-\mathrm{tr}\underline{X}H\big)-\frac{1}{2}\mathrm{tr}\underline{X}(\overline{H}\cdot H).
\end{aligned}
$$

Using the Teukolsky equation for $\mathfrak{B}$ given by $\mathcal{T}_1(\mathfrak{B}) = \mathbf{M}_1[\mathfrak{F}, \mathfrak{X}]$ and the expression for the Teukolsky operator (6.4) we can write

$$\begin{aligned}
&\frac{1}{2}\overline{{}^{(c)}\mathcal{D}}\cdot {}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\\
&\quad = {}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\mathfrak{B} + 3\overline{\mathrm{tr}X}\,{}^{(c)}\nabla_3\mathfrak{B} + \left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right){}^{(c)}\nabla_4\mathfrak{B}\\
&\qquad - \big(6H + \overline{H} + 3\overline{\underline{H}}\big)\cdot {}^{(c)}\nabla\mathfrak{B} + \left(\frac{9}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} - 9\overline{\underline{H}}\cdot H\right)\mathfrak{B}\\
&\qquad + \mathbf{M}_1[\mathfrak{F}, \mathfrak{X}]\\
&\quad = {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\mathfrak{B} + 3\overline{\mathrm{tr}X}\,{}^{(c)}\nabla_3\mathfrak{B} + \left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right){}^{(c)}\nabla_4\mathfrak{B}\\
&\qquad + \big(2(\eta - \underline{\eta}) - \big(6H + \overline{H} + 3\overline{\underline{H}}\big)\big)\cdot {}^{(c)}\nabla\mathfrak{B}\\
&\qquad + \left(\frac{9}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + 2\overline{P} + 6\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} - 10\overline{\underline{H}}\cdot H\right)\mathfrak{B} + \mathbf{M}_1[\mathfrak{F}, \mathfrak{X}],
\end{aligned}$$

where we used (A.18) to write $[\,{}^{(c)}\nabla_3, {}^{(c)}\nabla_4]\mathfrak{B} = 2(\eta - \underline{\eta})\cdot {}^{(c)}\nabla\mathfrak{B} + \big(2\overline{P} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} - (H\cdot\overline{\underline{H}})\big)\mathfrak{B}$. We therefore obtain

$$\begin{aligned}
J^{\mathfrak{B}} &= -\frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\mathfrak{B} + 2\eta\cdot{}^{(c)}\nabla\,{}^{(c)}\nabla_3\mathfrak{B}\\
&\quad + J^{\mathfrak{B}}_4\,{}^{(c)}\nabla_4\mathfrak{B} + J^{\mathfrak{B}}_3\,{}^{(c)}\nabla_3\mathfrak{B} + J^{\mathfrak{B}}_a\cdot{}^{(c)}\nabla\mathfrak{B} + J^{\mathfrak{B}}_0\mathfrak{B} - \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\mathbf{M}_1[\mathfrak{F}, \mathfrak{X}],
\end{aligned}\tag{C.13}$$

where

$$\begin{aligned}
J^{\mathfrak{B}}_3 &= -\frac{3}{2}\overline{\mathrm{tr}X}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big) + \tilde{J}^{\mathfrak{B}}_3,\\
J^{\mathfrak{B}}_4 &= -\frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right),\\
J^{\mathfrak{B}}_a &= -\frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\big(2(\eta - \underline{\eta}) - \big(6H + \overline{H} + 3\overline{\underline{H}}\big)\big) + \tilde{J}^{\mathfrak{B}}_a,\\
J^{\mathfrak{B}}_0 &= -\frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\left(\frac{9}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + 2\overline{P} + 6\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} - 10\overline{\underline{H}}\cdot H\right) + \tilde{J}^{\mathfrak{B}}_0.
\end{aligned}\tag{C.14}$$

We have

$$J^{\mathfrak{F}} = \frac{1}{2}[\mathcal{P}_{C_2}, {}^{(c)}\mathcal{D}\widehat{\otimes}](\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F}) + \frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}([\mathcal{P}_{C_2}, \overline{{}^{(c)}\mathcal{D}}\cdot]\mathfrak{F}).$$

Using Lemma C.1 applied to $F = \overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F}$ of conformal type $s = 1$, we have

$$\begin{aligned}
&[\mathcal{P}_{C_2}, {}^{(c)}\mathcal{D}\widehat{\otimes}](\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F}) \\
&= -\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F}) + H\widehat{\otimes}\,{}^{(c)}\nabla_3(\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F}) \\
&\quad + \big(-\,{}^{(c)}\mathcal{D}C_2 - \mathrm{tr}\underline{X}H\big)\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F}) \\
&= -\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F}) + H\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}} \cdot {}^{(c)}\nabla_3\mathfrak{F} - \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,(\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F} - \overline{H} \cdot \mathfrak{F}) \\
&\quad + \overline{H} \cdot \nabla_3\mathfrak{F}) + \big(-\,{}^{(c)}\mathcal{D}C_2 - \mathrm{tr}\underline{X}H\big)\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F}) \\
&= -\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F}) + 2H \cdot {}^{(c)}\nabla\,{}^{(c)}\nabla_3\mathfrak{F} + (H \cdot \overline{H})\nabla_3\mathfrak{F} \\
&\quad + \big(-2\,{}^{(c)}\mathcal{D}C_2 - (2\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})H\big) \cdot {}^{(c)}\nabla\mathfrak{F} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,(H \cdot \overline{H})\mathfrak{F}.
\end{aligned}$$

We also have

$$\begin{aligned}
&{}^{(c)}\mathcal{D}\widehat{\otimes}([\mathcal{P}_{C_2}, \overline{{}^{(c)}\mathcal{D}}\cdot]\mathfrak{F}) \\
&= {}^{(c)}\mathcal{D}\widehat{\otimes}\left(-\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F} + \overline{H} \cdot {}^{(c)}\nabla_3\mathfrak{F} + \left(-\overline{{}^{(c)}\mathcal{D}}C_2 + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\overline{H}\right) \cdot \mathfrak{F}\right) \\
&= -\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F}) + 2\overline{H} \cdot {}^{(c)}\nabla\,{}^{(c)}\nabla_3\mathfrak{F} \\
&\quad + \big(-2\,\overline{{}^{(c)}\mathcal{D}}C_2 + \overline{\mathrm{tr}\underline{X}}\,\overline{H} - \left(\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\right)\underline{H}\big) \cdot {}^{(c)}\nabla\mathfrak{F} \\
&\quad + ({}^{(c)}\mathcal{D} \cdot \overline{H})\,{}^{(c)}\nabla_3\mathfrak{F} + {}^{(c)}\mathcal{D} \cdot \left(-\overline{{}^{(c)}\mathcal{D}}C_2 + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\overline{H}\right)\mathfrak{F},
\end{aligned}$$

where we used ${}^{(c)}\mathcal{D}\overline{\mathrm{tr}\underline{X}} - (\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}})\,\underline{H} = 0$.

Putting the above together we obtain

$$\begin{aligned}
J^{\mathfrak{F}} &= -\frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\left(\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F})\right) + 2\eta \cdot {}^{(c)}\nabla\,{}^{(c)}\nabla_3\mathfrak{F} + \tilde{J}_3^{\mathfrak{F}}\nabla_3\mathfrak{F} \\
&\quad + \tilde{J}_a^{\mathfrak{F}} \cdot {}^{(c)}\nabla\mathfrak{F} + \tilde{J}_0^{\mathfrak{F}}\mathfrak{F},
\end{aligned}$$

where

$$\begin{aligned}
\tilde{J}_3^{\mathfrak{F}} &= \frac{1}{2}({}^{(c)}\mathcal{D} \cdot \overline{H}) + \frac{1}{2}(H \cdot \overline{H}), \\
\tilde{J}_a^{\mathfrak{F}} &= -2\,{}^{(c)}\nabla C_2 + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\overline{H} - \frac{1}{2}\left(\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\right)\underline{H} - \frac{1}{2}(2\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})H, \\
\tilde{J}_0^{\mathfrak{F}} &= \frac{1}{2}\,{}^{(c)}\mathcal{D} \cdot \left(-\overline{{}^{(c)}\mathcal{D}}C_2 + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\overline{H}\right) + \frac{1}{4}\overline{\mathrm{tr}\underline{X}}\,(H \cdot \overline{H}).
\end{aligned}$$

Using the Teukolsky equation for $\mathfrak{F}$ given by $\mathcal{T}_2(\mathfrak{F}) = \mathbf{M}_2[A, \mathfrak{X}, \mathfrak{B}]$ and the expression for the Teukolsky operator (6.5) we can write

$$\begin{aligned}
&\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F})\\
&\quad= {}^{(c)}\nabla_3\,{}^{(c)}\nabla_4\mathfrak{F} + \left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right){}^{(c)}\nabla_3\mathfrak{F} + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right){}^{(c)}\nabla_4\mathfrak{F}\\
&\qquad+ \left(\frac{3}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + \frac{1}{4}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}X - 3\overline{P} + P - 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} + \frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H\right)\mathfrak{F}\\
&\qquad- \frac{1}{2}\underline{H}\widehat{\otimes}(\overline{H}\cdot\mathfrak{F}) - \left(4H + \overline{H} + \underline{H}\right)\cdot{}^{(c)}\nabla\mathfrak{F} + \mathbf{M}_2[A, \mathfrak{X}, \mathfrak{B}]\\
&\quad= {}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\mathfrak{F} + \left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right){}^{(c)}\nabla_3\mathfrak{F} + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right){}^{(c)}\nabla_4\mathfrak{F}\\
&\qquad+ \left(2(\eta - \underline{\eta}) - \left(4H + \overline{H} + \underline{H}\right)\right)\cdot{}^{(c)}\nabla\mathfrak{F} + \mathbf{M}_2[A, \mathfrak{X}, \mathfrak{B}]\\
&\qquad+ \left(\frac{3}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + \frac{1}{4}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}X - 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} + \frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H - 3\eta\cdot\underline{\eta} + 3i\eta\wedge\underline{\eta}\right)\mathfrak{F},
\end{aligned}$$

where we used (A.18) to write $[{}^{(c)}\nabla_3, {}^{(c)}\nabla_4]\mathfrak{F} = 2(\eta - \underline{\eta})\cdot{}^{(c)}\nabla\mathfrak{F} + \big(-P + 3\overline{P} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} - 2\eta\cdot\underline{\eta} + 4i\eta\wedge\underline{\eta}\big)\mathfrak{F}$ and $\frac{1}{2}\underline{H}\widehat{\otimes}(\overline{H}\cdot\mathfrak{F}) = \big(\eta\cdot\underline{\eta} + i\eta\wedge\underline{\eta}\big)\mathfrak{F}$.

We therefore obtain

$$\begin{aligned}
J^{\mathfrak{F}} &= -\frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\mathfrak{F} + 2\eta\cdot{}^{(c)}\nabla\,{}^{(c)}\nabla_3\mathfrak{F}\\
&\quad+ J_4^{\mathfrak{F}}\,{}^{(c)}\nabla_4\mathfrak{F} + J_3^{\mathfrak{F}}\,{}^{(c)}\nabla_3\mathfrak{F} + J_a^{\mathfrak{F}}\cdot{}^{(c)}\nabla\mathfrak{F} + J_0^{\mathfrak{F}}\mathfrak{F}\\
&\quad- \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\mathbf{M}_2[A, \mathfrak{X}, \mathfrak{B}], \qquad \text{(C.15)}
\end{aligned}$$

where

$$\begin{aligned}
J_3^{\mathfrak{F}} &= -\frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right) + \tilde{J}_3^{\mathfrak{F}},\\
J_4^{\mathfrak{F}} &= -\frac{1}{4}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right),\\
J_a^{\mathfrak{F}} &= -\frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\big(2(\eta - \underline{\eta}) - \left(4H + \overline{H} + \underline{H}\right)\big) + \tilde{J}_a^{\mathfrak{F}},\\
J_0^{\mathfrak{F}} &= -\frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\left(\frac{3}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + \frac{1}{4}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}X - 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right.\\
&\qquad\left. + \frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H - 3\eta\cdot\underline{\eta} + 3i\eta\wedge\underline{\eta}\right) + \tilde{J}_0^{\mathfrak{F}}. \qquad \text{(C.16)}
\end{aligned}$$

C.1.5. *Expressions for $K^{\mathfrak{B}}$ and $K^{\mathfrak{F}}$*

Observe that

$$\begin{aligned}\mathcal{P}_C(gF) &= ({}^{(c)}\nabla_3 + C)(gF) = ({}^{(c)}\nabla_3 g)F + g\,{}^{(c)}\nabla_3 F + CgF\\ &= g\mathcal{P}_C(F) + ({}^{(c)}\nabla_3 g)F.\end{aligned} \tag{C.17}$$

We have, for $g = -3\overline{\mathrm{tr}X}$

$$\begin{aligned}K^{\mathfrak{B}} &= [\mathcal{P}_{C_1}, g\,{}^{(c)}\nabla_3]\mathfrak{B} = g[\mathcal{P}_{C_1}, {}^{(c)}\nabla_3]\mathfrak{B} + ({}^{(c)}\nabla_3 g)\,{}^{(c)}\nabla_3\mathfrak{B}\\ &= ({}^{(c)}\nabla_3 g)\,{}^{(c)}\nabla_3\mathfrak{B} - g\,{}^{(c)}\nabla_3 C_1\mathfrak{B}.\end{aligned}$$

We therefore obtain

$$K^{\mathfrak{B}} = K_3^{\mathfrak{B}}\,{}^{(c)}\nabla_3\mathfrak{B} + K_0^{\mathfrak{B}}\mathfrak{B}, \tag{C.18}$$

where

$$K_3^{\mathfrak{B}} = -3\,{}^{(c)}\nabla_3(\overline{\mathrm{tr}X}), \quad K_0^{\mathfrak{B}} = 3\overline{\mathrm{tr}X}\,{}^{(c)}\nabla_3 C_1.$$

Similarly,

$$K^{\mathfrak{F}} = K_3^{\mathfrak{F}}\,{}^{(c)}\nabla_3\mathfrak{F} + K_0^{\mathfrak{F}}\mathfrak{F}, \tag{C.19}$$

where

$$K_3^{\mathfrak{F}} = -\,{}^{(c)}\nabla_3\left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right), \quad K_0^{\mathfrak{F}} = \left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right){}^{(c)}\nabla_3 C_2.$$

C.1.6. *Expressions for $L^{\mathfrak{B}}$ and $L^{\mathfrak{F}}$*

Using (C.17), we obtain

$$\begin{aligned}L &= [\mathcal{P}_C, g\,{}^{(c)}\nabla_4]\Psi = \mathcal{P}_C(g\,{}^{(c)}\nabla_4\Psi) - g\,{}^{(c)}\nabla_4(\mathcal{P}_C\Psi)\\ &= g[\mathcal{P}_C, {}^{(c)}\nabla_4]\Psi + ({}^{(c)}\nabla_3 g)\,{}^{(c)}\nabla_4\Psi\end{aligned}$$

Using (C.7) with $g = -\left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right)$, we obtain

$$L^{\mathfrak{B}} = L_4^{\mathfrak{B}}\,{}^{(c)}\nabla_4\mathfrak{B} + L_a^{\mathfrak{B}}\cdot{}^{(c)}\nabla\mathfrak{B} + L_0^{\mathfrak{B}}\mathfrak{B}, \tag{C.20}$$

where

$$\begin{aligned}L_4^{\mathfrak{B}} &= -\,{}^{(c)}\nabla_3\left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right),\\ L_a^{\mathfrak{B}} &= -\left(3\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)(\eta - \underline{\eta}),\\ L_0^{\mathfrak{B}} &= -\left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right)\big(2\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\big)\\ &\quad + 2i\big(-{}^*\rho + \eta\wedge\underline{\eta}\big) - {}^{(c)}\nabla_4 C_1\big).\end{aligned} \tag{C.21}$$

Similarly, using (C.10) with $g = -\frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})$ we obtain

$$L^{\mathfrak{F}} = L_4^{\mathfrak{F}}\, {}^{(c)}\nabla_4 \mathfrak{F} + L_a^{\mathfrak{F}} \cdot {}^{(c)}\nabla \mathfrak{F} + L_0^{\mathfrak{F}} \mathfrak{F}, \tag{C.22}$$

where

$$\begin{aligned}
L_4^{\mathfrak{F}} &= -\frac{1}{2}\, {}^{(c)}\nabla_3 \big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big),\\
L_a^{\mathfrak{F}} &= -\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)(\eta - \underline{\eta}),\\
L_0^{\mathfrak{F}} &= -\frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\big(2\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^{*\,(\mathbf{F})}\rho^2 - \eta \cdot \underline{\eta}\big)\\
&\quad + 4i\big(- {}^{*}\rho + \eta \wedge \underline{\eta}\big) - {}^{(c)}\nabla_4 C_2\big).
\end{aligned} \tag{C.23}$$

C.1.7. *Expressions for $M^{\mathfrak{B}}$ and $M^{\mathfrak{F}}$*

Observe that

$$\begin{aligned}
\mathcal{P}_{C_1}(F \cdot U) &= ({}^{(c)}\nabla_3 + C_1)(F \cdot U) = {}^{(c)}\nabla_3 F \cdot U + F \cdot {}^{(c)}\nabla_3 U + C_1 F \cdot U\\
&= F \cdot \mathcal{P}_{C_1}(U) + ({}^{(c)}\nabla_3 F) \cdot U.
\end{aligned}$$

We have, for $F = (6H + \overline{H} + 3\overline{\underline{H}})$, and using Lemma C.1

$$\begin{aligned}
M^{\mathfrak{B}} &= [\mathcal{P}_{C_1}, F \cdot {}^{(c)}\nabla]\mathfrak{B} = F \cdot [\mathcal{P}_{C_1}, {}^{(c)}\nabla]\mathfrak{B} + ({}^{(c)}\nabla_3 F) \cdot {}^{(c)}\nabla \mathfrak{B}\\
&= -\frac{1}{2}\mathrm{tr}\,\underline{\chi}\, F \cdot {}^{(c)}\nabla \mathfrak{B} - \frac{1}{2}\, {}^{(a)}\mathrm{tr}\underline{\chi}\, F \cdot {}^{*\,(c)}\nabla \mathfrak{B} + F \cdot \eta\, {}^{(c)}\nabla_3 \mathfrak{B}\\
&\quad + F \cdot (\mathcal{V}^s_{[3,a]} - {}^{(c)}\nabla C_1)\mathfrak{B} + ({}^{(c)}\nabla_3 F) \cdot {}^{(c)}\nabla \mathfrak{B}.
\end{aligned}$$

We therefore obtain

$$M^{\mathfrak{B}} = M_3^{\mathfrak{B}}\, {}^{(c)}\nabla_3 \mathfrak{B} + M_a^{\mathfrak{B}} \cdot {}^{(c)}\nabla \mathfrak{B} + M_0^{\mathfrak{B}} \mathfrak{B}, \tag{C.24}$$

where

$$\begin{aligned}
M_3^{\mathfrak{B}} &= \eta \cdot \big(6H + \overline{H} + 3\overline{\underline{H}}\big),\\
M_a^{\mathfrak{B}} &= {}^{(c)}\nabla_3 \big(6H + \overline{H} + 3\overline{\underline{H}}\big) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\big(6H + \overline{H} + 3\overline{\underline{H}}\big)\\
&\quad + \frac{1}{2}\, {}^{(a)}\mathrm{tr}\underline{\chi}\, {}^{*}\big(6H + \overline{H} + 3\overline{\underline{H}}\big),\\
M_0^{\mathfrak{B}} &= \big(6H + \overline{H} + 3\overline{\underline{H}}\big) \cdot (\mathcal{V}^s_{[3,a]} - {}^{(c)}\nabla C_1).
\end{aligned} \tag{C.25}$$

Similarly, we obtain

$$M^{\mathfrak{F}} = M_3^{\mathfrak{B}}\, {}^{(c)}\nabla_3 \mathfrak{F} + M_a^{\mathfrak{F}} \cdot {}^{(c)}\nabla \mathfrak{F} + M_0^{\mathfrak{F}} \mathfrak{F}, \tag{C.26}$$

where

$$
\begin{aligned}
M_3^{\mathfrak{F}} &= \eta \cdot \big(4H + \overline{H} + \underline{H}\big),\\
M_a^{\mathfrak{F}} &= {}^{(c)}\nabla_3\big(4H + \overline{H} + \underline{H}\big) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\big(4H + \overline{H} + \underline{H}\big)\\
&\quad + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\ {}^*\big(4H + \overline{H} + \underline{H}\big),\\
M_0^{\mathfrak{F}} &= \big(4H + \overline{H} + \underline{H}\big) \cdot (\mathcal{V}^s_{[3,a]} - {}^{(c)}\nabla C_2).
\end{aligned}
\tag{C.27}
$$

C.1.8. *Expressions for $N^{\mathfrak{B}}$ and $N^{\mathfrak{F}}$*

Using (C.17) we have

$$N = [\mathcal{P}_C, g]\Psi = ({}^{(c)}\nabla_3 g)\Psi.$$

We therefore obtain

$$N^{\mathfrak{B}} = N_0^{\mathfrak{B}}\mathfrak{B}, \tag{C.28}$$

where

$$N_0^{\mathfrak{B}} = {}^{(c)}\nabla_3\left(-\frac{9}{2}\mathrm{tr}\underline{X}\,\overline{\mathrm{tr}X} - 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} + 9\,\overline{\underline{H}}\cdot H\right)$$

and

$$N^{\mathfrak{F}} = N_0^{\mathfrak{F}}\mathfrak{F}, \tag{C.29}$$

where

$$
\begin{aligned}
N_0^{\mathfrak{F}} = {}^{(c)}\nabla_3\bigg(&-\frac{3}{4}\mathrm{tr}\underline{X}\,\overline{\mathrm{tr}X} - \frac{1}{4}\overline{\mathrm{tr}\underline{X}}\,\mathrm{tr}X + 3\overline{P} - P + 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\\
&- \frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H + \eta\cdot\underline{\eta} + i\eta\wedge\underline{\eta}\bigg).
\end{aligned}
$$

C.1.9. *The sum*

From (C.5) and (C.6), we obtain

$$
\begin{aligned}
[\mathcal{P}_{C_1}, \mathcal{T}_1](\mathfrak{B}) &= \text{(C.8)} + \text{(C.13)} + \text{(C.18)} + \text{(C.20)} + \text{(C.24)} + \text{(C.28)}\\
&= 2\underline{\eta}\cdot\,{}^{(c)}\nabla\,{}^{(c)}\nabla_3\mathfrak{B} - \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\,{}^{(c)}\nabla_4\,{}^{(c)}\nabla_3\mathfrak{B}\\
&\quad + \big(I_4^{\mathfrak{B}} + J_4^{\mathfrak{B}} + L_4^{\mathfrak{B}}\big)\,{}^{(c)}\nabla_4\mathfrak{B} + \big(I_3^{\mathfrak{B}} + J_3^{\mathfrak{B}} + K_3^{\mathfrak{B}} + M_3^{\mathfrak{B}}\big)\,{}^{(c)}\nabla_3\mathfrak{B}\\
&\quad + \big(I_a^{\mathfrak{B}} + J_a^{\mathfrak{B}} + L_a^{\mathfrak{B}} + M_a^{\mathfrak{B}}\big)\cdot\,{}^{(c)}\nabla\mathfrak{B}\\
&\quad + \big(I_0^{\mathfrak{B}} + J_0^{\mathfrak{B}} + K_0^{\mathfrak{B}} + L_0^{\mathfrak{B}} + M_0^{\mathfrak{B}} + N_0^{\mathfrak{B}}\big)\mathfrak{B}\\
&\quad - \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\mathbf{M}_1[\mathfrak{F}, \mathfrak{X}]
\end{aligned}
$$

and

$$
\begin{aligned}
[\mathcal{P}_{C_2}, \mathcal{T}_2](\mathfrak{F}) &= (\text{C.11}) + (\text{C.15}) + (\text{C.19}) + (\text{C.22}) + (\text{C.26}) + (\text{C.29}) \\
&= 2\underline{\eta} \cdot {}^{(c)}\nabla\, {}^{(c)}\nabla_3 \mathfrak{F} - \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\, {}^{(c)}\nabla_4\, {}^{(c)}\nabla_3 \mathfrak{F} \\
&\quad + \left(I_4^{\mathfrak{F}} + J_4^{\mathfrak{F}} + L_4^{\mathfrak{F}}\right) {}^{(c)}\nabla_4 \mathfrak{F} + \left(I_3^{\mathfrak{F}} + J_3^{\mathfrak{F}} + K_3^{\mathfrak{F}} + M_3^{\mathfrak{F}}\right) {}^{(c)}\nabla_3 \mathfrak{F} \\
&\quad + \left(I_a^{\mathfrak{F}} + J_a^{\mathfrak{F}} + L_a^{\mathfrak{F}} + M_a^{\mathfrak{F}}\right) \cdot {}^{(c)}\nabla \mathfrak{F} + \left(I_0^{\mathfrak{F}} + J_0^{\mathfrak{F}} + K_0^{\mathfrak{F}} + L_0^{\mathfrak{F}} + M_0^{\mathfrak{F}}\right. \\
&\quad \left. + N_0^{\mathfrak{F}}\right)\mathfrak{F} - \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\mathbf{M}_2[A, \mathfrak{X}, \mathfrak{B}].
\end{aligned}
$$

Using (7.2) and (7.3) to write

$$
{}^{(c)}\nabla_3 \mathfrak{B} = \mathfrak{P} - C_1\, \mathfrak{B}, \qquad {}^{(c)}\nabla_3 \mathfrak{F} = \mathfrak{Q} - C_2\, \mathfrak{F}
$$

and therefore

$$
\begin{aligned}
{}^{(c)}\nabla_4\, {}^{(c)}\nabla_3 \mathfrak{B} &= {}^{(c)}\nabla_4 (\mathfrak{P} - C_1\, \mathfrak{B}) = {}^{(c)}\nabla_4 \mathfrak{P} - C_1\, {}^{(c)}\nabla_4 \mathfrak{B} - ({}^{(c)}\nabla_4 C_1)\mathfrak{B}, \\
{}^{(c)}\nabla\, {}^{(c)}\nabla_3 \mathfrak{B} &= {}^{(c)}\nabla (\mathfrak{P} - C_1\, \mathfrak{B}) = {}^{(c)}\nabla \mathfrak{P} - C_1\, {}^{(c)}\nabla \mathfrak{B} - ({}^{(c)}\nabla C_1)\mathfrak{B}, \\
{}^{(c)}\nabla_4\, {}^{(c)}\nabla_3 \mathfrak{F} &= {}^{(c)}\nabla_4 (\mathfrak{Q} - C_2\, \mathfrak{F}) = {}^{(c)}\nabla_4 \mathfrak{Q} - C_2\, {}^{(c)}\nabla_4 \mathfrak{F} - ({}^{(c)}\nabla_4 C_2)\mathfrak{F}, \\
{}^{(c)}\nabla\, {}^{(c)}\nabla_3 \mathfrak{F} &= {}^{(c)}\nabla (\mathfrak{Q} - C_2\, \mathfrak{F}) = {}^{(c)}\nabla \mathfrak{Q} - C_2\, {}^{(c)}\nabla \mathfrak{F} - ({}^{(c)}\nabla C_2)\mathfrak{F}.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
[\mathcal{P}_{C_1}, \mathcal{T}_1](\mathfrak{B}) &= 2\underline{\eta} \cdot {}^{(c)}\nabla \mathfrak{P} - \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\, {}^{(c)}\nabla_4 \mathfrak{P} \\
&\quad + \left(I_4^{\mathfrak{B}} + J_4^{\mathfrak{B}} + L_4^{\mathfrak{B}} + \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})C_1\right) {}^{(c)}\nabla_4 \mathfrak{B} \\
&\quad + \hat{V}_1 \mathfrak{P} + Z_a^{\mathfrak{B}} \cdot {}^{(c)}\nabla \mathfrak{B} + \tilde{Z}_0^{\mathfrak{B}} \mathfrak{B} - \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\mathbf{M}_1[\mathfrak{F}, \mathfrak{X}], \\
[\mathcal{P}_{C_1}, \mathcal{T}_1](\mathfrak{B}) &= 2\underline{\eta} \cdot {}^{(c)}\nabla \mathfrak{Q} - \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\, {}^{(c)}\nabla_4 \mathfrak{Q} \\
&\quad + \left(I_4^{\mathfrak{F}} + J_4^{\mathfrak{F}} + L_4^{\mathfrak{F}} + \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})C_2\right) {}^{(c)}\nabla_4 \mathfrak{F} \\
&\quad + \hat{V}_2 \mathfrak{P} + Z_a^{\mathfrak{F}} \cdot {}^{(c)}\nabla \mathfrak{F} + \tilde{Z}_0^{\mathfrak{F}} \mathfrak{F} - \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\mathbf{M}_2[A, \mathfrak{X}, \mathfrak{B}],
\end{aligned}
$$

where

$$
\hat{V}_1 = I_3^{\mathfrak{B}} + J_3^{\mathfrak{B}} + K_3^{\mathfrak{B}} + M_3^{\mathfrak{B}}, \tag{C.30}
$$

$$
Z_a^{\mathfrak{B}} = I_a^{\mathfrak{B}} + J_a^{\mathfrak{B}} + L_a^{\mathfrak{B}} + M_a^{\mathfrak{B}} - 2\underline{\eta} \cdot C_1, \tag{C.31}
$$

$$
\begin{aligned}
\tilde{Z}_0^{\mathfrak{B}} &= I_0^{\mathfrak{B}} + J_0^{\mathfrak{B}} + K_0^{\mathfrak{B}} + L_0^{\mathfrak{B}} + M_0^{\mathfrak{B}} + N_0^{\mathfrak{B}} + \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\, {}^{(c)}\nabla_4 C_1 \\
&\quad - C_1\left(I_3^{\mathfrak{B}} + J_3^{\mathfrak{B}} + K_3^{\mathfrak{B}} + M_3^{\mathfrak{B}}\right) - 2\underline{\eta} \cdot {}^{(c)}\nabla C_1
\end{aligned} \tag{C.32}
$$

and

$$\hat{V}_2 = I_3^{\mathfrak{F}} + J_3^{\mathfrak{F}} + K_3^{\mathfrak{F}} + M_3^{\mathfrak{F}}, \tag{C.33}$$

$$Z_a^{\mathfrak{F}} = I_a^{\mathfrak{F}} + J_a^{\mathfrak{F}} + L_a^{\mathfrak{F}} + M_a^{\mathfrak{F}} - 2\underline{\eta} \cdot C_2, \tag{C.34}$$

$$\begin{aligned}\tilde{Z}_0^{\mathfrak{F}} &= I_0^{\mathfrak{F}} + J_0^{\mathfrak{F}} + K_0^{\mathfrak{F}} + L_0^{\mathfrak{F}} + M_0^{\mathfrak{F}} + N_0^{\mathfrak{F}} + \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\, {}^{(c)}\nabla_4 C_2 \\ &\quad - C_2\big(I_3^{\mathfrak{F}} + J_3^{\mathfrak{F}} + K_3^{\mathfrak{F}} + M_3^{\mathfrak{F}}\big) - 2\underline{\eta} \cdot {}^{(c)}\nabla C_2.\end{aligned} \tag{C.35}$$

Observe that the coefficients of ${}^{(c)}\nabla_4\mathfrak{B}$ and ${}^{(c)}\nabla_4\mathfrak{F}$ are given by

$$\begin{aligned} & I_4^{\mathfrak{B}} + J_4^{\mathfrak{B}} + L_4^{\mathfrak{B}} + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right) C_1 \\ &= {}^{(c)}\nabla_3 C_1 - \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right) \\ & - {}^{(c)}\nabla_3\left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right) + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right) C_1 \\ &= {}^{(c)}\nabla_3 C_1 + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right) C_1 - \mathrm{tr}\underline{X}\,\overline{\mathrm{tr}\underline{X}} \end{aligned}$$

and

$$\begin{aligned} & I_4^{\mathfrak{F}} + J_4^{\mathfrak{F}} + L_4^{\mathfrak{F}} + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right) C_2 \\ &= {}^{(c)}\nabla_3 C_2 - \frac{1}{4}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right) - \frac{1}{2}\,{}^{(c)}\nabla_3\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right) \\ & + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right) C_2 \\ &= {}^{(c)}\nabla_3 C_2 + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right) C_2 - \frac{1}{2}\mathrm{tr}\underline{X}\,\overline{\mathrm{tr}\underline{X}}, \end{aligned}$$

which give conditions (7.10) and (7.11) for the vanishing of those coefficients.

C.1.10. *The lower order terms*

Defining

$$L_{\mathfrak{P}}[\mathfrak{B}, \mathfrak{F}] := -Z_a^{\mathfrak{B}} \cdot {}^{(c)}\nabla\mathfrak{B} - \tilde{Z}_0^{\mathfrak{B}}\mathfrak{B},$$

$$L_{\mathfrak{Q}}[\mathfrak{B}, \mathfrak{F}] := -Z_a^{\mathfrak{F}} \cdot {}^{(c)}\nabla\mathfrak{F} - \tilde{Z}_0^{\mathfrak{F}}\mathfrak{F}$$

to complete the proof of the proposition, we need to compute the terms $\tilde{Z}_0^{\mathfrak{B}}$ and $\tilde{Z}_0^{\mathfrak{F}}$.

Observe that, according to (7.16), we can write

$$C_1 = 2\mathrm{tr}\,\underline{\chi} + O\left(\frac{|a|}{r^2}\right) \qquad C_2 = \mathrm{tr}\,\underline{\chi} + O\left(\frac{|a|}{r^2}\right).$$

This gives

$$
{}^{(c)}\nabla_3 C_1 = -\mathrm{tr}\,\underline{\chi}^2 + O\left(\frac{|a|}{r^3}\right), \quad {}^{(c)}\nabla_3 C_2 = -\frac{1}{2}\mathrm{tr}\,\underline{\chi}^2 + O\left(\frac{|a|}{r^3}\right),
$$

$$
{}^{(c)}\nabla_4 C_1 = -\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} + 4\rho + O\left(\frac{|a|}{r^3}\right), \quad {}^{(c)}\nabla_4 C_2 = -\frac{1}{2}\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} + 2\rho + O\left(\frac{|a|}{r^3}\right).
$$

We compute

$$
\begin{aligned}
I_3^{\mathfrak{B}} &= -2\rho - 2\,{}^{(\mathbf{F})}\rho^2 - 2\,{}^*{}^{(\mathbf{F})}\rho^2 - 2\eta\cdot(\eta - 2\underline{\eta}) + i\left(2\ {}^*\rho - 2\eta\wedge\underline{\eta}\right)\\
&\quad + {}^{(c)}\nabla_4 C_1 \qquad (C.36)\\
&= -\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} + 2\rho - 2\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^3}\right),
\end{aligned}
$$

$$
\begin{aligned}
J_3^{\mathfrak{B}} &= -\frac{3}{2}\overline{\mathrm{tr}X}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right) + \frac{1}{2}(\overline{{}^{(c)}\mathcal{D}}\cdot H) + \frac{1}{2}(H\cdot\overline{H})\\
&= -3\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} + O\left(\frac{|a|}{r^3}\right), \qquad (C.37)
\end{aligned}
$$

$$
K_3^{\mathfrak{B}} = -3\,{}^{(c)}\nabla_3(\overline{\mathrm{tr}X}) = \frac{3}{2}\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} - 6\rho + O\left(\frac{|a|}{r^3}\right), \qquad (C.38)
$$

$$
M_3^{\mathfrak{B}} = \eta\cdot\left(6H + \overline{H} + 3\overline{\underline{H}}\right) = O\left(\frac{|a|}{r^4}\right), \qquad (C.39)
$$

which gives

$$
I_3^{\mathfrak{B}} + J_3^{\mathfrak{B}} + K_3^{\mathfrak{B}} + M_3^{\mathfrak{B}} = -\frac{5}{2}\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} - 4\rho - 2\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^3}\right).
$$

Similarly,

$$
\begin{aligned}
I_3^{\mathfrak{F}} &= -2\rho - 2\,{}^{(\mathbf{F})}\rho^2 - 2\,{}^*{}^{(\mathbf{F})}\rho^2 - 2\eta\cdot(\eta - 2\underline{\eta})\\
&\quad + i\left(4\ {}^*\rho - 4\eta\wedge\underline{\eta}\right) + {}^{(c)}\nabla_4 C_2 \qquad (C.40)\\
&= -\frac{1}{2}\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} - 2\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^3}\right),
\end{aligned}
$$

$$
\begin{aligned}
J_3^{\mathfrak{F}} &= -\frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right) + \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{H}) + \frac{1}{2}(H\cdot\overline{H}) \qquad (C.41)\\
&= -2\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} + O\left(\frac{|a|}{r^3}\right),
\end{aligned}
$$

$$
K_3^{\mathfrak{F}} = -\,{}^{(c)}\nabla_3\left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right) = \mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} - 4\rho + O\left(\frac{|a|}{r^3}\right). \qquad (C.42)
$$

$$
M_3^{\mathfrak{F}} = \eta\cdot\left(4H + \overline{H} + \underline{H}\right) = O\left(\frac{|a|}{r^4}\right), \qquad (C.43)
$$

which gives

$$I_3^{\mathfrak{F}} + J_3^{\mathfrak{F}} + K_3^{\mathfrak{F}} + M_3^{\mathfrak{F}} = -\frac{3}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - 4\rho - 2\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^3}\right).$$

We also compute

$$\begin{aligned}
I_0^{\mathfrak{B}} =\ & {}^{(c)}\nabla_3\left[-2\left(\rho + {}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\right) + 2i\left({}^*\rho - \eta\wedge\underline{\eta}\right) + {}^{(c)}\nabla_4 C_1\right]\\
& - 2(\eta - \underline{\eta})\cdot\mathcal{V}^{s=1}_{[3,a]}\\
=\ & -2\left(-\frac{3}{2}\mathrm{tr}\,\underline{\chi}\rho - \mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2\right) + 4\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 + {}^{(c)}\nabla_3\left(-\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + 4\rho\right) + O\left(\frac{|a|}{r^4}\right)\\
=\ & \mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2 - 5\mathrm{tr}\,\underline{\chi}\rho + 2\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^4}\right),\\
J_0^{\mathfrak{B}} =\ & -\frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\left(\frac{9}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + 2\overline{P} + 6\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} - 10\overline{\underline{H}}\cdot H\right) + O\left(\frac{|a|}{r^4}\right)\\
=\ & -\frac{9}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2 - 2\mathrm{tr}\,\underline{\chi}\rho - 6\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^4}\right),\\
K_0^{\mathfrak{B}} =\ & 3\overline{\mathrm{tr}X}\,{}^{(c)}\nabla_3 C_1 = -3\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2 + O\left(\frac{|a|}{r^4}\right),\\
L_0^{\mathfrak{B}} =\ & -\left(\frac{3}{2}\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right)\Big(2\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^{*(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\big)\\
& + 2i\big(-{}^*\rho + \eta\wedge\underline{\eta}\big) - {}^{(c)}\nabla_4 C_1\Big)\\
=\ & -2\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2 + 4\mathrm{tr}\,\underline{\chi}\rho - 4\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^4}\right),\\
M_0^{\mathfrak{B}} =\ & \big(6H + \overline{H} + 3\overline{\underline{H}}\big)\cdot\big(\mathcal{V}^s_{[3,a]} - {}^{(c)}\nabla C_1\big) = O\left(\frac{|a|}{r^4}\right),\\
N_0^{\mathfrak{B}} =\ & {}^{(c)}\nabla_3\left(-\frac{9}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} - 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} + 9\overline{\underline{H}}\cdot H\right)\\
=\ & \frac{9}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2 - 9\mathrm{tr}\,\underline{\chi}\rho + 8\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^4}\right),
\end{aligned}$$

which gives

$$I_0^{\mathfrak{B}} + J_0^{\mathfrak{B}} + K_0^{\mathfrak{B}} + L_0^{\mathfrak{B}} + M_0^{\mathfrak{B}} + N_0^{\mathfrak{B}} = -4\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2 - 12\mathrm{tr}\,\underline{\chi}\rho + O\left(\frac{|a|}{r^4}\right).$$

This finally implies

$$\begin{aligned}
\tilde{Z}_0^{\mathfrak{B}} =\ & I_0^{\mathfrak{B}} + J_0^{\mathfrak{B}} + K_0^{\mathfrak{B}} + L_0^{\mathfrak{B}} + M_0^{\mathfrak{B}} + N_0^{\mathfrak{B}} + \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\,{}^{(c)}\nabla_4 C_1\\
& - C_1\left(I_3^{\mathfrak{B}} + J_3^{\mathfrak{B}} + K_3^{\mathfrak{B}} + M_3^{\mathfrak{B}}\right) - 2\underline{\eta}\cdot{}^{(c)}\nabla C_1
\end{aligned}$$

$$
\begin{aligned}
&= -4\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2 - 12\mathrm{tr}\,\underline{\chi}\rho + \mathrm{tr}\,\underline{\chi}(-\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + 4\rho)\\
&\quad - 2\mathrm{tr}\,\underline{\chi}\left(-\frac{5}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - 4\rho - 2\,{}^{(\mathbf{F})}\rho^2\right) + O\left(\frac{|a|}{r^4}\right)\\
&= 4\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^4}\right).
\end{aligned}
$$

We compute

$$
\begin{aligned}
I_0^{\mathfrak{F}} &= {}^{(c)}\nabla_3\big[-2\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^{*\,(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\big) + 4i\big({}^*\rho - \eta\wedge\underline{\eta}\big) + {}^{(c)}\nabla_4 C_2\big]\\
&\quad - 2(\eta - \underline{\eta})\cdot\mathcal{V}^{s=1}_{[3,a]}\\
&= -2\left(-\frac{3}{2}\mathrm{tr}\,\underline{\chi}\rho - \mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2\right) + 4\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 + {}^{(c)}\nabla_3\left(-\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + 2\rho\right)\\
&\quad + O\left(\frac{|a|}{r^4}\right)\\
&= \frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2 - \mathrm{tr}\,\underline{\chi}\rho + 4\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^4}\right),\\
J_0^{\mathfrak{F}} &= -\frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\left(\frac{3}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} + \frac{1}{4}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}X - 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right.\\
&\quad\left. + \frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H - 3\eta\cdot\underline{\eta} + 3i\eta\wedge\underline{\eta}\right) + O\left(\frac{|a|}{r^4}\right)\\
&= -\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2 + 2\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^4}\right),\\
K_0^{\mathfrak{F}} &= \left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right){}^{(c)}\nabla_3 C_2 = -\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2 + O\left(\frac{|a|}{r^4}\right),\\
L_0^{\mathfrak{F}} &= -\frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\big(2\big(\rho + {}^{(\mathbf{F})}\rho^2 + {}^{*\,(\mathbf{F})}\rho^2 - \eta\cdot\underline{\eta}\big)\\
&\quad + 4i\big(-{}^*\rho + \eta\wedge\underline{\eta}\big) - {}^{(c)}\nabla_4 C_2\big)\\
&= -\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}^2 - 2\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^4}\right),\\
M_0^{\mathfrak{F}} &= \big(4H + \overline{H} + \underline{H}\big)\cdot\big(\mathcal{V}^s_{[3,a]} - {}^{(c)}\nabla C_2\big) = O\left(\frac{|a|}{r^4}\right),\\
N_0^{\mathfrak{F}} &= {}^{(c)}\nabla_3\left(-\frac{3}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} - \frac{1}{4}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}X + 3\overline{P} - P + 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right.\\
&\quad\left. - \frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H + \eta\cdot\underline{\eta} + i\eta\wedge\underline{\eta}\right)\\
&= \mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - 5\mathrm{tr}\,\underline{\chi}\rho - 10\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^4}\right),
\end{aligned}
$$

which gives

$$I_0^{\mathfrak{F}} + J_0^{\mathfrak{F}} + K_0^{\mathfrak{F}} + L_0^{\mathfrak{F}} + M_0^{\mathfrak{F}} + N_0^{\mathfrak{F}} = -\mathrm{tr}\,\chi \mathrm{tr}\,\underline{\chi} - 6\mathrm{tr}\,\underline{\chi}\rho - 6\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^4}\right).$$

This finally implies

$$\begin{aligned}\tilde{Z}_0^{\mathfrak{F}} &= I_0^{\mathfrak{F}} + J_0^{\mathfrak{F}} + K_0^{\mathfrak{F}} + L_0^{\mathfrak{F}} + M_0^{\mathfrak{F}} + N_0^{\mathfrak{F}} + \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\,{}^{(c)}\nabla_4 C_2 \\ &\quad - C_2\left(I_3^{\mathfrak{F}} + J_3^{\mathfrak{F}} + K_3^{\mathfrak{F}} + M_3^{\mathfrak{F}}\right) - 2\underline{\eta}\cdot{}^{(c)}\nabla C_2 \\ &= -\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - 6\mathrm{tr}\,\underline{\chi}\rho - 6\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 + \mathrm{tr}\,\underline{\chi}\left(-\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + 2\rho\right) \\ &\quad - \mathrm{tr}\,\underline{\chi}\left(-\frac{3}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - 4\rho - 2\,{}^{(\mathbf{F})}\rho^2\right) + O\left(\frac{|a|}{r^4}\right) \\ &= -4\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2 + O\left(\frac{|a|}{r^4}\right),\end{aligned}$$

which concludes the proof of Proposition 7.4.

C.2. *Proof of Proposition 7.5*

Let f be given by

$$f = (q)^n(\overline{q})^m.$$

Recall, see [19, Proposition 8.9],

$$\begin{aligned}\nabla_3(f) &= \left(\frac{n}{2}\overline{\mathrm{tr}\underline{X}} + \frac{m}{2}\mathrm{tr}\underline{X}\right)f, \\ \nabla_4(f) &= \left(\frac{n}{2}\mathrm{tr}X + \frac{m}{2}\overline{\mathrm{tr}X}\right)f, \\ 2\nabla f &= \left(mH + n\overline{H} + n\,\underline{H} + m\,\overline{\underline{H}}\right)f\end{aligned}$$

and for $\Psi \in \mathfrak{s}_k(\mathbb{C})$:

$$\dot{\square}_k(f\Psi) = \square(f)\Psi + f\dot{\square}_k\Psi - \nabla_3 f\nabla_4\Psi - \nabla_4 f\nabla_3\Psi + 2\nabla f\cdot\nabla\Psi. \qquad \text{(C.44)}$$

We then obtain for $\mathfrak{p} = f_1\mathfrak{P}$, using (7.17):

$$\begin{aligned}\dot{\square}_1\mathfrak{p} = f_1\Bigg[&\frac{5}{2}\overline{\mathrm{tr}X}\nabla_3\mathfrak{P} + \left(2\mathrm{tr}\underline{X} + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\right)\nabla_4\mathfrak{P} - \left(5H + \underline{H} + 4\overline{\underline{H}}\right)\cdot\nabla\mathfrak{P} + \tilde{V}_1\mathfrak{P} \\ &\mathcal{P}_{C_1}(\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]) + \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\mathbf{M}_1[\mathfrak{F},\mathfrak{X}] + L_{\mathfrak{P}}[\mathfrak{B},\mathfrak{F}]\Bigg] + \square(f_1)\mathfrak{P} \\ &- \left(\frac{n}{2}\overline{\mathrm{tr}\underline{X}} + \frac{m}{2}\mathrm{tr}\underline{X}\right)f_1\nabla_4\mathfrak{P} - \left(\frac{n}{2}\mathrm{tr}X + \frac{m}{2}\overline{\mathrm{tr}X}\right)f_1\nabla_3\mathfrak{P} \\ &+ \left(mH + n\overline{H} + n\,\underline{H} + m\,\overline{\underline{H}}\right)f_1\cdot\nabla\mathfrak{P},\end{aligned}$$

which gives

$$\dot{\square}_1 \mathfrak{p} = \left(\left(\frac{1}{2}-\frac{n}{2}\right)\overline{\mathrm{tr}\underline{X}} + \left(2-\frac{m}{2}\right)\mathrm{tr}\underline{X}\right) f_1 \nabla_4 \mathfrak{P}$$
$$+ \left(-\frac{n}{2}\mathrm{tr}X + \left(\frac{5}{2}-\frac{m}{2}\right)\overline{\mathrm{tr}X}\right) f_1 \nabla_3 \mathfrak{P}$$
$$+ \left((m-5)H + n\overline{H} + (n-1)\,\underline{H} + (m-4)\overline{\underline{H}}\right) f_1 \cdot \nabla \mathfrak{P} + \left(\tilde{V}_1 + f_1^{-1}\square(f_1)\right)\mathfrak{p}$$
$$+ f_1 \left[\mathcal{P}_{C_1}\left(\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]\right) + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\mathbf{M}_1[\mathfrak{F},\mathfrak{X}] + L_{\mathfrak{P}}[\mathfrak{B},\mathfrak{F}]\right].$$

Observe that the real part of the coefficients of all the first derivatives are multiple of $m+n-5$. To cancel their real part we then take $m = 5-n$, which implies $f_1 = (q)^n(\overline{q})^{5-n}$ and gives

$$\dot{\square}_1 \mathfrak{p} = i f_1 \left[(1-n)\,{}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4 \mathfrak{P} + n\,{}^{(a)}\mathrm{tr}\chi \nabla_3 \mathfrak{P} + \left(-2n\,{}^*\eta + 2(n-1)\,{}^*\underline{\eta}\right)\cdot \nabla \mathfrak{P}\right]$$
$$+\left(\tilde{V}_1 + f_1^{-1}\square(f_1)\right)\mathfrak{p}$$
$$+ f_1 \left[\mathcal{P}_{C_1}\left(\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]\right) + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\mathbf{M}_1[\mathfrak{F},\mathfrak{X}] + L_{\mathfrak{P}}[\mathfrak{B},\mathfrak{F}]\right].$$

Similarly, for $\mathfrak{q}^{\mathbf{F}} = f_2 \mathfrak{Q}$, using (7.19), we obtain

$$\dot{\square}_2 \mathfrak{q}^{\mathbf{F}} = f_2 \left[\frac{3}{2}\overline{\mathrm{tr}X}\nabla_3 \mathfrak{Q} + \left(\frac{1}{2}\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\nabla_4 \mathfrak{Q} - \left(3H + 2\,\underline{H} + \overline{\underline{H}}\right)\cdot \nabla \mathfrak{Q} + \tilde{V}_2 \mathfrak{Q}\right.$$
$$\left. + \mathcal{P}_{C_2}\left(\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\right) + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}] + L_{\mathfrak{Q}}[\mathfrak{B},\mathfrak{F}]\right] + \square(f_2)\mathfrak{Q}$$
$$- \left(\frac{n}{2}\overline{\mathrm{tr}\underline{X}} + \frac{m}{2}\mathrm{tr}\underline{X}\right) f_2 \nabla_4 \mathfrak{Q} - \left(\frac{n}{2}\mathrm{tr}X + \frac{m}{2}\overline{\mathrm{tr}X}\right) f_2 \nabla_3 \mathfrak{Q}$$
$$+ \left(mH + n\overline{H} + n\,\underline{H} + m\overline{\underline{H}}\right) f_2 \cdot \nabla \mathfrak{Q},$$

which gives

$$\dot{\square}_2 \mathfrak{q}^{\mathbf{F}} = \left(\left(1-\frac{n}{2}\right)\overline{\mathrm{tr}\underline{X}} + \left(\frac{1}{2}-\frac{m}{2}\right)\mathrm{tr}\underline{X}\right) f_2 \nabla_4 \mathfrak{Q}$$
$$+ \left(-\frac{n}{2}\mathrm{tr}X + \left(\frac{3}{2}-\frac{m}{2}\right)\overline{\mathrm{tr}X}\right) f_2 \nabla_3 \mathfrak{Q} + \left((m-3)H\right.$$
$$\left. + n\overline{H} + (n-2)\,\underline{H} + (m-1)\overline{\underline{H}}\right) f_2 \cdot \nabla \mathfrak{Q} + \left(\tilde{V}_2 + f_2^{-1}\square(f_2)\right)\mathfrak{q}^{\mathbf{F}}$$
$$+ f_2 \left[\mathcal{P}_{C_2}\left(\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\right) + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}] + L_{\mathfrak{Q}}[\mathfrak{B},\mathfrak{F}]\right].$$

Observe that the real part of the coefficients of all the first derivatives are multiple of $m+n-3$. To cancel their real part we then take $m = 3-n$, which implies

$f_2 = (q)^n(\overline{q})^{3-n}$, and gives

$$\begin{aligned}\dot{\square}_2\mathfrak{q}^{\mathbf{F}} &= if_2\big[(2-n)\,{}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4\mathfrak{Q} + n\,{}^{(a)}\mathrm{tr}\chi\nabla_3\mathfrak{Q} + \big(-2n\,{}^*\eta + 2(n-2)\,{}^*\underline{\eta}\big)\cdot\nabla\mathfrak{Q}\big]\\ &\quad + \big(\tilde{V}_2 + f_2^{-1}\square(f_2)\big)\mathfrak{q}^{\mathbf{F}} + f_2\Bigg[\mathcal{P}_{C_2}\big(\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\big)\\ &\quad + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}] + L_{\mathfrak{Q}}[\mathfrak{B},\mathfrak{F}]\Bigg].\end{aligned}$$

Using the values in Kerr–Newman:

$$\begin{aligned}{}^{(a)}\mathrm{tr}\underline{\chi}e_4 &= \frac{2a\Delta\cos\theta}{|q|^4}\nabla_r + \frac{2a\cos\theta(r^2+a^2)}{|q|^4}\nabla_t + \frac{2a^2\cos\theta}{|q|^4}\nabla_\varphi,\\ {}^{(a)}\mathrm{tr}\chi e_3 &= -\frac{2a\Delta\cos\theta}{|q|^4}\nabla_r + \frac{2a\cos\theta(r^2+a^2)}{|q|^4}\nabla_t + \frac{2a^2\cos\theta}{|q|^4}\nabla_\varphi\\ {}^*\eta_1 &= \frac{a\sin\theta r}{|q|^3}, \qquad {}^*\eta_2 = \frac{a^2\sin\theta\cos\theta}{|q|^3},\\ {}^*\underline{\eta}_1 &= -\frac{a\sin\theta(r)}{|q|^3}, \qquad {}^*\underline{\eta}_2 = \frac{a^2\sin\theta\cos\theta}{|q|^3}\end{aligned}$$

we respectively obtain

$$\begin{aligned}&(1-n)\,{}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4 + n\,{}^{(a)}\mathrm{tr}\chi\nabla_3 + \big(-2n\,{}^*\eta + 2(n-1)\,{}^*\underline{\eta}\big)\cdot\nabla\\ &\quad = (1-n)\left(\frac{2a\Delta\cos\theta}{|q|^4}\nabla_r + \frac{2a\cos\theta(r^2+a^2)}{|q|^4}\nabla_t + \frac{2a^2\cos\theta}{|q|^4}\nabla_\varphi\right)\\ &\qquad + n\left(-\frac{2a\Delta\cos\theta}{|q|^4}\nabla_r + \frac{2a\cos\theta(r^2+a^2)}{|q|^4}\nabla_t + \frac{2a^2\cos\theta}{|q|^4}\nabla_\varphi\right)\\ &\qquad + \big(-2n\,{}^*\eta_1 + 2(n-1)\,{}^*\underline{\eta}_1\big)\nabla_1 + \big(-2n\,{}^*\eta_2 + 2(n-1)\,{}^*\underline{\eta}_2\big)\nabla_2\\ &\quad = (1-2n)\frac{2a\Delta\cos\theta}{|q|^4}\nabla_r + \frac{2a\cos\theta(r^2+a^2)}{|q|^4}\nabla_t + \frac{2a^2\cos\theta}{|q|^4}\nabla_\varphi\\ &\qquad + 2(1-2n)\frac{a\sin\theta r}{|q|^3}\nabla_1 - 2\frac{a^2\sin\theta\cos\theta}{|q|^3}\nabla_2\end{aligned}$$

and

$$\begin{aligned}&(2-n)\,{}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4 + n\,{}^{(a)}\mathrm{tr}\chi\nabla_3 + \big(-2n\,{}^*\eta + 2(n-2)\,{}^*\underline{\eta}\big)\cdot\nabla\\ &\quad = (2-2n)\frac{2a\Delta\cos\theta}{|q|^4}\nabla_r + 2\frac{2a\cos\theta(r^2+a^2)}{|q|^4}\nabla_t + 2\frac{2a^2\cos\theta}{|q|^4}\nabla_\varphi\\ &\qquad + 2(2-2n)\frac{a\sin\theta r}{|q|^3}\nabla_1 - 4\frac{a^2\sin\theta\cos\theta}{|q|^3}\nabla_2.\end{aligned}$$

Writing that $\nabla_1 = \frac{1}{|q|}\nabla_\theta$ and $\nabla_2 = \frac{a\sin\theta}{|q|}\nabla_t + \frac{1}{|q|\sin\theta}\nabla_\varphi$, we finally, respectively, have

$$\begin{aligned}&(1-n)\,{}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4 + n\,{}^{(a)}\mathrm{tr}\chi\nabla_3 + \big(-2n\,{}^*\eta + 2(n-1)\,{}^*\underline{\eta}\big)\cdot\nabla\\ &\quad= \frac{2a\cos\theta}{|q|^2}\nabla_t + (1-2n)\left(\frac{2a\Delta\cos\theta}{|q|^4}\nabla_r + \frac{2a\sin\theta r}{|q|^4}\nabla_\theta\right)\end{aligned}$$

and

$$\begin{aligned}&(2-n)\,{}^{(a)}\mathrm{tr}\underline{\chi}\nabla_4 + n\,{}^{(a)}\mathrm{tr}\chi\nabla_3 + \big(-2n\,{}^*\eta + 2(n-2)\,{}^*\underline{\eta}\big)\cdot\nabla\\ &\quad= \frac{4a\cos\theta}{|q|^2}\nabla_t + (1-n)\left(\frac{4a\Delta\cos\theta}{|q|^4}\nabla_r + \frac{4a\sin\theta r}{|q|^4}\nabla_\theta\right),\end{aligned}$$

which completes the proof.

C.3. *Proof of Proposition 7.6*

We compute here the right-hand sides of the main equations.

C.3.1. *The right-hand side of the equation for* $\mathfrak{p}$

Using the definition (6.7) of $\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]$, we can write

$$\begin{aligned}\mathbf{M}_1[\mathfrak{F},\mathfrak{X}] &= (2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P})\tilde{\mathbf{M}}_1[\mathfrak{F},\mathfrak{X}],\\ \tilde{\mathbf{M}}_1[\mathfrak{F},\mathfrak{X}] &= 2\,\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + 4\,\overline{\underline{H}}\cdot\mathfrak{F} - \big(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\big)\mathfrak{X}.\end{aligned}$$

Using (C.17), we can therefore compute

$$\begin{aligned}&\mathcal{P}_{C_1}\big(\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]\big) + \frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\mathbf{M}_1[\mathfrak{F},\mathfrak{X}]\\ &\quad= \mathcal{P}_{C_1}\big(2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\tilde{\mathbf{M}}_1[\mathfrak{F},\mathfrak{X}]\big) + \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)(2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P})\tilde{\mathbf{M}}_1[\mathfrak{F},\mathfrak{X}]\\ &\quad= (2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P})\mathcal{P}_{C_1}\big(\tilde{\mathbf{M}}_1[\mathfrak{F},\mathfrak{X}]\big) + 2\,{}^{(c)}\nabla_3({}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P})\tilde{\mathbf{M}}_1[\mathfrak{F},\mathfrak{X}]\\ &\qquad+ \frac{1}{2}\left(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\right)(2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P})\tilde{\mathbf{M}}_1[\mathfrak{F},\mathfrak{X}]\\ &\qquad= 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\left[\mathcal{P}_{C_1}\big(\tilde{\mathbf{M}}_1[\mathfrak{F},\mathfrak{X}]\big) - \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\big(\tilde{\mathbf{M}}_1[\mathfrak{F},\mathfrak{X}]\big)\right],\end{aligned}$$

where we used that ${}^{(c)}\nabla_3({}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}) = -(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}$. We then compute $\mathcal{P}_{C_1}\big(\tilde{\mathbf{M}}_1[\mathfrak{F},\mathfrak{X}]\big)$, using Lemma C.1:

$$\begin{aligned}\mathcal{P}_{C_1}\big(\tilde{\mathbf{M}}_1[\mathfrak{F},\mathfrak{X}]\big) &= \mathcal{P}_{C_1}\big(2\,\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + 4\,\overline{\underline{H}}\cdot\mathfrak{F} - \big(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\big)\mathfrak{X}\big)\\ &= 2\,\overline{{}^{(c)}\mathcal{D}}\cdot(\mathcal{P}_{C_1}\mathfrak{F}) + 2[\mathcal{P}_{C_1}, \overline{{}^{(c)}\mathcal{D}}\cdot]\mathfrak{F} + 4\,\overline{\underline{H}}\cdot\mathcal{P}_{C_1}(\mathfrak{F}) + 4\,{}^{(c)}\nabla_3\overline{\underline{H}}\cdot\mathfrak{F}\end{aligned}$$

$$
\begin{aligned}
&- \big(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\big)\mathcal{P}_{C_1}(\mathfrak{X}) - {}^{(c)}\nabla_3\big(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\big)\mathfrak{X}\\
&= 2\overline{{}^{(c)}\mathcal{D}} \cdot (\mathcal{P}_{C_1}\mathfrak{F}) - \overline{\mathrm{tr}\underline{X}}\,\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F} + 4\,\overline{\underline{H}} \cdot \mathcal{P}_{C_1}(\mathfrak{F}) + 2\overline{H} \cdot {}^{(c)}\nabla_3\mathfrak{F}\\
&+ \big(-2\overline{{}^{(c)}\mathcal{D}}C_1 + 4\,{}^{(c)}\nabla_3\overline{\underline{H}} + \overline{\mathrm{tr}\underline{X}}\,\overline{H}\big) \cdot \mathfrak{F}\\
&- \big(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\big)\mathcal{P}_{C_1}(\mathfrak{X}) - {}^{(c)}\nabla_3\big(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\big)\mathfrak{X}.
\end{aligned}
$$

We now write:

$$
\begin{aligned}
\mathcal{P}_{C_1}\mathfrak{F} &= {}^{(c)}\nabla_3\mathfrak{F} + C_1\mathfrak{F} = ({}^{(c)}\nabla_3\mathfrak{F} + C_2\mathfrak{F}) + (C_1 - C_2)\mathfrak{F} = \mathfrak{Q} + (C_1 - C_2)\mathfrak{F}\\
\overline{{}^{(c)}\mathcal{D}} \cdot (\mathcal{P}_{C_1}\mathfrak{F}) &= \overline{{}^{(c)}\mathcal{D}} \cdot (\mathfrak{Q} + (C_1 - C_2)\mathfrak{F})\\
&= \overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{Q} + (C_1 - C_2)\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F} + (\overline{{}^{(c)}\mathcal{D}}C_1 - \overline{{}^{(c)}\mathcal{D}}C_2) \cdot \mathfrak{F}\\
{}^{(c)}\nabla_3\mathfrak{F} &= \mathfrak{Q} - C_2\mathfrak{F}.
\end{aligned}
$$

Using (5.10) we can write

$$
\mathcal{P}_{C_1}(\mathfrak{X}) = {}^{(c)}\nabla_3\mathfrak{X} + C_1\mathfrak{X} = \left(-\frac{1}{2}\overline{\mathrm{tr}\underline{X}} + C_1\right)\mathfrak{X} - \overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F} - \overline{H} \cdot \mathfrak{F} - 2\mathfrak{B}.
$$

By substituting in the above expression we obtain

$$
\begin{aligned}
&\mathcal{P}_{C_1}\big(\tilde{\mathbf{M}}_1[\mathfrak{F}, \mathfrak{X}]\big)\\
&= 2\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{Q} + \big(4\,\overline{\underline{H}} + 2\overline{H}\big) \cdot \mathfrak{Q} + (2C_1 - 2C_2 + 2\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}})\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F}\\
&+ \big(-2\overline{{}^{(c)}\mathcal{D}}C_2 + 4\,{}^{(c)}\nabla_3\overline{\underline{H}} + 4(C_1 - C_2)\overline{\underline{H}} + (2\mathrm{tr}\underline{X} - 2C_2)\overline{H}\big) \cdot \mathfrak{F}\\
&+ \big(4\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}}\big)\mathfrak{B}\\
&- \left({}^{(c)}\nabla_3\big(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\big) + \big(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\big)\left(-\frac{1}{2}\overline{\mathrm{tr}\underline{X}} + C_1\right)\right)\mathfrak{X}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
&\mathcal{P}_{C_1}\big(\tilde{\mathbf{M}}_1[\mathfrak{F}, \mathfrak{X}]\big) - \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\big(\tilde{\mathbf{M}}_1[\mathfrak{F}, \mathfrak{X}]\big)\\
&= 2\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{Q} + \big(4\,\overline{\underline{H}} + 2\overline{H}\big) \cdot \mathfrak{Q} + (2C_1 - 2C_2 + \mathrm{tr}\underline{X} - 3\overline{\mathrm{tr}\underline{X}})\overline{{}^{(c)}\mathcal{D}} \cdot \mathfrak{F}\\
&+ \big(-2\overline{{}^{(c)}\mathcal{D}}C_2 + 4\,{}^{(c)}\nabla_3\overline{\underline{H}} + (4C_1 - 4C_2 - 2\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}})\overline{\underline{H}}\\
&+ (2\mathrm{tr}\underline{X} - 2C_2)\overline{H}\big) \cdot \mathfrak{F} + \big(4\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}}\big)\mathfrak{B}\\
&- \left({}^{(c)}\nabla_3\big(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\big) + \big(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\big)\left(-\frac{1}{2}\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}} + C_1\right)\right)\mathfrak{X}.
\end{aligned}
$$

Using (4.5) and (4.9) we simplify the above to

$$
\begin{aligned}
&\mathcal{P}_{C_1}\big(\tilde{\mathbf{M}}_1[\mathfrak{F},\mathfrak{X}]\big) - \frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\big(\tilde{\mathbf{M}}_1[\mathfrak{F},\mathfrak{X}]\big)\\
&\quad = 2\left(\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{Q} + \big(2\,\underline{H} + \overline{H}\big)\cdot\mathfrak{Q}\right) + \big(4\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}}\big)\,\mathfrak{B}\\
&\qquad + Y_a^{\mathfrak{F}}\,\overline{{}^{(c)}\mathcal{D}}\cdot\mathfrak{F} + Y_0^{\mathfrak{F}}\cdot\mathfrak{F} + Y_0^{\mathfrak{X}}\ \mathfrak{X},
\end{aligned}
$$

where

$$
Y_a^{\mathfrak{F}} = 2C_1 - 2C_2 + \mathrm{tr}\underline{X} - 3\overline{\mathrm{tr}\underline{X}}, \tag{C.45}
$$

$$
Y_0^{\mathfrak{F}} = -2\,\overline{{}^{(c)}\mathcal{D}}C_2 + (4C_1 - 4C_2 - 4\mathrm{tr}\underline{X} - 2\overline{\mathrm{tr}\underline{X}})\,\underline{H} + (4\mathrm{tr}\underline{X} - 2C_2)\overline{H}, \tag{C.46}
$$

$$
Y_0^{\mathfrak{X}} = 2(\mathrm{tr}\underline{X})^2 - \frac{3}{2}\overline{\mathrm{tr}\underline{X}}^2 + \frac{3}{2}\mathrm{tr}\underline{X}\,\overline{\mathrm{tr}\underline{X}} - \big(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\big)C_1. \tag{C.47}
$$

To complete the proof of the first part of the proposition, we need to compute the terms $Y_a^{\mathfrak{F}}$ and $Y_0^{\mathfrak{X}}$. Recall that, according to (7.16), we can write

$$
C_1 = 2\mathrm{tr}\,\underline{\chi} + O\left(\frac{|a|}{r^2}\right) \qquad C_2 = \mathrm{tr}\,\underline{\chi} + O\left(\frac{|a|}{r^2}\right).
$$

We then have

$$
\begin{aligned}
Y_a^{\mathfrak{F}} &= 2C_1 - 2C_2 + \mathrm{tr}\underline{X} - 3\overline{\mathrm{tr}\underline{X}} = 4\mathrm{tr}\,\underline{\chi} - 2\mathrm{tr}\,\underline{\chi} + \mathrm{tr}\,\underline{\chi} - 3\mathrm{tr}\,\underline{\chi} + O\left(\frac{|a|}{r^2}\right)\\
&= O\left(\frac{|a|}{r^2}\right),\\
Y_0^{\mathfrak{X}} &= 2(\mathrm{tr}\underline{X})^2 - \frac{3}{2}\overline{\mathrm{tr}\underline{X}}^2 + \frac{3}{2}\mathrm{tr}\underline{X}\,\overline{\mathrm{tr}\underline{X}} - \big(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\big)C_1\\
&= 2\mathrm{tr}\,\underline{\chi}^2 - \frac{3}{2}\mathrm{tr}\,\underline{\chi}^2 + \frac{3}{2}\mathrm{tr}\,\underline{\chi}^2 - \big(2\mathrm{tr}\,\underline{\chi} - \mathrm{tr}\,\underline{\chi}\big)2\mathrm{tr}\,\underline{\chi} + O\left(\frac{|a|}{r^3}\right) = O\left(\frac{|a|}{r^3}\right).
\end{aligned}
$$

Finally by writing $4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\,\big(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\big)\,\mathfrak{B} = 4\mathrm{tr}\,\underline{\chi}\,{}^{(\mathbf{F})}\rho^2\mathfrak{B} + (2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P})Y_0^{\mathfrak{B}}\mathfrak{B}$, for $Y_0^{\mathfrak{B}} = O\big(\frac{|a|}{r^2}\big)$, we obtain the stated relation.

C.3.2. *The right-hand side of the equation for* $\mathfrak{q}^{\mathbf{F}}$

We have

$$
\begin{aligned}
&\mathcal{P}_{C_2}\big(\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\big) + \frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\\
&\quad = {}^{(c)}\nabla_3\big(\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\big) + \left(C_2 + \frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\right)\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}].
\end{aligned}
$$

Using the definition (6.8) of $\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]$, we compute

$$
\begin{aligned}
&{}^{(c)}\nabla_3\big(\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\big)\\
&\quad= {}^{(c)}\nabla_3\left(-{}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3A+\frac{1}{2}\big(3\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\big)A\right)\right.\\
&\qquad\left.+\left(\frac{3}{2}\,{}^{(c)}\nabla_3H\right)\widehat{\otimes}\mathfrak{X}+\big(2H-\underline{H}\big)\widehat{\otimes}\mathfrak{B}\right)\\
&\quad=-{}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A+\frac{1}{2}\big(3\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\big)\,{}^{(c)}\nabla_3A+\frac{1}{2}\,{}^{(c)}\nabla_3\big(3\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\big)A\right)\\
&\qquad+\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3A+\frac{1}{2}\big(3\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\big)A\right)+\left(\frac{3}{2}\,{}^{(c)}\nabla_3H\right)\widehat{\otimes}\,{}^{(c)}\nabla_3\mathfrak{X}\\
&\qquad+\left(\frac{3}{2}\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3H\right)\widehat{\otimes}\mathfrak{X}+\big(2H-\underline{H}\big)\widehat{\otimes}\,{}^{(c)}\nabla_3\mathfrak{B}+{}^{(c)}\nabla_3\big(2H-\underline{H}\big)\widehat{\otimes}\mathfrak{B}.
\end{aligned}
$$

Using (5.10) to express ${}^{(c)}\nabla_3\mathfrak{X}$, and writing ${}^{(c)}\nabla_3\mathfrak{B}=\mathfrak{P}-C_1\mathfrak{B}$, we obtain

$$
\begin{aligned}
&{}^{(c)}\nabla_3\big(\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\big)\\
&\quad=-{}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A+\frac{1}{2}\big(\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\big)\,{}^{(c)}\nabla_3A\right.\\
&\qquad\left.+\left(-\frac{9}{4}\mathrm{tr}\underline{X}^2+\frac{1}{2}\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}+\frac{1}{4}\overline{\mathrm{tr}\underline{X}}^2\right)A\right)\\
&\qquad-3\,{}^{(c)}\nabla_3H\cdot{}^{(c)}\nabla\mathfrak{F}-\left(\frac{3}{2}\,{}^{(c)}\nabla_3H\cdot\overline{H}\right)\mathfrak{F}\\
&\qquad+\left(\frac{3}{2}\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3H-\frac{3}{4}\overline{\mathrm{tr}\underline{X}}\,{}^{(c)}\nabla_3H\right)\widehat{\otimes}\mathfrak{X}\\
&\qquad+\big(2H-\underline{H}\big)\widehat{\otimes}\mathfrak{P}+\left(-{}^{(c)}\nabla_3\left(H+\underline{H}\right)-C_1\left(2H-\underline{H}\right)\right)\widehat{\otimes}\mathfrak{B}.
\end{aligned}
$$

We therefore obtain

$$
\begin{aligned}
&\mathcal{P}_{C_2}\big(\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\big)+\frac{1}{2}\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\\
&\quad=-{}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A+\left(C_2+\mathrm{tr}\underline{X}\right)\,{}^{(c)}\nabla_3A\right.\\
&\qquad\left.+\left(-\frac{3}{2}\mathrm{tr}\underline{X}^2+\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}+\frac{1}{2}\big(3\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\big)C_2\right)A\right)\\
&\qquad-3\,{}^{(c)}\nabla_3H\cdot{}^{(c)}\nabla\mathfrak{F}-\left(\frac{3}{2}\,{}^{(c)}\nabla_3H\cdot\overline{H}\right)\mathfrak{F}\\
&\qquad+\left(\frac{3}{2}\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3H+\frac{3}{2}\left(C_2+\frac{1}{2}\mathrm{tr}\underline{X}\right)\,{}^{(c)}\nabla_3H\right)\widehat{\otimes}\mathfrak{X}
\end{aligned}
$$

$$+\big(2H-\underline{H}\big)\widehat{\otimes}\mathfrak{P}$$
$$+\left(-{}^{(c)}\nabla_3\big(H+\underline{H}\big)+\big(C_2-C_1+\frac{1}{2}\big(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\big)\big)\,(2H-\underline{H})\right)\widehat{\otimes}\mathfrak{B}.$$

We now want to relate the first line of the above to the relation (5.7). Observe that

$$\begin{aligned}
&{}^{(c)}\nabla_3\left({}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3A+\frac{1}{2}\mathrm{tr}\underline{X}A\right)\right)\\
&\quad={}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A+\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_3A-\frac{1}{4}\mathrm{tr}\underline{X}^2A\right)\\
&\qquad-\mathrm{tr}\underline{X}\,{}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3A+\frac{1}{2}\mathrm{tr}\underline{X}A\right)\\
&\quad={}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_3A-\frac{3}{4}\mathrm{tr}\underline{X}^2A\right).
\end{aligned}$$

The first line of the above can then be written as

$$\begin{aligned}
&-{}^{(\mathbf{F})}P\bigg({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A+(C_2+\mathrm{tr}\underline{X})\,{}^{(c)}\nabla_3A\\
&\qquad+\left(-\frac{3}{2}\mathrm{tr}\underline{X}^2+\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}+\frac{1}{2}\left(3\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\right)C_2\right)A\bigg)\\
&\quad=-{}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3\,{}^{(c)}\nabla_3A-\frac{1}{2}\mathrm{tr}\underline{X}\,{}^{(c)}\nabla_3A-\frac{3}{4}\mathrm{tr}\underline{X}^2A\right)\\
&\qquad-\left(C_2+\frac{3}{2}\mathrm{tr}\underline{X}\right){}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3A+\frac{1}{2}\mathrm{tr}\underline{X}A\right)\\
&\qquad-{}^{(\mathbf{F})}P\left(-\frac{3}{2}\mathrm{tr}\underline{X}^2+\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}+\frac{1}{2}\left(2\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\right)C_2\right)A\\
&\quad=-{}^{(c)}\nabla_3\left({}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3A+\frac{1}{2}\mathrm{tr}\underline{X}A\right)\right)\\
&\qquad-\left(C_2+\frac{3}{2}\mathrm{tr}\underline{X}\right){}^{(\mathbf{F})}P\big({}^{(c)}\nabla_3A+\frac{1}{2}\mathrm{tr}\underline{X}A\big)\\
&\qquad-{}^{(\mathbf{F})}P\left(-\frac{3}{2}\mathrm{tr}\underline{X}^2+\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}+\frac{1}{2}\big(2\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\big)C_2\right)A.
\end{aligned}$$

Using (5.7), we obtain

$$\begin{aligned}
&-{}^{(c)}\nabla_3\left({}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3A+\frac{1}{2}\mathrm{tr}\underline{X}A\right)\right)\\
&\quad=-{}^{(c)}\nabla_3\left(\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}+3H\widehat{\otimes}\mathfrak{B}-\big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\mathfrak{F}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\,{}^{(c)}\nabla_3\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} - 3H\widehat{\otimes}\,{}^{(c)}\nabla_3\mathfrak{B} - 3\,{}^{(c)}\nabla_3 H\widehat{\otimes}\mathfrak{B}\\
&\quad+\big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\,{}^{(c)}\nabla_3\mathfrak{F} + {}^{(c)}\nabla_3\big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\mathfrak{F}.
\end{aligned}
$$

Using now (B.2) applied to $\mathfrak{B}$, we obtain

$$
\begin{aligned}
&-\,{}^{(c)}\nabla_3\left({}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}A\right)\right)\\
&= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_3\mathfrak{B} + \frac{1}{4}\mathrm{tr}\underline{X}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} - \frac{7}{2}H\widehat{\otimes}\,{}^{(c)}\nabla_3\mathfrak{B}\\
&\quad+\left(-3\,{}^{(c)}\nabla_3 H + \frac{1}{2}\mathrm{tr}\underline{X}H\right)\widehat{\otimes}\mathfrak{B} + \big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\,{}^{(c)}\nabla_3\mathfrak{F}\\
&\quad+\,{}^{(c)}\nabla_3\big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\mathfrak{F}.
\end{aligned}
$$

We therefore obtain, using once again (5.7):

$$
\begin{aligned}
&-\,{}^{(c)}\nabla_3\left({}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}A\right)\right) - \left(C_2+\frac{3}{2}\mathrm{tr}\underline{X}\right){}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}A\right)\\
&= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\,{}^{(c)}\nabla_3\mathfrak{B} - \frac{1}{2}\left(C_2+\mathrm{tr}\underline{X}\right){}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B} - \frac{7}{2}H\widehat{\otimes}\,{}^{(c)}\nabla_3\mathfrak{B}\\
&\quad+\big(-3\,{}^{(c)}\nabla_3 H + (-3C_2 - 4\mathrm{tr}\underline{X})H\big)\widehat{\otimes}\mathfrak{B} + \big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\,{}^{(c)}\nabla_3\mathfrak{F}\\
&\quad+\left({}^{(c)}\nabla_3\big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big) + \left(C_2+\frac{3}{2}\mathrm{tr}\underline{X}\right)\big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\right)\mathfrak{F}.
\end{aligned}
$$

Writing ${}^{(c)}\nabla_3\mathfrak{B} = \mathfrak{P} - C_1\mathfrak{B}$ and ${}^{(c)}\nabla_3\mathfrak{F} = \mathfrak{Q} - C_2\mathfrak{F}$, we obtain

$$
\begin{aligned}
&-\,{}^{(c)}\nabla_3\left({}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}A\right)\right) - \left(C_2+\frac{3}{2}\mathrm{tr}\underline{X}\right){}^{(\mathbf{F})}P\left({}^{(c)}\nabla_3 A + \frac{1}{2}\mathrm{tr}\underline{X}A\right)\\
&= -\frac{1}{2}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{P} - \frac{7}{2}H\widehat{\otimes}\mathfrak{P} + \frac{1}{2}\left(C_1 - C_2 - \mathrm{tr}\underline{X}\right){}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}\\
&\quad+\left(-3\,{}^{(c)}\nabla_3 H + \frac{1}{2}\,{}^{(c)}\mathcal{D}C_1 + (\frac{7}{2}C_1 - 3C_2 - 4\mathrm{tr}\underline{X})H\right)\widehat{\otimes}\mathfrak{B}\\
&\quad+\big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big)\mathfrak{Q} - 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\big)\mathfrak{F},
\end{aligned}
$$

where we used that ${}^{(c)}\nabla_3\big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big) + \big(\frac{3}{2}\mathrm{tr}\underline{X}\big)\big(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big) = -2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\big(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\big)$.

Finally, to express the last line we recall (5.8) and then write

$$
\begin{aligned}
&-{}^{(\mathbf{F})}P\left(-\frac{3}{2}\mathrm{tr}\underline{X}^2+\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}+\frac{1}{2}\left(2\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\right)C_2\right)A\\
&\quad=\left(-\frac{3}{2}\mathrm{tr}\underline{X}^2+\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}+\frac{1}{2}\left(2\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\right)C_2\right){}^{(c)}\nabla_4\mathfrak{F}\\
&\qquad\times\left(-\frac{3}{2}\mathrm{tr}\underline{X}^2+\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}+\frac{1}{2}\left(2\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\right)C_2\right)\left(\frac{3}{2}\overline{\mathrm{tr}X}+\frac{1}{2}\mathrm{tr}X\right)\mathfrak{F}\\
&\qquad\times\left(-\frac{3}{2}\mathrm{tr}\underline{X}^2+\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}+\frac{1}{2}\left(2\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\right)C_2\right)\frac{1}{2}{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X}\\
&\qquad\times\left(-\frac{3}{2}\mathrm{tr}\underline{X}^2+\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}+\frac{1}{2}\left(2\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\right)C_2\right)\frac{1}{2}\left(3H+\underline{H}\right)\widehat{\otimes}\mathfrak{X}.
\end{aligned}
$$

By putting everything together we finally obtain

$$
\begin{aligned}
&\mathcal{P}_{C_2}\left(\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\right)+\frac{1}{2}\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\mathbf{M}_2[A,\mathfrak{X},\mathfrak{B}]\\
&\quad=\left(3\overline{P}+2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\right)\mathfrak{Q}-\frac{1}{2}\left({}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{P}+(3H+2\,\underline{H})\widehat{\otimes}\mathfrak{P}\right)\\
&\qquad-2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\mathfrak{F}\\
&\qquad+W_4^{\mathfrak{F}}\,{}^{(c)}\nabla_4\mathfrak{F}+W_a^{\mathfrak{F}}\cdot{}^{(c)}\nabla\mathfrak{F}+W_0^{\mathfrak{F}}\mathfrak{F}+W_a^{\mathfrak{B}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{B}+W_0^{\mathfrak{B}}\widehat{\otimes}\mathfrak{B}\\
&\qquad+W_a^{\mathfrak{X}}\,{}^{(c)}\mathcal{D}\widehat{\otimes}\mathfrak{X}+W_0^{\mathfrak{X}}\widehat{\otimes}\mathfrak{X},
\end{aligned}
$$

where

$$
W_4^{\mathfrak{F}}=-\frac{3}{2}\mathrm{tr}\underline{X}^2+\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}+\frac{1}{2}\left(2\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\right)C_2, \tag{C.48}
$$

$$
W_a^{\mathfrak{F}}=-3\,{}^{(c)}\nabla_3H, \tag{C.49}
$$

$$
\begin{aligned}
W_0^{\mathfrak{F}}=&\left(-\frac{3}{2}\mathrm{tr}\underline{X}^2+\overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X}+\frac{1}{2}\left(2\mathrm{tr}\underline{X}-\overline{\mathrm{tr}\underline{X}}\right)C_2\right)\left(\frac{3}{2}\overline{\mathrm{tr}X}+\frac{1}{2}\mathrm{tr}X\right)\\
&-\frac{3}{2}\,{}^{(c)}\nabla_3H\cdot\overline{H},
\end{aligned} \tag{C.50}
$$

$$
W_a^{\mathfrak{B}}=\frac{1}{2}\left(C_1-C_2-\mathrm{tr}\underline{X}\right), \tag{C.51}
$$

$$
W_0^{\mathfrak{B}}=-3\,{}^{(c)}\nabla_3H+\frac{1}{2}\,{}^{(c)}\mathcal{D}C_1+\left(\frac{7}{2}C_1-3C_2-4\mathrm{tr}\underline{X}\right)H \tag{C.52}
$$

$$
-{}^{(c)}\nabla_3\left(H+\underline{H}\right)+\left(C_2-C_1+\frac{1}{2}\left(\mathrm{tr}\underline{X}+\overline{\mathrm{tr}\underline{X}}\right)\right)\left(2H-\underline{H}\right), \tag{C.53}
$$

$$W_a^{\mathfrak{X}} = \frac{1}{2}\left(-\frac{3}{2}\mathrm{tr}\underline{X}^2 + \overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X} + \frac{1}{2}\left(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\right)C_2\right), \tag{C.54}$$

$$W_0^{\mathfrak{X}} = \left(-\frac{3}{2}\mathrm{tr}\underline{X}^2 + \overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X} + \frac{1}{2}\left(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\right)C_2\right)\frac{1}{2}(3H + \underline{H}) \tag{C.55}$$

$$+\frac{3}{2}\,{}^{(c)}\nabla_3\,{}^{(c)}\nabla_3 H + \frac{3}{2}\left(C_2 + \frac{1}{2}\mathrm{tr}\underline{X}\right){}^{(c)}\nabla_3 H. \tag{C.56}$$

To complete the proof of the second part of the proposition, we need to compute the terms $W_4^{\mathfrak{F}}$, $W_0^{\mathfrak{F}}$, $W_a^{\mathfrak{B}}$, $W_a^{\mathfrak{X}}$ and $W_0^{\mathfrak{X}}$. Recall that, according to (7.16), we can write

$$C_1 = 2\mathrm{tr}\,\underline{\chi} + O\left(\frac{|a|}{r^2}\right) \qquad C_2 = \mathrm{tr}\,\underline{\chi} + O\left(\frac{|a|}{r^2}\right).$$

We then have

$$\begin{aligned}
W_4^{\mathfrak{F}} &= -\frac{3}{2}\mathrm{tr}\underline{X}^2 + \overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X} + \frac{1}{2}\left(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\right)C_2\\
&= -\frac{3}{2}\mathrm{tr}\,\underline{\chi}^2 + \mathrm{tr}\,\underline{\chi}^2 + \frac{1}{2}\mathrm{tr}\,\underline{\chi}^2 + O\left(\frac{|a|}{r^3}\right) = O\left(\frac{|a|}{r^3}\right),\\
W_a^{\mathfrak{B}} &= \frac{1}{2}\left(C_1 - C_2 - \mathrm{tr}\underline{X}\right) = \frac{1}{2}\left(2\mathrm{tr}\,\underline{\chi} - \mathrm{tr}\,\underline{\chi} - \mathrm{tr}\,\underline{\chi}\right) + O\left(\frac{|a|}{r^2}\right) = O\left(\frac{|a|}{r^2}\right).
\end{aligned}$$

Finally observe that $W_0^{\mathfrak{F}}$, $W_a^{\mathfrak{X}}$ and $W_0^{\mathfrak{X}}$ are $O(|a|)$ because they are multiplied by $W_4^{\mathfrak{F}} = O\left(\frac{|a|}{r^3}\right)$.

C.4. *Proof of Proposition 7.7*

Recall that

$$\begin{aligned}
V_1 &:= \tilde{V}_1 + f_1^{-1}\Box(f_1),\\
V_2 &:= \tilde{V}_2 + f_2^{-1}\Box(f_2) + 3\overline{P} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P}.
\end{aligned}$$

We start by computing the real and imaginary part of $f^{-1}\Box(f)$.

Lemma C.2. *For $f = q^n\overline{q}^m$, we have*

$$\Re(f^{-1}\Box f) = -\frac{(m+n)(m+n+1)}{4}tr\,\chi\mathrm{tr}\,\underline{\chi} - \frac{(n-m)^2+m+n}{4}\,{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi}$$
$$-(n+m)\rho + 2nm|\underline{\eta}|^2 + (m^2+n^2+m+n)\eta\cdot\underline{\eta}$$

and

$$\Im(f^{-1}\Box f) = (m-n)\left(-\frac{1}{2}(n+m+1)\mathrm{tr}\,\underline{\chi}\,{}^{(a)}tr\chi + {}^*\rho - (m+n+1)\eta\wedge\underline{\eta}\right).$$

In particular, for $f_1=(q)^{1/2}(\overline{q})^{9/2}$ *and* $f_2=q\overline{q}^2$*, we obtain*

$$\Re(f_1^{-1}\square f_1) = -\frac{15}{2}tr\,\chi\mathrm{tr}\,\underline{\chi}-\frac{21}{4}\,{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi}-5\rho+\frac{9}{2}|\underline{\eta}|^2+\frac{51}{2}\eta\cdot\underline{\eta},$$

$$\Im(f_1^{-1}\square f_1) = -12\mathrm{tr}\,\underline{\chi}\,{}^{(a)}tr\chi+4\ {}^*\rho-24\eta\wedge\underline{\eta}$$

and

$$\Re(f_2^{-1}\square f_2) = -3tr\,\chi\mathrm{tr}\,\underline{\chi}-\,{}^{(a)}tr\chi\,{}^{(a)}tr\underline{\chi}-3\rho+4|\underline{\eta}|^2+8\eta\cdot\underline{\eta},$$

$$\Im(f_2^{-1}\square f_2) = -2\mathrm{tr}\,\underline{\chi}\,{}^{(a)}tr\chi+\ {}^*\rho-4\eta\wedge\underline{\eta}.$$

Proof. Recall that for a scalar

$$f^{-1}\square f=-f^{-1}e_4e_3f-\frac{1}{2}\mathrm{tr}\,\underline{\chi}f^{-1}e_4f-\frac{1}{2}tr\,\chi f^{-1}e_3f+f^{-1}\Delta f+2\underline{\eta}\cdot f^{-1}\nabla f. \tag{C.57}$$

Using

$$f^{-1}\nabla_3(f)=\left(\frac{n}{2}\overline{\mathrm{tr}\underline{X}}+\frac{m}{2}\mathrm{tr}\underline{X}\right),$$

$$f^{-1}\nabla_4(f)=\left(\frac{n}{2}\mathrm{tr}X+\frac{m}{2}\overline{\mathrm{tr}X}\right),$$

$$2f^{-1}\nabla f=\left(mH+n\overline{H}+n\,\underline{H}+m\overline{\underline{H}}\right).$$

we have

$$\begin{aligned}\nabla_4\nabla_3 f &= \left(\frac{n}{2}\nabla_4\overline{\mathrm{tr}\underline{X}}+\frac{m}{2}\nabla_4\mathrm{tr}\underline{X}\right)f+\left(\frac{n}{2}\overline{\mathrm{tr}\underline{X}}+\frac{m}{2}\mathrm{tr}\underline{X}\right)\nabla_4 f\\ &= \left(\frac{n}{2}\left(-\frac{1}{2}\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}+\overline{\mathcal{D}}\cdot\ \underline{H}+\ \underline{H}\cdot\ \overline{\underline{H}}+2P\right)\right.\\ &\quad\left.+\frac{m}{2}\left(-\frac{1}{2}\mathrm{tr}X\mathrm{tr}\underline{X}+\mathcal{D}\cdot\ \overline{\underline{H}}+\ \underline{H}\cdot\ \overline{\underline{H}}+2\overline{P}\right)\right)f\\ &\quad+\left(\frac{n}{2}\overline{\mathrm{tr}\underline{X}}+\frac{m}{2}\mathrm{tr}\underline{X}\right)\left(\frac{n}{2}\mathrm{tr}X+\frac{m}{2}\overline{\mathrm{tr}X}\right)f,\end{aligned}$$

which gives

$$\begin{aligned}f^{-1}\nabla_4\nabla_3 f &= \frac{nm-n}{4}\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}+\frac{nm-m}{4}\mathrm{tr}X\mathrm{tr}\underline{X}+\frac{n^2}{4}\mathrm{tr}X\overline{\mathrm{tr}\underline{X}}+\frac{m^2}{4}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X}\\ &\quad+\frac{n}{2}\overline{\mathcal{D}}\cdot\ \underline{H}+\frac{m}{2}\mathcal{D}\cdot\overline{\underline{H}}+\frac{n+m}{2}\ \underline{H}\cdot\ \overline{\underline{H}}+nP+m\overline{P}.\end{aligned}$$

Observe that $\Re(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}})=tr\,\chi\mathrm{tr}\,\underline{\chi}-\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}$, while $\Re(\mathrm{tr}X\overline{\mathrm{tr}\underline{X}})=tr\,\chi\mathrm{tr}\,\underline{\chi}+{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}$. In particular

$$\begin{aligned}\Re(f^{-1}\nabla_4\nabla_3 f) &= \frac{2nm-m-n}{4}(tr\,\chi\mathrm{tr}\,\underline{\chi}-\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi})\\ &\quad+\frac{n^2+m^2}{4}(tr\,\chi\mathrm{tr}\,\underline{\chi}+\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi})\end{aligned}$$

$$
\begin{aligned}
&+(n+m)\mathrm{div}\underline{\eta}+(n+m)|\underline{\eta}|^2+(n+m)\rho\\
&=\frac{(m+n)(m+n-1)}{4}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi}+\frac{(n-m)^2+m+n}{4}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}\\
&+(n+m)\mathrm{div}\underline{\eta}+(n+m)|\underline{\eta}|^2+(n+m)\rho.
\end{aligned}
$$

Observe that $\Im(\overline{\mathrm{tr}X\mathrm{tr}\underline{X}}) = \mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}+\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi = 0$ and $\Im(\mathrm{tr}X\mathrm{tr}\underline{X}) = -\mathrm{tr}\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}-\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi=0$, while $\Im(\mathrm{tr}\overline{X}\,\overline{\mathrm{tr}\underline{X}})=-2\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi$. In particular

$$
\begin{aligned}
\Im(f^{-1}\nabla_4\nabla_3 f) &= -\frac{n^2}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi+\frac{m^2}{2}\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi+(n-m)\big({}^*\rho+\mathrm{curl}\underline{\eta}\big)\\
&=(n-m)\left(-\frac{1}{2}(n+m)\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi+{}^*\rho+\mathrm{curl}\underline{\eta}\right).
\end{aligned}
$$

Also

$$
\begin{aligned}
\Delta f &= \frac{1}{2}\nabla\cdot\big(\big(mH+n\overline{H}+n\,\underline{H}+m\overline{\underline{H}}\big)f\big)\\
&=\frac{1}{2}\nabla\cdot\big(mH+n\overline{H}+n\,\underline{H}+m\overline{\underline{H}}\big)f+\frac{1}{4}\big(mH+n\overline{H}+n\,\underline{H}+m\overline{\underline{H}}\big)\\
&\cdot\big(mH+n\overline{H}+n\,\underline{H}+m\overline{\underline{H}}\big)f.
\end{aligned}
$$

In particular

$$
\begin{aligned}
\Re(f^{-1}\Delta f) &= \frac{1}{2}\nabla\cdot\big((m+n)\eta+(m+n)\underline{\eta}\big)+\frac{1}{4}\big((m+n)\eta+(m+n)\underline{\eta}\big)\\
&\cdot\big((m+n)\eta+(m+n)\underline{\eta}\big)-\frac{1}{4}((m-n)\,{}^*\eta+(n-m)\,{}^*\underline{\eta})\\
&\cdot((m-n)\,{}^*\eta+(n-m)\,{}^*\underline{\eta})\\
&=\frac{1}{2}(m+n)\big(\mathrm{div}\eta+\mathrm{div}\underline{\eta}\big)+\frac{1}{4}(m+n)^2\big(\eta+\underline{\eta}\big)\cdot\big(\eta+\underline{\eta}\big)\\
&-\frac{1}{4}(m-n)^2(\eta-\underline{\eta})\cdot(\eta-\underline{\eta})\\
&=(m+n)\big(\mathrm{div}\underline{\eta}\big)+2nm|\underline{\eta}|^2+(m^2+n^2)\eta\cdot\underline{\eta}
\end{aligned}
$$

since $\mathrm{div}\eta=\mathrm{div}\underline{\eta}$ and $|\eta|^2=|\underline{\eta}|^2$, and

$$
\begin{aligned}
\Im(f^{-1}\Delta f) &= \frac{1}{2}\nabla\cdot\big((m-n)\,{}^*\eta+(n-m)\,{}^*\underline{\eta}\big)+\frac{1}{2}((m+n)\eta+(m+n)\underline{\eta})\\
&\cdot\big((m-n)\,{}^*\eta+(n-m)\,{}^*\underline{\eta}\big)\\
&=\frac{1}{2}(m-n)\big(\mathrm{curl}\eta-\mathrm{curl}\underline{\eta}+(m+n)(-2\eta\cdot{}^*\underline{\eta})\big)\\
&=(m-n)\big(-\mathrm{curl}\underline{\eta}-(m+n)\eta\wedge\underline{\eta}\big)
\end{aligned}
$$

since $\mathrm{curl}\eta=-\mathrm{curl}\underline{\eta}$.

We therefore have

$$\Re(f^{-1}\square f) = -\Re(f^{-1}e_4e_3 f) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\Re(f^{-1}e_4 f) - \frac{1}{2}tr\,\chi\Re(f^{-1}e_3 f) + \Re(f^{-1}\Delta f)$$

$$+ 2\underline{\eta}\cdot\Re(f^{-1}\nabla f)$$

$$= -\frac{(m+n)(m+n-1)}{4}tr\,\chi\mathrm{tr}\,\underline{\chi} - \frac{(n-m)^2+m+n}{4}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}$$

$$-(n+m)\mathrm{div}\underline{\eta} - (n+m)|\underline{\eta}|^2 - (n+m)\rho$$

$$-\frac{1}{2}\mathrm{tr}\,\underline{\chi}\frac{n+m}{2}tr\,\chi - \frac{1}{2}tr\,\chi\frac{n+m}{2}\mathrm{tr}\,\underline{\chi} + (m+n)\big(\mathrm{div}\underline{\eta}\big) + 2nm|\underline{\eta}|^2$$

$$+(m^2+n^2)\eta\cdot\underline{\eta} + \underline{\eta}\cdot((m+n)\eta + (m+n)\underline{\eta}),$$

which finally gives

$$\Re(f^{-1}\square f) = -\frac{(m+n)(m+n+1)}{4}tr\,\chi\mathrm{tr}\,\underline{\chi} - \frac{(n-m)^2+m+n}{4}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}$$

$$-(n+m)\rho + 2nm|\underline{\eta}|^2 + (m^2+n^2+m+n)\eta\cdot\underline{\eta}.$$

Also,

$$\Im(f^{-1}\square f) = -\Im(f^{-1}e_4e_3 f) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\Im(f^{-1}e_4 f) - \frac{1}{2}tr\,\chi\Im(f^{-1}e_3 f)$$

$$+\Im(f^{-1}\Delta f) + 2\underline{\eta}\cdot\Im(f^{-1}\nabla f)$$

$$= (m-n)\left(-\frac{1}{2}(n+m)\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi + {}^*\rho + \mathrm{curl}\underline{\eta} - \frac{1}{4}\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi\right.$$

$$\left. + \frac{1}{4}tr\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - \mathrm{curl}\underline{\eta} - (m+n)\eta\wedge\underline{\eta} + \underline{\eta}\cdot({}^*\eta - {}^*\underline{\eta})\right)$$

$$= (m-n)\left(-\frac{1}{2}(n+m+1)\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi + {}^*\rho - (m+n+1)\eta\wedge\underline{\eta}\right)$$

as stated. □

We now compute $\tilde{V}_1$. Using (7.18), we have

$$\Re(\tilde{V}_1) = \Re\left(\frac{9}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} - 9\overline{\underline{H}}\cdot H\right) + 4\,{}^{(\mathbf{F})}\rho^2 + 4\,{}^*{}^{(\mathbf{F})}\rho^2 - \Re(\hat{V}_1)$$

$$+\frac{1}{4}tr\,\chi\mathrm{tr}\,\underline{\chi} + \frac{1}{4}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + \rho - {}^{(\mathbf{F})}\rho^2 - {}^*{}^{(\mathbf{F})}\rho^2$$

$$= \frac{19}{4}tr\,\chi\mathrm{tr}\,\underline{\chi} + \frac{19}{4}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + \rho + 3\,{}^{(\mathbf{F})}\rho^2 + 3\,{}^*{}^{(\mathbf{F})}\rho^2 - 18\eta\cdot\underline{\eta} - \Re(\hat{V}_1)$$

and

$$\Im(\tilde{V}_1) = \Im\left(\frac{9}{2}\mathrm{tr}\underline{X}\overline{\mathrm{tr}X} - 9\overline{\underline{H}}\cdot H\right) - \Im(\hat{V}_1) - {}^*\rho + \eta\wedge\underline{\eta}$$

$$= 9\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi - {}^*\rho + 19\eta\wedge\underline{\eta} - \Im(\hat{V}_1).$$

Using (7.12), we have

$$\Re(\hat{V}_1) = \Re(I_3^{\mathfrak{B}}) + \Re(J_3^{\mathfrak{B}}) + \Re(K_3^{\mathfrak{B}}) + \Re(M_3^{\mathfrak{B}}),$$
$$\Im(\hat{V}_1) = \Im(I_3^{\mathfrak{B}}) + \Im(J_3^{\mathfrak{B}}) + \Im(K_3^{\mathfrak{B}}) + \Im(M_3^{\mathfrak{B}}).$$

Using (C.36), and writing $C_1 = 2\mathrm{tr}\,\underline{\chi} + ip_1\,{}^{(a)}\mathrm{tr}\underline{\chi}$, we obtain

$$\begin{aligned}
\Re(I_3^{\mathfrak{B}}) &= -2\rho - 2\,{}^{(\mathbf{F})}\rho^2 - 2\,{}^*{}^{(\mathbf{F})}\rho^2 - 2\eta\cdot(\eta - 2\underline{\eta}) + 2\,{}^{(c)}\nabla_4\mathrm{tr}\,\underline{\chi} \\
&= -2\rho - 2\,{}^{(\mathbf{F})}\rho^2 - 2\,{}^*{}^{(\mathbf{F})}\rho^2 - 2\eta\cdot(\eta - 2\underline{\eta}) \\
&\quad + 2\left(-\frac{1}{2}tr\,\chi\mathrm{tr}\,\underline{\chi} + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + 2\mathrm{div}\underline{\eta} + 2|\underline{\eta}|^2 + 2\rho\right) \\
&= -tr\,\chi\mathrm{tr}\,\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + 2\rho - 2\,{}^{(\mathbf{F})}\rho^2 - 2\,{}^*{}^{(\mathbf{F})}\rho^2 + 4\mathrm{div}\underline{\eta} + 4\eta\cdot\underline{\eta} + 2|\underline{\eta}|^2
\end{aligned}$$

and using that $\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi + tr\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} = 0$,

$$\begin{aligned}
\Im(I_3^{\mathfrak{B}}) &= 2\ {}^*\rho - 2\eta\wedge\underline{\eta} + p_1\,{}^{(c)}\nabla_4\,{}^{(a)}\mathrm{tr}\underline{\chi} \\
&= 2\ {}^*\rho - 2\eta\wedge\underline{\eta} + p_1\left(-\frac{1}{2}({}^{(a)}\mathrm{tr}\chi\mathrm{tr}\,\underline{\chi} + tr\,\chi\,{}^{(a)}\mathrm{tr}\underline{\chi}) + 2\mathrm{curl}\underline{\eta} + 2\ {}^*\rho\right) \\
&= (2 + 2p_1)\ {}^*\rho + 2p_1\mathrm{curl}\underline{\eta} - 2\eta\wedge\underline{\eta}.
\end{aligned}$$

Using (C.37), we have

$$\Re(J_3^{\mathfrak{B}}) = \Re\left(-3\overline{\mathrm{tr}X}\mathrm{tr}\,\underline{\chi} + \frac{1}{2}(\overline{{}^{(c)}\mathcal{D}}\cdot H) + \frac{1}{2}(H\cdot\overline{H})\right) = -3tr\,\chi\mathrm{tr}\,\underline{\chi} + \mathrm{div}\underline{\eta} + |\underline{\eta}|^2,$$

$$\Im(J_3^{\mathfrak{B}}) = \Im\left(-3\overline{\mathrm{tr}X}\mathrm{tr}\,\underline{\chi} + \frac{1}{2}(\overline{{}^{(c)}\mathcal{D}}\cdot H)\right) = -3\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi - \mathrm{curl}\underline{\eta}.$$

Using (C.38), we have

$$\Re(K_3^{\mathfrak{B}}) = -3\,{}^{(c)}\nabla_3(tr\,\chi) = \frac{3}{2}\mathrm{tr}\,\underline{\chi}tr\,\chi - \frac{3}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - 6\rho - 6\mathrm{div}\underline{\eta} - 6|\underline{\eta}|^2,$$
$$\Im(K_3^{\mathfrak{B}}) = -3\,{}^{(c)}\nabla_3({}^{(a)}\mathrm{tr}\chi) = 6\ {}^*\rho + 6\mathrm{curl}\underline{\eta}.$$

Using (C.39), we have

$$\Re(M_3^{\mathfrak{B}}) = \Re\big(\eta\cdot\big(6(\eta + i\ {}^*\eta) + \eta - i\ {}^*\eta + 3(\underline{\eta} - i\ {}^*\underline{\eta})\big)\big) = 7|\underline{\eta}|^2 + 3\eta\cdot\underline{\eta},$$
$$\Im(M_3^{\mathfrak{B}}) = \Im\big(\eta\cdot\big(6(\eta + i\ {}^*\eta) + \eta - i\ {}^*\eta + 3(\underline{\eta} - i\ {}^*\underline{\eta})\big)\big) = -3\eta\cdot\ {}^*\underline{\eta} = -3\eta\wedge\underline{\eta}.$$

This gives

$$\begin{aligned}
\Re(\hat{V}_1) &= -\frac{5}{2}tr\,\chi\mathrm{tr}\,\underline{\chi} - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - 4\rho - 2\,{}^{(\mathbf{F})}\rho^2 - 2\,{}^*{}^{(\mathbf{F})}\rho^2 - \mathrm{div}\underline{\eta} \\
&\qquad + 7\eta\cdot\underline{\eta} + 4|\underline{\eta}|^2, \\
\Im(\hat{V}_1) &= -3\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi + (8 + 2p_1)\ {}^*\rho + (2p_1 + 5)\mathrm{curl}\underline{\eta} - 5\eta\wedge\underline{\eta}
\end{aligned}$$

and therefore

$$\begin{aligned}\Re(\tilde{V}_1) &= \frac{29}{4}tr\,\chi\text{tr}\,\underline{\chi} + \frac{21}{4}\,{}^{(a)}\text{tr}\chi\,{}^{(a)}\text{tr}\underline{\chi} + 5\rho + 5\,{}^{(\mathbf{F})}\rho^2 + 5\,{}^*{}^{(\mathbf{F})}\rho^2 + \text{div}\underline{\eta}\\ &\quad - 25\eta\cdot\underline{\eta} - 4|\underline{\eta}|^2,\\ \Im(\tilde{V}_1) &= 12\text{tr}\,\underline{\chi}\,{}^{(a)}\text{tr}\chi - (9+2p_1)\,{}^*\rho - (2p_1+5)\text{curl}\underline{\eta} + 24\eta\wedge\underline{\eta}.\end{aligned}$$

Finally this gives:

$$\begin{aligned}\Re(V_1) &= \Re(\tilde{V}_1) + \Re(f_1^{-1}\Box(f_1))\\ &= \frac{29}{4}tr\,\chi\text{tr}\,\underline{\chi} + \frac{21}{4}\,{}^{(a)}\text{tr}\chi\,{}^{(a)}\text{tr}\underline{\chi} + 5\rho + 5\,{}^{(\mathbf{F})}\rho^2 + 5\,{}^*{}^{(\mathbf{F})}\rho^2 + \text{div}\underline{\eta} - 25\eta\cdot\underline{\eta}\\ &\quad - 4|\underline{\eta}|^2 - \frac{15}{2}tr\,\chi\text{tr}\,\underline{\chi} - \frac{21}{4}\,{}^{(a)}\text{tr}\chi\,{}^{(a)}\text{tr}\underline{\chi} - 5\rho + \frac{9}{2}|\underline{\eta}|^2 + \frac{51}{2}\eta\cdot\underline{\eta}\\ &= -\frac{1}{4}tr\,\chi\text{tr}\,\underline{\chi} + 5\,{}^{(\mathbf{F})}\rho^2 + 5\,{}^*{}^{(\mathbf{F})}\rho^2 + \text{div}\underline{\eta} + \frac{1}{2}\eta\cdot\underline{\eta} + \frac{1}{2}|\underline{\eta}|^2\end{aligned}$$

and

$$\begin{aligned}\Im(V_1) &= \Im(\tilde{V}_1) + \Im(f_1^{-1}\Box(f_1))\\ &= 12\text{tr}\,\underline{\chi}\,{}^{(a)}\text{tr}\chi - (9+2p_1)\,{}^*\rho - (2p_1+5)\text{curl}\underline{\eta} + 24\eta\wedge\underline{\eta} - 12\text{tr}\,\underline{\chi}\,{}^{(a)}\text{tr}\chi\\ &\quad + 4\,{}^*\rho - 24\eta\wedge\underline{\eta}\\ &= -(2p_1+5)\,{}^*\rho - (2p_1+5)\text{curl}\underline{\eta}.\end{aligned}$$

Observe that for $p_1 = -\frac{5}{2}$, we obtain $\Im(V_1) = 0$, as desired.

We now compute $\tilde{V}_2$. Using (7.20), we have

$$\begin{aligned}\Re(\tilde{V}_2) &= \Re\left(\frac{3}{4}\text{tr}\underline{X}\overline{\text{tr}X} + \frac{1}{4}\overline{\text{tr}\underline{X}}\text{tr}X - 3\overline{P} + P - 4\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P} + \frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H\right)\\ &\quad - \eta\cdot\underline{\eta} - \Re(\hat{V}_2) - \frac{1}{2}tr\,\chi\text{tr}\,\underline{\chi} - \frac{1}{2}\,{}^{(a)}\text{tr}\chi\,{}^{(a)}\text{tr}\underline{\chi} - 2\rho + 2\,{}^{(\mathbf{F})}\rho^2 + 2\,{}^*{}^{(\mathbf{F})}\rho^2\\ &= \frac{1}{2}tr\,\chi\text{tr}\,\underline{\chi} + \frac{1}{2}\,{}^{(a)}\text{tr}\chi\,{}^{(a)}\text{tr}\underline{\chi} - 4\rho - 2\,{}^{(\mathbf{F})}\rho^2 - 2\,{}^*{}^{(\mathbf{F})}\rho^2\\ &\quad + 3\text{div}\underline{\eta} - \eta\cdot\underline{\eta} - \Re(\hat{V}_2)\end{aligned}$$

and

$$\begin{aligned}\Im(\tilde{V}_2) &= \Im\left(\frac{3}{4}\text{tr}\underline{X}\overline{\text{tr}X} + \frac{1}{4}\overline{\text{tr}\underline{X}}\text{tr}X - 3\overline{P} + P + \frac{3}{2}\overline{{}^{(c)}\mathcal{D}}\cdot H\right)\\ &\quad - \eta\wedge\underline{\eta} - \Im(\hat{V}_2) - 2\,{}^*\rho + 2\eta\wedge\underline{\eta}\\ &= \text{tr}\,\underline{\chi}\,{}^{(a)}\text{tr}\chi + 2\,{}^*\rho - 3\text{curl}\underline{\eta} + \eta\wedge\underline{\eta} - \Im(\hat{V}_2),\end{aligned}$$

where

$$\Re(\hat{V}_2) = \Re(I_3^{\mathfrak{F}}) + \Re(J_3^{\mathfrak{F}}) + \Re(K_3^{\mathfrak{F}}) + \Re(M_3^{\mathfrak{F}}),$$
$$\Im(\hat{V}_2) = \Im(I_3^{\mathfrak{F}}) + \Im(J_3^{\mathfrak{F}}) + \Im(K_3^{\mathfrak{F}}) + \Im(M_3^{\mathfrak{F}}).$$

Using (C.40), and writing $C_2 = \mathrm{tr}\,\underline{\chi} + ip_2\,{}^{(a)}\mathrm{tr}\underline{\chi}$ we obtain

$$\begin{aligned}
\Re(I_3^{\mathfrak{F}}) &= -2\rho - 2\,{}^{(\mathbf{F})}\rho^2 - 2\,{}^{*\,(\mathbf{F})}\rho^2 - 2\eta\cdot(\eta - 2\underline{\eta}) + {}^{(c)}\nabla_4\mathrm{tr}\,\underline{\chi}\\
&= -\frac{1}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - 2\,{}^{(\mathbf{F})}\rho^2 - 2\,{}^{*\,(\mathbf{F})}\rho^2 + 2\mathrm{div}\underline{\eta} + 4\eta\cdot\underline{\eta},\\
\Im(I_3^{\mathfrak{F}}) &= 4\,{}^*\rho - 4\eta\wedge\underline{\eta} + p_2\,{}^{(c)}\nabla_4\,{}^{(a)}\mathrm{tr}\underline{\chi}\\
&= (4+2p_2)\,{}^*\rho + 2p_2\mathrm{curl}\underline{\eta} - 4\eta\wedge\underline{\eta}.
\end{aligned}$$

Using (C.41), we have

$$\begin{aligned}
\Re(J_3^{\mathfrak{F}}) &= \Re(-\mathrm{tr}\,\underline{\chi}\left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right) + \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{\underline{H}}) + \frac{1}{2}(H\cdot\overline{\underline{H}}))\\
&= -2\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + \mathrm{div}\underline{\eta} + |\underline{\eta}|^2,\\
\Im(J_3^{\mathfrak{F}}) &= \Im\left(-\frac{1}{2}\big(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}}\big)\left(\frac{3}{2}\overline{\mathrm{tr}X} + \frac{1}{2}\mathrm{tr}X\right) + \frac{1}{2}({}^{(c)}\mathcal{D}\cdot\overline{\underline{H}})\right)\\
&= -\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi + \mathrm{curl}\underline{\eta}.
\end{aligned}$$

Using (C.42), we have

$$\begin{aligned}
\Re(K_3^{\mathfrak{F}}) &= -2\,{}^{(c)}\nabla_3(\mathrm{tr}\,\chi) = \mathrm{tr}\,\underline{\chi}\mathrm{tr}\,\chi - {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - 4\rho - 4\mathrm{div}\underline{\eta} - 4|\underline{\eta}|^2\\
\Im(K_3^{\mathfrak{F}}) &= -{}^{(c)}\nabla_3\,{}^{(a)}\mathrm{tr}\chi = 2\,{}^*\rho + 2\mathrm{curl}\underline{\eta}.
\end{aligned}$$

Using (C.43), we have

$$\begin{aligned}
\Re(M_3^{\mathfrak{F}}) &= \Re\big(\eta\cdot\big(4(\eta + i\,{}^*\eta) + (\eta - i\,{}^*\eta) + \underline{\eta} + i\,{}^*\underline{\eta}\big)\big) = 5|\underline{\eta}|^2 + \eta\cdot\underline{\eta},\\
\Im(M_3^{\mathfrak{F}}) &= \Im\big(\eta\cdot\big(4(\eta + i\,{}^*\eta) + (\eta - i\,{}^*\eta) + \underline{\eta} + i\,{}^*\underline{\eta}\big)\big) = \eta\wedge\underline{\eta}.
\end{aligned}$$

This gives

$$\begin{aligned}
\Re(\hat{V}_2) &= -\frac{3}{2}\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} - \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - 4\rho - 2\,{}^{(\mathbf{F})}\rho^2 - 2\,{}^{*\,(\mathbf{F})}\rho^2 - 2\mathrm{div}\underline{\eta} + 5\eta\cdot\underline{\eta}\\
&\quad + \mathrm{div}\underline{\eta} + 2|\underline{\eta}|^2,\\
\Im(\hat{V}_2) &= -\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi + (2p_2+6)\,{}^*\rho + (2p_2+3)\mathrm{curl}\underline{\eta} - 3\eta\wedge\underline{\eta}
\end{aligned}$$

and therefore

$$\begin{aligned}
\Re(\tilde{V}_2) &= 2\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + 4\mathrm{div}\underline{\eta} - 6\eta\cdot\underline{\eta} - 2|\underline{\eta}|^2,\\
\Im(\tilde{V}_2) &= 2\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi - (2p_2+4)\,{}^*\rho - (2p_2+6)\mathrm{curl}\underline{\eta} + 4\eta\wedge\underline{\eta}.
\end{aligned}$$

Finally, this gives

$$\begin{aligned}\Re(V_2) &= \Re(\tilde{V}_2) + \Re(f_2^{-1}\Box(f_2)) + \Re(3\overline{P} + 2\,{}^{(\mathbf{F})}P\,\overline{{}^{(\mathbf{F})}P})\\ &= 2tr\,\chi\mathrm{tr}\,\underline{\chi} + {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} + 4\mathrm{div}\underline{\eta} - 6\eta\cdot\underline{\eta} - 2|\underline{\eta}|^2\\ &\quad -3tr\,\chi\mathrm{tr}\,\underline{\chi} - {}^{(a)}\mathrm{tr}\chi\,{}^{(a)}\mathrm{tr}\underline{\chi} - 3\rho + 4|\underline{\eta}|^2 + 8\eta\cdot\underline{\eta} + 3\rho + 2\,{}^{(\mathbf{F})}\rho^2 + 2\,{}^{*\,(\mathbf{F})}\rho^2\\ &= -\mathrm{tr}\,\chi\mathrm{tr}\,\underline{\chi} + 2\,{}^{(\mathbf{F})}\rho^2 + 2\,{}^{*\,(\mathbf{F})}\rho^2 + 4\mathrm{div}\underline{\eta} + 2\eta\cdot\underline{\eta} + 2|\underline{\eta}|^2\end{aligned}$$

and

$$\begin{aligned}\Im(V_2) &= \Im(\tilde{V}_2) + \Im(f_2^{-1}\Box(f_2)) + \Im(3\overline{P})\\ &= 2\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi - (2p_2+4)\,{}^*\rho - (2p_2+6)\mathrm{curl}\underline{\eta} + 4\eta\wedge\underline{\eta} - 2\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\chi\\ &\quad + {}^*\rho - 4\eta\wedge\underline{\eta} - 3\,{}^*\rho\\ &= -(2p_2+6)\,{}^*\rho - (2p_2+6)\mathrm{curl}\underline{\eta}.\end{aligned}$$

Observe that for $p_2 = -3$, we obtain $\Im(V_2) = 0$, as desired. This completes the proof of the proposition.

C.5. *Proof of Lemma 7.8*

From (C.48) and $C_2 = \mathrm{tr}\,\underline{\chi} - 3i\,{}^{(a)}\mathrm{tr}\underline{\chi}$ as in (7.38), we compute

$$\begin{aligned}\Im(W_4^{\mathfrak{F}}) &= \Im\left(-\frac{3}{2}\mathrm{tr}\underline{X}^2 + \overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X} + \frac{1}{2}\left(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\right)C_2\right)\\ &= 3\mathrm{tr}\,\underline{\chi}\,{}^{(a)}\mathrm{tr}\underline{\chi} + \Im\left(\frac{1}{2}(\mathrm{tr}\,\underline{\chi} - 3i\,{}^{(a)}\mathrm{tr}\underline{\chi})(\mathrm{tr}\,\underline{\chi} - 3i\,{}^{(a)}\mathrm{tr}\underline{\chi})\right) = 0\end{aligned}$$

and similarly, from (C.54)

$$\Im(W_a^{\mathfrak{X}}) = \frac{1}{2}\Im\left(-\frac{3}{2}\mathrm{tr}\underline{X}^2 + \overline{\mathrm{tr}\underline{X}}\mathrm{tr}\underline{X} + \frac{1}{2}(2\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}})C_2\right) = 0.$$

From (C.51), we obtain

$$\begin{aligned}W_a^{\mathfrak{B}} &= \frac{1}{2}(C_1 - C_2 - \mathrm{tr}\underline{X})\\ &= \frac{1}{2}\left(2\mathrm{tr}\,\underline{\chi} - \frac{5}{2}i\,{}^{(a)}\mathrm{tr}\underline{\chi} - \mathrm{tr}\,\underline{\chi} + 3i\,{}^{(a)}\mathrm{tr}\underline{\chi} - \mathrm{tr}\,\underline{\chi} + i\,{}^{(a)}\mathrm{tr}\underline{\chi}\right)\\ &= \frac{3}{4}i\,{}^{(a)}\mathrm{tr}\underline{\chi}.\end{aligned}$$

From (C.45)

$$\begin{aligned}Y_a^{\mathfrak{F}} &= 2C_1 - 2C_2 + \mathrm{tr}\underline{X} - 3\overline{\mathrm{tr}\underline{X}}\\ &= 2\left(2\mathrm{tr}\,\underline{\chi} - \frac{5}{2}i\,{}^{(a)}\mathrm{tr}\underline{\chi}\right) - 2(\mathrm{tr}\,\underline{\chi} - 3i\,{}^{(a)}\mathrm{tr}\underline{\chi}) + \mathrm{tr}\,\underline{\chi} - i\,{}^{(a)}\mathrm{tr}\underline{\chi} - 3(\mathrm{tr}\,\underline{\chi} + i\,{}^{(a)}\mathrm{tr}\underline{\chi})\\ &= -3i\,{}^{(a)}\mathrm{tr}\underline{\chi}.\end{aligned}$$

We now compute the imaginary parts of $Z_a^{\mathfrak{B}}$ and $W_a^{\mathfrak{F}} - Z_a^{\mathfrak{F}}$. From (C.31) and (C.34), we have

$$\Im(Z_a^{\mathfrak{B}}) = \Im(I_a^{\mathfrak{B}}) + \Im(J_a^{\mathfrak{B}}) + \Im(L_a^{\mathfrak{B}}) + \Im(M_a^{\mathfrak{B}}) - 2\underline{\eta}\cdot\Im(C_1),$$
$$\Im(Z_a^{\mathfrak{F}}) = \Im(I_a^{\mathfrak{F}}) + \Im(J_a^{\mathfrak{F}}) + \Im(L_a^{\mathfrak{F}}) + \Im(M_a^{\mathfrak{F}}) - 2\underline{\eta}\cdot\Im(C_2).$$

Using (C.9) and (C.12), we have

$$\Im(I_a^{\mathfrak{B}}) = \Im(I_a^{\mathfrak{F}}) = \Im\big(-2\,{}^{(c)}\nabla_3(\eta-\underline{\eta}) + \mathrm{tr}\,\underline{\chi}(\eta-\underline{\eta}) - {}^{(a)}\mathrm{tr}\underline{\chi}\,{}^*(\eta-\underline{\eta})\big) = 0.$$

Using (C.14) and (C.16), we write

$$J_a^{\mathfrak{B}} = -\frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\big(2(\eta-\underline{\eta}) - \big(6H + \overline{H} + 3\overline{\underline{H}}\big)\big)$$
$$-2\,{}^{(c)}\nabla C_1 - \mathrm{tr}\underline{X}H - \frac{1}{2}(\overline{\mathrm{tr}\underline{X}} - \mathrm{tr}\underline{X})\,\overline{\underline{H}} + \frac{1}{2}(\overline{\mathrm{tr}\underline{X}} - \mathrm{tr}\underline{X})\,\overline{H}$$

and

$$J_a^{\mathfrak{F}} = -\frac{1}{2}(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})\big(2(\eta-\underline{\eta}) - \big(4H + \overline{H} + \underline{H}\big)\big)$$
$$-2\,{}^{(c)}\nabla C_2 + \frac{1}{2}\overline{\mathrm{tr}\underline{X}}\,\overline{H} - \frac{1}{2}\big(\mathrm{tr}\underline{X} - \overline{\mathrm{tr}\underline{X}}\big)\,\underline{H} - \frac{1}{2}(2\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})H.$$

By writing ${}^{(c)}\nabla C_1 = \nabla C_1 - \zeta C_1$ and ${}^{(c)}\nabla C_2 = \nabla C_2 - \zeta C_2$, we obtain

$$\begin{aligned}
\Im(J_a^{\mathfrak{B}}) &= \mathrm{tr}\,\underline{\chi}\Im\big(6H + \overline{H} + 3\overline{\underline{H}}\big) - 2\nabla\Im(C_1) + 2\zeta\Im(C_1)\\
&\quad - {}^{(a)}\mathrm{tr}\underline{\chi}\underline{\eta} + {}^{(a)}\mathrm{tr}\underline{\chi}\eta - (\mathrm{tr}\,\underline{\chi}\,{}^*\eta - {}^{(a)}\mathrm{tr}\underline{\chi}\eta)\\
&= \mathrm{tr}\,\underline{\chi}\big(5\,{}^*\eta - 3\,{}^*\underline{\eta}\big) - 2\nabla\Im(C_1) + 2\zeta\Im(C_1) - {}^{(a)}\mathrm{tr}\underline{\chi}\underline{\eta} + 2\,{}^{(a)}\mathrm{tr}\underline{\chi}\eta - \mathrm{tr}\,\underline{\chi}\,{}^*\eta\\
&= -2\nabla\Im(C_1) + 2\zeta\Im(C_1) - {}^{(a)}\mathrm{tr}\underline{\chi}\underline{\eta} + 2\,{}^{(a)}\mathrm{tr}\underline{\chi}\eta + 4\mathrm{tr}\,\underline{\chi}\,{}^*\eta - 3\mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}
\end{aligned}$$

and

$$\begin{aligned}
\Im(J_a^{\mathfrak{F}}) &= \mathrm{tr}\,\underline{\chi}\Im\big(4H + \overline{H} + \underline{H}\big) - 2\nabla\Im(C_2) + 2\zeta\Im(C_2)\\
&\quad - \mathrm{tr}\,\underline{\chi}\,{}^*\eta + {}^{(a)}\mathrm{tr}\underline{\chi}\underline{\eta} - (\mathrm{tr}\,\underline{\chi}\,{}^*\eta - {}^{(a)}\mathrm{tr}\underline{\chi}\eta)\\
&= \mathrm{tr}\,\underline{\chi}\big(3\,{}^*\eta + {}^*\underline{\eta}\big) - 2\nabla\Im(C_2) + 2\zeta\Im(C_2) - 2\mathrm{tr}\,\underline{\chi}\,{}^*\eta + {}^{(a)}\mathrm{tr}\underline{\chi}\underline{\eta} + {}^{(a)}\mathrm{tr}\underline{\chi}\eta\\
&= -2\nabla\Im(C_2) + 2\zeta\Im(C_2) + {}^{(a)}\mathrm{tr}\underline{\chi}\underline{\eta} + {}^{(a)}\mathrm{tr}\underline{\chi}\eta + \mathrm{tr}\,\underline{\chi}\,{}^*\eta + \mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}.
\end{aligned}$$

Using (C.21) and (C.23), we have

$$\Im(L_a^{\mathfrak{B}}) = \Im(-(3\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})(\eta-\underline{\eta})) = 2\,{}^{(a)}\mathrm{tr}\underline{\chi}(\eta-\underline{\eta}),$$
$$\Im(L_a^{\mathfrak{F}}) = \Im(-(\mathrm{tr}\underline{X} + \overline{\mathrm{tr}\underline{X}})(\eta-\underline{\eta})) = 0.$$

Using (C.25) and (C.27), we have

$$\Im(M_a^{\mathfrak{B}}) = {}^{(c)}\nabla_3 \Im\big(6H + \overline{H} + 3\underline{\overline{H}}\big) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\,\Im\big(6H + \overline{H} + 3\underline{\overline{H}}\big)$$

$$+ \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\ \ {}^{*}\Im\big(6H + \overline{H} + 3\underline{\overline{H}}\big)$$

$$= \nabla_3\big(5\ {}^{*}\eta - 3\ {}^{*}\underline{\eta}\big) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\big(5\ {}^{*}\eta - 3\ {}^{*}\underline{\eta}\big) + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\big(- 5\eta + 3\underline{\eta}\big)$$

and

$$\Im(M_a^{\mathfrak{F}}) = {}^{(c)}\nabla_3 \Im\big(4H + \overline{H} + \underline{H}\big) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\,\Im\big(4H + \overline{H} + \underline{H}\big)$$

$$+ \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\ \ {}^{*}\Im\big(4H + \overline{H} + \underline{H}\big)$$

$$= \nabla_3\big(3\ {}^{*}\eta +\ {}^{*}\underline{\eta}\big) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\big(3\ {}^{*}\eta +\ {}^{*}\underline{\eta}\big) + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\big(- 3\eta - \underline{\eta}\big).$$

We therefore obtain

$$\Im(Z_a^{\mathfrak{B}}) = -2\nabla\Im(C_1) + 2\zeta\Im(C_1) - {}^{(a)}\mathrm{tr}\underline{\chi\eta} + 2\,{}^{(a)}\mathrm{tr}\underline{\chi}\eta + 4\mathrm{tr}\,\underline{\chi}\ {}^{*}\eta - 3\mathrm{tr}\,\underline{\chi}\ {}^{*}\underline{\eta}$$

$$+ 2\,{}^{(a)}\mathrm{tr}\underline{\chi}(\eta - \underline{\eta}) + \nabla_3\big(5\ {}^{*}\eta - 3\ {}^{*}\underline{\eta}\big) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\big(5\ {}^{*}\eta - 3\ {}^{*}\underline{\eta}\big)$$

$$+ \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\big(- 5\eta + 3\underline{\eta}\big) - 2\underline{\eta}\cdot\Im(C_1)$$

$$= -2\nabla\Im(C_1) + 2(\zeta - \underline{\eta})\Im(C_1) + \frac{3}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}(\eta - \underline{\eta}) + \frac{3}{2}\mathrm{tr}\,\underline{\chi}(\ {}^{*}\eta -\ {}^{*}\underline{\eta})$$

$$+ \nabla_3\big(5\ {}^{*}\eta - 3\ {}^{*}\underline{\eta}\big)$$

and, also using (C.49)

$$\Im(Z_a^{\mathfrak{F}} - W_a^{\mathfrak{F}}) = -2\nabla\Im(C_2) + 2\zeta\Im(C_2) + {}^{(a)}\mathrm{tr}\underline{\chi\eta} + {}^{(a)}\mathrm{tr}\underline{\chi}\eta + \mathrm{tr}\,\underline{\chi}\ {}^{*}\eta + \mathrm{tr}\,\underline{\chi}\ {}^{*}\underline{\eta}$$

$$+ \nabla_3\big(3\ {}^{*}\eta +\ {}^{*}\underline{\eta}\big) - \frac{1}{2}\mathrm{tr}\,\underline{\chi}\big(3\ {}^{*}\eta +\ {}^{*}\underline{\eta}\big) + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}\big(- 3\eta - \underline{\eta}\big)$$

$$- 2\underline{\eta}\cdot\Im(C_2) + 3\nabla_3\ {}^{*}\eta$$

$$= -2\nabla\Im(C_2) + 2(\zeta - \underline{\eta})\Im(C_2) + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}(\underline{\eta} - \eta) + \frac{1}{2}\mathrm{tr}\,\underline{\chi}(\ {}^{*}\underline{\eta} -\ {}^{*}\eta)$$

$$+ \nabla_3\big(6\ {}^{*}\eta +\ {}^{*}\underline{\eta}\big).$$

We now evaluate in the outgoing frame. Since $\zeta = -\underline{\eta}$, and by writing $\Im(C_1) = p_1\,{}^{(a)}\mathrm{tr}\underline{\chi}$ and $\Im(C_1) = p_2\,{}^{(a)}\mathrm{tr}\underline{\chi}$ we have

$$\Im(Z_a^{\mathfrak{B}}) = -2p_1\nabla\,{}^{(a)}\mathrm{tr}\underline{\chi} - 4p_1\,{}^{(a)}\mathrm{tr}\underline{\chi\eta} + \frac{3}{2}\,{}^{(a)}\mathrm{tr}\underline{\chi}(\eta - \underline{\eta}) + \frac{3}{2}\mathrm{tr}\,\underline{\chi}(\ {}^{*}\eta -\ {}^{*}\underline{\eta})$$

$$+ \nabla_3\big(5\ {}^{*}\eta - 3\ {}^{*}\underline{\eta}\big)$$

and

$$\Im(Z^{\mathfrak{F}}_a - W^{\mathfrak{F}}_a) = -2p_2\nabla\, {}^{(a)}\mathrm{tr}\underline{\chi} - 4p_2\, {}^{(a)}\mathrm{tr}\underline{\chi\eta} + \frac{1}{2}\, {}^{(a)}\mathrm{tr}\underline{\chi}(\underline{\eta} - \eta) + \frac{1}{2}\mathrm{tr}\,\underline{\chi}(\,{}^*\underline{\eta} - \,{}^*\eta)$$
$$+\nabla_3\big(6\,{}^*\eta + \,{}^*\underline{\eta}\big).$$

We start by evaluating at 2. Then since $\eta_2 = -\underline{\eta}_2$ and ${}^*\eta_2 = {}^*\underline{\eta}_2$, we have

$$\Im(Z^{\mathfrak{B}}_2) = -4p_1\, {}^{(a)}\mathrm{tr}\underline{\chi\eta}_2 - 3\, {}^{(a)}\mathrm{tr}\underline{\chi\eta}_2 + \nabla_3\big(5\,{}^*\eta - 3\,{}^*\underline{\eta}\big)_2.$$

Also using that ${}^{(c)}\nabla_3\,{}^*\eta_2 = -2\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_2$ and $({}^{(c)}\nabla_3\,{}^*\underline{\eta})_2 = -\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_2$, see [19], we have

$$\begin{aligned}\Im(Z^{\mathfrak{B}}_2) &= -4p_1\, {}^{(a)}\mathrm{tr}\underline{\chi\eta}_2 - 3\, {}^{(a)}\mathrm{tr}\underline{\chi\eta}_2 - 10\, {}^{(a)}\mathrm{tr}\underline{\chi\eta}_2 + 3\, {}^{(a)}\mathrm{tr}\underline{\chi\eta}_2\\ &= -2(2p_1+5)\, {}^{(a)}\mathrm{tr}\underline{\chi\eta}_2,\end{aligned}$$

which indeed vanish for $p_1 = -\frac{5}{2}$. We now also evaluate at 1. We have, using that $\nabla_1(\,{}^{(a)}\mathrm{tr}\underline{\chi}) = -3\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_1 - \mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1$ and $\eta_1 = \underline{\eta}_1$ and ${}^*\eta_1 = -\,{}^*\underline{\eta}_1$,

$$\begin{aligned}\Im(Z^{\mathfrak{B}}_1) &= -2p_1(-3\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_1 - \mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1) - 4p_1\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_1 - 3\mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1\\ &\quad + \nabla_3\big(5\,{}^*\eta - 3\,{}^*\underline{\eta}\big)_1\\ &= 2p_1\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_1 + (2p_1-3)\mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1 + \nabla_3\big(5\,{}^*\eta - 3\,{}^*\underline{\eta}\big)_1.\end{aligned}$$

Also using that ${}^{(c)}\nabla_3\,{}^*\eta_1 = \mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1 + {}^{(a)}\mathrm{tr}\underline{\chi\eta}_1$ and $({}^{(c)}\nabla_3\,{}^*\underline{\eta})_1 = -\mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1$, we have

$$\begin{aligned}\Im(Z^{\mathfrak{B}}_1) &= 2p_1\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_1 + (2p_1-3)\mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1 + 5\mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1 + 5\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_1 + 3\mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1\\ &= (2p_1+5)(\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_1 + \mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1),\end{aligned}$$

which vanishes for $p_1 = -\frac{5}{2}$.

Similarly, we have

$$\begin{aligned}\Im(Z^{\mathfrak{F}}_2 - W^{\mathfrak{F}}_2) &= -4p_2\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_2 + {}^{(a)}\mathrm{tr}\underline{\chi\eta}_2 + \nabla_3\big(6\,{}^*\eta + \,{}^*\underline{\eta}\big)_2\\ &= -4p_2\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_2 + {}^{(a)}\mathrm{tr}\underline{\chi\eta}_2 - 12\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_2 - {}^{(a)}\mathrm{tr}\underline{\chi\eta}_2\\ &= -(4p_2+12)\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_2\end{aligned}$$

and

$$\begin{aligned}\Im(Z^{\mathfrak{F}}_1 - W^{\mathfrak{F}}_1) &= -2p_2\nabla_1\,{}^{(a)}\mathrm{tr}\underline{\chi} - 4p_2\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_1 + \mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1 + \nabla_3\big(6\,{}^*\eta + \,{}^*\underline{\eta}\big)_1\\ &= 2p_2\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_1 + (2p_2+1)\mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1 + \nabla_3\big(6\,{}^*\eta + \,{}^*\underline{\eta}\big)_1\\ &= 2p_2\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_1 + (2p_2+1)\mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1 + 6\mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1 + 6\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_1 - \mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1\\ &= (2p_2+6)(\,{}^{(a)}\mathrm{tr}\underline{\chi\eta}_1 + \mathrm{tr}\,\underline{\chi}\,{}^*\underline{\eta}_1),\end{aligned}$$

which vanishes for $p_2 = -3$. This completes the proof of the lemma.

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