NEWELL-LITTLEWOOD NUMBERS

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ABSTRACT. The *Newell-Littlewood numbers* are defined in terms of their celebrated cousins, the *Littlewood-Richardson coefficients*. Both arise as tensor product multiplicities for a classical Lie group. They are the structure coefficients of the K. Koike-I. Terada basis of the ring of symmetric functions. Recent work of H. Hahn studies them, motivated by R. Langlands' *beyond endoscopy* proposal; we address her work with a simple characterization of *detection* of Weyl modules. This motivates further study of the combinatorics of the numbers. We consider analogues of ideas of J. De Loera-T. McAllister, H. Derksen-J. Weyman, S. Fomin–W. Fulton-C.-K. Li–Y.-T. Poon, W. Fulton, R. King-C. Tollu-F. Toumazet, M. Kleber, A. Klyachko, A. Knutson-T. Tao, T. Lam-A. Postnikov-P. Pylyavskyy, K. Mulmuley-H. Narayanan-M. Sohoni, H. Narayanan, A. Okounkov, J. Stembridge, and H. Weyl.

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1. INTRODUCTION

1.1. **Overview.** The *Newell-Littlewood numbers* [31, 35] are defined as

$$(1) \qquad N_{\mu,\nu,\lambda} = \sum_{\alpha,\beta,\gamma} c^{\mu}_{\alpha,\beta} c^{\nu}_{\alpha,\gamma} c^{\lambda}_{\beta,\gamma},$$

where the indices are partitions in

$$\mathsf{Par}_n = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0} : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}.$$

Here, $c^{\mu}_{\alpha,\beta}$ is the *Littlewood-Richardson coefficient*. The latter numbers are of interest in combinatorics, representation theory and algebraic geometry; see, *e.g.*, the

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books [9, 10, 41]. We study $N_{\mu,\nu,\lambda}$ by analogy with modern research on their better known constituents.

For an n-dimensional complex vector space V over $\mathbb{C}$ and $\lambda \in \mathsf{Par}_n$, the *Weyl module* (or *Schur functor*) $\mathbb{S}_\lambda(V)$ is an irreducible $\mathsf{GL}(V)$-module ([10, Lectures 6 and 15] is our reference). The Littlewood-Richardson coefficient is the tensor product multiplicity

$$(2) \qquad \mathbb{S}_\mu(V) \otimes \mathbb{S}_\nu(V) \cong \bigoplus_{\lambda \in \mathsf{Par}_n} \mathbb{S}_\lambda(V)^{\oplus c_{\mu,\nu}^\lambda}.$$

The Newell-Littlewood numbers arise in a similar manner, where $\mathsf{GL}(V)$ is replaced by one of the other classical Lie groups G. That is, suppose W is a complex vector space, with a nondegenerate symplectic or orthogonal form ω, where $\dim W = 2n + \delta$ and $\delta \in \{0, 1\}$. Fix a basis $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2n+\delta}\}$ such that

$$\omega(\varepsilon_k, \varepsilon_{2n+1+\delta-k}) = \pm\omega(\varepsilon_{2n+1+\delta-k}, \varepsilon_k) = 1, \quad \text{if } 1 \leq i \leq n + \delta$$

(other pairings are zero). Let G be the subgroup of $\mathsf{SL}(W)$ preserving ω. Then $\mathsf{G} = \mathsf{SO}_{2n+1}$ if $\dim W = 2n + 1$ and ω is orthogonal. It is $\mathsf{G} = \mathsf{Sp}_{2n}$ if $\dim W = 2n$ and ω is symplectic. Finally, $\mathsf{G} = \mathsf{SO}_{2n}$ if $\dim W = 2n$ and ω is once again orthogonal. These are, respectively, groups in the B_n, C_n, D_n series of the Cartan-Killing classification.

If $\lambda \in \mathsf{Par}_n$, H. Weyl's construction [47] (see also [10, Lectures 17 and 19]) gives a G-module $\mathbb{S}_{[\lambda]}(W)$. In the *stable range* $\ell(\mu) + \ell(\nu) \leq n$,

$$(3) \qquad \mathbb{S}_{[\mu]}(W) \otimes \mathbb{S}_{[\nu]}(W) \cong \bigoplus_{\lambda \in \mathsf{Par}_n} \mathbb{S}_{[\lambda]}(W)^{\oplus N_{\mu,\nu,\lambda}};$$

this is [24, Corollary 2.5.3]. $\mathbb{S}_{[\lambda]}(W)$ is an irreducible G-module, except in type D_n, where irreducibility holds if $\lambda_n = 0$ (otherwise it is the direct sum of two irreducible G-modules).

For any semisimple connected complex algebraic group G there is an irreducible G-module V_λ for each dominant weight λ. Uniform-type combinatorial frameworks for tensor product multiplicities (subsuming $c_{\lambda,\mu}^\nu$ and $N_{\mu,\nu,\lambda}$) are central in combinatorial representation theory; see, *e.g.*, the surveys [2, 25] for details and references. To compare and contrast, $N_{\mu,\nu,\lambda}$ is *itself* independent of the choice of G [24, Theorem 2.3.4].

Our thesis is that, like the Littlewood-Richardson coefficients, the Newell-Littlewood numbers form a subfamily of the general multiplicities whose combinatorics deserves separate study. Indeed, we reinforce the parallel with the Littlewood-Richardson coefficients by developing the topic from first principles and symmetric function basics.

1.2. **Earlier work.** Reading includes K. Koike-I. Terada's [24] which cites D. E. Littlewood's book [32] and R. C. King's [17, 18]. In turn, [17, 18] reference the papers of M. J. Newell [35] and D. E. Littlewood [31]. The Schur function s_λ, an element of the ring Λ of symmetric functions, is the "universal character" of $\mathbb{S}_\lambda(V)$. By analogy, [24, Section 2] establishes universal characters of $\mathbb{S}_{[\lambda]}(W)$ for the other classical groups.

In addition, [24, Theorem 2.3.4] shows that, in the stable range, the tensor product multiplicities coincide across the classical Lie groups (of types B, C, D).

For definiteness, we discuss Sp. It has a universal character basis $\{s_{[\lambda]}\}$ of Λ such that

$$(4) \qquad s_{[\mu]}s_{[\nu]} = \sum_\lambda N_{\mu,\nu,\lambda}s_{[\lambda]},$$

where μ, ν, λ are arbitrary partitions; we call this the *Koike-Terada* basis.[1] This basis specializes to the characters for fixed Sp_{2n}, just as the specialization

$$(5) \qquad s_\lambda \mapsto s_\lambda(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$$

does for GL_n. Their work discusses "modification rules" (cf. [17, 18]) to *non-positively* compute multiplicities outside the stable range. See [27] for recent work connecting the stable range combinatorics to crystal models in combinatorial representation theory.

This paper does not focus on the Koike-Terada basis *per se*. It is devoted to the inner logic of the Newell-Littlewood numbers. We were inspired by H. Hahn's [13] which concerns the case $\mu = \nu = \lambda$; we engage her work in Section 4.

1.3. **Summary of results.** Section 2 collects elementary facts about $N_{\mu,\nu,\lambda}$ (Lemma 2.2). We will need a Pieri-type rule (Proposition 2.4). This appears as S. Okada's [36, Proposition 3.1] with a short derivation from (1) (which we include for completeness); see also earlier work of A. Berele [3] and S. Sundaram [44].

In Section 3, we derive our initial result:

(I) Theorem 3.1 describes the "shape" of (4). It characterizes the sizes of λ that appear in (4) and gives a comparison result for partitions of different sizes. This result suggests the Unimodality Conjecture 3.7.

Section 4 is about the original stimulus for our work. We address a combinatorial question of H. Hahn [13] (who was motivated by R. Langlands' *beyond endoscopy* proposal [30] towards his functoriality conjecture [29]). More specifically, we prove

(II) Theorem 4.1, which is equivalent to showing

$$N_{\lambda,\lambda,\lambda} > 0 \text{ if and only if } |\lambda| \equiv 0 \pmod 2.$$

In [13], "$\Rightarrow$" was proved (see Lemma 2.2(V)) and the "$\Leftarrow$" implication was established for three infinite families of λ.

In Section 5, suggested by the simplicity of (II), we develop a broader framework by investigating "polytopal" aspects of (1).

(III) Theorem 5.1 shows that $N_{\mu,\nu,\lambda}$ counts the number of lattice points in a polytope $\mathcal{P}_{\mu,\nu,\lambda}$ that we directly construct (avoiding use of [4]). Its Corollary 5.3 says that

$$\mathsf{NL}_n := \{(\mu, \nu, \lambda) \in \mathsf{Par}_n^3 : N_{\mu,\nu,\lambda} > 0\}$$

is a semigroup.

(IV) We state two logically equivalent saturation conjectures about NL_n, *i.e.*, Conjectures 5.5 and 5.6. We prove special cases (Corollary 4.5, Theorem 5.8, Corollary 5.16). While saturation holds for the Littlewood-Richardson coefficients [22], it does *not* hold for the general tensor product multiplicities (although it is conjectured for simply-laced types). The

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[1][24] defines another basis, for SO. It also has $N_{\mu,\nu,\lambda}$ as its structure coefficients [24, Theorem 2.3.4 (3)]. Hence, for our purposes, discussing Sp rather than the SO basis is merely a matter of choice.

aforementioned results and conjectures provide a new view on this subject (compare, *e.g.*, [16, 26] and the references therein).

(V) Among the Horn inequalities [14] are the *Weyl inequalities* [46]. Our "extended Weyl inequalities" hold whenever $N_{\mu,\nu,\lambda} > 0$; this is Theorem 5.13. Theorem 5.15 is our justification of the nomenclature; it establishes that the (extended) Weyl inequalities are enough to characterize NL_2. Our proof uses a generalizable strategy; we will return to this in a sequel.

(VI) We also discuss limits of the analogy with $c^{\lambda}_{\mu,\nu}$. Theorem 5.27 shows that R. C. King-C. Tollu-F. Toumazet's Littlewood-Richardson polynomial conjecture [19] (proved by H. Derksen-J. Weyman [7]) has no naïve Newell-Littlewood version.

(VII) Section 5.5 sketches the computational complexity implications of Theorem 5.1.

The "nonvanishing" results of Section 5 are related to Section 6, where we prove:

(VIII) Theorem 6.1, which characterizes pairs (λ, μ) such that (4) is multiplicity-free. This is an analogue of J. R. Stembridge's [42, Theorem 3.1] for Schur functions, with a similar, self-contained proof.

Section 7 gathers some miscellaneous items. This includes two open problems, and

(IX) Theorem 7.4, which generalizes the results of T. Lam–A. Postnikov–P. Pylyavskyy [28] that solved conjectures of A. Okounkov [37] and of S. Fomin-W. Fulton-C.-K. Li-Y.-T. Poon [8].

The appendix gives a list of decompositions (4) for the reader's convenience.

2. Preliminaries

2.1. **The Littlewood-Richardson rule.** Let Par be the set of all partitions (with parts of size 0 being ignored). Identify $\lambda \in \mathsf{Par}$ with Young diagrams of shape λ (drawn in English convention). Let $\ell(\lambda)$ be the number of nonzero parts of λ and let $|\lambda| := \sum_{i=1}^{\ell(\lambda)} \lambda_i$ be the *size* of λ, that is, the number of boxes of λ. If $\mu \subseteq \lambda$, the *skew shape* λ/μ is the set-theoretic difference of the diagrams when aligned by their northwest most box.

A *semistandard filling* T of λ/μ assigns positive integers to each box of λ/μ such that the rows are weakly increasing from left to right, and the columns are strictly increasing from top to bottom. The *content* of T is $(c_1, c_2, \ldots)$ where $c_i = \#\{i \in T\}$. Let

$$\mathsf{rowword}(T) = (w_1, w_2, \ldots, w_{|\lambda/\mu|})$$

be the right to left, top to bottom, row reading word of T. We say $\mathsf{rowword}(T)$ is *ballot* if for each $i, k \geq 1$ we have

$$\#\{w_j = i : j \leq k\} \geq \#\{w_j = i + 1 : j \leq k\}.$$

T is *ballot* if $\mathsf{rowword}(T)$ is ballot. The *Littlewood-Richardson coefficient* $c^{\lambda}_{\mu,\nu}$ is the number of ballot, semistandard tableaux of shape λ/μ and content ν; we will call these *LR tableaux*.

Example 2.1. If $\mu = (3,1), \nu = (4,2,1), \lambda = (5,4,2)$ then $c_{\mu,\nu}^{\lambda} = 2$ because of these two tableaux:

$$T_1 = \begin{array}{|c|c|c|c|c|} \hline & & & 1 & 1 \\ \hline & 1 & 2 & 2 \\ \cline{1-3} 1 & 3 \\ \cline{1-2} \end{array} \quad \text{and} \quad T_2 = \begin{array}{|c|c|c|c|c|} \hline & & & 1 & 1 \\ \hline & 1 & 1 & 2 \\ \cline{1-3} 2 & 3 \\ \cline{1-2} \end{array}$$

Here $\mathsf{rowword}(T_1) = (1,1,2,2,1,3,1)$ and $\mathsf{rowword}(T_2) = (1,1,2,1,1,3,2)$. $\qquad \square$

The Littlewood-Richardson rule implies that $N_{\mu,\nu,\lambda}$ is well-defined for $\mu, \nu, \lambda \in$ Par.

2.2. **Facts about $N_{\mu,\nu,\lambda}$.** We gather some simple facts we will use; we make no claims of originality:

Lemma 2.2 (Facts about the Newell-Littlewood numbers).

  (I) $N_{\mu,\nu,\lambda}$ is invariant under any $\mathfrak{S}_3$-permutation of the indices (μ,ν,λ).
 (II) $N_{\mu,\nu,\lambda} = c_{\mu,\nu}^{\lambda}$ if $|\mu| + |\nu| = |\lambda|$.
(III) $N_{\mu,\nu,\lambda} = 0$ unless $|\mu|, |\nu|, |\lambda|$ satisfy the triangle inequalities (possibly with equality), i.e., $|\mu| + |\nu| \geq |\lambda|$, $|\mu| + |\lambda| \geq |\nu|$, and $|\lambda| + |\nu| \geq |\mu|$.[2]
(IV) $N_{\mu,\nu,\lambda} = 0$ if $|\nu \wedge \lambda| + |\mu \wedge \nu| < |\nu|$.[3]
 (V) $N_{\mu,\nu,\lambda} = 0$ unless $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$.
(VI) $N_{\mu,\nu,\lambda} = N_{\mu',\nu',\lambda'}$ where μ' is the conjugate partition of μ, etc.

Proof. (I) is immediate from (1).

By (1), $N_{\mu,\nu,\lambda} = 0$ unless there exist $\alpha, \beta, \gamma \in$ Par such that $c_{\alpha,\beta}^{\mu}, c_{\alpha,\gamma}^{\nu}, c_{\beta,\gamma}^{\lambda} > 0$. Henceforth we will call α, β, γ a *witness* for $N_{\mu,\nu,\lambda} > 0$. These Littlewood-Richardson coefficients are zero unless

$$|\alpha| + |\beta| = |\mu|, |\alpha| + |\gamma| = |\nu|, |\beta| + |\gamma| = |\lambda| \text{ (respectively).}$$

Therefore

(6) $$2|\alpha| + |\lambda| = |\mu| + |\nu|,$$

which implies $|\lambda| \leq |\mu| + |\nu|$. Now apply (I) to get (III). If $|\lambda| = |\mu| + |\nu|$ then (6) implies the only witness is $\alpha = \emptyset, \beta = \mu, \gamma = \nu$, hence $N_{\mu,\nu,\lambda} = c_{\mu,\nu}^{\lambda}$, as asserted by (II).

For (IV), any such γ satisfies $\gamma \subseteq \nu, \lambda$. Hence $|\gamma| \leq |\nu \wedge \lambda|$. Similarly, $|\alpha| \leq |\mu \wedge \nu|$. Now combine these inequalities with the fact that $|\alpha| + |\gamma| = |\nu|$.

(V) holds by (6).

Finally, (VI) holds by the standard fact $c_{\alpha,\beta}^{\mu} = c_{\alpha',\beta'}^{\mu'}, c_{\alpha,\gamma}^{\nu} = c_{\alpha',\gamma'}^{\nu'}$ and $c_{\beta,\gamma}^{\lambda} = c_{\beta',\gamma'}^{\lambda'}$. $\qquad \square$

2.3. **Symmetric functions.** Let Λ be the ring of symmetric functions in $x_1, x_2, \ldots$. Define the (skew) *Schur function*

$$s_{\mu/\lambda}(x_1, x_2, \ldots) := \sum_T x^T,$$

where the sum is over semistandard Young tableaux of skew shape μ/λ.

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[2]In the case of *reduced Kronecker coefficients* $\overline{g}_{\mu,\nu}^{\lambda}$ these are called *Murnaghan's inequalities*.
[3]Recall $\nu \wedge \lambda$ is the partition whose i-th part is $\min(\nu_i, \lambda_i)$.

It is true that $s_{\mu/\lambda} \in \Lambda$. Moreover, the $\{s_\lambda : \lambda \in \mathsf{Par}, |\lambda| = N\}$ is a basis of $\Lambda^{(N)}$, the degree N homogeneous component of $\Lambda = \bigoplus_N \Lambda^{(N)}$. In fact,

$$(7) \qquad s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu,\nu} s_\nu,$$

and

$$(8) \qquad s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda.$$

There is an inner product $\langle \cdot, \cdot \rangle : \Lambda \times \Lambda \to \mathbb{Q}$ such that $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$; see [41, Chapter 7].

We will make use of the following *asymmetric* formula for $N_{\mu,\nu,\lambda}$:

Proposition 2.3. $N_{\mu,\nu,\lambda} = \sum_\alpha \langle s_{\mu/\alpha} s_{\nu/\alpha}, s_\lambda \rangle$, *where the sum is over* $\alpha \subseteq \mu \wedge \nu$.

Proof. Combine (7), (8) and (1) with the fact that $s_{\mu/\alpha} = 0$ unless $\alpha \subseteq \mu$ and $s_{\nu/\alpha} = 0$ unless $\alpha \subseteq \nu$. $\qquad \square$

Although we will not need it in this paper, we recall the definition of $s_{[\lambda]}$ from [24, Definition 2.1.1]. Let $h_t = s_{(t)}$ be the homogeneous symmetric function of degree t. If $t < 0$ then by convention $h_t = 0$. Then if $\lambda \in \mathsf{Par}_n$, let $\lambda^* = (\lambda_1, \lambda_2 - 1, \dots, \lambda_n - (n-1))$. Below, h_{λ^*} denotes the column vector $(h_{\lambda_1}, h_{\lambda_2-1}, \dots, h_{\lambda_n-(n-1)})^t$ and $h_{\lambda^*+j(1^n)} + h_{\lambda^*-j(1^n)}$ means the column vector

$$(h_{\lambda_1+j}+h_{\lambda_1-j}, h_{\lambda_2-1+j}+h_{\lambda_2-1-j}, \dots, h_{\lambda_i-(i-1)+j}+h_{\lambda_i-(i-1)-j}, \dots, h_{\lambda_n-(n-1)+j}$$
$$+ h_{\lambda_n-(n-1)-j})^t.$$

With this notation,

$$s_{[\lambda]} := \left| h_{\lambda^*} \ h_{\lambda^*+(1^n)} + h_{\lambda^*-(1^n)} \ \cdots \ h_{\lambda^*+j(1^n)} + h_{\lambda^*-j(1^n)} \ \cdots \ h_{\lambda^*+(n-1)(1^n)} \right.$$
$$\left. + h_{\lambda^*-(n-1)(1^n)} \right|.$$

Hence, for example

$$s_{[4,2,1]} = \begin{vmatrix} h_4 & h_5 + h_3 & h_6 + h_2 \\ h_1 & h_2 + 1 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} = s_{4,2,1} - s_{4,1} - s_{3,2} - s_{3,1,1} + s_3 + s_{2,1}.$$

2.4. Pieri rules.
The Pieri rule for Schur functions [41, Theorem 7.5.17] states that

$$(9) \qquad s_\mu s_{(p)} = \sum_\lambda s_\lambda,$$

where the sum is over all λ such that λ/μ consists of p boxes, none of which are in the same column. We need the Newell-Littlewood analogue. It was known, and we include a proof which is the same as [36, Proposition 3.1] for completeness:

Proposition 2.4 (Pieri-type rule; Theorem 13.1 of [44] and Proposition 3.1 of [36])**.** $N_{\mu,(p),\lambda}$ *equals the number of ways to remove* $\frac{|\mu|+p-|\lambda|}{2}$ *boxes from* μ *(all from different columns), then add* $\frac{|\lambda|+p-|\mu|}{2}$ *boxes (all to different columns) to make* λ. *In other words,*

$$(10) \qquad s_{[\mu]} s_{[(p)]} = \sum_\lambda s_{[\lambda]},$$

where the sum is over the multiset of λ obtained from μ by removing a horizontal strip of j boxes where $0 \le j \le p$ and then adding a horizontal strip of length $p - j$ boxes.

Proof. Consider any α, β, γ such that $c_{\alpha,\beta}^{\mu} c_{\alpha,\gamma}^{(p)} c_{\beta,\gamma}^{\lambda} > 0$. By (6), $2|\alpha| = |\mu| + p - |\lambda|$, so $|\alpha| = \frac{|\mu| + p - |\lambda|}{2}$ and similarly $|\gamma| = \frac{|\lambda| + p - |\mu|}{2}$. Since $\alpha, \gamma \subseteq (p)$, we have that $\alpha = (\frac{|\mu| + p - |\lambda|}{2})$ and $\gamma = (\frac{|\lambda| + p - |\mu|}{2})$. Moreover, by (9), $c_{\alpha,\gamma}^{(p)} = 1$. Therefore,

$$(11) \qquad N_{\mu,(p),\lambda} = \sum_{\beta} c_{(\frac{|\mu|+p-|\lambda|}{2}),\beta}^{\mu} c_{\beta,(\frac{|\lambda|+p-|\mu|}{2})}^{\lambda}.$$

By (9), $c_{(\frac{|\mu|+p-|\lambda|}{2}),\beta}^{\mu} \in \{0,1\}$. It is 1 if and only if one can remove $\frac{|\mu|+p-|\lambda|}{2}$ boxes from different columns of μ to get β. Similarly, $c_{\beta,(\frac{|\lambda|+p-|\mu|}{2})}^{\lambda} \in \{0,1\}$, and is 1 if and only if one can add $\frac{|\mu|+p-|\lambda|}{2}$ boxes to different columns of β to get λ. We are done proving the $N_{\mu,(p),\lambda}$ claim by (11). The assertion (10) is a straightforward rephrasing of the first claim. $\qquad \square$

Example 2.5. We have

$$s_{[2,1]}s_{[3]} = s_{[1,1]} + s_{[2]} + s_{[2,1,1]} + s_{[2,2]} + 2s_{[3,1]} + s_{[4]} + s_{[3,2,1]} + s_{[4,1,1]} + s_{[4,2]} + s_{[5,1]}.$$

For example, $\lambda = (3,1)$ can be obtained in two ways from $\mu = (2,1)$ using $j = 1$:

 and .
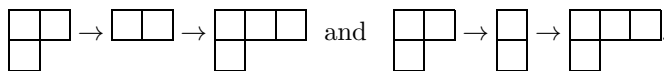
This explains the multiplicity in the computation. $\qquad \square$

Proposition 2.4 immediately implies a special case that we also use.

Corollary 2.6. $s_{[(1)]}s_{[\nu]} = \sum_{\lambda} s_{[\lambda]}$, *where the sum is over all partitions λ obtained by adding a box to ν or removing a box from ν.*[4]

3. Shape of $s_{[\mu]}s_{[\nu]}$

We describe some salient features of $s_{[\mu]}s_{[\nu]}$. Let $\mu \Delta \nu = (\mu \setminus \nu) \cup (\nu \setminus \mu)$ be the symmetric difference of λ and μ.

Theorem 3.1. *Fix $\mu, \nu \in \mathsf{Par}$.*

(I) *There exists $\lambda \in \mathsf{Par}$ with $|\lambda| = k$ and $N_{\mu,\nu,\lambda} > 0$ if and only if*

$$k \equiv |\mu \Delta \nu| \pmod{2} \quad and \quad |\mu \Delta \nu| \le k \le |\mu| + |\nu|.$$

(II) *If $N_{\mu,\nu,\lambda} > 0$ with $|\lambda| > |\mu \Delta \nu|$, there exists $\lambda^{\downarrow\downarrow}$ such that $N_{\mu,\nu,\lambda^{\downarrow\downarrow}} > 0$, $\lambda^{\downarrow\downarrow} \subset \lambda$ and $|\lambda^{\downarrow\downarrow}| = |\lambda| - 2$.*

(III) *If $N_{\mu,\nu,\lambda} > 0$ with $|\lambda| < |\mu| + |\nu|$, there exists $\lambda^{\uparrow\uparrow}$ such that $N_{\mu,\nu,\lambda^{\uparrow\uparrow}} > 0$, $\lambda \subset \lambda^{\uparrow\uparrow}$ and $|\lambda^{\uparrow\uparrow}| = |\lambda| + 2$.*

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[4]Let $(\mathbb{Y}, \le)$ be Young's poset. Standard tableaux biject with walks in $\mathbb{Y}$ from $\emptyset$ to λ, where each step is a covering relation. Iterating (9) shows $s_{(1)}^k = \sum_{\lambda} f^{\lambda} s_{\lambda}$, where f^{λ} counts standard Young tableaux of shape λ. An *oscillating tableau* of shape λ and length k is a walk in $\mathbb{Y}$ starting at $\emptyset$ and ending at λ with k edges such that each step $\theta \to \pi$ either has π/θ or θ/π being a single box. Let $o^{\lambda,k}$ be the number of these tableaux. It is known that $o^{\lambda,k} = \binom{k}{|\nu|}(k-1)!! f^{\nu}$. Iterating Corollary 2.6 gives $s_{[(1)]}^k = \sum_{\lambda} o^{\lambda,k} s_{[\lambda]}$; see [3,36,44].

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Proof.

(I) By Proposition 2.3, $N_{\mu,\nu,\lambda} > 0$ if and only if there exists $\alpha \subseteq \mu \wedge \nu$ such that $\langle s_{\mu/\alpha} s_{\nu/\alpha}, s_\lambda \rangle > 0$. Now,

$$(12) \qquad s_{\mu/\alpha} s_{\nu/\alpha} \neq 0 \iff \alpha \subseteq \mu \wedge \nu$$

Thus, by (7) and (8) combined, it suffices to characterize the possible values of $\deg(s_{\mu/\alpha} s_{\nu/\alpha})$. By taking $\alpha = \emptyset$ we obtain that $\deg(s_{\mu/\alpha} s_{\mu/\alpha}) \leq |\mu| + |\nu|$. Considering $\alpha = \mu \wedge \nu$ shows $|\mu \Delta \nu| \leq \deg(s_{\mu/\alpha} s_{\nu/\alpha})$. Also, it is clear that

$$(13) \qquad \deg(s_{\mu/\alpha} s_{\nu/\alpha}) \equiv \deg(s_{\mu/\theta} s_{\nu/\theta}) \pmod 2, \quad \forall \alpha, \theta \subseteq \mu \wedge \nu.$$

Thus (I) follows.

(II) We need two claims.

Claim 3.2. Suppose $c_{\alpha,\beta}^\mu > 0$ and $\alpha \subset \alpha^\uparrow \subseteq \mu$ with $|\alpha^\uparrow / \alpha| = 1$. Then there exists $\beta^\downarrow \subset \beta$ with $|\beta/\beta^\downarrow| = 1$ such that $c_{\alpha^\uparrow \beta^\downarrow}^\mu > 0$.

Proof of Claim 3.2. It is possible to prove this using the Littlewood-Richardson rule, however for brevity, we will use a result [1, Proposition 2.1] which concerns the *equivariant* generalization $C_{\lambda,\mu}^\nu$ of $c_{\lambda,\mu}^\nu$. For our purposes, it suffices to know that $C_{\lambda,\mu}^\nu$ is a polynomial that is nonzero only if $|\lambda| + |\mu| \geq |\nu|$ and moreover, $C_{\lambda,\mu}^\nu = c_{\lambda,\mu}^\nu$ if $|\lambda| + |\mu| = |\nu|$.

Given $c_{\alpha,\beta}^\mu > 0$, by part (A) of [1, Proposition 2.1] for any $\alpha \subset \alpha^\uparrow \subset \mu$ (where $\alpha^\uparrow$ is α with a box added) we have $C_{\alpha^\uparrow,\beta}^\mu \neq 0$ (as a polynomial). However, by part (B) of [1, Proposition 2.1], *there exists* $\beta^\downarrow \subset \beta$ (which is β with a box removed) such that $C_{\alpha^\uparrow,\beta^\downarrow}^\mu \neq 0$. Since $|\alpha^\uparrow| + |\beta^\downarrow| = |\mu|$, $C_{\alpha^\uparrow,\beta^\downarrow}^\mu = c_{\alpha^\uparrow,\beta^\downarrow}^\mu > 0$. $\qquad \square$

Claim 3.3. Suppose $\beta, \gamma, \beta^\uparrow, \gamma^\uparrow$ are partitions such that $\beta \subset \beta^\uparrow$ where $|\beta^\uparrow/\beta| = 1$, and $\gamma \subset \gamma^\uparrow$ where $|\gamma^\uparrow/\gamma| = 1$. If $c_{\beta^\uparrow,\gamma^\uparrow}^{\overline{\lambda}} > 0$ then there exists $\overline{\lambda}^{\downarrow\downarrow} \subset \overline{\lambda}$ with $|\overline{\lambda}/\overline{\lambda}^{\downarrow\downarrow}| = 2$ such that $c_{\beta,\gamma}^{\overline{\lambda}^{\downarrow\downarrow}} > 0$.

Proof of Claim 3.4. By Pieri's rule (9),

$$s_\beta s_{(1)} = s_{\beta^\uparrow} + (\text{positive sum of Schur functions})$$

and

$$s_\gamma s_{(1)} = s_{\gamma^\uparrow} + (\text{positive sum of Schur functions}).$$

Hence,

$$(14) \qquad s_\beta s_\gamma s_{(1)}^2 = s_{\beta^\uparrow} s_{\gamma^\uparrow} + (\text{positive sum of Schur functions})$$

Expanding the lefthand side of (14) into the basis of Schur functions, gives

$$s_\beta s_\gamma s_{(1)}^2 = \sum_\theta c_{\beta,\gamma}^\theta (s_\theta s_{(1)}^2).$$

Hence, by Pieri's rule (9),

$$[s_\kappa] s_\beta s_\gamma s_{(1)}^2 \neq 0$$

only if κ is obtained from θ with $c_{\beta,\gamma}^\theta > 0$ with $\theta \subset \kappa$ and $|\kappa/\theta| = 2$. Now, since the righthand side of (14) is Schur positive the same must be true of any κ such that $[s_\kappa] s_{\beta^\uparrow} s_{\gamma^\uparrow}$. In particular this is true of $\kappa = \overline{\lambda}$. $\qquad \square$

Since $N_{\mu,\nu,\lambda} > 0$, there exists (α, β, γ) such that $c_{\alpha,\beta}^{\mu} c_{\alpha,\gamma}^{\nu} c_{\beta,\gamma}^{\lambda} > 0$. Since $|\lambda| > |\mu \Delta \nu|$ we must have $\alpha \subsetneq \mu \wedge \nu$. Hence let

$$\alpha \subsetneq \alpha^{\uparrow} \subseteq \mu \wedge \nu$$

be α with a box added. By two applications of Claim 3.2, there exists $\beta^{\downarrow}$ and $\gamma^{\downarrow}$ which are respectively β and γ with a box removed such that $c_{\alpha^{\uparrow}\beta^{\downarrow}}^{\mu}, c_{\alpha^{\uparrow}\gamma^{\downarrow}}^{\nu} > 0$. Now apply Claim 3.3 with $\overline{\lambda} = \lambda$ and $\beta^{\downarrow}, \gamma^{\downarrow}, \beta, \gamma$. The conclusion is that $(\alpha^{\uparrow}, \beta^{\downarrow}, \gamma^{\downarrow})$ is a witness for $N_{\mu,\nu,\lambda^{\downarrow\downarrow}}$ and $\lambda^{\downarrow\downarrow} \subset \lambda$ of two smaller size, as desired.

(III) We need two additional claims.

Claim 3.4. Suppose $c_{\beta,\gamma}^{\lambda} > 0$. If $\gamma^{\uparrow} \supset \gamma$ with $|\gamma^{\uparrow}/\gamma| = 1$ then there exists $\lambda^{\uparrow} \supset \lambda$ with $|\lambda^{\uparrow}/\lambda| = 1$ such that $c_{\beta,\gamma^{\uparrow}}^{\lambda^{\uparrow}} > 0$.

Proof of Claim 3.4. Fix a rectangle $R = \ell \times (m - \ell)$ (for some positive integers ℓ, m) sufficiently large to contain β, γ, λ. Given a Young diagram $\theta \subseteq R$ let $\theta^{\vee}$ be the 180-degree rotation of $R \setminus \theta$. A Schubert calculus symmetry for the Grassmannian $\mathsf{Gr}_{\ell}(\mathbb{C}^m)$ states that

$$(15) \qquad\qquad c_{\beta,\gamma}^{\lambda} = c_{\lambda^{\vee},\gamma}^{\beta^{\vee}}.$$

Choose ℓ, m sufficiently large so that $\gamma^{\uparrow} \subset \beta^{\vee}$. By Claim 3.2, there exists $(\lambda^{\vee})^{\downarrow}$ which is $\lambda^{\vee}$ with a box removed such that $c_{(\lambda^{\vee})^{\downarrow},\gamma^{\uparrow}}^{\beta^{\vee}} > 0$. By (15),

$$0 < c_{(\lambda^{\vee})^{\downarrow},\gamma^{\uparrow}}^{\beta^{\vee}} = c_{\beta,\gamma^{\uparrow}}^{((\lambda^{\vee})^{\downarrow})^{\vee}}.$$

By definition of "$\vee$", $((\lambda^{\vee})^{\downarrow})^{\vee}$ is of the form $\lambda^{\uparrow}$ such that $c_{\beta,\gamma^{\uparrow}}^{\lambda^{\uparrow}} > 0$. $\qquad\square$

Claim 3.5. Suppose $c_{\alpha,\beta}^{\mu} > 0$. For any $\emptyset \subseteq \alpha^{\downarrow} \subset \alpha$ with $|\alpha/\alpha^{\downarrow}| = 1$ there exists $\beta^{\uparrow} \supset \beta$ with $|\beta^{\uparrow}/\beta| = 1$ such that $c_{\alpha^{\downarrow},\beta^{\uparrow}}^{\mu} > 0$.

Proof of Claim 3.5. Since $c_{\alpha,\beta}^{\mu} > 0$, there exists an LR tableau T of shape μ/α and content β. We are done once we modify T to give an LR tableau T' of shape $\mu/\alpha^{\downarrow}$ and content $\beta^{\uparrow}$, as follows: Place 1 in $b_1 = \alpha/\alpha^{\downarrow}$. Find the first 1 (if it exists, say in b_2) in the column reading (top to bottom, right to left) word order after b_1 and turn that into a 2. Next, find the first 2 (again, if it exists, say in b_3) in the column reading word order after b_2 and change that to a 3. We terminate and output T' when, after replacing the $k-1$ in b_k with k, there is no later k in the column reading order.

Since the number of boxes of T is finite, this process does end. T' is clearly of the desired shape. The content of T' is

$$\beta^{\uparrow} := (\beta_1, \beta_2, \ldots, \beta_k + 1, \beta_{k+1}, \ldots).$$

It remains to check two things:

(T' is semistandard): Since $T'(b_1) = 1$, we can only violate semistandardness if the box d_1 directly below b_1 has $T(d_1) = 1$. However, in that case $T'(d_1) = 2$, by construction. In general, since

$$T'(b_j) := T(b_j) + 1(= j) \text{ for } 2 \leq j \leq k,$$

the entry in b_j of T' can only cause a problem with semistandardness with the box d_j directly below, or the box r_j directly to the right. The former is only a concern if $T(d_j) = j$, but in that case $T'(d_j) = j + 1$.

The latter concern occurs if $T(r_j) = j - 1$. If b_{j-1} is in a column strictly to the right of b_j then $T(r_j) = j - 1$ cannot occur since the $j - 1$ in r_j occurs strictly between b_{j-1} and b_j in the column reading word. This contradicts the definition of b_j. So we may assume b_{j-1} is in the same column as b_j. Since

$$T(b_{j-1}) := j - 2 \text{ and } T(b_j) := j - 1,$$

in fact, b_{j-1} is immediately above b_j, i.e., $d_{j-1} = b_j$. Since we assume $T(r_j) = j-1$, semistandardness of T implies $T(r_{j-1}) = j-2$, which by the same argument implies b_{j-2} is directly above b_{j-1} (otherwise we would contradict the definition of b_{j-1}. Repeating this logic tells us that $b_2, b_3, \ldots, b_j$ are consecutive boxes in the same column with $T(b_2) := 1$ and $T(r_2) = 1$. However, this forces b_1 to be in a column strictly right of b_2. Since $T(r_2) = 1$ and r_2 is between b_1 and b_2, we contradict the definition of b_2. Thus, the situation $T(r_j) = j-1$ of this paragraph cannot actually occur.

(T' is ballot): It is well-known that any semistandard tableau is ballot with respect to the row reading if and only if it is ballot with respect to the column reading. For $j \geq 2$, we need to show that T' is $(j-1, j)$-ballot, that is, the number of $j - 1$'s appearing at any given point of the column reading word exceeds the number of j's at the same point. If $j > k + 1$ then the $j - 1$'s and j's in T' and T are in the exact same positions, and T' is $(j-1, j)$-ballot since T is. If $j = k+1$ the same is true except T' has an additional $j - 1 = k$ at b_k, and ballotness similarly follows.

Now suppose $j \leq k$. The only boxes b_t ($1 \leq t \leq k$) that contain $j - 1$ or j in T or T' are b_{j-1}, b_j and b_{j+1}. Hence consider four regions of T': (i) strictly before b_{j-1}; (ii) starting from b_{j-1} to before b_j; (iii) starting from b_j until before b_{j+1}; and (iv) b_{j+1} and thereafter (in the column reading order). Below, let $w[b]$ be a partial reading word of T that ends at a box b. Let $w'[b]$ be the word using the same boxes of T'.

In region (i), the j's and $(j - 1)$'s are in the same positions in both T and T'. Hence since $w[b]$ is $(j - 1, j)$-ballot, the same is true of $w'[b]$ for any b in (i). For any b in (ii), $w'[b]$ has one more $j - 1$ than $w[b]$ (since $T(b_{j-1}) = j - 2$ and $T'(b_{j-1}) = j - 1$. Hence, $w'[b]$ is $(j - 1, j)$-ballot because this is true of $w[b]$.

For any b in region (iii), $w'[b]$ and $w[b]$ have the same number of $(j - 1)$'s but w' has one more j. There are two cases.

Case 1 (b_{j+1} exists, *i.e.*, $j < k$ and region (iv) exists). If $w'[b]$ is not $(j-1, j)$-ballot, then it follows $w[b_{j+1}]$ is not $(j-1, j)$-ballot, a contradiction. Finally, if b is in (iv), $w[b]$ and $w[b']$ have the same number of $(j - 1)$'s and j's, so we are again done.

Case 2 (b_{j+1} does not exist, *i.e.*, $j = k$ and region (iv) does not exist). This case means there are no j's in T after b_j. Hence if $w'[b]$ fails to be $(j - 1, j)$-ballot for any b weakly after b_j, in fact $w'[b_j]$ is not $(j - 1, j)$-ballot. By definition, $w[b_j]$ has the same number of $(j - 1)'s$ but one less j than $w'[b_j]$. Since $w'[b_j]$ is not $(j-1, j)$-ballot, it must be that $w[b_j]$ has the same number of $(j - 1)$'s and j's. Let b° be the box immediately before b_j in the reading order. Since $T(b_j) = j - 1$ we conclude $w[b^\circ]$ is not $(j - 1, j)$-ballot, a contradiction. $\square$

Since $N_{\mu, \nu, \lambda} > 0$ there exists (α, β, γ) such that $c_{\alpha, \beta}^\mu c_{\alpha, \gamma}^\nu c_{\beta, \gamma}^\lambda > 0$. Remove any corner from α to obtain $\alpha^\downarrow$. By two applications of Claim 3.5 there exists $\beta^\uparrow$ and $\gamma^\uparrow$ such that $c_{\alpha^\downarrow, \beta^\uparrow}^\mu, c_{\alpha^\downarrow, \gamma^\uparrow}^\nu > 0$. By two applications of Claim 3.4, there exists $\lambda^{\uparrow\uparrow}$

(as in the theorem statement) such that $c_{\beta^\uparrow,\gamma^\uparrow}^{\lambda^{\uparrow\uparrow}} > 0$. Hence $(\alpha^\downarrow, \beta^\uparrow, \gamma^\uparrow)$ witnesses that $N_{\mu,\nu,\lambda^{\uparrow\uparrow}} > 0$. $\qquad\square$

Example 3.6. If $\mu = (3)$ and $\nu = (2,1)$ then $|\mu\Delta\nu| = 2$ and $|\mu| + |\nu| = 6$. We compute:

$$s_{[3]}s_{[2,1]} = s_{[1,1]} + s_{[2]} + s_{[2,1,1]} + s_{[2,2]} + 2s_{[3,1]} + s_{[4]} + s_{[3,2,1]} + s_{[4,1,1]} + s_{[4,2]} + s_{[5,1]}.$$

The reader can check agreement with Theorem 3.1. $\qquad\square$

There seems to be another "structural" aspect of (4). Define

$$h_t^{\mu,\nu} = \sum_{\lambda:|\lambda|=|\mu\Delta\nu|+2t} N_{\mu,\nu,\lambda}.$$

A sequence $(a_k)_{k=0}^N$ is *unimodal* if there exists $0 \leq m \leq N$ such that

$$0 \leq a_0 \leq a_1 \leq \ldots \leq a_m \geq a_{m+1} \geq \ldots a_{N-1} \geq a_N.$$

Conjecture 3.7 (Unimodality). *The sequence* $\{h_t^{\mu,\nu}\}_{t=0}^{|\mu\wedge\nu|}$ *is a unimodal sequence.*

We checked Conjecture 3.7 for all $s_{[\mu]}s_{[\nu]}$ where $0 \leq |\mu|, |\nu| \leq 7$, and many larger cases. Theorem 3.1 (II) and (III) suggest proving Conjecture 3.7 by constructing chains in Young's poset, each element λ appearing $N_{\mu,\nu,\lambda}$-many times, "centered" at m:

Example 3.8. Continuing the previous example, $\{h_t^{\mu,\nu}\}_{t=0}^3 = 2, 5, 4$. Here $m = 1$ and we are suggesting that the following chains demonstrate the unimodality:

$$(1,1) \subset (2,2) \subset (4,2)$$
$$(2) \subset (2,1,1) \subset (4,1,1)$$
$$(3,1)$$
$$(3,1) \subset (3,2,1)$$
$$(4) \subset (5,1)$$

There is choice in the chains; in the first and third chains we could interchange the roles of $(2,2)$ and $(3,1)$. $\qquad\square$

A sequence is *log-concave* if

$$a_t^2 \geq a_{t-1}a_{t+1} \quad \text{for } 0 < t < N.$$

Log-concavity implies unimodality. Thus, a warning against Conjecture 3.7 is this:

Example 3.9 (Log-concavity counterexample). $\{h_t^{(2,2),(2,2)}\}_{t=0}^4 = 1, 2, 6, 8, 6$ is unimodal but not log-concave. $\qquad\square$

4. H. HAHN'S NOTION OF DETECTION

Our study of NL_n was stimulated by work of H. Hahn [12,13]. Suppose H is an irreducible reductive subgroup of GL_N. H. Hahn [12] defines that a representation

$$(16) \qquad\qquad\qquad \rho : \mathsf{GL}_N \to \mathsf{GL}(V)$$

detects H if H stabilizes a line in V. She initiates a study of detection, motivated by R. Langlands' *beyond endoscopy* proposal [30] towards proving his functoriality conjecture [29] (see [12,13] for elucidation and further references).

The general question stated in [12] is to determine which algebraic subgroups of GL_N are detected by a representation (16). In [13], this question is studied using the classical groups $\mathsf{G} = \mathsf{SO}_{2n+1}, \mathsf{Sp}_{2n}, \mathsf{SO}_{2n}$ (where in the latter case n is assumed to be even) and where $\rho : \mathsf{GL}_N \to \mathsf{GL}_{N^3}$ is $\rho = \otimes^3$, i.e., the corresponding GL_N-module is $\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$ with the diagonal (standard) action of GL_N where $g \cdot (u \otimes v \otimes w) = gu \otimes gv \otimes gw$.

In each case, H. Hahn considers the (irreducible) G-module $\mathbb{S}_{[\lambda]}(W)$ from the introduction (in type D_n she assumes $\lambda_n = 0$). If $r : \mathsf{G} \to \mathsf{GL}_N$ is the G-representation corresponding to $\mathbb{S}_{[\lambda]}(W)$, then it makes sense to define H as the Zariski closure of $r(\mathsf{G})$ inside GL_N. That is, in the notation of [13], H is the irreducible subgroup of GL_N of interest.

Theorem 1.5 of *ibid.* proves that if $|\lambda|$ is odd then $\rho = \otimes^3$ does not detect $\mathbb{S}_{[\lambda]}(W)$. Conversely, when $|\lambda|$ is even. Theorem 1.6 of *ibid.* gives three infinite subfamilies of Par_n where $\rho = \otimes^3$ detects $\mathbb{S}_{[\lambda]}(W)$.

We give a short proof of a complete converse.

Theorem 4.1. *Let $\lambda \in \mathsf{Par}_n$. Then $\rho = \otimes^3$ detects $\mathbb{S}_{[\lambda]}(W)$ if $|\lambda| \equiv 0 \pmod 2$.*[5]

Proof of Theorem 4.1. Hahn's [13, Proposition 3.1] shows that

$$
(17) \qquad \rho = \otimes^3 \ \text{ detects } \mathbb{S}_{[\lambda]}(W) \text{ if and only if } N_{\lambda,\lambda,\lambda} > 0.
$$

In *ibid.* this is used to prove $(\Rightarrow)$.[6] Therefore, (1) shows

Lemma 4.2. $\rho = \otimes^3$ *detects $\mathbb{S}_{[\lambda]}(W)$ if there exists $\mu \in \mathsf{Par}_n$ such that $c_{\mu,\mu}^\lambda > 0$.*

Claim 4.3. For any $\lambda \in \mathsf{Par}_n$ with $|\lambda| = 2m$, there exists $\mu \in \mathsf{Par}_n$ such that $c_{\mu,\mu}^\lambda > 0$.

Proof of Claim 4.3. Since $|\lambda|$ is even, there are an even number of odd parts in λ. Let

$$
\lambda_{i_1} \geq \ldots \geq \lambda_{i_{2k}}
$$

be the odd parts of λ.

Define $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ to be a partition of m, where

$$
\mu_j = \begin{cases} \frac{\lambda_j}{2} & \lambda_j \text{ is even} \\ \frac{\lambda_j + 1}{2} & \lambda_j \text{ is odd and } j \leq i_k \\ \frac{\lambda_j - 1}{2} & \lambda_j \text{ is odd and } j > i_k \end{cases}
$$

We show $c_{\mu,\mu}^\lambda > 0$ by giving an explicit ballot filling of λ/μ with content μ (see Section 2.1).

For λ_i even, fill in the rightmost $\frac{\lambda_i}{2}$ boxes with i. For a row i_j of λ with an odd number of boxes, fill in the rightmost $\frac{\lambda_{i_j} - 1}{2}$ boxes in the row with i_j. There are $\frac{\lambda_{i_j} - 1}{2}$ boxes in each of the top k rows with odd parts. Hence those boxes are entirely filled. There are $\frac{\lambda_{i_j} + 1}{2}$ boxes in each of the bottom k rows of odd parts. For these rows, one box remains unfilled by the above step. Fill in the empty box in row i_{k+j} with i_j; for the purposes of discussion below, we will call this box *extraordinary*. It will also be convenient to call indices j λ-*even* if λ_j is even, λ-*top-odd* if λ_j is odd

---

[5]One might compare this parity characterization to [13, Theorem 1.5] which shows that $G :=$ $\mathrm{Sym}^{n-1}(\mathsf{SL}_2) \hookrightarrow \mathsf{GL}_n$ is detected by $\rho := \mathrm{Sym}^3$ if and only if $n \equiv 1 \pmod 4$.

[6]This follows from Proposition 2.2(IV), which just extends the argument made in [13].

and $j \leq i_k$, and λ-*bottom-odd* otherwise. Let T be this filling. (See Example 4.4 below.) We must check three things:

(T is semistandard): By construction, T is row-semistandard. It remains to show column strictness. This is clear when comparing adjacent rows j and $j+1$ that are either λ-even, or λ-top-odd, since those only use those labels in their respective rows. If either row is bottom-odd, notice that any extraordinary box is either directly beneath an empty square or another extraordinary box. Since extraordinary boxes are labeled in strictly increasing from top to bottom, we are done.

(T has content μ): If j is λ-even, then $\mu_j = \frac{\lambda_j}{2}$ and there are that many j's in row j of T (and nowhere else). Otherwise, if j is λ-top-odd then we are deficient one label of j in that row. By construction, this missing j appears in row i_{k+j}.

(T is ballot): If j is λ-even, the ballotness holds since all j's appear in row j and all $j+1$'s appear in the row $j+1$ or further south, and since $\mu_j \geq \mu_{j+1}$. Next, suppose $j+1$ (but not j) is λ-even. Hence $\lambda_{j+1} < \lambda_j$ and row j of T will contain $\frac{\lambda_j - 1}{2} \geq \frac{\lambda_{j+1}}{2}$ many j's; these j's will be read before the $\frac{\lambda_{j+1}}{2}$-many $j+1$'s of T, which appear only in row $j+1$. Similarly, we are done if j and $j+1$ are both λ-bottom-odd, or (since extraordinary boxes' labels increase top-down) if both are λ-top-odd. Finally, say j is λ-top-odd, $j+1$ is λ-bottom-odd. Then row j of T has $\frac{\lambda_{j+1} - 1}{2}$ many j's and all $\frac{\lambda_{j+1} - 1}{2}(\leq \frac{\lambda_j - 1}{2})$ many $j+1$'s appear in row $j+1$ of T, so ballotness follows. $\qquad\square$

In view of Lemma 4.2, Claim 4.3 completes the proof of the theorem. $\qquad\square$

Example 4.4. To illustrate the proof of Claim 4.3, let $\lambda = (14, 11, 10, 8, 8, 7, 6, 6, 5, 5, 4, 3, 2, 1)$. Hence $2k = 6$, $(i_1, i_2, i_3, i_4, i_5, i_6) = (2, 6, 9, 10, 12, 14)$, and $\mu = (7, 6, 5, 4, 4, 4, 3, 3, 3, 2, 2, 1, 1, 0)$. In this case, T is

where we have boldfaced the labels in the exceptional boxes. $\qquad\square$

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ let $k\lambda = (k\lambda_1, k\lambda_2, \ldots)$. Theorem 4.1 combined with (17) implies:

Corollary 4.5. *If $|\lambda| \equiv 0 \pmod 2$ then $N_{\lambda,\lambda,\lambda} > 0 \iff N_{k\lambda,k\lambda,k\lambda} > 0$ for all $k \in \mathbb{Z}_{\geq 1}$.*

The simplicity of this "saturation" statement suggested the ideas of the next section.

5. Polytopal results

5.1. Newell-Littlewood polytopes. Fix $\lambda, \mu, \nu \in \mathsf{Par}_n$. Let $a_i^j, b_i^j, \gamma_i^j \in \mathbb{R}$ for $1 \le i, j \le n$ and consider the linear constraints:

(1) *Non-negativity*: For all $1 \le i, j \le n$, $\alpha_i^j, \beta_i^j, \gamma_i^j \ge 0$
(2) *Shape constraints*: For all k,
  (a) $\sum_j \alpha_k^j + \sum_i \beta_i^k = \mu_k$
  (b) $\sum_j \gamma_k^j + \sum_i \alpha_i^k = \nu_k$
  (c) $\sum_j \beta_k^j + \sum_i \gamma_i^k = \lambda_k$
(3) *Tableau/semistandardness constraints*: For all k, l:
  (a) $\sum_j \alpha_{k+1}^j + \sum_{i \le l} \beta_i^{k+1} \le \sum_j \alpha_k^j + \sum_{i < l} \beta_i^k$
  (b) $\sum_j \gamma_{k+1}^j + \sum_{i \le l} \alpha_i^{k+1} \le \sum_j \gamma_k^j + \sum_{i < l} \alpha_i^k$
  (c) $\sum_j \beta_{k+1}^j + \sum_{i \le l} \gamma_i^{k+1} \le \sum_j \beta_k^j + \sum_{i < l} \gamma_i^k$
(4) *Ballot constraints*: For all k, l:
  (a) $\sum_{i < k} \alpha_l^i \ge \sum_{i \le k} \alpha_{l+1}^i$
  (b) $\sum_{i < k} \beta_l^i \ge \sum_{i \le k} \beta_{l+1}^i$
  (c) $\sum_{i < k} \gamma_l^i \ge \sum_{i \le k} \gamma_{l+1}^i$

We define the *Newell-Littlewood polytope* in $\mathbb{R}^{3n^2}$ by

$$\mathcal{P}_{\mu,\nu,\lambda} = \{(\alpha_i^j, \beta_i^j, \gamma_i^j) \in \mathbb{R}^{3n^2} : (1)\text{-}(4) \text{ hold}\}.$$

Theorem 5.1. $N_{\mu,\nu,\lambda} = \#(\mathcal{P}_{\mu,\nu,\lambda} \cap \mathbb{Z}^{3n^2})$.

Proof. By definition, $N_{\mu,\nu,\lambda}$ is the number of LR tableaux T, U and V of shape μ/α, ν/γ and λ/β respectively, and of content β, α, and γ respectively for any choice of α, β, and γ in Par_n. Given such a triple (T, U, V) let β_i^j be the number of i's in the jth row of the ballot filling of T. Similarly, α_i^j and γ_i^j are defined with respect to U and V respectively. It is straightforward that $(\alpha_i^j, \beta_i^j, \gamma_i^j)$ satisfies (1)-(4).

Conversely, suppose we are given $(\alpha_i^j, \beta_i^j, \gamma_i^j) \in \mathcal{P}_{\mu,\nu,\lambda}$. For $1 \le i \le n$, let

$$\alpha_i := \sum_j \alpha_i^j, \ \beta_i := \sum_j \beta_i^j, \text{ and } \gamma_i := \sum_j \gamma_i^j.$$

Notice $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathsf{Par}_n$ by 4(a). Similarly we define $\beta, \gamma \in \mathsf{Par}_n$. Now construct T by placing β_i^j many i's in row j (indented by α_i many boxes), and order the labels in the row to be in increasing from left to right. By 2(a), T is of skew shape μ/α. Conditions 3(a) and 4(b) guarantee that T is an LR tableau. In the same way, we construct appropriate LR tableaux U and V using α_i^j, γ_i^j and β. This correspondence $(T, U, V) \leftrightarrow (\alpha_i^j, \beta_i^j, \gamma_i^j)$ is clearly bijective. $\square$

Example 5.2. To illustrate the correspondence in the proof of Theorem 5.1, let $\mu = (2, 1)$, $\nu = (2, 2)$, and $\lambda = (3, 2)$. Let us write α_i^j, β_i^j and γ_i^j in terms of matrices $[\alpha], [\beta]$ and $[\gamma]$ so that $[\alpha]_{i,j} = \alpha_i^j$, $[\beta]_{i,j} = \beta_i^j$ and $[\gamma]_{i,j} = \gamma_i^j$. Then $\mathcal{P}_{\mu,\nu,\lambda} \cap \mathbb{Z}^{12}$ would be the two triples

$$([\alpha], [\beta], [\gamma]) = \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \text{ or } \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

implying $N_{\mu,\nu,\lambda} = 2$. The first triple corresponds to the triple of LR tableaux contributing, respectively, to $c_{\alpha,\beta}^{\mu}, c_{\gamma,\alpha}^{\nu}$ and $c_{\beta,\gamma}^{\lambda}$ where $\alpha = (1), \beta = (2), \gamma = (2,1)$:

$$\begin{array}{c}\begin{array}{|c|c|}\hline & 1 \\\hline 1 & \\\hline\end{array}\end{array}, \quad \begin{array}{|c|c|}\hline & \\\hline & 1 \\\hline\end{array}, \quad \begin{array}{|c|c|c|}\hline & & 1 \\\hline 1 & 2 & \\\hline\end{array}.$$

Similarly, the second triple corresponds to these of LR tableaux

$$\begin{array}{|c|c|}\hline & 1 \\\hline 2 & \\\hline\end{array}, \quad \begin{array}{|c|c|}\hline & \\\hline & 1 \\\hline\end{array}, \quad \begin{array}{|c|c|c|}\hline & 1 & 1 \\\hline & 2 & \\\hline\end{array},$$

which contribute, respectively, to $c_{\alpha,\beta}^{\mu}, c_{\gamma,\alpha}^{\nu}$ and $c_{\beta,\gamma}^{\lambda}$ where $\alpha = (1), \beta = (1,1), \gamma = (2,1)$. $\qquad \square$

That $N_{\lambda,\mu,\nu}$ counts lattice points in a polytope also follows from work of A. Berenstein–A. Zelevinsky [4, Section 2.2] on the more general tensor product multiplicities, together with [24, Corollary 2.5.3]. Their polytopes are described in terms of root-system datum. The above gives an *ab initio* approach, similar to one seen in a preprint version of [33] for the Littlewood-Richardson coefficients.

5.2. **Newell-Littlewood semigroups.** The *Littlewood-Richardson semigroup* is

$$\mathsf{LR}_n = \{(\mu, \nu, \lambda) \in \mathsf{Par}_n^3 : c_{\mu,\nu}^{\lambda} > 0\};$$

see, e.g., [48]. We define the *Newell-Littlewood semigroup* by

$$\mathsf{NL}_n = \{(\mu, \nu, \lambda) \in \mathsf{Par}_n^3 : N_{\mu,\nu,\lambda} > 0\}.$$

Corollary 5.3. NL_n *is a semigroup.* LR_n *is a subsemigroup of* NL_n.

Proof. Suppose (μ, ν, λ) and $(\overline{\mu}, \overline{\nu}, \overline{\lambda}) \in \mathsf{NL}_n$. By Theorem 5.1, there exist lattice points

$$(\alpha_i^j, \beta_i^j, \gamma_i^j) \in \mathcal{P}_{\mu,\nu,\lambda} \quad \text{and} \quad (\overline{\alpha}_i^j, \overline{\beta}_i^j, \overline{\gamma}_i^j) \in \mathcal{P}_{\overline{\mu},\overline{\nu},\overline{\lambda}}.$$

Observe

$$(\alpha_i^j, \beta_i^j, \gamma_i^j) + (\overline{\alpha}_i^j, \overline{\beta}_i^j, \overline{\gamma}_i^j) \in \mathcal{P}_{\mu+\overline{\mu},\nu+\overline{\nu},\lambda+\overline{\lambda};n}$$

is a lattice point. By Theorem 5.1, $N_{\mu+\overline{\mu},\nu+\overline{\nu},\lambda+\overline{\lambda}} > 0$ and so $(\mu+\overline{\mu}, \nu+\overline{\nu}, \lambda+\overline{\lambda}) \in \mathsf{NL}_n$. Hence NL_n is a semigroup.

The remaining assertion follows from Lemma 2.2(II). $\qquad \square$

In turn, Corollary 5.3 immediately implies

Corollary 5.4. *If* $N_{\mu,\nu,\lambda} > 0$ *then* $N_{k\mu,k\nu,k\lambda} > 0$ *for every* $k \geq 1$.

A. Knutson-T. Tao [22] established the *saturation property* of $c_{\mu,\nu}^{\lambda}$. That is

$$(18) \qquad\qquad c_{\mu,\nu}^{\lambda} > 0 \iff c_{k\mu,k\nu}^{k\lambda} > 0, \quad \forall k \in \mathbb{Z}_{\geq 1}.$$

Conjecture 5.5 (Newell-Littlewood Saturation I). *Suppose* $\lambda, \mu, \nu \in \mathsf{Par}$ *such that* $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$. *If* $N_{k\mu,k\nu,k\lambda} > 0$ *for some* $k \geq 1$ *then* $N_{\mu,\nu,\lambda} > 0$.

We checked Conjecture 5.5 exhaustively for λ, μ, ν with $1 \leq |\lambda|, |\mu|, |\nu| \leq 8$ and $k = 2, 3$ as well as many other examples. The necessity of the parity hypothesis is Lemma 2.2(V).

This is an *a priori* stronger version of Conjecture 5.5:

Conjecture 5.6 (Newell-Littlewood Saturation II). *Under the hypotheses of Conjecture 5.5, if* $N_{k\mu,k\nu,k\lambda} > 0$ *then there exists* $\alpha, \beta, \gamma \in \mathsf{Par}$ *such that* $c_{k\alpha,k\beta}^{k\mu} c_{k\alpha,k\gamma}^{k\nu} c_{k\beta,k\gamma}^{k\lambda} > 0$.

Proposition 5.7. *Conjectures* 5.5 *and* 5.6 *are equivalent.*

Proof. ($\Rightarrow$) Suppose $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$ and $N_{k\mu,k\nu,k\lambda} > 0$. By Conjecture 5.5, $N_{\mu,\nu,\lambda} > 0$. Hence by (1) there exists α, β, γ such that $c^{\mu}_{\alpha,\beta}, c^{\nu}_{\alpha,\gamma}, c^{\lambda}_{\beta,\gamma}$ are all nonzero. By the semigroup property for Littlewood-Richardson coefficients (Corollary 5.3), $c^{k\mu}_{k\alpha,k\beta}, c^{k\nu}_{k\alpha,k\gamma}, c^{k\lambda}_{k\beta,k\gamma}$ are also nonzero, as asserted by Conjecture 5.6.

($\Leftarrow$) This holds by (1) and saturation of the Littlewood-Richardson coefficients (18). $\qquad\square$

There has been significant interest in the saturation problem for tensor products of irreducibles for complex semisimple algebraic groups. Suppose μ, ν, λ are dominant weights corresponding to the irreducible modules V_μ, V_ν and V_λ. Let

$$V_\mu \otimes V_\nu = \bigoplus_\lambda V_\lambda^{\oplus m^\lambda_{\mu,\nu}}.$$

The aforementioned problem is, if we assume $\mu + \nu - w_0\lambda$ is in the root lattice, is

$$m^\lambda_{\mu,\nu} \neq 0 \iff m^{k\lambda}_{k\mu,k\nu} \neq 0, \quad \forall k \geq 1?$$

In type A, $m^\lambda_{\mu,\nu}$ is a Littlewood-Richardson coefficient, and (18) provides an affirmative answer. The answer is negative for types B and C, and is conjectured to be true for all simply-laced types, and in particular, type D. The state of the art is that the type D conjecture is proved for type D_4 by M. Kapovich-S. Kumar-J. J. Milson [15] and more recently by J. Kiers for D_5, D_6 [16] (which we refer to for more references).

Conjecture 5.5 suggests that saturation should hold in types B and C at least in the stable range and under the parity hypothesis. In view of [24, Theorem 2.3.4], the D_n conjecture should imply Conjecture 5.5 (taking into account the parity vs root-lattice hypotheses); we thank J. Kiers for pointing this out (private communication). We emphasize that Conjecture 5.6 permits a different approach than [15, 16] for the cases at hand. For example, in addition to the infinite family of cases provided by Corollary 4.5, we have:

Theorem 5.8. *Conjecture* 5.6 *is true if one of* λ, μ, ν *is a single row or a single column.*

Proof. Suppose one of λ, μ, ν is a single column. By Lemma 2.2(I), we may suppose $\mu = (1^t)$. By assumption, there exists α, β, γ such that $c^{(k^t)}_{\alpha,\beta}, c^{k\nu}_{\alpha,\gamma}, c^{k\lambda}_{\beta,\gamma} > 0$. For convenience, let $[\lambda/\mu]_i$ be the number of boxes of the i-th row of the skew shape λ/μ.

Lemma 5.9. *If* $c^\lambda_{\mu,\nu} > 0$, *then* $[\lambda/\mu]_i \leq \nu_1$ *for all* i.

Proof of Lemma 5.9. Since $c^{\lambda'}_{\mu',\nu'} = c^\lambda_{\mu,\nu} > 0$, there is an LR tableau T of λ'/μ' of content ν'. The labels of boxes in a given column C of T are distinct. Hence $\#C \leq \ell(\nu')$ and the lemma follows. $\qquad\square$

The fact $c^{(k^t)}_{\alpha,\beta} > 0$ implies that $\alpha, \beta \subseteq (k^t)$ and hence $\alpha_1, \beta_1 \leq k$. So by Lemma 5.9,

$$[(k\lambda)/\gamma]_i, [(k\nu)/\gamma]_i \leq k, \forall i. \tag{19}$$

Since $\gamma \subseteq k\nu \wedge k\lambda$, by (19), for all i:

$$[(k\lambda)/(k\nu \wedge k\lambda)]_i \leq [(k\lambda)/\gamma]_i \leq k, \quad \text{and} \quad [(k\nu)/(k\nu \wedge k\lambda)]_i \leq [(k\lambda)/\gamma]_i \leq k. \tag{20}$$

Also, (20) and $k\nu \wedge k\lambda = k(\nu \wedge \lambda)$ combined imply

$$[\lambda/(\nu \wedge \lambda)]_i, [\nu/(\nu \wedge \lambda)]_i \leq 1, \forall i;$$

that is,

(21) $|\nu_i - \lambda_i| \leq 1.$

By Theorem 3.1 (I),

$$k|\nu \Delta \lambda| = |k\nu \Delta k\lambda| \leq |(k^t)| = kt,$$

and so $|\nu \Delta \lambda| \leq t$. Since $|\nu \Delta \lambda| \equiv |\nu| + |\lambda| \pmod 2$ and (by hypothesis)

$$|\nu| + |\lambda| + |(1^t)| = |\nu| + |\lambda| + t \equiv 0 \ (\bmod\ 2),$$

we have that $\frac{t - |\nu \Delta \lambda|}{2} \in \mathbb{Z}_{\geq 0}$.

Claim 5.10. There are at least $\frac{t - |\nu \Delta \lambda|}{2}$ indices i such that $\nu_i = \lambda_i > 0$.

Proof of Claim 5.10. By definition of α, β, and γ,

$$\begin{aligned}
kt &= |\alpha| + |\beta| \\
&= |(k\nu)/\gamma| + |(k\lambda)/\gamma| \\
&= |(k\nu)/(k\nu \wedge k\lambda)| + |(k\nu \wedge k\lambda)/\gamma| + |(k\lambda)/(k\nu \wedge k\lambda)| + |(k\nu \wedge k\lambda)/\gamma| \\
&= |k\nu \Delta k\lambda| + 2|(k\nu \wedge k\lambda)/\gamma|.
\end{aligned}$$

This is equivalent to

(22) $k\left(\dfrac{t - |\nu \Delta \lambda|}{2}\right) = |(k\nu \wedge k\lambda)/\gamma|.$

By (19),

$$[(k\nu \wedge k\lambda)/\gamma]_i \leq [(k\nu)/\gamma]_i \leq k, \forall i.$$

Thus (22) and the Pigeonhole Principle shows

(23) $\#\{i : [(k\nu \wedge k\lambda)/\gamma]_i > 0\} \geq \dfrac{t - |\nu \Delta \lambda|}{2}.$

By (21), if $\nu_j \neq \lambda_j$ then $[k\nu \Delta k\lambda]_j = k$. By (19), $k\nu_j - \gamma_j, k\lambda_j - \gamma_j \leq k$. Hence

(24) $k \geq \max\{k\nu_j, k\lambda_j\} - \gamma_j = (\max\{k\nu_j, k\lambda_j\} - \min\{k\nu_j, k\lambda_j\})$
$$+ (\min\{k\nu_j, k\lambda_j\} - \gamma_j) = k + (\min\{k\nu_j, k\lambda_j\} - \gamma_j).$$

Therefore $\min\{k\nu_j, k\lambda_j\} - \gamma_j = 0$. That is,

$$[(k\nu \wedge k\lambda)/\gamma]_j = 0.$$

As a result, $[(k\nu \wedge k\lambda)/\gamma]_i > 0$ only if $\nu_i = \lambda_i > 0$. Hence by (23) there are at least $\frac{t - |\nu \Delta \lambda|}{2}$ many i with $\nu_i = \lambda_i > 0$.   □

By Claim 5.10, we may define $\overline{\gamma}$ to be $\nu \wedge \lambda$ with one box removed from the southmost $\frac{t - |\nu \Delta \lambda|}{2}$ rows i such that $\nu_i = \lambda_i > 0$. It follows from (21) that $\nu/\overline{\gamma}$ and

$\lambda/\overline{\gamma}$ are vertical strips. Now, since $|\nu| + |\lambda| = 2|\nu \wedge \lambda| + |\nu \Delta \lambda|$,

$$
\begin{aligned}
|\nu/\overline{\gamma}| &= |\nu| - |\nu \wedge \lambda| + \frac{t - |\nu \Delta \lambda|}{2} \\
&= \frac{2|\nu| - 2|\nu \wedge \lambda| + t - |\nu \Delta \lambda|}{2} \\
&= \frac{|\nu| - |\lambda| + t + (|\nu| + |\lambda| - 2|\nu \wedge \lambda| - |\nu \Delta \lambda|)}{2} \\
&= \frac{|\nu| - |\lambda| + t}{2}.
\end{aligned}
$$

Similarly, $|\lambda/\overline{\gamma}| = \frac{|\lambda| - |\nu| + t}{2}$. Therefore, the (column version) of the classical Pieri rule (9) shows that

$$
((1^{(t+|\nu|-|\lambda|)/2}), (1^{(t+|\lambda|-|\nu|)/2}), \overline{\gamma})
$$

is a witness for $N_{(1^t),\nu,\lambda} > 0$.

The proof where one of μ, ν, λ is a single row is similar to the above argument, except simpler. Therefore we only sketch the necessary changes and leave the details to the reader. By Proposition 2.4, we have $|\nu_i' - \lambda_i'| \leq 1$; this is the analogue of (21). By the same reasoning, $\frac{t - |\nu \Delta \lambda|}{2} \in \mathbb{Z}_{\geq 0}$. The column version of Claim 5.10 states that there are at least $\frac{t - |\nu \Delta \lambda|}{2}$ indices i such that $\nu_i' = \lambda_i' > 0$; it is proved using a different Pigeonhole argument. Given this claim, one defines $\widehat{\gamma}$ be removing a single box from the eastmost $\frac{t - |\nu \Delta \lambda|}{2}$ columns such that $\nu_i' = \lambda_i'$. Then one concludes in the same way.   □

5.3. Horn and (extended) Weyl inequalities. Let $[n] := \{1, 2, \ldots n\}$. For any

$$
I = \{i_1 < i_2 < \cdots < i_d\} \subseteq [n]
$$

define the partition

$$
\tau(I) := (i_d - d \geq \cdots \geq i_2 - 2 \geq i_1 - 1).
$$

This bijects subsets of $[n]$ of cardinality d with partitions whose Young diagrams are contained in a $d \times (n - d)$ rectangle. The following combines the main results of A. Klyachko [21] and A. Knutson–T. Tao [22].

Theorem 5.11 ([21], [22]). *Let* $\lambda, \mu, \nu \in \mathsf{Par}_n$ *such that* $|\lambda| + |\mu| = |\nu|$. *Then* $c_{\mu,\nu}^\lambda > 0$ *if and only if for every* $d < n$, *and every triple of subsets* $I, J, K \subseteq [n]$ *of cardinality* d *such that* $c_{\tau(I),\tau(J)}^{\tau(K)} > 0$,

$$
\tag{25} \sum_{k \in K} \lambda_k \leq \sum_{i \in I} \mu_i + \sum_{j \in J} \nu_j.
$$

The inequalities (25) are the *Horn inequalities* [14].

Proposition 5.12. *Let* $\mu, \nu, \lambda \in \mathsf{Par}_n$ *such that* $N_{\mu,\nu,\lambda} > 0$. *Then the Horn inequalities* (25) *hold.*

Proof. Since $N_{\mu,\nu,\lambda} > 0$, there exists α, β, γ such that $c_{\alpha,\beta}^\mu, c_{\alpha,\gamma}^\nu, c_{\beta,\gamma}^\lambda > 0$.

By Theorem 5.11, (μ, α, β) satisfies the Horn inequalities (25). Consider an arbitrary Horn inequality associated to a triple of subsets (I, J, K) as in Theorem 5.11.

$$
\sum_{k \in K} \lambda_k \leq \sum_{i \in I} \beta_i + \sum_{j \in J} \gamma_j.
$$

Since $c_{\alpha,\gamma}^{\nu} > 0$, $\gamma \subseteq \nu$ and so in particular $\gamma_j \le \nu_j$ for all j, and similarly $\beta_i \le \mu_i$, so

$$\sum_{k \in K} \lambda_k \le \sum_{i \in I} \mu_i + \sum_{j \in J} \nu_j.$$

Hence (μ, ν, λ) satisfies (25), as desired. □

Among the Horn inequalities are the *Weyl's inequalities* [46]. The latter inequalities state that a necessary condition for $c_{\mu,\nu}^{\lambda} > 0$ is

(26) $$\lambda_{i+j-1} \le \nu_i + \mu_j \quad \text{for } i+j-1 \le n;$$

we refer to [5] and the references therein for an expository account. When $n = 2$, the Horn inequalities (25) and Weyl inequalities (26) coincide:

(27) $$\lambda_1 \le \mu_1 + \nu_1, \ \lambda_2 \le \mu_1 + \nu_2, \ \lambda_2 \le \mu_2 + \nu_1.$$

Theorem 5.11 has been extended in a number of ways. For a recent example, see work of N. Ressayre [38], who gave inequalities valid whenever the *Kronecker coefficient* $g_{\mu,\nu,\lambda} > 0$.

Theorem 5.13 (Extended Weyl inequalities)**.** *Let* $\mu, \nu, \lambda \in \mathsf{Par}_n$ *and* $1 \le k \le i < j \le l \le n$, *let* $m = \min(i-k, l-j)$ *and* $M = \max(i-k, l-j)$. *If* $N_{\mu,\nu,\lambda} > 0$ *then*

(28) $$\mu_i - \mu_j \le \lambda_k - \lambda_l + \nu_{m-p+1} + \nu_{M+p+2} \quad \text{where } 0 \le p \le m.$$

Proof. Since $N_{\mu,\nu,\lambda} > 0$, there exists α, β, γ such that $c_{\alpha,\beta}^{\mu}, c_{\alpha,\gamma}^{\nu}, c_{\beta,\gamma}^{\lambda} > 0$. By Theorem 5.11, (μ, α, β), (ν, α, γ), (λ, β, γ) all satisfy the Horn inequalities. Therefore, by Weyl's inequalities (26), we have that

(29) $$\mu_i \le \alpha_{i-k+1} + \beta_k \quad \text{and} \quad \lambda_l \le \beta_j + \gamma_{l+1-j}.$$

Additionally,

$$c_{\tau([n]\setminus\{j\}),\tau([n-1])}^{\tau([n]\setminus\{j\})} = c_{(1^{n-j}),(0)}^{(1^{n-j})} = 1,$$

so by Theorem 5.11 applied to $c_{\alpha,\beta}^{\mu} > 0$,

(30) $$\sum_{a \neq j} \mu_a \le \sum_{b \neq n} \alpha_b + \sum_{c \neq j} \beta_c.$$

Subtracting (30) from

$$\sum_a \mu_a = \sum_b \alpha_b + \sum_c \beta_c,$$

gives

(31) $$\mu_j \ge \alpha_n + \beta_j.$$

By the same logic,

(32) $$\lambda_k \ge \beta_k + \gamma_n.$$

Also, by treating α, γ, and ν as partitions of $n+1$ rows with $\alpha_{n+1} = \gamma_{n+1} = \nu_{n+1} = 0$, we have that

$$c_{\tau([n+1]\setminus\{i-k+1,n+1\}),\tau([n+1]\setminus\{l-j+1,n+1\})}^{\tau([n+1]\setminus\{m-p+1,M+p+2\})} = c_{(1^{n-1-(i-k)}),(1^{n-1-(l-j)})}^{(2^{n-1-M-p})\cup(1^{2p+M-m})} = 1.$$

Thus, Theorem 5.11 applied to $c_{\alpha,\gamma}^{\nu} > 0$ gives

(33) $$\sum_{a \notin \{m-p+1,M+p+2\}} \nu_a \le \sum_{b \notin \{i-k+1,n+1\}} \alpha_b + \sum_{c \notin \{l-j+1,n+1\}} \gamma_c.$$

Subtracting (33) from

$$\sum_a \nu_a = \sum_b \alpha_b + \sum_c \gamma_c$$

gives

(34) $\quad \alpha_{i-k+1} + \gamma_{l-j+1} = \alpha_{i-k+1} + \alpha_{n+1} + \gamma_{l-j+1} + \gamma_{n+1} \leq \nu_{m-p+1} + \nu_{M+p+2}$

Therefore, combining (29), (31) and (32) gives the first inequality below:

$$\begin{aligned}
\mu_i - \mu_j + \lambda_l - \lambda_k &\leq (\alpha_{i-k+1} + \beta_k) - (\alpha_n + \beta_j) + (\beta_j + \gamma_{l+1-j}) - (\beta_k + \gamma_n) \\
&= \alpha_{i-k+1} - \alpha_n + \gamma_{l+1-j} - \gamma_n \\
&\leq \alpha_{i-k+1} + \gamma_{l+1-j} \\
&\leq \nu_{m-p+1} + \nu_{M+p+2},
\end{aligned}$$

where we have just applied (34). This completes the derivation of (28). $\qquad \square$

Corollary 5.14. *The inequalities* (25) *and* (28), *where the roles of* (μ, ν, λ) *are interchanged under all* $\mathfrak{S}_3$-*permutations, also hold whenever* $N_{\mu,\nu,\lambda} > 0$.

Proof. Combine Lemma 2.2(I) with Proposition 5.12 and Theorem 5.13. $\qquad \square$

Just as the Weyl inequalities are necessary and sufficient to characterize LR_2, we now show that the (extended) Weyl inequalities (together with symmetries given by Corollary 5.14) are necessary and sufficient to describe NL_2.

Theorem 5.15. *Suppose* $\lambda, \mu, \nu \in \mathsf{Par}_2$ *satisfies* $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$ *and the triangle inequalities. Then* $(\mu, \nu, \lambda) \in \mathsf{NL}_2$ *if and only if this list of linear inequalities holds:*

(35) $\qquad\qquad \lambda_1 \leq \mu_1 + \nu_1, \ \nu_1 \leq \lambda_1 + \mu_1, \ \mu_1 \leq \lambda_1 + \nu_1$

(36) $\qquad\qquad \lambda_2 \leq \mu_1 + \nu_2, \ \nu_2 \leq \lambda_1 + \mu_2, \ \mu_2 \leq \lambda_1 + \nu_2$

(37) $\qquad\qquad \lambda_2 \leq \mu_2 + \nu_1, \ \nu_2 \leq \lambda_2 + \mu_1, \ \mu_2 \leq \lambda_2 + \nu_1$

(38)
$\nu_1 - \nu_2 \leq \mu_1 + \mu_2 + \lambda_1 - \lambda_2, \ \mu_1 - \mu_2 \leq \lambda_1 + \lambda_2 + \nu_1 - \nu_2, \ \lambda_1 - \lambda_2 \leq \nu_1 + \nu_2 + \mu_1 - \mu_2$
$\lambda_1 - \lambda_2 \leq \mu_1 + \mu_2 + \nu_1 - \nu_2, \ \mu_1 - \mu_2 \leq \nu_1 + \nu_2 + \lambda_1 - \lambda_2, \ \nu_1 - \nu_2 \leq \lambda_1 + \lambda_2 + \mu_1 - \mu_2.$

Above, (35), (36), (37) are the $n = 2$ Horn/Weyl inequalities (27) and their symmetric analogues. (38) represents (up to symmetry) the unique inequality of the form (28) for this case.

Theorem 5.15 implies another case of Conjectures 5.5 and 5.6:

Corollary 5.16. *Conjectures* 5.5 *and* 5.6 *hold when* $n = 2$.

Proof. Suppose that $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$ and $N_{k\mu, k\nu, k\lambda} > 0$. By Theorem 5.15, $(k\mu, k\nu, k\lambda)$ satisfies (35), (36), (37) and (38) after the substitution

$$\mu \mapsto k\mu, \nu \mapsto k\nu, \lambda \mapsto k\lambda.$$

These inequalities are homogeneous in λ_i, μ_i, ν_i. Hence (μ, ν, λ) satisfies (35), (36), (37) and (38). Therefore by the "$\Leftarrow$" direction of Theorem 5.15, $N_{\mu,\nu,\lambda} > 0$, as required. $\qquad \square$

The classical Weyl inequalities do not characterize LR_3. Analogously, the extended Weyl inequalities (combined with Proposition 5.12 and Corollary 5.14) are not sufficient to characterize NL_3. An example is $\mu = (6, 0, 0), \nu = (4, 2, 2)$ and $\lambda = (4, 4, 0)$. However, we have an additional list of inequalities that should close the gap in this case. We plan to address this issue (and more) in a sequel. For now, we restrict to proving Theorem 5.15, to illustrate a general strategy.

Proof of Theorem 5.15. The "$\Rightarrow$" direction is by Proposition 5.12, Theorem 5.13, and Corollary 5.14. To prove the converse, let $(\lambda, \mu, \nu) \in \mathsf{Par}_2$ be such that $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$ and $N_{\mu,\nu,\lambda} = 0$. We now show that either one of the triangle inequalities, or an inequality from (35)-(38), is violated.

Claim 5.17. If $|\lambda| < |\mu \Delta \nu|$, either a triangle inequality or an inequality from (38) is violated.

Proof of Claim 5.17. By Lemma 2.2(I), we may assume without loss that $\nu_1 \geq \mu_1$. If $\nu_2 \geq \mu_2$, then $|\mu \Delta \nu| = |\nu| - |\mu|$. Combining this with the hypothesis $|\lambda| < |\mu \Delta \nu|$ we obtain a failure of the triangle inequality $|\lambda| + |\mu| \geq |\nu|$. If $\nu_2 < \mu_2$, then

$$|\mu \Delta \nu| = \nu_1 - \mu_1 + \mu_2 - \nu_2.$$

Now, $|\lambda| < |\mu \Delta \nu|$ implies that

$$\nu_1 - \nu_2 > \lambda_1 + \lambda_2 + \mu_1 - \mu_2$$

which violates the sixth equation of (38). $\qquad\square$

By Claim 5.17, we may henceforth assume that

$$\tag{39} |\mu \Delta \nu| \leq |\lambda| \leq |\mu| + |\nu|.$$

Let

$$\tag{40} k = \frac{|\mu| + |\nu| - |\lambda|}{2} \geq 0;$$

$k \in \mathbb{Z}$ by the hypothesis that $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$. For future use, we record this rewriting of (40):

$$\tag{41} \lambda_1 + \lambda_2 = \mu_1 + \mu_2 + \nu_1 + \nu_2 - 2k.$$

A pair $(\mu^{\downarrow k}, \nu^{\downarrow k}) \in \mathsf{Par}_2$ is *valid* if there exists $\alpha \in \mathsf{Par}_2$ with $|\alpha| = k$ such that $c^{\mu}_{\alpha, \mu^{\downarrow k}} > 0$ and $c^{\nu}_{\alpha, \nu^{\downarrow k}} > 0$ (equivalently, $\mu^{\downarrow k} \subset \mu$, $\nu^{\downarrow k} \subset \nu$ with $|\mu/\mu^{\downarrow k}| = |\nu/\nu^{\downarrow k}| = k$, and the two skew shapes $\mu/\mu^{\downarrow k}$ and $\nu/\nu^{\downarrow k}$ each have an LR tableau of the same content α).

Claim 5.18. A valid pair $(\mu^{\downarrow k}, \nu^{\downarrow k})$ exists. Moreover,

$$\tag{42} k \leq |\mu \wedge \nu| = \min(\mu_1, \nu_1) + \min(\mu_2, \nu_2).$$

Proof of Claim 5.18. By (39), $|\lambda| \geq |\mu \Delta \nu|$. Thus existence follows from Theorem 3.1(I) combined with (1). (42) holds since $|\mu \wedge \nu| = \frac{|\mu|+|\nu|-|\mu \Delta \nu|}{2} \geq \frac{|\mu|+|\nu|-|\lambda|}{2} := k$. $\qquad\square$

For $i = 1, 2$, let k_i and l_i be, respectively, the number of boxes in row i of the skew shapes $\mu/\mu^{\downarrow k}$ and $\nu/\nu^{\downarrow k}$.

Claim 5.19. If $(\mu^{\downarrow k}, \nu^{\downarrow k})$ is valid then at least one of the following inequalities holds:

$$(43) \qquad\qquad \lambda_1 > \mu_1 + \nu_1 - k_1 - l_1$$

$$(44) \qquad\qquad \lambda_2 > \mu_1 + \nu_2 - k_1 - l_2$$

$$(45) \qquad\qquad \lambda_2 > \mu_2 + \nu_1 - k_2 - l_1.$$

Proof of Claim 5.19. By (1), $N_{\mu,\nu,\lambda} = 0 \iff c^{\lambda}_{\mu^{\downarrow k}\nu^{\downarrow k}} = 0$ whenever $(\mu^{\downarrow k}, \nu^{\downarrow k})$ is a valid pair. Now the claim holds by the $n = 2$ case of Theorem 5.11 (see (27)). $\square$

Claim 5.20. Suppose $\mu^{\downarrow k} = (\mu_1 - k_1, \mu_2 - k_2), \nu^{\downarrow k} = (\nu_1 - l_1, \nu_2 - l_2)$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$. Then $(\mu^{\downarrow k}, \nu^{\downarrow k})$ is a valid pair of content α if and only if

 (I) $\mu^{\downarrow k}, \nu^{\downarrow k} \in \mathsf{Par}_2$;
 (II) $\alpha \in \mathsf{Par}_2$;
 (III) $k_1, k_2, l_1, l_2 \in \mathbb{Z}_{\geq 0}$;
 (IV) $k_1 + k_2 = l_1 + l_2 = \alpha_1 + \alpha_2 = k$;
 (V) $k_1, k_2 \geq \alpha_2$ and $l_1, l_2 \geq \alpha_2$; and
 (VI) $\alpha_2 + (\mu_1 - \mu_2) \geq k_1$ and $\alpha_2 + (\nu_1 - \nu_2) \geq l_1$.

Proof of Claim 5.20. ($\Leftarrow$) We construct an LR tableau T of shape $\mu/\mu^{\downarrow k}$ of content α. Conditions (I), (III) guarantee this is a skew-shape. Fill the k_1 boxes of the first row of $\mu/\mu^{\downarrow k}$ with 1's. Since by (V), $k_2 \geq \alpha_2$, we can fill the rightmost α_2 boxes of the second row of $\mu/\mu^{\downarrow k}$ with 2's. Then fill the remaining boxes of that row with 1's. T is clearly row semistandard. It is column semistandard because of (VI). It is ballot by (II) and the condition $k_1 \geq \alpha_2$ of (V). Finally the content of T is α by (IV). Thus $c^{\mu}_{\mu^{\downarrow k}, \alpha} > 0$. Similarly, we show $c^{\nu}_{\nu^{\downarrow k}, \alpha} > 0$.

 ($\Rightarrow$) If $(\mu^{\downarrow k}, \nu^{\downarrow k})$ is a valid pair of content α then there exists LR tableaux T, U of shapes $\mu/\mu^{\downarrow k}$ and $\nu/\nu^{\downarrow k}$ (respectively), and of common content α. Now the conditions follow by reversing the reasoning in the above paragraph. $\square$

Claim 5.21. If (43) holds for every valid pair $(\mu^{\downarrow k}, \nu^{\downarrow k})$ then an inequality from (35)-(38) is violated.

Proof of Claim 5.21. By Lemma 2.2(I), we may assume, without loss, that $\mu_2 \geq \nu_2$. In each case below, it is straightforward to verify the conditions (I)-(VI) of Claim 5.20, so this is left mostly to the reader.

Case 1 ($\min(\mu_2, \nu_1, k) = \nu_1$). Consider $\mu^{\downarrow k} = (\mu_1 - (k - \nu_1), \mu_2 - \nu_1)$ and $\nu^{\downarrow k} = (\nu_1 - (k - \nu_2))$. We point out that, here and elsewhere, (42) is relevant to checking Claim 5.20; in this case condition (I). Specifically, $\mu_1^{\downarrow k}, \nu_1^{\downarrow k} \geq 0$ by (42). In addition $\mu_1^{\downarrow k} \geq \mu_2^{\downarrow k}$ since

$$\mu_1 - (k - \nu_1) - (\mu_2 - \nu_1) \geq \mu_1 - \mu_2 + |\nu| - k \geq 0$$

(again by (42)). It follows that $(\mu^{\downarrow k}, \nu^{\downarrow k})$ is a valid pair of content $\alpha = (\nu_1, k - \nu_1)$. In this case we have $k_2 = \nu_1$ and $l_2 = \nu_2$ and thus $k_2 + l_2 = |\nu|$. Now by (41), (43), and Claim 5.20(IV),

$$(46) \qquad\qquad \lambda_2 < \mu_2 + \nu_2 - k_2 - l_2.$$

Hence, $\lambda_2 + \nu_1 < (\mu_2 + \nu_2 - k_2 - l_2) + \nu_1 = \mu_2 + \nu_2 - |\nu| + \nu_1 = \mu_2$. This violates the third inequality of (37).

Case 2a $(\min(\mu_2, \nu_1, k) = k$ and $\nu_2 \geq k)$. $\mu^{\downarrow k} = (\mu_1, \mu_2 - k)$ and $\nu^{\downarrow k} = (\nu_1, \nu_2 - k)$ is a valid pair of content $\alpha = (k)$. Here $k_1 = l_1 = 0$. Hence (43) states $\lambda_1 > \mu_1 + \nu_1$, violating (35).

Case 2b $(\min(\mu_2, \nu_1, k) = k$ and $k \geq \nu_2)$. $\mu^{\downarrow k} = (\mu_1, \mu_2 - k)$ and $\nu^{\downarrow k} = (\nu_1 - (k - \nu_2))$ is a valid pair with $\alpha = (k)$. Here $k_1 = 0, k_2 = k, l_1 = k - \nu_2$ and $l_2 = \nu_2$. By (43) and (46),

$$\lambda_1 - \lambda_2 > (\mu_1 + \nu_1 - k_1 - l_1) - (\mu_2 + \nu_2 - k_2 - l_2)$$
$$= \mu_1 + \nu_1 - (k - \nu_2) - \mu_2 - \nu_2 + k + \nu_2$$
$$= \nu_1 + \nu_2 + \mu_1 - \mu_2$$

which violates the third inequality from (38).

Case 3 $(\min(\mu_2, \nu_1, k) = \mu_2)$. Let $\mu^{\downarrow k} = (\mu_1 - (k - \mu_2))$, $\nu^{\downarrow k} = (\nu_1 - (k - \nu_2))$. By (42), $\mu_2 \geq \nu_2 \geq k - \min\{\mu_1, \nu_1\}$. Using this, one checks $(\mu^{\downarrow k}, \nu^{\downarrow k})$ is valid of content $\alpha = (\min\{\mu_1, \nu_1\}, k - \min\{\mu_1, \nu_1\})$. Here, $k_2 = \mu_2$ and $l_2 = \nu_2$. Hence by (46), $\lambda_2 < \mu_2 + \nu_2 - k_2 - l_2 = 0$ contradicts that $\lambda \in \mathsf{Par}_2$. $\qquad\square$

Introduce the quantity

$$\Delta(\mu^{\downarrow k}, \nu^{\downarrow k}) := (\mu_1 + \nu_2 - k_1 - l_2) - (\mu_2 + \nu_1 - k_2 - l_1).$$

Claim 5.22. Suppose $(\mu^{\downarrow k}, \nu^{\downarrow k})$ is a valid pair such that $|\Delta(\mu^{\downarrow k}, \nu^{\downarrow k})| \leq 1$. Then (44) and (45) are violated.

Proof of Claim 5.22. If (44) holds, by (41) and Claim 5.20(IV) we obtain

$$\lambda_1 \leq \mu_2 + \nu_1 - k_2 - l_1 - 1 \leq \mu_1 + \nu_2 - k_1 - l_2 < \lambda_2,$$

which is a contradiction of $\lambda \in \mathsf{Par}_2$. Similarly, if (45) holds then

$$\lambda_1 \leq \mu_1 + \nu_2 - k_1 - l_2 - 1 \leq \mu_2 + \nu_1 - k_2 - l_1 < \lambda_2,$$

giving the same contradiction. $\qquad\square$

Claim 5.23. Suppose $(\mu^{\downarrow k}, \nu^{\downarrow k})$ and $(\widetilde{\mu}^{\downarrow k}, \widetilde{\nu}^{\downarrow k})$ are valid pairs of content α and $\widetilde{\alpha}$, respectively. There is a sequence of valid pairs

$$(\mu_{(0)}^{\downarrow k}, \nu_{(0)}^{\downarrow k}) = (\mu^{\downarrow k}, \nu^{\downarrow k}), (\mu_{(1)}^{\downarrow k}, \nu_{(1)}^{\downarrow k}), \dots (\mu_{(m)}^{\downarrow k}, \nu_{(m)}^{\downarrow k}) = (\widetilde{\mu}^{\downarrow k}, \widetilde{\nu}^{\downarrow k})$$

of contents $\alpha^{(0)} = \alpha, \alpha^{(1)}, \dots, \alpha^{(m)} = \widetilde{\alpha}$ (respectively) such that for all $i \in [m]$,

(47) $$|\Delta(\mu_{(i)}^{\downarrow k}, \nu_{(i)}^{\downarrow k}) - \Delta(\mu_{(i-1)}^{\downarrow k}, \nu_{(i-1)}^{\downarrow k})| \leq 2.$$

Proof of Claim 5.23. First suppose that $\alpha = \widetilde{\alpha}$. By exchanging the roles of $(\mu^{\downarrow k}, \nu^{\downarrow k})$ and $(\widetilde{\mu}^{\downarrow k}, \widetilde{\nu}^{\downarrow k})$ if necessary, we may assume that $k_1 - \widetilde{k}_1 = j \geq 0$. Define

$$\mu_{(i+1)}^{\downarrow k} = (\mu_{(i)1}^{\downarrow k} + 1, \mu_{(i)2}^{\downarrow k} - 1)$$

$0 \leq i < j$. Also, set $\nu_{(i)}^{\downarrow k} = \nu_{(0)}^{\downarrow k}$ for all $0 < i \leq j$. By definition of j, $\mu_{(j)}^{\downarrow k} = \widetilde{\mu}^{\downarrow k}$. Moving a single box at a time, we construct $\nu_{(i)}^{\downarrow k}$ similarly for $i > j$ such that when $i = m$ we obtain $\nu^{\downarrow k}$ (and we set $\mu_i^{\downarrow k} = \widetilde{\mu}^{\downarrow k}$ for $j < i \leq m$). More precisely if $l_1 = \widetilde{l}_1$ then $j = m$. If $l_1 > \widetilde{l}_1$ then set $\nu_{(i+1)}^{\downarrow k} = (\nu_{(i)1}^{\downarrow k} + 1, \nu_{(i)2}^{\downarrow k} - 1)$ for $j \leq i < m$. Finally if $l_1 < \widetilde{l}_1$ we set $\nu_{(i+1)}^{\downarrow k} = (\nu_{(i)1}^{\downarrow k} - 1, \nu_{(i)2}^{\downarrow k} + 1)$ for $j \leq i < m$.

Set $\alpha^{(i)} = \alpha = \widetilde{\alpha}$ for $0 \leq i \leq m$. It is a straightforward induction argument to see that each $(\mu_{(i)}^{\downarrow k}, \nu_{(i)}^{\downarrow k})$ is valid of content $\alpha^{(i)}$. Finally, by construction,

$$(48) \qquad |(k_2^{(i)} - k_1^{(i)} + l_1^{(i)} - l_2^{(i)}) - (k_2^{(i-1)} - k_1^{(i-1)} + l_1^{(i-1)} - l_2^{(i-1)})| = 2,$$

which implies (47).

Now suppose that $\alpha \neq \widetilde{\alpha}$. We assume without loss of generality that $\alpha_2 > \widetilde{\alpha}_2$. Let $m^\star := \alpha_2 - \widetilde{\alpha}_2 > 0$. Then, for $0 \leq i \leq m^\star - 1$ set

$$(49) \qquad\qquad\qquad \alpha^{(i+1)} = (\alpha_1^{(i)} + 1, \alpha_2^{(i)} - 1),$$

$$(50) \qquad\qquad \mu_{(i+1)}^{\downarrow k} = \begin{cases} \mu_{(i)}^{\downarrow k} & \text{if } c_{\mu_{(i)}^{\downarrow k}, \alpha^{(i+1)}}^{\mu} > 0 \\ (\mu_{(i)1}^{\downarrow k} + 1, \mu_{(i)2}^{\downarrow k} - 1) & \text{otherwise,} \end{cases}$$

and

$$(51) \qquad\qquad \nu_{(i+1)}^{\downarrow k} = \begin{cases} \nu_{(i)}^{\downarrow k} & \text{if } c_{\nu_{(i)}^{\downarrow k}, \alpha^{(i+1)}}^{\nu} > 0 \\ (\nu_{(i)1}^{\downarrow k} + 1, \nu_{(i)2}^{\downarrow k} - 1) & \text{otherwise.} \end{cases}$$

It is straightforward to check

$$|(k_2^{(i)} - k_1^{(i)} + l_1^{(i)} - l_2^{(i)}) - (k_2^{(i-1)} - k_1^{(i-1)} + l_1^{(i-1)} - l_2^{(i-1)})| \in \{0, 2\}$$

and hence (47) holds.

Thus, it remains to show that $(\mu_{(i+1)}^{\downarrow k}, \nu_{(i+1)}^{\downarrow k})$ is a valid pair of content $\alpha^{(i+1)}$. By definition, the only concern is if $\mu_{(i+1)}^{\downarrow k}$ (respectively, $\nu_{(i+1)}^{\downarrow k}$) is obtained by applying the second case of (50) (respectively, (51)). Now, suppose we applied the second case of (50) to obtain $\mu_{(i+1)}^{\downarrow k}$. Since, by induction, $(\mu_{(i)}^{\downarrow k}, \nu_{(i)}^{\downarrow k})$ is valid of content $\alpha^{(i)}$, there exists an LR tableau T of shape $\mu/\mu_{(i)}^{\downarrow k}$ of content $\alpha^{(i)}$. The assumption $\alpha_2 > \widetilde{\alpha}_2$ implies $\alpha_1 < \widetilde{\alpha}_1$. This combined with the induction hypothesis, the fact that $\mu_{(i),2}^{\downarrow k} + \alpha_1^{(i)} = \mu_1$ holds when $c_{\mu_{(i)}^{\downarrow k}, \alpha^{(i+1)}}^{\mu} = 0$, and $\mu_1 \geq \widetilde{\alpha}_1 > \alpha_1^{(i)}$, shows

$$(\mu_{(i)1}^{\downarrow k} + 1, \mu_{(i)2}^{\downarrow k} - 1) \in \mathsf{Par}_2.$$

Now, define T' by modifying T as follows: Move the leftmost 1 in the first row and place it to the left of the leftmost entry of the second row. Then change the leftmost 2 in the second row into a 1.

By definition of $m^\star$, and the existence of T, there exists a (leftmost) 1 in the first row and a 2 in the second row. Hence the modification is well-defined for $0 \leq i < m$. Moreover, it is clear T' is semistandard, of content $\alpha^{(i+1)}$ and has shape $\mu/\mu_{(i+1)}^{\downarrow k}$. That T' is ballot follows easily from the fact T is ballot. Hence T' is an LR tableau of the desired type.

In the same way, if $\nu_{(i+1)}^{\downarrow k}$ is obtained from $\nu_{(i)}^{\downarrow k}$ using the second case of (51), we can modify an LR tableau U of shape $\nu/\nu_{(i)}^{\downarrow k}$ of content $\alpha^{(i)}$ into an LR tableau of shape $\nu/\nu_{(i+1)}^{\downarrow k}$ and content $\alpha^{(i+1)}$.

Summarizing, irregardless of which cases of (50) and (51) are used at each stage, by induction, $(\mu_{(i+1)}^{\downarrow k}, \nu_{(i+1)}^{\downarrow k})$ is valid of content $\alpha^{(i+1)}$. Moreover when $i + 1 = m^\star$, we arrive at $(\mu_{(m^\star)}^{\downarrow k}, \nu_{(m^\star)}^{\downarrow k})$ of content $\widetilde{\alpha}$. We have therefore reduced to the $\alpha = \widetilde{\alpha}$

case above. Applying the argument of that case we continue this sequence to $(\widetilde{\mu}^{\downarrow k}, \widetilde{\nu}^{\downarrow k})$. $\qquad\square$

Claim 5.24. No valid pair $(\mu^{\downarrow k}, \nu^{\downarrow k})$ can satisfy (44) and (45) simultaneously.

Proof of Claim 5.24. If some valid pair $(\mu^{\downarrow k}, \nu^{\downarrow k})$ satisfies both (44) and (45), then

$$\lambda_2 > \mu_1^{\downarrow k} + \nu_2^{\downarrow k} \text{ and } \lambda_2 > \mu_2^{\downarrow k} + \nu_1^{\downarrow k}.$$

Therefore we have

$$|\lambda| \geq 2\lambda_2 > |\mu^{\downarrow k}| + |\nu^{\downarrow k}| = |\lambda|,$$

a contradiction. $\qquad\square$

Claim 5.25. If all valid pairs $(\mu^{\downarrow k}, \nu^{\downarrow k})$ satisfy (44) or (45) then one of the inequalities from (35)-(38) is violated.

Proof of Claim 5.25. Claim 5.22 says that $|\Delta(\mu^{\downarrow k}, \nu^{\downarrow k})| \leq 1$ cannot occur.

If we have two valid pairs $(\mu^{\downarrow k}, \nu^{\downarrow k})$, $(\widetilde{\mu}^{\downarrow k}, \widetilde{\nu}^{\downarrow k})$ satisfying

$$\Delta(\mu^{\downarrow k}, \nu^{\downarrow k}) < -1 \text{ and } \Delta(\widetilde{\mu}^{\downarrow k}, \widetilde{\nu}^{\downarrow k}) > 1,$$

then by Claim 5.23 there is a sequence $(\mu_{(0)}^{\downarrow k}, \nu_{(0)}^{\downarrow k}) = (\mu^{\downarrow k}, \nu^{\downarrow k})$, $(\mu_{(1)}^{\downarrow k}, \nu_{(1)}^{\downarrow k})$, $\ldots$, $(\mu_{(m)}^{\downarrow k}, \nu_{(m)}^{\downarrow k}) = (\widetilde{\mu}^{\downarrow k}, \widetilde{\nu}^{\downarrow k})$ such that $|\Delta(\mu_{(i)}^{\downarrow k}, \nu_{(i)}^{\downarrow k}) - \Delta(\mu_{(i-1)}^{\downarrow k}, \nu_{(i-1)}^{\downarrow k})| \leq 2$ for all $i \in [m]$. Hence for some j, $\Delta(\mu_{(j)}^{\downarrow k}, \nu_{(j)}^{\downarrow k}) \in \{-1, 0, 1\}$. However, in that case, $(\mu_{(j)}^{\downarrow k}, \nu_{(j)}^{\downarrow k})$ contradicts our hypothesis, by Claim 5.22.

Since $\Delta(\mu^{\downarrow k}, \nu^{\downarrow k}) = -\Delta(\nu^{\downarrow k}, \mu^{\downarrow k})$, by Lemma 2.2(I), we may assume $\Delta(\mu^{\downarrow k}, \nu^{\downarrow k}) < -1$. By definition this means $\mu_1 + \nu_2 - k_1 - l_2 < \mu_2 + \nu_1 - k_2 - l_1$. If furthermore $\lambda_2 > \mu_2 + \nu_1 - k_2 - l_1$ then $\lambda_2 > \mu_1 + \nu_2 - k_1 - l_2$. That is, if $(\mu^{\downarrow k}, \nu^{\downarrow k})$ satisfies (45) then $(\mu^{\downarrow k}, \nu^{\downarrow k})$ satisfies (44). Now by Claim 5.24 we get a contradiction. Thus, henceforth we assume $(\mu^{\downarrow k}, \nu^{\downarrow k})$ satisfies

$$(52) \qquad\qquad \Delta(\mu^{\downarrow k}, \nu^{\downarrow k}) < -1 \text{ and } (44).$$

We have four cases, depending on k. We appeal to Claim 5.20 in each case.

Case 1 $(k \leq \mu_2, \nu_1 - \nu_2)$. Notice that $\mu^{\downarrow k} = (\mu_1, \mu_2 - k)$, $\nu^{\downarrow k} = (\nu_1 - k, \nu_2)$ is a valid pair with content $\alpha = (k)$. We have $k_1 = l_2 = 0$ and hence (44) says $\lambda_2 > \mu_1 + \nu_2$ violating (36).

Case 2 $(\mu_2 < k \leq \mu_1, \nu_1 - \nu_2)$. Observe that $\mu^{\downarrow k} = (\mu_1 - (k - \mu_2))$, $\nu^{\downarrow k} = (\nu_1 - k, \nu_2)$ is a valid pair with content $\alpha = (k)$. By (44) combined with (41),

$$(53) \qquad\qquad \lambda_1 < \mu_2 + \nu_1 - k_2 - l_1.$$

We will use this inequality here and in the cases below. In the present case, $k_2 = 0, l_1 = k$ and thus (53) says $\lambda_2 > \mu_1 + \nu_2 - k + \mu_2$. Combining with (44) gives

$$\lambda_1 - \lambda_2 < \nu_1 - \nu_2 - \mu_1 - \mu_2,$$

which violates (38).

Case 3 $(\mu_1 < k \leq \nu_1 - \nu_2)$. Since $\nu_2 \geq \alpha_2 \geq k - \mu_1$ and $\mu_1 \leq \nu_1 - \nu_2 + k - \mu_1$, we have a valid pair $\mu^{\downarrow k} = (\mu_1 - (k - \mu_2))$, $\nu^{\downarrow k} = (\nu_1 - \mu_1, \nu_2 - (k - \mu_1))$ with content $\alpha = (\mu_1, k - \mu_1)$. We have $k_2 = \mu_2$ and $l_1 = \mu_1$ and thus by (53),

$$\lambda_1 < \mu_2 + \nu_1 - \mu_2 - \mu_1 = \nu_1 - \mu_1,$$

which violates (35).

Case 4 $(k > \nu_1 - \nu_2)$. Let

$$\alpha = \left(\nu_1 - \nu_2 + \left\lceil \frac{k - \nu_1 + \nu_2}{2} \right\rceil, \left\lfloor \frac{k - \nu_1 + \nu_2}{2} \right\rfloor \right), \text{ and}$$

$$\nu^{\downarrow k} = \left(\nu_2 - \left\lfloor \frac{k - \nu_1 + \nu_2}{2} \right\rfloor, \nu_2 - \left\lceil \frac{k - \nu_1 + \nu_2}{2} \right\rceil \right).$$

One can check that there is an LR tableau of shape $\nu/\nu^{\downarrow k}$ and content α by verifying the conditions (I)-(VI) of Claim 5.20. In particular $\alpha \subseteq \nu$. If $\alpha \subseteq \mu$ as well then since $s_{\mu/\alpha} \neq 0$, by (7) we can find $\mu^{\downarrow k}$ such that $(\mu^{\downarrow k}, \nu^{\downarrow k})$ is valid of content α. However, in that case

$$(\nu_1 - l_1) - (\nu_2 - l_2) = \nu_1^{\downarrow k} - \nu_2^{\downarrow k} \leq 1,$$

and hence

$$\begin{aligned}
\Delta(\mu^{\downarrow k}, \nu^{\downarrow k}) :=& \mu_1 + \nu_2 - k_1 - l_2 - (\mu_2 + \nu_1 - k_2 - l_1) \\
=& \mu_1 - k_1 - (\mu_2 - k_2) + \nu_2 - \nu_1 + l_1 - l_2 \\
=& (\mu_1^{\downarrow k} - \mu_2^{\downarrow k}) - [(\nu_1 - l_1) - (\nu_2 - l_2)] \\
\geq& -1.
\end{aligned}$$

This would contradict the assumption $\Delta(\mu^{\downarrow k}, \nu^{\downarrow k}) < -1$. Therefore we may assume either $\mu_1 < \alpha_1$ or $\mu_2 < \alpha_2$.

First suppose $\mu_1 < \alpha_1$. Using this assumption, and the definition of α_1 one verifies the conditions (II) and (VI) Claim 5.20. It follows that

$$\mu_{(1)}^{\downarrow k} = (\mu_1 - k + \mu_2), \nu_{(1)}^{\downarrow k} = (\nu_1 - \mu_1, \nu_2 - (k - \mu_1))$$

is a valid pair with content $\overline{\alpha} = (\mu_1, k - \mu_1)$. Now we have $k_2 = \mu_2$ and $l_1 = \mu_1$ and thus (53) states

$$\lambda_1 < \mu_2 + \nu_1 - \mu_2 - \mu_1 = \nu_1 - \mu_1.$$

This violates the second inequality of (35).

Now suppose $\mu_2 < \alpha_2$. Using this assumption,

$$\mu_{(2)}^{\downarrow k} = (\mu_1 - k + \mu_2), \nu_{(2)}^{\downarrow k} = (\nu_1 - [\nu_1 - \nu_2 + \mu_2], \nu_2 - [\nu_2 - \nu_1 + k - \mu_2])$$

gives a valid pair of content $\overline{\alpha} = (k - \mu_2, \mu_2)$. Now we have $k_2 = \mu_2$ and $l_1 = \nu_1 - \nu_2 + \mu_2$ and so here (53) is

$$\lambda_1 < \mu_2 + \nu_1 - (\mu_2) - (\nu_1 - \nu_2 + \mu_2) = \nu_2 - \mu_2.$$

This gives a violation of the second equation of (36). $\qquad\square$

Conclusion of the proof of Theorem 5.15: If all valid pairs satisfy (44) or (45), we are done by Claim 5.25. Since by Claim 5.19, at least one of (43), (44) or (45) holds for valid pairs, we may assume there is a valid pair $(\mu^{\downarrow k}, \nu^{\downarrow k})$ such that (43) holds. If in fact, *all* valid pairs satisfy (43), we are done by Claim 5.21. Hence we may also suppose there is a valid pair $(\widetilde{\mu}^{\downarrow k}, \widetilde{\nu}^{\downarrow k})$ that does not satisfy (43).

Let us consider the sequence of valid pairs

$$(\mu_{(0)}^{\downarrow k}, \nu_{(0)}^{\downarrow k}) := (\mu^{\downarrow k}, \nu^{\downarrow k}), (\mu_{(1)}^{\downarrow k}, \nu_{(1)}^{\downarrow k}), \ldots, (\mu_{(m)}^{\downarrow k}, \nu_{(m)}^{\downarrow k}) := (\widetilde{\mu}^{\downarrow k}, \widetilde{\nu}^{\downarrow k})$$

where $(\mu_{(i)}^{\downarrow k}, \nu_{(i)}^{\downarrow k}) \mapsto (\mu_{(i+1)}^{\downarrow k}, \nu_{(i+1)}^{\downarrow k})$ by Claim 5.23's construction.

Combining the fact that $(\mu^{\downarrow k}_{(0)}, \nu^{\downarrow k}_{(0)}) = (\mu^{\downarrow k}, \nu^{\downarrow k})$ is a valid pair satisfying (43) with (41) and Claim 5.20(IV),

$$\lambda_2 < \mu^{\downarrow k}_{(0)2} + \nu^{\downarrow k}_{(0)2} - 2k + k_1 + l_1 < \mu^{\downarrow k}_{(0)2} + \nu^{\downarrow k}_{(0)2}.$$

Hence

$$(54) \qquad \lambda_2 < \mu^{\downarrow k}_{(0)2} + \nu^{\downarrow k}_{(0)2} \le \min\{\mu^{\downarrow k}_{(0)1} + \nu^{\downarrow k}_{(0)2}, \mu^{\downarrow k}_{(0)2} + \nu^{\downarrow k}_{(0)1}\}.$$

By examining Claim 5.23's construction (for both $\alpha = \widetilde{\alpha}$ and $\alpha \ne \widetilde{\alpha}$), it is straightforward to see that

$$(55) \ |\min\{\mu^{\downarrow k}_{(i)1} + \nu^{\downarrow k}_{(i)2}, \mu^{\downarrow k}_{(i)2} + \nu^{\downarrow k}_{(i)1}\} - \min\{\mu^{\downarrow k}_{(i+1)1} + \nu^{\downarrow k}_{(i+1)2}, \mu^{\downarrow k}_{(i+1)2} + \nu^{\downarrow k}_{(i+1)1}\}| \le 1.$$

Inductively, if (43) holds for $(\mu^{\downarrow k}_{(i)}, \nu^{\downarrow k}_{(i)})$, then by the same reasoning as for (54),

$$\lambda_2 \le \mu^{\downarrow k}_{(i)2} + \nu^{\downarrow k}_{(i)2} - 1$$
$$\le \min\{\mu^{\downarrow k}_{(i)1} + \nu^{\downarrow k}_{(i)2}, \mu^{\downarrow k}_{(i)2} + \nu^{\downarrow k}_{(i)1}\} - 1.$$

Combining with (55), we get

$$\lambda_2 \le \min\{\mu^{\downarrow k}_{(i+1)1} + \nu^{\downarrow k}_{(i+1)2}, \mu^{\downarrow k}_{(i+1)2} + \nu^{\downarrow k}_{(i+1)1}\}.$$

This means $(\mu^{\downarrow k}_{(i+1)}, \nu^{\downarrow k}_{(i+1)})$ violates (44) and (45); consequently, (43) holds for this valid pair. Therefore by induction, $(\mu^{\downarrow k}_{(m)}, \nu^{\downarrow k}_{(m)})$ satisfies (43), which contradicts the choice of $(\mu^{\downarrow k}_{(m)}, \nu^{\downarrow k}_{(m)})$. $\qquad\square$

$\square$

5.4. **Refinements?** A conjecture of W. Fulton (proved in [23]) states that

$$c^\lambda_{\mu,\nu} = 1 \implies c^{k\lambda}_{k\mu,k\nu} = 1, \quad \forall k \ge 1.$$

Example 5.26 (Counterexample to analogue of W. Fulton's conjecture)**.** One checks that

$$N_{(1,1),(1,1),(1,1)} = (c^{(1,1)}_{(1),(1)})^3 \text{ but } N_{(2,2),(2,2),(2,2)} = (c^{(2,2)}_{(1,1),(1,1)})^3 + (c^{(2,2)}_{(2),(2)})^3 = 2$$

Hence, the analogue of Fulton's conjecture for $N_{\nu,\mu,\lambda}$ is false. $\qquad\square$

Define a function

$$\mathfrak{c}^\lambda_{\mu,\nu} : \mathbb{Z}_{\ge 1} \to \mathbb{N} \ \text{ by } k \mapsto c^{k\lambda}_{k\mu,k\nu}.$$

A conjecture of R. C. King-C. Tollu-F. Toumazet [19] asserts that this function is interpolated by a polynomial with nonnegative rational coefficients. The polynomiality property was proved by H. Derksen-J. Weyman [7]. Consequently, $\mathfrak{c}^\lambda_{\mu,\nu}$ is called the *Littlewood-Richardson polynomial*. (The positivity conjecture remains open in general.)

Similarly, let us define the *Newell-Littlewood function*:

$$\mathfrak{N}_{\mu,\nu,\lambda} : \mathbb{Z}_{\ge 1} \to \mathbb{N} \text{ by } k \mapsto N_{k\mu,k\nu,k\lambda}.$$

The following shows that $\mathfrak{N}_{\mu,\nu,\lambda}(k)$ cannot always be interpolated by a single polynomial.

Theorem 5.27 (Non-polynomiality)**.** *There exist* λ, μ, ν *such that* $\mathfrak{N}_{\mu,\nu,\lambda}(k) \notin \mathbb{R}[k]$.

Proof. We will show $\mathfrak{N}_{(1,1),(1,1),(1,1)}(k) = \left\lceil \frac{k+1}{2} \right\rceil$, which is clearly non-polynomial.

Let $\mu, \nu, \lambda = (1,1)$ and suppose α, β, γ satisfy $c^{k\mu}_{\alpha,\beta} c^{k\nu}_{\alpha,\gamma} c^{k\lambda}_{\beta,\gamma} > 0$, *i.e.*, $c^{(k,k)}_{\alpha,\beta} c^{(k,k)}_{\alpha,\gamma} c^{(k,k)}_{\beta,\gamma} > 0$. The claim is that the only possible (α, β, γ) are

$$(56) \qquad \alpha = \beta = \gamma = (j, k-j) \text{ where } \left\lfloor \frac{k+1}{2} \right\rfloor \leq j \leq k,$$

and in this case the contribution to (1) is $(c^{(k,k)}_{(j,k-j),(j,k-j)})^3 = 1$. This would complete the proof as there are $\lceil \frac{k+1}{2} \rceil$ such j. That $c^{(k,k)}_{(j,k-j),(j,k-j)} = 1$ follows easily from the Littlewood-Richardson rule. Hence it only remains to rule out other possible (α, β, γ). Indeed, given such a triple, since $c^{(k,k)}_{\alpha,\beta} > 0$ we must have $|\alpha| + |\beta| = 2k$. Similarly, we obtain $|\alpha| + |\gamma| = 2k$ and $|\beta| + |\gamma| = 2k$ which together imply $|\alpha| = |\beta| = |\gamma| = k$. To conclude, we apply another fact about Littlewood-Richardson coefficients that has a Schubert calculus provenance. That is, $c^{(m-\ell)^\ell}_{\alpha,\beta} = \delta_{\beta,\alpha^\vee}$ where $\alpha^\vee$ is the 180-degree rotation of $(m-\ell)^\ell \setminus \beta$ (as used in Claim 3.4).[7] In our case $\ell = 2$ and $m = k+2$; moreover $(j, k-j)^\vee = (j, k-j)$. From this, the result follows. $\square$

Example 5.28. Let $\overline{\mathfrak{N}}_{\mu,\nu,\lambda}(k) := \mathfrak{N}^\lambda_{\mu,\nu}(2k-1)$, $\widetilde{\mathfrak{N}}_{\mu,\nu,\lambda}(k) := \mathfrak{N}_{\mu,\nu,\lambda}(2k)$. By Proposition 5.27,

$$\overline{\mathfrak{N}}_{(1,1),(1,1),(1,1)} = k \text{ and } \widetilde{\mathfrak{N}}_{(1,1),(1,1),(1,1)} = k+1.$$

For another example, it seems that

$$\overline{\mathfrak{N}}_{(2,1,1),(2,1,1),(1,1,1,1)} = \frac{1}{3}k(k+2)(k+1)$$

and

$$\widetilde{\mathfrak{N}}_{(2,1,1),(2,1,1),(1,1,1,1)} = \frac{1}{6}(2k+3)(k+2)(k+1).$$

This would suggest $\overline{\mathfrak{N}}_{\mu,\nu,\lambda}, \widetilde{\mathfrak{N}}_{\mu,\nu,\lambda} \in \mathbb{Q}_{\geq 0}[k]$. However, when $\lambda = \mu = \nu = (2,1,1)$, the values of $\mathfrak{N}_{\mu,\nu,\lambda}(k)$ for $k = 1, 2, \ldots, 11$ are $4, 18, 51, 141, 315, 676, 1288, 2370,$ $4047, 6720, 10605$. None of $\mathfrak{N}_{\mu,\nu,\lambda}, \overline{\mathfrak{N}}_{\mu,\nu,\lambda}, \widetilde{\mathfrak{N}}_{\mu,\nu,\lambda}$ seem to have a nice interpolation, although it is possible we do not have sufficiently many values.[8] $\square$

5.5. **Complexity of computing** $N_{\mu,\nu,\lambda}$. Following H. Narayanan [34], T. McAllister–J. De Loera [6], and K. D. Mulmuley-H. Narayanan-M. Sohoni [33], Theorem 5.1 and Conjecture 5.5 have some implications about the complexity of computing $N_{\mu,\nu,\lambda}$. For brevity, we limit ourselves to a sketch.

Given input $(\lambda, \mu, \nu) \in \mathsf{Par}_n$ (measured in terms of bit-size complexity) there is the counting problem NLvalue which outputs $N_{\mu,\nu,\lambda}$. By Lemma 2.2(II), a subproblem is LRvalue (computation of $c^\nu_{\lambda,\mu}$). H. Narayanan [34] shows LRvalue $\in$ #P-complete (thus, in particular, no polynomial time algorithm exists for this problem unless $P = NP$). This implies NLvalue is #P-hard. Theorem 5.1 shows that

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[7]Let σ_α denote the Schubert class for $\alpha \subset (m-\ell)^\ell$. The underlying Schubert calculus statement is that if $|\alpha| + |\beta| = \dim \mathsf{Gr}_\ell(\mathbb{C}^m)(= \ell \times (m-\ell))$ then $\sigma_\alpha \cup \sigma_\beta = \delta_{\beta,\alpha^\vee} \sigma_{(m-\ell)^\ell} \in H^*(\mathsf{Gr}_\ell(\mathbb{C}^m))$.

[8]After posting this work to the arXiv, R. C. King (private communication) informed us that this sequence of numbers fit the coefficients of the generating series $\frac{(1+x+5x^2+4x^3+8x^4+x^5+x^6)}{(1-x)^3(1-x^2)^4}$. From this he conjectures that $\mathfrak{N}_{(2,1,1),(2,1,1),(2,1,1)}(k) = (k+2)(k+4)(7k^4+57k^3+212k^2+492k+480)/3840$ if k is even and $\mathfrak{N}_{(2,1,1),(2,1,1),(2,1,1)}(k) = (k+1)(k+3)(7k^4+71k^3+305k^2+697k+840)/3840$ if k is odd. On the basis of this and other examples, he conjectures more generally that $\mathfrak{N}_{\lambda,\mu,\nu}(k)$ is a quasi-polynomial in k.

the problem is in $\#P$ since the vectors $(\alpha_i^j, \beta_i^j, \gamma_i^j)$ provide an efficient encoding of elements of a set counted by $N_{\mu,\nu,\lambda}$. Summarizing,

$$\texttt{NLvalue} \in \#P\text{-complete}.$$

The decision problem $\texttt{NLnonzero}$ decides if $N_{\mu,\nu,\lambda} > 0$. Theorem 5.1 implies $\texttt{NLnonzero} \in NP$. In [6, 33] it is shown that the analogous problem $\texttt{LRnonzero}$ (deciding $c_{\lambda,\mu}^\nu > 0$) can be done in polynomial time. Their proof relies on the Saturation Theorem for $c_{\lambda,\mu}^\nu$.

Conjecture 5.5 implies $\texttt{NLnonzero} \in P$ as well. In brief, Conjecture 5.5 actually shows

$$N_{\mu,\nu,\lambda} \neq 0 \iff \mathcal{P}_{\mu,\nu,\lambda} \neq \emptyset.$$

The "$\Rightarrow$" implication is by Theorem 5.1. For "$\Leftarrow$", we may assume, by Lemma 2.2(V), that $|\lambda| + |\mu| + |\nu| \equiv 0 \pmod 2$. Then $\mathcal{P}_{\mu,\nu,\lambda} \neq \emptyset$ implies $\mathcal{P}_{\mu,\nu,\lambda}$ contains a rational point $\vec{p}$. Then choose $k \in \mathbb{Z}_{>0}$ such that $k \cdot \vec{p} \in k\mathcal{P}_{\mu,\nu,\lambda}$ is a lattice point. By construction, $k\mathcal{P}_{\mu,\nu,\lambda} = \mathcal{P}_{k\mu,k\nu,k\lambda}$ and so by Theorem 5.1, $N_{k\mu,k\nu,k\lambda} > 0$. Conjecture 5.5 then says $N_{\mu,\nu,\lambda} > 0$. Finally, the inequalities defining the Newell-Littlewood polytope are of the form $Ax \leq b$ where the entries of A are $0, \pm 1$ whereas the entries of b are integers. Hence the polytope is *combinatorial*, and one can appeal É. Tardos' algorithm [11, 45] to decide if $\mathcal{P}_{\mu,\nu,\lambda}$ is feasible in strongly polynomial time. This completes the conditional argument.

6. Multiplicity-freeness

In Section 5 we studied when $N_{\lambda,\mu,\nu} = 0$. We now look at a related problem, proving an analogue of J. R. Stembridge's [42, Theorem 3.1] which characterizes pairs $(\mu,\nu) \in \mathsf{Par}$ such that (8) is multiplicity-free, *i.e.*, $c_{\mu,\nu}^\lambda \in \{0,1\}$ for all $\lambda \in \mathsf{Par}$.

Call a pair $(\mu,\nu) \in \mathsf{Par}^2$ *NL-multiplicity-free* if (4) contains no multiplicity, *i.e.*, each $N_{\mu,\nu,\lambda} \in \{0,1\}$ for all $\lambda \in \mathsf{Par}$.

Theorem 6.1. *A pair $(\mu,\nu) \in \mathsf{Par}^2$ is NL-multiplicity-free if and only if*

(I) *μ or ν is either a single box or $\emptyset$;*
(II) *μ is a single row and ν is a rectangle (or vice versa); or*
(III) *μ is a single column and ν is a rectangle (or vice versa).*

Before the proof, we pause to compare and contrast Theorem 6.1 with [42, Theorems 3.1, 4.1], and with J. R. Stembridge's later work [43]. Theorem 6.1 is an analogue of [42, Theorem 3.1] in the sense that the Schur functions $\{s_\lambda\}$ are universal characters for GL, whereas $\{s_{[\lambda]}\}$ are universal characters for Sp (we repeat that by [24, Theorem 2.3.4], Theorem 6.1 holds without change for SO). A generalization of [42, Theorem 3.1] is [42, Theorem 4.1], which characterizes when a product of Schur *polynomials* $s_\mu(x_1,\ldots,x_n)s_\nu(x_1,\ldots,x_n)$ is multiplicity-free. This is a generalization since (5) preserves multiplicity-freeness.

Since $s_\mu(x_1,\ldots,x_n)$ is the character of the (finite) $\mathsf{GL}(V)$-module $\mathbb{S}_\lambda(V)$, [43] provides the appropriate generalization to all other Weyl characters (associated to an irreducible representation of a complex semisimple Lie algebra). However, unlike the GL story, the modification rules are non-positive (see the discussion and references of Section 1.2). Nevertheless, by invoking [24, Corollary 2.5.3], it should be possible to derive Theorem 6.1 from [43] by translating the root-system language to partitions (we have not actually done this). That said, our proof is different

and self-contained, starting from (1). It is relatively short, and has a component (Lemma 6.2) which might be of some independent interest.

Proof. ($\Leftarrow$) Suppose we are in case (I). If $\mu = \emptyset$, then $c^\mu_{\alpha,\beta} > 0$ if and only if $\alpha = \beta = \emptyset$, in which case $c^\mu_{\alpha,\beta} = 1$. Hence, $c^\nu_{\alpha,\gamma} = \delta_{\gamma,\nu}$. Therefore $N_{\emptyset,\nu,\lambda} = \delta_{\nu,\lambda}$. As a result, $s_{[\emptyset]}s_{[\nu]} = s_{[\nu]}$ is multiplicity-free. Thus we may suppose $\mu = (1)$. This case is NL-multiplicity-free by Corollary 2.6.

(III) follows from (II) by Lemma 2.2(VI).

Thus suppose we are in case (II). Without loss, let $\mu = (k)$ and let $\nu = (c^d)$. We apply Proposition 2.4, and specifically (10). Since ν is a rectangle, for any $0 \leq j \leq k$ there is at most one way to remove a horizontal strip of size j from ν. The result is a shape $\theta_u = (c^{d-1}, u)$ where $0 \leq u \leq c$. Straightforwardly, if $u \neq u'$ then one cannot add a horizontal strip of $k - j$ boxes to θ_u and separately to $\theta_{u'}$ and obtain the same λ. NL-multiplicity-freeness follows from this analysis.

($\Rightarrow$) Our argument is similar to (and uses) the one used in J. Stembridge's work [42]. If $\alpha, \beta \in \mathsf{Par}$, by $\alpha \cup \beta$ we mean the partition obtained by sorting the (nonzero) parts in the multiset union of α and β.

Lemma 6.2. *For all triples of partitions μ, ν, λ and $t \in \mathbb{Z}_{\geq 0}$,*

$$N_{\mu \cup (t), \nu, \lambda \cup (t)} \geq N_{\mu,\nu,\lambda} \text{ and } N_{\mu+(1^t),\nu,\lambda+(1^t)} \geq N_{\mu,\nu,\lambda}.$$

Proof of Lemma 6.2. We will only prove the first assertion; the second follows by Lemma 2.2(VI). By [42, Lemma 2.2],

$$(57) \qquad c^{\kappa \cup (t)}_{\sigma \cup (t), \pi} \geq c^\kappa_{\sigma,\pi}.$$

Compare

$$(58) \qquad N_{\mu,\nu,\lambda} = \sum_{\alpha^\bullet, \beta^\bullet, \gamma^\bullet} c^\mu_{\alpha^\bullet, \beta^\bullet} c^\nu_{\alpha^\bullet, \gamma^\bullet} c^\lambda_{\beta^\bullet, \gamma^\bullet}$$

with

$$(59) \qquad N_{\mu \cup (t), \nu, \lambda \cup (t)} = \sum_{\alpha^\circ, \beta^\circ, \gamma^\circ} c^{\mu \cup (t)}_{\alpha^\circ, \beta^\circ} c^\nu_{\alpha^\circ, \gamma^\circ} c^{\lambda \cup (t)}_{\beta^\circ, \gamma^\circ}.$$

Notice that if $(\alpha^\bullet, \beta^\bullet, \gamma^\bullet)$ is a witness for $N_{\mu,\nu,\lambda}$ then by (57), $(\alpha^\circ, \beta^\circ, \gamma^\circ) := (\alpha^\bullet, \beta^\bullet \cup (t), \gamma^\bullet)$ is a witness for $N_{\mu \cup (t), \nu, \lambda \cup (t)}$, and moreover $N_{\mu \cup (t), \nu, \lambda \cup (t)} \geq N_{\mu,\nu,\lambda}$, as desired. □

Suppose $(\mu, \nu) \in \mathsf{Par}^2$ that do not fall into (I), (II), or (III). We break the argument into two cases, depending on whether either of μ or ν is a rectangle.

Case 1 (One of μ or ν is not a rectangle). Say that ν is not a rectangle. Since μ is not a single box, it has at least two rows or at least two columns. In view of Lemma 2.2(VI), we may assume without loss of generality that μ has at least two columns. We first establish:

Claim 6.3. For ν not a rectangle and $k \geq 2$, $N_{(k),\nu,\nu+(k-2)} \geq 2$.

Proof of Claim 6.3. Since ν is not a rectangle, it has two corners, so let $\alpha = (1)$, $\beta = (k-1)$, and γ and $\overline{\gamma}$ each be ν with a different corner removed. By (9),

$$c^{(k)}_{(1),(k-1)} = c^\nu_{\gamma,(1)} = c^\nu_{\overline{\gamma},(1)} = 1,$$

and since $(\nu + (k-2))/\gamma$ and $(\nu + (k-2))/\overline{\gamma}$ are horizontal strips of $k-1$ boxes,

$$c_{\gamma,(k-1)}^{\nu+(k-2)} = c_{\overline{\gamma},(k-1)}^{\nu+(k-2)} = 1.$$

Therefore,

$$N_{(k),\nu,\nu+(k-2)} \geq c_{(1),(k-1)}^{(k)} c_{\gamma,(1)}^{\nu} c_{\gamma,(k-1)}^{\nu+(k-2)} + c_{(1),(k-1)}^{(k)} c_{\overline{\gamma},(1)}^{\nu} c_{\overline{\gamma},(k-1)}^{\nu+(k-2)} = 2,$$

as asserted. $\square$

In general, consider μ and ν such that $\mu_1 \geq 2$, and ν is not a rectangle. Let $\lambda = (\nu + (\mu_1 - 2)) \cup (\mu_2, \mu_3, \dots)$. By repeated application of Lemma 6.2, followed by Claim 6.3:

$$N_{\mu,\nu,\lambda} = N_{\mu,\nu,(\nu+(\mu_1-2))\cup(\mu_2,\mu_3,\dots)}$$
$$\geq N_{(\mu_1,\mu_3,\mu_4,\dots),\nu,(\nu+(\mu_1-2))\cup(\mu_3,\dots)} \geq \cdots \geq N_{(\mu_1),\nu,\nu+(\mu_1-2)} \geq 2.$$

Hence (μ, ν) is not NL-multiplicity-free.

Case 2 (μ and ν are both rectangles with at least two rows and columns). We first consider the special case $\mu = (k^2)$ and $\nu = (c^d)$:

Claim 6.4. For $k, c, d \geq 2$, $N_{(k^2),(c^d),((c+k-2)^2)\cup(c^{d-2})} \geq 2$.

Proof of Claim 6.4. Let $\alpha = (1,1), \beta = (k-1,k-1), \gamma = (c^{d-2}) \cup ((c-1)^2)$. By the Littlewood-Richardson rule,

$$c_{\alpha,\beta}^{(k^2)} = c_{\alpha,\gamma}^{\nu} = c_{\beta,\gamma}^{((c+k-2)^2)\cup(c^{d-2})} = 1.$$

Similarly, letting $\overline{\alpha} = (2), \overline{\beta} = (k, k-2), \overline{\gamma} = (c^{d-1}) \cup (c-2)$, we obtain

$$c_{\overline{\alpha},\overline{\beta}}^{(k^2)} = c_{\overline{\alpha},\overline{\gamma}}^{\nu} = c_{\overline{\beta},\overline{\gamma}}^{((c+k-2)^2)\cup(c^{d-2})} = 1.$$

Therefore,

$$N_{(k^2),(c^d),((c+k-2)^2)\cup(c^{d-2})} \geq c_{\alpha,\beta}^{(k^2)} c_{\alpha,\gamma}^{\nu} c_{\beta,\gamma}^{((c+k-2)^2)\cup(c^{d-2})}$$
$$+ c_{\overline{\alpha},\overline{\beta}}^{(k^2)} c_{\overline{\alpha},\overline{\gamma}}^{\nu} c_{\overline{\beta},\overline{\gamma}}^{((c+k-2)^2)\cup(c^{d-2})} = 2,$$

as needed. $\square$

Consider arbitrary rectangles $\mu = (k^p)$ and $\nu = (c^d)$ that both contain at least two rows and columns; hence $k, p, c, d \geq 2$. Let $\lambda = ((c+k-2)^2) \cup (k^{p-2}) \cup (c^{d-2})$. By repeatedly applying Lemma 6.2, followed by Claim 6.4:

$$N_{\mu,\nu,\lambda} = N_{(k^p),(c^d),((c+k-2)^2)\cup(k^{p-2})\cup(c^{d-2})} \geq N_{(k^{p-1}),(c^d),((c+k-2)^2)\cup(k^{p-3})\cup(c^{d-2})}$$
$$\geq \cdots \geq N_{(k^2),(c^d),((c+k-2)^2)\cup(c^{d-2})} \geq 2.$$

Hence (μ, ν) is not NL-multiplicity-free in this case, either.

These two cases cover all possibilities for μ and ν not satisfying (I), (II), or (III). In both cases we established multiplicity. $\square$

7. Final remarks

7.1. **The associativity relation.** Since $N_{\mu,\nu,\lambda}$ are the structure constants for the Koike-Terada basis of Λ, the associativity relation

$$(s_{[\mu]}s_{[\nu]})s_{[\lambda]} = s_{[\mu]}(s_{[\nu]}s_{[\lambda]}),$$

implies for any $\mu, \nu, \lambda, \tau \in \mathsf{Par}$ that:

(60) $$\sum_{\theta} N_{\mu,\nu,\theta} N_{\theta,\lambda,\tau} = \sum_{\theta} N_{\nu,\lambda,\theta} N_{\mu,\theta,\tau}.$$

Problem 7.1. Give a bijective proof of (60) using the definition (1).

Now, $c_{\mu,\nu}^\lambda$ also "associative" in that it satisfies a relation of the form (60). However, (60) does not *formally* follow from this fact. To explain, we considered other associative structure coefficients $w_{\mu,\nu}^\lambda$ studied in algebraic combinatorics. For each of these one can define a "Newell-Littlewood" analogue:

$$O_{\mu,\nu,\lambda} := \sum_{\alpha,\beta,\gamma} w_{\alpha,\beta}^\mu w_{\alpha,\gamma}^\nu w_{\beta,\gamma}^\lambda.$$

Specifically, we looked at the K-theoretic Littlewood-Richardson coefficients for Grassmannians, the shifted Littlewood-Richardson coefficients for multiplication of Schur $P-$ or Schur $Q-$ functions, and the structure coefficients for Schubert polynomials (here we replace partitions with permutations). Small examples show $O_{\mu,\nu,\lambda}$ is not associative. Under what conditions/natural examples is $O_{\mu,\nu,\lambda}$ associative?

7.2. **An analogue of M. Kleber's conjecture.** Fix a rectangle $a \times b$ and consider all products $s_\lambda s_{\lambda^\vee}$ where $\lambda \subseteq a \times b$ and $\lambda^\vee$ is the 180-degree rotation of $(a \times b) \setminus \lambda$. M. Kleber [20, Section 3] conjectured that these products, ranging over unordered pairs $(\lambda, \lambda^\vee)$ are linearly independent in Λ.

Problem 7.2. Are the products $s_{[\lambda]}s_{[\lambda^\vee]}$, indexed over unordered pairs of partitions $(\lambda, \lambda^\vee)$ contained in $a \times b$, linearly independent in Λ?

By Lemma 2.2(II), M. Kleber's conjecture implies an affirmative answer to Problem 7.2. However, the extra terms in $s_{[\lambda]}s_{[\lambda^\vee]}$ versus $s_\lambda s_{\lambda^\vee}$ might make Problem 7.2 more tractable. (The interested reader can test ideas for $a = b = 2$ using the data in the Appendix.)

7.3. **Version of T. Lam-A. Postnikov-P. Pylyavskyy's theorems.** We give another implication of Proposition 2.3. This concerns results of T. Lam-A. Postnikov-P. Pylyavskyy [28]. Their paper solves (and generalizes) conjectures of A. Okounkov [37] and S. Fomin-W. Fulton-C.-K. Li-T.-Y. Poon [8]. It builds on work of B. Rhoades-M. Skandera [39, 40].

If $\alpha, \beta \in \mathsf{Par}$ then $\alpha \vee \beta \in \mathsf{Par}$ has parts $\max(\alpha_i, \beta_i)$ (where we have adjoined 0's to α or β as necessary). For any two skew shapes ν/α and μ/β, define

$$(\nu/\alpha) \wedge (\mu/\beta) := (\nu \wedge \mu)/(\alpha \wedge \beta) \text{ and } (\nu/\alpha) \vee (\mu/\beta) := (\nu \vee \mu)/(\alpha \vee \beta).$$

Let

$$\mathrm{sort}_1(\nu, \mu) := (\rho_1, \rho_3, \rho_5, \ldots) \text{ and } \mathrm{sort}_2(\nu, \mu) := (\rho_2, \rho_4, \rho_6, \ldots),$$

where $(\rho_1, \rho_2, \rho_3, \ldots) := \nu \cup \mu$. Below, $\frac{\nu+\mu}{2}$ means coordinate-wise addition and division. Also $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are taken coordinate-wise.

If $f \in \Lambda$ then f is said to be *Schur nonnegative* if $f = \sum_\lambda a_\lambda s_\lambda$ with $a_\lambda \geq 0$ for all $\lambda \in \mathsf{Par}$.

Theorem 7.3 ([28])**.** *Let ν/α and μ/β be skew shapes. The following are Schur nonnegative:*

(1) $s_{(\nu/\alpha)\wedge(\mu/\beta)}s_{(\nu/\alpha)\vee(\mu/\beta)} - s_{\nu/\alpha}s_{\mu/\beta}$

(2) $s_{\lfloor\frac{\nu+\mu}{2}\rfloor/\lfloor\frac{\alpha+\beta}{2}\rfloor}s_{\lceil\frac{\nu+\mu}{2}\rceil/\lceil\frac{\alpha+\beta}{2}\rceil} - s_{\nu/\alpha}s_{\mu/\beta}$

(3) $s_{\mathrm{sort}_1(\nu,\mu)/\mathrm{sort}_1(\alpha,\beta)}s_{\mathrm{sort}_2(\nu,\mu)/\mathrm{sort}_2(\alpha,\beta)} - s_{\nu/\alpha}s_{\mu/\beta}$

Define $f \in \Lambda$ to be *Koike-Terada nonnegative* if $f = \sum_\lambda b_\lambda s_{[\lambda]}$ has $b_\lambda \geq 0$ for every $\lambda \in \mathsf{Par}$.

Theorem 7.4. *The following are Koike-Terada nonnegative:*

(1) $s_{[\nu\wedge\mu]}s_{[\nu\vee\mu]} - s_{[\nu]}s_{[\mu]}$

(2) $s_{[\lfloor\frac{\nu+\mu}{2}\rfloor]}s_{[\lceil\frac{\nu+\mu}{2}\rceil]} - s_{[\nu]}s_{[\mu]}$

(3) $s_{[\mathrm{sort}_1(\nu,\mu)]}s_{[\mathrm{sort}_2(\nu,\mu)]} - s_{[\nu]}s_{[\mu]}$

Proof. We only prove the first statement; the others are similar. Fix any λ. Then

$$N_{\mu,\nu,\lambda} = [s_\lambda]\sum_\alpha s_{\mu/\alpha}s_{\nu/\alpha} \qquad \text{(Proposition 2.3)}$$

$$\leq [s_\lambda]\sum_\alpha s_{\mu\wedge\nu/\alpha}s_{\mu\vee\nu/\alpha} \qquad \text{(Theorem 7.3(1))}$$

$$= N_{\mu\wedge\nu,\mu\vee\nu,\lambda} \qquad \text{(Proposition 2.3)}$$

and the result follows. $\square$

Example 7.5. Let $\mu = (2), \nu = (1,1)$. Then

$$s_{[\mu]}s_{[\nu]} = s_{[2]}s_{[1,1]} = s_{[1,1]} + s_{[2]} + s_{[2,1,1]} + s_{[3,1]},$$

and

$$s_{[\mu\wedge\nu]}s_{[\mu\vee\nu]} = s_{[1]}s_{[2,1]} = s_{[1,1]} + s_{[2]} + s_{[2,1,1]} + s_{[3,1]} + s_{[2,2]}.$$

Hence $s_{[\mu\wedge\nu]}s_{[\mu\vee\nu]} - s_{[\mu]}s_{[\nu]} = s_{[2,2]}$, which is s-positive, as asserted by Theorem 7.4(1). The reader can verify that, in this case,

$$s_{[\mu\wedge\nu]}s_{[\mu\vee\nu]} = s_{[\lfloor\frac{\nu+\mu}{2}\rfloor]}s_{[\lceil\frac{\nu+\mu}{2}\rceil]} = s_{[\mathrm{sort}_1(\nu,\mu)]}s_{[\mathrm{sort}_2(\nu,\mu)]}.$$

Therefore the above also agrees with parts (2) and (3) of Theorem 7.4, as well. $\square$

APPENDIX A. A LIST OF PRODUCTS $s_{[\mu]}s_{[\nu]}$

We compute (4) for $\emptyset \neq \mu, \nu \subseteq 2 \times 2$.

$$s_{[1]}^2 = s_{[0]} + s_{[1,1]} + s_{[2]}$$

$$s_{[1]}s_{[2]} = s_{[1]} + s_{[2,1]} + s_{[3]}$$

$$s_{[1]}s_{[1,1]} = s_{[1]} + s_{[1,1,1]} + s_{[2,1]}$$

$$s_{[1]}s_{[2,1]} = s_{[1,1]} + s_{[2]} + s_{[2,1,1]} + s_{[2,2]} + s_{[3,1]}$$

$$s_{[1]}s_{[2,2]} = s_{[2,1]} + s_{[2,2,1]} + s_{[3,2]}$$

$$s_{[2]}^2 = s_{[0]} + s_{[1,1]} + s_{[2]} + s_{[2,2]} + s_{[3,1]} + s_{[4]}$$

$$s_{[2]}s_{[1,1]} = s_{[1,1]} + s_{[2]} + s_{[2,1,1]} + s_{[3,1]}$$

$$s_{[2]}s_{[2,1]} = s_{[1]} + s_{[1,1,1]} + 2s_{[2,1]} + s_{[3]} + s_{[2,2,1]} + s_{[3,1,1]} + s_{[3,2]} + s_{[4,1]}$$

$$s_{[2]}s_{[2,2]} = s_{[2]} + s_{[2,1,1]} + s_{[2,2]} + s_{[3,1]} + s_{[2,2,2]} + s_{[3,2,1]} + s_{[4,2]}$$

$$s_{[1,1]}^2 = s_{[0]} + s_{[1,1]} + s_{[2]} + s_{[1,1,1,1]} + s_{[2,1,1]} + s_{[2,2]}$$

$$s_{[1,1]}s_{[2,1]} = s_{[1]} + 2s_{[2,1]} + s_{[3]} + s_{[2,1,1,1]} + s_{[2,2,1]} + s_{[3,1,1]} + s_{[3,2]}$$

$$s_{[1,1]}s_{[2,2]} = s_{[1,1]} + s_{[2,1,1]} + s_{[2,2]} + s_{[3,1]} + s_{[2,2,1,1]} + s_{[3,2,1]} + s_{[3,3]}$$

$$s_{[2,1]}^2 = s_{[0]} + 2s_{[1,1]} + 2s_{[2]} + s_{[1,1,1,1]} + 3s_{[2,1,1]} + 2s_{[2,2]} + 3s_{[3,1]} + s_{[4]} + s_{[2,2,1,1]} + s_{[2,2,2]}$$
$$+ s_{[3,1,1,1]} + 2s_{[3,2,1]} + s_{[3,3]} + s_{[4,1,1]} + s_{[4,2]}$$

$$s_{[2,1]}s_{[2,2]} = s_{[1]} + s_{[1,1,1]} + 2s_{[2,1]} + s_{[3]} + s_{[2,1,1,1]} + 2s_{[2,2,1]} + 2s_{[3,1,1]} + 2s_{[3,2]} + s_{[4,1]}$$
$$+ s_{[2,2,2,1]} + s_{[3,2,1,1]} + s_{[3,2,2]} + s_{[3,3,1]} + s_{[4,2,1]} + s_{[4,3]}$$

$$s_{[2,2]}^2 = s_{[0]} + s_{[1,1]} + s_{[2]} + s_{[1,1,1,1]} + s_{[2,1,1]} + 2s_{[2,2]} + s_{[3,1]} + s_{[4]} + s_{[2,2,1,1]}$$
$$+ s_{[2,2,2]} + s_{[3,1,1,1]} + 2s_{[3,2,1]} + s_{[3,3]} + s_{[4,1,1]} + s_{[4,2]} + s_{[2,2,2,2]} + s_{[3,2,2,1]} + s_{[3,3,1,1]}$$
$$+ s_{[4,2,2]} + s_{[4,3,1]} + s_{[4,4]}$$

The computation $s_{[2]}s_{[2,2]}$ matches the multiplication $(2,2)_{\mathsf{Sp}} \times (2)_{\mathsf{Sp}}$ in [24, pg. 509]. This calculation is coincides with the tensor products in Sp_{2n} for any $n \geq 3$. However, when $n = 2$, as shown in *loc. cit.* the expansion differs from the one above (and from each other, among the classical groups).

Acknowledgments

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