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Injective edge-coloring of graphs with given maximum degree

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ABSTRACT

A coloring of edges of a graph G is *injective* if for any two distinct edges e_1 and e_2, the colors of e_1 and e_2 are distinct if they are at distance 1 in G or in a common triangle. Naturally, the injective chromatic index of G, $\chi'_{inj}(G)$, is the minimum number of colors needed for an injective edge-coloring of G. We study how large can be the injective chromatic index of G in terms of maximum degree of G when we have restrictions on girth and/or chromatic number of G. We also compare our bounds with analogous bounds on the strong chromatic index.

1. Introduction

1.1. Notation and definitions

For a positive integer n, we denote by $[n]$ the set $\{1, \dots, n\}$. By $\Delta(G)$ we denote the maximum degree of a graph G and by $\alpha(G)$ — the *independence number* of G.

For disjoint subsets A and B of vertices in a graph G, let $E_G(A, B)$ denote the set of edges in G with one end in A and one in B. Also $G[A]$ denotes the subgraph of G induced by A. By $N_G(v)$ we denote

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the neighborhood of vertex v in graph G, and let $d_G(v) = |N_G(v)|$. When the graph G is clear from the context, we can drop the subscript.

A vertex coloring of a graph G is *injective* if for every vertex v of G, all the neighbors of v have different colors. In other words, in injective coloring two vertices u and v must have distinct colors if there is a u, v-path of length exactly 2. The *injective chromatic number*, $\chi_{\rm inj}(G)$, of a graph G is the minimum k such that G admits injective coloring with k colors.

Similarly, an edge coloring of a graph G is *injective* if any two edges e and f that are at distance exactly 1 in G or are in a common triangle have distinct colors. The *injective chromatic index* of G, $\chi'_{\rm inj}(G)$, is the minimum number of colors needed for an injective edge coloring of G.

Note that an injective edge coloring is not necessarily a proper edge-coloring. Also, $\chi'_{\rm inj}(G)$ may significantly differ from the injective chromatic number of the line graph $L(G)$ of G. In fact, if G has no vertices of degree 2, the injective chromatic number of $L(G)$ equals the strong chromatic index of G. Recall that the *strong chromatic index*, $\chi'_s(G)$, is the minimum k such that one can color the edges of G with k colors so that every two edges at distance at most 1 have distinct colors. By definition, $\chi'_{\rm inj}(G) \le \chi'_s(G)$ for every graph G and the difference between them can be large. For example, for the star $K_{1,n}$ we have $\chi'_{\rm inj}(K_{1,n}) = 1$ and $\chi_{\rm inj}(L(K_{1,n})) = \chi'_s(K_{1,n}) = n$.

1.2. Previous results

Injective vertex coloring was introduced and studied by Hahn, Kratochvíl, Sotteau and Širáň [17] in 2002. In particular, they showed that for each graph G with maximum degree Δ,

$$\Delta \le \chi_{\rm inj}(G) \le \Delta(\Delta - 1) + 1,$$

and both bounds are sharp.

The notion of injective edge coloring was introduced in 2015 by Cardoso, Cerdeira, Cruz, and Dominic [12] motivated by a Packet Radio Network problem and independently in 2019 by Axenovich, Dörr, Rollin, and Ueckerdt [1] (they called it *induced star arboricity*).

Cardoso et al. [12] proved that computing $\chi'_{\rm inj}(G)$ of a graph G is NP-hard and determined the injective chromatic index for paths, cycles, wheels, Petersen graph and complete bipartite graphs. They also proved that $\chi'_{\rm inj}(T) \le 3$ for each tree T and that $\chi'_{\rm inj}(K_{\Delta+1}) = \frac{\Delta(\Delta+1)}{2}$. Axenovich et al. [1] concentrated more on another parameter, *induced arboricity*, but they also proved that the induced star arboricity of each planar graph is at most 30 and can be as large as 18. Apart from this, they presented bounds on the induced star arboricity of a graph in terms of its acyclic chromatic number and treewidth.

Bu and Qi [10] gave upper bounds on injective chromatic index of graphs with maximum degree 3 and 4 and low maximum average degree. In particular, they showed that the injective chromatic index of every subcubic graph with maximum average degree at most $\frac{18}{7}$ (respectively, at most $\frac{5}{2}$) is at most 6 (respectively, at most 5).

Ferdjallah, Kerdjoudj and Raspaud [16] used Proposition 2.2 of [12] to observe that the induced star arboricity of a graph is exactly its injective chromatic index and proved a series of bounds on injective chromatic index of "sparse" graphs. They proved that for every $\Delta \ge 3$ and any graph G with maximum degree at most Δ,

$$\chi'_{\rm inj}(G) \le 2(\Delta - 1)^2. \tag{1}$$

Ferdjallah, Kerdjoudj and Raspaud [16] posed the following conjecture.

Conjecture 1. *For every subcubic graph G, $\chi'_{\rm inj}(G) \le 6$.*

Furthermore, for bipartite graphs, Ferdjallah et al. [16] proved stronger bounds: for any bipartite graph G with maximum degree at most Δ,

$$\chi'_{\rm inj}(G) \le \Delta(\Delta - 1) + 1, \tag{2}$$

and for every subcubic bipartite graph G, $\chi'_{\rm inj}(G) \le 6$. They posed the following conjecture:

Conjecture 2. *For every subcubic bipartite graph,* $\chi'_{inj}(G) \leq 5$.

If this conjecture is true, then 5 is tight (see [16]).

Ferdjallah et al. [16] also gave the exact upper bound of 5 for the injective chromatic index of subcubic outerplanar graphs, and somewhat strengthened the result of Bu and Qi [10] for subcubic graphs mentioned above.

For graphs without 4-cycles Mahdian [22] proved stronger upper bounds even for strong chromatic index:

Theorem 3 (*Mahdian [22]*)**.** *For every C_4-free graph G,*

$$\chi'_s(G) \leq (2 + o(1))\frac{\Delta^2}{\ln \Delta}.$$

The strong chromatic index of bipartite graphs was also studied. In particular, Faudree, Gyárfás, Schelp and Tuza [15] in 1990 conjectured that $\chi'_s(G) \leq \Delta^2$ for every bipartite graph G with maximum degree Δ. Brualdi and Massey [9] posed in 1993 the refined conjecture that $\chi'_s(G) \leq \Delta(X)\Delta(Y)$ for every bipartite graph G with parts X and Y, where $\Delta(X)$ (resp. $\Delta(Y)$) is the maximum degree of a vertex of X (resp. Y). Partial cases of this conjecture were proved by Brualdi and Massey [9], Nakprasit [23] and Huang, Yu and Zhou [19].

1.3. Our results

The goal of this paper is to present new bounds on the injective chromatic index of graphs with given maximum degree involving chromatic number.

Our first results are two steps toward Conjecture 1:

Theorem 4. *For every subcubic graph G,* $\chi'_{inj}(G) \leq 7$.

Theorem 5. *For every planar subcubic graph G,* $\chi'_{inj}(G) \leq 6$.

The bound in Theorem 5 is exact: it is attained at K_4 and the 3-prism.

The proof of this theorem yields a stronger bound for bipartite graphs which is a step forward Conjecture 2: it implies that $\chi'_{inj}(G) \leq 4$ for every bipartite planar subcubic graph. This bound is attained at the 3-dimensional cube Q_3 with any edge deleted.

The main result of this paper is the following bound significantly improving (2):

Theorem 6. *Let* $2 \leq \chi \leq \Delta$. *If G is a graph with maximum degree* Δ *and chromatic number* χ, *then*

$$\chi'_{inj}(G) \leq (\chi - 1)\lceil 27\Delta \ln \Delta \rceil. \tag{3}$$

In particular, if G is bipartite, then $\chi'_{inj}(G) \leq \lceil 27\Delta \ln \Delta \rceil$.

We also discuss the bound of Theorem 6 and compare it with similar bounds for χ'_s. First, we show that without restrictions on the chromatic number, the bound is much weaker even for graphs with large girth. In fact, the order of magnitude of the bound in Theorem 3 cannot be improved not only for strong chromatic index but also for injective chromatic index even for graphs with arbitrary girth:

Proposition 7. *For every* $\Delta \geq 3$ *and* $g \geq 3$, *there exists a graph G with maximum degree* Δ *and girth at least g such that* $\chi'_{inj}(G) \geq \frac{\Delta(\Delta-1)}{4\ln \Delta}$.

Then we show that the bound in Theorem 6 cannot be made less than Δ even for bipartite graphs with any girth.

Proposition 8. *For every* Δ*-regular bipartite graph G,* $\chi'_{inj}(G) \geq \Delta$.

Finally, we show that the result of Theorem 6 does not hold for strong chromatic index. Moreover, we show that there are bipartite Δ-regular graphs of large girth that do not have "too large" induced matchings.

Proposition 9. *For every $\Delta \geq 21$ and $g \geq 3$, there exists a Δ-regular bipartite graph G with girth at least g such that the size of every induced matching in G is less than $k = k(\Delta, n) = \left\lceil \frac{3n \ln \Delta}{\Delta} \right\rceil$, where n is the number of vertices in each of the parts of G. In particular, for each such G,*

$$\chi'_s(G) \geq \frac{\Delta^2}{3 \ln \Delta}.$$

In the next section, we prove Theorems 4 and 5, in Section 3 prove the main result, Theorem 6, and in Section 4 Propositions 7–9 are proven.

2. Subcubic graphs

2.1. A bound for all subcubic graphs

Another way to define the injective chromatic index of a graph G is to consider the graph $G^{(*)}$ obtained from G as follows: $V(G^{(*)}) = E(G)$ and two vertices of $G^{(*)}$ are adjacent if the edges of G corresponding to these two vertices of $G^{(*)}$ are at distance 1 in G or in a triangle.

Then

$$\chi'_{\text{inj}}(G) = \chi(G^{(*)}). \tag{4}$$

Also,

$$\Delta(G^{(*)}) \leq 2(\Delta - 1)^2. \tag{5}$$

We will apply to $G^{(*)}$ the following theorem of Lovász [21].

Theorem 10 (*Lovász [21]*)**.** *Let G be a multigraph with maximum degree Δ. Let $t, k_1, k_2, \ldots, k_t$ be nonnegative integers such that*

$$k_1 + k_2 + \cdots + k_t = \Delta - t + 1.$$

Then the vertices of G can be partitioned into sets $V_1, V_2, \ldots, V_t$ so that the subgraph induced by each V_i has maximum degree at most k_i.

Our goal is to prove Theorem 4: *For every subcubic graph G, $\chi'_{\text{inj}}(G) \leq 7$.* We will study minimum counterexamples to the theorem and will show that they do not exist. We will exploit *partial injective 7-colorings of edges* of graphs when not every edge is colored. Given a partial injective edge coloring f of H with colors from [7] and an uncolored edge $e \in E(H)$, $C_f(e)$ denotes the set of colors in [7] not used on the edges at distance 1 from e or in a common triangle with e. Furthermore, $\overline{C_f}(e)$ stands for $[7] \setminus C_f(e)$.

We will use only Claims (d) and (f) of the lemma below, but it will be convenient to prove all of them in alphabetical order.

Lemma 11. *Let H be a counterexample to Theorem 4 with minimum $|E(H)| + |V(H)|$. Then*
(a) H is connected and has at least 8 edges;
(b) H is 3-regular;
(c) H does not contain a $K_4 - e$;
(d) H does not contain a triangle;
(e) H does not contain a $K_{2,3}$;
(f) H does not contain a 4-cycle.

Proof. Claim (a) immediately follows from the minimality of H and the fact that we have 7 available colors.

Suppose H has a vertex v of degree at most 2. The case when $d(v) \leq 1$ is trivial, so suppose $N(v) = \{u, w\}$. By the minimality of H, graph $H' = H - v$ has an injective edge coloring f with colors in [7]. Since the degrees of vu and vw in $H^{(*)}$ are at most 6, we can greedily extend f to vu and vw. This contradicts the choice of H and hence proves (b).

Suppose H contains a copy F of $K_4 - e$ with $V(F) = \{a, b, c, d\}$. If $H[V(F)] = K_4$, then either H is disconnected or $H = K_4$ and has 6 edges. Both possibilities contradict (a). Thus we may assume $bd \notin E(H)$. By the minimality of H, $H' = H - \{a, c\}$ has an injective edge coloring f using colors in [7]. Each $e \in \{ab, bc, cd, da\}$ has at most three colored edges in H' at distance one, so $|C_f(e)| \geq 4$. Furthermore, $|C_f(ac)| \geq 5$. Thus we can color greedily these edges in the order ab, bc, cd, da, ac. This proves (c).

Suppose H contains a copy F of K_3 with $V(F) = \{u, v, w\}$. Let u', v', and w' denote the neighbor of u, v and w outside F, respectively. By (c), u', v' and w' are pairwise distinct. By the minimality of H, $H' = H \setminus V(F)$ has an injective edge coloring f using colors in [7]. For each e in $E_0 = \{uv, vw, wu, uu', vv', ww'\}$, we have $|C_f(e)| \geq 3$. The maximum degree of $H^{(*)}[E_0]$ is 3, and $H^{(*)}[E_0]$ does not contain K_4. So by the list version of Brooks' Theorem (see e.g. [25] or [14]) $H^{(*)}[E_0]$ is 3-choosable. Thus we can extend f to whole H. This contradiction proves (d).

Suppose H contains a copy F of $K_{2,3}$ with parts $\{x, y, z\}$ and $\{u, v\}$. By (d), $H[V(F)] = F$. Let x', y' and z' be the neighbor of x, y and z, respectively, not in F. Some of them may coincide. By the minimality of H, $H' = H \setminus \{u, v\}$ has an injective edge coloring f using colors in [7]. For each e in $E_0 = \{xu, xv, yu, yv, zu, zv\}$, we have $|C_f(e)| \geq 3$: for example, the colored neighbors of xu are yy', zz' and two edges incident to x'. The maximum degree of $H^{(*)}[E_0]$ is 2. So by the list version of Brooks' Theorem, graph $H^{(*)}[E_0]$ is 3-choosable. Thus we can extend f to whole H. This contradiction proves (e).

Finally, suppose H has a 4-cycle $C = wxyzw$. Let w', x', y' and z' be the neighbor of w, x, y and z outside of C. By (d), C has no chords. By (e), all w', x', y', z' are distinct. By the minimality of H, $H' = H \setminus \{w, x, y, z\}$ has an injective edge coloring f using colors in [7]. Let $E_0 = \{ww', xx', yy', zz'\}$ and $E_1 = \{wx, xy, yz, zw\}$. Since each $e \in E_0$ has at most four colored edges at distance one, $|C_f(e)| \geq 3$.

Case 1: $C_f(ww') \cap C_f(yy') \neq \varnothing$, say $1 \in C_f(ww') \cap C_f(yy')$. Extend f to ww', yy' by $f(ww') = f(yy') = 1$. Choose distinct $f(xx') \in C_f(xx')$ and $f(zz') \in C_f(zz')$ from the colors available for them. By symmetry, assume $f(xx') = 2, f(zz') = 3$.

For each $e \in E_1$, $C_f(e) \neq \emptyset$ and e has only one neighbor in $H^{(*)}[E_1]$. So, if we cannot extend to E_1, then by symmetry we may assume

$$C_f(wx) = C_f(zy) = \{7\}. \tag{6}$$

Denote the set of edges incident to w' except ww' by $U(w')$, and define $U(x')$, $U(y')$, $U(z')$ similarly. Then in order for (6) to hold, we need $f(U(w') \cup U(x')) = \{2, 4, 5, 6\}$ and $f(U(z') \cup U(y')) = \{3, 4, 5, 6\}$. Now since $f(ww') = f(yy')$, there are at least two available colors for xx'. Let α be such color for xx' distinct from 2. Recolor xx' by α. Now both 2 and 7 are available for zy. Let $f(zy) = 2, f(wx) = 7$.

If $2 \in f(U(w'))$, then $2 \notin f(U(x'))$. We can then extend f by letting $f(xy) = 2, f(wz) = 7$, if $\alpha \neq 7$. If $\alpha = 7$, let $f(xy) = 7$. Now for wz we have at most 6 forbidden colors, so we can extend f to wz, as well.

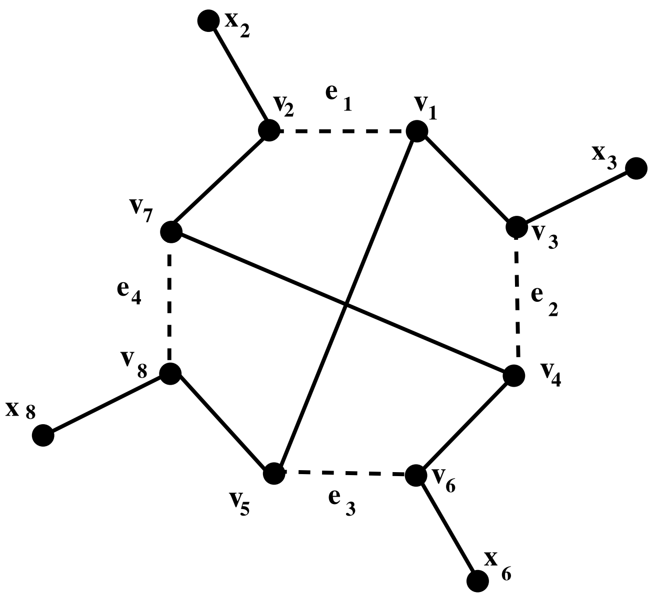
If $2 \in f(U(x'))$ then $2 \notin f(U(w'))$. In this case, we extend f letting $f(xy) = 7$ and $f(wz) = 2$. In either case f is an injective edge coloring of H, a contradiction.

Case 2: $C_f(ww') \cap C_f(yy') = \varnothing = C_f(xx') \cap C_f(zz')$. Since $|C_f(e)| \geq 3$ for each $e \in E_0$, we can extend f to the edges in E_0 so that $f(xx') \neq f(yy') \neq f(zz') \neq f(ww') \neq f(xx')$. By the case, the colors of all edges in E_0 are distinct. By symmetry, we may assume $f(ww') = 1, f(xx') = 2, f(yy') = 3, f(zz') = 4$, and $C_f(wx) = C_f(zy) = \{7\}$. Then $f(U(w') \cup U(x')) = \{1, 2, 5, 6\}$ and $f(U(z') \cup U(y')) = \{3, 4, 5, 6\}$. If we could recolor xx' with some $\alpha \in C_f(xx') - \{1, 2, 3\}$, then we let $f(xx') = \alpha, f(zy) = 2, f(wx) = 7$. As in Case 1, depending on whether $2 \in f(U(w'))$ or $2 \in f(U(x'))$, we can either color wz by 7 and xy by 2 or vice versa.

Hence there is no such α. Again by symmetry we may assume that initially $C_f(ww') = \{1, 2, 4\}, C_f(xx') = \{1, 2, 3\}, C_f(yy') = \{2, 3, 4\}$, and $C_f(zz') = \{1, 3, 4\}$. But this is a contradiction to $C_f(ww') \cap C_f(yy') = \varnothing$. □

Proof of Theorem 4. Let H be a minimal counterexample as in Lemma 11. By Theorem 10, the set of vertices of $H^{(*)}$ can be partitioned into two sets V_1 and V_2 so that $\Delta(H^{(*)}[V_1]) \leq 3$ and $\Delta(H^{(*)}[V_2]) \leq 4$. If $\chi(H^{(*)}[V_1]) \leq 3$ and $\chi(H^{(*)}[V_2]) \leq 4$, then we are done. By Brooks' Theorem, if $\chi(H^{(*)}[V_1]) \geq 4$, then $H^{(*)}[V_1]$ contains a K_4, and if $\chi(H^{(*)}[V_2]) \geq 5$, then $H^{(*)}[V_2]$ contains a K_5. So, we have two cases.

Case 1: $H^{(*)}[V_1]$ contains a K_4. Let e_1, e_2, e_3, e_4 be the vertices of this K_4 and let $e_i = v_{2i-1}v_{2i}$ for $i \in [4]$. Since H has no 3-cycles, all $v_1, \ldots, v_8$ are distinct and all edges e_1, e_2, e_3, e_4 are at distance exactly 1 from each other in H. By symmetry, we may assume that v_1 is adjacent to v_3 and v_5, and v_2 is adjacent to v_7. Then since H has no 3- and 4-cycles, in order to have e_2 and e_3 at distance 1, we need $v_4v_6 \in E(H)$. Now for the same reason, v_7 is not adjacent to v_3 or v_5 and neither of v_7 and v_8 can be adjacent to two vertices in the 5-cycle $v_1v_3v_4v_6v_5v_1$. So again by symmetry, we may assume that v_7 is adjacent to v_4, and v_8 is adjacent to v_5 (see the picture below).



Since $\Delta(H^{(*)}[V_1]) \leq 3$, all the edges incident to $\{v_1, \ldots, v_8\}$ apart from e_1, e_2, e_3 and e_4 are vertices in V_2 in $H^{(*)}$. In particular, vertex $v_1v_5 \in V_2$ is adjacent in $H^{(*)}$ to vertices v_2x_2, v_2v_7, v_3x_3, v_8x_8, v_6x_6, v_6v_4 in V_2, contradicting $\Delta(H^{(*)}[V_2]) \leq 4$.

Case 2: $H^{(*)}[V_2]$ contains a K_5. Let $e_1, \ldots, e_5$ be the vertices of this K_5 and let $e_i = v_{2i-1}v_{2i}$ for $i \in [5]$. Let $V_0 = \{v_1, \ldots, v_{10}\}$. As in Case 1, all $v_1, \ldots, v_{10}$ are distinct and all edges $e_1, \ldots, e_5$ are at distance exactly 1 from each other in H. In order to achieve this, each vertex in V_0 has neighbors only in V_0. So by the minimality of H, $|V(H)| = 10$. The only cubic 10-vertex graph with no 3- and 4-cycles is the Petersen graph $\mathbf{P}$ (see [3] or [18]). As it was mentioned in the introduction, Cardoso et al. [12] proved that $\chi'_{inj}(\mathbf{P}) = 5$. Hence $H^{(*)}[V_2]$ is not K_5, a contradiction again. □

2.2. A bound for planar subcubic graphs

We will use the following partial case of a generalization of Brooks' Theorem proved independently by Borodin [8] and Bollobás and Manvel [6].

Theorem 12 ([6,8])**.** *If G is a connected graph with $\Delta(G) \leq 3$ not containing K_4, then one can partition $V(G)$ into sets X and Y so that X is independent and $G[Y]$ is an acyclic graph with maximum degree at most 2.*

Proof of Theorem 5. Let G be a vertex-minimal subcubic plane graph with $\chi'_{\text{inj}}(G) \geq 7$. Then, $\delta(G) \geq 2$, G is connected, and has at least 7 edges; in particular, $G \neq K_4$. So by Theorem 12, $V(G) = X \cup Y$ where X is independent and $G[Y]$ is an acyclic graph with maximum degree at most 2.

Construct an auxiliary (multi)graph G' with vertex set Y as follows. For each $x \in X$, if $N(x) = \{y_1, y_2\}$, then delete x and add edge y_1y_2, and if $N(x) = \{y_1, y_2, y_3\}$, then delete x and add edges y_1y_2, y_1y_3 and y_2y_3. By construction, G' is a planar (multi)graph. By the Four Color Theorem, G' has a 4-coloring g.

We now color $E(G)$ in two steps. On Step 1 for each edge xy connecting X with Y color xy with $g(y)$. By construction, for each $x \in X$, the colors of all edges incident with x are distinct. Also, two edges of the same color cannot have an edge in Y connecting them. So, after Step 1, we have a partial injective edge coloring of G with colors in [4].

On Step 2 for each path component $y_1, y_2, \ldots, y_s$ in $G[Y]$, color the first two edges with 5, second pair of edges with 6, third pair again with 5 and so on. This yields an injective edge-coloring of G. □

If our planar subcubic graph G is also bipartite, then instead of the partition $V(G) = X \cup Y$ provided by Theorem 12, we can use the natural bipartition of G, and do not need to run Step 2 and use the extra colors 5 and 6. Thus our proof has the following implication.

Corollary 13. *For every planar subcubic bipartite graph G,* $\chi'_{\text{inj}}(G) \leq 4$.

The bound in the corollary is sharp: $\chi'_{\text{inj}}(Q_3) = 4$ where Q_3 is the graph of the unit 3-cube. Also, if we delete any edge from Q_3, the injective chromatic index of the remaining graph is still 4.

3. Graphs with high maximum degree

Our tool in this section is Lovász Local Lemma [13] in a slightly stronger form proved by Spencer [24]:

Theorem 14 (*Lovász Local Lemma [13,24]*)**.** *Let $A_1, \ldots, A_n$ be events such that $Pr[A_i] \leq p$, for $1 \leq i \leq n$. Suppose each event is independent of all the other events except for at most d of them. If $ep(d+1) < 1$, then $Pr[\bigwedge_{i=1}^{n} \overline{A_i}] > 0$.*

A subset F of edges of a graph G is *G-good* if no two edges in F are at distance 1 in G or in the same triangle. In other words, a G-good set is an independent set in the auxiliary graph $G^{(*)}$ defined in Section 3, so that an injective edge coloring is simply a partition of $E(G)$ into G-good sets.

For $\Delta \leq 40$ the bound of Theorem 6 is weaker than (1). So, it is enough to prove Theorem 6 for $\Delta \geq 41$. If $\chi(G) = \chi$, then there is a coloring of G with color classes $Y_1, Y_2, \ldots, Y_\chi$ such that for every $1 \leq j \leq \chi - 1$, set Y_j is a maximal (by inclusion) independent set in the graph $G[Y_j \cup Y_{j+1} \cup \cdots \cup Y_\chi]$. The theorem below allows us for each $1 \leq j \leq \chi - 1$ to color the edges connecting Y_j to $G[Y_{j+1} \cup Y_{j+2} \cup \cdots \cup Y_\chi]$ with $\lceil 27\Delta \ln \Delta \rceil$ colors. Thus, applying this theorem $\chi - 1$ times will imply Theorem 6.

Theorem 15. *Let $\Delta \geq 41$ and $k = \lceil 27\Delta \ln \Delta \rceil$. If G is a graph with maximum degree Δ and Y is a maximal independent set in G, then we can partition $E(Y, V(G) - Y)$ into k G-good sets.*

Proof of Theorem 15. Let $X = N(Y) = V(G) - Y$. We will construct k random G-good sets $J_1, \ldots, J_k$, using the following algorithm for $j = 1, \ldots, k$.

Step 1: Construct a random subset X_j of X by including each vertex of X into X_j with probability $\frac{1}{\Delta}$ independently of each other.

Step 2: Delete from X_j every vertex that after Step 1 has a neighbor in X_j.

Step 3: Let J_j be the set of the edges connecting X_j with Y.

Step 4: For each $y \in Y$, if y had at least two neighbors in X_j *after Step 1*, then remove all these edges from J_j incident to y.

We claim that

$$J_j \text{ is } G\text{-good.} \tag{7}$$

Indeed, since after Step 2, X_j is independent, no two edges in J_j are in a common triangle. Suppose that edges x_1y_1 and x_2y_2 in J_j are at distance 1, say $x_1y_2 \in E(G)$. But then after Step 1 vertex y_2 has at least two neighbors in X_j and hence on Step 4 edges y_2x_1 and y_2x_2 would be deleted from J_j. This proves (7).

By (7), it remains to prove that with positive probability each edge in $E(X, Y)$ will belong to at least one J_j. For this, we introduce several events and estimate their probabilities.

Denote the event that an $x \in X$ *is in* X_j *after Step 1* by $F_{1,j}(x)$. By definition, $p[F_{1,j}(x)] = 1/\Delta$ for each $x \in X$.

Denote the event that an $x \in X$ *is in* X_j *after Step 2* by $F_{2,j}(x)$. Then for each $x \in X$,

$$\frac{1}{\Delta} = p[F_{1,j}(x)] \geq p[F_{2,j}(x)] = \frac{1}{\Delta} \cdot \left(\frac{\Delta - 1}{\Delta}\right)^{|N(x)\cap X|} \geq \frac{1}{\Delta} \cdot \left(\frac{\Delta - 1}{\Delta}\right)^{\Delta - 1} > \frac{1}{e \cdot \Delta}. \tag{8}$$

For $xy \in E(X, Y)$ let $F_j(xy)$ be the event that xy *is in* J_j *after Step 4*.

Observe that for each $xy \in E(X, Y)$,

$$p[F_j(xy)] = p[F_{2,j}(x)] \prod_{x' \in N(y)-x} (1 - p[F_{1,j}(x')]).$$

Hence by (8),

$$p[F_j(xy)] \geq \frac{1}{e\Delta}\left(1 - \frac{1}{\Delta}\right)^{d(y)-1} \geq \frac{1}{e\Delta}\left(1 - \frac{1}{\Delta}\right)^{\Delta-1} > \frac{1}{e^2\Delta}. \tag{9}$$

For $xy \in E(X, Y)$, let $A(xy)$ denote the event that none of $F_j(xy)$ happened. We want to show that with positive probability none of $A(xy)$ occurs, because in this case we can assign to each edge one of the k colors. We plan to apply Theorem 14, so we need to give upper bounds on the probability of each $A(xy)$ and on the number of events $A(x'y')$ on which depends $A(xy)$.

The first part is easy, since for distinct j the events $F_j(xy)$ are independent, and hence by (9)

$$p[A(xy)] = \prod_{j=1}^{k}(1 - p[F_j(xy)]) \leq \left(1 - \frac{1}{e^2\Delta}\right)^{27\Delta \ln \Delta} < \exp\{-\frac{27 \ln \Delta}{e^2}\} = \Delta^{-27/e^2}. \tag{10}$$

For the second part, we define $Q(xy)$ as the set of edges $x'y'$ such that the distance from $\{x', y'\}$ to $N(y) \cup (N(x) \cap X)$ is at most 1 and will prove that for each $xy \in E(X, Y)$

$$A(xy) \textit{ is independent of all } A(x'y') \textit{ such that } x'y' \in E(X, Y) - Q(xy). \tag{11}$$

Recall that $A(xy)$ is fully defined by the events $F_1(xy), F_2(xy), \ldots, F_k(xy)$, and that for all $j' \neq j$ event $F_j(xy)$ is independent of all $F_{j'}(x'y')$. Thus to get (11), it is enough to prove that for each $xy \in E(X, Y)$ and each $j \in [k]$,

$$F_j(xy) \textit{ is independent of all } F_j(x'y') \textit{ such that } x'y' \in E(G) - Q(xy). \tag{12}$$

By definition, for each $xy \in E(X, Y)$, event $F_j(xy)$ occurs if and only if $F_{1,j}(x)$ occurs but none of $F_{1,j}(x')$ over all $x' \in (N(x) \cap X) \cup (N(y) - x)$ occurs. Since the events $F_{1,j}(x)$ are independent for all $x \in X$, $F_j(xy)$ is fully defined by the event

$$R(xy) = \bigcup_{x' \in N(y)\cup(N(x)\cap X)} F_{1,j}(x').$$

It follows that $F_j(xy)$ is independent of all $F_j(x_1, y_1)$ such that $R(xy) \cap R(x_1, y_1) = \emptyset$. Since each $F_{1,j}(x')$ belongs to $R(xy)$ only for the edges xy such that $\{x, y\}$ is at distance at most 1 from x', (12) follows. This in turn yields (11).

Since for each $x' \in X$ there are at most Δ^2 edges at distance at most 1 from x', (11) implies that $F_j(xy)$ is independent of all but at most

$$\left| \bigcup_{x'\in N(y)} F_{1,j}(x') \cup \bigcup_{x''\in N(x)\cap X} F_{1,j}(x'') \right| \Delta^2 \le (2\Delta - 1)\Delta^2$$

other events $F_j(x'y')$.

Therefore, by Theorem 14 with $d < 2\Delta^3$ and $p = \Delta^{-27/e^2}$, it is enough to prove that

$$e \cdot 2\Delta^3 \cdot \Delta^{-27/e^2} < 1. \tag{13}$$

To derive (13), it is enough to check that $\Delta^{-3+27/e^2} > 2e$. Since $-3 + 27/e^2 > 0.654$, this holds for $\Delta > 40$. □

4. Lower bounds

4.1. Proofs of Propositions 7 and 8

We will use the following result obtained by Kostochka and Mazurova [20] and independently by Bollobás [5]:

Theorem 16 ([5,20]). *For every Δ and g, there is a graph $G_{\Delta,g}$ with maximum degree Δ of girth at least g such that $\alpha(G_{\Delta,g}) \le \frac{2\ln\Delta}{\Delta}|V(G)|$ and $|E(G_{\Delta,g})| > (\Delta - 1)|V(G)|/2$.*

Proof of Proposition 7. Fix any $\Delta \ge 3$ and $g \ge 3$. Let $G_{\Delta,g}$ be a graph satisfying Theorem 16. Let $n = |V(G_{\Delta,g})|$. Assume that G has an injective edge coloring with k colors. Let $\{I_1, \dots, I_k\}$ be a partition of $E(G)$ into k color classes, and let I_1 be a largest color class. Then

$$|I_1| \ge \frac{(\Delta - 1)n}{2k}. \tag{14}$$

Let $V(I_1)$ denote the union of the vertex sets of all edges in I_1. By definition, every component of $G[V(I_1)]$ is a star. Hence, if we delete from $V(I_1)$ the center of each star (when the star has only two vertices, then we assign exactly one of them as the center), then we obtain an independent set $J(I_1)$ of vertices in $G_{\Delta,g}$ with $|J(I_1)| = |I_1|$. Thus by (14) and the choice of $G_{\Delta,g}$,

$$\frac{(\Delta - 1)n}{2k} \le \frac{2\ln\Delta}{\Delta} n.$$

This yields the proposition. □

Proof of Proposition 8. The argument is similar to the proof above (and simpler). For each Δ, let B be any Δ-regular bipartite graph. By Marriage Theorem, $\alpha(B) = 0.5|V(B)|$. As in the proof of Proposition 7, the number of edges in any color class of an injective edge coloring of B is at most

$$\alpha(B) = 0.5|V(B)| = \frac{|E(B)|}{\Delta},$$

so we need at least Δ colors for injective edge coloring of B. □

4.2. The bipartite configuration model

In order to prove Proposition 9, we will use a bipartite version of the configuration model. This model in different versions is due to Bender and Canfield [2] and Bollobás [4]. The bipartite version is considered in several papers. We follow the convention and use the results described in survey [27](Section 3.2) by Wormald.

Let n and D be positive integers, and

$$V_n = V_n(D) = \{v_1, \dots, v_{nD}\} \text{ and } W_n = W_n(D) = \{w_1, \dots, w_{nD}\} \text{ be disjoint sets.} \tag{15}$$

A *configuration* (of order n and degree D) is a perfect matching from V_n to W_n (each edge has one end in V_n and one in W_n). Let $\mathcal{F}_D(n)$ denote the collection of all $(Dn)!$ such matchings.

For every $F \in \mathcal{F}_D(n)$ we define the D-regular bipartite multigraph $\pi(F)$ with parts $X_n = \{x_1, \ldots, x_n\}$ and $Y_n = \{y_1, \ldots, y_n\}$ as follows: For every $j \in [n]$ we glue the D vertices $v_{D\cdot(j-1)+1}$, $v_{D\cdot(j-1)+2}, \ldots, v_{D\cdot j}$ into a new vertex x_j and the D vertices $w_{D\cdot(j-1)+1}, w_{D\cdot(j-1)+2}, \ldots, w_{D\cdot j}$ into a new vertex y_j.

Definition 1. Let $\mathcal{G}_{D,g}(n)$ be the set of all D-regular bipartite graphs with parts $X_n = \{x_1, \ldots, x_n\}$ and $Y_n = \{y_1, \ldots, y_n\}$ and girth at least g, and $\mathcal{G}'_{D,g}(n) = \{F \in \mathcal{F}_D(n) : \pi(F) \in \mathcal{G}_{D,g}(n)\}$.

Bollobás [4] and Wormald [26] proved that for each fixed g and D, there is $\epsilon(g, D) > 0$ such that

$$\frac{|\mathcal{G}'_{D,g}(n)|}{|\mathcal{F}_D(n)|} > \epsilon(g, D).$$

As discussed in [26] and [7], the same phenomenon holds for bipartite configurations. So, we will use the following fact:

Theorem 17. *For each fixed $D, g \geq 3$, if a property holds for $\pi(F)$ for almost all configurations $F \in \mathcal{F}_D(n)$, then it also holds for $\pi(G)$ for almost all $G \in \mathcal{G}'_{D,g}(n)$.*

4.3. Proof of Proposition 9

Lemma 18. *For each fixed $D \geq 21$ and almost all configurations $F \in \mathcal{F}_D(n)$, $\pi(F)$ does not have induced matchings of size*

$$k = k(D) = \left\lceil \frac{3n \ln D}{D} \right\rceil. \tag{16}$$

Proof. Given V_n and W_n as in (15), the number of matchings of size k between V_n and W_n corresponding to matchings between X_n and Y_n defined above is $\binom{n}{k}^2 \cdot D^{2k} \cdot k!$: There are $\binom{n}{k}^2$ ways to choose the k-element subsets of X_n and Y_n joined by a matching, then there are D^{2k} ways to choose the vertices in V_n and W_n to represent the chosen $2k$ vertices from X_n and Y_n, and finally there are $k!$ ways to match the k chosen vertices of V_n with the k chosen vertices in W_n.

For each such matching M, the number of configurations $F \in \mathcal{F}_D(n)$ in which M is an induced matching in the multigraph $\pi(F)$ is exactly

$$\left(\prod_{j=1}^{k(D-1)} (D(n-k)+1-j)\right)(D(n-k))!.$$

Hence the portion of $F \in \mathcal{F}_D(n)$ such that $\pi(F)$ has at least one induced matching of size k is at most

$$\begin{aligned}
&\binom{n}{k}^2 D^{2k} \cdot k! \left(\prod_{j=1}^{k(D-1)} (D(n-k)+1-j)\right) \frac{(D(n-k))!}{(Dn)!} \\
&\leq \left(\frac{n^k}{k!}\right)^2 \left(D^{2k} \cdot k!\right) \left(\prod_{j=1}^{k(D-1)} \frac{D(n-k)+1-j}{Dn+1-j}\right) \frac{1}{(D(n-k))^k} \\
&\leq \frac{(nD)^{2k}}{k!} \cdot \frac{1}{(D(n-k))^k} \left(\frac{D(n-k)}{Dn}\right)^{k(D-1)} = \frac{(nD)^k}{k!} \cdot \left(\frac{n-k}{n}\right)^{k(D-2)} \\
&\leq \left[\frac{nD \cdot e}{k}\left(1-\frac{k}{n}\right)^{D-2}\right]^k \leq \left[\frac{nD}{k} \cdot e^{1-(D-2)k/n}\right]^k.
\end{aligned}$$

By (16), $k \geq \frac{3n \ln D}{D}$, so the last expression in the brackets is at most

$$\frac{nD^2}{3n \ln D} \exp\left\{1 - \frac{3(D-2)\ln D}{D}\right\} < \frac{D^2}{\ln D} \exp\left\{-\frac{3(D-2)\ln D}{D}\right\}.$$

Since $D \geq 21$, $D - 2 > 0.9D$ and $\ln D > 3$. So

$$\frac{D^2}{\ln D} \exp\left\{-\frac{3(D-2)\ln D}{D}\right\} < \frac{D^2}{3} \exp\{-2.7 \ln D\} < \frac{1}{3D^{0.7}}.$$

It follows that the portion of $F \in \mathcal{F}_D(n)$ such that $\pi(F)$ has at least one induced matching of size k is at most $(3D^{0.7})^{-k} \to_{n\to\infty} 0$. □

Now we are ready to prove Proposition 9. By Lemma 18 together with Theorem 17, for every $\Delta \geq 21$ and $g \geq 3$, there is a Δ-regular bigraph G with girth at least g with the maximum size of an induced matching less than $\frac{3|V(G)| \ln \Delta}{2\Delta}$. Then $\chi'_s(G) > \frac{\Delta^2}{3 \ln \Delta}$.

5. Concluding remarks

1. Our proof of Theorem 15 does not work for injective list edge-coloring. We do not know how to prove the list analog of this theorem.
2. On the other hand, several parts of the proof of Theorem 4 do work for list coloring.
3. Recall that an $L(h, k)$-coloring of a graph H is a coloring f of the vertices of H with colors $1, 2, \ldots$ such that for every adjacent vertices $x, y \in V(G)$, $|f(x) - f(y)| \geq h$ and for each $u, v \in V(G)$ at distance exactly 2, $|f(u) - f(v)| \geq k$. Such colorings arose from several applications and attracted some attention, see survey [11]. In these terms, if a graph G is triangle-free (in particular, if G is bipartite), then each injective edge-coloring of G corresponds to an $L(0, 1)$-coloring of $L(G)$ and vice versa.

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