Distributed Learning-based Stability Assessment for Large Scale Networks of Dissipative Systems

Amit Jena, Tong Huang, S. Sivaranjani, Dileep Kalathil, and Le Xie

Abstract— **We propose a new distributed learning-based framework for stability assessment of a class of networked nonlinear systems, where each subsystem is dissipative. The aim is to learn, in a distributed manner, a Lyapunov function and associated region of attraction for the networked system. We begin by using a neural network function approximation to learn a storage function for each subsystem such that the subsystem satisfies a local dissipativity property. We next use a satisfiability modulo theories (SMT) solver based falsifier that verifies the local dissipativity of each subsystem by determining an absence of counterexamples that violate the local dissipativity property, as established by the neural network approximation. Finally, we verify network-level stability by using an alternating direction method of multipliers (ADMM) approach to update the storage function of each subsystem in a distributed manner until a global stability condition for the network of dissipative subsystems is satisfied. This step also leads to a network-level Lyapunov function that we then use to estimate a region of attraction. We illustrate the proposed algorithm and its advantages on a microgrid interconnection with power electronics interfaces.**

I. INTRODUCTION

Large-scale networks of nonlinear systems are frequently encountered in modeling several infrastructure networks. For example, in power grids, clusters of renewable generators, storage, and loads, modeled as nonlinear systems, are typically aggregated to form microgrids, which are networked to exchange power with each other [1]. Due to the large size and continually expanding scale of these networked systems, it is desirable to develop distributed approaches to assess the stability margins of these systems.

In this context, we consider the problem of assessing the Lyapunov stability and estimating an associated region of attraction (stability region) for a networked system comprised of nonlinear subsystems in a distributed manner. There are two key challenges in this regard. First, the problem of finding Lyapunov functions for individual subsystems is in itself non-trivial due to the nonlinear dynamics of the subsystems. Second, the stability of a network of nonlinear subsystems cannot be guaranteed merely by assessing the stability of individual subsystems.

Typically, the former challenge of finding a Lyapunov function and estimating a region of attraction has been addressed using sum-of-squares (SOS) tools [2]- [6]. However, these approaches typically do not scale well to large systems, since a large number of semidefinite programs need to be solved for SOS decomposition of polynomial systems with even a few states [2]. Another approach is to employ local linearizations and adopt quadratic Lyapunov functions to assess stability. However, this approach is typically conservative in the sense that stability can only be certified in a small vicinity of an equilibrium point of a nonlinear system, which may be insufficient to cover the normal range of operation in several practical applications such as microgrids in power systems [7], [8]. Recently, neural network based approximations of Lyapunov functions for nonlinear systems have been proposed [9], [10], and demonstrated to be less conservative than quadratic Lyapunov functions using local linearizations and sum-of-squares approaches.

The latter challenge of assessing the stability of networked nonlinear systems, based on stability of individual subsystems remains a complex open problem. However, for a special class of networked systems where each subsystem is passive or dissipative [11], [12], it is possible to assess stability of the networked system based on the dissipativity of individual subsystems and some conditions on the coupling between subsystems [13], [14]. Furthermore, fully distributed approaches to stability assessment have also been developed for such networks of passive or dissipative systems [15], [16]. However, these results are typically confined to local stability assessments based on linearized system dynamics.

In this paper, we consider the problem of assessing the Lyapunov stability and region of attraction for a class of networked systems, where each subsystem is nonlinear and satisfies a dissipativity property. For this class of systems, we propose a distributed learning-based stability assessment approach, summarized as follows. First, we use a neural network approximation to learn a storage function for each subsystem such that the subsystem satisfies a local dissipativity property. We use a satisfiability modulo theories (SMT) solver based falsifier that finds counterexamples to verify this local dissipativity property using the Lyapunov function learned by the neural network; when no counterexample is found by the falsifier, the local system is provably dissipative. Then, the stability of the networked system can be guaranteed by verifying standard conditions on the local dissipativity and coupling between dissipative subsystems [12]. For this purpose, we use an alternating direction method of multipliers (ADMM) to iteratively update the storage function for each subsystem until the network-level stability condition is satisfied. This step also allows us to compute the network-level Lyapunov function and estimate a region of attraction for the networked system.

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The key contribution of this work is the combination of neural network based approaches to learn Lyapunov functions for nonlinear systems with dissipativity-based stability analysis for networked systems to develop a scalable approach for stability assessment of large-scale networked nonlinear systems. Through a case study on networked microgrids, we demonstrate that the proposed approach has the advantage of yielding a less conservative region of attraction as compared to stability assessement based on local linearization and quadratic Lyapunov functions, as well as being more computationally tractable than SOS based assessments. Further, exploiting the dissipativity property of the subsystems also makes the approach distributed, and hence scalable to large-scale networks.

II. PROBLEM FORMULATION

We consider a large scale nonlinear system constituted by the interconnection of n subsystems. The dynamics of the i-th subsystem is described by

$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i, \mathbf{u}_i), \quad \mathbf{y}_i = \mathbf{h}_i(\mathbf{x}_i, \mathbf{u}_i), \tag{1}$$

where $\mathbf{x}_i \in \mathbb{R}^{p_i}$, $\mathbf{u}_i \in \mathbb{R}^{q_i}$, and $\mathbf{y}_i \in \mathbb{R}^{w_i}$ are the state, input, and output of the subsystem i, respectively, and the system dynamics and system output are specified by the continuously differentiable mappings $\mathbf{f}_i : \mathbb{R}^{p_i} \times \mathbb{R}^{q_i} \mapsto \mathbb{R}^{p_i}$ and $\mathbf{h}_i : \mathbb{R}^{p_i} \times \mathbb{R}^{q_i} \mapsto \mathbb{R}^{w_i}$, respectively.

We then define the state, input, and output vectors of the networked system as $\mathbf{x} = [\mathbf{x}_1^\top, \dots, \mathbf{x}_n^\top]^\top$, $\mathbf{u} = [\mathbf{u}_1^\top, \mathbf{u}_2^\top, \dots, \mathbf{u}_n^\top]^\top$ and $\mathbf{y} = [\mathbf{y}_1^\top, \mathbf{y}_2^\top, \dots, \mathbf{y}_n^\top]^\top$, respectively. Then, the dynamics of the networked system constituted by the interconnection of these n subsystems can be written as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad \mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}), \tag{2}$$

where $\mathbf{f} = [\mathbf{f}_1^\top, \dots, \mathbf{f}_n^\top]^\top$, and $\mathbf{h} = [\mathbf{h}_1^\top, \dots, \mathbf{h}_n^\top]^\top$, and the subsystems are coupled with each other through a nonlinear map

$$\mathbf{u} = \mathbf{g}(\mathbf{y}), \tag{3}$$

where the nonlinear map $\mathbf{g}(\cdot)$ describes the coupling between the subsystems.

Without loss of generality, assume that the equilibrium point of (2) is given by $(x^*, u^*, y^*) = \mathbf{o}$, where $\mathbf{o}$ is the origin of the state space. For simplicity in the formulation, we further restrict consideration to a linear approximation of the coupling between subsystems (3), given by

$$\mathbf{u} = M\mathbf{y} \tag{4}$$

where $M \in \mathbb{R}^{q_i \times w_i}$ is the Jacobian matrix of $\mathbf{g}$ evaluated at $\mathbf{o}$. Then, the dynamics of the networked system constituted by the interconnection of these n subsystems can be represented as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}), \quad \mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}), \quad \mathbf{u} = M\mathbf{y}, \tag{5}$$

for all $(x, u, y) \in B_{\mathbf{o}}$, where $B_{\mathbf{o}}$ is a neighborhood around the origin $\mathbf{o}$.

Further, we assume that each subsystem satisfies a local dissipativity property, as follows.

Assumption 1. (Dissipative System [12]) Every subsystem $i \in \{1, 2, \dots, n\}$, described by (1) is ***locally dissipative*** with respect to a *supply rate function* $r_i(\mathbf{u}_i, \mathbf{y}_i) = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{y}_i \end{bmatrix}^T R_i \begin{bmatrix} \mathbf{u}_i \\ \mathbf{y}_i \end{bmatrix}$, where $R_i = \begin{bmatrix} R_i^{11} & R_i^{12} \\ R_i^{21} & R_i^{22} \end{bmatrix}$, if there exists a continuously differentiable *storage function* $\tilde{V}_i : \mathbb{R}^{p_i} \mapsto \mathbb{R}$ that satisfies the following conditions:

$$\tilde{V}_i(\mathbf{x}_i) \geq 0, \quad \tilde{V}_i(\mathbf{o}) = 0, \quad \nabla \tilde{V}_i \mathbf{f}_i(\mathbf{x}_i, \mathbf{u}_i) < r_i(\mathbf{u}_i, \mathbf{y}_i), \tag{6}$$

for all $(\mathbf{x}_i, \mathbf{u}_i, \mathbf{y}_i) \in B_{\mathbf{o}_i}$, where $B_{\mathbf{o}_i}$ is a neighborhood around the origin $\mathbf{o}$.

Next, we introduce the following definitions.

Definition 1 (Lyapunov function and Region of Attraction [17])**.** For the system (5), a continuous differentiable scalar function $V(\mathbf{x})$ is a *strict Lyapunov function* valid in a region $\mathcal{V}_\mathcal{S} := \{\mathbf{x} : \|\mathbf{x}\|_2 \leq \mathcal{S}\}$ if the following conditions are satisfied: (i) $V(\mathbf{o}) = 0$, (ii) $V(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathcal{V}_\mathcal{S}$, (iii) $\dot{V} = \nabla V(x)\dot{\mathbf{x}} < 0$, for all $\mathbf{x} \in \mathcal{V}_\mathcal{S} \backslash \mathbf{o}$. Further, the region $\mathcal{V}_\mathcal{S}$ is defined as the *region of attraction* of the equilibrium point $\mathbf{o}$, that is, $\mathbf{x}(0) \in \mathcal{V}_\mathcal{S} \implies \lim_{t\to\infty} \mathbf{x}(t) = \mathbf{o}$.

The stability of the networked system (5) can be assessed by identifying a Lyapunov function $V(\mathbf{x})$ satisfying Definition 1. If the subsystems are dissipative as described in Assumption 1, then such a Lyapunov function for the networked system can be obtained from the local storage functions of each subsystem as follows.

Proposition 1 (Proposition 2.1, [12])**.** Consider the networked system (5) where each subsystem $i \in \{1, 2, \dots, n\}$ given by (1) satisfies Assumption 1. If

$$\begin{bmatrix} M \\ I \end{bmatrix}^T \mathbf{R}\left(R_1, R_2 \cdots, R_n\right) \begin{bmatrix} M \\ I \end{bmatrix} \preceq 0, \tag{7}$$

where $\mathbf{R}\left(R_1, R_2 \cdots, R_n\right) =$

$$\begin{bmatrix} R_1^{11} & & & R_1^{12} & & \\ & \ddots & & & \ddots & \\ & & R_n^{11} & & & R_n^{12} \\ R_1^{21} & & & R_1^{22} & & \\ & \ddots & & & \ddots & \\ & & R_n^{21} & & & R_n^{22} \end{bmatrix}, \tag{8}$$

then, (i) the origin $\mathbf{o}$ is stable equilibrium of the networked system (5), and (ii) $V(\mathbf{x}) = \sum_{i=1}^{n} \tilde{V}_i(x_i)$ is a Lyapunov function of the networked system (5).

In general, for nonlinear systems, sum-of-squares approaches [2]- [6] can be employed to assess stability, either directly or by determining storage functions in the case of dissipative systems [18]. Alternatively, local linearizations of (5) may be used to compute a quadratic Lyapunov function. However, these approaches are typically conservative in their estimation of the region of attraction [7], [8] and computational tractability [2]. Further, they are not typically scalable to large networked systems. In this context, we address the following problem.

Problem 1. Given the networked system (5) where each subsystem is dissipative and satisfies Assumption 1, we aim to develop a distributed approach to assess the stability of the networked system (5) by (i) learning local storage functions $\tilde{V}_i(x_i)$ at each subsystem $i \in \{1, 2, \ldots, n\}$, (ii) verifying the stability condition (7), and (iii) computing the Lyapunov function $V(\mathbf{x})$ and the associated region of attraction using Proposition 1.

We describe our solution approach and detailed algorithm in the next section.

III. Distributed Learning of Neural Lyapunov Function

We propose a distributed approach for learning a Lyapunov function for the networked system described in (5) and characterizing its stability region. As mentioned before, finding a Lyapunov function through the SOS method does not scale well to large systems. Though we can leverage on the recently proposed idea of learning a Lyapunov function using neural networks [8]–[10], the training time and computational complexity of this approach will also scale exponentially with the dimension of the system. In particular, this can be prohibitively high for networked systems constituted by the interconnection of a large number of subsystems.

The key idea of our proposed approach is to decompose the problem of learning a Lyapunov function for the whole system into a distributed learning problem, where the dimension of each subproblem is much smaller than that of the centralized problem. In particular, given a supply rate function r_i, we will learn a local neural storage function for each subsystem i by adapting the approach proposed in [9]. We will verify the dissipativity condition (6) with respect to the learned local neural storage function and the assumed supply rate function using an SMT solver. We will then verify the global stability condition given in (7). If this condition is not satisfied, we will update the supply rate function of each subsystem using an ADMM approach and repeat the above procedure. Once this procedure converges, by Proposition 1, we will have a neural Lyapunov function for the whole system obtained as the sum of local neural storage functions. The stability region of the system can also be computed using this neural Lyapunov function. The training time and computation complexity of this approach will scale only linearly in the number of subsystem as opposed to the exponential scaling of any centralized approach.

We now give the details of each step below.

A. Learning Local Neural Storage Function

We will follow the neural Lyapunov framework proposed in [9] with some modification for learning a local neural storage function for each subsystem. We will design a loss function that penalizes any violation of the dissipativity condition given in (6). In particular, we will use the following empirical loss function for each subsystem i:

$$L(\theta_i) = \tilde{V}^2_{\theta_i}(0) + \frac{1}{N}\sum_{i=1}^{N}\Big[\max(-\tilde{V}_{\theta_i}(x_i), 0) + \max(0, \nabla V_{\theta_i}(x_i) f_i(x_i, u_i) - r_i(u_i, y_i))\Big], \quad (9)$$

where θ_i is the parameter of the local neural storage function $\tilde{V}_{\theta_i}$, r_i is the supply function, and N is the number of training samples. We will train a neural network for each subsystem using the above loss function until convergence.

Any deep learning algorithm using neural networks can only give an empirical guarantee with respect to its loss function. So, the parameters of the neural network obtained by minimizing the above empirical loss function may not *provably* yield a storage function that satisfies the condition for dissipativity (6). Following the approach used in [9], we construct a falsifier which searches for a counterexample that violates the dissipativity condition. If the falsifier finds a counterexample, that is added to the training data and the neural network is re-trained.

The falsification constraint is specified using the following first-order logic formula over real numbers:

$$\Phi(x_i, u_i) := \Big(\varepsilon_{u_i} \geq \|x_i\|^2 \geq \varepsilon_{l_i}\Big) \wedge \Big(\varepsilon_{u_i} \geq \|u_i\|^2 \geq \varepsilon_{l_i}\Big) \wedge \Big(\tilde{V}_{\theta_i}(x_i) \leq 0 \ \vee \ \nabla V_{\theta_i}(x_i) f_i(x_i, u_i) - r_i(u_i, y_i)) \geq 0\Big), \quad (10)$$

where ε_{l_i} is a small positive constant to avoid issues such as arithmetic underflow in numerical verification algorithms, and ε_{u_i} is a general upper bound. We use the SMT solver dReal [19] to solve the falsification constraint above.

Under the conditions given [9], [19], [20] the SMT solver is guaranteed to return counterexamples if there are any. If the falsifier is not able to find any counterexample, we stop the training and accept the local neural storage function. More precisely, at the end of this learning phase, we will get a local neural storage function $\tilde{V}_{\theta_i}$ for each subsystem with respect to the storage function r_i we considered.

B. Distributed Learning of Lyapunov Function

After learning the local neural storage function for each subsystems, we form a candidate neural Lyapunov function $V(x)$ for the whole system as, $V(x) = \sum_i V_{\theta_i}(x_i)$. We can then verify if this candidate is indeed a valid Lyapunov function by checking the condition (7). If this condition is not satisfied, we will update the earlier used supply rate function r_i of each system and repeat the procedure described in the previous subsection. Since the supply rate function r_i is completely specified by the matrix R_i, we will use an alternating direction method of multipliers (ADMM) method for updating the supply rate functions as suggested in [12, Chapter 6]. We describe the procedure below.

To bring the condition (7) to the canonical optimization form, we will define the indicator function $\mathbb{I}_{\text{global}}$ as

$$\mathbb{I}_{\text{global}}(R_1, \ldots, R_n) := \begin{cases} 0 & \text{if (7) holds true} \\ \infty & \text{otherwise.} \end{cases} \quad (11)$$

Similarly, to bring the dissipativity condition (6) of each subsystem to the optimizations form, we define n indicator functions $\mathbb{I}_{\text{local},i}$ as

$$\mathbb{I}_{\text{local},i}(R_i, \tilde{V}_{\theta_i}) := \begin{cases} 0 & \text{if (6) holds true} \\ \infty & \text{otherwise.} \end{cases} \tag{12}$$

Now, the problem of finding a Lyapunov function for the whole system can be written as the following optimization problem:

$$\begin{aligned} \min_{(R_i, \tilde{V}_{\theta_i}, Z_i)_{i=1}^n} \quad & \sum_{i=1}^n \mathbb{I}_{\text{local},i}\left(R_i, \tilde{V}_{\theta_i}\right) + \mathbb{I}_{\text{global}}\ (Z_1, \ldots, Z_n) \\ \text{s.t.} \quad & R_i - Z_i = 0 \quad \text{for } i = 1, \ldots, n. \end{aligned} \tag{13}$$

Here Z_is are auxiliary variables of R_is that allow the entire optimization problem to be partitioned into $n+1$ simpler ones, and then to be solved in a distributed manner. The iterative steps of the algorithm are as follows:

$$R_i^{k+1} = \underset{R_i \text{ s.t. } (6)}{\arg\min} \|R_i - Z_i^k + S_i^k\|_F^2, \tag{14a}$$

$$(Z_i^{k+1})_{i=1}^n = \underset{(Z_i)_{i=1}^n \text{ s.t. } (7)}{\arg\min} \sum_{i=1}^n \|R_i^{k+1} - Z_i + S_i^k\|_F^2, \tag{14b}$$

$$S_i^{k+1} = R_i^{k+1} - Z_i^{k+1} + S_i^k, \tag{14c}$$

where S_is are matrices to establish consensus among the all the R_is and $\|.\|_F$ is the Frobenius norm.

For each subsystem i, at each iteration k, the above described optimization approach gives a supply rate function R_i^{k+1} close to Z_i^k suggested by the global problem. For the newly obtained supply rate function R_i^{k+1}, we retrain the neural network to get a new local storage function $\tilde{V}_{\theta_i}$. We repeat this procedure until convergence or until a termination condition is reached.

We formally present our algorithm in Algorithm 1.

Algorithm 1 Distributed Neural Lyapunov Function Learning Algorithm

1: **Input:** Initial supply rate matrices $(R_i)_{i=1}^n$, tolerance level δ, initial consensus metrics $(S_i)_{i=1}^n$
2: **Output:** Local neural storage functions $(\tilde{V}_{\theta_i})_{i=1}^n$, global Lyapunov function $V = \sum_i^n \tilde{V}_{\theta_i}$
3: **repeat**
4: Learn $(\tilde{V}_{\theta_i})_{i=1}^n$ according to the procedure described in Section III-A, for the current supply rate functions $(R_i)_{i=1}^n$
5: Update $(Z_i)_{i=1}^n$ according to (14b)
6: Update $(S_i)_{i=1}^n$ according to (14c)
7: Update $(R_i)_{i=1}^n$ according to (14a)
8: **until** $(\sum_i ||R_i - Z_i|| > \delta)$

IV. CASE STUDIES

In this section, we illustrate the proposed algorithm in the context of networked microgrids, and compare its results with the a centralized neural Lyapunov based approach [7], [8], and a local linearization approach.

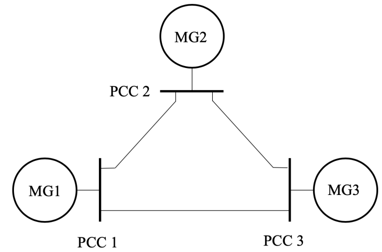

Fig. 1. Networked Microgrids [8]

A. Test System Description and Simulation Setup

We consider three networked microgrids (MGs) with power electronics (PE) interfaces shown in Fig. 1. These MGs are connected in a distribution network. They can share power through their PE interfaces. The dynamics of each MG depends on the control scheme deployed in the PE interface [8]. Without loss of generality, suppose that a voltage angle droop control scheme [7], [8], [21], is deployed in all PE interfaces. With the time-scale separation assumption [7], the fast dynamics of the phase angle of the voltage at the i-th interface can be described by [7], [22], [23]

$$\alpha_i \dot{\delta}_i = \beta_i (P_i^* - P_i) - \delta_i, i \in \{1, 2, 3\},$$

where δ_i is the deviation of the voltage phase angle at the i-th PCC (Point of Common Coupling) from its nominal value δ_i^*; P_i is the real power injection to the i-th PCC; P_i^* is the nominal value of P_i; and α_i and β_i are programmable control parameters of the power electronic interface. Nominal values δ_i^* and P_i^* are determined by the system operator according to economic dispatch [24]. The i-th microgrid is coupled with the other microgrids for power sharing through distribution lines. The net power output of each microgrid can be written as the sum of the power exchanged by the microgrid with other microgrids, and the power used by the microgrid itself locally, using standard AC power flow equations as follows. The net power output of the i-th microgrid is given by:

$$P_i = \sum_{k \neq i} E_i E_k Y_{ik} \cos(\delta_{ik}^* + \delta_{ik} - \gamma_{ik}) + E_i^2 G_{ii}, i \in \{1, 2, 3\},$$

where E_i is the voltage magnitude at the i-th microgrid, assumed to be constant because of the time-scale separation assumption [24]; $\delta_{ik} = \delta_i - \delta_k$; $\delta_{ik}^* = \delta_i^* - \delta_k^*$, and Y_{ik}, G_{ii}, and α_{ik} are distribution line parameters. The dynamics of the networked microgrids can then be written as

$$\begin{aligned} &\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{g}(C\mathbf{x}), \\ &\mathbf{x} = [\delta_1, \delta_2, \delta_3]^\top, \quad A = \text{diag}\left(-\frac{1}{\alpha_1}, -\frac{1}{\alpha_2}, -\frac{1}{\alpha_3}\right), \\ &B = \text{diag}\left(\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \frac{\beta_3}{\alpha_3}\right), \quad C = I \\ &\mathbf{g}(\mathbf{x}) = [g_1(\mathbf{x}), g_2(\mathbf{x}), g_3(\mathbf{x})]^\top \\ &g_i(\mathbf{x}) = P_i^* - \sum_{k \neq i} E_i E_k Y_{ik} \cos(\delta_{ik}^* + \delta_{ik} - \gamma_{ik}) - E_i^2 G_{ii}. \end{aligned}$$

As discussed in Section II, we would first obtain a linear approximation of the map g_i to implement the proposed approach in Section III. We obtain such a linear approximation by assuming δ_{ik} to be small, which leads to $\sin(\delta_{ik}) \approx \delta_{ik}$ and $\cos(\delta_{ik}) \approx 1$. Using this approximation, after some basic algebra, we can express $\mathbf{g}(\mathbf{x}) = Mx$.

The control parameters and reference values for the voltages, angles, and power outputs are reported in Table. I.

B. Distributed Stability Assessment

Following the approach in Section III-A, we begin by learning an approximation of the local storage function $\tilde{V}_i(\mathbf{x}_i)$ for each microgrid. For this purpose, we use a neural network with a single hidden layer of size 6, and $\tanh$ activation functions. The training sample size and learning rate used are 2000 and 0.01 respectively. We then verify the local dissipativity of each subsystem using the learned storage function using an SMT-based falsifier with uniform lower and upper bounds of $\epsilon_{l_i} = 0.05$ and $\epsilon_{u_i} = 0.8$ respectively for all $i \in \{1,2,3\}$. Finally, we employ the ADMM based distributed approach in Algorithm 1 to adjust the learned storage functions to guarantee the global stability condition 7. In Algorithm 1, the initial R_i and S_i for each i are chosen as $\begin{bmatrix} -0.5 & 0.1 \\ 0.1 & -0.5 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ respectively. The tolerance δ to decide the convergence of our distributed algorithm is set to be 10^{-6}.

With the above parameters, Algorithm 1 returns a neural Lyapunov function as well as three local neural storage functions. The Lyapunov function and its time derivative are visualized in Fig. 2 (some plots are omitted due to space constraints). Given $\delta_3 = 0$, Fig. 2-(a) visualizes the learned Lyapunov function in the δ_1-δ_2-V space. Fig. 2-(b) also shows a similar plot. Fig. 2-(c) visualizes the time derivative of the neural Lyapunov function. We note that while the algorithm learns a Lyapunov function with a linear approximation of the coupling relationship, the time derivative is evaluated based on the original nonlinear dynamics without any approximation. From the figures, it is clear that the neural Lyapunov function is positive definite and its time derivative is non-positive in the region $\mathcal{V}_{0.8}$, which makes it a valid Lyapunov function satisfying Definition 1.

Based on the network-level Lyapunov function (labelled as DNL) learned using Algorithm 1, we estimate the ROA for the nonlinear networked microgrids, and compare it with the ROA estimated by the neural Lyapunov function learned by the centralized method (labeled as CNL) proposed in [7], as well as the ROA using a quadratic Lyapunov function (labeled as QL) corresponding to linearization of the dynamics [25]. As seen in Fig. 3 and Table II, the ROA estimated by DNL method is comparable to the ROA estimated by CNL method and is much less conservative than the ROA estimated by the QL method.

TABLE I
CONTROL PARAMETERS AND REFERENCE SETPOINTS OF THE THREE MICROGRID NETWORK [7]

	MG 1	MG 2	MG 3
α_i	1.2	1.0	0.8
β_i	1.2	1.2	1.2
δ_i^*	0	$55.67°$	$-45.37°$
E_i(p.u.)	1	1.05	0.95
P_i^*(p.u.)	0.1706	1.4578	-0.0013

TABLE II
AREA OF PROJECTED ROA ESTIMATED BY THE THREE METHODS

Plane	DNL	CNL [7]	QL
δ_1, δ_2	1.374	2.058	0.178
δ_2, δ_3	1.325	1.965	0.154
δ_1, δ_3	1.379	2.074	0.180

To summarize the case study, we observe that the proposed distributed algorithm can effectively assess the stability of networked nonlinear systems in a scalable manner, while being less conservative than typical SOS or linearization based approaches.

V. CONCLUSION

This paper proposes a novel distributed learning-based framework for assessing Lyapunov stability of a class of networked nonlinear systems, where each subsystem is dissipative. The objective of the proposed framework is to construct a Lyapunov function in a distributed manner and to estimate the associated region of attraction for the networked system. We begin by leveraging a neural network function approximation to learn a storage function for each subsystem such that a local dissipativity property is satisfied by each subsystem. We next use a satisfiability modulo theories (SMT) solver based falsifier that verifies the local dissipativity of each subsystem by determining an absence of counterexamples that violate the local dissipativity property, as established by the neural network approximation. Finally, we verify network-level stability by using an alternating direction method of multipliers (ADMM) approach to update the storage function of each subsystem in a distributed manner until a global stability condition for the network of dissipative subsystems is satisfied. This step also leads to a network-level Lyapunov function that we then use to estimate the region of attraction. We illustrate the proposed algorithm and its advantages on a three networked microgrids with power-electronic interfaces. Future work will test and validate the proposed algorithm in larger networked systems and investigate how to learn controllers with provable performance for large-scale networked systems.

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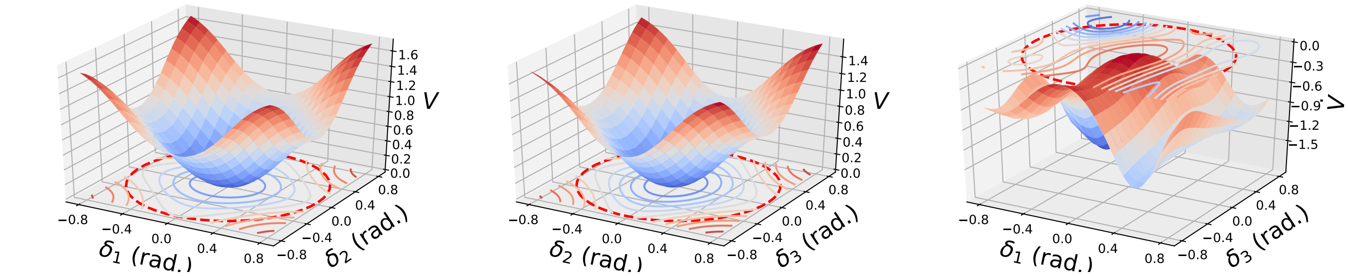
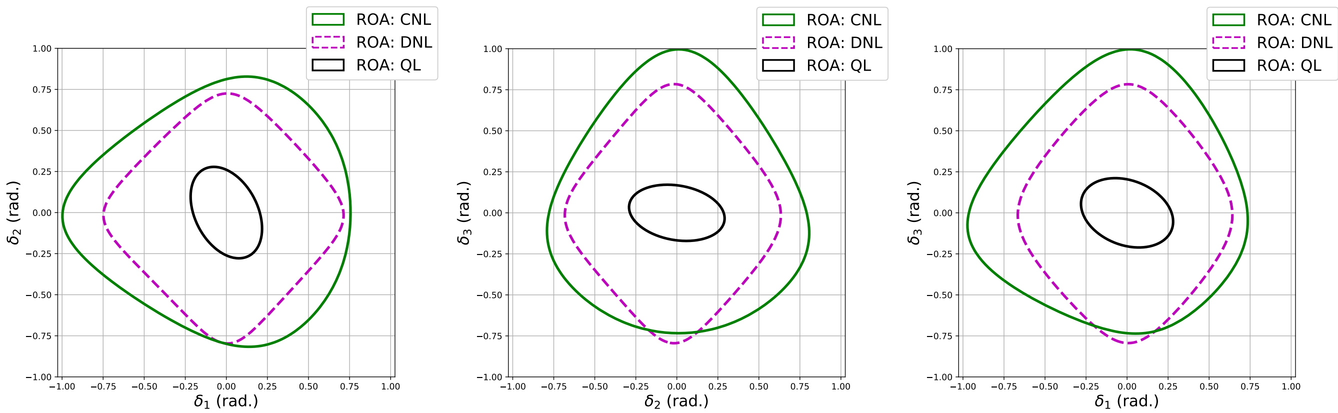


(a) (b) (c)

Fig. 2. Visualization of the distributed neural Lyapunov function (V), and its time derivative ($\dot{V}$) for the nonlinear system described in Section IV-A. (a) projection of V on the δ_1-δ_2 space with $\delta_3 = 0$; (b) projection of V on the δ_2-δ_3 space with $\delta_1 = 0$; and (c) projection of $\dot{V}$ on the δ_1-δ_3 space with $\delta_2 = 0$. Rest of the projection plots of V and $\dot{V}$ have been omitted due to space constraints. The non-positive values of the time derivative shows that this is a valid local Lyapunov function for the nonlinear dynamics.



(a) (b) (c)

Fig. 3. Visualization of regions of attraction (ROA) associated with the proposed neural Lyapunov function learned in a distributed manner (DNL), the neural Lyapunov function learned in a centralized manner (CNL), and a quadratic Lyapunov function (QL): (a) projection of ROAs on the δ_1-δ_2 space with $\delta_3 = 0$; (b) projection of ROAs on the δ_2-δ_3 space with $\delta_1 = 0$; and (c) projection of ROAs on the δ_1-δ_3 space with $\delta_2 = 0$.

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