# Optimal Time Decay Rates for a Chemotaxis Model with Logarithmic Sensitivity

**Yanni Zeng and Kun Zhao**

**Abstract** We consider a Keller-Segel type chemotaxis model with logarithmic sensitivity and density-dependent production/consumption rate. It is a $2 \times 2$ reaction-diffusion system describing the interaction of cells and a chemical signal. We study Cauchy problem for the original system and its transformed system, which is one of hyperbolic-parabolic conservation laws. In both cases of diffusive and non-diffusive chemical, we obtain optimal $L^2$ time decay rates for the solution. Our results improve those in Li et al. (Nonlinearity 28:2181-2210, 2015 [5]), Martinez et al. (Indiana Univ Math J 67:1383-1424, 2018 [7]).

**Keywords** Conservation laws · Hyperbolic-parabolic · Reaction-diffusion · Asymptotic behavior · Time decay

## 1 Introduction

In this paper we consider Cauchy problem of a Keller-Segel type chemotaxis model:

$$\begin{cases} s_t = \varepsilon s_{xx} - \mu u s - \sigma s, \\ u_t = D u_{xx} - \chi [u (\ln s)_x]_x, \end{cases} \qquad x \in \mathbb{R}, \ \ t > 0, \tag{1}$$

$$(s, u)(x, 0) = (s_0, u_0)(x), \qquad x \in \mathbb{R}. \tag{2}$$

Here the unknown functions $s = s(x, t)$ and $u = u(x, t)$ are the concentration of a chemical signal and the density of a cellular population, respectively. The constant system parameters are $\varepsilon \geq 0$, $\mu \neq 0$, $\sigma \geq 0$, $D > 0$ and $\chi \neq 0$, standing for

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the diffusion coefficient of chemical signal, coefficient of density-dependent production/consumption rate of chemical signal, natural degradation rate of chemical signal, diffusion coefficient of cellular population, and coefficient of chemotactic sensitivity, respectively. Equation (1) describes the movement of a cellular population in response to a chemical signal, while both entities are naturally diffusing and producing/degrading in the local environment.

Equation (1) is a system of reaction-diffusion equations. It can be transformed into a system of hyperbolic-parabolic conservation laws by the inverse Hopf-Cole transformation [4]:

$$v = (\ln s)_x = \frac{s_x}{s}. \tag{3}$$

The new system under the variables $v$ and $u$ reads:

$$\begin{cases} v_t + (\mu u - \varepsilon v^2)_x = \varepsilon v_{xx}, \\ u_t + \chi (uv)_x = D u_{xx}. \end{cases} \tag{4}$$

Throughout this paper we assume

$$\chi\mu > 0, \tag{5}$$

which includes two scenarios: $\chi > 0$ and $\mu > 0$, or $\chi < 0$ and $\mu < 0$. The former is interpreted as cells are attracted to and consume the chemical. The latter describes cells depositing the chemical to modify the local environment for succeeding passages [8]. Further discussion on (5) can be found in [9].

Under assumption (5), (4) can be simplified by using rescaled variables [9]:

$$\tilde{t} = \frac{\chi\mu}{D} t, \qquad \tilde{x} = \frac{\sqrt{\chi\mu}}{D} x, \qquad \tilde{v} = \text{sign}(\chi)\sqrt{\frac{\chi}{\mu}}\, v, \qquad \tilde{u} = u. \tag{6}$$

This simplifies (4) to

$$\begin{cases} v_t + (u - \varepsilon_2 v^2)_x = \varepsilon_1 v_{xx}, \\ u_t + (uv)_x = u_{xx}, \end{cases} \qquad x \in \mathbb{R}, \quad t > 0 \tag{7}$$

after dropping the tilde accent. Here the new parameters are

$$\varepsilon_1 = \frac{\varepsilon}{D} \geq 0, \qquad \varepsilon_2 = \frac{\varepsilon}{\chi}. \tag{8}$$

The initial condition for (7) is

$$(v, u)(x, 0) = (v_0, u_0)(x), \quad x \in \mathbb{R}. \tag{9}$$

For a general background on (1), (7) and related models, readers are referred to [5, 7, 9] and references therein. Here we focus on results directly related to this paper. Equation (7) is a system of hyperbolic-parabolic conservation laws. If Cauchy data are small perturbations of a constant state $(0, \bar{u})$ with $\bar{u} > 0$, the $L^2$ theory of (7), (9) is well understood. This includes local existence, global existence, asymptotic decay rates, and convergence to an asymptotic solution, as a direct application of Kawashima's theory [2, 3]. Similarly, pointwise estimates hence $L^p$ theory with $p \geq 1$ are available, also as an application of the general theory [6].

If $(v_0, u_0)$ is prescribed around $(0, \bar{u})$ but $(v_0, u_0 - \bar{u})$ has finite $H^2$ norm that is not necessary small, global existence of solution to (7), (9) has been established in [1, 5, 11] for the case $\varepsilon = 0$. In particular, under the additional zero-mass assumption on the perturbation and the smallness assumption on the initial perturbation and its anti-derivative, algebraic time decay rates in the $L^2$ framework are established in [5]. For the case $\varepsilon > 0$, similar results are obtained in [7].

The time decay rates in [5, 7] are obtained by energy and weighted energy methods. Although the rates are one can possibly have via those methods, they are not optimal. Here our purpose is to improve those rates to optimal ones through an iteration scheme based on spectral analysis, Green's function and Duhamel's principle. We also obtain corresponding rates for the original variables, i.e., the solution to (1), (2). In particular, we establish optimal rates of $s$ and its derivatives in the border case $-\mu\bar{u} = \sigma$. This answers a question posted in [5], see Remark 1.2 therein. We further comment that similar results are obtained recently when (1) or (7) has a logistic growth term in the equation for cells [10].

Next we formulate the results from [5, 7], as they are the starting point of our analysis. We consider the scenario that $(s_0, u_0)$ in (2) is prescribed around a constant state $(\bar{s}, \bar{u})$, where $\bar{s}$ and $\bar{u}$ are positive. Correspondingly, from (3) and (6) we have $(v_0, u_0)$ in (9) as prescribed around $(0, \bar{u})$. From (7), both $v$ and $u - \bar{u}$ are conserved quantities. In particular, from (3) and (6),

$$\int_{\mathbb{R}} v(x,t)\,dx = \int_{\mathbb{R}} v_0(x)\,dx = \operatorname{sign}(\chi)\sqrt{\frac{\chi}{\mu}}\int_{\mathbb{R}} \frac{d}{dx}(\ln s_0(x))\,dx = 0.$$

If we assume $\int_{\mathbb{R}}[u_0(x) - \bar{u}]\,dx = 0$, we also have

$$\int_{\mathbb{R}}[u(x,t) - \bar{u}]\,dx = 0.$$

These allow us to define anti-derivatives:

$$\psi(x,t) \equiv \int_{-\infty}^{x} v(y,t)\,dy, \quad \phi(x,t) \equiv \int_{-\infty}^{x} [u(y,t)-\bar{u}]\,dy. \tag{10}$$

$$\psi_0(x) \equiv \psi(x,0) = \int_{-\infty}^{x} v_0(y)\,dy, \quad \phi_0(x) \equiv \phi(x,0) = \int_{-\infty}^{x} [u_0(y)-\bar{u}]\,dy. \tag{11}$$

We introduce some notations. Throughout this paper we use $C$ to denote a universal positive constant, depending only on the system parameters and initial data. We also use the following notations to abbreviate the norms of Sobolev spaces with respect to $x$:

$$\|\cdot\|_k = \|\cdot\|_{H^k(\mathbb{R})}, \quad \|\cdot\| = \|\cdot\|_{L^2(\mathbb{R})}.$$

**Theorem 1** ([5]) *Suppose that $u_0 \geq 0$, $\bar{u} > 0$, $(\psi_0, \phi_0) \in H^3(\mathbb{R})$ and there exists a sufficiently small constant $\eta_0 > 0$ such that $\|\psi_0\|_1^2 + \|\phi_0\|^2 \leq \eta_0$. Then there exists a unique global solution to (7)–(9) withl $\varepsilon = 0$, satisfying $v \in C([0,\infty); H^2(\mathbb{R})) \cap L^2([0,\infty); H^2(\mathbb{R}))$ and $u - \bar{u} \in C([0,\infty); H^2(\mathbb{R})) \cap L^2([0,\infty); H^3(\mathbb{R}))$. Moreover, the solution has the decay estimate:*

$$\begin{aligned}&\sum_{k=0}^{2}(t+1)^{k+1}\|D_x^k(v,u-\bar{u})\|^2(t) + \sum_{k=1}^{2}\int_0^t (\tau+1)^k \|D_x^k v\|^2(\tau)\,d\tau \\ &+ \sum_{k=1}^{3}\int_0^t (\tau+1)^k \|D_x^k u\|^2(\tau)\,d\tau \leq C \qquad t > 0.\end{aligned} \tag{12}$$

We comment that the statement of Theorem 1 is slightly different from Theorem 1.3 of [5]. This can be justified by a simple iteration, using Theorem 1.1 in [5]. See a similar argument for the model with logistic growth in [10].

**Theorem 2** ([7]) *Suppose that $u_0 \geq 0$, $\bar{u} > 0$, $(\psi_0, \phi_0) \in H^3(\mathbb{R})$ and there exists a sufficiently small constant $\eta_0 > 0$ such that $\|(\psi_0, \phi_0)\|^2 \leq \eta_0$. Then there exists a unique global solution to (7)–(9) with $\varepsilon > 0$, satisfying $(v, u-\bar{u}) \in C([0,\infty); H^2(\mathbb{R})) \cap L^2([0,\infty); H^3(\mathbb{R}))$. Moreover, the solution has the decay estimate: For $t > 0$,*

$$\sum_{k=0}^{2}(t+1)^{k+1}\|D_x^k(v,u-\bar{u})\|^2(t) + \sum_{k=0}^{3}\int_0^t (\tau+1)^k \|D_x^k(v,u-\bar{u})\|^2(\tau)\,d\tau \leq C. \tag{13}$$

Our main results are the following theorems. The first one improves the $L^2$ decay rates of $(v, u-\bar{u})$ and its derivatives in (12) and (13) to optimal ones. The second one concerns the original variables $s$ and $u$, or the solution to (1), (2).

**Theorem 3** *Assume that $u_0 \geq 0$, $\bar{u} > 0$, and $(\psi_0, \phi_0) \in H^3(\mathbb{R}) \cap L^1(\mathbb{R})$.*

- *There exists a sufficiently small constant* $\eta_0 > 0$ *such that if* $\|\psi_0\|_1^2 + \|\phi_0\|^2 \le \eta_0$, *the unique global solution to* (7)–(9) *with* $\varepsilon = 0$, *given in Theorem 1, satisfies*

$$\sum_{k=0}^{1}(t+1)^{\frac{3}{4}+\frac{k}{2}}\|D_x^k(v, u-\bar{u})\|(t) \le C, \qquad t > 0. \tag{14}$$

- *There exists a sufficiently small constant* $\eta_0 > 0$ *such that if* $\|(\psi_0, \phi_0)\|^2 \le \eta_0$, *the unique global solution to* (7)–(9) *with* $\varepsilon > 0$, *given in Theorem 2, satisfies*

$$\sum_{k=0}^{2}(t+1)^{\frac{3}{4}+\frac{k}{2}}\|D_x^k(v, u-\bar{u})\|(t) \le C, \qquad t > 0. \tag{15}$$

**Theorem 4** *Assume that* $s_0 > 0$, $\bar{s} > 0$, $u_0 \ge 0$, $\bar{u} > 0$, *and* $\phi_0$ *be defined in* (11). *Let* $(s_0 - \bar{s}, \phi_0) \in H^3(\mathbb{R}) \cap L^1(\mathbb{R})$. *Then there exists a sufficiently small constant* $\eta_0 > 0$ *such that if* $\|s_0 - \bar{s}\|_1^2 + \|\phi_0\|^2 \le \eta_0$, *the Cauchy problem* (1), (2) *with* $\varepsilon \ge 0$ *has a unique classical solution for* $t \ge 0$, *satisfying* $s(x,t) > 0$ *and* $u(x,t) \ge 0$. *We write*

$$s(x,t) = e^{-(\mu\bar{u}+\sigma)t}\tilde{s}(x,t). \tag{16}$$

*Then the solution has the decay property for* $t > 0$ *as follows: If* $\varepsilon = 0$,

$$\sum_{k=0}^{2}(t+1)^{\frac{1}{4}+\frac{k}{2}}\|D_x^k(\tilde{s}-\bar{s})\|(t) + \sum_{k=0}^{1}(t+1)^{\frac{3}{4}+\frac{k}{2}}\|D_x^k(u-\bar{u})\|(t) \le C. \tag{17}$$

*If* $\varepsilon > 0$,

$$\sum_{k=0}^{3}(t+1)^{\frac{1}{4}+\frac{k}{2}}\|D_x^k(\tilde{s}-\bar{s})\|(t) + \sum_{k=0}^{2}(t+1)^{\frac{3}{4}+\frac{k}{2}}\|D_x^k(u-\bar{u})\|(t) \le C. \tag{18}$$

We prove Theorem 3 in Sect. 2, and Theorem 4 in Sect. 3.

## 2 Decay Rates for the Transformed System

We write (7) in terms of the perturbation. Let

$$\begin{aligned}
&\tilde{u} = u - \bar{u}, \quad \tilde{u}_0 = u_0 - \bar{u},\\
&w(x,t) = \begin{pmatrix} v \\ \tilde{u} \end{pmatrix}(x,t) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}(x,t), \;\; \Phi(x,t) = \int_{-\infty}^{x} w(y,t)\,dy = \begin{pmatrix} \psi \\ \phi \end{pmatrix}(x,t), \qquad (19)\\
&w_0(x) = \begin{pmatrix} v_0 \\ \tilde{u}_0 \end{pmatrix}(x) = \begin{pmatrix} w_{01} \\ w_{02} \end{pmatrix}(x), \;\; \Phi_0(x) = \Phi(x,0) = \begin{pmatrix} \psi_0 \\ \phi_0 \end{pmatrix}(x). \qquad (20)
\end{aligned}$$

Then (7), (9) can be written as

$$\begin{cases} w_t + Aw_x = Bw_{xx} + R \\ w(x,0) = w_0(x) \end{cases}, \tag{21}$$

$$A = \begin{pmatrix} 0 & 1 \\ \bar{u} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = \tilde{R}_x, \quad \tilde{R} = \begin{pmatrix} \varepsilon_2 w_1^2 \\ -w_1 w_2 \end{pmatrix}. \tag{22}$$

Denote the Fourier transform of $w(x,t)$ with respect to $x$ as $\hat{w}(\xi,t)$, etc. Then taking Fourier transform of (21) gives us

$$\begin{aligned} \hat{w}_t &= E(i\xi)\hat{w} + \hat{R}, \\ E(i\xi) &= -i\xi A - \xi^2 B. \end{aligned} \tag{23}$$

The solution of (23) is

$$\hat{w}(\xi,t) = e^{tE(i\xi)}\hat{w}(\xi,0) + \int_0^t e^{(t-\tau)E(i\xi)}\hat{R}(\xi,\tau)\,d\tau. \tag{24}$$

To study the solution operator in (24), we perform spectral analysis for

$$E(i\xi) = \begin{pmatrix} -\varepsilon_1\xi^2 & -i\xi \\ -\bar{u}i\xi & -\xi^2 \end{pmatrix} = \lambda_1(i\xi)P_1(i\xi) + \lambda_2(i\xi)P_2(i\xi),$$

where by direct calculation, the eigenvalues are

$$\lambda_{1,2}(i\xi) = -\frac{1}{2}(\varepsilon_1+1)\xi^2 \pm \sqrt{\frac{1}{4}(\varepsilon_1+1)^2\xi^4 - \xi^2(\varepsilon_1\xi^2+\bar{u})}, \tag{25}$$

and the corresponding eigenprojections are

$$P_{1,2}(i\xi) = \frac{1}{-\bar{u}\xi^2 + (\lambda_{1,2}+\varepsilon_1\xi^2)^2} \begin{pmatrix} -\bar{u}\xi^2 & -i\xi(\lambda_{1,2}+\varepsilon_1\xi^2) \\ -\bar{u}i\xi(\lambda_{1,2}+\varepsilon_1\xi^2) & (\lambda_{1,2}+\varepsilon_1\xi^2)^2 \end{pmatrix}. \tag{26}$$

The solution operator in (24) is

$$e^{tE(i\xi)} = e^{\lambda_1(i\xi)t}P_1(i\xi) + e^{\lambda_2(i\xi)t}P_2(i\xi). \tag{27}$$

Noting that time decay rates are mainly determined by the behavior of solution operator for small $\xi$, we take Taylor expansions in (25) and (26): For $|\xi| \ll 1$,

$$\begin{aligned} &\lambda_{1,2}(i\xi) = -\frac{1}{2}(\varepsilon_1+1)\xi^2 \pm i\xi g(\xi), \quad g(\xi) = \sqrt{\bar{u}} + O(\xi^2) \in \mathbb{R}, \\ &P_{1,2}(i\xi) = \frac{1}{2}\begin{pmatrix} 1+O(\xi) & \mp\frac{1}{\sqrt{\bar{u}}} + O(\xi^2) \\ \mp\sqrt{\bar{u}} + O(\xi^2) & 1+O(\xi) \end{pmatrix}. \end{aligned} \tag{28}$$

We also need an estimate on the solution operator for $\xi \in \mathbb{R}$:

**Lemma 1** *The solution operator in* (24) *satisfies*

$$|e^{tE(i\xi)}| \le Ce^{-\frac{c\xi^2 t}{1+\xi^2}}, \quad \xi \in \mathbb{R}. \quad t \ge 0, \tag{29}$$

*where $C$ and $c$ are two positive constants depending on $\varepsilon_1 \ge 0$ and $\bar{u} > 0$ only.*

Lemma 1 is an application of Kawashima's theory [2]. A discussion of it can be found in [10], where a direct proof of Lemma 1 is also given. With (28) and Lemma 1, we are ready to prove the following decay estimate on the flow:

**Lemma 2** *Let $\varepsilon_1 \ge 0$, $k \ge 0$ be an integer, $h = (h_1, h_2)^t \in L^1(\mathbb{R})$, $D_x^k h \in L^2(\mathbb{R})$. Then*

$$\|e^{tE(i\xi)}(i\xi)^k \hat{h}(\xi)\| \le C(t+1)^{-\frac{1}{4}-\frac{k}{2}}(\|h_1\|_{L^1} + \|h_2\|_{L^1}) + Ce^{-ct}\|D_x^k h\|, \quad t \ge 0, \tag{30}$$

*where $C$ and $c$ are positive constants depending only on $\varepsilon_1 \ge 0$ and $\bar{u} > 0$.*

***Proof*** Let $\eta > 0$ be small such that (28) holds for $|\xi| \le \eta$. We write

$$I \equiv \|e^{tE(i\xi)}(i\xi)^k \hat{h}(\xi)\|^2 = \Big(\int_{|\xi|\le\eta} + \int_{|\xi|\ge\eta}\Big)|e^{tE(i\xi)}(i\xi)^k \hat{h}(\xi)|^2\, d\xi.$$

Applying (27) and (28) to the first integral and (29) to the second one, we have

$$\begin{aligned} I &\le \int_{|\xi|\le\eta} C|\xi|^{2k} e^{-\xi^2 t}|\hat{h}(\xi)|^2\, d\xi + \int_{|\xi|\ge\eta} Ce^{-\frac{2c\eta^2 t}{1+\eta^2}}|(i\xi)^k \hat{h}(\xi)|^2\, d\xi \\ &\le C(t+1)^{-k-\frac{1}{2}}\|\hat{h}\|_{L^\infty}^2 + Ce^{-\tilde{c}t}\|(i\xi)^k\hat{h}\|^2 \le C(t+1)^{-k-\frac{1}{2}}\|h\|_{L^1}^2 + Ce^{-\tilde{c}t}\|D_x^k h\|^2, \end{aligned}$$

where $\tilde{c} > 0$ is a constant, and we have used Plancherel theorem. Taking the square root we obtain (30). □

To prove Theorem 3, we only need to prove (14) for $\varepsilon = 0$ while (12) is valid, and prove (15) for $\varepsilon > 0$ while (13) is true. For this we use (21), which is equivalent to (7), (9). We consider $\varepsilon \ge 0$. By Plancherel theorem, (24) and the triangle inequality, for an integer $k \ge 0$ we have

$$\begin{aligned} \|D_x^k w\|(t) = \|(i\xi)^k \hat{w}\|(t) &\le \|(i\xi)^k e^{tE(i\xi)}\hat{w}(\xi, 0)\| \\ &\quad + \int_0^t \|(i\xi)^k e^{(t-\tau)E(i\xi)}\hat{R}(\xi,\tau)\|\, d\tau \equiv I_1 + I_2. \end{aligned} \tag{31}$$

Noting (11) and (20), we have $w_0(x) = (\psi_0', \phi_0')^t(x)$. Thus applying (30) gives us

$$\begin{aligned} I_1 =& \|(i\xi)^{k+1} e^{tE(i\xi)}(\hat{\psi}_0, \hat{\phi}_0)^t\| \le C[(t+1)^{-\frac{3}{4}-\frac{k}{2}}(\|\psi_0\|_{L^1} + \|\phi_0\|]_{L^1}) \\ &+ Ce^{-ct}\|D_x^{k+1}(\psi_0, \phi_0)^t\| \le C(t+1)^{-\frac{3}{4}-\frac{k}{2}}, \quad 0 \le k \le 2. \end{aligned} \tag{32}$$

Similarly, with (22) we have

$$\begin{aligned} I_2 \le \int_0^t & [C(t-\tau+1)^{-\frac{3}{4}-\frac{k}{2}}(\|w_1^2\|_{L^1} + \|w_1 w_2\|_{L^1})(\tau) \\ & + Ce^{-c(t-\tau)}(\|D_x^{k+1}(w_1^2)\| + \|D_x^{k+1}(w_1 w_2)\|)(\tau)]\, d\tau. \end{aligned} \tag{33}$$

For the case $k = 0$, we define

$$M(t) = \sup_{0 \le \tau \le t} [(\tau+1)^{\frac{3}{4}}\|w\|(\tau)], \tag{34}$$

which implies $\|w\|(t) \le M(t)(t+1)^{-\frac{3}{4}}$ for $t \ge 0$. With (12) and (13) we have

$$\begin{aligned} &(\|w_1^2\|_{L^1} + \|w_1 w_2\|_{L^1})(\tau) \le (\|w_1\|^2 + \|w_1\|\|w_2\|)(\tau) \\ &= \|w_1\|^{\frac{1}{2}}(\tau)(\|w_1\|^{\frac{3}{2}} + \|w_1\|^{\frac{1}{2}}\|w_2\|)(\tau) \le CM(\tau)^{\frac{1}{2}}(\tau+1)^{-\frac{9}{8}}. \end{aligned} \tag{35}$$

By Sobolev inequality, (12) and (13), we also have

$$\begin{aligned} (\|D_x(w_1^2)\| + \|D_x(w_1 w_2)\|)(\tau) &\le C(\|w\|_{L^\infty}\|w_x\|)(\tau) \\ &\le C(\|w\|^{\frac{1}{2}}\|w_x\|^{\frac{3}{2}})(\tau) \le C(\tau+1)^{-\frac{7}{4}}. \end{aligned} \tag{36}$$

Substituting (35) and (36) into (33), for $k = 0$ we have

$$\begin{aligned} I_2 &\le C\int_0^t [M(\tau)^{\frac{1}{2}}(t-\tau+1)^{-\frac{3}{4}}(\tau+1)^{-\frac{9}{8}} + e^{-c(t-\tau)}(\tau+1)^{-\frac{7}{4}}]\, d\tau \\ &\le C[M(t)^{\frac{1}{2}}(t+1)^{-\frac{3}{4}} + (t+1)^{-\frac{7}{4}}]. \end{aligned} \tag{37}$$

Substituting (32) and (37) into (31) with $k = 0$, we have

$$\|w\|(t) \le C(t+1)^{-\frac{3}{4}} + CM(t)^{\frac{1}{2}}(t+1)^{-\frac{3}{4}}.$$

Thus by (34) and Young inequality,

$$M(t) \le C + CM(t)^{\frac{1}{2}} \le C + \frac{1}{2}M(t),$$

which implies $M(t) \le C$, hence

$$(t+1)^{\frac{3}{4}}\|w\|(t) \le C, \qquad t \ge 0. \tag{38}$$

The case $k = 1$ is simpler as we are able to use the updated estimate (38) in (35) to give $(\|w_1^2\|_{L^1} + \|w_1 w_2\|_{L^1})(\tau) \le C(\tau+1)^{-\frac{3}{2}}$. Thus for this case,

$$I_2 \le C\int_0^t [(t-\tau+1)^{-\frac{5}{4}}(\tau+1)^{-\frac{3}{2}} + e^{-c(t-\tau)}(\tau+1)^{-\frac{9}{4}}]\,d\tau \le C(t+1)^{-\frac{5}{4}}. \tag{39}$$

Substituting (32) and (39) into (31) gives us $(t+1)^{\frac{5}{4}}\|D_x w\|(t) \le C$.

We only need to justify the term $k = 2$ in (15), which is for $\varepsilon > 0$. In this case, we replace (33) by

$$\begin{aligned} I_2 \le& \int_0^{\frac{t}{2}} C(t-\tau+1)^{-\frac{7}{4}}(\|w_1^2\|_{L^1} + \|w_1 w_2\|_{L^1})(\tau)\,d\tau \\ &+ \int_{\frac{t}{2}}^t C(t-\tau+1)^{-\frac{5}{4}}(\|D_x(w_1^2)\|_{L^1} + \|D_x(w_1 w_2)\|_{L^1})(\tau)\,d\tau \\ &+ \int_0^t Ce^{-c(t-\tau)}(\|D_x^3(w_1^2)\| + \|D_x^3(w_1 w_2)\|)(\tau)]\,d\tau. \end{aligned}$$

With the updated estimates on $\|D_x^k w\|$, $k = 0, 1$, we have

$$\begin{aligned} I_2 \le& C\int_0^{\frac{t}{2}} (t-\tau+1)^{-\frac{7}{4}}(\tau+1)^{-\frac{3}{2}}\,d\tau + C\int_{\frac{t}{2}}^t (t-\tau+1)^{-\frac{5}{4}}(\tau+1)^{-2}\,d\tau \\ &+ C\int_0^t e^{-c(t-\tau)}[(\tau+1)^{-\frac{23}{8}} + (\tau+1)^{-1}\|D_x^3 w\|(\tau)]\,d\tau \\ \le& C(t+1)^{-\frac{7}{4}} + C[\int_0^t e^{-2c(t-\tau)}(\tau+1)^{-5}\,d\tau]^{\frac{1}{2}}[\int_0^t (\tau+1)^3\|D_x^3 w\|^2(\tau)\,d\tau]^{\frac{1}{2}} \\ \le& C(t+1)^{-\frac{7}{4}}, \end{aligned} \tag{40}$$

where we have used Cauchy-Schwarz inequality and (13). Now combining (31), (32) and (40) gives $(t+1)^{\frac{7}{4}}\|D_x^2 w\|(t) \le C$. Thus we have proved Theorem 3.

The following is a natural extension of this section, and is needed in next section. Using notations in (19), (20) and (22), we integrate (21) to have

$$\Phi_t + A\Phi_x = B\Phi_{xx} + \tilde{R}.$$

Thus similar to (24),

$$\hat{\Phi}(\xi, t) = e^{tE(i\xi)}\hat{\Phi}(\xi, 0) + \int_0^t e^{(t-\tau)E(i\xi)}\hat{\tilde{R}}(\xi,\tau)\,d\tau.$$

Following (31) – (33) and applying Theorem 3, we have

$$\begin{aligned}
\|\Phi\|(t) = \|\hat{\Phi}\|(t) \le \|e^{tE(i\xi)}\hat{\Phi}_0\| + \int_0^t \|e^{(t-\tau)E(i\xi)}\hat{\hat{R}}(\xi,\tau)\|\, d\tau \\
\le & C(t+1)^{-\frac{1}{4}} + C\int_0^t [(t-\tau+1)^{-\frac{1}{4}}(\|w_1^2\|_{L^1} + \|w_1 w_2\|_{L^1})(\tau) \\
& + e^{-c(t-\tau)}(\|w_1^2\| + \|w_1 w_2\|)(\tau)]\, d\tau \\
\le & C(t+1)^{-\frac{1}{4}}.
\end{aligned} \tag{41}$$

## 3 Decay Rates for the Original System

To simplify our notations and without loss of generality, we assume $\tilde{t} = t$, $\tilde{x} = x$, and $\tilde{v} = v$ in (6). To prove Theorem 4 we first note that under the hypotheses of the theorem, the assumptions in Theorem 3 are satisfied for each of the cases $\varepsilon = 0$ and $\varepsilon > 0$. This is in view of (3) and (11), which imply $\psi_0(x) = \ln s_0(x) - \ln \bar{s}$, hence $|\psi_0(x)| \le \frac{2}{\bar{s}}|s_0(x) - \bar{s}|$ and $|\psi_0'(x)| \le \frac{2}{\bar{s}}|s_0'(x)|$ for small $\|s_0 - \bar{s}\|_1$. Thus (7)–(9) has a unique global solution, satisfying (14) and (15) for $\varepsilon = 0$ and $\varepsilon > 0$, respectively. The inverse transform of (3),

$$s(x,t) = e^{-(\mu\bar{u}+\sigma)t}\tilde{s}(x,t), \qquad \tilde{s}(x,t) = \bar{s}e^{\psi(x,t)}, \tag{42}$$

then gives us a unique, global solution to (1), (2).

The inverse transform (42) implies $s(x,t) > 0$ for all $x \in \mathbb{R}$ and $t \ge 0$. Applying the maximum principle to the second equation in (7), one concludes that $u(x,t) \ge 0$ as well, provided $u_0(x) \ge 0$. A similar, detailed discussion can be found in [9] for the model with logistic growth. As the estimates for $u - \bar{u}$ in (17) and (18) are inherited from (14) and (15), respectively, we obtain those for $\tilde{s} - \bar{s}$ below.

From (19) and (41), we have

$$\|\psi\|(t) \le C(t+1)^{-\frac{1}{4}}. \tag{43}$$

Since $\psi_x = v$, by Sobolev inequality, (14) and (15), we further have

$$\|\psi\|_{L^\infty}(t) \le C\|\psi\|^{\frac{1}{2}}(t)\|v\|^{\frac{1}{2}}(t) \le C(t+1)^{-\frac{1}{2}}. \tag{44}$$

Therefore,

$$\|\tilde{s}\|_{L^\infty}(t) \le \bar{s}e^{\|\psi\|_{L^\infty}(t)} \le C. \tag{45}$$

From (42), (45) and the mean value theorem, we have

$$\begin{aligned}
&|\tilde{s}(x,t) - \bar{s}| = \bar{s}|e^{\psi(x,t)} - 1| \le \bar{s}e^{\|\psi\|_{L^\infty}(t)}|\psi(x,t)| \le C|\psi(x,t)|, \\
&\tilde{s}_x(x,t) = (\tilde{s}v)(x,t), \quad \tilde{s}_{xx}(x,t) = (\tilde{s}v^2 + \tilde{s}v_x)(x,t), \\
&\tilde{s}_{xxx}(x,t) = (\tilde{s}v^3 + 3\tilde{s}vv_x + \tilde{s}v_{xx})(x,t).
\end{aligned}$$

Together with (43), (45), (14) and (15), these give us

$$\|\tilde{s} - \bar{s}\|(t) \le C\|\psi\|(t) \le C(t+1)^{-\frac{1}{4}}, \quad \|\tilde{s}_x\|(t) \le (\|\tilde{s}\|_{L^\infty}\|v\|)(t) \le C(t+1)^{-\frac{3}{4}},$$
$$\|\tilde{s}_{xx}\|(t) \le \|\tilde{s}\|_{L^\infty}(t)(\|v\|_{L^\infty}\|v\| + \|v_x\|)(t) \le C(t+1)^{-\frac{5}{4}}.$$

In the case $\varepsilon > 0$ we also have

$$\|D_x^3\tilde{s}\|(t) \le C\|\tilde{s}\|_{L^\infty}(t)(\|v\|^2\|v_x\| + \|v\|^{\frac{1}{2}}\|v_x\|^{\frac{3}{2}} + \|v_{xx}\|)(t) \le C(t+1)^{-\frac{7}{4}}.$$

We thus settle (17) and (18).

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