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Bounds on heat flux for Rayleigh–Bénard convection between Navier-slip fixed-temperature boundaries

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We study two-dimensional Rayleigh–Bénard convection with Navier-slip, fixed temperature boundary conditions and establish bounds on the Nusselt number. As the slip-length varies with Rayleigh number Ra, this estimate interpolates between the Whitehead–Doering bound by $\mathrm{Ra}^{\frac{5}{12}}$ for free-slip conditions (Whitehead & Doering. 2011 Ultimate state of two-dimensional Rayleigh–Bénard convection between free-slip fixed-temperature boundaries. *Phys. Rev. Lett.* **106**, 244501) and the classical Doering–Constantin $\mathrm{Ra}^{\frac{1}{2}}$ bound (Doering & Constantin. 1996 Variational bounds on energy dissipation in incompressible flows. III. Convection. *Phys. Rev. E* **53**, 5957–5981).

This article is part of the theme issue 'Mathematical problems in physical fluid dynamics (part 1)'.

1. Introduction

The standard Rayleigh–Bénard convection model describes the dynamics of a fluid layer confined between two rigid plates held at different uniform temperatures: the lower plate is hot and the upper plate is cool. This temperature difference triggers density variations of the fluid layers and instability ensues,

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leading to a convective fluid motion and, as the control parameter Rayleigh number Ra increases, eventually becomes turbulent. Rayleigh–Bénard convection is a paradigm of nonlinear dynamics, including pattern formation and fully developed turbulence, and has important applications in meteorology, oceanography and industry. A principal quantity of interest due to its relevance in geophysical and industrial applications is the vertical heat transport across the domain. This is usually expressed through the non-dimensional Nusselt number Nu, which is the ratio between the total heat flux and the flux due to thermal conduction. Famously, experiment and numerical simulation suggest a power-law scaling for the Nusselt number Nu

$$\mathsf{Nu} \sim \mathsf{Pr}^{\alpha}\mathsf{Ra}^{\beta} \quad \text{for some } \alpha, \beta \in \mathbb{R},$$

where Ra and Pr are the non-dimensional Rayleigh and Prandtl number, respectively. In [1] a systematic theory for the scaling of the Nusselt number Nu is proposed, based on the decomposition of the global thermal and kinetic energy dissipation rates into their boundary layer and bulk contributions. As such, it is of interest to provide mathematical constraints on allowed exponents from the equations of motion.

In physical theories, scaling laws are based, in part, on the structure of (thermal and viscous) boundary layers. It is therefore interesting to understand how the heat transport properties change with respect to different choice of boundary conditions for the velocity. Most research has focused on the cases where the velocity field satisfies the no-slip [2–5] and free-slip boundary conditions [6–9]. In this paper we consider the non-dimensional Rayleigh–Bénard convection model subject to Navier-slip boundary conditions. We note that, in contrast to the free-slip boundary conditions studied by Whitehead–Doering, the Navier-slip boundary conditions allow for vorticity to be produced at the boundary. In a sense, these conditions interpolate between the no-slip and free-slip conditions as the slip length is increased from 0 to ∞. As such, our bounds degenerate to those available for no-slip in the small slip length regime. As we show later in this paper, the bound $\mathsf{Nu} \lesssim \mathsf{Ra}^{1/2}$ holds uniformly in Prandtl number in any dimension and for any boundary conditions such that the vertical component of the velocity is zero at the (upper and lower) boundaries. At fixed Pr, this bound corresponds to the classical Spiegel–Kraichnan scaling and has since been termed the ‘ultimate regime’. To this day, there is active debate regarding the validity of the ultimate regime insofar as it can be inferred from data [10–12]. We remark that the bound holds in any dimension and for any of the three types of boundary conditions mentioned above and its estimation uses only non-penetration of the velocity at the walls.

We now describe our setup precisely. Let $\Omega = [0, \Gamma] \times [0, 1]$ be the channel with boundaries at $\{x_2 = 0\}$ and $\{x_2 = 1\}$ and periodic in x_1. We consider the Rayleigh–Bénard system [13]

$$\frac{1}{\mathsf{Pr}}(\partial_t u + u \cdot \nabla u) + \nabla p - \Delta u = \mathsf{Ra}Te_2, \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega, \tag{1.2}$$

$$\partial_t T + u \cdot \nabla T = \Delta T, \quad \text{in } \Omega, \tag{1.3}$$

$$\partial_2 u_1 = \frac{1}{\mathsf{L_s}} u_1, \quad \text{on } \{x_2 = 0\}, \tag{1.4}$$

$$-\partial_2 u_1 = \frac{1}{\mathsf{L_s}} u_1, \quad \text{on } \{x_2 = 1\}, \tag{1.5}$$

$$u_2 = 0, \quad \text{on } \{x_2 = 0\} \cup \{x_2 = 1\}, \tag{1.6}$$

$$T = 1, \quad \text{on } \{x_2 = 0\}, \tag{1.7}$$

$$T = 0, \quad \text{on } \{x_2 = 1\}. \tag{1.8}$$

In the horizontal direction x_1, all the unknowns are Γ-periodic. See figure 1 for a depiction of the setup in two dimensions. For higher dimensions, e_2 in equation (1.1) becomes e_d and the boundary conditions are (1.5) and (1.6) in all tangential components. There are two non-dimensional parameters appearing in the system: the Rayleigh number Ra which expresses the

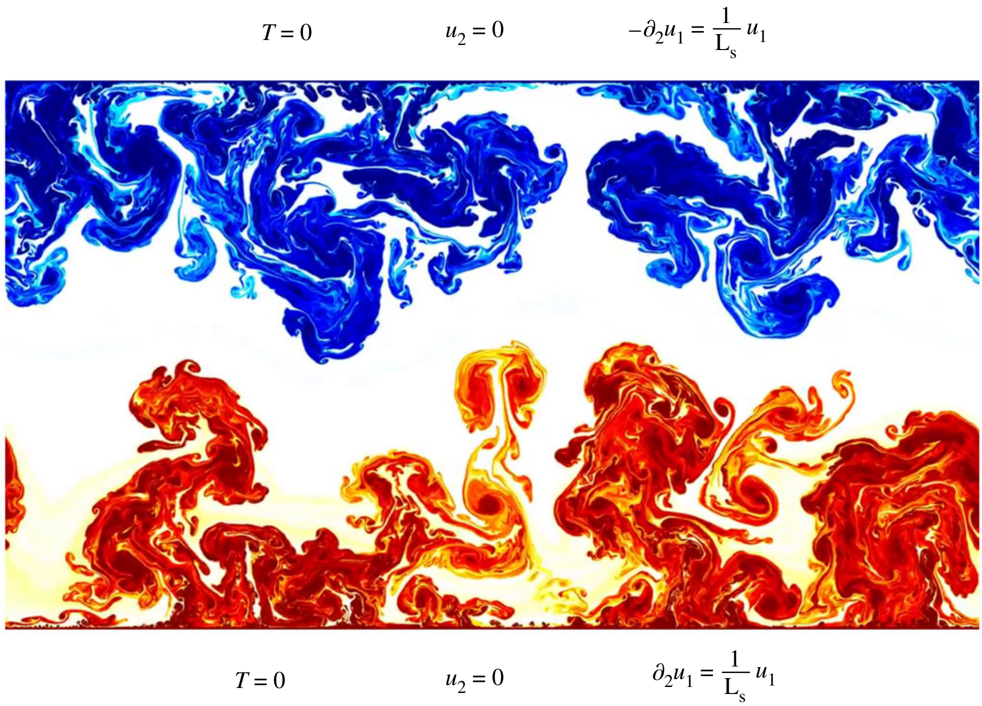

Figure 1. Visualization (with data from no-slip convection [14]) of temperature field. (Online version in colour.)

strength of the thermal forcing and the Prandtl number Pr which represents the ratio of kinematic viscosity to thermal diffusivity.

As (1.1)–(1.8) is already non-dimensional, the Nusselt number is defined simply by

$$\mathsf{Nu} := \langle u_2 T - \partial_2 T \rangle, \tag{1.9}$$

where we have introduced notation for the long-time, global-in-space average

$$\langle \varphi \rangle = \limsup_{T\to\infty} \frac{1}{T}\int_0^T \frac{1}{\Gamma}\int_0^\Gamma \int_0^1 \varphi(x_1, x_2, t)\,\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}t. \tag{1.10}$$

We shall also write $\langle \varphi \rangle_{x_j}$ for the long-time and x_j average. Our main result is the following:

Theorem 1.1. *Let* $\mathsf{L_s} > 0$. *Then*

— *For any* $d \geq 2$, *we have*

$$\mathsf{Nu} \lesssim \mathsf{Ra}^{1/2}. \tag{1.11}$$

— *For* $d = 2$, *if* Pr *satisfies* $\mathsf{L_s}^2\mathsf{Pr}^2 \geq \mathsf{Ra}^{3/2}$, *then for all* $\mathsf{Ra} > 1$ *it holds*

$$\mathsf{Nu} \lesssim \mathsf{Ra}^{\frac{5}{12}} + \mathsf{L_s}^{-2}\mathsf{Ra}^{1/2}. \tag{1.12}$$

The implicit constants depend only on Γ, $||T_0||_{L^\infty}$ *and* $||u_0||_{W^{1,r}}$ *for any fixed* $r \in (2, \infty)$.

Note that when $\mathsf{L_s} = c_s\mathsf{Ra}^\alpha$ with $c_s > 0$ then for $\mathsf{Pr} \geq c_s{}^{-1}\mathsf{Ra}^{(3/4)-\alpha}$ the bound (1.12) reads

$$\mathsf{Nu} \lesssim \mathsf{Ra}^{p(\alpha)}, \quad p(\alpha) := \begin{cases} \dfrac{5}{12} & \text{if } \alpha \geq \dfrac{1}{24} \\ \dfrac{1}{2} - 2\alpha & \text{if } 0 \leq \alpha \leq \dfrac{1}{24} \end{cases}. \tag{1.13}$$

Theorem 1.1 recovers the Whitehead–Doering bound of [8] in two dimensions with $\mathsf{L_s} = \infty$ and of [9] in three dimensions with $\mathsf{L_s} = \mathsf{Pr} = \infty$. For smaller slip-lengths, the bound (1.13) approaches the classical result of Doering–Constantin [4]. Our result improves upon available bounds at fixed Prandtl numbers when the system is equipped with no-slip boundary conditions instead of (1.4)–(1.5) provided that the slip-length is sufficiently large $\mathsf{L_s} \geq c_s \mathsf{Ra}^{3/4}$, suggesting that the Navier-slip conditions may slightly inhibit turbulent heat transport. We remark that the work of Choffrut–Nobili–Otto [2] for no-slip boundaries (in arbitrary dimensions) gives $\mathsf{Nu} \lesssim \mathsf{Ra}^{1/3}$ for $\mathrm{Pr} \gtrsim \mathsf{Ra}^{1/3}$, which improves the bound over Doering–Constantin in that regime. Similar arguments may improve our estimates in that case. Moreover we observe that for the three-dimensional model with free-slip boundary conditions, Wang and Whitehead in [15] proved the estimate $\mathsf{Nu} \lesssim \mathsf{Ra}^{\frac{5}{12}} + \mathrm{Gr}^2 \mathsf{Ra}^{1/4}$ where the Grashof number $\mathrm{Gr} = \mathsf{Ra}/\mathrm{Pr}$ is small.

Remark 1.2 (Infinite Prandtl number). For $d \geq 2$, $\mathsf{Pr} = \infty$, J. Whitehead (unpublished) proved $\mathsf{Nu} \lesssim \mathsf{Ra}^{\frac{5}{12}}$ for all $\mathsf{L_s} > 0$. In remark 3.6, we show how this follows from our argument.

Inspired by [8], we employ the background field method with the simple ansatz of a background profile $\tau(x_2)$ being constant in the bulk and linear in the boundary layers of size δ. Since the Navier-slip conditions allow vorticity production at the walls, our argument is delicate in a number of places compared to that for free-slip conditions. A consequence of the vorticity production at the walls is the lack of conservation of the mean of u_1. As a result, our uniform-in-time bound for the kinetic energy grows linearly with the slip-length $\mathsf{L_s}$ (see lemma 2.3 and remark 2.4). Another consequence is that the uniform-in-time bound for the enstrophy does not follow directly from an energy estimate for the vorticity equation. Here, following an idea in [16], we establish the uniform L^p bounds

$$||\omega(t)||_{L^p} \leq C\left(||\omega_0||_{L^p} + \frac{1}{\mathsf{L_s}}||u_0||_{L^2} + \mathsf{Ra}\right) \quad \forall t > 0,\ p \in [1, \infty). \tag{1.14}$$

Firstly, (1.14) yields the long-time average enstrophy balance (2.22). Secondly, (1.14) is carefully combined with an appropriate pressure estimate (see (2.10)) to handle the bad boundary term in (3.16) in such a way that our Nusselt bound (1.12) recovers the result in [8] when $\mathsf{L_s} \to \infty$.

Following [8], we use the long-time average energy/enstrophy balances and reduce the proof of (1.12) to establishing the positivity of certain quadratic functional $\mathcal{Q}$ (see proposition 3.3) when parameters are suitably chosen. By obtaining a new estimate for the term $\langle \tau' u_2 \theta \rangle$ generated by the background field, we bypass a Fourier argument in [8] and base the proof entirely in physical space.

2. Energy identities and uniform bounds

In what follows, we always consider smooth initial data so that the system (1.1)–(1.8) has a unique global smooth solution. See e.g. [17,18]. We will repeatedly use that $||T(t)||_{L^\infty(\Omega)} \leq \max\{1, ||T_0||_{L^\infty(\Omega)}\}$ for all $t \geq 0$ by the maximum principle. Without loss of generality, we consider initial data $||T_0||_{L^\infty(\Omega)} \leq 1$ so that

$$||T||_{L^\infty(\Omega)} \leq 1. \tag{2.1}$$

Now we recall the well-known (e.g. [4]) identification of the Nusselt number with the heating rate

Proposition 2.1. *The Nusselt number satisfies* $\mathsf{Nu} = \langle |\nabla T|^2 \rangle$.

Proof. Multiplying the temperature equation (1.3) by T, integrating by part in space, and using the incompressibility condition (1.2) and the boundary conditions for u_2 and T, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||T||^2_{L^2(\Omega)} = -||\nabla T||^2_{L^2(\Omega)} - \int_0^\Gamma \partial_2 T\big|_{x_2=0}\,\mathrm{d}x_1.$$

Since $||T(t)||_{L^2(\Omega)}$ is uniformly bounded in t, averaging in time yields

$$\langle |\nabla T|^2 \rangle = -\langle \partial_2 T\big|_{x_2=0} \rangle_{x_1},$$

where $\langle\cdot\rangle_{x_1}$ denotes the long time and x_1 average. On the other hand, if we integrate (1.3) in x_1 and time average, we find $\partial_2\langle u_2 T - \partial_2 T\rangle_{x_1} = 0$. Integrating in x_2 gives

$$\langle u_2 T - \partial_2 T\rangle_{x_1} = \langle (u_2 T - \partial_2 T)\big|_{x_2=0}\rangle_{x_1} = \langle -\partial_2 T\big|_{x_2=0}\rangle_{x_1}.$$

In view of the definition (1.9), we deduce that $\mathsf{Nu} = -\langle \partial_2 T|_{x_2=0}\rangle_{x_1} = \langle |\nabla T|^2\rangle$. ■

Proposition 2.2 (Energy Balance). *Strong solutions of (1.1)–(1.8) satisfy the balance*

$$\frac{1}{2\mathsf{Pr}}\frac{\mathrm{d}}{\mathrm{d}t}||u||_{L^2}^2 + ||\nabla u||_{L^2}^2 + \frac{1}{\mathsf{L_s}}(||u_1||^2_{L^2(\{x_2=1\})} + ||u_1||^2_{L^2(\{x_2=0\})}) = \mathsf{Ra}\int_\Omega u_2 T\,\mathrm{d}x. \tag{2.2}$$

Proof. Dotting equation (1.1) with u, integrating over Ω and using (1.2) and (1.6), we find

$$\frac{1}{2\mathsf{Pr}}\frac{\mathrm{d}}{\mathrm{d}t}||u||_{L^2}^2 = \int_\Omega u\cdot\Delta u + \mathsf{Ra}\int_\Omega u_2 T\,\mathrm{d}x.$$

Using the periodicity and (1.4), (1.5) and (1.6) gives

$$\begin{aligned}\int_\Omega u\cdot\Delta u\,\mathrm{d}x &= -||\nabla u||_{L^2}^2 + \int_0^\Gamma \left(u\cdot\partial_2 u\Big|_{x_2=1} - u\cdot\partial_2 u\Big|_{x_2=0}\right)\mathrm{d}x_1\\ &= -||\nabla u||_{L^2}^2 + \int_0^\Gamma \left(\partial_2 u_1 u_1\Big|_{x_2=1} - \partial_2 u_1 u_1\Big|_{x_2=0}\right)\mathrm{d}x_1\\ &= -||\nabla u||_{L^2}^2 - \frac{1}{\mathsf{L_s}}\left(||u_1||^2_{L^2(\{x_2=1\})} + ||u_1||^2_{L^2(\{x_2=0\})}\right).\end{aligned}$$

■

From the energy balance, we find that the kinetic energy is bounded for all times.

Lemma 2.3. *The energy of u satisfies the following bound:*

$$||u(t)||_{L^2} \le ||u_0||_{L^2}\,\mathrm{e}^{-t\frac{2}{3}\mathsf{Pr}\min\{1,\frac{1}{\mathsf{L_s}}\}} + 3\Gamma\max\{1,\mathsf{L_s}\}\mathsf{Ra},\quad \forall t>0. \tag{2.3}$$

Proof. From the fundamental theorem of calculus, we have

$$\begin{aligned}|u_1(x_1,x_2)|^2 &\le 2|u_1(x_1,0)|^2 + 2\left(\int_0^{x_2}|\partial_2 u_1(x_1,y)|\,\mathrm{d}y\right)^2\\ &\le 2|u_1(x_1,0)|^2 + 2x_2\int_0^1 |\partial_2 u_1(x_1,y)|^2\,\mathrm{d}y\end{aligned}$$

and thus upon integrating over Ω, we obtain

$$||u_1||^2_{L^2(\Omega)} \le 2||u_1(\cdot,0)||^2_{L^2(0,\Gamma)} + 2||\partial_2 u_1||^2_{L^2(\Omega)}.$$

Combining this with the Poincaré inequality $||u_2||_{L^2(\Omega)} \le ||\partial_2 u_2||_{L^2(\Omega)}$, we obtain

$$||u||^2_{L^2(\Omega)} \le 2||u_1(\cdot,0)||^2_{L^2(0,\Gamma)} + 3||\nabla u||^2_{L^2(\Omega)}.$$

In addition, the temperature T obeys the maximum principle (2.1); hence $||T(t)||_{L^2} \leq |\Omega|||T(t)||_{L^\infty} \leq \Gamma$. Then, proposition 2.2 gives

$$\frac{1}{\mathsf{Pr}}\frac{\mathrm{d}}{\mathrm{d}t}||u||_{L^2} \leq 2\mathsf{Ra}\Gamma - \frac{2}{3}\min\{1, \tfrac{1}{\mathsf{L_s}}\}||u||_{L^2}. \tag{2.4}$$

■

Remark 2.4. Consider the free-slip boundary conditions $u_2 = 0$ and $\partial_2 u_1 = 0$ on $x_2 = 0, 1$, which can be formally obtained by setting $\mathsf{L_s} = \infty$ in (1.6)–(1.7). The (spatial) mean of u_1 is conserved upon integrating the first component of (1.1). Appealing to the Galilean symmetry of the system, one can assume without loss of generality that the mean of u_1 is zero for all time. Consequently, the Poincaré inequality $||u||_{L^2} \leq C||\nabla u||_{L^2}$ holds. Then, the energy balance

$$\frac{1}{2\mathsf{Pr}}\frac{\mathrm{d}}{\mathrm{d}t}||u||_{L^2}^2 = -||\nabla u||_{L^2}^2 + \mathsf{Ra}\int_\Omega u_2 T\, \mathrm{d}x,$$

yields the uniform bound $||u(t)||_{L^2} \leq \mathrm{e}^{-(t/C)}||u_0||_{L^2} + C||T_0||_{L^\infty}\mathsf{Ra}$. This bound is better than (2.3) by the factor $\mathsf{L_s}$ in front of Ra. On the other hand, for the Navier-slip boundary condition, the mean of u_1 is not conserved due to the generation of vorticity at the walls.

Corollary 2.5 (Average energy balance). *The following balance holds*

$$\langle|\nabla u|^2\rangle + \frac{1}{\mathsf{L_s}}\left(\langle u_1^2|_{x_2=1}\rangle + \langle u_1^2|_{x_2=0}\rangle\right) = \mathsf{Ra}(\mathsf{Nu} - 1). \tag{2.5}$$

Proof. Using the boundary conditions for the temperature (1.7) and (1.8), one finds

$$\mathsf{Nu} = 1 + \langle u_2 T\rangle, \tag{2.6}$$

from definition (1.9). Then the claim follows upon integrating (2.2) in time and taking the long time limit using the uniform bound for $||u(t)||_{L^2}$ given by lemma 2.3. ■

Proposition 2.6 (Pressure-Poisson equation). *The pressure in* (1.1) *satisfies*

$$\Delta p = -\frac{1}{\mathsf{Pr}}\nabla u^T : \nabla u + \mathsf{Ra}\partial_2 T \quad \text{in } \Omega, \tag{2.7}$$

$$-\partial_2 p = \frac{1}{\mathsf{L_s}}\partial_1 u_1 - \mathsf{Ra} \quad \text{on } \{x_2 = 0\}, \tag{2.8}$$

and

$$\partial_2 p = \frac{1}{\mathsf{L_s}}\partial_1 u_1 \quad \text{on } \{x_2 = 1\}. \tag{2.9}$$

Proof. Equation (2.7) follows from taking the divergence of the momentum equation. The boundary conditions come from tracing the second component of the momentum equation along the boundaries. Specifically, one has

$$\partial_2 p = \partial_2^2 u_2 + \mathsf{Ra}T = -\partial_1\partial_2 u_1 + \mathsf{Ra}T,$$

where $\partial_2 u_1$ is given by (1.4) and (1.5). ■

Proposition 2.7. *For any $r \in (2, \infty)$, there exists $C = C(r, \Gamma)$ such that*

$$||p||_{H^1(\Omega)} \leq C\Big(\frac{1}{\mathsf{L_s}}||\partial_1\omega||_{L^2(\Omega)} + \mathsf{Ra}||T||_{L^2(\Omega)} + \frac{1}{\mathsf{Pr}}||\omega||_{L^2(\Omega)}||\omega||_{L^r(\Omega)}\Big). \tag{2.10}$$

Proof. On one hand, using the boundary conditions (2.8) and (2.9) gives

$$\int_\Omega p\Delta p\,\mathrm{d}x = -||p||_{\dot H^1}^2 + \int_0^\Gamma p\partial_2 p|_{x_2=0}^{x_2=1}\,\mathrm{d}x_1$$
$$= -||\nabla p||_{L^2}^2 + \frac{1}{\mathsf{L_s}}\int_0^\Gamma (p\partial_1 u_1|_{x_2=1} + p\partial_1 u_1|_{x_2=0})\mathrm{d}x_1 - \mathsf{Ra}\int_0^\Gamma p|_{x_2=0}\,\mathrm{d}x_1.$$

On the other hand, using (2.7), (1.7) and (1.8), we find

$$\int_\Omega p\Delta p\,\mathrm{d}x = -\frac{1}{\mathrm{Pr}}\int_\Omega p\nabla u^T : \nabla u\,\mathrm{d}x - \mathsf{Ra}\int_\Omega \partial_2 pT\,\mathrm{d}x - \mathsf{Ra}\int_0^\Gamma p|_{x_2=0}\,\mathrm{d}x_1.$$

Consequently,

$$||\nabla p||_{L^2}^2 = \frac{1}{\mathsf{L_s}}\int_0^\Gamma (p\partial_1 u_1|_{x_2=1} + p\partial_1 u_1|_{x_2=0})\,\mathrm{d}x_1 + \frac{1}{\mathrm{Pr}}\int_\Omega p\nabla u^T : \nabla u\,\mathrm{d}x + \mathsf{Ra}\int_\Omega \partial_2 pT\,\mathrm{d}x.$$

By virtue of the Sobolev trace inequality and Hölder's inequality, it follows that

$$||\nabla p||_{L^2}^2 \lesssim \frac{1}{\mathsf{L_s}}||p||_{H^1}||\partial_1 u_1||_{H^1} + \mathsf{Ra}||p||_{H^1}||T||_{L^2} + \frac{1}{\mathrm{Pr}}||p\nabla u^T : \nabla u||_{L^1}.$$

For any $r \in (2,\infty)$, letting $1/q = 1/2 - 1/r$, we have $q \in (2,\infty)$ and

$$||p\nabla u^T : \nabla u||_{L^1} \le ||p||_{L^q}||\nabla u||_{L^2}||\nabla u||_{L^r} \le C||p||_{H^1}||\omega||_{L^2}||\omega||_{L^r},$$

where we use the Sobolev embedding and (A 4).

Since p has mean zero, we have $||p||_{H^1} \le C||\nabla p||_{L^2}$, so that upon using $\partial_1 u_1 = -\partial_2 u_2$ we get

$$||p||_{H^1} \lesssim \frac{1}{\mathsf{L_s}}||\partial_2 u_2||_{H^1} + \mathsf{Ra}||T||_{L^2} + \frac{C}{\mathrm{Pr}}||\omega||_{L^2}||\omega||_{L^r}.$$

From lemma A.1 and (A 4), $||\nabla u||_{L^2} = ||\omega||_{L^2}$ and $||\partial_2 u_2||_{H^1} \le C||\partial_1\omega||_{L^2}$, whence (2.10) follows. ∎

Proposition 2.8 (Vorticity formulation). *The vorticity* $\omega = \nabla^\perp\cdot u$ *where* $\nabla^\perp = (-\partial_2, \partial_1)$ *satisfies*

$$\frac{1}{\mathsf{Pr}}(\partial_t\omega + u\cdot\nabla\omega) - \Delta\omega = \mathsf{Ra}\partial_1 T \quad \text{in } \Omega, \tag{2.11}$$

$$-\omega = \frac{1}{\mathsf{L_s}}u_1 \quad \text{on } \{x_2 = 0\} \tag{2.12}$$

and

$$\omega = \frac{1}{\mathsf{L_s}}u_1 \quad \text{on } \{x_2 = 1\}. \tag{2.13}$$

Proof. Equation (2.11) follows from taking the curl of the momentum equation (1.1). The boundary conditions (2.12) and (2.13) follow from the conditions (1.4) and (1.5) since the vorticity on the boundary is simply $\omega = -\partial_2 u_1$ upon recalling (1.6). ∎

Lemma 2.9. *The normal derivative of vorticity satisfies*

$$-\partial_2\omega = \frac{1}{\mathsf{Pr}}(\partial_t u_1 + u_1\partial_1 u_1) + \partial_1 p, \quad \text{on} \quad \{x_2 = 0\} \text{ and } \{x_2 = 1\}. \tag{2.14}$$

Proof. Using incompressibility of u, we find

$$\partial_2\omega = -\partial_2^2 u_1 + \partial_1\partial_2 u_2 = -\Delta u_1. \tag{2.15}$$

From the first component of (1.1) traced on the boundary (using $u_2 = 0$ there), we have

$$\Delta u_1 = \frac{1}{\mathsf{Pr}}(\partial_t u_1 + u_1\partial_1 u_1) + \partial_1 p. \tag{2.16}$$

∎

Proposition 2.10 (Enstrophy balance). *The following identity holds*

$$\frac{1}{2\mathsf{Pr}}\frac{\mathrm{d}}{\mathrm{d}t}||\omega||_{L^2}^2+\frac{1}{2\mathsf{L_s}\mathsf{Pr}}\frac{\mathrm{d}}{\mathrm{d}t}\left(||u_1||^2_{L^2(\{x_2=1\})}+||u_1||^2_{L^2(\{x_2=0\})}\right)+||\nabla\omega||^2_{L^2}$$
$$=\frac{1}{\mathsf{L_s}}\left(\int_0^\Gamma p\partial_1 u_1\Big|_{x_2=1}\mathrm{d}x_1+\int_0^\Gamma p\partial_1 u_1\Big|_{x_2=0}\mathrm{d}x_1\right)+\mathsf{Ra}\int_\Omega \omega\partial_1 T\,\mathrm{d}x. \tag{2.17}$$

Proof. Multiplying (2.11) by ω and integrating over the domain, we obtain

$$\frac{1}{2\mathsf{Pr}}\frac{\mathrm{d}}{\mathrm{d}t}||\omega||_{L^2}^2=\int_\Omega \omega\Delta\omega\,\mathrm{d}x+\mathsf{Ra}\int_\Omega\omega\partial_1 T\,\mathrm{d}x, \tag{2.18}$$

where we have use the non-penetration boundary conditions for the velocity (1.6). Now note that

$$\begin{aligned}\int_\Omega \omega\Delta\omega\,\mathrm{d}x&=-||\nabla\omega||^2_{L^2}+\int_0^\Gamma \omega\partial_2\omega\Big|_{x_2=1}\mathrm{d}x_1-\int_0^\Gamma \omega\partial_2\omega\Big|_{x_2=0}\mathrm{d}x_1\\
&=-||\nabla\omega||^2_{L^2}+\frac{1}{\mathsf{L_s}}\int_0^\Gamma u_1\partial_2\omega\Big|_{x_2=1}\mathrm{d}x_1+\frac{1}{\mathsf{L_s}}\int_0^\Gamma u_1\partial_2\omega\Big|_{x_2=0}\mathrm{d}x_1\\
&=-||\nabla\omega||^2_{L^2}-\frac{1}{2\mathsf{L_s}\mathsf{Pr}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^\Gamma u_1^2\Big|_{x_2=1}\mathrm{d}x_1+\int_0^\Gamma u_1^2\Big|_{x_2=0}\mathrm{d}x_1\right)\\
&\quad+\frac{1}{\mathsf{L_s}}\left(\int_0^\Gamma \partial_1 u_1 p\Big|_{x_2=1}\mathrm{d}x_1+\int_0^\Gamma \partial_1 u_1 p\Big|_{x_2=0}\mathrm{d}x_1\right),\end{aligned}$$

where we have used lemma 2.9 together with periodicity of the function u_1 in x_1. ■

Next we provide uniform in time bounds for the vorticity

Lemma 2.11 (L^p vorticity bounds). *Let* $\mathsf{L_s}\geq 1$, $p\in[1,\infty)$. *There is* $C=C(p,\Gamma)<\infty$ *so that*

$$||\omega(t)||_{L^p}\leq C\left(||\omega_0||_{L^p}+\frac{1}{\mathsf{L_s}}||u_0||_{L^2}+\mathsf{Ra}\right)\quad\forall t>0. \tag{2.19}$$

Proof. Since Ω is bounded it suffices to prove (2.19) for $p\in(2,\infty)$. To this end, we follow a strategy used in [16]. For arbitrary $T>0$ set

$$\Lambda:=\frac{1}{\mathsf{L_s}}||u_1||_{L^\infty(\{x_2=0,1\}\times(0,T))},$$

and consider the problems

$$\begin{aligned}&\frac{1}{\mathsf{Pr}}(\partial_t\tilde\omega_\pm+u\cdot\nabla\tilde\omega_\pm)-\Delta\tilde\omega_\pm=\mathsf{Ra}\partial_1 T\quad\text{in }\Omega,\\
&\tilde\omega_\pm|_{t=0}=\pm|\omega_0|\quad\text{in }\Omega\\
\text{and}\quad&\tilde\omega_\pm=\pm\Lambda\quad\text{on }\{x_2=0\}\cup\{x_2=1\}.\end{aligned}$$

Now let $\omega'_\pm:=\omega-\tilde\omega_\pm$. This quantity satisfies

$$\begin{aligned}&\frac{1}{\mathsf{Pr}}(\partial_t\omega'_\pm+u\cdot\nabla\omega'_\pm)-\Delta\omega'_\pm=0\quad\text{in }\Omega,\\
&\omega'_\pm|_{t=0}=\omega_0\mp|\omega_0|\quad\text{in }\Omega\\
&-\omega'_\pm=\frac{1}{\mathsf{L_s}}u_1\pm\Lambda\quad\text{on }\{x_2=0\}\\
\text{and}\quad&\omega'_\pm=\frac{1}{\mathsf{L_s}}u_1\mp\Lambda\quad\text{on }\{x_2=1\}.\end{aligned}$$

By the maximum principle, we have $\omega'_+\leq 0$ and $\omega'_-\geq 0$ a.e. $\Omega\times[0,T)$. Thus we obtain $\tilde\omega_-\leq\omega\leq\tilde\omega_+$ and hence

$$|\omega|\leq\max\{|\tilde\omega_+|,|\tilde\omega_-|\}\quad\text{a.e. }\Omega\times[0,T). \tag{2.20}$$

We now bound $\tilde{\omega}_\pm$ in L^p. We focus on $\tilde{\omega}=\tilde{\omega}_+$, the other is similar. Let $\hat{\omega}:=\tilde{\omega}-\Lambda$. This solves

$$\begin{aligned}
&\frac{1}{\mathsf{Pr}}(\partial_t\hat{\omega}+u\cdot\nabla\hat{\omega})-\Delta\hat{\omega}=\mathsf{Ra}\partial_1 T \quad \text{in } \Omega,\\
&\hat{\omega}|_{t=0}=|\omega_0|-\Lambda \quad \text{in } \Omega,\\
&\omega=0 \quad \text{on } \{x_2=0\}\cup\{x_2=1\}.
\end{aligned}$$

We now perform L^p estimates; multiplying by $\hat{\omega}|\hat{\omega}|^{p-2}$ where $p>2$ we find

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}||\hat{\omega}||_{L^p}^p+(p-1)\int_\Omega|\nabla\hat{\omega}|^2|\hat{\omega}|^{p-2}\,\mathrm{d}x=-\mathsf{Ra}\int_\Omega\partial_1(\hat{\omega}|\hat{\omega}|^{p-2})T\,\mathrm{d}x.$$

We bound using Cauchy–Schwarz and Young's inequality

$$\begin{aligned}
\mathsf{Ra}\left|\int_\Omega\partial_1(\hat{\omega}|\hat{\omega}|^{p-2})T\,\mathrm{d}x\right| &\le (p-1)\left(\int_\Omega|\nabla\hat{\omega}|^2|\hat{\omega}|^{p-2}\,\mathrm{d}x\right)^{1/2}\left(\mathsf{Ra}^2\int_\Omega|\hat{\omega}|^{p-2}T^2\,\mathrm{d}x\right)^{1/2}\\
&\le \frac{p-1}{2}\int_\Omega|\nabla\hat{\omega}|^2|\hat{\omega}|^{p-2}\,\mathrm{d}x+\frac{p-1}{2}|\Omega|^{2/p}\mathsf{Ra}^2||\hat{\omega}||_{L^p}^{p-2},
\end{aligned}$$

where we used that $||T||_{L^\infty}=1$. Thus we obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}||\hat{\omega}||_{L^p}^p+\frac{p-1}{2}\int_\Omega|\nabla\hat{\omega}|^2|\hat{\omega}|^{p-2}\,\mathrm{d}x\le\frac{p-1}{2}|\Omega|^{2/p}\mathsf{Ra}^2||\hat{\omega}||_{L^p}^{p-2}.$$

Finally, since $\hat{\omega}$ vanishes on the boundary, we have the Poincaré inequality

$$\int_\Omega|\nabla\hat{\omega}|^2|\hat{\omega}|^{p-2}\,\mathrm{d}x=\frac{4}{p^2}||\nabla|\omega|^{p/2}||_{L^2}^2\ge\frac{4}{p^2C_p^2}|||\omega|^{p/2}||_{L^2}^2=\frac{4}{p^2C_p^2}||\omega||_{L^p}^p$$

Thus we obtain (dividing through by $||\hat{\omega}||_{L^p}^{p-2}$) the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}||\hat{\omega}||_{L^p}^2\le-\frac{p-1}{2}\frac{4}{p^2C_p^2}||\hat{\omega}||_{L^p}^2+\frac{p-1}{2}|\Omega|^{2/p}\mathsf{Ra}^2.$$

It follows that for all $t\ge 0$

$$||\hat{\omega}(t)||_{L^p}\le||\hat{\omega}_0||_{L^p}\,\mathrm{e}^{-t((p-1)/p^2C_p^2)}+\frac{pC_p}{2}|\Omega|^{1/p}\mathsf{Ra}\le C\Big(||\omega_0||_{L^p}\,\mathrm{e}^{-t/C}+\Lambda+\mathsf{Ra}\Big),\quad C=C(p,\Gamma). \tag{2.21}$$

Given this bound, we estimate Λ using interpolation as follows:

$$\begin{aligned}
\Lambda&\le\frac{1}{\mathsf{L_s}}||u||_{L^\infty(\Omega\times(0,T))}\\
&\le\frac{C}{\mathsf{L_s}}||u||^\theta_{L^\infty([0,T];L^2_x)}||\nabla u||^{1-\theta}_{L^\infty([0,T];L^p_x)}+\frac{C}{\mathsf{L_s}}||u||_{L^\infty([0,T];L^2_x)}\\
&\le\frac{C}{\mathsf{L_s}}||u||^\theta_{L^\infty([0,T];L^2_x)}||\omega||^{1-\theta}_{L^\infty([0,T];L^p_x)}+\frac{C}{\mathsf{L_s}}||u||_{L^\infty([0,T];L^2_x)}\\
&\le C_\varepsilon\left(\frac{1}{\mathsf{L_s}^{1/\theta}}+\frac{1}{\mathsf{L_s}}\right)||u||_{L^\infty([0,T];L^2_x)}+\varepsilon||\omega||_{L^\infty([0,T];L^p_x)},
\end{aligned}$$

where $\theta=(p-2)/(2p-2)\in(0,1)$, $\varepsilon>0$ is arbitrary and, appealing to lemma A.2, we used $||\nabla u||_{L^p}\lesssim||\omega||_{L^p}$. By virtue of lemma 2.3, for $\mathsf{L_s}\ge 1$ we obtain

$$\Lambda\le C_\varepsilon\left(\frac{1}{\mathsf{L_s}}||u_0||_{L^2}+\mathsf{Ra}\right)+\varepsilon||\omega||_{L^\infty([0,T];L^p_x)}.$$

In view of this, (2.20) and (2.21), choosing ε small enough gives

$$||\omega||_{L^\infty([0,T];L^p)}\le C\Big(||\omega_0||_{L^p}+\frac{1}{\mathsf{L_s}}||u_0||_{L^2}+\mathsf{Ra}\Big),$$

where C is independent of T. Since $T>0$ is arbitrary, this completes the proof. ■

An immediate consequence of the enstrophy balance (2.17) and the uniform vorticity bound (2.19) is the following global balance

Corollary 2.12 (Average enstrophy balance). *We have the balance for long-time averages*

$$\langle|\nabla\omega|^2\rangle = \frac{1}{\mathsf{L_s}}\left(\langle p\partial_1 u_1\big|_{x_2=1}\rangle + \langle p\partial_1 u_1\big|_{x_2=0}\rangle\right) + \mathsf{Ra}\langle\omega\partial_1 T\rangle. \tag{2.22}$$

3. Proof of theorem 1.1

The theorem follows by an application of the background field method [4]. This method is based on adopting the ansatz

$$T(x_1,x_2,t) =: \tau(x_2) + \theta(x_1,x_2,t). \tag{3.1}$$

We choose the 'background' profile $\tau:[0,1]\to[0,1]$ to be the continuous function given by

$$\tau(z) := 1 - \frac{1}{2\delta}\begin{cases} z & z\in[0,\delta] \\ \delta & z\in(\delta,1-\delta) \\ z+2\delta-1 & z\in[1-\delta,1] \end{cases}, \tag{3.2}$$

for some $\delta>0$ to be chosen later in the proof. Note that

$$\tau'(z) = -\frac{1}{2\delta}\begin{cases} 1 & z\in[0,\delta) \\ 0 & z\in(\delta,1-\delta) \\ 1 & z\in(1-\delta,1] \end{cases}. \tag{3.3}$$

Note that $||\tau'||^2_{L^2([0,1])} = 1/2\delta$. Note that θ vanishes at the boundaries $x_2=\{0,1\}$.

Proposition 3.1. *With θ and τ defined by (3.1) and (3.2), the following identity holds*

$$\mathsf{Nu} - \frac{1}{2\delta} = -\langle|\nabla\theta|^2\rangle - 2\langle\tau' u_2\theta\rangle. \tag{3.4}$$

Proof. According to proposition 2.1, the decomposition (3.1) and the profile (3.3), we have

$$\mathsf{Nu} = \langle|\nabla\theta|^2\rangle + ||\tau'||^2_{L^2([0,1])} + 2\langle\tau'\partial_2\theta\rangle. \tag{3.5}$$

Inserting now the ansatz (3.1) into (1.3), we find the fluctuation θ satisfies

$$\partial_t\theta + u_2\tau' + u\cdot\nabla\theta - \Delta\theta - \tau'' = 0 \quad \text{in } \Omega, \tag{3.6}$$

$$\theta = 0 \quad \text{on } \{x_2=0\}\cup\{x_2=1\}. \tag{3.7}$$

Integrating (3.6) against θ and taking the long-time average (using the fact that θ, like T, is uniformly bounded in time), we obtain

$$\langle\tau'\partial_2\theta\rangle = -\langle|\nabla\theta|^2\rangle - \langle\tau' u_2\theta\rangle. \tag{3.8}$$

This argument can be made rigorous by smooth approximation of the profile τ. Inserting this equality above yields the claimed identity. ∎

Similarly to the bound of Doering–Constantin for the no-slip boundary condition [4], we have

Lemma 3.2. *For any $\mathsf{L_s}>0$, we have $\mathsf{Nu}\lesssim\mathsf{Ra}^{1/2}$.*

Proof. Equation (3.4) implies $\mathsf{Nu} \le 1/2\delta - 2\langle \tau' u_2 \theta \rangle$. Since $\tau' = 1/2\delta$ on its support $(0, \delta) \cup (1, 1-\delta)$ and θ and u_2 vanish on $x_2 = 0,\ 1$, we have

$$|\theta(x_1, x_2)| \le \sqrt{\delta} ||\partial_2 \theta(x_1, \cdot)||_{L^2(0,1)} \quad \forall x_2 \in (0, \delta) \cup (1, 1-\delta)$$

and similarly for u_2. Consequently,

$$\frac{1}{\Gamma} \int_0^{\Gamma} \int_0^1 2|\tau' u_2 \theta| \, \mathrm{d}x_2 \, \mathrm{d}x_1 \le \delta \frac{1}{\Gamma} ||\partial_2 u_2||_{L^2(\Omega)} ||\partial_2 \theta||_{L^2(\Omega)}.$$

Integrating in time and applying the Cauchy–Schwarz inequality gives

$$|\langle -2\tau' u_2 \theta \rangle| \le 2\delta \langle |\partial_2 u_2|^2 \rangle^{1/2} \langle |\partial_2 \theta|^2 \rangle^{1/2}. \tag{3.9}$$

Appealing to proposition 2.1 and corollary 2.5 we deduce

$$\mathsf{Nu} \le \frac{1}{2\delta} + 2\delta (\mathsf{Nu})^{1/2} ((\mathsf{Nu} - 1)\mathsf{Ra})^{1/2} \lesssim \frac{1}{2\delta} + 2\delta \mathsf{Nu} \mathsf{Ra}^{1/2}. \tag{3.10}$$

Choosing $\delta \sim \mathsf{Nu}^{-1/2} \mathsf{Ra}^{-1/4}$ by balancing the contributions of each term yields $\mathsf{Nu} \lesssim \mathsf{Ra}^{1/2}$. ■

To improve the bound, we follow [8] by using the energy and enstrophy balances

$$\textbf{(a)} := \langle |\nabla \omega|^2 \rangle - \frac{1}{\mathsf{L_s}} \left(\langle p \partial_1 u_1 |_{x_2=1} \rangle + \langle p \partial_1 u_1 |_{x_2=0} \rangle \right) - \mathsf{Ra} \langle \omega \partial_1 T \rangle,$$

$$\textbf{(b)} := \langle |\nabla u|^2 \rangle + \frac{1}{\mathsf{L_s}} \left(\langle u_1^2 |_{x_2=1} \rangle + \langle u_1^2 |_{x_2=0} \rangle \right) - \mathsf{Ra}(\mathsf{Nu} - 1).$$

Note that **(a)** $=$ **(b)** $= 0$ by corollary 2.5 and 2.12. Thus in view of (3.4) we have

$$\mathsf{Nu} = \frac{1}{2\delta} - \langle |\nabla \theta|^2 \rangle - 2\langle \tau' u_2 \theta \rangle - \frac{b}{\mathsf{Ra}} \textbf{(b)} - a\textbf{(a)}, \tag{3.11}$$

for all $b \in [0, 1)$ and $a \in \mathbb{R}$.

Proposition 3.3. *Let $\delta > 0$, $b \in [0, 1)$, $a > 0$ and $M > 0$. Then the following identity holds*

$$(1 - b)\mathsf{Nu} + b = \frac{1}{2\delta} + M\mathsf{Ra}^2 - \mathcal{Q}[\theta, u, \tau], \tag{3.12}$$

where $\mathcal{Q}[\theta, u, \tau]$ is defined by

$$\mathcal{Q}[\theta, u, \tau] := M\mathsf{Ra}^2 + \langle |\partial_1 \theta|^2 \rangle + \langle |\partial_2 \theta|^2 \rangle + 2\langle \tau' u_2 \theta \rangle + \frac{b}{\mathsf{Ra}} \langle |\omega|^2 \rangle + \frac{b}{\mathsf{Ra L_s}} \left(\langle u_1^2 |_{x_2=1} \rangle + \langle u_1^2 |_{x_2=0} \rangle \right)$$
$$+ a \langle |\nabla \omega|^2 \rangle - \frac{a}{\mathsf{L_s}} \left(\langle p \partial_1 u_1 |_{x_2=1} \rangle + \langle p \partial_1 u_1 |_{x_2=0} \rangle \right) - a\mathsf{Ra} \langle \omega \partial_1 \theta \rangle. \tag{3.13}$$

The strategy is to show that $\mathcal{Q}$ is non-negative for an appropriate choice of $\delta := \delta(\mathsf{Ra})$. Then (3.12) will yield the desired bound on the Nusselt number. This requires bounds for the pressure and for $2\langle \tau' u_2 \theta \rangle$, where the former is handled by virtue of (2.10) and the latter requires a bound different from (3.9). The main result is

Proposition 3.4. *There exists a universal constant $L_0 > 0$ such that for all $\mathsf{L_s} \ge L_0$ and* Pr *such that $\mathsf{L_s}^2 \mathsf{Pr}^2 \ge \mathsf{Ra}^{3/2}$, we have*

$$\mathsf{Nu} \lesssim \mathsf{Ra}^{\frac{5}{12}} + \mathsf{L_s}^{-2} \mathsf{Ra}^{1/2}, \quad \forall\, \mathsf{Ra} > 1. \tag{3.14}$$

Here, the implicit constant depends only on Γ, $||T_0||_{L^\infty}$ and $||u_0||_{W^{1,r}}$ for any fixed $r \in (2, \infty)$.

Proof. First we use Cauchy–Schwarz and Young's inequality to get

$$|a\mathsf{Ra}\langle\omega\partial_1\theta\rangle| \le \frac{a^2\mathsf{Ra}^2}{2}\langle|\omega|^2\rangle + \frac{1}{2}\langle|\partial_1\theta|^2\rangle, \tag{3.15}$$

so that $\mathcal{Q}$ of proposition 3.3 enjoys the lower bound

$$\begin{aligned}\mathcal{Q}[\theta,u,\tau] \ge{}& M\mathsf{Ra}^2 + \frac{1}{2}\langle|\partial_1\theta|^2\rangle + \langle|\partial_2\theta|^2\rangle + 2\langle\tau' u_2\theta\rangle + \left(\frac{b}{\mathsf{Ra}} - \frac{a^2\mathsf{Ra}^2}{2}\right)\langle|\omega|^2\rangle + a\langle|\nabla\omega|^2\rangle \\ &+ \frac{b}{\mathsf{Ra}\mathsf{L_s}}\left(\langle u_1^2|_{x_2=1}\rangle + \langle u_1^2|_{x_2=0}\rangle\right) - \frac{a}{\mathsf{L_s}}\left(\langle p\partial_1 u_1|_{x_2=1}\rangle + \langle p\partial_1 u_1|_{x_2=0}\rangle\right).\end{aligned} \tag{3.16}$$

Note that from the Sobolev trace inequality and the incompressibility, we have

$$\frac{a}{\mathsf{L_s}}\left|\langle p\partial_1 u_1|_{x_2=1}\rangle + \langle p\partial_1 u_1|_{x_2=0}\rangle\right| \le \frac{C_1 a}{\mathsf{L_s}}\langle ||p||_{H^1}||\partial_2 u_2||_{H^1}\rangle \le \frac{C_1 a}{\mathsf{L_s}}\langle ||\nabla p||_{L^2}||\partial_1\omega||_{L^2}\rangle,$$

where we used (A 4) and $C_1 = C_1(\Gamma)$. To bound the pressure, we recall from (2.10) that for any $r \in (2,\infty)$,

$$||p||_{H^1(\Omega)} \le C\Big(\frac{1}{\mathsf{L_s}}||\partial_1\omega||_{L^2(\Omega)} + \mathsf{Ra}||T||_{L^2(\Omega)} + \frac{1}{\mathrm{Pr}}||\omega||_{L^2(\Omega)}||\omega||_{L^r(\Omega)}\Big).$$

Recall also from lemma 2.11 that $||\omega||_{L^r} \le C(||u_0||_{W^{1,r}} + \mathsf{Ra})$ and hence

$$C_1||p||_{H^1(\Omega)} \le C_2\Big(\frac{1}{\mathsf{L_s}}||\partial_1\omega||_{L^2(\Omega)} + \mathsf{Ra} + \frac{||u_0||_{W^{1,r}} + \mathsf{Ra}}{\mathrm{Pr}}||\omega||_{L^2(\Omega)}\Big).$$

Using Young's inequality yields

$$\begin{aligned}\frac{aC_1}{\mathsf{L_s}}||\nabla p||_{L^2}||\partial_1\omega||_{L^2} &\le \frac{aC_2}{\mathsf{L_s}^2}||\partial_1\omega||_{L^2}^2 + \frac{aC_2}{\mathsf{L_s}}||\partial_1\omega||_{L^2}\Big(\mathsf{Ra} + \frac{||u_0||_{W^{1,r}} + \mathsf{Ra}}{\mathrm{Pr}}||\omega||_{L^2}\Big) \\ &\le \frac{aC_2}{\mathsf{L_s}^2}||\partial_1\omega||_{L^2}^2 + \frac{a}{2}||\partial_1\omega||_{L^2}^2 + \frac{aC_2^2}{2\mathsf{L_s}^2}\left(\mathsf{Ra}^2 + \frac{||u_0||_{W^{1,r}}^2}{\mathrm{Pr}^2}||\omega||_{L^2}^2 + \frac{\mathsf{Ra}^2}{\mathrm{Pr}^2}||\omega||_{L^2}^2\right).\end{aligned}$$

Choosing $M = aC_2^2/2\mathsf{L_s}^2$ in the definition on $\mathcal{Q}$, we find

$$\begin{aligned}\mathcal{Q}[\theta,u,\tau] \ge{}& \frac{1}{2}\langle|\partial_1\theta|^2\rangle + \langle|\partial_2\theta|^2\rangle + 2\langle\tau' u_2\theta\rangle \\ &+ \left(\frac{b}{\mathsf{Ra}} - \frac{a^2\mathsf{Ra}^2}{2} - \frac{aC_2^2||u_0||_{W^{1,r}}^2}{2\mathsf{L_s}^2\,\mathrm{Pr}^2} - \frac{aC_2^2\mathsf{Ra}^2}{2\mathsf{L_s}^2\,\mathrm{Pr}^2}\right)\langle|\omega|^2\rangle + a\Big(\frac{1}{2} - \frac{C_2}{\mathsf{L_s}^2}\Big)\langle|\nabla\omega|^2\rangle.\end{aligned} \tag{3.17}$$

Lemma 3.5. *For some $C_0 > 0$ and any $\varepsilon > 0$, we have*

(a)

$$|2\langle\tau' u_2\theta\rangle| \le \frac{1}{2}\langle|\partial_2\theta|^2\rangle + C_0\delta^6\varepsilon^{-1}\langle|\omega|^2\rangle + \frac{\varepsilon}{4}\langle|\partial_1\omega|^2\rangle, \tag{3.18}$$

(b)

$$|2\langle\tau' u_2\theta\rangle| \le \frac{1}{2}\langle|\partial_2\theta|^2\rangle + C_0\delta^4\varepsilon^{-\frac{2}{3}}\langle|\omega|^2\rangle + \frac{\varepsilon^2}{4}\langle|\partial_1^2\omega|^2\rangle. \tag{3.19}$$

Proof of lemma 3.5. Note that

$$2\int_0^1 \tau' u_2\theta\,\mathrm{d}x_2 = \frac{1}{\delta}\left(\int_0^\delta u_2\theta\,\mathrm{d}x_2 + \int_{1-\delta}^1 u_2\theta\,\mathrm{d}x_2\right).$$

We shall consider the first integral; the second one is treated similarly. Since θ and u_2 vanish on $x_2 = 0$, we have

$$|\theta(x_1,x_2)| \le \sqrt{x_2}||\partial_2\theta(x_1,\cdot)||_{L^2(0,x_2)}, \quad |u_2(x_1,x_2)| \le x_2||\partial_2 u_2(x_1,\cdot)||_{L^\infty(0,1)}\ \forall x_2 \in (0,1),$$

where, for the second bound, we used the fundamental theorem of calculus to have $u_2(x_1,x_2) = \int_0^{x_2}\partial_2 u_2(x_1,z)\,\mathrm{d}z \le x_2\sup_{0\le z\le x_2}|\partial_2 u_2(x_1,\cdot)|$. Noting that $\int_0^1 \partial_2 u_2(x_1,x_2)\,\mathrm{d}x_2 = 0$, we deduce $\partial_2 u_2(x_1, z_0) = 0$ for some $z_0 = z_0(x_1) \in (0, 1)$. Then by the fundamental theorem of calculus and Hölder's inequality, we obtain

$$|\partial_2 u_2(x_1, x_2)|^2 = 2\left|\int_{z_0}^{x_2} \partial_2 u_2(x_1, z)\partial_2^2 u_2(x_1, z)\mathrm{d}z\right| \lesssim ||\partial_2 u_2(x_1, \cdot)||_{L^2(0,1)}||\partial_2^2 u_2(x_1, \cdot)||_{L^2(0,1)}. \tag{3.20}$$

Applying Hölder's inequality for x_1 yields

$$\begin{aligned} I &:= \frac{1}{\delta}\frac{1}{\Gamma}\left|\int_0^\Gamma \int_0^\delta u_2\theta\, \mathrm{d}x_2\, \mathrm{d}x_1\right| \\ &\lesssim \delta^{3/2}\frac{1}{\Gamma}||\partial_2\theta||_{L^2(\Omega)}||\partial_2 u_2||_{L^2(\Omega)}^{1/2}||\partial_2^2 u_2||_{L^2(\Omega)}^{1/2} \\ &\le \frac{C}{\Gamma}\delta^{3/2}||\partial_2\theta||_{L^2(\Omega)}||\omega||_{L^2(\Omega)}^{1/2}||\partial_1\omega||_{L^2(\Omega)}^{1/2}, \end{aligned}$$

where we have used lemma A.1 and (A 4).

Proof of (a): From the above we have

$$I \le \frac{C}{\Gamma}||\partial_2\theta||_{L^2(\Omega)}\{\delta^{3/2}\varepsilon^{-1/4}||\omega||_{L^2(\Omega)}^{1/2}\}\{\varepsilon^{1/4}||\partial_1\omega||_{L^2(\Omega)}^{1/2}\}.$$

Taking the time average and using the Hölder and Young inequalities, we deduce

$$\langle I\rangle \le \frac{1}{4}\langle|\partial_2\theta|^2\rangle + C\delta^6\varepsilon^{-1}\langle|\omega|^2\rangle + \frac{\varepsilon}{8}\langle|\partial_1\omega|^2\rangle.$$

Proof of (b): As in (3.20), we have the interpolation inequality $||\partial_1\omega||_{L^2(\Omega)}^2 \le ||\omega||_{L^2(\Omega)}||\partial_1^2\omega||_{L^2(\Omega)}$. Thus we obtain the bound

$$\begin{aligned} I &\le \frac{C}{\Gamma}||\partial_2\theta||_{L^2(\Omega)}\{\delta^{3/2}\varepsilon^{-1/4}||\omega||_{L^2(\Omega)}^{3/4}\}\{\varepsilon^{1/4}||\partial_1^2\omega||_{L^2(\Omega)}^{1/4}\} \\ &\le \frac{1}{4}||\partial_2\theta||_{L^2(\Omega)}^2 + C_0\{\delta^{3/2}\varepsilon^{-1/4}||\omega||_{L^2(\Omega)}^{3/4}\}^{8/3} + \frac{1}{8}\{\varepsilon^{1/4}||\partial_1^2\omega||_{L^2(\Omega)}^{1/4}\}^8. \end{aligned}$$

The proof is complete. ∎

Applying lemma 3.5 (a) with $\varepsilon = a$ to (3.17), we find

$$\begin{aligned} \mathcal{Q}[\theta, u, \tau] \ge{}& \frac{1}{2}\langle|\partial_1\theta|^2\rangle + \frac{1}{2}\langle|\partial_2\theta|^2\rangle \\ &+ \left(\frac{b}{\mathsf{Ra}} - \frac{a^2\mathsf{Ra}^2}{2} - \frac{aC_2^2||u_0||_{W^{1,r}}^2}{2\mathsf{L_s}^2\,\mathrm{Pr}^2} - \frac{aC_2^2\mathsf{Ra}^2}{2\mathsf{L_s}^2\,\mathrm{Pr}^2} - C_0\delta^6 a^{-1}\right)\langle|\omega|^2\rangle + a\left(\frac{1}{4} - \frac{C_2}{\mathsf{L_s}^2}\right)\langle|\nabla\omega|^2\rangle. \end{aligned} \tag{3.21}$$

Clearly, the coefficient of $\langle|\nabla\omega|^2\rangle$ in (3.21) is positive for sufficiently large $\mathsf{L_s}$. Fixing an arbitrary $b \in (0, 1)$ and imposing $\mathsf{L_s}^2\,\mathrm{Pr}^2 \ge \mathsf{Ra}^{3/2}$ and $a = a_0\mathsf{Ra}^{-\frac{3}{2}}$ gives

$$A := \frac{b}{\mathsf{Ra}} - \frac{a^2\mathsf{Ra}^2}{2} - \frac{aC_2^2||u_0||_{W^{1,r}}^2}{2\mathsf{L_s}^2\,\mathrm{Pr}^2} - \frac{aC_2^2\mathsf{Ra}^2}{2\mathsf{L_s}^2\,\mathrm{Pr}^2} \ge \frac{b}{\mathsf{Ra}} - \frac{a_0^2}{2\mathsf{Ra}} - \frac{a_0C_2^2||u_0||_{W^{1,r}}^2}{\mathsf{Ra}^3} - \frac{a_0C_2^2}{2\mathsf{Ra}}.$$

We choose

$$a_0 = \frac{b}{100C_2^2}\min\left\{1, \frac{\mathsf{Ra}^2}{||u_0||_{W^{1,r}}^2}\right\}$$

so that $A \ge \frac{b}{2\mathsf{Ra}}$. Letting δ solve $b/2\mathsf{Ra} = 2C_0\delta^6 a_0^{-1}\mathsf{Ra}^{3/2}$, the coefficient of $\langle|\omega|^2\rangle$ in (3.21) is positive and hence $\mathcal{Q}$ is positive. This gives

$$\delta = \left(\frac{a_0 b}{4C_0}\right)^{1/6}\mathsf{Ra}^{-\frac{5}{12}}.$$

In view of (3.12) with $M = aC_2^2/2\mathsf{L_s}^2$, we obtain $\mathsf{Nu} \le 1/2(4C_0/a_0b)^{1/6}\mathsf{Ra}^{\frac{5}{12}} + (a_0C_2^2/2)\mathsf{L_s}^{-2}\mathsf{Ra}^{1/2}$. Inserting a_0 we finally arrive at (3.14).

For $\mathsf{L_s} \in (0, L_0)$, we have $\mathsf{Nu} \lesssim \mathsf{Ra}^{1/2}$ according to lemma 3.2, and hence the bound (3.14) is still valid. If $\mathsf{L_s} = \infty$, the entire argument follows the same way in view of remark 2.4.

Remark 3.6 (A proof of the $\Pr = \infty$ result of Whitehead). If $\Pr = \infty$, the inertial term in the momentum equation vanishes. We work in 2*d* for the sake of simplicity. The key observation of Whitehead is that from (2.11) with $\Pr = \infty$ we have

$$\langle |\partial_1 \theta|^2 \rangle = \frac{1}{\mathsf{Ra}^2} \langle |\Delta \omega|^2 \rangle \geq \frac{1}{C} \langle |\partial_1^2 \omega|^2 \rangle, \tag{3.22}$$

since $\partial_1 \theta = \partial_1 T$ and according to lemma A.3, we have $\langle |\partial_1^2 \omega|^2 \rangle \leq C \langle |\Delta \omega|^2 \rangle$ for some $C > 0$ for any $\mathsf{L_s} > 0$. Applying lemma 3.5 (b) to (3.17) with $M = a = 0$, we find

$$\mathcal{Q}[\theta, u, \tau] \geq \left(\frac{b}{\mathsf{Ra}} - C_0 \delta^4 \varepsilon^{-2/3} \right) \langle |\omega|^2 \rangle + \left(\frac{1}{2C\mathsf{Ra}^2} - \frac{\varepsilon^2}{8} \right) \langle |\partial_1^2 \omega|^2 \rangle.$$

The bound $\mathcal{Q}[\theta, u, \tau] \geq 0$ follows by choosing $\varepsilon = C^{-1/2} \mathsf{Ra}^{-1}$ and $\delta \sim \mathsf{Ra}^{-5/12}$. ■

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Appendix A. Some elliptic estimates

Here we record some useful identities/inequalities involving the vorticity.

Lemma A.1. *With $\omega = \nabla^\perp \cdot u$, the following identities hold*

— $||\nabla u||_{L^2} = ||\omega||_{L^2}$,
— $||\Delta u||_{L^2} = ||\nabla \omega||_{L^2}$.

Proof. The second identity is a consequence of $\Delta u = \nabla^\perp \omega$. Next we prove the first identity. By the periodicity in x_1 and the boundary condition $u_2 = 0$ on $\{x_2 = 0\} \cup \{x_2 = 1\}$, we have

$$\begin{aligned}
\sum_{i,j=1,2} \int_\Omega \partial_j u_i \partial_j u_i \, \mathrm{d}x &= -\int_\Omega u \cdot \Delta u \, \mathrm{d}x + \int_0^\Gamma u_1 \partial_2 u_1 \Big|_{x_2=0}^{x_2=1} \mathrm{d}x_1 \\
&= -\int_\Omega u \cdot \nabla^\perp \omega \, \mathrm{d}x + \int_0^\Gamma u_1 \partial_2 u_1 \Big|_{x_2=0}^{x_2=1} \mathrm{d}x_1 \\
&= \int_\Omega |\omega|^2 \, \mathrm{d}x + \int_0^\Gamma u_1 (\partial_2 u_1 + \omega) \Big|_{x_2=0}^{x_2=1} \mathrm{d}x_1 = \int_\Omega |\omega|^2 \, \mathrm{d}x,
\end{aligned}$$

where we have used that $\partial_2 u_1 + \omega = \partial_1 u_2 = 0$ on $\partial \Omega$. ■

Lemma A.2. *For any $m \geq 1$ and $p \in (1, \infty)$, there exists C such that $||\nabla u||_{W^{m,p}} \leq C ||\omega||_{W^{m,p}}$.*

Proof. Let ψ be the streamfunction for u, i.e. $u = \nabla^{\perp}\psi$ such that

$$\Delta\psi = \omega \quad \text{in } \Omega,$$

$$\psi = 0 \quad \text{on}\{x_2 = 0\},$$

and

$$\psi = c(t) \quad \text{on}\{x_2 = 1\},$$

for some possibly time dependent but spatially constant $c(t)$. Consequently, $\partial_1\psi$ satisfies

$$\Delta\partial_1\psi = \partial_1\omega \quad \text{in } \Omega \tag{A 1}$$

and

$$\partial_1\psi = 0 \quad \text{on } \{x_2 = 0\} \cup \{x_2 = 1\}. \tag{A 2}$$

Fix $k \geq 1$ and $p \in (1, \infty)$. By elliptic regularity, we have

$$||\nabla u_2||_{L^p} = ||\nabla\partial_1\psi||_{L^p} \leq C||\omega||_{L^p} \tag{A 3}$$

and

$$||u_2||_{W^{1+k,p}} = ||\partial_1\psi||_{W^{1+k,p}} \leq C||\partial_1\omega||_{W^{k-1,p}}. \tag{A 4}$$

Now note that by divergence-free and the definition of the vorticity we have $\partial_1 u_1 = -\partial_2 u_2$ and $\partial_2 u_1 = \partial_1 u_2 - \omega$. Therefore, for any $m \geq 0$, we have the bound

$$||\nabla u_1||_{W^{m,p}} \leq C(||\nabla u_2||_{W^{m,p}} + ||\omega||_{W^{m,p}}) \leq C||\omega||_{W^{m,p}}.$$

■

Lemma A.3. *With $\omega = \nabla^{\perp} \cdot u$, we have $||\partial_1\omega||_{L^2} \leq C||\Delta\omega||_{L^2}$ for some $C > 0$.*

Proof. From (A 1)–(A 2) we have $\Delta\partial_1 u_2 = \partial_1^2\omega$ in Ω and $\partial_1 u_2 = 0$ on $\{x_2 = 0\} \cup \{x_2 = 1\}$ since ∂_1 is a tangential derivative. It follows

$$\int_{\Omega} \Delta^2\partial_1 u_2 \partial_1 u_2 \, dx_1 \, dx_2 = \int_{\Omega} \Delta\partial_1^2\omega\partial_1 u_2 \, dx_1 \, dx_2.$$

First note

$$\begin{aligned}
\int_{\Omega} \Delta^2\partial_1 u_2\partial_1 u_2 \, dx_1 \, dx_2 &= -\int_{\Omega} \nabla\Delta\partial_1 u_2 \cdot \nabla\partial_1 u_2 \, dx_1 \, dx_2 \\
&= ||\Delta\partial_1 u_2||^2_{L^2(\Omega)} - \int_0^{\Gamma} \partial_2^2\partial_1 u_2 \partial_2\partial_1 u_2 \, dx_1 \, dx_2\Big|_{x_2=0}^{1} \\
&= ||\Delta\partial_1 u_2||^2_{L^2(\Omega)} - \int_0^{\Gamma} \partial_1^2\partial_2 u_1 \partial_1^2 u_1 \, dx_1 \, dx_2\Big|_{x_2=0}^{1} \\
&= ||\Delta\partial_1 u_2||^2_{L^2(\Omega)} + \frac{1}{L_s}\int_0^{\Gamma} (\partial_1^2 u_1)^2 \, dx_1 \, dx_2\Big|_{x_2=1} + \frac{1}{L_s}\int_0^{\Gamma} (\partial_1^2 u_1)^2 \, dx_1 \, dx_2\Big|_{x_2=0} \\
&\geq ||\Delta\partial_1 u_2||^2_{L^2(\Omega)},
\end{aligned}$$

where we used incompressibility, the fact that $\partial_1^3 u_2$ is zero on the boundary and the boundary conditions (1.4) and (1.5). On the other hand,

$$\begin{aligned}
\int_{\Omega} \Delta\partial_1^2\omega\partial_1 u_2 \, dx_1 \, dx_2 &= \int_{\Omega} \Delta\omega\partial_1^3 u_2 \, dx_1 \, dx_2 \\
&\leq ||\Delta\omega||_{L^2(\Omega)}||\partial_1^3 u_2||_{L^2(\Omega)} \leq C||\Delta\omega||_{L^2(\Omega)}||\Delta\partial_1 u_2||_{L^2(\Omega)},
\end{aligned}$$

where we used that, since $\partial_1 u_2 = 0$ on the boundary, elliptic regularity tells us $||\partial_1^3 u_2||_{L^2(\Omega)} \leq ||\partial_1 u_2||_{H^2(\Omega)} \leq C||\Delta\partial_1 u_2||_{L^2(\Omega)}$. Finally since $\Delta\partial_1 u_2\partial_1^2\omega$, we are done. ■

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