Research

Author for correspondence:
Theodore D. Drivas
e-mail: tdrivas@math.stonybrook.edu

Self-regularization in turbulence from the Kolmogorov 4/5-law and alignment

Theodore D. Drivas

Department of Mathematics, Stony Brook University, Stony Brook, NY 11794, USA

TDD, 0000-0003-2818-0376

A defining feature of three-dimensional hydro-dynamic turbulence is that the rate of energy dissipation is bounded away from zero as viscosity is decreased (Reynolds number increased). This phenomenon—anomalous dissipation—is sometimes called the 'zeroth law of turbulence' as it underpins many celebrated theoretical predictions. Another robust feature observed in turbulence is that velocity structure functions $S_p(\ell) := \langle |\delta_\ell u|^p \rangle$ exhibit persistent power-law scaling in the inertial range, namely $S_p(\ell) \sim |\ell|^{\zeta_p}$ for exponents $\zeta_p > 0$ over an ever increasing (with Reynolds) range of scales. This behaviour indicates that the velocity field retains some fractional differentiability uniformly in the Reynolds number. The Kolmogorov 1941 theory of turbulence predicts that $\zeta_p = p/3$ for all p and Onsager's 1949 theory establishes the requirement that $\zeta_p \leq p/3$ for $p \geq 3$ for consistency with the zeroth law. Empirically, $\zeta_2 \gtrsim 2/3$ and $\zeta_3 \lesssim 1$, suggesting that turbulent Navier–Stokes solutions approximate dissipative weak solutions of the Euler equations possessing (nearly) the minimal degree of singularity required to sustain anomalous dissipation. In this note, we adopt an experimentally supported hypothesis on the anti-alignment of velocity increments with their separation vectors and demonstrate that the inertial dissipation provides a regularization mechanism via the Kolmogorov 4/5-law.

This article is part of the theme issue 'Mathematical problems in physical fluid dynamics (part 2)'.

1. Introduction

We consider spatially periodic, incompressible viscous fluids governed by the Navier–Stokes equations

$$\partial_t u^\nu + \nabla \cdot (u^\nu \otimes u^\nu) = -\nabla p^\nu + \nu \Delta u^\nu + f^\nu \tag{1.1}$$

and

$$\nabla \cdot u^\nu = 0, \tag{1.2}$$

with solenoidal initial data $u^\nu|_{t=0} = u_0^\nu \in L^2(\mathbb{T}^d)$ and body forcing $f^\nu \in L^2(0,T;L^2(\mathbb{T}^d))$. The parameter $\nu > 0$ is the kinematic viscosity of the fluid. Upon non-dimensionalization, it is replaced by the inverse Reynolds number $\mathrm{Re}^{-1} = \nu/\mathrm{UL}$, where U is a characteristic velocity and L a characteristic length. If equations (1.1) and (1.2) are understood as holding in the sense of distributions on $[0,T] \times \mathbb{T}^d$, then solutions of class $L^\infty(0,T;L^2(\mathbb{T}^d)) \cap L^2(0,T;H^1(\mathbb{T}^d))$ known as Leray solutions [1], exist for all time $T > 0$ but are not known to be unique. A fundamental property of these solutions is that they satisfy a global energy inequality. This means that energy dissipation due to the viscosity of the fluid cannot exceed the difference in initial and final kinetic energies plus the energy input by forcing. This inequality can be restated as an equality by accounting for the dissipation arising from an inertial cascade to small scales caused by (hypothetical) singularities in the Leray weak solutions [2]:

$$\int_0^T \int_{\mathbb{T}^d} \varepsilon^\nu[u^\nu]\,dx dt = \frac{1}{2}\int_{\mathbb{T}^d} |u_0^\nu|^2\,dx - \frac{1}{2}\int_{\mathbb{T}^d} |u^\nu(\cdot,T)|^2\,dx + \int_0^T \int_{\mathbb{T}^d} u^\nu \cdot f^\nu\,dx dt, \tag{1.3}$$

for almost every $T \geq 0$, where the total energy dissipation rate is

$$\varepsilon^\nu[u^\nu] := \nu|\nabla u^\nu|^2 + D[u^\nu]. \tag{1.4}$$

The dissipation due to possible singularities, $D[u^\nu]$, is a non-negative distribution (Radon measure). A consequence of (1.3) is that the cumulative energy dissipation $\varepsilon[u^\nu]$ is bounded by norms of data and forcing.

A striking feature of high-*Re* turbulence is that energy dissipation does not vanish in the limit of viscosity going to zero. Namely, that there exists a number $\varepsilon > 0$ independent of viscosity ν such that

$$\int_0^T \int_{\mathbb{T}^d} \varepsilon^\nu[u^\nu]\,dx dt \geq \varepsilon > 0. \tag{1.5}$$

See, e.g. [3–8]. This phenomenon, known as anomalous dissipation, is so fundamental to our modern understanding of turbulence that it has been termed the 'zeroth law' [9]. It should be emphasized however that, to this day, no single mathematical example of (1.5) is available, although there has been great progress in understanding similar behaviour in some model problems such as one-dimensional conservation laws and compressible flows [10–13], shell models [14–17] and passive scalars [18–21].

Despite its conjectural status from the point of view of mathematics, under the experimentally corroborated assumption that behaviour (1.5) occurs together with some heuristic assumptions on statistical properties (homogeneity, isotropy, monofractal scaling), Kolmogorov [22] made a remarkable prediction about the structure of turbulent velocity fields at high Reynolds number, namely that

$$S_p^{||}(\ell) := \langle (\delta_\ell u^\nu \cdot \hat{\ell})^p \rangle \sim (\varepsilon|\ell|)^{p/3} \quad \text{for } \ell_\nu \ll \ell \ll L \tag{1.6}$$

where $\delta_\ell u^\nu(x,t) := u^\nu(x+\ell,t) - u^\nu(x,t)$, $\hat{\ell} = \ell/|\ell|$ and where $\langle \cdot \rangle$ represents some suitable combination of space, time and ensemble averages. The length ℓ_ν, known as the Kolmogorov scale, represents a small-scale dissipative cutoff and the integral scale L represents the size of the largest eddy in the flow. The range $\ell_\nu \ll \ell \ll L$ over which the scaling (1.6) holds is known as the *inertial range*. The objects $S_p^{||}(\ell)$ are called *p*th-order longitudinal structure functions since they measure the (signed) variation of *p*th powers of the velocity increments in the direction of their separation vectors. See figure 1 for evidence of such persistent inertial-range scaling from numerical simulations of homogenous isotropic Navier–Stokes turbulence [23].
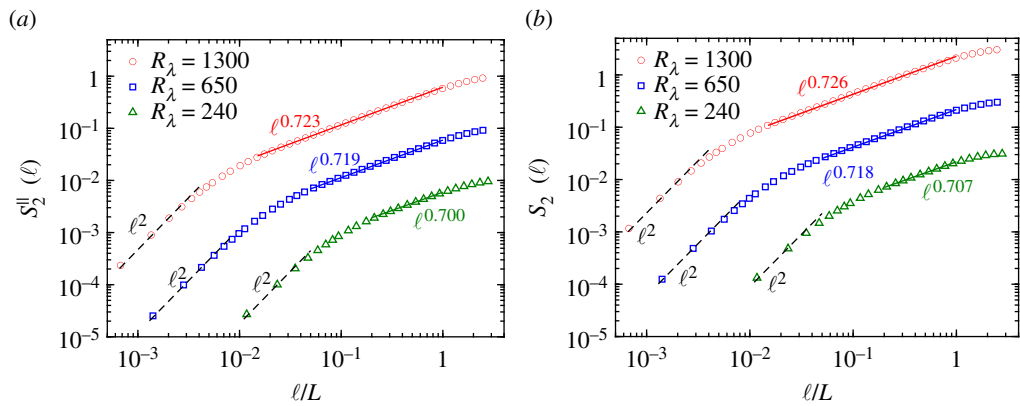
Figure 1. Second-order *longitudinal* (*a*) and *absolute* (*b*) structure functions computed from direct numerical simulation of forced homogenous isotropic turbulence with Taylor scale Reynolds numbers ($R_\lambda := U\lambda/\nu$ where $U := \langle|u|^2\rangle^{1/2}$ and $\lambda := \langle|u|^2\rangle^{1/2}/\langle|\nabla u|^2\rangle^{1/2}$) ranging from $R_\lambda = 240$ (green), 650 (blue), 1300 (red). They exhibit scaling over an inertial range which extends as Reynolds increases. A best-fit power-law exponent ζ_2 for the power-law $|r|^{\zeta_2}$ in this range is included. Data from [23], figure 3*a*. (Online version in colour.)

Onsager took a further step by recognizing that the behaviour (1.5) requires the fluid to develop singularities as $\nu \to 0$ in a mathematically precise sense. Specifically, for (1.5) to occur on sequences of Navier–Stokes solutions, the *p*th-order absolute structure functions *cannot* satisfy a bound of the type

$$S_p(\ell) = \int_0^T \int_{\mathbb{T}^d} |u^\nu(x+\ell,t) - u^\nu(x,t)|^p \, dx dt \leq C|\ell|^{\zeta_p}, \quad \forall|\ell| \leq L, \qquad (1.7)$$

for any $\zeta_p > p/3$, $p \geq 3$ and a constant C independent of viscosity. This assertion, originally stated by Onsager [24] about weak solutions of the Euler equation and in the slightly more restrictive setting of Hölder spaces, has since been rigorously proved [25–27]. In fact, energy dissipation must vanish as viscosity goes to zero for any family of solutions $\{u^\nu\}_{\nu>0}$ which are uniformly bounded in the Besov space[1] $L^p(0,T; B_p^{1/3+,\infty}(\mathbb{T}^d))$ for $p \geq 3$ [28]. Thus, Kolmogorov's 1941 theory corresponds to turbulent solutions possessing the maximal degree of smoothness consistent with their ability to anomalously dissipate energy.

It is well known that real fluids do not conform exactly to Kolmogorov's prediction.[2] Intermittency, or spottiness/non-uniformity of the velocity's roughness and the energy dissipation rate, result in deviations of the scaling exponents $\zeta_p^{||}$ (and ζ_p) from a linear behaviour in p [35–40].[3] Experiments do however indicate that for p near three, the formula $\zeta_p \approx p/3$

---

[1]A vector field v belongs to the Besov space $B_p^{\sigma,\infty}(\mathbb{T}^d)$ for $p \geq 1$, $\sigma \in (0,1)$ at time t if and only if

$$||v(\cdot,t)||_{L^p}^p < C_0(t), \quad S_p(\ell,t) \leq C_1(t) \left|\frac{\ell}{L}\right|^{\zeta_p}, \quad \forall|\ell| \leq L \qquad (1.8)$$

with $\zeta_p = \sigma p$, $L > 0$ and $C_0, C_1 \in L^1(0,T)$. Uniform boundedness of the family $\{u^\nu\}_{\nu>0}$ in $L^p(0,T; B_p^{\sigma,\infty}(\mathbb{T}^d))$ is equivalent to the condition that coefficients $C_0(t), C_1(t)$ independent of $\nu > 0$ exist so that the bounds (1.8) are satisfied for a.e. $t \in [0,T]$.

[2]However, weak Euler solutions with less regularity $u \in C^{1/3-}([0,T] \times \mathbb{T}^d)$ and which do not conserve energy have been constructed [29] after a long series of works [30–32] and they can be made strictly (globally) dissipative [33]. See [34] for a very nice recent review of the subject. In a sense, these solutions exhibit exact K41-type behaviour, although they are not known to arise as physical limits of Leray solutions of Navier–Stokes as required to make contact with real-world high-*Re* flows.

[3]In fact, there is a rigorous connection between these two irregularities: Isett proved [41] that if $\zeta_p \geq p/3$ for some $p > 3$, then the dissipation would have to take place on a full measure set. As experiments indicate that the dissipation takes place on measure zero set of (spatial) fractal dimension ≈ 2.87 [42], this is consistent with velocity intermittency $\zeta_p < p/3$ for all $p > 3$. On the other hand, velocity irregularity is not enough to sustain anomalous dissipation: Shvydkoy [43] proved that 'ordered singularities' with $\zeta_p = 1$ for $p \geq 1$ such as tangential velocity discontinuities across smooth co-dimension one hypersurfaces (regular vortex sheets) conserve energy.
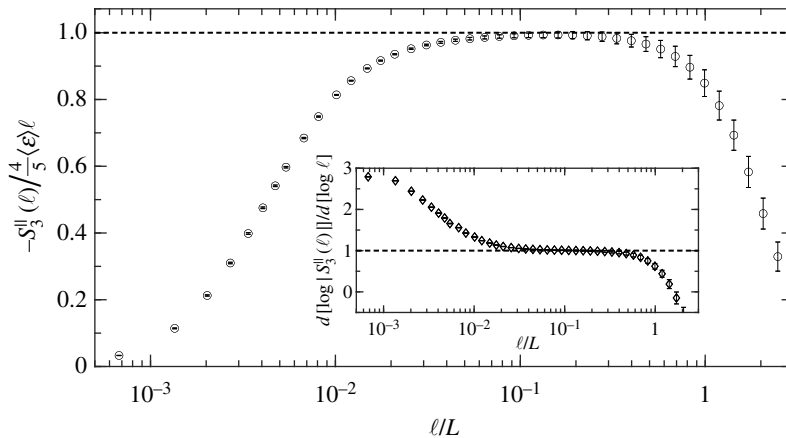
Figure 2. The quantity $-S_3^{||}(\ell)/(\frac{4}{5}\ell\langle\varepsilon\rangle)$, is plotted for Reynolds numbers $R_\lambda = 1300$. The range of scales over which a value near unity is achieved visibly extends as Reynolds number increases. Data from [23], fig. 1.

approximately holds with $\zeta_2 \in \frac{2}{3} + [0.03, 0.06]$ and $\zeta_3 \approx 1$. For example, in flow past a sphere $\zeta_2 \approx 0.701$ is reported in [36] and $\zeta_2 \approx 0.71$ in [35] (see table 2 therein). Recent high-resolution numerical simulations report $\zeta_2 \approx 0.725$ (figures 1 and 3). Although there are slight variations, all these results conform to $\zeta_2 \gtrapprox 2/3$ and $\zeta_3 \lessapprox 1$. These observations motivate:

Question 1.1. Why does high-Reynolds number turbulence seems to be as rough as required to support anomalous dissipation of energy but not much rougher?

In this direction, we note that Kolmogovov's prediction (1.6) in the case $p = 3$ has a privileged status in that it can be derived (under certain technical assumptions, see Prop. 2.2) from the equations of motion (1.1)–(1.2) rather than being merely a consequence of statistical hypotheses. Specifically, Kolmogorov established the '4/5-law' (in dimension three) under only the assumption of anomalous dissipation (1.5):

$$S_3^{||}(\ell) := \langle(\delta_\ell u \cdot \hat{\ell})^3\rangle \approx -\frac{12}{d(d+2)}\varepsilon\ell, \tag{1.9}$$

which holds in the limit of large Reynolds number $\nu \to 0$ and subsequently small scales $\ell \to 0$. In practice, (1.9) is observed to hold approximately over the inertial range; see figure 2 for evidence from [23]. The 4/5-law captures some aspects of the turbulent cascade: energy is transferred through scale by a cubic nonlinear flux term related to $S_3^{||}(\ell)$ until it is removed, $\varepsilon > 0$, from the system by infinitesimal viscosity.

In fact, the 4/5-law (1.9) fixes the scaling exponent $p/3$ in (1.6) via the statistical assumption of monofractal scaling in Kolmogorov's theory. On its face, this indicates that the turbulent fluid velocity satisfies $\delta_\ell u \sim \ell^{1/3}$ and so is '1/3 differentiable', at least in some averaged sense. More precisely, the fact that ε is *a priori* bounded by initial data and forcing via equation (1.3) ensures that $S_3^{||}(\ell)/\ell$ is controlled uniformly for small ℓ. This suggests that some kind of *a priori* regularity information—a 'turbulent energy estimate'—might be extracted from the 4/5-law. Unfortunately, aside from justifying the assumptions necessary for a rigorous derivation of (1.9), there are two obstructions to realizing this hope: (i) the longitudinal structure function does not measure velocity variations in all directions, only those aligned with the separation vector and (ii) the integrand is not sign-definite. In particular, although a certain skewness is implied by the 4/5-law (positive of ε means negativity of $\delta_\ell u \cdot \hat{\ell}$ in an averaged sense), it is conceivable that there are large fluctuations which cancel in the integral to yield (1.9) but would disturb this relation if the increments were replaced by their absolute values. Both of these issues prevent the control on
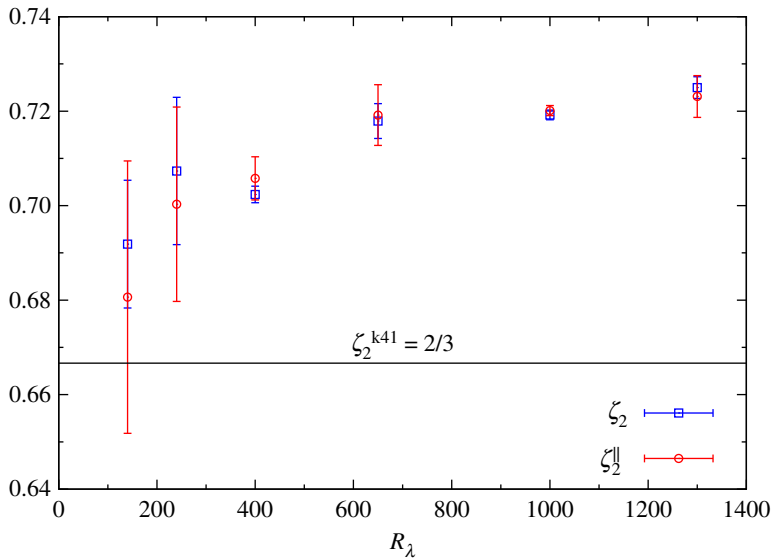
Figure 3. Best-fit exponents within the inertial range are plotted for absolute ζ_2 and longitudinal $\zeta_2^{||}$ structure functions. The K41 value of $\zeta_2^{k41} := 2/3$ is given for reference. The exponents ζ_2, $\zeta_2^{||}$ appear to saturate at 0.725, which serves as the exponent which provides the uniform (in *Re*) bounds (1.7) and (1.12). Data from [23], fig. 3*b*. (Online version in colour.)

third order longitudinal structure function afforded by (1.9) from being coercive and so it seems that no direct information about regularity of the velocity can be immediately extracted.

Here we explore possible nonlinear mechanisms to extract regularity from 4/5-law. Our results will be of a conditional nature, involving hypotheses which are unproved but which are corroborated by experiment and simulations of turbulence. Roughly, our main assumptions (hypothesis 2.1) are that

    (a) the Kolmogorov 4/5-ths law holds
    (b) the Kolmogorov 4/3-rds law holds

as well as (hypothesis 2.3).

    (a) there exists an $\alpha \in [0, 1)$ and $C > 0$ independent of ν such that for all scales $\ell > 0$

$$-\int_0^T \int_{\mathbb{T}^d} \left\langle (\delta_\ell u^\nu \cdot \hat{\ell})^3 \right\rangle_{\text{ang}} \mathrm{d}x\mathrm{d}t \geq C|\ell|^\alpha \int_0^T \int_{\mathbb{T}^d} \left\langle |\delta_\ell u^\nu \cdot \hat{\ell}|^3 \right\rangle_{\text{ang}} \mathrm{d}x\mathrm{d}t, \qquad (1.10)$$

    (b) there exists an $\beta \in [0, 1)$ and $C' > 0$ independent of ν such that for all scales $\ell > 0$

$$-\int_0^T \int_{\mathbb{T}^d} \left\langle (\delta_\ell u^\nu \cdot \hat{\ell})|\delta_\ell u^\nu|^2 \right\rangle_{\text{ang}} \mathrm{d}x\mathrm{d}t \geq C'|\ell|^\beta \int_0^T \int_{\mathbb{T}^d} \left\langle |\delta_\ell u^\nu|^3 \right\rangle_{\text{ang}} \mathrm{d}x\mathrm{d}t. \qquad (1.11)$$

The first hypotheses assert the validity of the Kolmogorov laws for weak solutions. Although anticipated to be true (the 4/5-ths law is sometimes referred to as the only 'exact' law in turbulence), to this day it has not been unconditionally established (see [44,45] for conditional validations and figure 2 for empirical evidence). The second hypotheses concern some effective alignment properties of the velocity increments with their separation vectors. A detailed discussion is deferred to the subsequent section. As stated above, they are slightly stronger than hypothesis 2.3 required in the proof, but may be more convenient to verify numerically or in experiment. In fact, there has already been direct evidence of the behaviour (1.10) with $\alpha \approx 0.03$

from experiment [38–40] (see extended discussion in remark 2.5). In practice, (1.10) and (1.11) need only be checked over the finite range of scales in the inertial range $\ell_\nu \ll \ell \ll L$. We prove

Theorem 1.2. *Let u be a weak solution of the Euler equations of class $L^3(0, T; L^3(\mathbb{T}^d))$. Then*

(a) *if hypotheses 2.1(a) and 2.3(a) hold, then $u \in L^2(0, T; B_2^{(1-\alpha)/3,\infty}(\mathbb{T}^d))$,*
(b) *if hypotheses 2.1(b) and 2.3(b) hold, then $u \in L^3(0, T; B_3^{(1-\beta)/3,\infty}(\mathbb{T}^d))$.*

Theorem 1.2(b) is not very surprising since hypothesis 2.3(b) (nearly) assumes control on the absolute structure function by the longitudinal. We include it because there is experimental evidence of such control, because L^3 seems to be the natural scale for regularization by the energy cascade. It also applies unconditionally to the entropic solutions of the Burgers equation (see remark 2.7). On the other hand, theorem 1.2(a) produces regularity in L^2 *without* assuming that the full velocity increment can be controlled by the longitudinal component. This is due to the following Lemma which shows that the dynamical law (Euler or Navier–Stokes equations) of the fluid can be used to deduce information on the full velocity increment from partial information on the behaviour of the component in the direction of its separation vector.

Lemma 1.3. *A weak solution of the incompressible Euler equations is of class $L^2(0, T; B_2^{\zeta_2^{\|}/2,\infty}(\mathbb{T}^d))$ with $\zeta_2^{\|} \in (0, 2]$ if and only if the longitudinal structure function defined by (3.7) satisfies*

$$\int_0^T \left\langle S_2^{\|}(\ell) \right\rangle_{ang} \, \mathrm{d}t \lesssim |\ell|^{\zeta_2^{\|}}, \quad \forall |\ell| > 0. \tag{1.12}$$

Remark 1.4 (Weak solutions as zero-viscosity limits). Lemma 1.3 has some implications for the weak inviscid limit. In particular, uniform boundedness of the family $\{u^\nu\}_{\nu>0}$ in $L^2(0, T; B_2^{\zeta_2^{\|}/2,\infty}(\mathbb{T}^d))$ is equivalent to a bound of the form (1.12) independent of viscosity. In fact, as in lemma 1 of [46], for Leray solutions $u^\nu \in L^\infty(0, T; L^2(\mathbb{T}^d)) \cap L^2(0, T; H^1(\mathbb{T}^d))$ the condition (1.12) is equivalent to

$$\int_0^T \left\langle S_2^{\|}(\ell; \nu) \right\rangle_{ang} \, \mathrm{d}t \lesssim |\ell|^{\zeta_2^{\|}}, \quad \eta(\nu) \leq |\ell| \leq L, \tag{1.13}$$

where $\eta(\nu) = \nu^{1/2(1-s)}$. Thus, a uniform scaling with any positive exponent of the longitudinal structure function in the 'inertial range' suffices to obtain weak Euler solutions in the inviscid limit (see Thm 1 of [46]). We emphasize that the bound (1.13) is not naively a compactness statement, although for equations structurally similar to Navier–Stokes, lemma 1.12 transforms it into one. See figure 3 for empirical verification of lemma 1.3 as it applies to inviscid limits of Navier–Stokes turbulence, relating the bounds (scalings) of the absolute and longitudinal structure functions.

2. Kolmogorov 4/5-law and alignment hypotheses

We here recall a rigorous formulation of the Kolmogorov 4/5-law for weak solutions of the Euler equations arising as zero viscosity limits and introduce the precise hypotheses under which our theorem is established. As discussed above, Onsager conjectured [24] that sufficiently rough, dissipative weak solutions of the Euler equations are candidate descriptions of high-Reynolds number flows exhibiting the behaviour (1.5). Onsager's vision of weak Euler solutions as a framework to study zero-viscosity limits follows from sufficient compactness. Indeed, if a family of Leray solutions $\{u^\nu\}_{\nu>0}$ is precompact in $L^3(0, T; L^3(\mathbb{T}^d))$, then strong space–time L^3–limits exist $u^\nu \to u \in L^3(0, T; L^3(\mathbb{T}^d))$ and are weak solutions of the Euler equations.[4] In fact, this assumption

---

[4]Such compactness is implied if the family of Leray solutions $\{u^\nu\}_{\nu>0}$ is uniformly in ν bounded in $L^3(0, T; B_3^{s,\infty}(\mathbb{T}^d))$ *for any s > 0* [28,46]. As discussed above, this is robustly observed in experiments and simulations [36–40].

also guarantees the existence of a limiting dissipation measure $\varepsilon[u]$,

$$\mathcal{D}'\text{-}\lim_{\nu\to 0}\varepsilon^\nu[u^\nu] = \varepsilon[u] \geq 0, \tag{2.1}$$

where the limit is understood in the sense of distributions (it holds upon pairing with any smooth test function and is denoted by $\mathcal{D}'$-lim). Furthermore, Duchon & Robert [2] showed that any weak solution of the Euler equations u of class $L^3(0, T; L^3(\mathbb{T}^d))$ satisfies a (weak) energy balance

$$\partial_t\left(\frac{1}{2}|u|^2\right) + \nabla\cdot\left[\left(\frac{1}{2}|u|^2 + p\right)u\right] = -D[u], \tag{2.2}$$

where the 'inertial dissipation' $D[u]$ is defined by the distributional limit

$$D[u] := \mathcal{D}'\text{-}\lim_{\ell\to 0}\frac{1}{4\ell}\int_{\mathbb{T}^d}(\nabla\varphi)_\ell(r)\cdot\delta_r u(x,t)|\delta_r u(x,t)|^2\,\mathrm{d}r \tag{2.3}$$

with φ an arbitrary standard mollifier, $(\nabla\varphi)_\ell(r) = \ell^{-d}\nabla\varphi(r/\ell)$ and $\delta_r u(x,t) = u(x+r,t) - u(x,t)$. The distribution defined by (2.3) represents the flux of energy into or out of the fluid due to a nonlinear inertial cascade to zero length-scale facilitated, as Onsager envisioned, by sufficiently irregular velocity fields. As a consequence of (2.2), the inertial dissipation matches onto the viscous dissipation anomaly

$$D[u] = \varepsilon[u], \tag{2.4}$$

and the distribution $D[u]$ must be non-negative and independent of the mollifier φ. This independence can be seen directly provided that u has some additional spatial continuity which is made precise by the following

Hypothesis 2.1. *Let u be any weak solution of the Euler equation of class $L^3(0, T; L^3(\mathbb{T}^d))$. Suppose*

(a) *the following version of the Kolmogorov 4/5-law holds*

$$\mathcal{D}'\text{-}\lim_{|\ell|\to 0}\frac{1}{|\ell|}\left\langle(\delta_\ell u\cdot\hat{\ell})^3\right\rangle_{ang} = D^*_{4/5}[u], \tag{2.5}$$

(b) *the following version of the Kolmogorov 4/3-law holds*

$$\mathcal{D}'\text{-}\lim_{|\ell|\to 0}\frac{1}{|\ell|}\left\langle(\delta_\ell u\cdot\hat{e})|\delta_\ell u|^2\right\rangle_{ang} = D^*_{4/3}[u], \tag{2.6}$$

where the angle average denotes $\langle f(\ell)\rangle_{ang} := \int_{S^{d-1}} f(\ell)\,\mathrm{d}\omega(\hat{\ell})$ and $d\omega$ is the measure on solid angles.

With hypothesis 2.1 in hand, we see explicitly that $D[u]$ does not depend on the arbitrary mollifier and the standard versions of the Kolmogorov laws hold:

Proposition 2.2 (Duchon & Robert [2] and Eyink [45]). *Under hypothesis* 2.1, *the following distributional equalities hold*

$$D^*_{4/5}[u] = -\frac{12}{d(d+2)}D[u], \quad D^*_{4/3}[u] = -\frac{4}{d}D[u]. \tag{2.7}$$

See also discussion in [47], which gives a Lagrangian interpretation of these distributions. In combination with (2.4) which holds for strong vanishing viscosity limits, proposition 2.2 constitutes precise versions of the celebrated Kolmogorov 4/5 and 4/3-laws (upon setting $d = 3$ in (2.7)):

$$D^*_{4/5}[u] = -\frac{12}{d(d+2)}\varepsilon[u], \qquad D^*_{4/3}[u] = -\frac{4}{d}\varepsilon[u]. \tag{2.8}$$

Equation (2.8) is a rigorous version of (1.9). It should be noted that the above relationships are *local* in that they hold in the sense of space–time distributions. Moreover, they show that the fluxes $D^*_{4/5}[u]$ and $D^*_{4/3}[u]$ are, in fact, non-positive as distributions, implying a form of skewness of the velocity field.

Now, for any weak Euler solution of class $L^3(0, T; L^3(\mathbb{T}^d))$, the inertial dissipation $D[u]$ must be finite upon averaging in space time. However, it need not be signed. One consequence of (2.4) is that the inertial dissipation inherits an *a priori* bound in terms of initial data and forcing, and is non-negative for high-Reynolds number flows exhibiting anomalous dissipation (1.5):

$$0 < \int_0^T \int_{\mathbb{T}^d} D[u] \, dx dx < \infty. \tag{2.9}$$

The balance (2.4), which leads to (2.9), is related to the direct energy cascade and can be interpreted as the statement that a nonlinear transfer of energy can be sustained even to infinitesimally small scales, where an infinitesimal viscosity can efficiently remove energy from the system. Our main thesis is that (2.9) can provide a partial explanation for the discussed smoothness of the weak Euler solutions, provided that the solutions additionally possess certain structural properties. More precisely, we adopt the assumption that

Hypothesis 2.3. *Let u be a weak solution of Euler satisfying hypothesis 2.1. Suppose in addition that*

(a) *there exists an $\alpha \in [0, 1)$ and $C := C(d, T, u_0, f) > 0$ such that*

$$-\int_0^T \int_{\mathbb{T}^d} D^*_{4/5}[u] \, dx dt \geq \limsup_{|\ell| \to 0} \frac{C}{|\ell|^{1-\alpha}} \int_0^T \int_{\mathbb{T}^d} \left\langle |\delta_\ell u \cdot \hat{\ell}|^3 \right\rangle_{ang} \, dx dt. \tag{2.10}$$

(b) *there exists an $\beta \in [0, 1)$ and $C' := C'(d, T, u_0, f) > 0$ such that*

$$-\int_0^T \int_{\mathbb{T}^d} D^*_{4/3}[u] \, dx dt \geq \limsup_{|\ell| \to 0} \frac{C'}{|\ell|^{1-\beta}} \int_0^T \int_{\mathbb{T}^d} \left\langle |\delta_\ell u|^3 \right\rangle_{ang} \, dx dt. \tag{2.11}$$

Remark 2.4. Clearly hypothesis 2.3(b) is the stronger assumption since $|\delta_\ell u \cdot \hat{\ell}|^3 \leq |\delta_\ell u|^3$. In turn, we will show that it leads to a stronger form of regularization on the limit Euler solution. It should be noted that hypothesis 2.3(a) is an assumption on the possible cancellations of the angle average rather than a brute force control of a piece of the velocity increment by the full increment as in hypothesis 2.3(b). Such statements are assumptions on the average *anti-alignment* of velocity increments with their separation vectors. Additionally, we remark that the choice of $D^*_{4/3}[u]$ in hypothesis 2.3(b) was not important. For our purpose (theorem 1.2(b)), it is sufficient that it holds for either distribution $D^*_{4/3}[u]$ or $D^*_{4/5}[u]$—the important properties are the finiteness and positivity upon space-time averaging. In any case, if hypothesis 2.1 holds, with both limits (2.5) and (2.6) existing, then the distributions are interchangeable in the statement of the (2.10) and (2.11).

Remark 2.5 (Evidence of anti-alignment). There is some experimental evidence [38–40] for the type of alignment assumed in hypothesis 2.3(a), in particular that absolute differences do differ in scaling as in (2.10), but very slightly. Specifically, for absolutely third-order longitudinal structure functions, experiments find [40] that (a slightly strengthened version of) hypothesis 2.3(a) holds with $\alpha \approx 0.03$, i.e. $-\langle (\delta_\ell u)^3 \rangle \sim \ell^\alpha \langle |\delta_\ell u|^3 \rangle$ in the inertial range; see table 1 therein. This behaviour of $\alpha \ll 1$ has also been observed in a number of other experiments [36,37]. It should be noted that the experimental measurements are inferred from data of the velocity field along a one-dimensional cut and the longitudinal structure was computed by appealing to Taylor's hypothesis, ergodicity and statistical isotropy and homogeneity.

Remark 2.6 (Stochastic setting). Rigorously establishing alignment properties such as those appearing in hypothesis 2.3 seems to be a very difficult task. Moreover, even if true generically, it is quite conceivable that it is false pathwise the setting of the deterministic Navier–Stokes solutions due to non-generic events. Thus, such properties might be easier established in the stochastically forced or random data setting. For instance, one might be able to prove the existence of statistically stationary, homogenous isotropic martingale solutions. It is then plausible that (2.10) and (2.11) hold for such solutions upon ensemble averaging. See [44] for some interesting developments concerning the validity of hypothesis 2.1.

The fact that inertial dissipation and the direct energy cascade can provide a regularization mechanism for the weak solutions is well understood in some model problems, such as one-dimensional conservation laws [48–50] as well the dyadic (Desnyansky–Novikov) shell model of turbulence [51]. For three-dimensional Navier–Stokes, no form of uniform fractional regularity or self-regularization has ever been rigorously established from first principles, however experiments and simulations ubiquitously indicate that solutions do possess some form of these phenomena [36–40]. In particular, as discussed above, measurements of multifractal structure function scaling exponents from over the last 60 years indicate that some turbulent solutions of Navier–Stokes enjoy some limited uniform fractional regularity in L^p spaces. Under the hypotheses 2.1 and 2.3, the latter being of a quantitative nature, we capture some of the smoothing effect of the nonlinearity and obtain a self-regularizing property of dissipative weak Euler solutions by theorem 1.2. Thus, the 4/5-law together some alignment properties implies regularization for any such weak Euler solution with a finite positive inertial dissipation (in particular, vanishing viscosity limits). Of course, our theorem 1.2 is of a conditional nature in that it relies on two major hypotheses 2.1 and 2.3, both of which seem very difficult to prove *a priori*. However, the validity of these hypotheses can be checked in Nature through controlled experiment and in direct numerical simulation (DNS) of the Navier–Stokes equations at high Reynolds number. In remark 2.5, we recalled some existing experimental results concerning the validity hypothesis 2.3(a). We hope that our hypotheses 2.1 and 2.3 will be subject to much further testing and scrutiny.

Remark 2.7 (Burgers equation). The two hypotheses 2.1 and 2.3 are true for entropy solutions of the one-dimensional Burgers equation. In particular, the so-called 1/12th law states

$$\lim_{|\ell| \to 0} \frac{1}{12} \int_0^T \int_{\mathbb{T}^d} \frac{1}{|\ell|} \left\langle (\delta_\ell u)^3 \right\rangle_{\text{ang}} \, \mathrm{d}x \mathrm{d}t = -\int_0^T \int_{\mathbb{T}^d} \varepsilon(x,t) \, \mathrm{d}x \mathrm{d}t, \tag{2.12}$$

where the one-dimensional angle average is defined by $\langle (\delta_\ell u)^3 \rangle_{\text{ang}} = \frac{1}{2}[(\delta_\ell u)^3(|\ell|) + (\delta_\ell u)^3(-|\ell|)]$. Equation (2.12) is the analogue of the 4/5-law in the setting of Burgers and is rigorously established for vanishing viscosity limits. If there are countably many shocks, the following can be explicitly computed

$$\lim_{|\ell| \to 0} \int_0^T \int_{\mathbb{T}^d} \frac{1}{|\ell|} \left\langle (\delta_\ell u)^3 \right\rangle_{\text{ang}} \, \mathrm{d}x \mathrm{d}t = \sum_i (\Delta u_i(t))^3,$$

where $\Delta u_i(t)$ is the jump at the ith shock. The Lax entropy condition is that $u^- > u^+$ at shocks, or $\Delta u_i(t) < 0$. This means that $\Delta u_i(t) = -|\Delta u_i(t)|$ and our hypothesis 2.3 holds with $\alpha = \beta = 0$. This is an example of perfect 'anti–alignment'. In accord with theorem 1.2(b), we obtain $u \in L^3(0,T;B_3^{1/3,\infty}(\mathbb{T}))$. In the light of the inclusion $(L^\infty \cap BV)(\mathbb{T}^d) \subset B_p^{1/p,\infty}(\mathbb{T}^d)$, this is consistent with the well-known BV regularity of entropy solutions of one-dimensional hyperbolic conservation laws [48–50].

3. Proofs

Proof of lemma 1.3. Let $s = \zeta_2^{||}/2$. **(1)** $\implies$ **(2)**. This direction is trivial since

$$\left\langle S_2^{||}(\ell) \right\rangle_{\text{ang}} \leq \sup_{|\ell'| \leq |\ell|} \int_{\mathbb{T}^d} |\delta_{\ell'} u \cdot \hat{\ell}'|^2 \, \mathrm{d}x \leq \sup_{|\ell'| \leq |\ell|} S_2(\ell') \leq C|\ell|^{2s}, \tag{3.1}$$

where the Besov regularity $u \in L^2(0,T;B_2^{s,\infty}(\mathbb{T}^d))$ was used in the final inequality.

(2) $\implies$ **(1)**. Let $u \in L^2(0,T;L^2(\mathbb{T}^d))$ be a weak solution of the incompressible Euler equations:

$$\int_0^T \int_{\mathbb{T}^d} u \partial_t \varphi \, \mathrm{d}x \mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} u \otimes u : \nabla \varphi \, \mathrm{d}x \mathrm{d}t = 0. \tag{3.2}$$

where $\varphi := \varphi(x,t) \in C_0^\infty([0,T] \times \mathbb{T}^d)$ is a compactly supported divergence-free test function. Defining $\varphi_\ell := \varphi(x - \ell)$. Introducing the increment field $\delta_\ell u := u(x + \ell) - u(x)$ and choosing the

test function $\varphi_\ell - \varphi$ in (3.2) shows that

$$(\partial_t + u \cdot \nabla)\delta_\ell u = -\nabla_x \delta_\ell p - \delta_\ell u \cdot \nabla_\ell \delta_\ell u + \delta_\ell f \tag{3.3}$$

holds in the sense of distributions. To obtain this we denote $u' = u(x + \ell)$ and $u = u(x)$ and derive a weak form of the 'doubling variables' identity

$$\int_{\mathbb{T}^d} (u \otimes u : \nabla\varphi_\ell - u \otimes u : \nabla\varphi)\,\mathrm{d}x = \int_{\mathbb{T}^d} (\delta_\ell u \otimes u' - u \otimes \delta_\ell u) : \nabla\varphi\,\mathrm{d}x$$

$$= \int_{\mathbb{T}^d} (\delta_{-\ell} u \otimes u : \nabla\varphi_\ell - u \otimes \delta_\ell u : \nabla\varphi)\,\mathrm{d}x$$

$$= \int_{\mathbb{T}^d} (\delta_{-\ell} u \otimes u : \nabla_\ell\varphi_\ell - u \otimes \delta_\ell u : \nabla\varphi)\,\mathrm{d}x$$

$$= \nabla_\ell \cdot \int_{\mathbb{T}^d} \delta_\ell u \otimes u' \cdot \varphi\,\mathrm{d}x - \int_{\mathbb{T}^d} u \otimes \delta_\ell u : \nabla\varphi\,\mathrm{d}x$$

$$= \nabla_\ell \cdot \int_{\mathbb{T}^d} \delta_\ell u \otimes \delta_\ell u \cdot \varphi\,\mathrm{d}x - \int_{\mathbb{T}^d} u \otimes \delta_\ell u : \nabla\varphi\,\mathrm{d}x,$$

where we used the fact that u is distributionally divergence-free. This establishes that (3.3) holds in the sense of distributions. Dotting the above equation with ℓ, we find

$$(\partial_t + u \cdot \nabla)(\delta_\ell u \cdot \ell) = -\ell \cdot \nabla_x \delta_\ell p - \delta_\ell u \cdot \nabla_\ell(\delta_\ell u \cdot \ell) + |\delta_\ell u|^2 + \delta_\ell f. \tag{3.4}$$

Integrating this balance over the torus, we have

$$\int_{\mathbb{T}^d} |\delta_\ell u|^2\,\mathrm{d}x = -\nabla_\ell \cdot \int_{\mathbb{T}^d} \delta_\ell u(\delta_\ell u \cdot \ell)\,\mathrm{d}x, \tag{3.5}$$

where we used only periodicity of the solution fields. Averaging (in the separation vector ℓ) equation (3.5) over a ball of radius L centred at zero, we find

$$\fint_{B_L(0)} \int_{\mathbb{T}^d} |\delta_\ell u|^2\,\mathrm{d}x\mathrm{d}\ell = \int_{S^{d-1}} \left(\int_{\mathbb{T}^d} |\delta_\ell u \cdot \hat{\ell}|^2\,\mathrm{d}x \right)\Bigg|_{|\ell|=L}\,\mathrm{d}\omega(\hat{\ell}), \tag{3.6}$$

where S^{d-1} is the unit sphere in d-dimensions and $\mathrm{d}\omega$ is the measure on solid angles (unit Haar measure on S^{d-1}) and $\fint_A := \frac{1}{|A|}\int_A$. For $p \geq 1$, we define the absolute and longitudinal structure functions to be

$$S_p(\ell) := \int_{\mathbb{T}^d} |\delta_\ell u|^p\,\mathrm{d}x, \quad S_p^{||}(\ell) := \int_{\mathbb{T}^d} (\delta_\ell u \cdot \hat{\ell})^p\,\mathrm{d}x. \tag{3.7}$$

Introducing the angle-averaging operation

$$\langle f(\ell) \rangle_{\mathrm{ang}} := \int_{S^{d-1}} f(\ell)\,\mathrm{d}\omega(\hat{\ell}), \tag{3.8}$$

from (3.6) we have deduced the identity

$$\fint_{B_L(0)} S_2(\ell)\,\mathrm{d}\ell = \left\langle S_2^{||}(L\hat{\ell}) \right\rangle_{\mathrm{ang}}. \tag{3.9}$$

Remark 3.1. In general, the tensor product with ℓ yields

$$(\partial_t + u^\nu \cdot \nabla_x)(\delta_\ell u \otimes \ell) + \delta_\ell u \cdot \nabla_\ell(\delta_\ell u \otimes \ell) = \delta_\ell u \otimes \delta_\ell u - \ell \otimes \nabla_x \delta_\ell p.$$

Thus, integrating over space and separation vectors then yields the tensorial identity

$$\left\langle \int_{\mathbb{T}^d} (\delta_\ell u \cdot \hat{\ell})(\delta_\ell u \otimes \hat{\ell})\,\mathrm{d}x \right\rangle_{\mathrm{ang}} = \fint_{B_\ell(0)} \int_{\mathbb{T}^d} (\delta_{\ell'} u \otimes \delta_{\ell'} u)\,\mathrm{d}x\mathrm{d}\ell'. \tag{3.10}$$

Taking the trace gives the scalar identity (3.9). We also obtain as a special case in three dimensions

$$\left\langle \int_{\mathbb{T}^d} (\delta_\ell u \cdot \hat{\ell})(\delta_\ell u \times \hat{\ell})\,\mathrm{d}x \right\rangle_{\mathrm{ang}} = 0. \tag{3.11}$$

Using the identity (3.9) together with the assumed bound (1.12), we have

$$\int_{B_L(0)} S_2(\ell) \, \mathrm{d}\ell \le C L^{2s}. \tag{3.12}$$

This inequality holds for all $L \ge 0$. Let $f(\ell) := ||\delta_\ell u||_{L^2}$ and note that for any $\ell' \in \mathbb{T}^d$, we have

$$|f(\ell) - f(\ell')| = |\,||\delta_\ell u||_{L^2} - ||\delta_{\ell'} u||_{L^2}\,| \le ||\delta_\ell u - \delta_{\ell'} u||_{L^2}$$
$$= \sqrt{\int_{\mathbb{T}^d} |u(x+\ell) - u(x+\ell')|^2 \, \mathrm{d}x} = ||\delta_{\ell'-\ell} u||_{L^2}.$$

Thus we have the bound

$$\fint_{B_L(\ell')} |f(\ell) - f(\ell')|^2 \, \mathrm{d}\ell \le \fint_{B_L(\ell')} |f(\ell' - \ell)|^2 \, \mathrm{d}\ell = \fint_{B_L(0)} |f(\ell)|^2 \, \mathrm{d}\ell. \tag{3.13}$$

We conclude that for any $\ell' \in \mathbb{T}^d$ and $L > 0$, the following inequality holds

$$(\fint_{B_L(\ell')} |f(\ell) - f(\ell')| \, \mathrm{d}\ell)^2 \le \fint_{B_L(\ell')} |f(\ell) - f(\ell')|^2 \, \mathrm{d}\ell \le C L^{2s}, \tag{3.14}$$

which follows by Jensen's inequality. We finally appeal to the following basic fact ∎

Lemma 3.2. *Assume that there exists $C > 0$ and $\alpha \in (0,1]$ such that for every $x_0 \in \mathbb{T}^d$ and $r > 0$,*

$$\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |f(x) - f(x_0)| \, dx < C r^\alpha. \tag{3.15}$$

Then f is Hölder continuous with exponent α.

Proof. Let $x_0, y_0 \in \mathbb{T}^d$ and set $r = |x_0 - y_0|$. Then we have $B(y_0, r) \subset B(x_0, 2r)$. Thus,

$$2Cr^\alpha \ge \frac{1}{|B_{2r}(x_0)|} \int_{B_{2r}(x_0)} |f(x) - f(x_0)| \, \mathrm{d}x + \frac{1}{|B_{2r}(y_0)|} \int_{B_{2r}(y_0)} |f(x) - f(y_0)| \, \mathrm{d}x$$
$$= (c_0(2r)^d)^{-1} \left\{ \int_{B_{2r}(x_0)} |f(x) - f(x_0)| \, \mathrm{d}x + \int_{B_{2r}(y_0)} |f(x) - f(y_0)| \, \mathrm{d}x \right\}$$
$$\ge (c_0(2r)^d)^{-1} \int_{B_r(x_0)} |f(x) - f(x_0)| + |f(x) - f(y_0)| \, \mathrm{d}x$$
$$\ge (c_0(2r)^d)^{-1} \int_{B_r(x_0)} |f(x_0) - f(y_0)| dx = 2^{-d} |f(x_0) - f(y_0)|$$

where c_0 denotes the volume of the unit ball. It follows that

$$|f(x_0) - f(y_0)| \le 2^{d+1} C |x_0 - y_0|^\alpha$$

for any $x_0, y_0 \in \mathbb{T}^d$. This completes the proof. ∎

Lemma 3.3. *allows us to conclude that $S_2(\ell)$ is Hölder continuous in ℓ with exponent $2s$ and*

$$||u||_{L^2(0,T;L^2(\mathbb{T}^d))} < C_0, \quad ||\delta_\ell u||_{L^2(0,T;L^2(\mathbb{T}^d))} \le C_1 |\ell|^s \implies ||u||_{L^2(0,T;B_2^{s,\infty}(\mathbb{T}^d))} \le C_2.$$

Proof of theorem 1.2(a). By Jensen's inequality, we have

$$\left(\frac{1}{|\ell|^{2(1-\alpha)/3}} \int_0^T \left\langle S_2^{||}(\ell) \right\rangle_{\mathrm{ang}} \mathrm{d}t \right)^{3/2} \le \frac{1}{|\ell|^{1-\alpha}} \int_0^T \left\langle \int_{\mathbb{T}^d} |\delta_\ell u \cdot \hat{\ell}|^3 \, \mathrm{d}x \right\rangle_{\mathrm{ang}} \mathrm{d}t. \tag{3.16}$$

Now, for any $\epsilon > 0$, we can choose ℓ_ϵ sufficiently small such that for all $\ell \le \ell_\epsilon$, we have

$$\left(\frac{1}{|\ell|^{2(1-\alpha)/3}} \int_0^T \left\langle S_2^{||}(\ell) \right\rangle_{\text{ang}} \mathrm{d}t\right)^{3/2} \le \sup_{\ell \le \ell_\epsilon} \frac{1}{|\ell|^{1-\alpha}} \int_0^T \int_{\mathbb{T}^d} \left\langle |\delta_\ell u \cdot \hat{\ell}|^3 \right\rangle_{\text{ang}} \mathrm{d}x \, \mathrm{d}t$$

$$\le \limsup_{|\ell| \to 0} \frac{1}{|\ell|^{1-\alpha}} \int_0^T \int_{\mathbb{T}^d} \left\langle |\delta_\ell u \cdot \hat{\ell}|^3 \right\rangle_{\text{ang}} \mathrm{d}x \mathrm{d}t + \epsilon$$

$$\le -\int_0^T \int_{\mathbb{T}^d} D_{4/5}^*[u] \, \mathrm{d}x \mathrm{d}t + \epsilon, \tag{3.17}$$

which results from hypothesis 2.3(a). It follows from hypothesis 2.1 and proposition 2.2 that

$$\left(\frac{1}{|\ell|^{2(1-\alpha)/3}} \int_0^T \left\langle S_2^{||}(\ell) \right\rangle_{\text{ang}} \mathrm{d}t\right)^{3/2} \le \frac{12}{\mathrm{d}(d+2)} \int_0^T \int_{\mathbb{T}^d} D[u] \, \mathrm{d}x \mathrm{d}t + \epsilon.$$

Since $f(x) = x^{2/3}$, $x > 0$ is monotone increasing, we have for all ℓ sufficiently small that

$$\int_0^T \left\langle S_2^{||}(\ell) \right\rangle_{\text{ang}} \mathrm{d}t \le C_0 |\ell|^{2(1-\alpha)/3}, \tag{3.18}$$

where $C_0 := C_0(d, T, u_0, f)$ depends on magnitude of the inertial energy dissipation (correspondingly the anomalous viscous dissipation if the Euler solution is obtained as a vanishing viscosity limit). The claimed regularity follows by applying lemma 1.3. ∎

Proof of theorem 1.2(b). We employ the following definition of Besov spaces. Fix $\ell > 0$ and consider the 'truncated ball' $B_T := \{n : 1/2 < |n| < 1\}$. The truncated-ball mean is then

$$(\mathcal{V}_\ell f)(x) := \frac{1}{|B_T|} \int_{B_T} \delta f(\ell n; x) \, \mathrm{d}n. \tag{3.19}$$

The norm for the Besov space $B_p^{s,\infty}(\mathbb{T}^d)$ can then be defined as

$$||f||_{B_p^{s,\infty}} := ||f||_p + |f|_{B_p^{s,\infty}}, \quad |f|_{B_p^{s,\infty}} := \sup_{N \ge 0} 2^{sN} ||\mathcal{V}_{2^{-N}} f||_p. \tag{3.20}$$

See appendix C of [52] and §2.5.11–12 of [53]. Our aim is to obtain a non-trivial L^3-integrable upper bound for the Besov norm $||u(t)||_{B_3^{s,\infty}}$ under for some $s > 0$. By assumption $u \in L^3(0, T; L^3(\mathbb{T}^d))$, so we need only to find an integrable upper bound for the Besov semi-norm $|u(t)|_{B_3^{s,\infty}}^3$. First, by Jensen's inequality, the pth-power of ball-averages is bounded by the ball-average of the pth power:

$$||\mathcal{V}_\ell f||_p^p \le \frac{1}{|B_T|} \int_{B_T} ||\delta f(\ell n; \cdot)||_p^p \, \mathrm{d}n, \quad p \ge 1. \tag{3.21}$$

Thus, letting $\ell_N := 2^{-N}$, we have

$$|u(t)|_{B_3^{s,\infty}}^3 = (\sup_{N \ge 0} \ell_N^{-s} ||\mathcal{V}_{2^{-N}} f||_3)^3$$

$$\le \sup_{N \ge 0} \ell_N^{-3s} \frac{1}{|B_T|} \int_{B_T} ||\delta u(\ell_N n; \cdot, t)||_3^3 \, \mathrm{d}n$$

$$= \sup_{N \ge 0} \ell_N^{-3s} \frac{\ell_N^{-d}}{|B_T|} \int_{\ell_{N-1}}^{\ell_N} \mathrm{d}\rho \rho^{d-1} \int_{S^{d-1}} ||\delta u(\rho \hat{r}; \cdot, t)||_3^3 \, \mathrm{d}\omega(\hat{\ell}) \tag{3.22}$$

where $\rho = |r|$ and where we have used the upper bound (3.21). Fix any scale ℓ_0 and split into small and large

$$|u(t)|^3_{B^{s,\infty}_3} \le \sup_{|\ell| \le \ell_0} \ell^{-3s} \int_{\mathbb{T}^d} \left\langle |\delta_\ell u(x,t)|^3 \right\rangle_{\text{ang}} \, \mathrm{d}x + \sup_{|\ell| > \ell_0} \ell^{-3s} \int_{\mathbb{T}^d} \left\langle |\delta_\ell u(x,t)|^3 \right\rangle_{\text{ang}} \, \mathrm{d}x$$

$$\le \sup_{|\ell| \le \ell_0} \ell^{-3s} \int_{\mathbb{T}^d} \left\langle |\delta_\ell u(x,t)|^3 \right\rangle_{\text{ang}} \, \mathrm{d}x + 2\ell_0^{-3s} ||u||^3_{L^3}. \tag{3.23}$$

For $\beta \in (0,1)$, we have

$$\int_0^T |u(t)|^3_{B^{s,\infty}_3} \, \mathrm{d}t \le \sup_{|\ell| \le \ell_0} \ell^{(1-\beta)-3s} \left(\frac{1}{|\ell|^{1-\beta}} \int_0^T \int_{\mathbb{T}^d} \left\langle |\delta_\ell u(x,t)|^3 \right\rangle_{\text{ang}} \, \mathrm{d}x\mathrm{d}t \right) \tag{3.24}$$

$$+ 2\ell_0^{-3s} ||u||^3_{L^3(0,T;L^3(\mathbb{T}^d))}. \tag{3.25}$$

It follows that for any $s \le (1-\beta)/3$, we have for any $\epsilon > 0$ that there exists an ℓ_ϵ such that for all $\ell_0 \le \ell_\epsilon$,

$$\int_0^T |u(t)|^3_{B^{s,\infty}_3} \, \mathrm{d}t \le \ell_0^{(1-\beta)-3s} \sup_{|\ell| \le \ell_0} \frac{1}{|\ell|^{1-\beta}} \int_0^T \int_{\mathbb{T}^d} \left\langle |\delta_\ell u(x,t)|^3 \right\rangle_{\text{ang}} \, \mathrm{d}x\mathrm{d}t + 2\ell_0^{-3s} ||u||^3_{L^3(0,T;L^3(\mathbb{T}^d))}$$

$$\le \ell_0^{-3s} \left(\limsup_{|\ell| \to 0} \frac{1}{|\ell|^{1-\beta}} \int_0^T \int_{\mathbb{T}^d} \left\langle |\delta_\ell u(x,t)|^3 \right\rangle_{\text{ang}} \, \mathrm{d}x\mathrm{d}t + \epsilon + 2||u||^3_{L^3(0,T;L^3(\mathbb{T}^d))} \right)$$

$$\le \ell_0^{-3s} \left(-\int_0^T \int_{\mathbb{T}^d} D^*_{4/3}[u] \, \mathrm{d}x\mathrm{d}t + \epsilon + 2||u||^3_{L^3(0,T;L^3(\mathbb{T}^d))} \right)$$

$$= \ell_0^{-3s} \left(\frac{4}{d} \int_0^T \int_{\mathbb{T}^d} D[u] \, \mathrm{d}x\mathrm{d}t + \epsilon + 2||u||^3_{L^3(0,T;L^3(\mathbb{T}^d))} \right), \tag{3.26}$$

where we used the fact that the distributional limit exists by hypothesis 2.1 and employed our main hypothesis 2.3(b) in passing to the second to last line. We remark that the bound on the Besov semi-norm depends on magnitude of the inertial (or anomalous) dissipation. ∎

References

1. Leray J. 1934 Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Math.* **63**, 193–248. (doi:10.1007/BF02547354)

2. Duchon J, Robert R. 2000 Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations. *Nonlinearity* **13**, 249–255. (doi:10.1088/0951-7715/13/1/312)

3. Borue V, Orszag SA. 1995 Self-similar decay of three-dimensional homogeneous turbulence with hyperviscosity. *Phys. Rev. E* **51**, R856–R859. (doi:10.1103/PhysRevE.51.R856)

4. Kaneda Y, Ishihara T, Yokokawa M, Itakura K, Uno A. 2003 Energy dissipation rate and energy spectrum in high resolution direct numerical simulations of turbulence in a periodic box. *Phys. Fluids* **15**, L21–L24. (doi:10.1063/1.1539855)

5. Pearson BR, Krogstad PA, van de Water W. 2002 Measurements of the turbulent energy dissipation rate. *Phys. Fluids* **14**, 1288–1290. (doi:10.1063/1.1445422)
6. Sreenivasan KR. 1984 On the scaling of the turbulence energy dissipation rate. *Phys. Fluids* **27**, 1048–1051. (doi:10.1063/1.864731)
7. Sreenivasan KR. 1998 An update on the energy dissipation rate in isotropic turbulence. *Phys. Fluids* **10**, 528–529. (doi:10.1063/1.869575)
8. Touil H, Bertoglio J-P, Shao L. 2002 The decay of turbulence in a bounded domain. *J. Turb.* **3**, N49. (doi:10.1088/1468-5248/3/1/049)
9. Frisch U. 1995 *Turbulence: the legacy of AN Kolmogorov*. Cambridge, UK: Cambridge University Press.
10. Buckmaster T, Drivas TD, Shkoller S, Vicol V. 2021 Simultaneous development of shocks and cusps for 2D Euler with azimuthal symmetry from smooth data. (http://arxiv.org/abs/2106.02143).
11. Drivas TD, Eyink GL. 2018 An Onsager singularity theorem for turbulent solutions of compressible Euler equations. *Comm. Math. Phys.* **359**, 733–763. (doi:10.1007/s00220-017-3078-4)
12. Khanin E, Khanin K, Mazel A, Sinai Y. 2000 Invariant measures for Burgers equation with stochastic forcing. *Ann. Math.* **151**, 817–960. (doi:10.2307/121049)
13. Eyink GL, Drivas TD. 2015 Spontaneous stochasticity and anomalous dissipation for Burgers equation. *J. Stat. Phys.* **158**, 386–432. (doi:10.1007/s10955-014-1135-3)
14. Cheskidov A, Friedlander S, Friedlander N. 2007 Inviscid dyadic model of turbulence: the fixed point and Onsager's conjecture. *J. Math. Phys.* **48**, 065503. (doi:10.1063/1.2395917)
15. Friedlander S, Glatt-Holtz N, Vicol V. 2016 Inviscid limits for a stochastically forced shell model of turbulent flow. *Ann. Inst. H. Poincaré Probab. Statist.* **52**, 1217–1247. (doi:10.1214/14-AIHP663)
16. Mattingly JC, Suidan T, Vanden-Eijnden E. 2007 Simple systems with anomalous dissipation and energy cascade. *Commun. Math. Phys.* **276**, 189–220. (doi:10.1007/s00220-007-0333-0)
17. Mailybaev AA. 2016 Spontaneously stochastic solutions in one-dimensional inviscid systems. *Nonlinearity* **29**, 2238. (doi:10.1088/0951-7715/29/8/2238)
18. Bedrossian J, Blumenthal A, Punshon-Smith S. 2021 The Batchelor spectrum of passive scalar turbulence in stochastic fluid mechanics. *Commun. Pure Appl. Math.* (https://arxiv.org/pdf/1911.11014.pdf)
19. Bernard D, Gawedzki K, Kupiainen A. 1998 Slow modes in passive advection. *J. Statist. Phys.* **90**, 519–569. (doi:10.1023/A:1023212600779)
20. Drivas TD, Elgindi TM, Iyer G, Jeong IJ. 2019 Anomalous dissipation in passive scalar transport. (http://arxiv.org/abs/1911.03271).
21. Le Jan Y, Raimond O. 2002 Integration of Brownian vector fields. *Ann. Probab.* **30**, 826–873. (doi:10.1214/aop/1023481009)
22. Kolmogorov AN. 1941 Dissipation of energy in locally isotropic turbulence. *Dokl. Akad. Nauk SSSR.* **32**, 16–18.
23. Iyer KP, Sreenivasan KR, Yeung PK. 2020 Scaling exponents saturate in three-dimensional isotropic turbulence. *Phys. Rev. Fluids* **5**, 054605. (doi:10.1103/PhysRevFluids.5.054605)
24. Onsager L. 1949 Statistical hydrodynamics. *Il Nuovo Cimento (Supplemento)* **6**, 279–287. (doi:10.1007/BF02780991)
25. Cheskidov A, Constantin P, Friedlander S, Shvydkoy R. 2008 Energy conservation and Onsager's conjecture for the Euler equations. *Nonlinearity* **21**, 1233. (doi:10.1088/0951-7715/21/6/005)
26. Constantin P, Weinan E, Titi E. 1994 Onsager's conjecture on the energy conservation for solutions of Euler's equation. *Comm. Math. Phys.* **165**, 207–209. (doi:10.1007/BF02099744)
27. Eyink GL. 1994 Energy dissipation without viscosity in ideal hydrodynamics I. Fourier analysis and local energy transfer. *Physica D* **78**, 222–240. (doi:10.1016/0167-2789(94)90117-1)
28. Drivas TD, Eyink GL. 2019 An Onsager singularity theorem for Leray solutions of incompressible Navier–Stokes. *Nonlinearity* **32**, 4465. (doi:10.1088/1361-6544/ab2f42)
29. Isett P. 2018 A proof of Onsager's conjecture. *Ann. Math.* **188**, 871–963. (doi:10.4007/annals.2018.188.3.4)
30. De Lellis C, Székelyhidi Jr L. 2012 The *h*-principle and the equations of fluid dynamics, B. *Am. Math. Soc.* **49**, 347–375. (doi:10.1090/S0273-0979-2012-01376-9)

31. Scheffer V. 1993 An inviscid flow with compact support in space-time. *J. Geom. Anal.* **3**, 343–401. (doi:10.1007/BF02921318)
32. Shnirelman A. 1997 On the nonuniqueness of weak solution of the Euler equation. *Comm. Pure Appl. Math.* **50**, 1261–1286. (doi:10.1002/(SICI)1097-0312(199712)50:12<1261::AID-CPA3>3.0.CO;2-6)
33. Buckmaster T, De Lellis C, Székelyhidi Jr L, Vicol V. 2019 Onsager's conjecture for admissible weak solutions. *Comm. Pure Appl. Math.* **72**, 227–448. (doi:10.1002/cpa.21768)
34. Buckmaster T, Vicol V. 2020 Convex integration and phenomenologies in turbulence. *EMS Surv. Math. Sci.* **6**, 173–263. (doi:10.4171/EMSS/34)
35. Anselmet F, Gagne Y, Hopfinger EJ, Antonia RA. 1984 High-order velocity structure functions in turbulent shear flows. *J. Fluid Mech.* **140**, 63–89. (doi:10.1017/S0022112084000513)
36. Benzi R, Ciliberto S, Tripiccione R, Baudet C, Massaioli F, Succi S. 1993 Extended self-similarity in turbulent flows. *Phys. Rev. E* **48**, R29. (doi:10.1103/PhysRevE.48.R29)
37. Stolovitzky G, Sreenivasan KR, Juneja A. 1993 Scaling functions and scaling exponents in turbulence. *Phys. Rev. E* **48**, R3217. (doi:10.1103/PhysRevE.48.R3217)
38. Stolovitzky G, Sreenivasan KR. 1993 Scaling of structure functions. *Phys. Rev. E* **48**, R33. (doi:10.1103/PhysRevE.48.R33)
39. Sreenivasan KR, Vainshtein SI, Bhiladvala R, San Gil I, Chen S, Cao N. 1996 Asymmetry of velocity increments in fully developed turbulence and the scaling of low-order moments. *Phys. Rev. Lett.* **77**, 1488. (doi:10.1103/PhysRevLett.77.1488)
40. Vainshtein SI, Sreenivasan KR. 1994 Kolmogorov's 4/5th Law and Intermittency in Turbulence. *Phys. Rev. Lett.* **73**, 3085. (doi:10.1103/PhysRevLett.73.3085)
41. Isett P. 2017 On the endpoint regularity in Onsager's conjecture. (http://arxiv.org/abs/1706.01549).
42. Meneveau C, Sreenivasan KR. 1991 The multifractal nature of turbulent energy dissipation. *J. Fluid Mech.* **224**, 429–484. (doi:10.1017/S0022112091001830)
43. Shvydkoy R. 2009 On the energy of inviscid singular flows. *J. Math. Anal. Appl.* **349**, 583–595. (doi:10.1016/j.jmaa.2008.09.007)
44. Bedrossian J, Zelati MC, Punshon-Smith S, Weber F. 2019 A sufficient condition for the Kolmogorov 4/5 law for stationary martingale solutions to the 3D Navier–Stokes equations. *Commun. Math. Phys.* **367**, 1045–1075. (doi:10.1007/s00220-019-03396-6)
45. Eyink GL. 2002 Local 4/5-law and energy dissipation anomaly in turbulence. *Nonlinearity* **16**, 137. (doi:10.1088/0951-7715/16/1/309)
46. Drivas TD, Nguyen HQ. 2018 Remarks on the emergence of weak Euler solutions in the vanishing viscosity limit. *J. Nonlinear Sci.* **2**, 1–13.
47. Drivas TD. 2019 Turbulent cascade direction and Lagrangian time-asymmetry. *J. Nonlinear Sci.* **29**, 65–88. (doi:10.1007/s00332-018-9476-8)
48. Golse F, Perthame B. 2013 Optimal regularizing effect for scalar conservation laws. *Revista Matemática Iberoamericana* **29**, 1477–1504. (doi:10.4171/RMI/765)
49. Jabin P-E. 2009 Some regularizing methods for transport equations and the regularity of solutions to scalar conservation laws. *Séminaire Équations aux dérivées partielles (Polytechnique)* **2008**, 1–15.
50. Tadmor E, Tao T. 2007 Velocity averaging, kinetic formulations, and regularizing effects in quasi–linear PDEs. *Commun. Pure Appl. Math.* **60**, 1488–1521. (doi:10.1002/cpa.20180)
51. Cheskidov A, Zaya K. 2016 Regularizing effect of the forward energy cascade in the inviscid dyadic model. *Proc. Amer. Math. Soc.* **144**, 73–85. (doi:10.1090/proc/12494)
52. Eyink GL. 1996 Exact results on stationary turbulence in 2D: consequences of vorticity conservation. *Physica D* **91**, 97–142. (doi:10.1016/0167-2789(95)00250-2)
53. Treibel H. 1983 *Theory of function spaces*. Monographs in Mathematics, vol. 78. Basel, Switzerland: Birkhäiuser.