Advances in Mathematics 393 (2021) 108078

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Smooth loops and loop bundles

Sergey Grigorian

School of Mathematical & Statistical Sciences, University of Texas Rio Grande Valley, Edinburg, TX 78539, USA

A R T I C L E I N F O

Article history:
Received 19 September 2020
Received in revised form 1 September 2021
Accepted 3 October 2021
Available online xxxx
Communicated by Tony Pantev

Keywords:
Loops
Smooth loops
Holonomy
Gauge theory
Non-associative
Chern-Simons

A B S T R A C T

A loop is a rather general algebraic structure that has an identity element and division, but is not necessarily associative. Smooth loops are a direct generalization of Lie groups. A key example of a non-Lie smooth loop is the loop of unit octonions. In this paper, we study properties of smooth loops and their associated tangent algebras, including a loop analog of the Maurer-Cartan equation. Then, given a manifold, we introduce a loop bundle as an associated bundle to a particular principal bundle. Given a connection on the principal bundle, we define the torsion of a loop bundle structure and show how it relates to the curvature, and also develop aspects of a non-associative gauge theory. Throughout, we see how some of the known properties of G_2-structures can be seen from this more general setting.

Contents

E-mail address: sergey.grigorian@utrgv.edu.

https://doi.org/10.1016/j.aim.2021.108078

1. Introduction

A major direction in differential geometry is the study of Riemannian manifolds with exceptional holonomy, i.e. 7-dimensional G_2-manifolds and 8-dimensional Spin (7)-manifolds, as well as more generally, G_2-structures and Spin (7)-structures. As it turns out, both of these structure groups are closely related to the octonions [20], which is the 8-dimensional nonassociative normed division algebra $\mathbb{O}$ over $\mathbb{R}$. A number of properties of G_2-structures and Spin (7)-structures are hence artifacts of the octonionic origin of these groups. In particular, in [15], the author has explicitly used an octonion formalism to investigate properties of isometric G_2-structures. In that setting, it emerged that objects such as the torsion of a G_2-structure are naturally expressed in terms of sections of a unit octonion bundle. The set of unit octonions $U\mathbb{O} \cong S^7$, has the algebraic structure of a *Moufang loop*. Indeed, a closer look at the properties of octonions that were used in [15] shows that it was not really the algebra structure of $\mathbb{O}$ that played the key role, but rather the loop structure on $U\mathbb{O}$ and the corresponding cross-product structure on the tangent space at the identity $T_1U\mathbb{O} \cong \operatorname{Im}\mathbb{O}$, the pure imaginary octonions. This suggests that there is room for generalization by considering bundles of other smooth loops. As far as possible, we will minimize assumptions made on the loops. Although smooth loops at first sight may seem like an exotic structure, in fact, there is a large supply of smooth loops, because given a Lie group G, a Lie subgroup H, and a smooth section $\sigma : G/H \longrightarrow G$ (i.e. a smooth collection of coset representatives), we may define a loop structure on G/H if σ satisfies certain conditions, such as $\sigma(H) = 1$, and for any cosets xH and yH, there exists a unique element $z \in \sigma(G/H)$ such that $zxH = yH$ [35]. A classical example of a smooth loop obtained directly from a Lie group quotient is the space of positive definite hermitian matrices [24]. Conversely, any smooth loop can also be described in terms of a section of a quotient of Lie groups. Special kinds of smooth loops, such as Moufang loops have been classified [35], however for broader classes, such as Bol loops, there exists only a partial classification [11].

The main purpose of this paper is develop substantial generalizations of Lie theory, principal bundles, and gauge theory to the non-associative setting. In the process, we carefully build up the theory of loop bundles starting with all the necessary algebraic preliminaries and properties of smooth loops. There are several anticipated applications of this theory. Firstly, this will help define a unified framework through which special geometric structures may be studied. In this sense, this can be considered as an extension of the normed division algebra approach to various special structures in Riemannian geometry as developed by Leung [29]. The long-term goal in G_2-geometry and Spin (7)-geometry is to obtain an analogue of Yau's celebrated theorem on existence of Calabi-Yau metrics [52], and thus a key theme in the study of such special geometries is to try to compare and contrast the corresponding theory of Kähler and Calabi-Yau manifolds. This requires putting the complex and octonionic geometries into the same framework. In [15], the octonion bundle is constructed out of the tangent bundle, and is hence very specific, one could say canonical. However to understand properties of the bundle, it is helpful to decouple the bundle structure and the properties of the base manifold. This leads directly to consider loop bundles over arbitrary manifolds. In particular, such an approach will also clarify which properties of the octonion bundle in the G_2 setting are generic, in the sense that they hold true for any loop bundle, and which are specific to G_2-structures. This leads directly to the second expected application, namely using the non-associative version of Chern-Simons theory to study connections with special properties on bundles. Indeed, nonassociativity allows to define new nontrivial functionals in different dimensions with nontrivial critical points, and this should give rise to a new invariant theory, in the spirit of Floer [12] in 3-dimensions. Finally, it is also expected that the ideas developed in this paper will find applications in physics. It is already known that octonions play a role in supersymmetric theories such as String Theory and M-theory (for example, [2,4,16]), so a better developed non-associative theory will help advance in these directions.

In Section 2 we give an overview of the key algebraic properties of loops. While many basic properties of loops may be known to algebraists, they may be new to geometers. Moreover, we adopt a point of view where we emphasize the pseudoautomorphism group of a loop, which is a generalization of the automorphism group, and properties of modified products defined on loops. These are the key objects that are required to define loop bundles, even though in algebraic literature they typically take the backstage. In particular, we show how the pseudoautomorphism group, the automorphism group, the nucleus of a loop are related and how these relationships manifest themselves in the octonion case as well-known relationships between the groups Spin (7), $SO(7)$, and G_2.

In Section 3, we then restrict attention to smooth loops, which are the not necessarily associative analogs of Lie groups. We also make the assumption that the pseudoautomorphism group acts on the smooth loop via diffeomorphisms and is hence itself a Lie group. This is an important assumption and it is not known whether this is always true. The key example of a non-associative smooth loop is precisely the loop of unit octonions. We first define the concept of an exponential function, which is similar to that on Lie groups. This is certainly not a new concept - it first defined by Malcev in 1955 [32], but here we show that in fact, generally, there may be different exponential maps, based on the initial conditions of the flow equation. This then relates to the concept of the modified product as defined in Section 2. Then, in Section 3.2, we define an algebra structure on tangent spaces of the loop. The key difference with Lie algebras is that in the non-associative case, there is a bracket defined at each point of the loop.

In Section 3.3, Theorem 3.59 gives us a loop version of the Maurer-Cartan structural equation. Namely, for any point p in the loop, the right Maurer-Cartan form satisfies the following equation:

$$(d\theta)_p - \frac{1}{2}[\theta,\theta]^{(p)} = 0, \tag{1.1}$$

where $[\cdot,\cdot]^{(p)}$ is the bracket at point p. To the best of the author's knowledge, this is a new result. Further, we show how the differential of the bracket depends on the associator, which of course vanishes on Lie algebras, but is non-trivial on tangent algebras of non-associative loops. Differentiating the structural equation then gives the Akivis identity [21], which is a non-associative generalization of the Jacobi identity. Indeed, in Lie theory, the Jacobi identity is the integrability condition for the Maurer-Cartan equation, however in the non-associative case, derivatives of the bracket give rise to the additional terms.

Then, we define another key component in the theory of smooth loops. As discussed above, each element s of the loop $\mathbb{L}$ defines a bracket b_s on the tangent algebra $\mathfrak{l}$. Moreover, we also define a map φ_s that maps the Lie algebra $\mathfrak{p}$ of the pseudoautomorphism group to the loop tangent algebra. The kernel of this map is precisely the Lie algebra $\mathfrak{h}_s$ of the stabilizer of s in the pseudoautomorphism group. In the case of unit octonions, we know $\mathfrak{p} \cong \mathfrak{so}(7) \cong \Lambda^2(\mathbb{R}^7)^*$ and $\mathfrak{l} = \operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$, so φ_s can be regarded as an element of $\mathbb{R}^7 \otimes \Lambda^2\mathbb{R}^7$, and this is (up to a constant factor) a dualized version of the G_2-invariant 3-form φ, as used to project from $\Lambda^2(\mathbb{R}^7)^*$ to $\mathbb{R}^7$. The kernel of this map is then the Lie algebra $\mathfrak{g}_2$. The 3-form φ also defines the bracket on $\operatorname{Im}\mathbb{O}$, so in this case, both b_s and φ_s are determined by the same object, but in general they have different roles. By considering the action of $U(n)$ on $U(1)$ (i.e. the unit complex numbers) and $Sp(n)\,Sp(1)$ on $Sp(1)$ (i.e. the unit quaternions), we find that Hermitian and hyperHermitian structures fit into the same framework. Namely, a complex Hermitian form, a quaternionic triple of Hermitian forms, and the G_2-invariant 3-form have the same origin as 2-forms with values in imaginary complex numbers, quaternions, and octonions, respectively.

In Section 3.4 we define an analog of the Killing form on $\mathfrak{l}$ and give conditions for it to be invariant under both the action of $\mathfrak{p}$ and the bracket on $\mathfrak{l}$. In particular, using the Killing form, we define the adjoint φ_s^t of φ_s. This allows one to use the Lie bracket on $\mathfrak{p}$ to define another bracket on $\mathfrak{l}$. In the case of octonions, it's proportional to the standard bracket on $\mathfrak{l}$, but in general it could be a distinct object.

In Section 3.5, we consider maps from some smooth manifold M to a smooth loop. Given a fixed map s, we can then define the corresponding products of loop-valued maps and correspondingly a bracket of $\mathfrak{l}$-valued maps. Similarly as for maps to Lie groups, we define the Darboux derivative [45] of s - this is just $s^*\theta$ - the pullback of the Maurer-Cartan form on $\mathbb{L}$. This now satisfies a structural equation, which is just the pullback of the loop Maurer-Cartan equation, as derived in Section 3.3, with respect to the bracket defined by s. For maps to Lie groups, there holds a non-abelian "Fundamental Theorem of Calculus" [45, Theorem 7.14], namely that if a Lie algebra-valued 1-form on M satisfies the structural equation, then it is the Darboux derivative of some Lie group-valued function. Here, we prove an analog for $\mathfrak{l}$-valued 1-forms (Theorem 3.59). However, since in the non-associative case, the bracket in the structural equation depends on s, Theorem 3.59 requires that such a map already exists and some additional conditions are also needed, so as expected, it's not as powerful as for Lie groups. However, in the case the loop is associative, it does reduce to the theorem for Lie groups.

Further, in Section 4, we turn our attention to loop bundles over a smooth manifold M. In fact, since it's not a single bundle, it's best to refer to a *loop structure* over a manifold. The key component is Ψ-principal bundle $\mathcal{P}$ where Ψ is a group that acts via pseudoautomorphisms on the loop $\mathbb{L}$. Then, several bundles associated to $\mathcal{P}$ are defined: two bundles $\mathcal{Q}$ and $\mathring{\mathcal{Q}}$ with fibers diffeomorphic to $\mathbb{L}$, but with the bundle structure with respect to different actions of Ψ; the vector bundle $\mathcal{A}$ with fibers isomorphic to $\mathfrak{l}$, as well as some others. Crucially, a section s of the bundle $\mathring{\mathcal{Q}}$ then defines a fiberwise product structure on sections of $\mathcal{Q}$, a fiberwise bracket structure, and a map φ_s from sections of the adjoint bundle $\mathfrak{p}_{\mathcal{P}}$ to sections of $\mathcal{A}$. In the key example of a G_2-structure on a 7-manifold M, the bundle $\mathcal{P}$ is then the $\mathrm{Spin}\,(7)$-bundle that is the lifting of the orthonormal frame bundle. The bundles $\mathcal{Q}$ and $\mathring{\mathcal{Q}}$ are unit octonion bundles, similarly as defined in [15], but $\mathcal{Q}$ transforms under $SO\,(7)$, and hence corresponds to the unit subbundle of $\mathbb{R}\oplus TM$, while $\mathring{\mathcal{Q}}$ transforms under $\mathrm{Spin}\,(7)$, and hence corresponds to the unit subbundle of the spinor bundle. The section s then defines a global unit spinor, and hence defines a reduction of the $\mathrm{Spin}\,(7)$-structure group to G_2, and thus defines a G_2-structure. In the complex and quaternionic examples, the corresponding bundle $\mathcal{P}$ then has $U\,(n)$ and $Sp\,(n)\,Sp\,(1)$ structure group, respectively, and the section s defines a reduction to $SU\,(n)$ and $Sp\,(n)$, respectively. Thus, as noted in [29], indeed the octonionic analog of a reduction from Kähler structure to Calabi-Yau structure and from quaternionic Kähler to HyperKähler, is the reduction from $\mathrm{Spin}\,(7)$ to G_2.

Using the equivalence between sections of bundles associated to $\mathcal{P}$ and corresponding equivariant maps, we generally work with equivariant maps. Indeed, in that case, $s : \mathcal{P} \longrightarrow \mathbb{L}$ is an equivariant map, and given a connection ω on $\mathcal{P}$, we find that the Darboux derivative of s decomposes as

$$s^*\theta = T^{(s,\omega)} - \hat{\omega}^{(s)}, \tag{1.2}$$

where $\hat{\omega}^{(s)} = \varphi_s\,(\omega)$ and $T^{(s,\omega)}$ is the *torsion of s with respect to the connection* ω, which is defined as the horizontal part of $s^*\theta$. The quantity $T^{(s,\omega)}$ is called the torsion because in the case of G_2-structures on a 7-manifold, if we take $\mathcal{P}$ to be the spin bundle and ω the Levi-Civita connection for a fixed metric, then $T^{(s,\omega)}$ is precisely (up to the chosen sign convention) the torsion of the G_2-structure defined by the section s. Moreover, vanishing of $T^{(s,\omega)}$ implies a reduction of the holonomy group of ω. As shown in [15], the torsion of a G_2-structure may be considered as a 1-form with values in the bundle of imaginary octonions. Indeed, in general, $T^{(s,\omega)}$ is a basic (i.e. horizontal and equivariant) $\mathfrak{l}$-valued 1-form on $\mathcal{P}$, so it corresponds to an $\mathcal{A}$-valued 1-form on M. It also enters expressions for covariant derivatives of products of sections of $\mathcal{Q}$ and the bracket on $\mathcal{A}$.

The relation (1.2) is significant because it shows that the torsion vanishes if, and only if, $-\hat{\omega}^{(s)}$ is equal to the $\mathfrak{l}$-valued Darboux derivative $s^*\theta$. In particular, a necessary condition is then that $-\hat{\omega}^{(s)}$ satisfies the loop structural equation. In Theorem 4.25, we give a partial converse under certain assumptions on $\mathbb{L}$.

In Section 4.2, we then also consider the projection of the curvature F of ω to $\mathfrak{l}$. We define $\hat{F} = \varphi_s(F)$, which is then equal to the horizontal part of $d\hat{\omega}$, and show in Theorem 4.19 that $\hat{F}$ and T are related via a structural equation:

$$\hat{F} = d^{\mathcal{H}}T - \frac{1}{2}[T,T]^{(s)}, \tag{1.3}$$

where $[\cdot,\cdot]^{(s)}$ is the bracket defined by s. Again, such a relationship is recognizable from G_2-geometry, where the projection $\pi_7 \operatorname{Riem}$ of the Riemann curvature to the 7-dimensional representation of G_2 satisfies the "G_2 Bianchi identity" [15,23]. We also consider gauge transformations. In this setting, we have two quantities - the connection and the section s. We show that under a simultaneous gauge transformation of the pair (s,ω), $\hat{F}$ and T transform equivariantly.

Finally, in Section 5, we establish a non-associative generalization of some aspects of gauge theory. In Section 5.1 we consider the loop bundle structure over a compact 3-dimensional manifold and on it, define a loop Chern-Simons functional. In Theorem 5.4 we show that the critical points over the space of connections, but with a fixed section s, are connections for which $\hat{F} = 0$, i.e. the curvature lies in $\mathfrak{h}_s$ everywhere. So unlike the flat connections which are critical points of the standard Chern-Simons functional, here the condition is less restrictive, and the Lie algebra part of the curvature is required to lie in a particular subalgebra. Similarly, we define a generalized loop Chern-Simons functional on a compact n-dimensional manifold, in the presence of closed $(n-3)$-form ψ. This the analogue of the higher-dimensional Chern-Simons functional as defined in [9,40]. In this case, Theorem 5.6 shows that the critical points satisfy $\hat{F} \wedge \psi = 0$.

An additional feature of gauge theory in the non-associative setting is that apart from the choice of connection, we also have a choice of the defining section s. Hence, the loop Chern-Simons functional may be considered as functionals on pairs (s,ω). Indeed, if we consider the critical points over pairs (s,ω), then in Corollary 5.10 we get an additional condition on the torsion, namely that $[T,T,T]^{(s)} = 0$, where $[\cdot,\cdot,\cdot]^{(s)}$ is the associator defined by s and wedge products of 1-forms are implied.

In Section 5.2 we define a loop analog of the Yang-Mills functional on a compact Riemannian manifold. This is essentially the L^2-norm of $\hat{F}$. The corresponding Yang-Mills equations, as derived in Theorem 5.12, involve the torsion, as it would be expected.

In particular, in 4 dimensions, we find that self-dual and anti-self-dual connections satisfy the new Yang-Mills equation if and only if they satisfy an additional property - namely that the torsion is invariant under the action of the $\mathfrak{h}_s$-component of the curvature.

Another functional that we consider (in Section 5.3) is the L^2-norm squared of the torsion $\int_M |T|^2$. In this case, we fix the connection, and consider critical points over the space of sections s, or equivalently, equivariant loop-valued maps from $\mathcal{P}$. In the G_2 setting, similar functionals have been considered in [5,10,15,17,19,31]. This is then closely related to the Dirichlet energy functional, but restricted to equivariant maps. The critical points then are maps s, for which the torsion is divergence-free.

Acknowledgments

This work was supported by the National Science Foundation [DMS-1811754]. The author also thanks Henrique Sá Earp, Jonathan D.H. Smith, and the anonymous referee for helpful suggestions.

2. Loops

2.1. Definitions

The main object of study in this paper is a *loop*. Roughly, this can be thought of as a non-associative analog of a group, but with a few caveats. According to [37], this term was coined by the group of Abraham Albert in Chicago in 1940's, as rhyming with *group* and also referring to the Chicago Loop. Unfortunately however, for non-algebraists, and especially in geometry and topology, this term may cause confusion. A less ambiguous term would be something like a *unital quasigroup* or *quasigroup with identity*, however this would be nonstandard terminology and also much longer than a loop. In general, non-associative algebra requires a large number of definitions and concepts that become unnecessary in the more standard associative setting. In this section we go over some of the terminology and notation that we will be using. The reader can also refer to [21,24,35,41,48] for the various concepts, although, as far as the author knows, much of the notation in this setting is not standardized.

Definition 2.1. A *quasigroup* $\mathbb{L}$ is a set together with the following operations $\mathbb{L}\times\mathbb{L} \longrightarrow \mathbb{L}$

1. Product $(p,q)\mapsto pq$
2. Right quotient $(p,q)\mapsto p/q$
3. Left quotient $(p,q)\mapsto q\backslash p$,

that satisfy the following properties

1. $(p/q)\,q = p$
2. $q\,(q\backslash p) = p$

3. $\frac{pq}{q} = p$
4. $p\backslash pq = q$.

We will interchangeably denote the product operation by $p \cdot q$. To avoid multiple parentheses, at times we will use the convention $a \cdot bc = a\,(bc)$ and $ab/c = (ab)\,/c$. If the same underlying set $\mathbb{L}$ is equipped with a different product operation $\circ_r$ (to be defined later), then the corresponding quasigroup will be denoted by $(\mathbb{L}, \circ_r)$ and the corresponding quotient operation by $\backslash_r$.

Definition 2.2. Let $\mathbb{L}$ be a quasigroup. The *right nucleus* $\mathcal{N}^R(\mathbb{L})$ *of* $\mathbb{L}$ is the set of all $r \in \mathbb{L}$, such that for any $p, q \in \mathbb{L}$,

$$pq \cdot r = p \cdot qr. \tag{2.1}$$

Similarly, define the *left nucleus* $\mathcal{N}^L(\mathbb{L})$ and the *middle nucleus* $\mathcal{N}^M(\mathbb{L})$.

Elements of $\mathcal{N}^R(\mathbb{L})$ satisfy several other useful properties.

Lemma 2.3. *If $r \in \mathcal{N}^R(\mathbb{L})$, then for any $p, q \in \mathbb{L}$,*

1. $\frac{pr}{qr} = p/q$
2. $p \cdot q/r = \frac{pq}{r}$
3. $p\backslash qr = p\backslash q \cdot r$.

Lemma 2.4. *The first property follows from (2.1) using*

$$p/q \cdot qr = (p/q \cdot q)\,r.$$

The second property follows similarly using

$$p\,(q/r \cdot r) = (p \cdot q/r)\,r.$$

The third property follows using

$$(p \cdot p\backslash q)\,r = p\,(p\backslash q \cdot r)\,.$$

In group theory the only reasonable morphism between groups is a group homomorphism, however for quasigroups there is significantly more flexibility.

Definition 2.5. Suppose $\mathbb{L}_1, \mathbb{L}_2$ are quasigroups. Then a triple (α, β, γ) of maps from $\mathbb{L}_1$ to $\mathbb{L}_2$ is a *homotopy* from $\mathbb{L}_1$ to $\mathbb{L}_2$ if for any $p, q \in \mathbb{L}_1$,

$$\alpha(p)\,\beta(q) = \gamma(pq)\,. \tag{2.2}$$

If (α, α, α) is a homotopy, then α is a *quasigroup homomorphism*. If each of the maps α, β, γ is a bijection, then (α, β, γ) is an *isotopy*. An isotopy from a quasigroup to itself is an *autotopy*. The set of all autotopies of a quasigroup $\mathbb{L}$ is clearly a group under composition. If (α, α, α) is an autotopy, then α is an automorphism of $\mathbb{L}$, and the group of automorphisms is denoted by $\mathrm{Aut}(\mathbb{L})$.

We will only be concerned with quasigroups that have an identity element, i.e. loops.

Definition 2.6. A *loop* $\mathbb{L}$ is a quasigroup that has a unique identity element $1 \in \mathbb{L}$ such that for any $q \in \mathbb{L}$,

$$1 \cdot q = q \cdot 1 = q. \tag{2.3}$$

Definition 2.7. Let $\mathbb{L}$ be a loop. Then, for any $q \in \mathbb{L}$ define

1. The *right inverse* $q^{\rho} = q \backslash 1$.
2. The *left inverse* $q^{\lambda} = 1/q$.
   In particular, they satisfy

$$qq^{\rho} = q^{\lambda} q = 1. \tag{2.4}$$

For a general quasigroup, the nuclei may be empty, however if $\mathbb{L}$ is a loop, the identity element 1 associates with any other element, so the nuclei are non-empty. Moreover, it is easy to show that $\mathcal{N}^R(\mathbb{L})$ (and similarly, $\mathcal{N}^L(\mathbb{L})$ and $\mathcal{N}^M(\mathbb{L})$) is a group [24].

Loops may be endowed with additional properties that bestow various weaker forms of associativity and inverse properties.

1. *Two-sided inverse*: for any $p \in \mathbb{L}$, $p^{\rho} = p^{\lambda}$. Then we can define a unique two-sided inverse p^{-1}.
2. *Right inverse property*: for any $p, q \in \mathbb{L}$, $pq \cdot q^{\rho} = p$. In particular, this implies that the inverses are two-sided, so we can set $p^{-1} = p^{\rho} = p^{\lambda}$, and moreover $p/q = pq^{-1}$. The *left* inverse property is defined similarly. A loop with both the left and right inverse properties is said to be an *inverse loop*.
3. *Power-associativity* (or *monoassociativity*): any element $p \in \mathbb{L}$ generates a subgroup of $\mathbb{L}$. In particular, this implies that $\mathbb{L}$ has two-sided inverses. Power-associativity allows to unambiguously define integer powers p^n of elements. Note that some authors use monoassociativity as a more restrictive property, namely only that $pp \cdot p = p \cdot pp$.
4. *(Left)-alternative*: for any $p, q \in \mathbb{L}$, $p \cdot pq = pp \cdot q$. Similarly we can define the right-alternative property (i.e. $q \cdot pp = qp \cdot p$). In each of these cases, $\mathbb{L}$ has two-sided inverses. If $\mathbb{L}$ is both left-alternative and right-alternative, then it is said to be *alternative*. A loop with a similar property that $p \cdot qp = pq \cdot p$ is known as a *flexible loop*.

5. *Diassociative:* any two elements $p, q \in \mathbb{L}$ generate a subgroup of $\mathbb{L}$. Clearly, a diassociative loop has the inverse property, is power-associative, alternative, and flexible.
6. *(Left) Bol loop*: for any $p, q, r \in \mathbb{L}$,

$$p\left(q \cdot pr\right) = \left(p \cdot qp\right) r. \tag{2.5}$$

   It is easy to see that a left Bol loop has the left inverse property and is left-alternative and flexible [38]. It is also power-associative. Similarly, define a right Bol loop: for any $p, q, r \in \mathbb{L}$

$$\left(pq \cdot r\right) q = p\left(qr \cdot q\right). \tag{2.6}$$

7. *Moufang loop:* a loop is a Moufang loop if it satisfies both the left and right Bol identities. In particular, Moufang loops are diassociative.
8. *Group*: clearly any associative loop is a group.

Example 2.8. The best-known example of a non-associative loop is the Moufang loop of unit octonions.

Example 2.9. Suppose G is a group with a subgroup H. Suppose $\sigma : G/H \longrightarrow G$ is a section of G, regarded as a bundle over G/H. Then, let $\mathbb{L} = \sigma\left(G/H\right)$, known as a *transversal* to H in G. Suppose $\sigma\left(H\right) = 1$. Then, define a product structure on $\mathbb{L}$, given by

$$a \circ b = \sigma\left(abH\right). \tag{2.7}$$

Equivalently, we can define a product on cosets of G/H: $\left(aH\right) \circ \left(bH\right) = \sigma\left(aH\right) bH$. Consider the equation $a \circ x = b$. Since σ is a section, we can see right away that we have a unique solution $x = a^{-1} \circ b = \sigma\left(a^{-1}bH\right)$. Thus, $\left(\mathbb{L}, \circ\right)$ has *left* division, and is thus a *left loop* [24,35]. To define right division, and hence to obtain a full loop structure, more structure is needed. It is known that a left loop that satisfies the Bol condition (2.5) is in fact a Bol loop [24, 3.11].

Example 2.10. Consider the set $\mathcal{P}_n^+$ of $n \times n$ positive-definite hermitian matrices, then by the polarization, any $A \in GL\left(n, \mathbb{C}\right)$ can be written uniquely as $A = PU$ where $P \in \mathcal{P}_n^+$ and $U \in U\left(n\right)$, with $P = \left(AA^{\dagger}\right)^{\frac{1}{2}}$. We can then definite a product $\circ$ on $\mathcal{P}_n^+$ given by

$$A \circ B = \left(AB^2A\right)^{\frac{1}{2}} \tag{2.8}$$

for any $A, B \in \mathcal{P}_n^+$. Note that $AB^2A = \left(AB\right)\left(AB\right)^{\dagger}$, so the square root is well-defined. Clearly, the identity matrix is the identity element in $\left(\mathcal{P}_n^+, \circ\right)$. In fact, the map $A \mapsto \left(AA^{\dagger}\right)^{\frac{1}{2}}$ gives rise to a section σ: $GL\left(n, \mathbb{C}\right)/U\left(n\right) \longrightarrow GL\left(n, \mathbb{C}\right)$, with

$\mathcal{P}_n^+ = \sigma(GL(n,\mathbb{C})/U(n))$ and so by Example 2.9, this is a transversal of $U(n)$ in $GL(n,\mathbb{C})$ and thus $(\mathcal{P}_n^+, \circ)$ is a left loop. It is not difficult to check that it also admits right division, and is hence a loop. Moreover, it can be shown that $(\mathcal{P}_n^+, \circ)$ is a Bol loop [24, Theorem 9.1]. Similarly, one can construct loops on $GL(n,\mathbb{R})/O(n)$, or other quotients of general linear or special linear groups.

Example 2.11. Using similar ideas as above, it is possible to see that in special relativity, the set of boosts forms a loop transversal to the subgroup of spatial rotations in the Lorentz group. More specifically, in this case $\mathbb{L} \cong O(n,1)/(O(n) \times O(1))$, and the loop operation corresponds to relativistic addition of velocities [24].

2.2. *Pseudoautomorphisms*

Suppose now $\mathbb{L}$ is a loop and (α, β, γ) is an autotopy of $\mathbb{L}$. Let $B = \alpha(1)$, $A = \beta(1)$, $C = \gamma(1)$. It is clear that $BA = C$. Moreover, from (2.2) we see that

$$\begin{aligned} \alpha(p) &= \gamma(p)/A \\ \beta(p) &= B\backslash\gamma(p). \end{aligned}$$

We can rewrite (2.2) as

$$\alpha(p) \cdot {}_B\backslash^{\alpha(q)A} = \alpha(pq)A$$

If $B = 1$, then, we obtain a *right pseudoautomorphism* α *of* $\mathbb{L}$ *with companion* A, which we'll denote by the pair (α, A), and which satisfies

$$\alpha(p) \cdot \alpha(q)A = \alpha(pq)A. \tag{2.9}$$

We have the following useful relations for quotients:

$$\alpha(q\backslash p)A = {}_{\alpha(q)}\backslash^{\alpha(p)A} \tag{2.10a}$$

$$\alpha(p/q) \cdot \alpha(q)A = \alpha(p)A \tag{2.10b}$$

There are several equivalent ways of characterizing *right pseudoautomorphisms*.

Theorem 2.12. *Let* $\mathbb{L}$ *be a loop and suppose* $\alpha : \mathbb{L} \longrightarrow \mathbb{L}$. *Also, let* $A \in \mathbb{L}$ *and* $\gamma = R_A \circ \alpha$. *Then the following are equivalent:*

1. (α, A) *is a* right pseudoautomorphism of $\mathbb{L}$ with companion A.
2. (α, β, γ) *is an autotopy of* $\mathbb{L}$ *with* $\alpha(1) = 1$ *and* $\beta(1) = \gamma(1) = A$.
3. $\gamma(1) = A$ *and* γ *satisfies*

$$\gamma(p)\gamma\left(q\gamma^{-1}(1)\right) = \gamma(pq). \tag{2.11}$$

Remark 2.13. Similarly, if $A = 1$, then we can rewrite (2.2) as

$$B\beta(p) \cdot \beta(q) = B\beta(pq)$$

and in this case, β is a *left pseudoautomorphism* with companion B. Finally, suppose $C = 1$, so that then $A = B^{\rho}$, and we can rewrite (2.2)

$$\gamma(p) / B^{\rho} \cdot B \backslash \gamma(q) = \gamma(pq)$$

so that in this case, γ is a *middle pseudoautomorphism* with companion B.

Example 2.14. In a Moufang loop, consider the map Ad_q, given by $p \longmapsto qpq^{-1}$. Note that this can be written unambiguously due to diassociativity. Then, this is a right pseudoautomorphism with companion q^3 [35, Lemma 1.2]. Indeed, using diassociativity for $\{q, xy\}$, we have

$$q(xy)q^{-1} \cdot q^3 = q(xy)q^2.$$

On the other hand,

$$\begin{aligned} qxq^{-1} \cdot qyq^2 =& q\left(xq^{-1}\right) \cdot (qyq)q \\ =& \left(q\left(xq^{-1} \cdot qyq\right)\right)q \\ =& \left(q(xy \cdot q)\right)q \\ =& q(xy)q^2, \end{aligned}$$

where we have use appropriate Moufang identities. Hence, indeed,

$$q(xy)q^{-1} \cdot q^3 = \left(qxq^{-1}\right)\left(qyq^{-1} \cdot q^3\right).$$

In general, the adjoint map on a loop is *not* a pseudoautomorphism or a loop homomorphism. For each $q \in \mathbb{L}$, Ad_q is just a bijection that preserves $1 \in \mathbb{L}$. However, as we see above, it is a pseudoautomorphism if the loop is Moufang. Keeping the same terminology as for groups, we'll say that Ad defines an adjoint action of $\mathbb{L}$ on itself, although for a non-associative loop, this is not an action in the usual sense of a group action.

We can easily see that the right pseudoautomorphisms of $\mathbb{L}$ form a group under composition. Denote this group by $\mathrm{PsAut}^R(\mathbb{L})$. Clearly, $\mathrm{Aut}(\mathbb{L}) \subset \mathrm{PsAut}^R(\mathbb{L})$. Similarly for left and middle pseudoautomorphisms. More precisely, $\alpha \in \mathrm{PsAut}^R(\mathbb{L})$ if there exists $A \in \mathbb{L}$ such that (2.9) holds. Here we are not fixing the companion. On the other hand, consider the set $\Psi^R(\mathbb{L})$ of all pairs (α, A) of *right pseudoautomorphisms with fixed companions*. This then also forms a group.

Lemma 2.15. *The set* $\Psi^R(\mathbb{L})$ *of all pairs* (α, A), *where* $\alpha \in \mathrm{PsAut}^R(\mathbb{L})$ *and* $A \in \mathbb{L}$ *is its companion, is a group with identity element* $(\mathrm{id}, 1)$ *and the following group operations:*

$$\textit{product: } (\alpha_1, A_1)(\alpha_2, A_2) = (\alpha_1 \circ \alpha_2, \alpha_1(A_2) A_1) \tag{2.12a}$$

$$\textit{inverse: } (\alpha, A)^{-1} = \left(\alpha^{-1}, \alpha^{-1}\left(A^\lambda\right)\right) = \left(\alpha^{-1}, \left(\alpha^{-1}(A)\right)^\rho\right). \tag{2.12b}$$

Proof. Indeed, it is easy to see that $\alpha_1(A_2) A_1$ is a companion of $\alpha_1 \circ \alpha_2$, that (2.12a) is associative, and that $(\mathrm{id}, 1)$ is the identity element with respect to it. Also, it is easy to see that

$$(\alpha, A)\left(\alpha^{-1}, \alpha^{-1}\left(A^\lambda\right)\right) = (\mathrm{id}, 1).$$

On the other hand, setting $B = \alpha^{-1}\left(A^\lambda\right)$, we have

$$\begin{aligned} B =& \alpha^{-1}(1) B = \alpha^{-1}\left(A^\lambda A\right) B \\ =& \alpha^{-1}\left(A^\lambda\right) \cdot \alpha^{-1}(A) B \\ =& B \cdot \alpha^{-1}(A) B. \end{aligned}$$

Canceling B on both sides on the left, we see that $B = \left(\alpha^{-1}(A)\right)^\rho$. □

Let $\mathcal{C}^R(\mathbb{L})$ be the set of elements of $\mathbb{L}$ that are a companion for a right pseudoautomorphism. Then, (2.12a) shows that there is a left action of $\Psi^R(\mathbb{L})$ on $\mathcal{C}^R(\mathbb{L})$ given by:

$$\Psi^R(\mathbb{L}) \times \mathcal{C}^R(\mathbb{L}) \longrightarrow \mathcal{C}^R(\mathbb{L}) \tag{2.13a}$$

$$((\alpha, A), B) \mapsto (\alpha, A) B = \alpha(B) A. \tag{2.13b}$$

This action is transitive, because if $A, B \in \mathcal{C}^R(\mathbb{L})$, then exist $\alpha, \beta \in \mathrm{PsAut}^R(\mathbb{L})$, such that $(\alpha, A), (\beta, B) \in \Psi^R(\mathbb{L})$, and hence $\left((\beta, B)(\alpha, A)^{-1}\right) A = B$. Similarly, $\Psi^R(\mathbb{L})$ also acts on all of $\mathbb{L}$. Let $h = (\alpha, A) \in \Psi^R(\mathbb{L})$, then for any $p \in \mathbb{L}$, $h(p) = \alpha(p) A$. This is in general non-transitive, but a faithful action (assuming $\mathbb{L}$ is non-trivial). Using this, the definition of (2.9) can be rewritten as

$$h(pq) = \alpha(p) h(q) \tag{2.14}$$

and hence the quotient relations (2.10) may be rewritten as

$$h(q \backslash p) = \alpha(q) \backslash h(p) \tag{2.15a}$$

$$\alpha(p/q) = h(p)/h(q). \tag{2.15b}$$

If $\Psi^R(\mathbb{L})$ acts transitively on $\mathbb{L}$, then $\mathcal{C}^R(\mathbb{L}) \cong \mathbb{L}$, since every element of $\mathbb{L}$ will be a companion for some right pseudoautomorphism. In that case, $\mathbb{L}$ is known as a *(right) G-loop*. Note that usually a loop is known as a G-loop if every element of $\mathbb{L}$ is a companion for a right pseudoautomorphism and for a left pseudoautomorphism [26]. However, in this paper we will only be concerned with right pseudoautomorphisms, so for brevity we will say $\mathbb{L}$ is a G-loop if $\Psi^R(\mathbb{L})$ acts transitively on it.

There is another action of $\Psi^R(\mathbb{L})$ on $\mathbb{L}$ - which is the action by the pseudoautomorphism. This is a non-faithful action of $\Psi^R(\mathbb{L})$, but corresponds to a faithful action of $\mathrm{PsAut}^R(\mathbb{L})$. Namely, let $h = (\alpha, A) \in \Psi^R(\mathbb{L})$, then h acts on $p \in \mathbb{L}$ by $p \mapsto \alpha(p)$. To distinguish these two actions, we make the following definitions.

Definition 2.16. A loop $\mathbb{L}$ admits two left actions of the group of right pseudoautomorphism pairs $\Psi^R(\mathbb{L})$.

1. The *full* action is given by $(h, p) \mapsto h(p) = \alpha(p) A$. The set $\mathbb{L}$ together with this action of $\Psi^R(\mathbb{L})$ will be denoted by $\mathring{\mathbb{L}}$.
2. The *partial* action, given by $(h, p) \mapsto h'(p) = \alpha(p)$. The set $\mathbb{L}$ together with this action of $\Psi^R(\mathbb{L})$ will be denoted by $\mathbb{L}$ again.

Remark 2.17. The relation (2.14) between these two actions suggests that the loop product on $\mathbb{L}$ can be regarded as a map $\cdot : \mathbb{L} \times \mathring{\mathbb{L}} \longrightarrow \mathring{\mathbb{L}}$. When $\Psi^R(\mathbb{L})$ acts via the full action on a product pq, the left factor p admits the partial action of $\Psi^R(\mathbb{L})$, while the right factor q admits the full action. The pairing between $\mathbb{L}$ and $\mathring{\mathbb{L}}$ is to some extent analogous to the Clifford action of a vector space V on the corresponding spinor space S, with $v \cdot s \in S$ for $v \in V$ and $s \in S$, where V and S admit different representation of the same group. The major difference is that V and S are vector spaces, and in general have different dimensions, whereas $\mathbb{L}$ and $\mathring{\mathbb{L}}$ have no linear structure, but are identical as sets.

Now let us consider several relationships between the different groups associated to $\mathbb{L}$. First of all define the following maps:

$$\begin{array}{cccc} \iota_1 : & \mathrm{Aut}(\mathbb{L}) & \hookrightarrow & \Psi^R(\mathbb{L}) \\ & \gamma & \mapsto & (\gamma, 1) \end{array} \tag{2.16}$$

and

$$\begin{array}{cccc} \iota_2 : & \mathcal{N}^R(\mathbb{L}) & \hookrightarrow & \Psi^R(\mathbb{L}) \\ & C & \mapsto & (\mathrm{id}, C) . \end{array} \tag{2.17}$$

The map ι_1 is clearly injective and is a group homomorphism, so $\iota_1(\mathrm{Aut}(\mathbb{L}))$ is a subgroup of $\Psi^R(\mathbb{L})$. On the other hand, if $A, B \in \mathcal{N}^R(\mathbb{L})$, then in $\Psi^R(\mathbb{L})$, $(\mathrm{id}, A)(\mathrm{id}, B) =$

(id, BA), so ι_2 is an antihomomorphism from $\mathcal{N}^R(\mathbb{L})$ to $\Psi^R(\mathbb{L})$ and thus a homomorphism from the opposite group $\mathcal{N}^R(\mathbb{L})^{\mathrm{op}}$. So, $\iota_2\left(\mathcal{N}^R(\mathbb{L})\right)$ is a subgroup of $\Psi^R(\mathbb{L})$ that is isomorphic to $\mathcal{N}^R(\mathbb{L})^{\mathrm{op}}$.

Using (2.16) let us define a right action of $\mathrm{Aut}(\mathbb{L})$ on $\Psi^R(\mathbb{L})$. Given $\gamma \in \mathrm{Aut}(\mathbb{L})$ and $(\alpha, A) \in \Psi^R(\mathbb{L})$, we define

$$(\alpha, A) \cdot \gamma = (\alpha, A)\,\iota_1(\gamma) = (\alpha \circ \gamma, A)\,. \tag{2.18}$$

Similarly, (2.17) induces a left action of $\mathcal{N}^R(\mathbb{L})^{\mathrm{op}}$, and hence a right action of $\mathcal{N}^R(\mathbb{L})$, on $\Psi^R(\mathbb{L})$:

$$C \cdot (\alpha, A) = \iota_2(C)\,(\alpha, A) = (\alpha, AC)\,. \tag{2.19}$$

The actions (2.18) and (2.19) commute, so we can combine them to define a left action of $\mathrm{Aut}(\mathbb{L}) \times \mathcal{N}^R(\mathbb{L})^{\mathrm{op}}$. Indeed, given $\gamma \in \mathrm{Aut}(\mathbb{L})$ and $C \in \mathcal{N}^R(\mathbb{L})$,

$$(\alpha, A) \cdot (\gamma, C) = \iota_2(C)\,(\alpha, A)\,\iota_1(\gamma) = (\alpha \circ \gamma, AC)\,. \tag{2.20}$$

Remark 2.18. Since any element of $\mathcal{N}^R(\mathbb{L})$ is a right companion for any automorphism, we can also define the semi-direct product subgroup $\iota_1(\mathrm{Aut}(\mathbb{L})) \ltimes \iota_2\left(\mathcal{N}^R(\mathbb{L})\right) \subset \Psi^R(\mathbb{L})$. Suppose $\beta, \gamma \in \mathrm{Aut}(\mathbb{L})$ and $B, C \in \mathcal{N}^R(\mathbb{L})$, then in this semi-direct product,

$$(\beta, B)(\gamma, C) = (\beta \circ \gamma, \beta(C)\,B)\,.$$

Lemma 2.19. *Given the actions of* $\mathrm{Aut}(\mathbb{L})$ *and* $\mathcal{N}^R(\mathbb{L})$ *on* $\Psi^R(\mathbb{L})$ *as in (2.18) and (2.19), respectively, we have the following properties.*

1. $\Psi^R(\mathbb{L}) / \mathrm{Aut}(\mathbb{L}) \cong \mathcal{C}^R(\mathbb{L})$ *as* $\Psi^R(\mathbb{L})$*-sets.*
2. *The image* $\iota_2\left(\mathcal{N}^R(\mathbb{L})\right)$ *is a normal subgroup of* $\Psi^R(\mathbb{L})$ *and hence*
$$\Psi^R(\mathbb{L}) / \mathcal{N}^R(\mathbb{L}) \cong \mathrm{PsAut}^R(\mathbb{L})\,.$$
3. *Moreover,*
$$\Psi^R(\mathbb{L}) / \mathrm{Aut}(\mathbb{L}) \times \mathcal{N}^R(\mathbb{L}) \cong \mathrm{PsAut}^R(\mathbb{L}) / \mathrm{Aut}(\mathbb{L}) \cong \mathcal{C}^R(\mathbb{L}) / \mathcal{N}^R(\mathbb{L})$$
where equivalence is as $\mathrm{Aut}(\mathbb{L}) \times \mathcal{N}^R(\mathbb{L})$*-sets.*

Proof. Suppose $\mathbb{L}$ is a loop.

1. Consider the projection on the second component $\mathrm{prj}_2 : \Psi^R(\mathbb{L}) \longrightarrow \mathcal{C}^R(\mathbb{L})$ under which $(\alpha, A) \mapsto A$. Both $\Psi^R(\mathbb{L})$ and $\mathcal{C}^R(\mathbb{L})$ are left $\Psi^R(\mathbb{L})$-sets, since both admit a

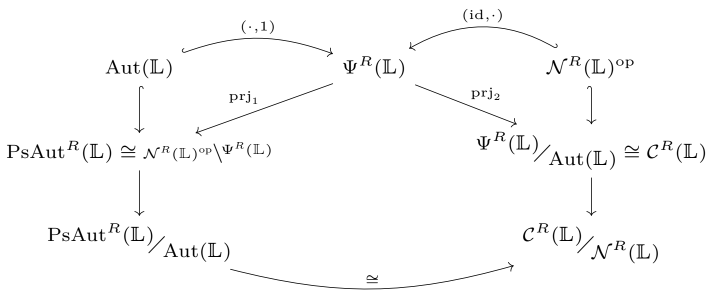

Fig. 1. Groups related to the loop $\mathbb{L}$.

left $\Psi^R(\mathbb{L})$ action - $\Psi^R(\mathbb{L})$ acts on itself by left multiplication and acts on $\mathcal{C}^R(\mathbb{L})$ via the action (2.13). Hence, prj_2 is a $\Psi^R(\mathbb{L})$-equivariant map (i.e. a G-set homomorphism). On the other hand, given the action (2.18) of $\mathrm{Aut}(\mathbb{L})$ on $\Psi^R(\mathbb{L})$, we easily see that two pseudoautomorphisms have the same companion if, and only if, they lie in the same orbit of $\mathrm{Aut}(\mathbb{L})$. Thus, prj_2 descends to a $\Psi^R(\mathbb{L})$-equivariant bijection $\Psi^R(\mathbb{L}) / \mathrm{Aut}(\mathbb{L}) \longrightarrow \mathcal{C}^R(\mathbb{L})$, so that $\Psi^R(\mathbb{L}) / \mathrm{Aut}(\mathbb{L}) \cong \mathcal{C}^R(\mathbb{L})$ as $\Psi^R(\mathbb{L})$-sets.

2. It is clear that $C \in \mathcal{C}^R(\mathbb{L})$ is a right companion of the identity map id if, and only if, $C \in \mathcal{N}^R(\mathbb{L})$. Now, let $\nu = (\mathrm{id}, C) \in \iota_2\left(\mathcal{N}^R(\mathbb{L})\right)$ and $g = (\alpha, A) \in \Psi^R(\mathbb{L})$. Then,

$$g\nu g^{-1} = (\alpha, A)(\mathrm{id}, C)\left(\alpha^{-1}, \alpha^{-1}\left(A^{\lambda}\right)\right) = \left(\mathrm{id}, A^{\lambda} \cdot \alpha(C) A\right). \tag{2.21}$$

In particular, this shows that $g\nu g^{-1} \in \iota_2\left(\mathcal{N}^R(\mathbb{L})\right)$ since $A^{\lambda} \cdot \alpha(C) A$ is the right companion of id. Thus indeed, $\iota_2\left(\mathcal{N}^R(\mathbb{L})\right)$ is a normal subgroup of $\Psi^R(\mathbb{L})$. Now consider the projection on the first component $\mathrm{prj}_1 : \Psi^R(\mathbb{L}) \longrightarrow \mathrm{PsAut}^R(\mathbb{L})$ under which $(\alpha, A) \mapsto \alpha$. This is clearly a group homomorphism with kernel $\iota_2\left(\mathcal{N}^R(\mathbb{L})\right)$. Thus, ${}^R(\mathbb{L})^{\mathrm{op}} \backslash \Psi^R(\mathbb{L}) \cong \Psi^R(\mathbb{L}) / \mathcal{N}^R(\mathbb{L}) \cong \mathrm{PsAut}^R(\mathbb{L})$.

3. Since the actions of $\mathcal{N}^R(\mathbb{L})$ and $\mathrm{Aut}(\mathbb{L})$ on $\Psi^R(\mathbb{L})$ commute, the action of $\mathrm{Aut}(\mathbb{L})$ descends to $\mathcal{N}^R(\mathbb{L})^{\mathrm{op}} \backslash \Psi^R(\mathbb{L}) \cong \mathrm{PsAut}^R(\mathbb{L})$ and the action of $\mathcal{N}^R(\mathbb{L})^{\mathrm{op}}$ descends to $\Psi^R(\mathbb{L}) / \mathrm{Aut}(\mathbb{L}) \cong \mathcal{C}^R(\mathbb{L})$. Since the left action of $\mathcal{N}^R(\mathbb{L})^{\mathrm{op}}$ on $\Psi^R(\mathbb{L})$ corresponds to an action by right multiplication on $\mathcal{C}^R(\mathbb{L})$, we find that there is a bijection $\mathrm{PsAut}^R(\mathbb{L}) / \mathrm{Aut}(\mathbb{L}) \longrightarrow \mathcal{C}^R(\mathbb{L}) / \mathcal{N}^R(\mathbb{L})$.
Suppose $(\alpha, A) \in \Psi^R(\mathbb{L})$ and let $[\alpha]_{\mathrm{Aut}(\mathbb{L})} \in \mathrm{PsAut}^R(\mathbb{L}) / \mathrm{Aut}(\mathbb{L})$ be the orbit of α under the action of $\mathrm{Aut}(\mathbb{L})$ and let $[A]_{\mathcal{N}^R(\mathbb{L})} \in \mathcal{C}^R(\mathbb{L}) / \mathcal{N}^R(\mathbb{L})$ be the orbit of A under the action of $\mathcal{N}^R(\mathbb{L})$. Then the bijection is given by $[\alpha]_{\mathrm{Aut}(\mathbb{L})} \mapsto [A]_{\mathcal{N}^R(\mathbb{L})}$. Moreover, each of these orbits also corresponds to the orbit of (α, A) under the right action of $\mathrm{Aut}(\mathbb{L}) \times \mathcal{N}^R(\mathbb{L})$ on $\Psi^R(\mathbb{L})$. These quotients preserve actions of $\mathrm{Aut}(\mathbb{L}) \times \mathcal{N}^R(\mathbb{L})$ on corresponding sets and thus these coset spaces are equivalent as $\mathrm{Aut}(\mathbb{L}) \times \mathcal{N}^R(\mathbb{L})$-sets. □

The above relationships between the different groups are summarized in Fig. 1.

Example 2.20. Suppose $\mathbb{L} = U\mathbb{H} \cong S^3$ - the group of unit quaternions. Then, since this is associative, $\mathcal{N}^R(U\mathbb{H}) = U\mathbb{H} \cong Sp(1)$. We also know that $\operatorname{Aut}(U\mathbb{H}) \cong SO(3)$. Now however, $\Psi^R(U\mathbb{H})$ consists of all pairs $(\alpha, A) \in SO(3) \times U\mathbb{H}$ with the group structure defined by (2.12a), which is the semi-direct product

$$\Psi^R(U\mathbb{H}) \cong SO(3) \ltimes Sp(1) \cong Sp(1)\,Sp(1) \cong SO(4). \tag{2.22}$$

In this case, $\operatorname{PsAut}^R(U\mathbb{H}) \cong \operatorname{Aut}(U\mathbb{H}) \cong SO(3)$. Here $(p, q) \sim (-p, -q)$ acts on $U\mathbb{H}$ via $r \mapsto prq^{-1}$.

Example 2.21. More generally, suppose $\mathbb{L} = G$ is a group. Then, $\operatorname{PsAut}^R(G) \cong \operatorname{Aut}(G)$ and $\Psi^R(G) \cong \operatorname{Aut}(G) \ltimes G^{\mathrm{op}}$, with $h = (\alpha, A) \in \Psi^R(G)$ acting on G by

$$h(g) = \alpha(g)\,A \tag{2.23}$$

Note that the group $\operatorname{Aut}(G) \ltimes G$ is known as the *holomorph* of G.

Example 2.22. Suppose $\mathbb{L} = U\mathbb{O}$ - the Moufang loop of unit octonions, which is homeomorphic to the 7-sphere S^7. From [20, Lemma 14.61] we know that $g \in O(\mathbb{O})$ belongs to $\operatorname{Spin}(7)$ if, and only if,

$$g(uv) = \chi_g(u)\,g(v) \tag{2.24}$$

for all $u, v \in \mathbb{O}$ where $\chi_g(u) = g\left(ug^{-1}(1)\right)$ gives the vector representation of $\operatorname{Spin}(7)$ on $\operatorname{Im}\mathbb{O}$. We may as well restrict everything to the non-zero octonions $\mathbb{O}^*$ or the unit octonions $U\mathbb{O}$, so that we have a loop. Now,

$$\begin{aligned} g(u) &= g(u \cdot 1) = \chi_g(u)\,g(1) \\ g(uv) &= g(uv \cdot 1) = \chi_g(uv)\,g(1) \end{aligned}$$

Hence, we find that (2.24) implies

$$\chi_g(uv)\,g(1) = \chi_g(u) \cdot \chi_g(v)\,g(1).$$

Thus, $(\chi_g, g(1))$ is a right pseudoautomorphism of $U\mathbb{O}$ with companion $g(1)$. Thus, in this case we find that $\Psi^R(U\mathbb{O}) \cong \operatorname{Spin}(7)$. We also know that $\mathcal{N}^R(U\mathbb{O}) = \{\pm 1\} \cong \mathbb{Z}_2$ and thus the projection $(\chi, A) \mapsto \chi$ corresponds to the double cover $\operatorname{Spin}(7) \longrightarrow SO(7)$. Hence, $\operatorname{PsAut}^R(U\mathbb{O}) \cong SO(7)$ and as we know, $\operatorname{Aut}(U\mathbb{O}) \cong G_2$. Since $U\mathbb{O}$ is a Moufang loop, and we know that for any q, the map Ad_q is a right pseudoautomorphism with companion q, we see that $\mathcal{C}^R(U\mathbb{O}) = U\mathbb{O}$, and indeed as we know, $\operatorname{Spin}(7)/G_2 \cong S^7$.

Remark 2.23. We have defined the group $\Psi^R(\mathbb{L})$ as the set of *all* right pseudoautomorphism pairs (α, A), however we could consistently truncate $\Psi^R(\mathbb{L})$ to a subgroup, or more generally, if G is some group with a homomorphism $\rho : G \longrightarrow \Psi^R(\mathbb{L})$, we can use this homomorphism to define a *pseudoautomorphism action* of G on $\mathbb{L}$. For example, if $G = \operatorname{Aut}(\mathbb{L}) \ltimes \mathcal{N}^R(\mathbb{L})^{\mathrm{op}}$, then we know that $\iota_1 \times \iota_2 : G \longrightarrow \Psi^R(\mathbb{L})$ is a homomorphism. With respect to the action of G, the companions would be just the elements of $\mathcal{N}^R(\mathbb{L})$.

Example 2.24. In [29], Leung developed a general framework for structures in Riemannian geometry based on division algebras - $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. As a first step, this involved representations of unitary groups with values in each of these algebras on the algebras themselves. The unitary groups, $O(n)$, $U(n)$, $Sp(n)\,Sp(1)$, and $\operatorname{Spin}(7)$, as well as the corresponding special unitary groups $SO(n)$, $SU(n)$, $Sp(n)$, and G_2, are precisely the possible Riemannian holonomy groups for irreducible, not locally symmetric smooth manifolds [6]. By considering the corresponding loops (groups for the associative cases) we can look at the pseudoautomorphism actions. The octonionic case is already covered in Example 2.22.

1. In the case of $\mathbb{R}$, consider instead the group of "unit reals" $U\mathbb{R} = \{\pm 1\} \cong \mathbb{Z}_2$. Then, $\Psi^R(U\mathbb{R}) = \{1\} \ltimes \{\pm 1\} \cong \mathbb{Z}_2$, however consider now for some positive integer n, the homomorphism $\det : O(n) \longrightarrow \mathbb{Z}_2$. Thus, $O(n)$ acts on $\mathbb{Z}_2$ via this homomorphism: $(g, x) \mapsto x \det g$, where $x \in \mathbb{Z}_2$ and $g \in O(n)$. The preimage $\operatorname{Aut}(\mathbb{Z}_2) = \{1\}$ is then just $\ker \det = SO(n)$. Thus, we can now define the group $\Psi_n^R(U\mathbb{R}) = O(n)$. The full action of $\Psi_n^R(U\mathbb{R})$ on $U\mathbb{R}$ is transitive, while the partial action is trivial. Similarly, we can also define $\operatorname{Aut}_n(U\mathbb{R}) = SO(n)$.
2. In the complex case, the group of unit complex numbers $U\mathbb{C} = U(1) \cong S^1$. Similarly, as above, $\Psi^R(U\mathbb{C}) = \{1\} \ltimes U(1) \cong U(1)$. Now however, we also have the homomorphism $\det_{\mathbb{C}} : U(n) \longrightarrow U(1)$. Then, $U(n)$ acts on $U(1)$ via $(g, z) \mapsto z \det g$, where $z \in U(1)$ and $g \in U(n)$. The preimage of $\operatorname{Aut}(U(1)) = \{1\}$ is then just $\ker \det_{\mathbb{C}} = SU(n)$. Thus, similarly as above, we can now define the group $\Psi_n^R(U\mathbb{C}) = U(n)$. The full action of $\Psi_n^R(U\mathbb{R})$ on $U\mathbb{C}$ is transitive, while the partial action is trivial. Similarly, we can also define $\operatorname{Aut}_n(U\mathbb{C}) = SU(n)$.
3. In the quaternionic case, we have already seen the case $n = 1$ in Example 2.20. The n-dimensional quaternionic unitary group is in general $Sp(n)\,Sp(1)$, where $Sp(n)$ is the compact symplectic group or equivalently, the quaternion special unitary group. The group $Sp(n)\,Sp(1)$ acts on $\mathbb{H}^n$ by $Sp(n)$ on the left, and multiplication by a unit quaternion on the right, and hence can be represented by pairs $h = (\alpha, q) \in Sp(n) \times Sp(1)$, with the identification $(-\alpha, -q) \sim (\alpha, q)$. For $n \geq 2$, define the homomorphism $\rho_{\mathbb{H}} : Sp(n)\,Sp(1) \longrightarrow Sp(1)\,Sp(1)$ given by $[\alpha, q] \mapsto [1, q]$. The image of this homomorphism simply corresponds to elements of $\Psi^R(U\mathbb{H})$ that are of the form (id, q), i.e. act by right multiplication of $U\mathbb{H}$ on itself. The preimage of $\operatorname{Aut}(U\mathbb{H}) \cong SO(3)$ is then $\ker \rho_{\mathbb{H}} \cong Sp(n)$. Overall, we may define the group $\Psi_n^R(U\mathbb{H}) = Sp(n)\,Sp(1)$ and $\operatorname{Aut}_n(U\mathbb{H}) = Sp(n)$. As in the previous examples, the full action of $\Psi_n^R(U\mathbb{H})$ on $U\mathbb{H}$ is transitive, whereas the partial action is again trivial. We will refer to this example later on, with the assumption that $n \geq 2$.

Thus, in each of the above cases, we may regard Ψ_n^R ($O(n)$, $U(n)$, or $Sp(n)\,Sp(1)$) as *a* group of pseudoautomorphism pairs acting on the unit real numbers, unit complex numbers, and unit quaternions with a trivial partial action and will the full action just given by right multiplication. The corresponding automorphism subgroups are then the "special" unitary subgroups $SO(n)$, $SU(n)$, $Sp(n)$.

2.3. Modified product

Let $r \in \mathbb{L}$, and define the modified product $\circ_r$ on $\mathbb{L}$ via

$$p \circ_r q = {}^{(p \cdot qr)}\!/_{r}. \tag{2.25}$$

Then, $p \circ_r q = p \cdot q$ if, and only if, $p \cdot qr = pq \cdot r$. This is true for all p, q if, and only if, $r \in \mathcal{N}^R(\mathbb{L})$. However, this will not hold for all r unless $\mathbb{L}$ is associative (and is thus a group). If $\mathbb{L}$ is a right Bol loop, and $a \in \mathbb{L}$, setting $r = q\backslash a$ in the right Bol identity (2.6), gives us

$$pq \cdot q\backslash a = {}^{(p \cdot aq)}\!/_{q} = p \circ_q a. \tag{2.26}$$

On octonions, the left-hand side of (2.26) is precisely the "modified octonion product" defined in [15] and also used in [16]. Since unit octonions are in particular a right Bol loop, the two products are equal on octonions.

The product (2.25) gives us a convenient definition of the *loop associator*.

Definition 2.25. Given $p, q, r \in \mathbb{L}$, the *loop associator* of p, q, r is given by

$$[p, q, r] = {}^{(p \circ_r q)}\!/_{pq}. \tag{2.27}$$

The *loop commutator* of p and q is given by

$$[p, q] = {}^{(pq/p)}\!/_{q}. \tag{2.28}$$

From the definition (2.27), we see that $[p, q, r] = 1$ if, and only if, $p(qr) = (pq)r$. There are several possible equivalent definitions of the associator, but from our point of view, (2.27) will be the most convenient. Similarly, the loop commutator can be defined in different ways, however (2.28) has an advantage, because if we define $\mathrm{Ad}_p(q) = pq/p$, then $[p, q] = (\mathrm{Ad}_p(q))/q$, which is a similar relation as for the group commutator.

We can easily see that $(\mathbb{L}, \circ_r)$ is a loop.

Lemma 2.26. *Consider the pair $(\mathbb{L}, \circ_r)$ of the set $\mathbb{L}$ equipped with the binary operation $\circ_r$.*

1. *The right quotient* $/_r$ *and the left quotient* $\backslash_r$ *on* $(\mathbb{L}, \circ_r)$ *are given by*

$$p/_r q = {}^{pr}\!/_{qr} \tag{2.29a}$$

$$p\backslash_r q = {}^{(p\backslash qr)}\!/_{r}, \tag{2.29b}$$

   and hence, $(\mathbb{L}, \circ_r)$ *is a quasigroup.*
2. $1 \in \mathbb{L}$ *is the identity element for* $(\mathbb{L}, \circ_r)$*, and hence* $(\mathbb{L}, \circ_r)$ *is a loop.*
3. *Let* $q \in \mathbb{L}$*, the left and right inverses with respect to* $\circ_r$ *are given by*

$$q^{\lambda_{(r)}} = {}^{r}\!/_{qr} \tag{2.30a}$$

$$q^{\rho_{(r)}} = {}^{(q\backslash r)}\!/_{r}. \tag{2.30b}$$

4. $(\mathbb{L}, \circ_r)$ *is isomorphic to* $(\mathbb{L}, \cdot)$ *if, and only if,* $r \in \mathcal{C}^R(\mathbb{L})$*. In particular,* $\alpha : (\mathbb{L}, \cdot) \longrightarrow (\mathbb{L}, \circ_r)$ *is an isomorphism, i.e. for any* $p, q \in \mathbb{L}$*,*

$$\alpha(pq) = \alpha(p) \circ_r \alpha(q), \tag{2.31}$$

   if, and only if, α *is a right pseudoautomorphism on* $(\mathbb{L}, \cdot)$ *with companion* r*.*

Proof. Let $x, p, q, r \in \mathbb{L}$.

1. Suppose

$$x \circ_r q = p.$$

   Using (2.25),

$$x \cdot qr = pr,$$

   and thus

$$x = pr/qr := p/_r q.$$

   Similarly, suppose

$$p \circ_r x = q,$$

   so that

$$p \cdot xr = qr,$$

   and thus

$$x = (p\backslash (qr))/r := p\backslash_r q.$$

Since the left and right quotients are both defined, $(\mathbb{L}, \circ_r)$ is a quasigroup.

2. We have

$$p \circ_r 1 = (p \cdot r)/r = p$$
$$1 \circ_r p = (1 \cdot pr)/r = p.$$

Hence, 1 is indeed the identity element for $(\mathbb{L}, \circ_r)$, and thus $(\mathbb{L}, \circ_r)$ is a loop.

3. Setting $p = 1$ in (2.29) we get the desired expressions.
4. Suppose $(\alpha, r) \in \Psi^R(\mathbb{L})$. Then, by definition, for any $p, q \in \mathbb{L}$,

$$\alpha(pq) = {}^{(\alpha(p) \cdot \alpha(q) r)}\!\!\diagup_{r}$$

Hence, from (2.25),

$$\alpha(pq) = \alpha(p) \circ_r \alpha(q). \tag{2.32}$$

Thus, α is an isomorphism from $(\mathbb{L}, \cdot)$ to $(\mathbb{L}, \circ_r)$. Clearly the converse is also true: if α is an isomorphism from $(\mathbb{L}, \cdot)$ to $(\mathbb{L}, \circ_r)$, then r is companion for α. Hence, $(\mathbb{L}, \cdot)$ and $(\mathbb{L}, \circ_r)$ are isomorphic if, and only if, r is a companion for some right pseudoautomorphism. □

Suppose $r, x \in \mathbb{L}$, then the next lemma shows the relationship between products $\circ_x$ and $\circ_{rx}$.

Lemma 2.27. *Let $r, x \in \mathbb{L}$, then*

$$p \circ_{rx} q = (p \circ_x (q \circ_x r))/_x r. \tag{2.33}$$

Proof. Let $r, x \in \mathbb{L}$, and suppose $y = rx$. Then, by (2.25),

$$\begin{aligned} p \cdot qy &= (p \circ_y q) \cdot y \\ &= (p \circ_y q) \cdot rx \\ &= ((p \circ_y q) \circ_x r) \cdot x. \end{aligned}$$

On the other hand, using (2.25) in a different way, we get

$$\begin{aligned} p \cdot qy &= p \cdot q(rx) \\ &= p \cdot ((q \circ_x r) x) \\ &= (p \circ_x (q \circ_x r)) \cdot x \end{aligned}$$

Hence,

$$(p \circ_y q) \circ_x r = p \circ_x (q \circ_x r) .$$

Dividing by r on the right using $/_x$ gives (2.33). □

Remark 2.28. Lemma 2.27 shows that the rx-product is equivalent to the r-product, *but defined on* $(\mathbb{L}, \circ_x)$. That is, if we start with $\circ_x$ and define the r-product using $\circ_x$, then we obtain the rx-product *on* $(\mathbb{L}, \cdot)$. If $x \in \mathcal{C}^R (\mathbb{L}, \cdot)$, then $(\mathbb{L}, \circ_x)$ is isomorphic to $(\mathbb{L}, \cdot)$. Similarly, if $r \in \mathcal{C}^R (\mathbb{L}, \circ_x)$, then $(\mathbb{L}, \circ_{rx})$ is isomorphic to $(\mathbb{L}, \circ_x)$.

On $(\mathbb{L}, \circ_x)$ we can define the associator and commutator. Given $p, q, r \in \mathbb{L}$, the *loop associator* on $(\mathbb{L}, \circ_x)$ is given by

$$[p, q, r]^{(x)} = (p \circ_{rx} q) /_x (p \circ_x q) . \tag{2.34}$$

The *loop commutator* on $(\mathbb{L}, \circ_x)$ is given by

$$[p, q]^{(x)} = ((p \circ_x q) /_x p) /_x q. \tag{2.35}$$

For any $x \in \mathbb{L}$, the adjoint map $\mathrm{Ad}^{(x)}\colon \mathbb{L} \times \mathbb{L} \longrightarrow \mathbb{L}$ with respect to $\circ_x$ is given by

$$\mathrm{Ad}_p^{(x)} (q) = \left(\left(R_p^{(x)} \right)^{-1} \circ L_p^{(x)} \right) q = (p \circ_x q) /_x p \tag{2.36}$$

for any $p, q \in \mathbb{L}$, and its inverse for a fixed p is

$$\left(\mathrm{Ad}_p^{(x)} \right)^{-1} (q) = \left(\left(L_p^{(x)} \right)^{-1} \circ R_p^{(x)} \right) q = p \backslash_x (q \circ_x p) . \tag{2.37}$$

Let us now consider how pseudoautomorphisms of $(\mathbb{L}, \cdot)$ act on $(\mathbb{L}, \circ_r)$.

Lemma 2.29. *Let $h = (\beta, B) \in \Psi^R (\mathbb{L}, \cdot)$. Then, for any $p, q, r \in \mathbb{L}$,*

$$\beta (p \circ_r q) = \beta (p) \circ_{h(r)} \beta (q) \tag{2.38}$$

and β is a right pseudoautomorphism of $(\mathbb{L}, \circ_r)$ with companion $h (r) /r$. It also follows that

$$\beta (p/_r q) = \beta (p) /_{h(r)} \beta (q) . \tag{2.39}$$

Proof. Consider $\beta (p \circ_r q)$. Then, using (2.12a) and (2.15),

$$\beta (p \circ_r q) = \beta ((p \cdot qr) /r)$$

$$
\begin{aligned}
&= h(p \cdot qr)/h(r) \\
&= (\beta(p) \cdot h(qr))/h(r) \\
&= (\beta(p) \cdot \beta(q)\, h(r))/h(r) \\
&= \beta(p) \circ_{h(r)} \beta(q),
\end{aligned}
$$

and hence we get (2.38). Alternatively, using (2.29a),

$$
\begin{aligned}
\beta(p \circ_r q) &= {}^{(\beta(p) \cdot \beta(q)\, h(r))}\!\diagup_{h(r)} \\
&= \left({}^{(\beta(p) \cdot \beta(q)\, h(r))}\!\diagup_{r}\right) /_r \left({}^{h(r)}\!\diagup_{r}\right).
\end{aligned}
$$

Now, let $C = h(r)/r$. Thus,

$$
\begin{aligned}
\beta(p \circ_r q) &= \left({}^{(\beta(p)(\beta(q) \cdot Cr))}\!\diagup_{r}\right) /_r C \\
&= (\beta(p) \circ_r (\beta(q) \circ_r C)) /_r C
\end{aligned}
$$

Thus, indeed, β is a right pseudoautomorphism of $(\mathbb{L}, \circ_r)$ with companion $C = h(r)/r$.
Now using (2.38) with $p/_r q$ and q, we find

$$
\beta(p) = \beta(p/_r q \circ_r q) = \beta(p/_r q) \circ_{h(r)} \beta(q)
$$

and hence we get (2.39). $\square$

Remark 2.30. We will use the notation $(\beta, C)_r$ to denote that $(\beta, C)_r$ is considered as a pseudoautomorphism pair on $(\mathbb{L}, \circ_r)$, i.e. $(\beta, C)_r \in \Psi^R(\mathbb{L}, \circ_r)$. Of course, the product of C with any element in $\mathcal{N}^R(\mathbb{L}, \circ_r)$ on the right will also give a companion of β on $(\mathbb{L}, \circ_r)$. Any right pseudoautomorphism of $(\mathbb{L}, \cdot)$ is also a right pseudoautomorphism of $(\mathbb{L}, \circ_r)$, however their companions may be different. In particular, $\mathrm{PsAut}^R(\mathbb{L}, \cdot) = \mathrm{PsAut}^R(\mathbb{L}, \circ_r)$. For $\Psi^R(\mathbb{L}, \cdot)$ and $\Psi^R(\mathbb{L}, \circ_r)$ we have a group isomorphism

$$
\begin{array}{ccc}
\Psi^R(\mathbb{L}, \cdot) & \longrightarrow & \Psi^R(\mathbb{L}, \circ_r) \\
h = (\beta, B) & \mapsto & h_r = \left(\beta, {}^{h(r)}\!\diagup_{r}\right)_r.
\end{array}
\tag{2.40}
$$

Conversely, if we have $h_r = (\beta, C)_r \in \Psi^R(\mathbb{L}, \circ_r)$, then this corresponds to $h = (\beta, B) \in \Psi^R(\mathbb{L}, \cdot)$ where

$$
B = \beta(r) \backslash (Cr). \tag{2.41}
$$

The group isomorphism (2.40) together with R_r^{-1} (right division by r) induces a G-set isomorphism between $\left(\mathring{\mathbb{L}}, \cdot\right)$ with the action of $\Psi^R(\mathbb{L}, \cdot)$ and $\left(\mathring{\mathbb{L}}, \circ_r\right)$ with the action of $\Psi^R(\mathbb{L}, \circ_r)$.

Lemma 2.31. *Let $r \in \mathbb{L}$, then the mapping (2.40) $h \mapsto h_r$ from $\Psi^R(\mathbb{L}, \cdot)$ to $\Psi^R(\mathbb{L}, \circ_r)$ together with the map $R_r^{-1} : \left(\mathring{\mathbb{L}}, \cdot\right) \longrightarrow \left(\mathring{\mathbb{L}}, \circ_r\right)$ gives a G-set isomorphism. In particular, for any $A \in \mathring{\mathbb{L}}$ and $h \in \Psi^R(\mathbb{L}, \cdot)$,*

$$h(A)/r = h_r(A/r). \tag{2.42}$$

Proof. Suppose $h = (\beta, B)$ and correspondingly, from (2.40), $h_r = \left(\beta, {}^{h(r)}\!/_{r}\right)$. Then, we have,

$$\begin{aligned} h_r(A/r) &= \beta(A/r) \circ_r {}^{h(r)}\!/_{r} \\ &= {}^{(h(A)/h(r) \cdot h(r))}\!/_{r} \\ &= h(A)/r, \end{aligned}$$

where we have also used (2.15b). □

Using (2.40), we now have the following characterizations of $\mathcal{C}^R(\mathbb{L}, \circ_r)$, $\mathcal{N}^R(\mathbb{L}, \circ_r)$, and $\operatorname{Aut}(\mathbb{L}, \circ_r)$.

Lemma 2.32. *Let $r, C \in \mathbb{L}$, then*

$$C \in \mathcal{C}^R(\mathbb{L}, \circ_r) \iff C = A/r \text{ for some } A \in \operatorname{Orb}_{\Psi^R(\mathbb{L},\cdot)}(r) \tag{2.43a}$$

$$C \in \mathcal{N}^R(\mathbb{L}, \circ_r) \iff C = \operatorname{Ad}_r(A) \text{ for some } A \in \mathcal{N}^R(\mathbb{L}, \cdot) \tag{2.43b}$$

and

$$\operatorname{Aut}(\mathbb{L}, \circ_r) \cong \operatorname{Stab}_{\Psi^R(\mathbb{L},\cdot)}(r). \tag{2.44}$$

If $r \in \mathcal{C}^R(\mathbb{L}, \cdot)$, so that there exists a right pseudoautomorphism pair $h = (\alpha, r) \in \Psi^R(\mathbb{L}, \cdot)$, then $\operatorname{Aut}(\mathbb{L}, \circ_r) \cong h \operatorname{Aut}(\mathbb{L}, \cdot) h^{-1}$.

Proof. From (2.40) we see that any companion in $(\mathbb{L}, \circ_r)$ is of the form $h(r)/r$ for some $h \in \Psi^R(\mathbb{L}, \cdot)$. Therefore, $C \in \mathbb{L}$ is a companion in $(\mathbb{L}, \circ_r)$ if, and only if, it is of the form $C = A/r$ for some $A \in \operatorname{Orb}_{\Psi^R(\mathbb{L},\cdot)}(r)$.

The right nucleus $\mathcal{N}^R(\mathbb{L}, \circ_r)$ corresponds to the companions of the identity map id on $\mathbb{L}$, hence taking $\beta = \mathrm{id}$ in (2.40), we find that companions of id in $(\mathbb{L}, \circ_r)$ must be of the form $C = (rA)/r = \operatorname{Ad}_r(A)$ for some $A \in \mathcal{N}^R(\mathbb{L}, \cdot)$. Conversely, suppose $C = (rA)/r$ for some $A \in \mathcal{N}^R(\mathbb{L}, \cdot)$, then we can explicitly check that for any $p, q \in \mathbb{L}$, we have

$$\begin{aligned} (p \circ_r q) \circ_r C &= ((p \cdot qr)/r \cdot rA)/r \\ &= ((p \cdot qr) \cdot A)/r \\ &= (p \cdot (qr \cdot A))/r = (p \cdot (q \cdot rA))/r \end{aligned}$$

$$=(p\cdot(q\cdot Cr))/r=(p\cdot(q\circ_r C)r)/r$$
$$=p\circ_r(q\circ_r C)$$

and hence, $C\in\mathcal{N}^R(\mathbb{L},\circ_r)$.

The group $\operatorname{Aut}(\mathbb{L},\circ_r)$ is isomorphic to the preimage $\mathrm{prj}_2^{-1}(1)$ with respect to the projection map $\mathrm{prj}_2\colon\Psi^R(\mathbb{L},\circ_r)\longrightarrow\mathcal{C}^R(\mathbb{L},\circ_r)$. From (2.40), this corresponds precisely to the maps $h\in\Psi^R(\mathbb{L},\cdot)$ for which $h(r)=r$. If r is in the $\Psi^R(\mathbb{L},\cdot)$-orbit of 1, then clearly $\operatorname{Aut}(\mathbb{L},\circ_r)$ is conjugate to $\operatorname{Aut}(\mathbb{L},\cdot)$. □

Remark 2.33. Suppose $r\in\mathcal{C}^R(\mathbb{L})$, then from (2.43a), we see that if $A\in\mathcal{C}^R(\mathbb{L},\circ_r)$, then $Ar\in\mathcal{C}^R(\mathbb{L})$. Also, using the isomorphism (2.40), we can define the left action of $\Psi^R(\mathbb{L},\circ_r)$ on $\Psi^R(\mathbb{L},\cdot)$ just by composition on the left by the corresponding element in $\Psi^R(\mathbb{L},\cdot)$. Now recall that

$$\mathcal{C}^R(\mathbb{L},\circ_r)\cong\Psi^R(\mathbb{L},\circ_r)/\operatorname{Aut}(\mathbb{L},\circ_r)\text{ and }\mathcal{C}^R(\mathbb{L})\cong\Psi^R(\mathbb{L},\cdot)/\operatorname{Aut}(\mathbb{L},\cdot).$$

Then, for any equivalence classes $\lfloor\alpha,A\rfloor_r\in\Psi^R(\mathbb{L},\circ_r)/\operatorname{Aut}(\mathbb{L},\circ_r)$ and $\lfloor\beta,r\rfloor\in\Psi^R(\mathbb{L},\cdot)/\operatorname{Aut}(\mathbb{L},\cdot)$, we find that

$$\lfloor\alpha,A\rfloor_r\cdot\lfloor\beta,r\rfloor=\lfloor\alpha\circ\beta,Ar\rfloor.\tag{2.45}$$

Another way to see this is the following. From (2.41), the element in $\Psi^R(\mathbb{L},\cdot)$ that corresponds to $(\alpha,A)_r\in\Psi^R(\mathbb{L},\circ_r)$ is $(\alpha,{}_{\alpha(r)}\backslash Ar)$. The composition of this with (β,r) is then $(\alpha\circ\beta,Ar)$. Then, it is easy to see that this reduces to cosets.

Example 2.34. Recall that in a Moufang loop $\mathbb{L}$, the map Ad_q is a right pseudoautomorphism with companion q^3. The relation (2.45) then shows that for any $r\in\mathbb{L}$,

$$\mathrm{Ad}_q^{(r^3)}\circ\mathrm{Ad}_r=\mathrm{Ad}_{(q^3r^3)^{\frac{1}{3}}}\circ h\tag{2.46}$$

where $h\in\operatorname{Aut}(\mathbb{L})$. This follows because $\mathrm{Ad}_q^{(r^3)}$ has companion q^3 in $\Psi^R(\mathbb{L},\circ_{r^3})$ and Ad_r has companion r^3 in $\Psi^R(\mathbb{L})$, thus the composition has companion q^3r^3, so up to composition with $\operatorname{Aut}(\mathbb{L})$, it is given by $\mathrm{Ad}_{(q^3r^3)^{\frac{1}{3}}}$. A similar expression for octonions has been derived in [15].

As we have seen, $\Psi^R(\mathbb{L})$ acts transitively on $\mathcal{C}^R(\mathbb{L})$ and moreover, for each $r\in\mathcal{C}^R(\mathbb{L})$, the loops $(\mathbb{L},\circ_r)$ are all isomorphic to one another, and related via elements of $\Psi^R(\mathbb{L})$. Concretely, consider $(\mathbb{L},\circ_r)$ and suppose $h=(\alpha,A)\in\Psi^R(\mathbb{L})$. Then, define the map

$$h:(\mathbb{L},\circ_r)\longrightarrow\left(\mathbb{L},\circ_{h(r)}\right),$$

where h acts on $\mathbb{L}$ via the partial action (i.e. via α). Indeed, from (2.31), we have for $p, q \in h(\mathbb{L})$

$$\alpha\left(\alpha^{-1}(p) \circ_r \alpha^{-1}(q)\right) = p \circ_{h(r)} q. \tag{2.47}$$

Moreover, if we instead consider the action of $\Psi^R(\mathbb{L}, \circ_r)$, then given $h_r = (\alpha, x)_r \in \Psi^R(\mathbb{L}, \circ_r)$, $h_r(\mathbb{L}) \cong (\mathbb{L}, \circ_{xr})$. This is summarized in the theorem below.

Theorem 2.35. *Let $\mathbb{L}$ be a loop with the set of right companions $\mathcal{C}^R(\mathbb{L})$. For every $r \in \mathcal{C}^R(\mathbb{L})$ and every $h \in \Psi^R(\mathbb{L})$, the loop $(\mathbb{L}, \circ_r)$ is isomorphic to $\left(\mathbb{L}, \circ_{h(r)}\right)$. Moreover, if instead, the action of $\Psi^R(\mathbb{L}, \circ_r)$ is considered, then an element of $\Psi^R(\mathbb{L}, \circ_r)$ with companion x induces a loop isomorphism from $(\mathbb{L}, \circ_r)$ to $(\mathbb{L}, \circ_{xr})$.*

Now again, let $h = (\alpha, A) \in \Psi^R(\mathbb{L})$, and we will consider the action of h on the nucleus. It is easy to see how the loop associator transforms under this map. Using (2.34) and (2.39), we have

$$\begin{aligned}\alpha\left([p, q, r]^{(x)}\right) &= \alpha\left((p \circ_{rx} q) /_x (p \circ_x q)\right) \\ &= \left(\alpha(p) \circ_{\alpha(r)h(x)} \alpha(q)\right) /_{h(x)} \left(\alpha(p) \circ_{h(x)} \alpha(q)\right) \\ &= [\alpha(p), \alpha(q), \alpha(r)]^{(h(x))}.\end{aligned} \tag{2.48}$$

So in particular, taking $x = 1$, $C \in \mathcal{N}^R(\mathbb{L})$ if, and only if, $\alpha(C) \in \mathcal{N}^R(\mathbb{L}, \circ_A)$. However from (2.43b), we know that $C \in \mathcal{N}^R(\mathbb{L})$ if, and only if, $(\mathrm{Ad}_A) C \in \mathcal{N}^R(\mathbb{L}, \circ_A)$. In particular, this means that $C \in \mathcal{N}^R(\mathbb{L})$ if, and only if,

$$\alpha^{-1}(\mathrm{Ad}_A C) \in \mathcal{N}^R(\mathbb{L}).$$

This defines a left action of $\Psi^R(\mathbb{L})$ on $\mathcal{N}^R(\mathbb{L})$:

$$h''(C) = \mathrm{Ad}_A^{-1}(\alpha(C)) = {}_A\backslash h(C) \tag{2.49}$$

for $h = (\alpha, A) \in \Psi^R(\mathbb{L})$ and $C \in \mathcal{N}^R(\mathbb{L})$. The action (2.49) can be seen from the following considerations. Recall $\mathcal{N}^R(\mathbb{L})^{\mathrm{op}}$ embeds in $\Psi^R(\mathbb{L})$ via the map $C \mapsto \iota_2(C) = (\mathrm{id}, C)$. The group $\Psi^R(\mathbb{L})$ acts on itself via the adjoint action, so let $h = (\alpha, A) \in \Psi^R(\mathbb{L})$, then from (2.21) recall,

$$h(\iota_2(C))h^{-1} = (\alpha, h(C))h^{-1} = \left(\mathrm{id}, A^{\lambda} \cdot h(C)\right). \tag{2.50}$$

On the other hand, suppose

$$(\alpha, h(C))h^{-1} = (\mathrm{id}, x),$$

so that

$$(\alpha, h(C)) = (\mathrm{id}, x)(\alpha, A) = (\alpha, Ax)$$

Therefore, $x = A\backslash h(C)$. In particular, $A\backslash h(C) \in \mathcal{N}^R(\mathbb{L})$. Thus the induced action on $\mathcal{N}^R(\mathbb{L})$ is precisely $C \mapsto A\backslash h(C) = \mathrm{Ad}_A^{-1}(\alpha(C))$. Moreover, right multiplication of elements in $\mathring{\mathbb{L}}$ by elements of $\mathcal{N}^R(\mathbb{L})$ is compatible with the corresponding actions of $\Psi^R(\mathbb{L})$.

Lemma 2.36. *For any $s \in \mathring{\mathbb{L}}, C \in \mathcal{N}^R(\mathbb{L})$, and $h \in \Psi^R(\mathbb{L})$, we have*

$$h(sC) = h(s)\, h''(C), \tag{2.51}$$

where h'' is the action (2.49).

Proof. Indeed, to show (2.51), we have

$$\begin{aligned} h(sC) =& \alpha(s)\, h(C) \\ =& h(s)/A \cdot Ah''(C) \\ =& (h(s)/A \cdot A)\, h''(C) \\ =& h(s) \cdot h''(C), \end{aligned}$$

since $h''(C) \in \mathcal{N}^R(\mathbb{L})$. □

3. Smooth loops

Suppose the loop $\mathbb{L}$ is a smooth finite-dimensional manifold such that the loop multiplication and division are smooth functions. Define maps

$$\begin{array}{rccc} L_r: & \mathbb{L} & \longrightarrow & \mathbb{L} \\ & q & \longmapsto & rq \end{array} \tag{3.1}$$

and

$$\begin{array}{rccc} R_r: & \mathbb{L} & \longrightarrow & \mathbb{L} \\ & q & \longmapsto & qr. \end{array} \tag{3.2}$$

These are diffeomorphisms of $\mathbb{L}$ with smooth inverses L_r^{-1} and R_r^{-1} that correspond to left division and right division by r, respectively. Also, assume that $\Psi^R(\mathbb{L})$ acts smoothly on $\mathbb{L}$ (as before, $\mathbb{L}$ together with the full action of $\Psi^R(\mathbb{L})$ will be denoted by $\mathring{\mathbb{L}}$). Thus, the action of $\Psi^R(\mathbb{L})$ is a group homomorphism from $\Psi^R(\mathbb{L})$ to $\mathrm{Diff}(\mathbb{L})$. In particular, this induces a Lie group structure on $\Psi^R(\mathbb{L})$. Similarly, $\mathrm{PsAut}^R(\mathbb{L})$ is then also a Lie group, and for any $s \in \mathring{\mathbb{L}}$, $\mathrm{Aut}\,(\mathbb{L}, \circ_s) \cong \mathrm{Stab}_{\Psi^R(\mathbb{L})}\,(s)$ is then a Lie subgroup of $\Psi^R\,(\mathbb{L})$, and indeed of $\mathrm{PsAut}^R\,(\mathbb{L})$ as well. The assumption that pseudoautomorphisms act smoothly on $\mathbb{L}$ may be nontrivial. To the best of the author's knowledge, it is an open question whether this is always true. However, for the loop $U\mathbb{O}$ of unit octonions, this is indeed true, as can be seen from Example 2.22.

Define X to be a *right fundamental vector field* if for any $q \in \mathbb{L}$, it is determined by a tangent vector at 1 via right translations. That is, given a tangent vector $\xi \in T_1\mathbb{L}$, we define a corresponding right fundamental vector field $\rho\,(\xi)$ given by

$$\rho\,(\xi)_q = (R_q)_*\,\xi \tag{3.3}$$

at any $q \in \mathbb{L}$. If $\mathbb{L}$ is a Lie group, then this definition is equivalent to the standard definition of a right-invariant vector field X such that $(R_q)_*\,X_p = X_{pq}$, however in the non-associative case, $R_q \circ R_p \neq R_{pq}$, so the standard definition wouldn't work, so a right fundamental vector field is not actually right-invariant in the usual sense. We can still say that the vector space of right fundamental vector fields has dimension $\dim \mathbb{L}$, and at any point, they still form a basis for the tangent space. In particular, any smooth loop is parallelizable. However this vector space is now in general not a Lie algebra under the Lie bracket of vector fields, which is to be expected, since $T_1\mathbb{L}$ doesn't necessarily have the Lie algebra structure either.

Instead of right invariance, we see that given a right fundamental vector field $X = \rho\,(\xi)$,

$$\begin{aligned}\left(R_p^{-1}\right)_* X_q &= \left(R_p^{-1} \circ R_q\right)_* \xi \\ &= \left(R_{q/p}^{(p)}\right)_* \xi\end{aligned} \tag{3.4}$$

where $R^{(p)}$ is the right product with respect to the operation $\circ_p$. This is because

$$\begin{aligned}\left(R_p^{-1} \circ R_q\right) r &= (rq)\,/p \\ &= (r \cdot (q/p \cdot p))\,/p \\ &= r \circ_p (q/p) = R_{q/p}^{(p)} r,\end{aligned} \tag{3.5}$$

where we have used (2.25). In (3.4) we are using the standard chain rule, because we are regarding R_p^{-1} and R_q simply as differentiable maps, so non-associativity does not cause any problems for the chain rule.

3.1. Exponential map

For some $\xi \in T_1\mathbb{L}$, define a flow p_ξ on $\mathbb{L}$ given by

$$\begin{cases} \frac{dp_\xi(t)}{dt} = \left(R_{p_\xi(t)}\right)_* \xi \\ p_\xi\,(0) = 1 \end{cases} \tag{3.6}$$

This generally has a solution for some sufficiently small time interval $(-\varepsilon, \varepsilon)$, and is only a local 1-parameter subgroup. However it is shown in [28,32] that if $\mathbb{L}$ is at least power-associative, then $p_{\xi}(t+s) = p_{\xi}(t)\, p_{\xi}(s)$ for all t, s, and hence the solution can extended for all t. Recall that power-associativity means that powers of the same element of $\mathbb{L}$ associate, and thus this assumption is required in order to be able to define $p_{\xi}(nh) = p_{\xi}(h)^n$ unambiguously.

The solutions of (3.6) define the (local) exponential map: $\exp(t\xi) := p_{\xi}(t)$. The corresponding diffeomorphisms are then the right translations $R_{\exp(t\xi)}$. We will generally only need this locally, so the power-associativity assumption will not be necessary. Now consider a similar flow but with a different initial condition:

$$\begin{cases} \frac{dp_{\xi,q}(t)}{dt} = \left(R_{p_{\xi,q}(t)}\right)_* \xi \\ p_{\xi,q}(0) = q \end{cases} \tag{3.7}$$

where $q \in \mathbb{L}$. Applying R_q^{-1}, and setting $\tilde{p}(t) = {p_{\xi,q}(t)}/{q}$, we obtain

$$\begin{cases} \frac{d\tilde{p}(t)}{dt} = \left(R_q^{-1} \circ R_{p_{\xi,q}(t)}\right)_* \xi \\ \tilde{p}(0) = 1 \end{cases}. \tag{3.8}$$

If $\mathbb{L}$ is associative, then $R_q^{-1} \circ R_{p_{\xi,q}(t)} = R_{(p_{\xi,q}(t))/q}$, and thus $\tilde{p}(t)$ would satisfy (3.6), and we could conclude that $p_{\xi,q}(t) = \exp(t\xi)\, q$. However, in the general case, we have (3.5) and hence, $\tilde{p}(t)$ satisfies the following equation

$$\begin{cases} \frac{d\tilde{p}(t)}{dt} = \left(R^{(q)}_{\tilde{p}(t)}\right)_* \xi \\ \tilde{p}(0) = 1 \end{cases}. \tag{3.9}$$

This is now an integral curve equation for ξ on $(\mathbb{L}, \circ_q)$, and hence for sufficiently small t we can define a local exponential map $\exp_q$ for $(\mathbb{L}, \circ_q)$:

$$\tilde{p}(t) = \exp_q(t\xi), \tag{3.10}$$

so, that

$$p_{\xi,q}(t) = \exp_q(t\xi)\, q. \tag{3.11}$$

If $q \in \mathcal{C}^R(\mathbb{L})$, then $(\mathbb{L}, \circ_q)$ is isomorphic to $\mathbb{L}$, so if $\mathbb{L}$ is power-associative, then so is $(\mathbb{L}, \circ_q)$, and hence, the solutions (3.10) are defined for all t.

Suppose $h = (\alpha, q) \in \Psi^R(\mathbb{L})$, then let $\hat{p}(t) = \alpha^{-1}(\tilde{p}(t))$. This then satisfies $\hat{p}(0) = 1$ and

$$\frac{d\hat{p}(t)}{dt} = \left(\alpha^{-1}\right)_* \left(R^{(q)}_{\tilde{p}(t)}\right)_* \xi. \tag{3.12}$$

However, let $r \in \mathbb{L}$ and consider $R_p^{(q)}$:

$$\begin{aligned} R_p^{(q)} r &= r \circ_q p = \alpha\left(\alpha^{-1}(r) \cdot \alpha^{-1}(p)\right) \\ &= \left(\alpha \circ R_{\alpha^{-1}(p)} \circ \alpha^{-1}\right)(r) . \end{aligned}$$

Thus,

$$R_p^{(q)} = \alpha \circ R_{\alpha^{-1}(p)} \circ \alpha^{-1}, \tag{3.13}$$

and hence, (3.12) becomes

$$\frac{d\hat{p}(t)}{dt} = \left(R_{\hat{p}(t)}\right)_* \left(\left(\alpha^{-1}\right)_* \xi\right). \tag{3.14}$$

This shows that $\hat{p}$ is a solution of (3.6) with initial velocity vector $\left(\alpha^{-1}\right)_* \xi \in T_1\mathbb{L}$, and is hence given by $\hat{p} = \exp\left(t\left(\alpha^{-1}\right)_* \xi\right)$. Comparing with (3.10) we see that in this case,

$$\exp_q(t\xi) = \alpha\left(\exp\left(t\left(\alpha^{-1}\right)_* \xi\right)\right), \tag{3.15}$$

and hence the solution $p_{\xi,q}(t)$ of (3.7) can be written as

$$p_{\xi,q}(t) = h\left(\exp\left(t\left(\alpha^{-1}\right)_* \xi\right)\right). \tag{3.16}$$

We can summarize these findings in the theorem below.

Theorem 3.1. *Suppose $\mathbb{L}$ is a smooth loop and suppose $q \in \mathcal{C}^R(\mathbb{L})$. Then, given $\xi \in T_1\mathbb{L}$, the equation*

$$\begin{cases} \frac{dp(t)}{dt} = \left(R_{p(t)}\right)_* \xi \\ p(0) = q \end{cases} \tag{3.17}$$

has the solution

$$p(t) = \exp_q(t\xi)\, q \tag{3.18}$$

for sufficiently small t, where

$$\exp_q(t\xi) = \alpha\left(\exp\left(t\left(\alpha^{-1}\right)_* \xi\right)\right)$$

where α is a right pseudoautomorphism of $\mathbb{L}$ that has companion q and $\exp(t\xi)$ is defined as the solution of (3.17) with initial condition $p(t) = 1$. In particular, ξ defines a flow $\Phi_{\xi,t}$, given by

$$\Phi_{\xi,t}(q) = \exp_q(t\xi)\, q. \tag{3.19}$$

Remark 3.2. The expression (3.15) can be made a bit more general. Suppose $\mathbb{L}_1$ and $\mathbb{L}_2$ are two loops and $\alpha : \mathbb{L}_1 \longrightarrow \mathbb{L}_2$ is a loop homomorphism. If we suppose $\exp_{(1)}$ and $\exp_{(2)}$ are the exponential maps on $\mathbb{L}_1$ and $\mathbb{L}_2$, respectively, then the following diagram commutes (Fig. 2).

$$\begin{array}{ccc} T_1\mathbb{L}_1 & \xrightarrow{\alpha_*} & T_1\mathbb{L}_2 \\ \downarrow{\scriptstyle \exp_{(1)}} & & \downarrow{\scriptstyle \exp_{(2)}} \\ \mathbb{L}_1 & \xrightarrow{\alpha} & \mathbb{L}_2 \end{array}$$

Fig. 2. Loop exponential maps.

Remark 3.3. The action of $\Phi_{\xi,t}$ given by (3.19) looks like it depends on q, however we easily see that for sufficiently small t, $\exp_q(t\xi) = \exp_r(t\xi)$ whenever q and r are on the same integral curve generated by ξ (equivalently in the same orbit of Φ_ξ). This is consistent with the 1-parameter subgroup property $\Phi_{\xi,t}(\Phi_{\xi,s}(q)) = \Phi_{\xi,t+s}(q)$.

Indeed, consider $r = \exp_q(s\xi)\, q$ and $\tilde{r} = \exp_q((t+s)\xi)\, q$. These are points that lie along the solution curve of (3.17). On the other hand, consider the solution of (3.17) at with $p(0) = r$. This is then given by $\hat{r} = \exp_r(t\xi)\, r$. However, clearly by uniqueness of solutions of ODEs, $\hat{r} = \tilde{r}$. So now,

$$\begin{aligned} \hat{r} &= \tilde{r} \\ &= \exp_q((t+s)\xi)\, q = \left(\exp_q(t\xi) \circ_q \exp_q(s\xi)\right) q \\ &= \exp_q(t\xi)\left(\exp_q(s\xi)\, q\right) \\ &= \exp_q(t\xi)\, r \end{aligned}$$

Hence, we conclude that $\exp_q(t\xi) = \exp_r(t\xi)$.

Remark 3.4. Suppose $(\mathbb{L}, \cdot)$ power left-alternative, i.e. $x^k\left(x^l q\right) = x^{k+l} q$ for all $x, q \in \mathbb{L}$ and any integers k, l. In particular this also means that $(\mathbb{L}, \cdot)$ is power-associative and has the left inverse property. In particular, powers of $x \in \mathbb{L}$ with respect to $\circ_q$ are equal to powers of x with respect to $\cdot$. For any $q \in \mathbb{L}$, $(\mathbb{L}, \circ_q)$ is then also power left-alternative. Now the right-hand side of (3.9) can be written as

$$\left(R^{(q)}_{\tilde{p}(t)}\right)_* \xi = \frac{d}{ds}\left(r(s) \circ_q \tilde{p}(t)\right)\Big|_{s=0} \tag{3.20}$$

where $r(s)$ is a curve with $r(0) = 1$ and $r'(0) = \xi$, so we may take $r(s) = \tilde{p}(s)$. Suppose there exist integers n, k and a real number h, such that $t = nh$ and $s = kh$. Then

$$\begin{aligned} \tilde{p}(s) \circ_q \tilde{p}(t) &= \tilde{p}(kh) \circ_q \tilde{p}(nh) \\ &= \left(\tilde{p}(h)^k \cdot \tilde{p}(h)^n q\right)/q \end{aligned}$$

$$\begin{aligned}&=\tilde{p}(h)^{k+n}=\tilde{p}(kh)\,\tilde{p}(nh)\\&=\tilde{p}(s)\,\tilde{p}(t).\end{aligned}$$

This is independent of n and k, and is hence true for any s,t. Thus we find that (3.20) is equal to the right-hand side of (3.6), so $\tilde{p}$ actually satisfies the same equation as p, so by uniqueness of solutions $\tilde{p}=p$. Hence, in this case, $\exp_q=\exp$. In general however, the exponential map will not be unique and will depend on the choice of q.

3.2. Tangent algebra

Suppose $\xi,\gamma\in T_1\mathbb{L}$ and let $X=\rho(\xi)$ and $Y=\rho(\gamma)$ be the corresponding right fundamental vector fields on $\mathbb{L}$. Then, recall that the vector field Lie bracket of X and Y is given by

$$[X,Y]_p=\frac{d}{dt}\left(\left(\Phi_t^{-1}\right)_*\left(Y_{\Phi_t(p)}\right)\right)\Big|_{t=0}, \tag{3.21}$$

where $\Phi_t=\Phi(\xi,t)$ is the flow generated by X. For sufficiently small t, we have $\Phi_t(p)=\exp_p(t\xi)\,p$, and thus

$$Y_{\Phi_t(p)}=\left(R_{\exp_p(t\xi)p}\right)_*\gamma.$$

Hence

$$\left(\Phi_t^{-1}\right)_*\left(Y_{\Phi_t(p)}\right)=\left(L^{-1}_{\exp_p(t\xi)}\circ R_{\exp_p(t\xi)p}\right)_*\gamma. \tag{3.22}$$

Now right translating back to $T_1\mathbb{L}$, we obtain

$$\left(R_p^{-1}\right)_*[X,Y]_p=\frac{d}{dt}\left(\left(R_p^{-1}\circ L^{-1}_{\exp_p(t\xi)}\circ R_{\exp_p(t\xi)p}\right)_*\gamma\right)\Big|_{t=0}. \tag{3.23}$$

In general, let $q,x,y\in\mathbb{L}$, then

$$\begin{aligned}\left(R_p^{-1}\circ L_x^{-1}\circ R_{yp}\right)q&=\left(x\backslash\left(q\cdot yp\right)\right)/p\\&=\left(x\backslash\left(\left(q\cdot yp\right)/p\cdot p\right)\right)/p\\&=x\backslash_p\left(q\circ_p y\right)\\&=\left(\left(L_x^{(p)}\right)^{-1}\circ R_y^{(p)}\right)q,\end{aligned}$$

where we have used (2.29b). Hence (3.23) becomes

$$\begin{aligned}\left(R_p^{-1}\right)_* [X,Y]_p &= \frac{d}{dt}\left(\left(\left(L^{(p)}_{\exp_p(t\xi)}\right)^{-1}\circ R^{(p)}_{\exp_p(t\xi)}\right)_*\gamma\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(\left(\mathrm{Ad}^{(p)}_{\exp_p(t\xi)}\right)_*^{-1}\gamma\right)\Big|;_{t=0} \\ &= -\frac{d}{dt}\left(\left(\mathrm{Ad}^{(p)}_{\exp_p(t\xi)}\right)_*\gamma\right)\Big|_{t=0} \\ &= -d_\xi\left(\mathrm{Ad}^{(p)}\right)_*\Big|_1(\gamma)\end{aligned} \tag{3.24}$$

Here, $\left(\mathrm{Ad}^{(p)}_x\right)_*$ denotes the induced adjoint action of $\mathbb{L}$ on $T_1\mathbb{L}$. As remarked earlier, this is not an action in the sense of group actions. Similarly, as for Lie groups and Lie algebras, we can also think of $\left(\mathrm{Ad}^{(p)}\right)_*:\mathbb{L}\longrightarrow \mathrm{End}\left(T_1\mathbb{L}\right)$, and then (3.24) is just the differential of this map at $1\in\mathbb{L}$ in the direction $\xi\in T_1\mathbb{L}$. The differential of $\left(\mathrm{Ad}^{(p)}\right)_*$ at an arbitrary point in $\mathbb{L}$ is given in Lemma A.3. This now allows us to define the tangent adjoint map $\mathrm{ad}^{(p)}$ on $T_1\mathbb{L}$.

Definition 3.5. For any $\xi,\gamma\in T_1\mathbb{L}$, the tangent adjoint map $\mathrm{ad}^{(p)}_\xi : T_1\mathbb{L}\longrightarrow T_1\mathbb{L}$ is defined as

$$\mathrm{ad}^{(p)}_\xi(\gamma) = d_\xi\left(\mathrm{Ad}^{(p)}\right)_*\Big|_1(\gamma) = -\left(R_p^{-1}\right)_*[X,Y]_p. \tag{3.25}$$

The negative sign in (3.25) is there to be consistent with the corresponding definitions for Lie groups for right-invariant vector fields. We then define the p-bracket $[\cdot,\cdot]^{(p)}$ on $T_1\mathbb{L}$ as

$$[\xi,\gamma]^{(p)} = \mathrm{ad}^{(p)}_\xi(\gamma). \tag{3.26}$$

From (3.25) it is clear that it's skew-symmetric in ξ and γ. Equivalently, we can say

$$\left[\left(R_p^{-1}\right)_* X_p, \left(R_p^{-1}\right)_* Y_p\right]^{(p)} = -\left(R_p^{-1}\right)_*[X,Y]_p. \tag{3.27}$$

Definition 3.6. The vector space $T_1\mathbb{L}$ together with the bracket $[\cdot,\cdot]^{(p)}$ is the *tangent algebra* or $\mathbb{L}$-*algebra* $\mathfrak{l}^{(p)}$ of $(\mathbb{L},\circ_p)$.

This is obviously a generalization of a Lie algebra. However, since now there is a bracket $[\cdot,\cdot]^{(p)}$ corresponding to each point $p\in\mathbb{L}$, it does not make sense to try and express $\left[[\cdot,\cdot]^{(p)},\cdot\right]^{(p)}$ in terms of Lie brackets of corresponding vector fields. Hence, the Jacobi identity for $[\cdot,\cdot]^{(p)}$ cannot be inferred, as expected. From (3.27), we cannot even infer that the bracket of two right fundamental vector fields is again a right fundamental vector field. In fact, at each point p it will be the pushforward of the bracket on $T_1\mathbb{L}$

with respect to p. Overall, we can summarize properties of the bracket in the theorem below.

Theorem 3.7. *Let $\xi, \gamma \in T_1\mathbb{L}$ and suppose $X = \rho(\xi)$ and $Y = \rho(\gamma)$ are the corresponding right fundamental vector fields on $\mathbb{L}$. Then, for any $p \in \mathbb{L}$,*

$$[\xi,\gamma]^{(p)} = \mathrm{ad}_{\xi}^{(p)}(\gamma) = \frac{d}{dt}\left(\left(\mathrm{Ad}_{\exp(t\xi)}^{(p)}\right)_* \gamma\right)\bigg|_{t=0} = -\left(R_p^{-1}\right)_* [X,Y]_p\,, \tag{3.28}$$

and moreover,

$$\begin{aligned}[\xi,\gamma]^{(p)} &= \frac{d^2}{dtd\tau}\left[\exp(t\xi),\exp(\tau\gamma)\right]^{(\mathbb{L},\circ_p)}\bigg|_{t,\tau=0} \\ &= \frac{d^2}{dtd\tau}\exp(t\xi)\circ_p\exp(\tau\gamma)\bigg|_{t,\tau=0} \\ &\quad - \frac{d^2}{dtd\tau}\exp(\tau\gamma)\circ_p\exp(t\xi)\bigg|_{t,\tau=0}.\end{aligned} \tag{3.29}$$

Here $[\cdot,\cdot]^{(p)}$ is the $\mathbb{L}$-algebra bracket on $\mathfrak{l}^{(p)}$, $[\cdot,\cdot]_p$ refers to the value of the vector field Lie bracket at $p \in \mathbb{L}$, and $[\cdot,\cdot]^{(\mathbb{L},\circ_p)}$ is the loop commutator (2.35) on $(\mathbb{L},\circ_p)$.

Proof. We have already shown (3.28), so let us prove (3.29). Recall from (2.35) that

$$\left[\exp(t\xi),\exp(\tau\gamma)\right]^{(\mathbb{L},\circ_p)} = \mathrm{Ad}_{\exp(t\xi)}^{(p)}\left(\exp(\tau\gamma)\right)/_p\exp(\tau\gamma)\,. \tag{3.30}$$

Differentiating (3.30) with respect to τ and evaluating at $\tau = 0$ using Lemma A.1 gives

$$\begin{aligned}\frac{d}{d\tau}\left[\exp(t\xi),\exp(\tau\gamma)\right]^{(\mathbb{L},\circ_p)}\bigg|_{\tau=0} &= \frac{d}{d\tau}\mathrm{Ad}_{\exp(t\xi)}^{(p)}\left(\exp(\tau\gamma)\right)\bigg|_{\tau=0} \\ &\quad - \frac{d}{d\tau}\exp(\tau\gamma)\bigg|_{\tau=0} \\ &= \left(\mathrm{Ad}_{\exp(t\xi)}^{(p)}\right)_*\gamma - \tau\end{aligned} \tag{3.31}$$

where we have also used the definition of $\exp_p$ via (3.9). This gives us the first part of (3.29). Now, using Lemma A.1 again, we can differentiate $\left(\mathrm{Ad}_{\exp(t\xi)}^{(p)}\right)_*\gamma$ with respect to t to get the second part:

$$\begin{aligned}\frac{d}{dt}\left(\left(\mathrm{Ad}_{\exp(t\xi)}^{(p)}\right)_*\gamma\right)\bigg|_{t=0} &= \frac{d^2}{dtd\tau}\left(\left(\exp(t\xi)\circ_p\exp(\tau\gamma)\right)/_p\exp(t\xi)\right)\bigg|_{t,\tau=0} \\ &= \frac{d^2}{dtd\tau}\left(\exp(t\xi)\circ_p\exp(\tau\gamma)\right)\bigg|_{t,\tau=0}\end{aligned}$$

$$- \frac{d^2}{dtd\tau} \exp(\tau\gamma) \circ_p \exp(t\xi)\bigg|_{t,\tau=0} . \quad \square$$

Remark 3.8. Applying (3.29) to the Moufang loop of unit octonions and the corresponding $\mathbb{L}$-algebra of imaginary octonions shows that as expected, the bracket on the $\mathbb{L}$-algebra coincides with the commutator of imaginary octonions in the algebra of octonions.

Although $\mathbb{L}$ and $\mathfrak{l}$ are not in general a Lie group and a Lie algebra, there are analogs of actions of these spaces on one another, which we summarize below.

Let $s \in \mathring{\mathbb{L}}$, $A \in \mathbb{L}$, and $\xi, \eta \in \mathfrak{l}$, then we have the following:

1. Action of $\mathbb{L}$ on $\mathring{\mathbb{L}}$: $A \cdot s = As$.
2. Adjoint action of $(\mathbb{L}, \circ_s)$ on $\mathbb{L}$: $A \cdot B = \mathrm{Ad}_A^{(s)}(B) = (A \circ_s B) /_s A$.
3. Action of $(\mathbb{L}, \circ_s)$ on $\mathfrak{l}$: $A \cdot \xi = \left(\mathrm{Ad}_A^{(s)}\right)_* \xi$.
4. Action of $\mathfrak{l}^{(s)}$ on itself: $\xi \cdot_s \eta = [\xi, \eta]^{(s)}$.
5. Action of $\mathfrak{l}$ on $\mathring{\mathbb{L}}$: $\xi \cdot s = (R_s)_* \xi = \frac{d}{dt} \exp_s(t\xi) s\big|_{t=0}$.

Remark 3.9. There may be some confusion about notation because we will sometimes consider the same objects but in different categories. Generally, for the loop $\mathbb{L}$, the notation "$\mathbb{L}$" will denote the underlying set, the underlying smooth manifold, the loop, and the G-set with the partial action of $\Psi^R(\mathbb{L})$. Similarly, $\mathring{\mathbb{L}}$ will denote the same underlying set, the same underlying smooth manifold, but will be different as a G-set - it has the full action of $\Psi^R(\mathbb{L})$. Since $\mathbb{L}$ and $\mathring{\mathbb{L}}$ are identical as smooth manifolds, they have the same tangent space at 1. Generally, we will only refer to $\mathring{\mathbb{L}}$ if we need to emphasize the group action. For the $\mathbb{L}$-algebra, the notation "$\mathfrak{l}$" will denote both the underlying vector space, and the vector space with the algebra structure on $T_1\mathbb{L}$ induced from the loop $\mathbb{L}$. For different values of $p \in \mathbb{L}$, $\mathfrak{l}^{(p)}$ is identical to $\mathfrak{l}$ as a vector space, but has a different algebra structure. We will use the notation $\mathfrak{l}^{(p)}$ to emphasize the algebra structure.

3.3. Structural equation

Let us now define an analog of the Maurer-Cartan form on right fundamental vector fields. Given $p \in \mathbb{L}$ and $\xi \in \mathfrak{l}$, define θ_p to be

$$\theta_p\left(\rho(\xi)_p\right) = \left(R_p^{-1}\right)_* \rho(\xi)_p = \xi. \tag{3.32}$$

Thus, this is an $\mathfrak{l}$-valued 1-form. The right fundamental vector fields still form a global frame for $T\mathbb{L}$, so this is sufficient to define the 1-form θ. Just as the right fundamental vector field $\rho(\xi)$ is generally not right-invariant, neither is θ. Indeed, let $q \in \mathbb{L}$ and consider $\left(R_q^{-1}\right)^* \theta$. Then, given $X_p = (R_p)_* \xi \in T_p\mathbb{L}$

$$\begin{aligned}\left(\left(R_q^{-1}\right)^* \theta\right)_p (X_p) &= \theta_{p/q}\left(\left(R_q^{-1} \circ R_p\right)_* \xi\right) \\ &= \left(R_{p/q}^{-1} \circ R_q^{-1} \circ R_p\right)_* \xi \\ &= \left(R_{p/q}^{-1} \circ R_{p/q}^{(q)}\right)_* \xi \end{aligned} \tag{3.33}$$

where same idea as in (3.4) was used.

Now consider $d\theta$. Generally, for a 1-form, we have

$$d\theta (X, Y) = X\theta (Y) - Y\theta (X) - \theta ([X, Y]) . \tag{3.34}$$

Suppose X, Y are right fundamental, then from (3.27), we get

$$(d\theta)_p (X, Y) - [\theta (X), \theta (Y)]^{(p)} = 0. \tag{3.35}$$

However, since right fundamental vector fields span the space of vector fields on $\mathbb{L}$, (3.35) is true for any vector fields, and we obtain the following analogue of the Maurer-Cartan equation.

Theorem 3.10. *Let $p \in \mathbb{L}$ and let $[\cdot, \cdot]^{(p)}$ be bracket on $\mathfrak{l}^{(p)}$. Then, θ satisfies the following equation at p:*

$$(d\theta)_p - \frac{1}{2} [\theta, \theta]^{(p)} = 0, \tag{3.36}$$

where $[\theta, \theta]^{(p)}$ is the bracket of $\mathbb{L}$-algebra-valued 1-forms such that for any $X, Y \in T_p\mathbb{L}$, $\frac{1}{2} [\theta, \theta]^{(p)} (X, Y) = [\theta (X), \theta (Y)]^{(p)}$.

Let $q \in \mathbb{L}$ and $\theta^{(q)} = (R_q)^* \theta$, then $\theta^{(q)}$ satisfies

$$\left(d\theta^{(q)}\right)_p - \frac{1}{2} \left[\theta^{(q)}, \theta^{(q)}\right]^{(pq)} = 0, \tag{3.37}$$

where $[\cdot, \cdot]^{(pq)}$ is the bracket on $\mathfrak{l}^{(pq)}$.

Proof. The first part already follows from (3.35). For the second part, by applying $(R_q)^*$ to (3.36) we easily see that $\theta^{(q)}$ satisfies (3.36) with the translated bracket $[\cdot, \cdot]^{(pq)}$, and hence we get (3.37). □

Remark 3.11. The 1-form $\theta^{(q)}$ can be seen as translating a vector in $T_p\mathbb{L}$ by R_q to $T_{pq}\mathbb{L}$, and then by R_{pq}^{-1} back to $\mathfrak{l}$. However, given the identity $xq/pq = x/_q p$, we see that $\theta^{(q)}$ is just the loop Maurer-Cartan form in $(\mathbb{L}, \circ_q)$.

The obvious key difference with the Lie group picture here is that the bracket in (3.36) is non-constant on $\mathbb{L}$, i.e. given a basis, the structure "constants" would vary.

In particular, the Jacobi identity is the integrability condition for the Maurer-Cartan equation on Lie groups, however here we see that the right-hand side of the Jacobi identity is related to a ternary form given by the derivative of the bracket. For any $\xi, \eta, \gamma \in \mathfrak{l}^{(p)}$, define

$$\mathrm{Jac}^{(p)}(\xi, \eta, \gamma) = \left[\xi, [\eta, \gamma]^{(p)}\right]^{(p)} + \left[\eta, [\gamma, \xi]^{(p)}\right]^{(p)} + \left[\gamma, [\xi, \eta]^{(p)}\right]^{(p)}. \tag{3.38}$$

We also need the following definition.

Definition 3.12. Define the *bracket function* $b : \mathring{\mathbb{L}} \longrightarrow \mathfrak{l} \otimes \Lambda^2 \mathfrak{l}^*$ to be the map that takes $p \mapsto [\cdot, \cdot]^{(p)} \in \mathfrak{l} \otimes \Lambda^2 \mathfrak{l}^*$, so that $b(\theta, \theta)$ is an $\mathfrak{l}$-valued 2-form on $\mathbb{L}$, i.e. $b(\theta, \theta) \in \Omega^2(\mathfrak{l})$.

Lemma 3.13 below will give the differential of b. The proof is given in Appendix A.

Lemma 3.13. *For fixed $\eta, \gamma \in \mathfrak{l}$,*

$$db|_p(\eta, \gamma) = [\eta, \gamma, \theta_p]^{(p)} - [\gamma, \eta, \theta_p]^{(p)}, \tag{3.39}$$

where $[\cdot, \cdot, \cdot]^{(p)}$ is the $\mathbb{L}$-algebra associator *on $\mathfrak{l}^{(p)}$ given by*

$$\begin{aligned}[\eta, \gamma, \xi]^{(p)} = {} & \frac{d^3}{dt d\tau d\tau'} \exp(\tau\eta) \circ_p (\exp(\tau'\gamma) \circ_p \exp(t\xi))\Big|_{t,\tau,\tau'=0} \\ & - \frac{d^3}{dt d\tau d\tau'} (\exp(\tau\eta) \circ_p \exp(\tau'\gamma)) \circ_p \exp(t\xi)\Big|_{t,\tau,\tau'=0}.\end{aligned} \tag{3.40}$$

Moreover,

$$[\eta, \gamma, \xi]^{(p)} = \frac{d^3}{dt d\tau d\tau'} [\exp(\tau\eta), \exp(\tau'\gamma), \exp(t\xi)]^{(\mathbb{L}, \circ_p)}\Big|_{t,\tau,\tau'=0} \tag{3.41}$$

where $[\cdot, \cdot, \cdot]^{(\mathbb{L}, \circ_p)}$ is the loop associator on $(\mathbb{L}, \circ_p)$ as defined by (2.34).

The skew-symmetric combination of associators, as in (3.39) will frequently occur later on, so let us define for convenience

$$a_p(\eta, \gamma, \xi) = [\eta, \gamma, \xi]^{(p)} - [\gamma, \eta, \xi]^{(p)}, \tag{3.42}$$

which we can call the *left-alternating associator*, so in particular, (3.39) becomes

$$db|_p(\eta, \gamma) = a_p(\eta, \gamma, \theta_p). \tag{3.43}$$

The loop Maurer-Cartan equation can be rewritten as

$$d\theta = \frac{1}{2} b(\theta, \theta), \tag{3.44}$$

and hence we see that $b(\theta,\theta)$ is an exact form, so in particular, $d(b(\theta,\theta)) = 0$. We will now use this to derive a generalization of the Jacobi identity.

Theorem 3.14. *The maps a and b satisfy the relation*

$$b(\theta, b(\theta,\theta)) = a(\theta,\theta,\theta), \tag{3.45}$$

where wedge products are assumed. Equivalently, if $\xi,\eta,\gamma \in \mathfrak{l}$ and $p \in \mathbb{L}$, then

$$\operatorname{Jac}^{(p)}(\xi,\eta,\gamma) = a_p(\xi,\eta,\gamma) + a_p(\eta,\gamma,\xi) + a_p(\gamma,\xi,\eta). \tag{3.46}$$

Proof. We know that $d(b(\theta,\theta)) = 0$, and thus, using (3.39) and (3.44), we have

$$\begin{aligned} 0 =& d(b(\theta,\theta)) \\ =& (db)(\theta,\theta) + b(d\theta,\theta) - b(\theta,d\theta) \\ =& a(\theta,\theta,\theta) - b(\theta,b(\theta,\theta)). \end{aligned}$$

So indeed, (3.45) holds. Now let X, Y, Z be vector fields on $\mathbb{L}$, such that $X = \rho(\xi)$, $Y = \rho(\eta)$, $Z = \rho(\gamma)$. Then, $a(\theta,\theta,\theta)_p(X,Y,Z) = 2\operatorname{Jac}^{(p)}(\xi,\eta,\gamma)$ and $\frac{1}{2}b(\theta,b(\theta,\theta))_p(X,Y,Z)$ gives the right-hand side of (3.46). □

Remark 3.15. An algebra $(A, [\cdot,\cdot], [\cdot,\cdot,\cdot])$ with a skew-symmetric bracket $[\cdot,\cdot]$ and ternary multilinear bracket $[\cdot,\cdot,\cdot]$ that satisfies (3.46) is known as an *Akivis algebra* [1,46]. If $(\mathbb{L},\circ_p)$ is left-alternative, we find from (3.40) that for any $\xi,\eta \in \mathfrak{l}$, $[\xi,\xi,\eta]^{(p)} = 0$, that is, the $\mathbb{L}$-algebra associator on $\mathfrak{l}^{(p)}$ is skew-symmetric in the first two entries, and thus $a_p = 2[\cdot,\cdot,\cdot]^{(p)}$. If the algebra is alternative, then $\operatorname{Jac}^{(p)}(\xi,\eta,\gamma) = 6[\xi,\eta,\gamma]^{(p)}$. Conversely, to an alternative Akivis algebra, there corresponds a unique local analytic alternative loop, up to local isomorphism [21]. If $(\mathbb{L},\circ_p)$ is a left Bol loop (so that it is left-alternative) then the corresponding algebra on $\mathfrak{l}^{(p)}$ will be a *Bol algebra*, where $[\cdot,\cdot]^{(p)}$ and $[\cdot,\cdot,\cdot]^{(p)}$ satisfy additional identities [1,36,42]. If $(\mathbb{L},\circ_p)$ is a Moufang loop (so in particular it is alternative), then the associator is totally skew-symmetric and the algebra on $\mathfrak{l}^{(p)}$ is then a *Malcev algebra*. It then satisfies in addition the following identity [28,32]:

$$\left[\xi,\eta,[\xi,\gamma]^{(p)}\right]^{(p)} = \left[[\xi,\eta,\gamma]^{(p)},\xi\right]^{(p)}. \tag{3.47}$$

Moreover, all non-Lie simple Malcev algebras have been classified [27] - these are either the imaginary octonions over the real number, imaginary octonions over the complex numbers, or split octonions over the real numbers.

We generally will not distinguish the notation between loop associators and $\mathbb{L}$-algebra associators. It should be clear from the context which one is being used. Moreover, it will be convenient to define mixed associators between elements of $\mathbb{L}$ and $\mathfrak{l}$. For example, an $(\mathbb{L},\mathbb{L},\mathfrak{l})$-associator is defined for any $p,q\in\mathbb{L}$ and $\xi\in\mathfrak{l}$ as

$$[p,q,\xi]^{(s)} = \left(L_p^{(s)}\circ L_q^{(s)}\right)_*\xi - \left(L_{p\circ_s q}^{(s)}\right)_*\xi \in T_{p\circ_s q}\mathbb{L} \tag{3.48}$$

and an $(\mathbb{L},\mathfrak{l},\mathfrak{l})$-associator is defined for an $p\in\mathbb{L}$ and $\eta,\xi\in\mathfrak{l}$ as

$$\begin{aligned}[p,\eta,\xi]^{(s)} = {} & \frac{d}{dtd\tau}\left(p\circ_s\left(\exp(t\eta)\circ_s\exp(\tau\xi)\right)\right)\Big|_{t,\tau=0} \\ & - \frac{d}{dtd\tau}\left(\left(p\circ_s\exp(t\eta)\right)\circ_s\exp(\tau\xi)\right)\Big|_{t,\tau=0},\end{aligned} \tag{3.49}$$

where we see that $[p,\eta,\xi]^{(s)}\in T_p\mathbb{L}$. Similarly, for other combinations.

Remark 3.16. To avoid long expressions with derivatives and exponentials, let us formally define the notation

$$p\circ_s\xi = \left(L_p^{(s)}\right)_*\xi = \frac{d}{dt}\left(p\circ_s\exp(t\xi)\right)\Big|_{t=0} \tag{3.50}$$

$$\xi\circ_s p = \left(R_p^{(s)}\right)_*\xi = \frac{d}{dt}\left(\exp(t\xi)\circ_s p\right)\Big|_{t=0} \tag{3.51}$$

$$\eta\circ_s\xi = \frac{d}{dtd\tau}\left(\exp(t\eta)\circ_s\exp(\tau\xi)\right)\Big|_{t=0} \tag{3.52}$$

for $p\in\mathbb{L}$ and $\eta,\xi\in\mathfrak{l}$, and similarly for quotients and pushforwards of tangent vectors at other points of $\mathbb{L}$. Using this convention, (3.48) and (3.49) can be written as

$$[p,q,\xi]^{(s)} = p\circ_s(q\circ_s\xi) - (p\circ_s q)\circ\xi \tag{3.53a}$$

$$[p,\eta,\xi]^{(s)} = p\circ_s(\eta\circ_s\xi) - (p\circ_s\eta)\circ\xi.$$

Let us now consider the action of loop homomorphisms on $\mathbb{L}$-algebras.

Lemma 3.17. *Suppose $\mathbb{L}_1$ and $\mathbb{L}_2$ are two smooth loops with tangent algebras at identity $\mathfrak{l}_1$ and $\mathfrak{l}_2$, respectively. Let $\alpha:\mathbb{L}_1\longrightarrow\mathbb{L}_2$ be a smooth loop homomorphism. Then, α_*: $\mathfrak{l}_1\longrightarrow\mathfrak{l}_2$ is an $\mathbb{L}$-algebra homomorphism, i.e., for any $\xi,\gamma\in\mathfrak{l}_1$,*

$$\alpha_*[\xi,\gamma]^{(1)} = [\alpha_*\xi,\alpha_*\gamma]^{(2)}, \tag{3.54}$$

where $[\cdot,\cdot]^{(1)}$ and $[\cdot,\cdot]^{(2)}$ are the corresponding brackets on $\mathfrak{l}_1$ and $\mathfrak{l}_2$, respectively. Moreover, α_* is an associator homomorphism, i.e., for any $\xi,\gamma,\eta\in\mathfrak{l}_1$,

$$\alpha_*[\xi,\gamma,\eta]^{(1)} = [\alpha_*\xi,\alpha_*\gamma,\alpha_*\eta]^{(2)} \tag{3.55}$$

where $[\cdot,\cdot,\cdot]^{(1)}$ *and* $[\cdot,\cdot,\cdot]^{(2)}$ *are the corresponding ternary brackets on* $\mathfrak{l}_1$ *and* $\mathfrak{l}_2$*, respectively.*

Proof. Suppose $\exp_{(1)}\colon \mathfrak{l}_1 \longrightarrow \mathbb{L}_1$ and $\exp_{(2)} : \mathfrak{l}_2 \longrightarrow \mathbb{L}_2$ are the corresponding exponential maps. Let $\xi, \gamma \in \mathfrak{l}_1$. We know from (2) that

$$\alpha\left(\exp_{(1)} \xi\right) = \exp_{(2)}\left(\alpha_* \xi\right). \tag{3.56}$$

From (3.29), we have

$$\begin{aligned} [\xi,\gamma]^{(1)} &= \frac{d^2}{dtd\tau} \exp_{(1)}(t\xi) \exp_{(1)}(\tau\gamma)\Big|_{t,\tau=0} \\ &\quad - \frac{d^2}{dtd\tau} \exp_{(1)}(\tau\gamma) \exp_{(1)}(t\xi)\Big|_{t,\tau=0}. \end{aligned}$$

Applying α_* to $[\xi,\gamma]^{(1)}$, we find

$$\begin{aligned} \alpha_*[\xi,\gamma]^{(1)} &= \frac{d^2}{dtd\tau} \alpha\left(\exp_{(1)}(t\xi) \exp_{(1)}(\tau\gamma)\right)\Big|_{t,\tau=0} \\ &\quad - \frac{d^2}{dtd\tau} \alpha\left(\exp_{(1)}(\tau\gamma) \exp_{(1)}(t\xi)\right)\Big|_{t,\tau=0}. \end{aligned}$$

However, since α is a loop homomorphism, and using (3.56), we have,

$$\begin{aligned} \alpha_*[\xi,\gamma]^{(1)} &= \frac{d^2}{dtd\tau} \exp_{(2)}(t\alpha_*\xi) \exp_{(1)}(\tau\alpha_*\gamma)\Big|_{t,\tau=0} \\ &\quad - \frac{d^2}{dtd\tau} \exp_{(1)}(\tau\alpha_*\gamma) \exp_{(1)}(t\alpha_*\xi)\Big|_{t,\tau=0} \\ &= [\alpha_*\xi, \alpha_*\gamma]^{(2)}. \end{aligned}$$

Similarly, using the definition (3.40) for the $\mathbb{L}$-algebra associator, we obtain (3.55). □

In particular, if $(\alpha, p) \in \Psi^R(\mathbb{L})$, then α induces an $\mathbb{L}$-algebra isomorphism $\alpha_* : (\mathfrak{l}, [\cdot,\cdot]) \longrightarrow \left(\mathfrak{l}, [\cdot,\cdot]^{(p)}\right)$. This shows that as long as p is a companion of some smooth right pseudoautomorphism, the corresponding algebras are isomorphic. More generally, we have the following.

Corollary 3.18. *Suppose* $h = (\alpha, p) \in \Psi^R(\mathbb{L})$*, and* $q \in \mathring{\mathbb{L}}$*, then, for any* $\xi, \eta, \gamma \in \mathfrak{l}$*,*

$$\alpha_*[\xi,\eta]^{(q)} = [\alpha_*\xi, \alpha_*\eta]^{h(q)} \tag{3.57a}$$

$$\alpha_*[\xi,\eta,\gamma]^{(q)} = [\alpha_*\xi, \alpha_*\eta, \alpha_*\gamma]^{h(q)}. \tag{3.57b}$$

Proof. Since $h = (\alpha, p)$ is right pseudo-automorphism of $\mathbb{L}$, by Lemma 2.29, it induces a loop homomorphism $\alpha : (\mathbb{L}, q) \longrightarrow (\mathbb{L}, h(q))$, and thus by Lemma 3.17, $\alpha_* : \mathfrak{l}^{(q)} \longrightarrow \mathfrak{l}^{(h(q))}$ is a loop algebra homomorphism. Thus (3.57) follows. □

Remark 3.19. In general, Akivis algebras are not fully defined by the binary and ternary brackets, as shown in [47]. Indeed, for a fuller picture, a more complicated structure of a *Sabinin algebra* is needed [41].

Generally, we see that $\Psi^R(\mathbb{L})$ acts on $\mathfrak{l}$ via pushforwards of the action on $\mathbb{L}$, i.e. for $h \in \Psi^R(\mathbb{L})$ and $\xi \in \mathfrak{l}$, we have $h \cdot \xi = (h')_* \xi$.

The expressions (3.57) show that the maps $b \in C^\infty\left(\mathring{\mathbb{L}}, \Lambda^2\mathfrak{l}^* \otimes \mathfrak{l}\right)$ and $a \in C^\infty\left(\mathring{\mathbb{L}}, \left(\otimes^3\mathfrak{l}^*\right) \otimes \mathfrak{l}\right)$ that correspond to the brackets are equivariant maps with respect to the action of $\Psi^R(\mathbb{L})$. Now suppose $s \in \mathring{\mathbb{L}}$, and denote $b_s = b(s) \in \Lambda^2\mathfrak{l}^* \otimes \mathfrak{l}$. Then the equivariance of b means that the stabilizer $\mathrm{Stab}_{\Psi^R(\mathbb{L})}(b_s)$ in $\Psi^R(\mathbb{L})$ of b_s is equivalent to the set of all $h \in \Psi^R(\mathbb{L})$ for which $b_{h(s)} = b_s$. In particular, $\mathrm{Stab}_{\Psi^R(\mathbb{L})}(b_s)$ is a Lie subgroup of $\Psi^R(\mathbb{L})$, and clearly $\mathrm{Aut}(\mathbb{L}, \circ_s) = \mathrm{Stab}_{\Psi^R(\mathbb{L})}(s) \subset \mathrm{Stab}_{\Psi^R(\mathbb{L})}(b_s)$. Moreover, note that if $h = (\gamma, C) \in \mathrm{Aut}(\mathbb{L}, \circ_s) \times \mathcal{N}^R(\mathbb{L}, \circ_s)$, then we still have $b_{h(s)} = b_s$. So, we can say that the corresponding subgroup $\iota_1(\mathrm{Aut}(\mathbb{L}, \circ_s)) \ltimes \iota_2\left(\mathcal{N}^R(\mathbb{L}, \circ_s)\right) \subset \Psi^R(\mathbb{L})$ is contained in $\mathrm{Stab}_{\Psi^R(\mathbb{L})}(b_s)$. Hence, as long as $\mathcal{N}^R(\mathbb{L}, \circ_s)$ is non-trivial, $\mathrm{Stab}_{\Psi^R(\mathbb{L})}(b_s)$ strictly contains $\mathrm{Aut}(\mathbb{L}, \circ_s)$. Similarly for a.

Let us now also consider how the bracket $[\cdot,\cdot]$ is transformed by $\left(\mathrm{Ad}_p^{(s)}\right)_*$.

Theorem 3.20. *Suppose $s \in \mathring{\mathbb{L}}$, $p \in \mathbb{L}$, and $\xi, \eta, \gamma \in \mathfrak{l}$. Then*

$$\begin{aligned}\left(\mathrm{Ad}_p^{(s)}\right)_* [\xi,\eta]^{(s)} = &\left[\left(\mathrm{Ad}_p^{(s)}\right)_* \xi, \left(\mathrm{Ad}_p^{(s)}\right)_* \eta\right]^{(ps)} \\ &- \left(R_p^{(s)}\right)_*^{-1} \left[\left(\mathrm{Ad}_p^{(s)}\right)_* \xi, p, \eta\right]^{(s)} + \left(R_p^{(s)}\right)_*^{-1} \left[\left(\mathrm{Ad}_p^{(s)}\right)_* \eta, p, \xi\right]^{(s)} \\ &+ \left(R_p^{(s)}\right)_*^{-1} [p,\xi,\eta]^{(s)} - \left(R_p^{(s)}\right)_*^{-1} [p,\eta,\xi]^{(s)}.\end{aligned} \tag{3.58}$$

The bracket $[\cdot,\cdot]^{(ps)}$ is related to $[\cdot,\cdot]^{(s)}$ via the expression

$$[\xi,\eta]^{(ps)} = [\xi,\eta]^{(s)} + \left(R_p^{(s)}\right)_*^{-1} a_s(\xi,\eta,p). \tag{3.59}$$

Proof. Consider

$$\begin{aligned}\left(\mathrm{Ad}_p^{(s)}\right)_* [\xi,\eta]^{(s)} = &\frac{d}{dtd\tau}\left(p \circ_s \left(\exp(t\xi) \circ_s \exp(\tau\eta)\right)\right)/_s p\Big|_{t,\tau=0} \\ &- \frac{d}{dtd\tau}\left(p \circ_s \left(\exp(t\eta) \circ_s \exp(\tau\xi)\right)\right)/_s p\Big|_{t,\tau=0}.\end{aligned} \tag{3.60}$$

For brevity and clarity, let us use the notation from Remark 3.16. Then, we can write (3.60) as

$$\left(\mathrm{Ad}_p^{(s)}\right)_* [\xi,\eta]^{(s)} = (p \circ_s (\xi \circ_s \eta))/_s p - (p \circ_s (\eta \circ_s \xi))/_s p. \tag{3.61}$$

Using mixed associators from (3.53), we can write

$$\begin{aligned}(p \circ_s (\xi \circ_s \eta))/_s p &= ((p \circ_s \xi) \circ_s \eta)/_s p + [p,\xi,\eta]^{(s)}/_s p \\ &= (((p \circ_s \xi)/_s p \circ_s p) \circ_s \eta)/_s p + [p,\xi,\eta]^{(s)}/_s p \\ &= \left(\mathrm{Ad}_p^{(s)}\xi \circ_s (p \circ_s \eta)\right)/_s p - \left[\mathrm{Ad}_p^{(s)}\xi, p, \eta\right]^{(s)}/_s p \\ &\quad + [p,\xi,\eta]^{(s)}/_s p.\end{aligned}$$

Applying (2.33), we get

$$(p \circ_s (\xi \circ_s \eta))/_s p = \mathrm{Ad}_p^{(s)}\xi \circ_{ps} \mathrm{Ad}_p^{(s)}\eta - \left[\mathrm{Ad}_p^{(s)}\xi, p, \eta\right]^{(s)}/_s p + [p,\xi,\eta]^{(s)}/_s p. \tag{3.62}$$

Subtracting the same expression with ξ and η reversed, (3.61) becomes

$$\begin{aligned}\left(\mathrm{Ad}_p^{(s)}\right)_* [\xi,\eta]^{(s)} &= \left[\left(\mathrm{Ad}_p^{(s)}\right)_*\xi, \left(\mathrm{Ad}_p^{(s)}\right)_*\eta\right]^{(ps)} \\ &\quad - \left(R_p^{(s)}\right)_*^{-1}\left[\left(\mathrm{Ad}_p^{(s)}\right)_*\xi, p, \eta\right]^{(s)} + \left(R_p^{(s)}\right)_*^{-1}\left[\left(\mathrm{Ad}_p^{(s)}\right)_*\eta, p, \xi\right]^{(s)} \\ &\quad + \left(R_p^{(s)}\right)_*^{-1}[p,\xi,\eta]^{(s)} - \left(R_p^{(s)}\right)_*^{-1}[p,\eta,\xi]^{(s)}.\end{aligned} \tag{3.63}$$

To obtain (3.59), using (3.29), we can write

$$\begin{aligned}[\xi,\eta]^{(ps)} &= \frac{d^2}{dtd\tau}\exp(t\xi) \circ_{ps} \exp(\tau\eta)\Big|_{t,\tau=0} \\ &\quad - \frac{d^2}{dtd\tau}\exp(\tau\xi) \circ_{ps} \exp(t\eta)\Big|_{t,\tau=0}.\end{aligned} \tag{3.64}$$

However, from (2.33),

$$\exp(t\xi) \circ_{ps} \exp(\tau\eta) = (\exp(t\xi) \circ_s (\exp(\tau\eta) \circ_s p))/_s p,$$

thus

$$\frac{d^2}{dtd\tau}\exp(t\xi) \circ_{ps} \exp(\tau\eta)\Big|_{t,\tau=0} = \left(R_p^{(s)}\right)_*^{-1}\frac{d^2}{dtd\tau}\exp(t\xi) \circ_s (\exp(\tau\eta) \circ_s p)\Big|_{t,\tau=0}$$

$$= \left(R_p^{(s)}\right)_*^{-1} [\xi, \eta, p]^{(s)} + \frac{d^2}{dtd\tau} \exp(t\xi) \circ_s \exp(\tau\eta)\Big|_{t,\tau=0}$$

and similarly for the second term in (3.64). Hence, we obtain (3.59). □

From (3.59) and noting that for any $h \in \Psi^R(\mathbb{L})$, $h(s) = h(s)/s \cdot s$, we find that $[\cdot,\cdot]^{(s)} = [\cdot,\cdot]^{(h(s))}$ if, and only if,

$$a_s\left(\xi, \eta, {}^{h(s)}\!/_{s}\right)^{(s)} = 0 \tag{3.65}$$

for any $\xi, \eta \in \mathfrak{l}$. From (2.40) recall that $h(s)/s$ is the companion that corresponds to h in $(\mathbb{L}, \circ_s)$.

Also, note that from (3.59), we have

$$[\theta, \theta]^{(p)} = [\theta, \theta]^{(1)} + (R_p)_*^{-1} a_1(\theta, \theta, p), \tag{3.66}$$

so the left-alternating associator with p is the obstruction for the brackets $[\cdot,\cdot]^{(p)}$ and $[\cdot,\cdot]^{(1)}$ to be equal. Moreover, the structural equation (3.36) can be rewritten as

$$d\theta - \frac{1}{2}[\theta, \theta]^{(1)} = \frac{1}{2}(R_p)_*^{-1} a_1(\theta, \theta, p). \tag{3.67}$$

This makes the dependence on the associator more explicit.

Using the associator on $\mathfrak{l}^{(p)}$ we can define the right nucleus $\mathcal{N}^R\left(\mathfrak{l}^{(p)}\right)$ of $\mathfrak{l}^{(p)}$.

Definition 3.21. Let $p \in \mathring{\mathbb{L}}$, then, the right nucleus $\mathcal{N}^R\left(\mathfrak{l}^{(p)}\right)$ is defined as

$$\mathcal{N}^R\left(\mathfrak{l}^{(p)}\right) = \{\xi \in \mathfrak{l} : a_p(\eta, \gamma, \xi) = 0 \text{ for all } \eta, \gamma \in \mathfrak{l}\}. \tag{3.68}$$

It may seem that $\mathcal{N}^R\left(\mathfrak{l}^{(p)}\right)$ could be defined more naturally as the set of all $\xi \in \mathfrak{l}$ such that $[\eta, \gamma, \xi]^{(p)} = 0$ for any $\eta, \gamma \in \mathfrak{l}$. However, the advantage of (3.68) is that, as we will see, it will always be a Lie subalgebra of $\mathfrak{l}^{(p)}$. For a left-alternative algebra, the skew-symmetrization in (3.68) would be unnecessary of course.

Theorem 3.22. *The right nucleus $\mathcal{N}^R\left(\mathfrak{l}^{(p)}\right)$ is a Lie subalgebra of $\mathfrak{l}^{(p)}$.*

Proof. We first need to show that $\mathcal{N}^R\left(\mathfrak{l}^{(p)}\right)$ is closed under $[\cdot,\cdot]^{(p)}$. Indeed, taking the exterior derivative of (3.43), for vector fields X, Y on $\mathbb{L}$ we have

$$\begin{aligned} 0 = \left(d^2 b(\beta, \gamma)\right)(X, Y) &= X(d_Y b(\beta, \gamma)) - Y(d_X b(\beta, \gamma)) - d_{[X,Y]} b(\beta, \gamma) \\ &= X(a(\beta, \gamma, \theta(Y))) - Y(a(\beta, \gamma, \theta(X))) - a(\beta, \gamma, \theta([X, Y])). \end{aligned}$$

Suppose now $\xi,\eta\in\mathfrak{l}^{(p)}$ and let $X=\rho(\xi)$, $Y=\rho(\eta)$ be the corresponding right fundamental vector fields, then using (3.26), we have

$$a(\beta,\gamma,b(\xi,\eta))=-X(a(\beta,\gamma,\eta))+Y(a(\beta,\gamma,\xi)) \tag{3.69}$$

Suppose now $\xi,\eta\in\mathcal{N}^R\left(\mathfrak{l}^{(p)}\right)$. Then, the right-hand side of (3.69) vanishes, and at $p\in\mathbb{L}$,

$$a_p\left(\beta,\gamma,[\xi,\eta]^{(p)}\right)=0, \tag{3.70}$$

and thus $[\xi,\eta]^{(p)}\in\mathcal{N}^R\left(\mathfrak{l}^{(p)}\right)$.

To conclude that $\mathcal{N}^R\left(\mathfrak{l}^{(p)}\right)$ is a Lie subalgebra, we also need to verify that Lie algebra Jacobi identity holds. That is, for any $\xi,\eta,\gamma\in\mathcal{N}^R\left(\mathfrak{l}^{(p)}\right)$, $\mathrm{Jac}^{(p)}(\xi,\eta,\gamma)=0$. Indeed, from the Akivis identity (3.46),

$$\mathrm{Jac}^{(p)}(\xi,\eta,\gamma)=a_p(\xi,\eta,\gamma)+a_p(\eta,\gamma,\xi)+a_p(\gamma,\xi,\eta)=0, \tag{3.71}$$

by definition of $\mathcal{N}^R\left(\mathfrak{l}^{(p)}\right)$. □

For any smooth loop, consider the loop right nucleus $\mathcal{N}^R(\mathbb{L},\circ_p)$ as a submanifold of $\mathbb{L}$. Then,

$$T_1\mathcal{N}^R(\mathbb{L},\circ_p)=\left\{\xi\in\mathfrak{l}:[q,r,\xi]^{(p)}=0\text{ for all }q,r\in\mathbb{L}\right\}, \tag{3.72}$$

where here we are using the mixed associator as defined by (3.48). Then, (3.41) implies that $T_1\mathcal{N}^R(\mathbb{L},\circ_p)\subset\mathcal{N}^R\left(\mathfrak{l}^{(p)}\right)$. It is unclear what are the conditions for the converse, and hence equality, of the two spaces.

Recall from (2.43b) that $A\in\mathcal{N}^R(\mathbb{L})$ if, and only if, $\mathrm{Ad}_p(A)\in\mathcal{N}^R(\mathbb{L},\circ_p)$, so in particular, $\eta\in T_1\mathcal{N}^R(\mathbb{L})$ if, and only if, $(\mathrm{Ad}_p)_*\eta\in T_1\mathcal{N}^R(\mathbb{L},\circ_p)$. In (3.58) we then see that for $\eta,\gamma\in T_1\mathcal{N}^R(\mathbb{L})$, the associators vanish, and we get

$$(\mathrm{Ad}_p)_*[\eta,\gamma]=\left[(\mathrm{Ad}_p)_*\eta,(\mathrm{Ad}_p)_*\gamma\right]^{(p)}. \tag{3.73}$$

Hence, for each $p\in\mathring{\mathbb{L}}$, $T_1\mathcal{N}^R(\mathbb{L})\cong T_1\mathcal{N}^R(\mathbb{L},\circ_p)$ as Lie algebras.

Example 3.23. Consider the Moufang loop of unit octonions $U\mathbb{O}$. Then, $T_1U\mathbb{O}\cong\operatorname{Im}\mathbb{O}$ - the space of imaginary octonions, with the bracket given by the commutator on $\operatorname{Im}\mathbb{O}$: for any $\xi,\eta\in\operatorname{Im}\mathbb{O}$, $[\xi,\eta]=\xi\eta-\eta\xi$. We also know that $\mathcal{N}(U\mathbb{O})\cong\mathbb{Z}_2$ and $\mathcal{N}(\operatorname{Im}\mathbb{O})=\{0\}$. On the other hand, taking a direct product $G\times U\mathbb{O}$ with any Lie group G will give a non-trivial nucleus.

Let $s\in\mathring{\mathbb{L}}$. Suppose the Lie algebras of $\Psi^R(\mathbb{L})$ and $\mathrm{Aut}(\mathbb{L},\circ_s)$ are $\mathfrak{p}$ and $\mathfrak{h}_s$, respectively. In particular, $\mathfrak{h}_s$ is a Lie subalgebra of $\mathfrak{p}$. Define $\mathfrak{q}_s=T_1\mathcal{C}^R(\mathbb{L},\circ_s)$, then since

$\mathcal{C}^R(\mathbb{L}, \circ_s) \subset \mathbb{L}$, so $\mathfrak{q}_s \subset \mathfrak{l}^{(s)} \cong T_1\mathbb{L}$. On the other hand, $\mathcal{C}^R(\mathbb{L}, \circ_s) \cong {}^{\Psi^R(\mathbb{L})}\!\diagup_{\mathrm{Aut}(\mathbb{L}, \circ_s)}$, and the tangent space at the coset $1 = \lfloor \mathrm{Aut}(\mathbb{L}, \circ_s) \rfloor$ is $\mathfrak{p}/\mathfrak{h}_s$. Hence, we see that $\mathfrak{q}_s \cong \mathfrak{p}/\mathfrak{h}_s$, at least as vector spaces. The groups $\Psi^R(\mathbb{L})$ and $\mathrm{Aut}(\mathbb{L}, \circ_s)$ act on $\mathfrak{p}$ and $\mathfrak{h}_s$ via their respective adjoint actions and hence $\mathrm{Aut}(\mathbb{L}, \circ_s)$ acts on $\mathfrak{q}_s$ via a restriction of the adjoint action of $\Psi^R(\mathbb{L})$. Now note that given $h = (\alpha, A) \in \Psi^R(\mathbb{L})$ and $\beta \in \mathrm{Aut}(\mathbb{L}, \circ_s)$, the conjugation of h by β is given by

$$(\beta, 1)(\alpha, A)\left(\beta^{-1}, 1\right) = \left(\beta \circ \alpha \circ \beta^{-1}, \beta(A)\right)$$

and hence the corresponding action on the companion A is via standard action of β on $\mathbb{L}$. The differentials of these actions give the corresponding actions on the tangent spaces. We thus see that the adjoint action of $\mathrm{Aut}(\mathbb{L}, \circ_s)$ on $\mathfrak{p}/\mathfrak{h}_s$ is equivalent to the standard tangent action of $\mathrm{Aut}(\mathbb{L}, \circ_s)$ on $\mathfrak{q}_s$. Hence, $\mathfrak{q}_s$ and $\mathfrak{p}/\mathfrak{h}_s$ are isomorphic as linear representations of $\mathrm{Aut}(\mathbb{L}, \circ_s)$. We can make the isomorphism from $\mathfrak{p}/\mathfrak{h}_s$ to $\mathfrak{q}_s$ more explicit in the following way.

Definition 3.24. Define the map $\varphi : \mathring{\mathbb{L}} \longrightarrow \mathfrak{l} \otimes \mathfrak{p}^*$ such that for each $s \in \mathring{\mathbb{L}}$ and $\gamma \in \mathfrak{p}$,

$$\varphi_s(\gamma) = \frac{d}{dt}\left(\exp(t\gamma)(s)\right)\diagup s\Big|_{t=0} \in \mathfrak{l}. \tag{3.74}$$

Thus, for each $s \in \mathring{\mathbb{L}}$, φ_s gives a map from $\mathfrak{p}$ to $\mathfrak{l}^{(s)}$.

Theorem 3.25. *The map φ as in (3.74) is equivariant with respect to corresponding actions of $\Psi^R(\mathbb{L})$, in particular for $h \in \Psi^R(\mathbb{L})$, $s \in \mathring{\mathbb{L}}$, $\gamma \in \mathfrak{p}$, we have*

$$\varphi_{h(s)}\left((\mathrm{Ad}_h)_* \gamma\right) = (h')_* \varphi_s(\gamma). \tag{3.75}$$

Moreover, the image of φ_s is $\mathfrak{q}_s$ and the kernel is $\mathfrak{h}_s$, and hence,

$$\mathfrak{p} \cong \mathfrak{h}_s \oplus \mathfrak{q}_s. \tag{3.76}$$

Proof. Consider $h \in \Psi^R(\mathbb{L})$. Then, using (2.15b), we have

$$\begin{aligned}
\varphi_{h(s)}(\gamma) &= \frac{d}{dt}\left[\exp(t\gamma)(h(s))\right]\diagup h(s)\Big|_{t=0} \\
&= \frac{d}{dt} h'\left[\mathrm{Ad}_{h^{-1}}(\exp(t\gamma))(s)\diagup s\right]\Big|_{t=0} \\
&= (h')_* \frac{d}{dt}\exp\left(t\left(\mathrm{Ad}_{h^{-1}}\right)_* \gamma\right)(s)\diagup s\Big|_{t=0}.
\end{aligned}$$

Since $\Psi^R(\mathbb{L})$ acts on $\mathfrak{l}$ via $(h')_*$ and on $\mathfrak{p}$ via $(\mathrm{Ad}_h)_*$ we see that φ is equivariant.

Since $\operatorname{Aut}(\mathbb{L}, \circ_s)$ is a Lie subgroup of $\Psi^R(\mathbb{L})$, the projection map $\pi : \Psi^R(\mathbb{L}) \longrightarrow \Psi^R(\mathbb{L}) / \operatorname{Aut}(\mathbb{L}, \circ_s) \cong \mathcal{C}^R(\mathbb{L}, \circ_s)$ is a smooth submersion given by $\pi(h) = h(s)/s$ for each $h \in \Psi^R(\mathbb{L})$. Thus, $\pi_*|_{\mathrm{id}} : \mathfrak{p} \longrightarrow \mathfrak{q}_s$ is surjective. However, since exp is a surjective map from $\mathfrak{p}$ to a neighborhood of id $\in \Psi^R(\mathbb{L})$, we find that $\pi_*|_{\mathrm{id}}(\gamma) = \varphi_s(\gamma)$. So indeed, the image of the map φ_s is $\mathfrak{q}_s$. Clearly the kernel is $\mathfrak{h}_s$. Then, (3.76) follows immediately. □

Theorem 3.25 implies that $\varphi : \mathring{\mathbb{L}} \longrightarrow \mathfrak{l} \otimes \mathfrak{p}^*$ is equivariant with respect to the action of $\Psi^R(\mathbb{L})$, and similarly as for b, we can define $\operatorname{Stab}_{\Psi^R(\mathbb{L})}(\varphi_s) = \{h \in \Psi^R(\mathbb{L}) : \varphi_{h(s)} = \varphi_s\}$. This is then a Lie subgroup of $\Psi^R(\mathbb{L})$, and $\operatorname{Aut}(\mathbb{L}, \circ_s) \subset \operatorname{Stab}_{\Psi^R(\mathbb{L})}(\varphi^{(s)})$. Suppose $h = (\alpha, A) \in \operatorname{Stab}_{\Psi^R(\mathbb{L})}(\varphi^{(s)})$, then

$$\varphi_s(\gamma) = \varphi_{h(s)}(\gamma) = \frac{d}{dt}\left[\exp(t\gamma)(\alpha(s)A)\right] / (\alpha(s)A)\Big|_{t=0}$$

We can also see the effect on φ of left multiplication of s by elements of $\mathbb{L}$.

Lemma 3.26. *Suppose $A \in \mathbb{L}$ and $s \in \mathring{\mathbb{L}}$, then for any $\gamma \in \mathfrak{p}$,*

$$\varphi_{As}(\gamma) = \left(R_A^{(s)}\right)_*^{-1}(\gamma' \cdot A) + \left(\operatorname{Ad}_A^{(s)}\right)_* \varphi_s(\gamma), \tag{3.77}$$

where $\gamma' \cdot A = \frac{d}{dt}(\exp t\gamma)'(A)\big|_{t=0}$ represents the infinitesimal action of $\mathfrak{p}$ on $\mathbb{L}$.

Proof. This follows from a direct computation:

$$\begin{aligned}
\varphi_{As}(\gamma) &= \frac{d}{dt}\exp(t\gamma)(As)/As\Big|_{t=0} \\
&= \frac{d}{dt}\left[\exp(t\gamma)'(A)\exp(t\gamma)(s)\right]/As\Big|_{t=0} \\
&= \frac{d}{dt}\left[A\exp(t\gamma)(s)\right]/As\Big|_{t=0} + \frac{d}{dt}\left(\left[\exp(t\gamma)'(A)\right]s\right)/As\Big|_{t=0} \\
&= \left(\operatorname{Ad}_A^{(s)}\right)_* \varphi_s(\gamma) + \left(R_A^{(s)}\right)_*^{-1}(\gamma' \cdot A),
\end{aligned}$$

where we have used (2.29a). □

Example 3.27. If $\mathbb{L}$ is the loop of unit octonions, then we know $\mathfrak{p} \cong \mathfrak{so}(7) \cong \Lambda^2(\mathbb{R}^7)^*$ and $\mathfrak{l} \cong \mathbb{R}^7$, so φ_s can be regarded as an element of $\mathbb{R}^7 \otimes \Lambda^2\mathbb{R}^7$, and this is precisely a dualized version of the G_2-invariant 3-form φ. The kernel is isomorphic to $\mathfrak{g}_2$.

Example 3.28. Suppose $\mathbb{L} = U\mathbb{C} \cong S^1$ - the unit complex numbers, so that $\mathfrak{l} \cong \mathbb{R}$. From Example 2.24, we may take $\Psi_n^R(U\mathbb{C}) = U(n)$, with a trivial partial action on $U\mathbb{C}$. The corresponding Lie algebra is $\mathfrak{p}_n \cong \mathfrak{u}(n) \cong \mathfrak{su}(n) \oplus i\mathbb{R}$. The map $\varphi_s : \mathfrak{p}_n \longrightarrow i\mathbb{R}$ is then just the projection $\mathfrak{su}(n) \oplus i\mathbb{R} \longrightarrow i\mathbb{R}$ (i.e. trace). It is independent of s. The kernel is $\mathfrak{su}(n)$. Suppose V is a n-dimensional real vector space, and $V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$. Then, the group $U(n)$ acts via unitary transformations on the complex vector space $V^{1,0}$, and correspondingly $\mathfrak{u}(n) \cong V^{1,1}$ (i.e. the space of $(1,1)$-forms). Then, we see that φ_s is just the dualized version of a Hermitian form on $V \otimes \mathbb{C}$.

Example 3.29. Suppose $\mathbb{L} = U\mathbb{H} \cong S^3$ - the unit quaternions, so that $\mathfrak{l} \cong \mathfrak{sp}(1)$. From Example 2.24, we may take $\Psi_n^R(U\mathbb{H}) = Sp(n)\,Sp(1)$, with $n \geq 2$, with a trivial partial action on $U\mathbb{H}$. The corresponding Lie algebra is $\mathfrak{p}_n \cong \mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$. The map φ_s : $\mathfrak{p}_n \longrightarrow \mathfrak{sp}(1)$ is then given by $(a, \xi) \mapsto (\mathrm{Ad}_s)_* \xi$. The kernel is then $\mathfrak{sp}(n)$. Suppose $Sp(n)\,Sp(1)$ acts on a $4n$-dimensional real vector space V, $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1) \subset \Lambda^2 V^*$. Given that $\mathfrak{sp}(1) \cong \operatorname{Im}\mathbb{H}$, we can then write $\varphi_s = i\omega_1^* + j\omega_2^* + k\omega_3^*$, where the ω_i^* are dualized versions of the 3 linearly independent Hermitian forms that span the $\mathfrak{sp}(1)$ subspace of $\Lambda^2 V^*$ [44].

Remark 3.30. The above examples clearly show that one interpretation of the G_2-structure 3-form φ is as $\operatorname{Im}\mathbb{O}$-valued 2-form. A complex Hermitian form is then an $\operatorname{Im}\mathbb{C}$-valued 2-form, and a quaternionic Hermitian form is an $\operatorname{Im}\mathbb{H}$-valued 2-form.

Now let us summarize the actions of different spaces on one another. For a fixed γ, define the map $\hat{\gamma} : \mathring{\mathbb{L}} \longrightarrow \mathfrak{l}$ given by $s \mapsto \hat{\gamma}^{(s)} = \varphi_s(\gamma)$.

Theorem 3.31. *Suppose $\mathbb{L}$ is a smooth loop with tangent algebra $\mathfrak{l}$ and suppose $\Psi^R(\mathbb{L})$ is a Lie group with Lie algebra $\mathfrak{p}$. Let $A \in \mathbb{L}$, $s \in \mathring{\mathbb{L}}$, $\xi \in \mathfrak{l}$, and $\gamma \in \mathfrak{p}$. Then, denoting by $\cdot$ the relevant action, we have the following:*

1. *Infinitesimal action of $\mathfrak{p}$ on $\mathring{\mathbb{L}}$:*

$$\gamma \cdot s = \frac{d}{dt} \exp(t\gamma)(s)\Big|_{t=0} = (R_s)_* \hat{\gamma}^{(s)} \in T_s\mathbb{L} \tag{3.78}$$

2. *Infinitesimal action of $\mathfrak{p}$ on $\mathbb{L}$, for any $s \in \mathring{\mathbb{L}}$:*

$$\gamma \cdot A = \frac{d}{dt} \exp(t\gamma)'(A)\Big|_{t=0} = \left(R_A^{(s)}\right)_* \hat{\gamma}^{(As)} - \left(L_A^{(s)}\right)_* \hat{\gamma}^{(s)} \in T_A\mathbb{L}. \tag{3.79}$$

   In particular, if $s = 1$,

$$\gamma \cdot A = (R_A)_* \hat{\gamma}^{(A)} - (L_A)_* \hat{\gamma}^{(1)}. \tag{3.80}$$

3. *Action of $\mathfrak{p}$ on $\mathfrak{l}$ for any $s \in \mathring{\mathbb{L}}$:*

$$\gamma \cdot \xi = \frac{d}{dt} \left(\exp(t\gamma)'\right)_* (\xi)\Big|_{t=0}$$

$$= d\hat{\gamma}|_s (\rho_s (\xi)) + \left[\hat{\gamma}^{(s)}, \xi\right]^{(s)}. \tag{3.81}$$

In particular, for $s = 1$, we have

$$\gamma \cdot \xi = d\hat{\gamma}|_1 (\xi) + \left[\hat{\gamma}^{(1)}, \xi\right]. \tag{3.82}$$

Proof. Let $A, B \in \mathbb{L}$, $s \in \mathring{\mathbb{L}}$, $\xi, \eta \in \mathfrak{l}$, $h \in \Psi^R (\mathbb{L})$, and $\gamma \in \mathfrak{p}$. Then we have the following.

1. The infinitesimal action of a Lie algebra is a standard definition.
2. Consider now the action of $\mathfrak{p}$ on $\mathbb{L}$. Suppose now $\gamma \in \mathfrak{p}$ and $A \in \mathbb{L}$

$$\gamma' \cdot A = \frac{d}{dt} \left(\exp (t\gamma)'\right) (A)\Big|_{t=0}. \tag{3.83}$$

Suppose $h \in \Psi^R (\mathbb{L}, \circ_s)$, then by (2.40), the action of h on $A \in \mathbb{L}$ is

$$h (A) = h' (A) \circ_s \left(h (s) /s\right)$$

Thus, the partial action $h' (A)$ is given by

$$h' (A) = \left(h (As) /s\right) /_s \left(h (s) /s\right). \tag{3.84}$$

Moreover,

$$h (As) /s = \left(h (As) /As\right) \circ_s A. \tag{3.85}$$

Hence, substituting into (3.83), we have

$$\begin{aligned} \gamma' \cdot A &= \frac{d}{dt} \left(\exp (t\gamma (As)) /As \circ_s A\right) /_s \left(\exp (t\gamma) (s) /s\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(\exp (t\gamma (As)) /As \circ_s A\right)\Big|_{t=0} - \frac{d}{dt} A \circ_s \left(\exp (t\gamma) (s) /s\right)\Big|_{t=0} \\ &= \left(R_A^{(s)}\right)_* \hat{\gamma}^{(As)} - \left(L_A^{(s)}\right)_* \hat{\gamma}^{(s)}. \end{aligned} \tag{3.86}$$

Setting $s = 1$ immediately gives (3.80).

3. Suppose now $\gamma \in \mathfrak{p}$ and $\xi \in \mathfrak{l}$, then we have

$$\begin{aligned} \gamma \cdot \xi &= \frac{d}{dt} \left(\exp (t\gamma)'\right)_* (\xi)\Big|_{t=0} \\ &= \frac{d^2}{dtd\tau} \exp (t\gamma)' (\exp_s \tau\xi)\Big|_{t,\tau=0}. \end{aligned} \tag{3.87}$$

Let $\Xi = \exp_s \tau\xi \in \mathbb{L}$, then using (3.84) and (3.85), we can write

$$\exp(t\gamma)'(\exp_s \tau\xi) = \exp(t\gamma)'(\Xi)$$
$$(\exp(t\gamma)(\Xi s/\Xi s \circ_s \Xi))/_s \left(\exp(t\gamma)(s)/s\right).$$

Using this, (3.87) becomes

$$\begin{aligned}
\gamma' \cdot \xi &= \frac{d^2}{dtd\tau}(\exp(t\gamma)((\exp_s \tau\xi)s))/((\exp_s \tau\xi)s \circ_s \exp_s \tau\xi)\Big|_{t,\tau=0} \\
&\quad - \frac{d^2}{dtd\tau}\exp_s \tau\xi \circ_s \left(\exp(t\gamma)(s)/s\right)\Big|_{t,\tau=0} \\
&= \frac{d^2}{dtd\tau}\exp(t\gamma)((\exp_s \tau\xi)s)/(\exp_s \tau\xi)s\Big|_{t,\tau=0} + \\
&\quad + \frac{d^2}{dtd\tau}\left(\exp(t\gamma)(s)/s\right)\circ_s \exp_s \tau\xi\Big|_{t,\tau=0} \\
&\quad - \frac{d^2}{dtd\tau}\exp_s \tau\xi \circ_s \left(\exp(t\gamma)(s)/s\right)\Big|_{t,\tau=0}
\end{aligned} \tag{3.88}$$

However $\hat{\gamma}^{(s)} = \frac{d}{dt}\exp(t\gamma)(s)/s\big|_{t=0} \in \mathfrak{l}$, and thus

$$\begin{aligned}
\frac{d}{d\tau}\left(L^{(s)}_{\exp_s \tau\xi}\right)_* \hat{\gamma}^{(s)}\Big|_{\tau=0} &= \frac{d^2}{dtd\tau}(\exp_s \tau\xi)\circ_s \exp_s\left(t\hat{\gamma}^{(s)}\right)\Big|_{t,\tau=0} \\
\frac{d}{d\tau}\left(R^{(s)}_{\exp_s \tau\xi}\right)_* \hat{\gamma}^{(s)}\Big|_{\tau=0} &= \frac{d^2}{dtd\tau}\exp_s\left(t\hat{\gamma}^{(s)}\right)\circ_s \exp_s \tau\xi\Big|_{t,\tau=0}.
\end{aligned}$$

Hence, using the expression (3.29) for $[\cdot,\cdot]^{(s)}$, we get

$$\gamma' \cdot \xi = \frac{d}{d\tau}\hat{\gamma}^{(\exp_s \tau\xi)s}\Big|_{\tau=0} + \left[\hat{\gamma}^{(s)}, \xi\right]^{(s)}. \tag{3.89}$$

The first term in (3.89) is then precisely the differential of $\hat{\gamma}$ at $s \in \mathbb{L}$ in the direction $(R_s)_* \xi$. Setting $s = 1$ we get (3.82). □

Remark 3.32. Since the full action of $\Psi^R(\mathbb{L})$ does not preserve 1, the pushforward of the action of some $h \in \Psi^R(\mathbb{L})$ sends $T_1\mathbb{L}$ to $T_A\mathbb{L}$, where $A = h(1)$ is the companion of $\mathbb{L}$. To actually obtain an action on $T_1\mathbb{L}$, translation back to 1 is needed. This can be achieved either by right or left division by A. Dividing by A on the right reduces to the partial action of $\Psi^R(\mathbb{L})$, i.e. action by h'. This is how the action of $\mathfrak{p}$ on $\mathfrak{l}$ in (3.81) is defined. Dividing by A on the left, gives the map $h'' = \mathrm{Ad}_{A^{-1}} \circ h'$, as defined in (2.49). In that setting, it was defined on the nucleus, and hence gave an actual group action of $\Psi^R(\mathbb{L})$, however in a non-associative setting, in general this will not be a group action.

Combining some of the above results, we also have the following useful relationship.

Lemma 3.33. *Suppose $\xi \in \mathfrak{p}$ and $\eta, \gamma \in \mathfrak{l}$, then*

$$\xi \cdot [\eta, \gamma]^{(s)} = [\xi \cdot \eta, \gamma]^{(s)} + [\eta, \xi \cdot \gamma]^{(s)} + a_s\left(\eta, \gamma, \varphi_s\left(\xi\right)\right). \tag{3.90}$$

Proof. Using the definition (3.81) of the action of $\mathfrak{p}$ on $\mathfrak{l}$, we have

$$\begin{aligned}\xi \cdot [\eta, \gamma]^{(s)} &= \frac{d}{dt}\left(\exp\left(t\xi\right)'\right)_* [\eta, \gamma]^{(s)}\Big|_{t=0} \\ &= \frac{d}{dt}\left[\left(\exp\left(t\xi\right)'\right)_* \eta, \left(\exp\left(t\xi\right)'\right)_* \gamma\right]^{\exp(t\xi)(s)}\Big|_{t=0}\end{aligned}$$

where we have also used (3.57a). Hence,

$$\xi \cdot [\eta, \gamma]^{(s)} = [\xi \cdot \eta, \gamma]^{(s)} + [\eta, \xi \cdot \gamma]^{(s)} + \frac{d}{dt}[\eta, \gamma]^{\exp(t\xi)(s)}\Big|_{t=0}. \tag{3.91}$$

We can rewrite the last term in (3.91) as

$$\frac{d}{dt}[\eta, \gamma]^{\exp(t\xi)(s)}\Big|_{t=0} = \frac{d}{dt}[\eta, \gamma]^{\exp_s(t\varphi_s(\xi))s}\Big|_{t=0} = d_{\rho\left(\hat{\xi}\right)}b\Big|_s (\eta, \gamma)$$

where $\hat{\xi} = \varphi_s\left(\xi\right)$. Then, from (3.39), we see that

$$d_{\rho\left(\hat{\xi}\right)}b\Big|_s (\eta, \gamma) = a_s\left(\eta, \gamma, \hat{\xi}\right) \tag{3.92}$$

and overall, we obtain (3.90). □

Recall that for each $s \in \mathring{\mathbb{L}}$, the bracket function b_s is in $\Lambda^2\mathfrak{l}^* \otimes \mathfrak{l}$, which is a tensor product of $\mathfrak{p}$-modules, so (3.90) can be used to define the action of $\xi \in \mathfrak{p}$ on b_s. Using the derivation property of Lie algebra representations on tensor products, we find that for $\eta, \gamma \in \mathfrak{l}$,

$$\begin{aligned}\left(\xi \cdot b_s\right)(\eta, \gamma) =& \xi \cdot \left(b_s\left(\eta, \gamma\right)\right) - b_s\left(\xi \cdot \eta, \gamma\right) - b_s\left(\eta, \xi \cdot \gamma\right) \\ =& a_s\left(\eta, \gamma, \varphi_s\left(\xi\right)\right).\end{aligned} \tag{3.93}$$

Definition 3.34. Suppose $\mathfrak{g}$ is a Lie algebra with a representation on a vector space M, so that $(M, \mathfrak{g})$ is a $\mathfrak{g}$-module. Then if $x \in M$, define the *annihilator subalgebra* $\mathrm{Ann}_{\mathfrak{g}}(x)$ in $\mathfrak{g}$ of x as

$$\mathrm{Ann}_{\mathfrak{g}}\left(x\right) = \left\{\xi \in \mathfrak{g} : \xi \cdot x = 0\right\}. \tag{3.94}$$

From (3.93), we see that

$$\mathrm{Ann}_{\mathfrak{p}}(b_s) = \{\xi \in \mathfrak{p} : a_s(\eta, \gamma, \varphi_s(\xi)) = 0 \text{ for all } \eta, \gamma \in \mathfrak{l}\}. \tag{3.95}$$

The definition (3.95) is simply that $\xi \in \mathrm{Ann}_{\mathfrak{p}}(b_s)$ if, and only if, $\varphi_s(\xi) \in \mathcal{N}^R(\mathfrak{l}^{(s)})$, so that $\mathrm{Ann}_{\mathfrak{p}}(b_s) = \varphi_s^{-1}(\mathcal{N}^R(\mathfrak{l}^{(s)}))$. This is the Lie algebra that corresponds to the Lie group $\mathrm{Stab}_{\Psi^R(\mathbb{L})}(b_s)$. Indeed, the condition (3.95) is precisely the infinitesimal version of (3.65). If $\mathbb{L}$ is a G-loop, so that $\varphi_s(\mathfrak{p}) = \mathfrak{l}^{(s)}$, then $\varphi_s(\mathrm{Ann}_{\mathfrak{p}}(b_s)) = \mathcal{N}^R(\mathfrak{l}^{(s)})$. Hence, in this case, $\mathrm{Ann}_{\mathfrak{p}}(b_s) \cong \mathfrak{h}_s \oplus \mathcal{N}^R(\mathfrak{l}^{(s)})$.

Using the definition (3.74) of φ_s, let us consider the action of $\mathfrak{p}$ on φ_s.

Lemma 3.35. *Suppose $\xi, \eta \in \mathfrak{p}$, then for any $s \in \mathbb{L}$, we have*

$$\xi \cdot \varphi_s(\eta) - \eta \cdot \varphi_s(\xi) = \varphi_s\left([\xi, \eta]_{\mathfrak{p}}\right) + [\varphi_s(\xi), \varphi_s(\eta)]^{(s)}, \tag{3.96}$$

where $\cdot$ means the action of $\mathfrak{p}$ on $\mathfrak{l}$.

Proof. Using (3.81) and the definition (3.74) of φ_s, we have

$$\begin{aligned}
\xi \cdot \varphi_s(\eta) &= \frac{d^2}{dtd\tau} \exp(t\xi)' \left(\exp(\tau\eta)(s) / s\right)\Big|_{t,\tau=0} \\
&= \frac{d^2}{dtd\tau} \exp(t\xi)(\exp(\tau\eta)(s)) / \exp(t\xi)(s)\Big|_{t,\tau=0} \\
&= \frac{d^2}{dtd\tau} \exp(t\xi)(\exp(\tau\eta)(s)) / s\Big|_{t,\tau=0} \\
&\quad - \frac{d^2}{dtd\tau} \left(\exp(\tau\eta)(s) / s \cdot \exp(t\xi)(s)\right) / s\Big|_{t,\tau=0} \\
&= \frac{d^2}{dtd\tau} (\exp(t\xi)\exp(\tau\eta))(s) / s\Big|_{t,\tau=0} \\
&\quad - \frac{d^2}{dtd\tau} \exp(\tau\eta)(s) / s \circ_s \exp(t\xi)(s) / s\Big|_{t,\tau=0},
\end{aligned} \tag{3.97}$$

where we have used (2.15b) and Lemma A.1. Now subtracting the same expression but with ξ and η switched around, we obtain (3.96). □

Remark 3.36. In terms of the Chevalley-Eilenberg complex of $\mathfrak{p}$ with values in $\mathfrak{l}$, the relation (3.96) shows that if we regard $\varphi_s \in C^1(\mathfrak{p}; \mathfrak{l})$, i.e. a 1-form on $\mathfrak{p}$ with values in $\mathfrak{l}$, then the Chevalley-Eilenberg differential d_{CE} of φ_s is given by

$$(d_{CE}\varphi_s)(\xi, \eta) = [\varphi_s(\xi), \varphi_s(\eta)]^{(s)} \tag{3.98}$$

for any $\xi, \eta \in \mathfrak{p}$. It is interesting that, at least on $\mathfrak{q}_s$, the bracket $[\cdot, \cdot]^{(s)}$ corresponds to an exact 2-cochain.

Similarly, from (3.96), we then see that the action of $\xi \in \mathfrak{p}$ on φ_s as an $\mathfrak{p}^* \otimes \mathfrak{l}$-valued map. Indeed, given $\xi, \eta \in \mathfrak{p}$, we have

$$\begin{aligned}(\xi \cdot \varphi_s)(\eta) =& \xi \cdot \varphi_s(\eta) - \varphi_s\left([\xi, \eta]_{\mathfrak{p}}\right) \\ =& \eta \cdot \varphi_s(\xi) - [\varphi_s(\eta), \varphi_s(\xi)]^{(s)} \end{aligned} \tag{3.99}$$

where we have first used the fact that $\mathfrak{p}$ acts on itself via the adjoint representation and then (3.96) in the second line.

Let us now consider $\operatorname{Ann}_{\mathfrak{p}}(\varphi_s)$. From (3.99), we see that we have two equivalent characterizations of $\operatorname{Ann}_{\mathfrak{p}}(\varphi_s)$. In particular, $\xi \in \operatorname{Ann}_{\mathfrak{p}}(\varphi_s)$ if, and only if,

$$\xi \cdot \hat{\eta} = \varphi_s\left([\xi, \eta]_{\mathfrak{p}}\right) \tag{3.100}$$

or equivalently, for $\xi \notin \mathfrak{h}_s$, if, and only if,

$$\eta \cdot \hat{\xi} = \left[\hat{\eta}, \hat{\xi}\right]^{(s)}, \tag{3.101}$$

for any $\eta \in \mathfrak{p}$. Here we are again setting $\hat{\xi} = \varphi_s(\xi)$ and $\hat{\eta} = \varphi_s(\eta)$. In particular, (3.100) shows that $\mathfrak{q}_s$ is a representation of $\operatorname{Ann}_{\mathfrak{p}}(\varphi_s)$. Suppose now, $\xi_1, \xi_2 \in \operatorname{Ann}_{\mathfrak{p}}(\varphi_s)$, then using (3.100) and (3.101), we find that

$$\varphi_s\left([\xi_1, \xi_2]_{\mathfrak{p}}\right) = \xi_1 \cdot \hat{\xi}_2 = \left[\hat{\xi}_1, \hat{\xi}_2\right]^{(s)}. \tag{3.102}$$

Therefore, $\varphi_s(\operatorname{Ann}_{\mathfrak{p}}(\varphi_s))$ is a Lie subalgebra of $\mathfrak{l}^{(s)}$ with φ_s being a Lie algebra homomorphism. The kernel $\mathfrak{h}_s = \ker \varphi_s$ is then of course an ideal of $\operatorname{Ann}_{\mathfrak{p}}(\varphi_s)$. Thus, the quotient $\operatorname{Ann}_{\mathfrak{p}}(\varphi_s)/\mathfrak{h}_s$ is again a Lie algebra, and hence $\operatorname{Ann}_{\mathfrak{p}}(\varphi_s)$ is a trivial Lie algebra extension of $\mathfrak{h}_s$. Moreover, note that the Lie algebra $\operatorname{Ann}_{\mathfrak{p}}(\varphi_s)$ corresponds to the Lie group $\operatorname{Stab}_{\Psi^R(\mathbb{L})}(\varphi_s)$, and thus if $\operatorname{Aut}(\mathbb{L}, \circ_s)$ and $\operatorname{Stab}_{\Psi^R(\mathbb{L})}(\varphi_s)$ are both connected, then we see that $\operatorname{Aut}(\mathbb{L}, \circ_s)$ is a normal subgroup of $\operatorname{Stab}_{\Psi^R(\mathbb{L})}(\varphi_s)$.

In the special case when $\mathbb{L}$ is a G-loop, we get a nice property of $\operatorname{Ann}_{\mathfrak{p}}(\varphi_s)$.

Theorem 3.37. *Suppose $\mathbb{L}$ is a G-loop, then $\operatorname{Ann}_{\mathfrak{p}}(\varphi_s) \subset \operatorname{Ann}_{\mathfrak{p}}(b_s)$.*

Proof. Suppose $\xi \in \operatorname{Ann}_{\mathfrak{p}}(\varphi_s)$ and let $\eta, \gamma \in \mathfrak{p}$. Consider

$$\begin{aligned}[\gamma, \eta]_{\mathfrak{p}} \cdot \hat{\xi} =& \gamma \cdot \left(\eta \cdot \hat{\xi}\right) - \eta \cdot \left(\gamma \cdot \hat{\xi}\right) \\ =& \gamma \cdot \left[\hat{\eta}, \hat{\xi}\right]^{(s)} - \eta \cdot \left[\hat{\gamma}, \hat{\xi}\right]^{(s)}\end{aligned}$$

$$
\begin{aligned}
&= \left[\gamma \cdot \hat{\eta}, \hat{\xi}\right]^{(s)} + \left[\hat{\eta}, \gamma \cdot \hat{\xi}\right]^{(s)} + a_s\left(\hat{\eta}, \hat{\xi}, \hat{\gamma}\right) \\
&\quad - \left[\eta \cdot \hat{\gamma}, \hat{\xi}\right]^{(s)} - \left[\hat{\gamma}, \eta \cdot \hat{\xi}\right]^{(s)} - a_s\left(\hat{\gamma}, \hat{\xi}, \hat{\eta}\right)^{(s)} \\
&= \left[\varphi_s\left([\gamma, \eta]_{\mathfrak{p}}\right), \hat{\xi}\right]^{(s)} + \left[[\hat{\gamma}, \hat{\eta}]^{(s)}, \hat{\xi}\right] + \left[\hat{\eta}, \left[\hat{\gamma}, \hat{\xi}\right]^{(s)}\right]^{(s)} \\
&\quad - \left[\hat{\gamma}, \left[\hat{\eta}, \hat{\xi}\right]^{(s)}\right]^{(s)} \\
&\quad + a_s\left(\hat{\eta}, \hat{\xi}, \hat{\gamma}\right) - a_s\left(\hat{\gamma}, \hat{\xi}, \hat{\eta}\right) \\
&= [\gamma, \eta]_{\mathfrak{p}} \cdot \hat{\xi} - a_s\left(\hat{\gamma}, \hat{\xi}, \hat{\eta}\right)
\end{aligned}
$$

where we have used (3.101), (3.90), (3.96), and the Akivis identity (3.46). We hence find that

$$
a_s\left(\hat{\gamma}, \hat{\xi}, \hat{\eta}\right) = 0. \tag{3.103}
$$

We know that if $\mathbb{L}$ is a G-loop, then $\mathfrak{l}^{(s)} = \varphi_s(\mathfrak{p})$, and thus the condition (3.103) is the same as (3.95), that is $\xi \in \mathrm{Ann}_{\mathfrak{p}}(b_s)$. $\square$

Remark 3.38. Overall, if $\mathbb{L}$ is a G-loop, we have the following inclusions of Lie algebras

$$
\ker \varphi_s = \mathfrak{h}_s \underset{\text{ideal}}{\subset} \mathrm{Ann}_{\mathfrak{p}}(\varphi_s) \subset \mathrm{Ann}_{\mathfrak{p}}(b_s) \cong \mathfrak{h}_s \oplus \mathcal{N}^R\left(\mathfrak{l}^{(s)}\right) \subset \mathfrak{p}. \tag{3.104}
$$

If we look at the octonion case, with $\mathbb{L} = U\mathbb{O}$, then $\mathfrak{p} = \mathfrak{so}(7)$, $\mathfrak{h}_s \cong \mathfrak{g}_2$. Moreover, in this case, $\mathcal{N}^R(\mathfrak{l}) = \{0\}$, so we must have $\mathfrak{h}_s = \mathrm{Ann}_{\mathfrak{p}}(\varphi_s) = \mathrm{Ann}_{\mathfrak{p}}(b_s)$. This also makes sense because in this case, φ_s and b_s are essentially the same objects, and moreover, almost uniquely determine s (up to ± 1). At the other extreme, if $\mathbb{L}$ is associative, so that $\mathcal{N}^R(\mathfrak{l}) = \mathfrak{l}$, then $\mathrm{Ann}_{\mathfrak{p}}(b_s) = \mathfrak{p}$, but $\mathrm{Ann}_{\mathfrak{p}}(\varphi_s)$ does not have to equal $\mathrm{Ann}_{\mathfrak{p}}(b_s)$.

Example 3.39. Using the setup from Examples 2.24, 3.28, and 3.29, if $\mathbb{L} = U\mathbb{C}$ with $\Psi_n^R(U\mathbb{C}) = U(n)$ or $\mathbb{L} = U\mathbb{H}$ with $\Psi_n^R(U\mathbb{H}) = Sp(n)\, Sp(1)$, since the partial action of Ψ_n^R in each case here is trivial, from (3.87), we see that the action of each Lie algebra $\mathfrak{p}_n$ on $\mathfrak{l}$ is trivial. In the complex case, $\mathfrak{l} \cong \mathbb{R}$, and is thus abelian. Hence, from (3.99), we see that in this case $\xi \cdot \varphi_s = 0$ for each $\xi \in \mathfrak{p}_n$. This makes because in Example 3.28 we noted that φ_s does not depend on s in the complex case. In the quaternion case, (3.99) shows that if $\xi, \eta \in \mathfrak{sp}(n) \oplus \mathfrak{sp}(1) = \mathfrak{p}_n$, then

$$
\begin{aligned}
(\xi \cdot \varphi_s)(\eta) &= -\,\varphi_s\left([\xi, \eta]_{\mathfrak{p}_n}\right) \\
&= -\,[\xi_1, \eta_1]_{\mathrm{Im}\,\mathbb{H}}
\end{aligned} \tag{3.105}
$$

where ξ_1, η_1 are the $\mathfrak{sp}(1)$ components of ξ and η, and $[\cdot,\cdot]_{\operatorname{Im}\mathbb{H}}$ is the bracket on $\operatorname{Im}\mathbb{H}$ (and equivalently on $\mathfrak{sp}(1)$). In particular, $\operatorname{Ann}_{\mathfrak{p}_n}(\varphi_s) = \mathfrak{sp}(n)$.

Note that, while it is known that any simple (i.e. has no nontrivial proper normal subloops) Moufang loop is a G-loop, it is not known whether there are simple Bol loops that are neither G-loops nor isotopic to Bruck loops [34] (a Bruck loop is a Bol loop for which the inverse is an automorphism). On the other hand, there is an example of a Bol loop that is a G-loop but is not a Moufang loop [39]. That particular example is constructed from an alternative division ring, but if that is taken to be $\mathbb{O}$, we obtain a smooth loop.

3.4. Killing form

Similarly as for Lie groups, we may define a Killing form $K^{(s)}$ on $\mathfrak{l}^{(s)}$. For $\xi, \eta \in \mathfrak{l}$, we have

$$K^{(s)}(\xi,\eta) = \operatorname{Tr}\left(\operatorname{ad}_{\xi}^{(s)} \circ \operatorname{ad}_{\eta}^{(s)}\right), \tag{3.106}$$

where $\circ$ is just composition of linear maps on $\mathfrak{l}$ and $\operatorname{ad}_{\xi}^{(s)}(\cdot) = [\xi,\cdot]^{(s)}$, as in (3.25). Clearly $K^{(s)}$ is a symmetric bilinear form on $\mathfrak{l}$. Given the form $K^{(s)}$ on $\mathfrak{l}$, we can extend it to a "right-invariant" form $\langle\cdot,\cdot\rangle^{(s)}$ on $\mathbb{L}$ via right translation, so that for vector fields X, Y on $\mathbb{L}$,

$$\langle X, Y\rangle_{\mathbb{L}}^{(s)} = K^{(s)}(\theta(X), \theta(Y)). \tag{3.107}$$

Theorem 3.40. *The bilinear form $K^{(s)}$ (3.106) on $\mathfrak{l}$ has the following properties.*

1. *Let $h \in \Psi^R(\mathbb{L})$, then for any $\xi, \eta \in \mathfrak{l}$,*

$$K^{(h(s))}(h'_*\xi, h'_*\eta) = K^{(s)}(\xi,\eta). \tag{3.108}$$

2. *Suppose also $\gamma \in \mathfrak{l}$, then*

$$\begin{aligned} K^{(s)}\left(\operatorname{ad}_{\gamma}^{(s)}\eta,\xi\right) = &-K^{(s)}\left(\eta,\operatorname{ad}_{\gamma}^{(s)}\xi\right) + \operatorname{Tr}\left(\operatorname{Jac}_{\xi,\gamma}^{(s)} \circ \operatorname{ad}_{\eta}^{(s)}\right) \\ &+ \operatorname{Tr}\left(\operatorname{Jac}_{\eta,\gamma}^{(s)} \circ \operatorname{ad}_{\xi}^{(s)}\right), \end{aligned} \tag{3.109}$$

   where $\operatorname{Jac}_{\gamma,\xi}^{(s)} : \mathfrak{l} \longrightarrow \mathfrak{l}$ is given by $\operatorname{Jac}_{\eta,\gamma}^{(s)}(\xi) = \operatorname{Jac}^{(s)}(\xi,\eta,\gamma)$.
3. *Let $\alpha \in \mathfrak{p}$, then*

$$\begin{aligned} K^{(s)}(\alpha\cdot\xi,\eta) = &-K^{(s)}(\xi,\alpha\cdot\eta) + \operatorname{Tr}\left(a_{\eta,\hat{\alpha}}^{(s)} \circ \operatorname{ad}_{\xi}^{(s)}\right) \\ &+ \operatorname{Tr}\left(a_{\xi,\hat{\alpha}}^{(s)} \circ \operatorname{ad}_{\eta}^{(s)}\right), \end{aligned} \tag{3.110}$$

where $a_{\xi,\eta}^{(s)} : \mathfrak{l} \longrightarrow \mathfrak{l}$ *is given by* $a_{\xi,\eta}^{(s)}(\gamma) = [\gamma, \xi, \eta]^{(s)} - [\xi, \gamma, \eta]^{(s)}$ *and* $\hat{\alpha} = \varphi_s(\alpha)$.

The proof of Theorem (3.40) is given in Appendix A.

Remark 3.41. If $(\mathbb{L}, \circ_s)$ is an alternative loop, we know that $\mathrm{Jac}_{\eta,\gamma}^{(s)} = 3a^{(s)}$, so in that in case, $K^{(s)}$ is invariant with respect to both $\mathrm{ad}^{(s)}$ and the action of $\mathfrak{p}$ if, and only if,

$$\mathrm{Tr}\left(a_{\eta,\hat{\alpha}}^{(s)} \circ \mathrm{ad}_{\xi}^{(s)}\right) + \mathrm{Tr}\left(a_{\xi,\hat{\alpha}}^{(s)} \circ \mathrm{ad}_{\eta}^{(s)}\right) = 0. \tag{3.111}$$

Indeed, in [43], it is shown that for a Malcev algebra, the Killing form is ad-invariant. A Malcev algebra is alternative and hence the Killing form is also $\mathfrak{p}$-invariant in that case. Moreover, it shown in [30] that for a *semisimple* Malcev algebra, the Killing form is non-degenerate. Here the definition of "semisimple" is the same as for Lie algebras, namely that the maximal solvable ideal is zero. Indeed, given the algebra of imaginary octonions on $\mathbb{R}^7$, it is known that the corresponding Killing form is negative-definite [3]. Moreover, since in this case, the pseudoautomorphism group is $SO(7)$, so (3.108) actually shows that $K^{h(s)} = K^s$ for every h, and thus is independent of s. General criteria for a loop algebra to admit an invariant definite (or even just non-degenerate) Killing form do not seem to appear in the literature, and could be the subject of further study. At least for well-behaved loops, such as Malcev loops, it is likely that there is significant similarity to Lie groups.

Suppose now $K^{(s)}$ is nondegenerate and both $\mathrm{ad}^{(s)}$- and $\mathfrak{p}$-invariant, and moreover suppose $\mathfrak{p}$ is semisimple itself, so that it has a nondegenerate, invariant Killing form $K_{\mathfrak{p}}$. We will use $\langle \cdot, \cdot \rangle^{(s)}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$ to denote the inner products using $K^{(s)}$ and $K_{\mathfrak{p}}$, respectively. Then, given the map $\varphi_s : \mathfrak{p} \longrightarrow \mathfrak{l}^{(s)}$, we can define its adjoint with respect to these two bilinear maps.

Definition 3.42. Define the map $\varphi_s^t : \mathfrak{l}^{(s)} \longrightarrow \mathfrak{p}$ such that for any $\xi \in \mathfrak{l}^{(s)}$ and $\eta \in \mathfrak{p}$,

$$\left\langle \varphi_s^t(\xi), \eta \right\rangle_{\mathfrak{p}} = \langle \xi, \varphi_s(\eta) \rangle^{(s)}. \tag{3.112}$$

Since $\mathfrak{h}_s \cong \ker \varphi_s$, we then clearly have $\mathfrak{p} \cong \mathfrak{h}_s \oplus \mathrm{Im}\, \varphi_s^t$, so that $\mathfrak{h}_s^{\perp} = \mathrm{Im}\, \varphi_s^t$. On the other hand, we also have $\mathfrak{l}^{(s)} \cong \ker \varphi_s^t \oplus \mathfrak{q}_s$, since $\mathfrak{q}_s = \mathrm{Im}\, \varphi_s$. Define the corresponding projections $\pi_{\mathfrak{h}_s}, \pi_{\mathfrak{h}_s^{\perp}}$ and $\pi_{\mathfrak{q}_s}, \pi_{\mathfrak{q}_s^{\perp}}$. We then have the following properties.

Lemma 3.43. *Suppose* $\mathfrak{q}_s$ *is an irreducible representation of* $\mathfrak{h}$ *and suppose the base field of* $\mathfrak{p}$ *is* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$. *Then, there exists a* $\lambda_s \in \mathbb{F}$ *such that*

$$\varphi_s \varphi_s^t = \lambda_s \pi_{\mathfrak{q}^{(s)}} \quad \textit{and } \varphi_s^t \varphi_s = \lambda_s \pi_{\mathfrak{h}_s^{\perp}}. \tag{3.113}$$

Moreover, for any $h \in \Psi^R(\mathbb{L})$, $\lambda_s = \lambda_{h(s)}$.

Proof. Let $\gamma, \eta \in \mathfrak{p}$ and $\xi \in \mathfrak{l}^{(s)}$, then using (3.99),

$$\begin{aligned}\left\langle \left(\gamma \cdot \varphi_s^t\right)(\xi), \eta\right\rangle_{\mathfrak{p}} &= \left\langle \left[\gamma, \varphi_s^t(\xi)\right]_{\mathfrak{p}}, \eta\right\rangle_{\mathfrak{p}} - \left\langle \varphi_s^t(\gamma\cdot\xi), \eta\right\rangle_{\mathfrak{p}} \\ &= -\left\langle \varphi_s^t(\xi), [\gamma,\eta]_{\mathfrak{p}}\right\rangle - \left\langle \gamma\cdot\xi, \varphi_s(\eta)\right\rangle^{(s)} \\ &= \left\langle \xi, \gamma\cdot\varphi_s(\eta) - \varphi_s\left([\gamma,\eta]_{\mathfrak{p}}\right)\right\rangle^{(s)} \\ &= \left\langle \xi, (\gamma\cdot\varphi_s)(\eta)\right\rangle^{(s)}, \end{aligned} \tag{3.114}$$

so in particular, $\operatorname{Ann}_{\mathfrak{p}}(\varphi_s) = \operatorname{Ann}_{\mathfrak{p}}(\varphi_s^t)$. Thus, the map $\varphi_s\varphi_s^t : \mathfrak{l}^{(s)} \longrightarrow \mathfrak{l}^{(s)}$ is an equivariant map of representations of the Lie subalgebra $\operatorname{Ann}_{\mathfrak{p}}(\varphi_s) \subset \mathfrak{p}$ and is moreover self-adjoint with respect to $\langle\cdot,\cdot\rangle^{(s)}$. We can also restrict this map to $\mathfrak{q}_s$, which is also a representation of $\operatorname{Ann}_{\mathfrak{p}}(\varphi_s)$, and in particular of $\mathfrak{h}_s$. Hence, if $\mathfrak{q}_s$ is an irreducible representation of $\mathfrak{h}_s$, since $\varphi_s\varphi_s^t$ is diagonalizable (in general, if $\mathbb{C}$ is the base field, or because it symmetric if the base field is $\mathbb{R}$), by Schur's Lemma, there exists some number $\lambda_s \neq 0$ such that

$$\varphi_s\varphi_s^t\big|_{\mathfrak{q}^{(s)}} = \lambda_s \operatorname{id}_{\mathfrak{q}^{(s)}}. \tag{3.115}$$

Applying φ_s^t to (3.115), we also obtain.

$$\varphi_s^t\varphi_s\big|_{\mathfrak{h}_s^\perp} = \lambda_s \operatorname{id}_{\mathfrak{h}_s^\perp}. \tag{3.116}$$

Since φ_s^t and φ_s vanish on $\mathfrak{q}_s^\perp$ and $\mathfrak{h}_s$, respectively, we obtain (3.113).

Let $h \in \Psi^R(\mathbb{L})$, then from (3.75), recall that

$$\varphi_{h(s)} = (h')_* \circ \varphi_s \circ \left(\operatorname{Ad}_h^{-1}\right)_*. \tag{3.117}$$

It is then easy to see using (3.108) and the invariance of the Killing form on $\mathfrak{p}$ that

$$\varphi_{h(s)}^t = (\operatorname{Ad}_h)_* \circ \varphi_s^t \circ (h')_*^{-1}. \tag{3.118}$$

In particular, we see that

$$(h')_* \mathfrak{q}_s = \mathfrak{q}_{h(s)} \text{ and } (\operatorname{Ad}_h)_* \mathfrak{h}_s^\perp = \mathfrak{h}_{h(s)}^\perp.$$

Hence,

$$\begin{aligned}\varphi_{h(s)}\varphi_{h(s)}^t\Big|_{\mathfrak{q}_{h(s)}} &= (h')_* \circ \varphi_s\varphi_s^t \circ (h')_*^{-1}\Big|_{\mathfrak{q}_{h(s)}} \\ &= \lambda_s \operatorname{id}_{\mathfrak{q}_{h(s)}}\end{aligned}$$

and so indeed, $\lambda_s = \lambda_{h(s)}$. □

Example 3.44. In the case of octonions, suppose we set $\varphi_s(\eta)_a = k\varphi_{abc}\eta^{bc}$ where $\eta \in \mathfrak{so}(7) \cong \Lambda^2 \left(\mathbb{R}^7\right)^*$, φ is the defining 3-form on $\mathbb{R}^7$, and $k \in \mathbb{R}$ is some constant. Then, $\varphi_s^t(\gamma)_{ab} = k\varphi_{abc}\gamma^c$ where $\gamma \in \mathbb{R}^7 \cong \operatorname{Im}\mathbb{O}$. Now, $\mathbb{R}^7$ is an irreducible representation of $\mathfrak{g}_2$, so the hypothesis of Lemma 3.43 is satisfied. In this case, $\lambda_s = 6k^2$ due to the contraction identities for φ [14,22].

Consider the action of $\varphi_s^t\left(\mathfrak{l}^{(s)}\right) \subset \mathfrak{p}$ on $\mathfrak{q}_s$. Let $\xi, \eta \in \mathfrak{q}_s$, then from (3.96),

$$\varphi_s^t(\xi) \cdot \varphi_s\varphi_s^t(\eta) - \varphi_s^t(\eta) \cdot \varphi_s\varphi_s^t(\xi) = \varphi_s\left(\left[\varphi_s^t(\xi), \varphi_s^t(\eta)\right]_{\mathfrak{p}}\right) + \left[\varphi_s\varphi_s^t(\xi), \varphi_s\varphi_s^t(\eta)\right]^{(s)}, \tag{3.119}$$

and thus,

$$\varphi_s^t(\xi) \cdot \eta - \varphi_s^t(\eta) \cdot \xi = \frac{1}{\lambda_s}\varphi_s\left(\left[\varphi_s^t(\xi), \varphi_s^t(\eta)\right]_{\mathfrak{p}}\right) + \lambda_s[\xi, \eta]^{(s)}. \tag{3.120}$$

We now show that $\varphi_s^t(\xi) \cdot \eta$ is skew-symmetric when restricted to $\mathfrak{q}_s$ and then projected back to $\mathfrak{q}_s$.

Lemma 3.45. *Suppose $\mathbb{L}$ is a loop and $s \in \mathbb{L}$, such that the Killing form is non-degenerate and $\mathrm{ad}^{(s)}$- and $\mathfrak{p}$-invariant. Then, for any $\xi, \eta \in \mathfrak{q}_s$,*

$$\pi_{\mathfrak{q}_s}\left(\varphi_s^t(\xi) \cdot \eta\right) = -\pi_{\mathfrak{q}_s}\left(\varphi_s^t(\eta) \cdot \xi\right). \tag{3.121}$$

Proof. Suppose $\xi, \eta \in \mathfrak{q}_s$, then using the $\mathrm{ad}^{(s)}$- and $\mathfrak{p}$-invariance of the Killing form on $\mathfrak{l}^{(s)}$ and (3.120) we have

$$\begin{aligned}
\left\langle \varphi_s^t(\eta) \cdot \eta, \xi\right\rangle^{(s)} &= -\left\langle \eta, \varphi_s^t(\eta) \cdot \xi\right\rangle^{(s)} \\
&= -\left\langle \eta, \varphi_s^t(\xi) \cdot \eta - \frac{1}{\lambda_s}\varphi_s\left(\left[\varphi_s^t(\xi), \varphi_s^t(\eta)\right]_{\mathfrak{p}}\right) - \lambda_s[\xi, \eta]^{(s)}\right\rangle^{(s)} \\
&= -\left\langle \eta, \varphi_s^t(\xi) \cdot \eta\right\rangle^{(s)} + \frac{1}{\lambda_s}\left\langle \varphi_s^t(\eta), \left[\varphi_s^t(\xi), \varphi_s^t(\eta)\right]_{\mathfrak{p}}\right\rangle \\
&\quad - \lambda_s\left\langle [\eta, \eta]^{(s)}, \xi\right\rangle^{(s)} \\
&= -\left\langle \eta, \varphi_s^t(\xi) \cdot \eta\right\rangle^{(s)} = \left\langle \varphi_s^t(\xi) \cdot \eta, \eta\right\rangle^{(s)} \\
&= 0.
\end{aligned}$$

Thus, we see that $\pi_{\mathfrak{q}_s}\left(\varphi_s^t(\eta) \cdot \eta\right) = 0$, and hence (3.121) holds. □

Taking the $\pi_{\mathfrak{q}_s}$ projection of (3.120) gives

$$\pi_{\mathfrak{q}_s}\left(\varphi_s^t(\xi) \cdot \eta\right) = \frac{1}{2\lambda_s}\varphi_s\left(\left[\varphi_s^t(\xi), \varphi_s^t(\eta)\right]_{\mathfrak{p}} + \lambda_s\varphi_s^t\left([\xi, \eta]^{(s)}\right)\right). \tag{3.122}$$

The relation (3.122) suggests that we can define a new bracket $[\cdot,\cdot]_{\varphi_s}$ on $\mathfrak{l}^{(s)}$ using φ_s.

Definition 3.46. Suppose $\mathbb{L}$ satisfies the assumptions of Lemma 3.45. Then, for $\xi,\eta\in\mathfrak{l}^{(s)}$, define

$$[\xi,\eta]_{\varphi_s}=\varphi_s\left(\left[\varphi_s^t(\xi),\varphi_s^t(\eta)\right]_{\mathfrak{p}}\right). \tag{3.123}$$

This bracket restricts to $\mathfrak{q}_s$ and vanishes on $\mathfrak{q}_s^{\perp}$, so that $\mathfrak{q}_s^{\perp}$ is an abelian ideal with respect to it. We can rewrite (3.122) as

$$\pi_{\mathfrak{q}_s}\left(\varphi_s^t(\xi)\cdot\eta\right)=\frac{1}{2\lambda_s}[\xi,\eta]_{\varphi_s}+\frac{\lambda_s}{2}\pi_{\mathfrak{q}_s}\left([\xi,\eta]^{(s)}\right). \tag{3.124}$$

Example 3.47. In the case of octonions, if, as before, we set $\varphi_s(\eta)_a=k\varphi_{abc}\eta^{bc}$ and $\left([\xi,\gamma]^{(s)}\right)_a=2\varphi_{abc}\xi^b\gamma^c$, we find that $[\cdot,\cdot]_{\varphi_s}=3k^3[\cdot,\cdot]^{(s)}$. Then, recalling that $\lambda_s=6k^2$, (3.124) shows that in this case

$$\varphi_s^t(\xi)\cdot\gamma=\left(\frac{k}{4}+3k^2\right)[\xi,\gamma]^{(s)},$$

and to be consistent with the standard action of $\mathfrak{so}(7)$ on $\mathbb{R}^7$, we must have

$$k\varphi_{abc}\xi^c\gamma^b=\left(\frac{k}{2}+6k^2\right)\varphi_{abc}\xi^b\gamma^c,$$

which means that $6k^2+\frac{3}{2}k=0$ and therefore, $k=-\frac{1}{4}$. This also implies that $\lambda_s=\frac{3}{8}$ in this case. We also thus obtain

$$\varphi_s^t(\xi)\cdot\gamma=\frac{1}{8}[\xi,\gamma]^{(s)}. \tag{3.125}$$

Example 3.48. If $\mathbb{L}$ is a Lie group, and $\Psi^R(\mathbb{L})$ is the full group of pseudoautomorphism pairs, then $\mathfrak{p}\cong\mathfrak{aut}(\mathbb{L})\oplus\mathfrak{l}$, where $\mathfrak{aut}(\mathbb{L})$ is the Lie algebra of $\mathrm{Aut}(\mathbb{L})$ and $\mathfrak{l}$ is the Lie algebra of $\mathbb{L}$. In this case, $\varphi_s^t\varphi_s$ is just the projection to $\mathfrak{l}\subset\mathfrak{p}$, and thus $\lambda_s=1$ and $[\cdot,\cdot]_{\varphi_s}=[\cdot,\cdot]^{(s)}$. Then (3.124) just shows that $\mathfrak{l}$ acts on itself via the adjoint representation, i.e.

$$\varphi_s^t(\xi)\cdot\eta=[\xi,\eta]. \tag{3.126}$$

Remark 3.49. Both of the above examples have the two brackets $[\cdot,\cdot]_{\varphi_s}$ and $[\cdot,\cdot]^{(s)}$ proportional to one another. This really means that $\mathfrak{l}^{(s)}$ and $\mathfrak{h}_s^{\perp}$ have equivalent $\mathbb{L}$-algebra structures with φ_s and φ_s^t (up to a constant factor) being the corresponding isomorphisms. It is not clear if this is always the case.

The bracket $[\cdot,\cdot]_{\varphi_s}$ has some reasonable properties.

Lemma 3.50. *Under the assumptions of Lemma 3.45, the bracket* $[\cdot,\cdot]_{\varphi_s}$ *satisfies the following properties. Let* $\xi,\eta,\gamma\in\mathfrak{l}$, *then*

1. $\left\langle [\xi,\eta]_{\varphi_s},\gamma\right\rangle^{(s)} = -\left\langle \eta,[\xi,\gamma]_{\varphi_s}\right\rangle^{(s)}$.
2. *For any* $h\in\Psi^R(\mathbb{L})$, $[\xi,\eta]_{\varphi_{h(s)}} = (h')_*\left[(h')_*^{-1}\xi,(h')_*^{-1}\eta\right]_{\varphi_s}$.

Proof. The first property follows directly from the definition (3.123) and the ad-invariance of the Killing form on $\mathfrak{p}$. Indeed,

$$\begin{aligned}\left\langle [\xi,\eta]_{\varphi_s},\gamma\right\rangle^{(s)} &= \left\langle \varphi_s\left(\left[\varphi_s^t(\xi),\varphi_s^t(\eta)\right]_{\mathfrak{p}}\right),\gamma\right\rangle^{(s)}\\ &= \left\langle \left[\varphi_s^t(\xi),\varphi_s^t(\eta)\right]_{\mathfrak{p}},\varphi_s^t(\gamma)\right\rangle^{(s)}\\ &= -\left\langle \varphi_s^t(\eta),\left[\varphi_s^t(\xi),\varphi_s^t(\gamma)\right]_{\mathfrak{p}}\right\rangle^{(s)}\\ &= -\left\langle \eta,[\xi,\gamma]_{\varphi_s}\right\rangle^{(s)}.\end{aligned}$$

Now let $h\in\Psi^R(\mathbb{L})$, and then since $(\mathrm{Ad}_h)_*$ is a Lie algebra automorphism of $\mathfrak{p}$, we have

$$\begin{aligned}&[\xi,\eta]_{\varphi_{h(s)}}\\ &=\varphi_{h(s)}\left(\left[\varphi_{h(s)}^t(\xi),\varphi_{h(s)}^t(\eta)\right]_{\mathfrak{p}}\right)\\ &=(h')_*\circ\varphi_s\circ\left(\mathrm{Ad}_h^{-1}\right)_*\left(\left[(\mathrm{Ad}_h)_*\left(\varphi_s^t\left((h')_*^{-1}(\xi)\right)\right),(\mathrm{Ad}_h)_*\left(\varphi_s^t\left((h')_*^{-1}(\eta)\right)\right)\right]_{\mathfrak{p}}\right)\\ &=(h')_*\circ\varphi_s\left(\left[\varphi_s^t\left((h')_*^{-1}(\xi)\right),\varphi_s^t\left((h')_*^{-1}(\eta)\right)\right]_{\mathfrak{p}}\right)\\ &=(h')_*\left[(h')_*^{-1}\xi,(h')_*^{-1}\eta\right]_{\varphi_s}.\end{aligned} \tag{3.127}$$

Therefore, $[\cdot,\cdot]_{\varphi_s}$ is equivariant with respect to transformations of s. □

3.5. Darboux derivative

Let M be a smooth manifold and suppose $s: M\longrightarrow\mathbb{L}$ is a smooth map. The map s can be used to define a product on $\mathbb{L}$-valued maps from M and a corresponding bracket on $\mathfrak{l}$-valued maps. Indeed, let $A,B: M\longrightarrow\mathbb{L}$ and $\xi,\eta: M\longrightarrow\mathfrak{l}$ be smooth maps, then at each $x\in M$, define

$$A\circ_s B|_x = A_x\circ_{s_x} B_x\in\mathbb{L} \tag{3.128a}$$

$$A/_s B|_x = A_x/_{s_x} B_x\in\mathbb{L} \tag{3.128b}$$

$$A\backslash_s B|_x = A_x \backslash_s B_x \in \mathbb{L} \tag{3.128c}$$

$$[\xi, \eta]^{(s)}\Big|_x = [\xi_x, \eta_x]^{(s_x)} \in \mathfrak{l}. \tag{3.128d}$$

In particular, the bracket $[\cdot, \cdot]^{(s)}$ defines the map $b_s : M \longrightarrow \Lambda^2\mathfrak{l}^* \otimes \mathfrak{l}$. We also have the corresponding associator $[\cdot, \cdot, \cdot]^{(s)}$ and the left-alternating associator map $a_s : M \longrightarrow \Lambda^2\mathfrak{l}^* \otimes \mathfrak{l}^* \otimes \mathfrak{l}$. Similarly, define the map $\varphi_s : M \longrightarrow \mathfrak{p}^* \otimes \mathfrak{l}$.

Then, similarly as for maps to Lie groups, we may define the (right) *Darboux derivative* θ_s of s, which is an $\mathfrak{l}$-valued 1-form on M given by $s^*\theta$ [45]. In particular, at every $x \in M$,

$$(\theta_s)|_x = \left(R_{s(x)}^{-1}\right)_* ds|_x . \tag{3.129}$$

It is then clear that θ_s, being a pullback of θ, satisfies the loop Maurer-Cartan structural equation (3.35). In particular, for any vectors $X, Y \in T_xM$,

$$d\theta_s(X, Y) - [\theta_s(X), \theta_s(Y)]^{(s)} = 0. \tag{3.130}$$

We can then calculate the derivatives of these maps. For clarity, we will be using notation from Remark 3.16, in that we will suppress the pushforwards of right multiplication and their inverses (i.e. quotients) on $T\mathbb{L}$, so that if $X \in T_q\mathbb{L}$, then we will write $X \circ_s A$ for $\left(R_A^{(s)}\right)_* X$.

Theorem 3.51. *Let M be a smooth manifold and let $x \in M$. Suppose $A, B, s \in C^\infty(M, \mathbb{L})$, then*

$$d(A \circ_s B) = (dA) \circ_s B + A \circ_s (dB) + [A, B, \theta_s]^{(s)} \tag{3.131}$$

and

$$\begin{aligned} d(A/_s B) = & dA/_s B - (A/_s B \circ_s dB)/_s B \\ & - [A/_s B, B, \theta_s]^{(s)}/_s B \end{aligned} \tag{3.132a}$$

$$\begin{aligned} d(B\backslash_s A) = & B\backslash_s dA - B\backslash_s (dB \circ_s (B\backslash_s A)) \\ & - B\backslash_s [B, B\backslash_s A, \theta_s]^{(s)} . \end{aligned} \tag{3.132b}$$

Suppose now $\xi, \eta \in C^\infty(M, \mathfrak{l})$, then

$$d[\xi, \eta]^{(s)} = [d\xi, \eta]^{(s)} + [\xi, d\eta]^{(s)} + a_s(\xi, \eta, \theta_s). \tag{3.133}$$

The $\mathfrak{l} \otimes \mathfrak{p}^*$-valued map $\varphi_s : M \longrightarrow \mathfrak{l} \otimes \mathfrak{p}^*$ satisfies

$$d\varphi_s = \mathrm{id}_{\mathfrak{p}} \cdot \theta_s - [\varphi_s, \theta_s]^{(s)}, \tag{3.134}$$

where $\mathrm{id}_{\mathfrak{p}}$ *is the identity map of* $\mathfrak{p}$ *and* $\cdot$ *denotes the action of the Lie algebra* $\mathfrak{p}$ *on* $\mathfrak{l}$ *given by (3.87)*

Proof. Let $V \in T_xM$ and let $x(t)$ be a curve on M with $x(0) = x$ and $\dot{x}(0) = V$. To show (3.131), first note that

$$d\left(A \circ_s B\right)|_x (V) = \left.\frac{d}{dt}\left(A_{x(t)} \circ_{s_{x(t)}} B_{x(t)}\right)\right|_{t=0}. \tag{3.135}$$

However,

$$\begin{aligned}\left.\frac{d}{dt}\left(A_{x(t)} \circ_{s_{x(t)}} B_{x(t)}\right)\right|_{t=0} &= \left.\frac{d}{dt}\left(A_{x(t)} \circ_{s_x} B_x\right)\right|_{t=0} + \left.\frac{d}{dt}\left(A_x \circ_{s_x} B_{x(t)}\right)\right|_{t=0} \\ &\quad + \left.\frac{d}{dt}\left(A_x \circ_{s_{x(t)}} B_x\right)\right|_{t=0} \\ &= \left(R_{B_x}^{(s_x)}\right)_* dA|_x (V) + \left(L_{A_x}^{(s_x)}\right)_* dB|_x (V) \\ &\quad + \left.\frac{d}{dt}\left(A_x \circ_{s_{x(t)}} B_x\right)\right|_{t=0}\end{aligned} \tag{3.136}$$

and then, using Lemma A.1,

$$\begin{aligned}\left.\frac{d}{dt}\left(A_x \circ_{s_{x(t)}} B_x\right)\right|_{t=0} &= \left.\frac{d}{dt}\left(\left(A_x \cdot B_x s_{x(t)}\right)/s_{x(t)}\right)\right|_{t=0} \\ &= \left.\frac{d}{dt}\left(\left(A_x \cdot B_x s_{x(t)}\right)/s_x\right)\right|_{t=0} \\ &\quad + \left.\frac{d}{dt}\left(\left(A_x \cdot B_x s_x\right)/s_x \cdot s_{x(t)}\right)/s_x\right|_{t=0}.\end{aligned} \tag{3.137}$$

Looking at each term in (3.137), we have

$$\begin{aligned}\left(A_x \cdot B_x s_{x(t)}\right)/s_x &= \left(A_x \cdot B_x \left(s_{x(t)}/s_x \cdot s_x\right)\right)/s_x \\ &= A_x \circ_{s_x} \left(B_x \circ_{s_x} \left(s_{x(t)}/s_x\right)\right)\end{aligned}$$

and

$$\left(\left(A_x \cdot B_x s_x\right)/s_x \cdot s_{x(t)}\right)/s_x = \left(A_x \circ_{s_x} B_x\right) \circ_{s_x} \left(s_{x(t)}/s_x\right).$$

Overall (3.136) becomes,

$$\left.\frac{d}{dt}\left(A_x \circ_{s_{x(t)}} B_x\right)\right|_{t=0} = \left(\left(L_{A_x}^{(s_x)} \circ L_{B_x}^{(s_x)}\right)_* - \left(L_{A_x \circ_{s_x} B_x}^{(s_x)}\right)_*\right)\left(R_{s_x}^{-1}\right)_* ds|_x (V_x) \tag{3.138}$$

and hence we get (3.131) using the definitions of θ_s and the mixed associator (3.11).

Let us now show (3.132). From Lemma A.1, we find

$$d(A/B) = (dA)/B - (A/B \cdot dB)/B \tag{3.139a}$$

$$d(B\backslash A) = B\backslash(dA) - B\backslash(dB \cdot B\backslash A). \tag{3.139b}$$

Now if we instead have the quotient defined by s, using (2.29a), we have a modification:

$$\begin{aligned} d(A/_sB) &= d(As/Bs) = d(As)/(Bs) - (A/_sB \cdot d(Bs))/(Bs) \\ &= dA/_sB + A(ds)/(Bs) - (A/_sB \cdot (dB)s)/(Bs) \\ &\quad - (A/_sB \cdot B(ds))/(Bs) \\ &= dA/_sB - (A/_sB \circ_s dB)/_sB + (A \circ_s \theta_s)/_sB \\ &\quad - (A/_sB \circ_s (B \circ_s \theta_s))/_sB \\ &= dA/_sB - (A/_sB \circ_s dB)/_sB - [A/_sB, B, \theta_s]^{(s)}/_sB. \end{aligned} \tag{3.140}$$

Similarly, for the left quotient, using (2.29b), we have

$$\begin{aligned} d(B\backslash_sA) &= d((B\backslash As)/s) \\ &= d(B\backslash As)/s - (((B\backslash As)/s) \cdot ds)/s \\ &= (B\backslash d(As))/s - (B\backslash(dB \cdot B\backslash As))/s - (B\backslash_sA) \circ_s \theta_s \\ &= B\backslash_s dA + (B\backslash(A(ds)))/s - B\backslash_s((dB \cdot B\backslash As)/s) \\ &\quad - (B\backslash_sA) \circ_s \theta_s \\ &= B\backslash_s dA - B\backslash_s(dB \circ_s (B\backslash_sA)) + B\backslash_s(A \circ_s \theta_s) \\ &\quad - (B\backslash_sA) \circ_s \theta_s \end{aligned} \tag{3.141}$$

However, using the mixed associator (3.11),

$$\begin{aligned} A \circ_s \theta_s &= (B \circ_s (B\backslash_sA)) \circ_s \theta_s \\ &= B \circ_s ((B\backslash_sA) \circ_s \theta_s) - [B, B\backslash_sA, \theta_s]^{(s)}, \end{aligned} \tag{3.142}$$

and thus,

$$d(B\backslash_sA) = B\backslash_s dA - B\backslash_s(dB \circ_s (B\backslash_sA)) - B\backslash_s[B, B\backslash_sA, \theta_s]^{(s)}.$$

To show (3.133), note that

$$d\left([\xi, \eta]^{(s)}\right)\Big|_x(V) = \frac{d}{dt}\left[\xi_{x(t)}, \eta_{x(t)}\right]^{(s_{x(t)})}\Big|_{t=0}$$

$$= \left[d\xi|_x (V), \eta_x \right]^{(s_x)} + \left[\xi_x, d\eta|_x \right]^{(s_x)} + \frac{d}{dt} \left[\xi_x, \eta_x \right]^{(s_{x(t)})} \bigg|_{t=0}$$

However, using (3.39), the last term becomes

$$\frac{d}{dt} \left[\xi_x, \eta_x \right]^{(s_{x(t)})} \bigg|_{t=0} = a_{s_x} \left(\xi_x, \eta_x, \theta_s|_x \right)$$

and hence we obtain (3.133).

Let us now show (3.134). From (3.81), given $\gamma \in \mathfrak{p}$, setting $\hat{\gamma}(r) = \varphi_r(\gamma)$ for each $r \in \mathbb{L}$, we have

$$d\hat{\gamma}|_r \left(\rho_r (\xi) \right) = \gamma \cdot \xi - \left[\hat{\gamma}(r), \xi \right]^{(r)} \tag{3.143}$$

for some $\xi \in \mathfrak{l}$. Now for at each $x \in M$ we have

$$\begin{aligned} d\left(\varphi_s (\gamma) \right)|_x (V) &= d\hat{\gamma}|_{s_x} \circ ds|_x (V) \\ &= d\hat{\gamma}|_{s_x} \left(\rho_{s_x} \left(\theta_s (V) \right) \right) \\ &= \gamma \cdot \theta_s (V) - \left[\varphi_{s_x} (\gamma), \theta_s (V) \right]^{(s_x)}. \end{aligned} \tag{3.144}$$

Therefore, $d\varphi_s$ is given by

$$d\varphi_s (\gamma) = \gamma \cdot \theta_s - \left[\varphi_s (\gamma), \theta_s \right]^{(s)}. \quad \square \tag{3.145}$$

Remark 3.52. Suppose A and B are now smooth maps from M to $\mathbb{L}$. In the case when $\mathbb{L}$ has the right inverse property, i.e. $A/B = AB^{-1}$ for any $A, B \in \mathbb{L}$, (3.139a) becomes

$$d\left(AB^{-1} \right) = (dA) B^{-1} - \left(AB^{-1} \cdot dB \right) B^{-1}. \tag{3.146}$$

However, from $d\left(BB^{-1} \right) = 0$, we find that $d\left(B^{-1} \right) = -B^{-1} \left(dB \cdot B^{-1} \right)$, and then expanding $d\left(AB^{-1} \right)$ using the product rule, and comparing with (3.146), we find

$$\left(AB^{-1} \cdot dB \right) B^{-1} = A \left(B^{-1} \left(dB \cdot B^{-1} \right) \right), \tag{3.147}$$

which is an infinitesimal version of the right Bol identity (2.6). In particular,

$$\left(B^{-1} \cdot dB \right) B^{-1} = B^{-1} \left(dB \cdot B^{-1} \right). \tag{3.148}$$

Similarly, using (3.132b), the left inverse property then implies an infinitesimal left Bol identity.

At each point $x \in M$, the map s defines a stabilizer subgroup $\mathrm{Stab}\left(s_x\right) = \mathrm{Aut}\left(\mathbb{L}, \circ_s\right) \subset \Psi^R\left(\mathbb{L}\right)$ with the corresponding Lie algebra $\mathfrak{h}_{s_x}$. Similarly, we also have the orbit of s_x given by $\mathcal{C}^R\left(\mathbb{L}, \circ_{s_x}\right) \cong {}^{\Psi^R\left(\mathbb{L}\right)}\!\big/_{\mathrm{Aut}\left(\mathbb{L}, \circ_{s_x}\right)}$, and the corresponding tangent space $\mathfrak{q}_{s_x} \cong \mathfrak{p}/\mathfrak{h}_{s_x}$. Suppose $\theta_s|_x \in \mathfrak{q}_{s_x}$ for each $x \in M$. This of course always holds if $\mathbb{L}$ is a G-loop, in which case $\mathfrak{q}_{s_x} = \mathfrak{l}^{(s_x)}$. In this case, there exists a $\mathfrak{p}$-valued 1-form Θ on M such that $\theta_s = \varphi_s\left(\Theta\right)$. We can then characterize Θ in the following way.

Theorem 3.53. *Suppose there exists $\Theta \in \Omega^1\left(M, \mathfrak{p}\right)$ such that $\theta_s = \varphi_s\left(\Theta\right)$. Then, for each $x \in M$, $d\Theta - \frac{1}{2}\left[\Theta, \Theta\right]_{\mathfrak{p}}\Big|_x \in \mathfrak{h}_{s_x}$, where $\left[\cdot, \cdot\right]_{\mathfrak{p}}$ is the Lie bracket on $\mathfrak{p}$.*

Proof. Consider $d\theta_s$ in this case. Using (3.145), we have

$$\begin{aligned} d\theta_s =& d\left(\varphi_s\left(\Theta\right)\right) = \left(d\varphi_s\right)\left(\Theta\right) + \varphi_s\left(d\Theta\right) \\ =& -\Theta \cdot \theta_s + \left[\varphi_s\left(\Theta\right), \theta_s\right]^{(s)}. \end{aligned} \tag{3.149}$$

Note that the signs are switched in (3.149) because we also have an implied wedge product of 1-forms. Overall, we have

$$d\left(\varphi_s\left(\Theta\right)\right) = \varphi_s\left(d\Theta\right) - \Theta \cdot \varphi_s\left(\Theta\right) + \left[\varphi_s\left(\Theta\right), \varphi_s\left(\Theta\right)\right]^{(s)}, \tag{3.150}$$

however since $\theta_s = \varphi_s\left(\Theta\right)$, it satisfies the Maurer-Cartan structural equation (3.130), so we also have

$$d\left(\varphi_s\left(\Theta\right)\right) = \frac{1}{2}\left[\varphi_s\left(\Theta\right), \varphi_s\left(\Theta\right)\right]. \tag{3.151}$$

Equating (3.150) and (3.151), we find

$$\varphi_s\left(d\Theta\right) = \Theta \cdot \varphi_s\left(\Theta\right) - \frac{1}{2}\left[\varphi_s\left(\Theta\right), \varphi_s\left(\Theta\right)\right]^{(s)}. \tag{3.152}$$

However, from (3.96), we find that

$$\Theta \cdot \varphi_s\left(\Theta\right) - \frac{1}{2}\left[\varphi_s\left(\Theta\right), \varphi_s\left(\Theta\right)\right] = \frac{1}{2}\varphi_s\left(\left[\Theta, \Theta\right]_{\mathfrak{p}}\right). \tag{3.153}$$

Thus, we see that

$$\varphi_s\left(d\Theta - \frac{1}{2}\left[\Theta, \Theta\right]_{\mathfrak{p}}\right) = 0. \quad \square \tag{3.154}$$

Remark 3.54. In general, we can think of $d - \Theta$ as a connection on the trivial Lie algebra bundle $M \times \mathfrak{p}$ with curvature contained in $\mathfrak{h}_{s(x)}$ for each $x \in M$. In general the spaces $\mathfrak{h}_{s(x)}$ need not be all of the same dimension, and thus may this may not give a vector subbundle. On the other hand, if $\mathbb{L}$ is a G-loop, then we do get a subbundle.

Now consider how θ_s behaves under the action of $\Psi^R(\mathbb{L})$.

Lemma 3.55. *Suppose $h : M \longrightarrow \Psi^R(\mathbb{L})$ is a smooth map, then*

$$\theta_{h(s)} = (h')_* \left(\varphi_s \left(\theta_h^{(\mathfrak{p})} \right) + \theta_s \right), \tag{3.155}$$

where $\theta_h^{(\mathfrak{p})} = h^* \theta^{(\mathfrak{p})}$ is the pullback of the left-invariant Maurer-Cartan form $\theta^{(\mathfrak{p})}$ on $\Psi^R(\mathbb{L})$.

Proof. Suppose $h : M \longrightarrow \Psi^R(\mathbb{L})$ is a smooth map, then consider $\theta_{h(s)}$. We then have

$$\begin{aligned} \left(\theta_{h(s)}\right)\big|_x &= \left(R^{-1}_{h(s(x))}\right)_* d\left(h\left(s\right)\right)\big|_x \\ &= \left(R^{-1}_{h(s(x))}\right)_* \left(\left(dh\right)\left(s\right) + h\left(ds\right)\right)\big|_x . \end{aligned}$$

Consider each term. Using simplified notation, we have

$$\begin{aligned} (dh)(s)/h(s) &= (h')_* \left(\left(h^{-1} dh \right)(s)/s \right) \\ \left(R^{-1}_{h(s(x))}\right)_* (h(ds))\big|_x &= (h')_* (\theta_s) . \end{aligned}$$

Thus,

$$\left(R^{-1}_{h(s(x))}\right)_* (dh)(s)\big|_x = \left(h(x)'\right)_* \varphi_{s(x)} \left(\theta_h^{(\mathfrak{p})}\Big|_x \right),$$

and hence we get (3.155). □

If we have another smooth map $f : M \longrightarrow \mathbb{L}$, using right multiplication with respect to $\circ_{s(x)}$, we can define a modified Darboux derivative $\theta_f^{(s)}$ with respect to s:

$$\left(\theta_f^{(s)}\right)\Big|_x = \left(R_{f(x)}^{(s(x))}\right)_*^{-1} df|_x . \tag{3.156}$$

Note that this is now no longer necessarily a pullback of θ and hence may not satisfy the Maurer-Cartan equation. Adopting simplified notation, we have the following:

$$\begin{aligned} d(fs)/fs &= (df \cdot s + f \cdot ds)/fs \\ &= df/_s f + \mathrm{Ad}_f^{(s)} \theta_s \end{aligned} \tag{3.157}$$

Hence,

$$\theta_f^{(s)} = \theta_{fs} - \left(\mathrm{Ad}_f^{(s)}\right)_* \theta_s. \tag{3.158}$$

Lemma 3.56. *Suppose* $f, s \in C^{\infty}(M, \mathbb{L})$, *then*

$$d\theta_f^{(s)} = \frac{1}{2}\left[\theta_f^{(s)}, \theta_f^{(s)}\right]^{(fs)} - \left(R_f^{(s)}\right)_*^{-1}\left[\theta_f^{(s)}, f, \theta_s\right]^{(s)}. \tag{3.159}$$

Proof. Applying the exterior derivative to (3.158) and then the structural equation for θ_{fs}, we have

$$d\theta_f^{(s)} = \frac{1}{2}\left[\theta_{fs}, \theta_{fs}\right]^{(fs)} - d\left(\left(\mathrm{Ad}_f^{(s)}\right)_* \theta_s\right). \tag{3.160}$$

From Lemma A.3, we can see that for $\xi \in \mathfrak{l}$,

$$\begin{aligned} d\left(\mathrm{Ad}_f^{(s)}\right)_* \xi = &\left[\theta_f^{(s)}, \left(\mathrm{Ad}_f^{(s)}\right)_* \xi\right]^{(fs)} - \left(R_f^{(s)}\right)_*^{-1}\left[\theta_f^{(s)}, f, \xi\right]^{(s)} \\ &+ \left(R_f^{(s)}\right)_*^{-1}\left[f, \xi, \theta_s\right]^{(s)} \\ &- \left(R_f^{(s)}\right)_*^{-1}\left[\left(\mathrm{Ad}_f^{(s)}\right)_* \xi, f, \theta_s\right]^{(s)}, \end{aligned} \tag{3.161}$$

and hence

$$\begin{aligned} d\left(\mathrm{Ad}_f^{(s)}\right)_* \wedge \theta_s = &\left[\theta_f^{(s)}, \left(\mathrm{Ad}_f^{(s)}\right)_* \theta_s\right]^{(fs)} - \left(R_f^{(s)}\right)_*^{-1}\left[\theta_f^{(s)}, f, \theta_s\right]^{(s)} \\ &- \left(R_f^{(s)}\right)_*^{-1}\left[f, \theta_s, \theta_s\right]^{(s)} \\ &+ \left(R_f^{(s)}\right)_*^{-1}\left[\left(\mathrm{Ad}_f^{(s)}\right)_* \theta_s, f, \theta_s\right]^{(s)}, \end{aligned} \tag{3.162}$$

where wedge products are implied. Now, using the structural equation and (3.58), we find

$$\begin{aligned} \left(\mathrm{Ad}_f^{(s)}\right)_* d\theta_s = &\frac{1}{2}\left(\mathrm{Ad}_f^{(s)}\right)_*\left[\theta_s, \theta_s\right]^{(s)} \\ = &\frac{1}{2}\left[\left(\mathrm{Ad}_f^{(s)}\right)_* \theta_s, \left(\mathrm{Ad}\, f\right)_* \theta_s\right]^{(fs)} \\ &- \left(R_f^{(s)}\right)_*^{-1}\left[\left(\mathrm{Ad}_f^{(s)}\right)_* \theta_s, f, \theta_s\right]^{(s)} \\ &+ \left(R_f^{(s)}\right)_*^{-1}\left[f, \theta_s, \theta_s\right]^{(s)}. \end{aligned} \tag{3.163}$$

Combining (3.162) and (3.163), we see that

$$\begin{aligned} d\left(\left(\mathrm{Ad}_f^{(s)}\right)_* \theta_s\right) = &d\left(\mathrm{Ad}_f^{(s)}\right)_* \wedge \theta_s + \left(\mathrm{Ad}_f^{(s)}\right)_* d\theta_s \\ = &\left[\theta_f^{(s)}, \left(\mathrm{Ad}\, f\right)_* \theta_s\right]^{(fs)} + \frac{1}{2}\left[\left(\mathrm{Ad}_f^{(s)}\right)_* \theta_s, \left(\mathrm{Ad}\, f\right)_* \theta_s\right]^{(fs)} \end{aligned}$$

$$
\begin{aligned}
& - \left(R_f^{(s)}\right)_*^{-1} \left[\theta_f^{(s)}, f, \theta_s\right]^{(s)} \\
= & \frac{1}{2} \left[\theta_{fs}, \theta_{fs}\right]^{(fs)} - \frac{1}{2} \left[\theta_f^{(s)}, \theta_f^{(s)}\right]^{(fs)} \\
& - \left(R_f^{(s)}\right)_*^{-1} \left[\theta_f^{(s)}, f, \theta_s\right]^{(s)}.
\end{aligned} \tag{3.164}
$$

Thus, overall, substituting (3.164) into (3.160), we obtain (3.159). □

For Lie groups, θ_f determines f up to right translation by a constant element, however in the non-associative case this is not necessarily true.

Lemma 3.57. *Let M be a connected manifold and suppose $A, B : M \longrightarrow \mathbb{L}$ be smooth maps. Then, $A = BC$ for some constant $C \in \mathbb{L}$ if, and only if,*

$$
\theta_A = \theta_B^{(B \backslash A)}. \tag{3.165}
$$

Proof. From (3.158),

$$
\theta_A - \theta_B^{(B \backslash A)} = \left(\mathrm{Ad}_B^{(B \backslash A)}\right)_* \theta_{B \backslash A},
$$

and thus, $B \backslash A$ is constant if, and only if, (3.165) holds. □

In particular, if $B \backslash A \in \mathcal{N}^R(\mathbb{L})$, then $\theta_B^{(B \backslash A)} = \theta_B$, and hence $\theta_A = \theta_B$. If $\mathbb{L}$ is associative, then of course $\theta_B^{(A)} = \theta_B$ for any A, B, and we get the standard result [45].

We can also get a version of the structural equation integration theorem. In particular, the question is whether an $\mathfrak{l}$-valued 1-form that satisfies the structural equation is the Darboux derivative of some $\mathbb{L}$-valued function.

Lemma 3.58. *Suppose M is a smooth manifold and $\mathbb{L}$ a smooth loop. Let $s \in C^\infty(M, \mathbb{L})$ and $\alpha \in \Omega^1(M, \mathfrak{l})$ satisfy the structural equation*

$$
d\alpha - \frac{1}{2} [\alpha, \alpha]^{(s)} = 0, \tag{3.166}
$$

then

$$
[\alpha, \alpha, \alpha - \theta_s]^{(s)} = 0, \tag{3.167}
$$

where wedge products are implied.

Proof. Applying d to (3.166) we have

$$
0 = d [\alpha, \alpha]^{(s)}
$$

$$
\begin{aligned}
&= [d\alpha, \alpha]^{(s)} - [\alpha, d\alpha]^{(s)} + [\alpha, \alpha, \theta_s]^{(s)} \\
&= [[\alpha, \alpha], \alpha] + [\alpha, \alpha, \theta_s]^{(s)} \\
&= - [\alpha, \alpha, \alpha]^{(s)} + [\alpha, \alpha, \theta_s]^{(s)},
\end{aligned}
$$

where we have used (3.39) and in the last line an analog of (3.45). □

Theorem 3.59. *Suppose M be a 1-connected (i.e. connected and simply-connected) smooth manifold and $\mathbb{L}$ a smooth loop. Let $s \in C^\infty(M, \mathbb{L})$ and $\alpha \in \Omega^1(M, \mathfrak{l})$ is such that*

$$d\alpha - \frac{1}{2}[\alpha, \alpha]^{(s)} = 0, \tag{3.168}$$

and

$$\left(\mathrm{Ad}_s^{-1}\right)_* (\alpha - \theta_s) \in \Omega^1\left(M, T_1\mathcal{N}^R(\mathbb{L})\right). \tag{3.169}$$

Then, there exists a function $f \in C^\infty\left(M, \mathcal{N}^R(\mathbb{L})\right)$ such that $\alpha = \theta_{sf}$. Moreover, f is unique up to right multiplication by a constant element of $\mathcal{N}^R(\mathbb{L})$.

Proof. Modifying the standard technique [45,51], let $N = M \times \mathcal{N}^R(\mathbb{L}) \subset M \times \mathbb{L}$. Define the projection map $\pi_M : N \longrightarrow M$ and the map

$$
\begin{aligned}
L_s : & N \longrightarrow \mathbb{L} \\
& (x, p) \mapsto s(x)\, p
\end{aligned}
$$

Given the Maurer-Cartan form θ on $\mathbb{L}$ and $\alpha \in \Omega^1(M, \mathfrak{l})$, define $\beta \in \Omega^1(N, \mathfrak{l})$ by

$$\beta = \pi_M^* \alpha - (L_s)^* \theta. \tag{3.170}$$

Then, at each point $(x, p) \in N$, define $\mathcal{D}_{(x,p)} = \ker \beta|_{(x,p)}$. We can then see that this is a distribution on N of rank $\dim M$. Let $(v, w) \in T_{(x,p)}N$, where we consider $w \in T_p\mathcal{N}^R(\mathbb{L}) \subset T_p\mathbb{L}$. Then,

$$\beta_{(x,p)}(v, w) = \alpha_x(v) - \theta_{s(x)p}\left((L_s)_*(v, w)\right). \tag{3.171}$$

Now, let $x(t)$ be a curve on M with $x(0) = x$ and $\dot{x}(0) = v$, and $p(t)$ a curve in $\mathcal{N}^R(\mathbb{L}) \subset \mathbb{L}$ with $p(0) = p$ and $\dot{p}(0) = w$. Then, using the fact that p is in the right nucleus,

$$
\begin{aligned}
\theta_{s(x)p}\left((L_s)_*(v, w)\right) &= \frac{d}{dt}(s(x(t))\,p(t)) / (s(x)\,p)\Big|_{t=0} \\
&= \frac{d}{dt} s(x(t)) / s(x)\Big|_{t=0} + \frac{d}{dt}\left(s(x)\left(p(t)/p\right)\right) / s(x)\Big|_{t=0}
\end{aligned}
$$

$$= \theta_s (v)|_x + \left(\mathrm{Ad}_{s(x)}\right)_* w.$$

So overall,

$$\beta_{(x,p)} (v, w) = (\alpha - \theta_s)_x (v) - \left(\mathrm{Ad}_{s(x)}\right)_* w. \tag{3.172}$$

Hence, $(v, w) \in \mathcal{D}_{(x,p)}$ if, and only if, $(\alpha - \theta_s)_x (v) = \left(\mathrm{Ad}_{s(x)}\right)_* w$. Now, consider $(\pi_M)_*|_{(x,p)} : \mathcal{D}_{(x,p)} \longrightarrow T_x M$. Suppose $(\pi_M)_*|_{(x,p)} (v, w) = 0$. Then, $v = 0$, and since $(\alpha - \theta_s)_x (v) = \left(\mathrm{Ad}_{s(x)}\right)_* w$, we have $w = 0$. Thus $(\pi_M)_*|_{(x,p)}$ is injective on $\mathcal{D}_{(x,p)}$. On the other hand, it is also clearly surjective, since if given $v \in T_x M$, then $\left(v, \left(\mathrm{Ad}^{-1}_{s(x)}\right)_* ((\alpha - \theta_s)_x (v))\right) \in \mathcal{D}_{(x,p)}$. Overall, $(\pi_M)_*|_{(x,p)}$ is a bijection from $\mathcal{D}_{(x,p)}$ to $T_x M$, so in particular, $\dim \mathcal{D}_{(x,p)} = \dim M$ and thus $\mathcal{D}$ is a distribution of rank $\dim M$.

Now let us show that $\mathcal{D}$ is involutive. We have

$$\begin{aligned} d\beta|_{(x,p)} &= \pi_M^* d\alpha|_{(x,p)} - (L_s)^* \, d\theta|_{(x,p)} \\ &= \frac{1}{2} \, \pi_M^* \left[\alpha, \alpha\right]^{(s)}\Big|_{(x,p)} - \frac{1}{2} \, (L_s)^* \left[\theta, \theta\right]|_{(x,p)} \\ &= \frac{1}{2} \left[\pi_M^* \alpha|_{(x,p)}, \pi_M^* \alpha|_{(x,p)} \right]^{s(x)} \\ &\quad - \frac{1}{2} \left[(L_s)^* \theta|_{(x,p)}, (L_s)^* \theta|_{(x,p)} \right]^{s(x)p}. \end{aligned} \tag{3.173}$$

Note however that because $p \in \mathcal{N}^R (\mathbb{L})$, we have $[\cdot, \cdot]^{s(x)} = [\cdot, \cdot]^{s(x)p}$. So overall, using (3.170), we get

$$d\beta|_{(x,p)} = \frac{1}{2} \left[\beta|_{(x,p)}, \beta|_{(x,p)} \right]^{s(x)} + \left[\beta|_{(x,p)}, (L_s)^* \theta|_{(x,p)} \right]^{s(x)}.$$

Thus, $d\beta = 0$ whenever $\beta = 0$, and hence $\mathcal{D} = \ker \beta$ is involutive, and by the Frobenius Theorem, $\mathcal{D}$ is integrable. Let $\mathcal{L}$ be a leaf through the point $(x, p) \in N$. Then, π_M induced a local diffeomorphism from a neighborhood to (x, p) to some neighborhood of $x \in M$. Then, let $F : U \longrightarrow \mathcal{L}$ be the inverse map, such that $F(y) = (y, f(y))$ for some $f : U \longrightarrow \mathcal{N}^R (\mathbb{L})$. By definition, $F^* \beta = 0$, so

$$\begin{aligned} 0 &= F^* \beta \\ &= F^* \left(\pi_M^* \alpha - (L_s)^* \theta \right) \\ &= \alpha - (L_s \circ f)^* \theta \end{aligned}$$

Hence, on U, $\alpha = \theta_{sf}$.

It is obvious that the distribution $\mathcal{D}$ is right-invariant with respect to $\mathcal{N}^R (\mathbb{L})$, then proceeding in the same way as for Lie groups, we find that in fact that when M is 1-connected, the function f extends to the whole manifold.

Now suppose $f, g \in C^{\infty}\left(M, \mathcal{N}^R(\mathbb{L})\right)$ such that $\theta_{sf} = \theta_{sg}$. Then using (3.157), but with roles of s and f reversed, we find

$$\theta_{sf} = \theta_s + (\mathrm{Ad}_s)_* \theta_f,$$

and similarly for g. Hence, we see that $\theta_f = \theta_g$. Using Lemma 3.57 for Lie groups, we find that $f = gC$ for some constant $C \in \mathcal{N}^R(\mathbb{L})$. □

Remark 3.60. In the case when $\mathbb{L}$ is a group, Theorem 3.59 reduces to the well-known analogous result for groups since the function s can be taken to be arbitrary. In particular, the hypothesis (3.169) is automatically satisfied in that case. On the other hand, for the loop of unit octonions, this theorem becomes trivial. In this case, $\mathcal{N}^R(\mathbb{L}) \cong \mathbb{Z}_2$, so the hypothesis (3.169) immediately implies that $\alpha = \theta_s$, even without using the equation (3.168). However, under certain additional assumptions about α and s, (3.168) may actually imply (3.169). Generally, (3.169) is stronger than (3.167), which we know holds for any $\alpha \in \Omega^1(M, \mathfrak{l})$ that satisfies (3.168). To bridge the gap between (3.167) and (3.169), additional properties of $\mathbb{L}$ and α are needed.

Corollary 3.61. *Suppose M be a 1-connected smooth manifold and $\mathbb{L}$ a smooth loop such that $\dim\left(\mathcal{N}^R(\mathbb{L})\right) = \dim\left(\mathcal{N}^R(\mathfrak{l})\right)$. Also suppose that $s \in C^{\infty}(M, \mathbb{L})$ and $\alpha \in \Omega^1(M, \mathfrak{l})$ are such that*

1. $d\alpha - \frac{1}{2}[\alpha, \alpha]^{(s)} = 0$,
2. $\alpha|_x : T_x M \longrightarrow \mathfrak{l}$ *is surjective for every* $x \in M$,
3. $T_x M \cong \ker \alpha|_x + \ker\left(\theta_s|_x - \alpha|_x\right)$ *for every* $x \in M$,
4. $s_x \in \mathcal{C}^R(\mathbb{L})$ *for every* $x \in M$.

Then, there exists a function $f \in C^{\infty}\left(M, \mathcal{N}^R(\mathbb{L})\right)$ such that $\alpha = \theta_{sf}$ with f unique up to right multiplication by a constant element of $\mathcal{N}^R(\mathbb{L})$.

Proof. Since α satisfies (3.168), from Lemma 3.58 we know that it also satisfies (3.167). Suppose $X, Y, Z \in T_x M$, such that $Z \in \ker \alpha|_x$. Then, from (3.167) we obtain

$$[\alpha(X), \alpha(Y), (\alpha - \theta_{s_x}) Z]^{(s_x)} - [\alpha(Y), \alpha(X), (\alpha - \theta_{s_x}) Z]^{(s_x)} = 0. \tag{3.174}$$

However, since $T_x M \cong \ker \alpha|_x + \ker\left(\theta_s|_x - \alpha|_x\right)$, this is true for any $Z \in T_x M$. Since $\alpha|_x$ is surjective, we hence find that for any $\xi, \eta \in \mathfrak{l}$,

$$[\xi, \eta, (\alpha - \theta_{s_x}) Z]^{(s_x)} - [\eta, \xi, (\alpha - \theta_{s_x}) Z]^{(s_x)} = 0. \tag{3.175}$$

Now, since $s_x \in \mathcal{C}^R(\mathbb{L})$, it is the right companion of some $h \in \Psi^R(\mathbb{L})$, thus applying $(h')_*^{-1}$ to (3.175), and using (3.57b), we find that for any $\xi, \eta \in \mathfrak{l}$,

$$\left[\xi,\eta,(h')_*^{-1}((\alpha-\theta_{s_x})Z)\right]^{(1)}-\left[\eta,\xi,(h')_*^{-1}((\alpha-\theta_{s_x})Z)\right]^{(1)}=0.$$

Thus, we see that for any $Z\in T_xM$, $(h')_*^{-1}((\alpha-\theta_{s_x})Z)\in\mathcal{N}^R(\mathfrak{l})$. We know that $T_1\mathcal{N}^R(\mathbb{L})\subset\mathcal{N}^R(\mathfrak{l})$, however by hypothesis, their dimensions are equal, so in fact, $T_1\mathcal{N}^R(\mathbb{L})=\mathcal{N}^R(\mathfrak{l})$. Thus, $(h')_*^{-1}((\alpha-\theta_{s_x})Z)\in T_1\mathcal{N}^R(\mathbb{L})$ and hence, from (2.49), $\left(\mathrm{Ad}_{s(x)}^{-1}\right)_*(\alpha-\theta_{s_x})\in\Omega^1\left(M,T_1\mathcal{N}^R(\mathbb{L})\right)$. This fulfils the hypothesis (3.169) for Theorem 3.59, and thus there exists a function $f\in C^\infty\left(M,\mathcal{N}^R(\mathbb{L})\right)$ such that $\alpha=\theta_{sf}$. □

Remark 3.62. Since α is assumed to be surjective in Corollary 3.61 and $\alpha=\theta_{sf}$, we see that $sf:M\longrightarrow\mathbb{L}$ is a smooth submersion.

4. Loop bundles

Let $\mathbb{L}$ be a smooth loop with the $\mathbb{L}$-algebra $\mathfrak{l}$, and let us define for brevity $\Psi^R(\mathbb{L})=\Psi$, $\mathrm{Aut}(\mathbb{L})=H$, and $\mathrm{PsAut}^R(\mathbb{L})=G\supset H$, and $\mathcal{N}^R(\mathbb{L})=\mathcal{N}$. Suppose $\Psi,H,G,\mathcal{N}$ are Lie groups. Recall that we also have $\Psi/\mathcal{N}\cong G$.

Let M be a smooth, finite-dimensional manifold with a Ψ-principal bundle $\mathcal{P}$. Then we will define several associated bundles. In general, recall that there is a one-to-one correspondence between equivariant maps from a principal bundle and sections of associated bundles. More precisely, suppose we have a manifold S with a left action $l:\Psi\times S\longrightarrow S$. Consider the associated bundle $E=\mathcal{P}\times_\Psi S$. Suppose we have a section $\tilde{f}:M\longrightarrow E$, then this defines a unique equivariant map $f:\mathcal{P}\longrightarrow S$, that is, such that for any $h\in\Psi$,

$$f_{ph}=l_{h^{-1}}(f_p).\tag{4.1}$$

Conversely, any equivariant map $f:\mathcal{P}\longrightarrow S$ defines a section $(\mathrm{id},f):\mathcal{P}\longrightarrow\mathcal{P}\times S$, and then via the quotient map $q:\mathcal{P}\times S\longrightarrow\mathcal{P}\times_\Psi S=E$, it defines a section $\tilde{f}:M\longrightarrow E$. In particular, for each $x\in M$, $\tilde{f}(x)=\lfloor p,f_p\rfloor_\Psi$ where $p\in\pi^{-1}(x)\subset\mathcal{P}$ and $\lfloor\cdot,\cdot\rfloor_\Psi$ is the equivalence class with respect to the action of Ψ:

$$(p,f_p)\sim(ph,l_{h^{-1}}(f_p))=(ph,f_{ph})\quad\text{for any }h\in\Psi.\tag{4.2}$$

For our purposes we will have the following associated bundles. Let $h\in\Psi$ and, as before, denote by h' the partial action of h.

Bundle	**Equivariant map**	**Equivariance property**
$\mathcal{P}$	$k:\mathcal{P}\longrightarrow\Psi$	$k_{ph}=h^{-1}k_p$
$\mathcal{Q}=\mathcal{P}\times_{\Psi'}\mathbb{L}$	$q:\mathcal{P}\longrightarrow\mathbb{L}$	$q_{ph}=(h')^{-1}q_p$
$\mathring{\mathcal{Q}}=\mathcal{P}\times_\Psi\mathring{\mathbb{L}}$	$r:\mathcal{P}\longrightarrow\mathring{\mathbb{L}}$	$r_{ph}=h^{-1}(r_p)$
$\mathcal{N}\cong\mathcal{P}\times_\Psi(\Psi/H)$	$s:\mathcal{P}\longrightarrow\Psi/H\cong\mathcal{C}\subset\mathring{\mathbb{L}}$	$s_{ph}=h^{-1}(s_p)$
$\mathcal{A}=\mathcal{P}\times_{\Psi'_*}\mathfrak{l}$	$\eta:\mathcal{P}\longrightarrow\mathfrak{l}$	$\eta_{ph}=(h')_*^{-1}\eta_p$
$\mathfrak{p}_{\mathcal{P}}=\mathcal{P}\times_{(\mathrm{Ad}_\xi)_*}\mathfrak{p}$	$\xi:\mathcal{P}\longrightarrow\mathfrak{p}$	$\xi_{ph}=\left(\mathrm{Ad}_h^{-1}\right)_*\xi_p$
$\mathcal{G}=\mathcal{P}\times_{\Psi'}G$	$\gamma:\mathcal{P}\longrightarrow G$	$\gamma_{ph}=(h')^{-1}\gamma_p$
$\mathrm{Ad}(\mathcal{P})=\mathcal{P}\times_{\mathrm{Ad}_\Psi}\Psi$	$u:\mathcal{P}\longrightarrow\Psi$	$u_{ph}=h^{-1}u_ph$

(4.3)

The bundle $\mathcal{Q}$ is the loop bundle with respect to the partial action Ψ' and the bundle $\mathring{\mathcal{Q}}$ is the loop bundle with respect to the full action of Ψ. The bundle $\mathcal{N}$ has fibers isomorphic to $\Psi/H \cong \mathcal{C}$, which is the set of right companions $\mathcal{C}^R(\mathbb{L}) \subset \mathring{\mathbb{L}}$. Thus it is a subbundle of $\mathring{\mathcal{Q}}$. Equivalently, $\mathcal{N} = \mathcal{P}/H$ is the orbit space of the right H-action on $\mathcal{P}$. Recall that the structure group of $\mathcal{P}$ reduces to H if, and only if, the bundle N has a global section. If this is the case, then we can reduce the bundle $\mathcal{P}$ to a principal H-bundle $\mathcal{H}$ over M, and then since $H \subset G$, lift to a principal G-bundle $\mathcal{G}$. Also, let $\mathcal{Q} = \mathcal{P} \times_{\Psi'} \mathbb{L}$ be the bundle associated to $\mathcal{P}$ with fiber $\mathbb{L}$, where Ψ' denotes that the left action on $\mathbb{L}$ is via the partial action of Ψ.

We also have some associated vector bundles - namely the vector bundle $\mathcal{A}$ with fibers isomorphic to the $\mathbb{L}$-algebra $\mathfrak{l}$ with the tangent partial action of Ψ and the vector bundle $\mathfrak{p}_{\mathcal{P}}$ with fibers isomorphic to the Lie algebra $\mathfrak{p}$, with the adjoint action of Ψ.

Example 4.1. Let $\mathbb{L} = U\mathbb{O}$ be the Moufang loop of unit octonions. In this case, $\Psi = \mathrm{Spin}(7)$, $H = G_2$, $G = SO(7)$, $\mathcal{N} = \mathbb{Z}_2$, and then we have the well-known relations

$$\begin{aligned} SO(7) &\cong \mathrm{Spin}(7)/\mathbb{Z}_2 \\ \mathrm{Spin}(7)/G_2 &\cong U\mathbb{O} \cong S^7 \\ SO(7)/G_2 &\cong S^7/\mathbb{Z}_2. \end{aligned}$$

Then, if an orientable 7-manifold has spin structure, we have a principal $\mathrm{Spin}(7)$-bundle $\mathcal{P}$ over M and the corresponding $\mathrm{Spin}(7)/G_2$-bundle always has a smooth section, hence allowing the $\mathrm{Spin}(7)$-bundle to reduce to a G_2-principal bundle, which in turn lifts to an $SO(7)$-bundle. The associated bundle $\mathcal{Q}$ in this case transforms under $SO(7)$, and is precisely the unit subbundle of the octonion bundle $\mathbb{R} \oplus TM$ defined in [15]. The bundle $\mathring{\mathcal{Q}}$ then transforms under $\mathrm{Spin}(7)$ and corresponds to the bundle of unit spinors. The associated vector bundle $\mathcal{A}$ in this case has fibers isomorphic to $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$, and then the bundle itself is isomorphic to the tangent bundle TM.

Let $s\colon \mathcal{P} \longrightarrow \mathring{\mathbb{L}}$ be an equivariant map. In particular, the equivalence class $\lfloor p, s_p \rfloor_{\Psi}$ defines a section of the bundle $\mathring{\mathcal{Q}}$. We will refer to s as *the defining map* (or *section*). It should be noted that such a map may not always exist globally. If $\mathbb{L}$ is a G-loop**,** then $\mathcal{Q} \overset{\circ}{\cong} N$ and hence existence of a global section of $\mathring{\mathcal{Q}}$ is equivalent to the reduction of the structure group of $\mathcal{P}$. There may be topological obstructions for this.

Example 4.2. As in Example 2.24, let $\mathbb{L} = U\mathbb{C} \cong U(1)$ be the unit complex numbers, and $\Psi = U(n)$, $H = G = SU(n)$. Then in this setting, $\mathcal{P}$ is a principal $U(n)$-bundle over M and $\mathcal{Q}$ is a circle bundle. Existence of a section of $\mathcal{Q}$ is equivalent to the reduction of the structure group of $\mathcal{P}$ to $SU(n)$. The obstruction for this is the first Chern class of $\mathcal{Q}$ [33]. In the quaternionic case, at least for the tangent bundle, obstruction to structure group reduction from $Sp(n)\,Sp(1)$ to $Sp(n)$ is also given by Chern class.

Given equivariant maps $q, r : \mathcal{P} \longrightarrow \mathbb{L}$, we can define an equivariant product using s, such that for any $p \in \mathcal{P}$,

$$q \circ_s r|_p = q_p \circ_{s_p} r_p. \tag{4.4}$$

Indeed, using (2.38),

$$\begin{aligned} q \circ_s r|_{ph} &= q_{ph} \circ_{s_{ph}} r_{ph} \\ &= (h')^{-1} q_p \circ_{h^{-1}(s_p)} (h')^{-1} r_p \\ &= (h')^{-1} \left(q \circ_s r|_p \right). \end{aligned} \tag{4.5}$$

In particular, this induces a fiberwise product on sections of $\mathcal{Q}$. Similarly, we define equivariant left and right quotients, and thus well-defined fiberwise quotients of sections of $\mathcal{Q}$.

Remark 4.3. The map s is required to define an equivariant product of two $\mathbb{L}$-valued maps. In the G_2-structure case, as discussed above, sections of $\mathring{\mathcal{Q}}$ correspond to unit spinors, and each unit spinor defines a G_2-structure, and hence a product on the corresponding octonion bundle [15]. On the other hand, a product of an equivariant $\mathbb{L}$-valued map and an equivariant $\mathring{\mathbb{L}}$-valued map will be always equivariant, using (2.12a). In the G_2-structure case, this corresponds to the Clifford product of a unit octonion, interpreted as an element of $\mathbb{R} \oplus T_x M$ at each point, and a unit spinor. The result is then again a unit spinor. This does not require any additional structure beyond the spinor bundle.

Given equivariant maps $\xi, \eta : \mathcal{P} \longrightarrow \mathfrak{l}$, we can define an equivariant bracket using s. For any $p \in \mathcal{P}$:

$$[\xi, \eta]^{(s)}\Big|_p = [\xi_p, \eta_p]^{(s_p)}. \tag{4.6}$$

Here the equivariance follows from (3.57). Using (3.75) we then also have an equivariant map φ_s from equivariant $\mathfrak{p}$-valued maps to equivariant $\mathfrak{l}$-valued maps:

$$\varphi_s(\gamma)|_p = \varphi_{s_p}(\gamma_p). \tag{4.7}$$

Other related objects such as the Killing form $K^{(s)}$ and the adjoint φ_s^t to φ_s are then similarly also equivariant.

Overall, we can condense the above discussion into the following definition and theorem.

Definition 4.4. A *loop bundle structure* over a smooth manifold M is a quadruple $(\mathbb{L}, \Psi, \mathcal{P}, s)$ where

1. $\mathbb{L}$ is a finite-dimensional smooth loop with a smoothly acting group of right pseudoautomorphism pairs Ψ.
2. $\mathcal{P}$ is a principal Ψ-bundle over M.
3. $s : \mathcal{P} \longrightarrow \mathring{\mathbb{L}}$ is a smooth equivariant map.

Theorem 4.5. *Given a loop bundle structure $(\mathbb{L}, \Psi, \mathcal{P}, s)$ over a manifold M, and associated bundles $\mathcal{Q} = \mathcal{P} \times_{\Psi'} \mathbb{L}$, $\mathring{\mathcal{Q}} = \mathcal{P} \times_{\Psi} \mathring{\mathbb{L}}$, $\mathcal{A} = \mathcal{P} \times_{\Psi'_*} \mathfrak{l}$, and $\mathfrak{p}_{\mathcal{P}} = \mathcal{P} \times_{(\mathrm{Ad}_\xi)_*} \mathfrak{p}$, where $\mathfrak{l}$ is the $\mathbb{L}$-algebra of $\mathbb{L}$ and $\mathfrak{p}$ the Lie algebra of Ψ,*

1. *s determines a smooth section $\sigma \in \Gamma\left(\mathring{\mathcal{Q}}\right)$.*
2. *For any $A, B \in \Gamma(\mathcal{Q})$, σ defines a fiberwise product $A \circ_\sigma B$, via (4.4).*
3. *For any $X, Y \in \Gamma(\mathcal{A})$, σ defines a fiberwise bracket $[X, Y]^{(\sigma)}$, via (4.6).*
4. *σ defines a fiberwise map $\varphi_\sigma : \Gamma(\mathfrak{p}_{\mathcal{P}}) \longrightarrow \Gamma(\mathcal{A})$, via (4.7).*

4.1. Connections and torsion

Suppose the principal Ψ-bundle $\mathcal{P}$ has a principal Ehresmann connection given by the decomposition

$$T\mathcal{P} = \mathcal{HP} \oplus \mathcal{VP} \tag{4.8}$$

with $\mathcal{H}_{ph}\mathcal{P} = (R_h)_* \mathcal{H}_p\mathcal{P}$ for any $p \in \mathcal{P}$ and $h \in \Psi$ and $\mathcal{VP} = \ker d\pi$, where $\pi : \mathcal{P} \longrightarrow M$ is the bundle projection map. Let the projection

$$v : T\mathcal{P} \longrightarrow \mathcal{VP}$$

be the Ehresmann connection 1-form. Similarly, define the projection $\mathrm{proj}_{\mathcal{H}} : T\mathcal{P} \longrightarrow \mathcal{HP}$.

Let $\mathfrak{p}$ be the Lie algebra of Ψ. Then, as it is well-known, we have an isomorphism

$$\begin{aligned} \sigma : &\mathcal{P} \times \mathfrak{p} \longrightarrow \mathcal{VP} \\ (p, \xi) &\mapsto \frac{d}{dt}\left(p \exp(t\xi)\right)\Big|_{t=0}. \end{aligned} \tag{4.9}$$

For any $\xi \in \mathfrak{p}$, this defines a vertical vector field $\sigma(\xi)$ on $\mathcal{P}$. Given the Ehresmann connection 1-form v, define the $\mathfrak{p}$-valued connection 1-form ω via

$$(\pi, \omega) = \sigma^{-1} \circ v : T\mathcal{P} \longrightarrow \mathcal{P} \times \mathfrak{p}$$

and recall that for any $h \in \Psi$,

$$(R_h)^* \omega = \mathrm{Ad}_{h^{-1}} \circ \omega.$$

As before, suppose S is a manifold with a left action l of Ψ. Given an equivariant map $f : \mathcal{P} \longrightarrow S$, define

$$d^{\mathcal{H}} f := f_* \circ \mathrm{proj}_{\mathcal{H}} : T\mathcal{P} \longrightarrow \mathcal{H}\mathcal{P} \longrightarrow TS. \tag{4.10}$$

This is then a horizontal map since it vanishes on any vertical vectors. Equivalently, for any $X_p \in T_p\mathcal{P}$, if $\gamma(t)$ is a curve on $\mathcal{P}$ with $\gamma(0) = 0$ and $\dot{\gamma}(0) = \mathrm{proj}_{\mathcal{H}} X_p \in \mathcal{H}_p\mathcal{P}$, then

$$d^{\mathcal{H}} f\big|_p (X_p) = \frac{d}{dt} f(\gamma(t))\bigg|_{t=0}. \tag{4.11}$$

The map $d^{\mathcal{H}} f$ is moreover still equivariant. The group Ψ acts on $T\mathcal{P}$ via pushforwards of the right action of Ψ on $\mathcal{P}$. Let $h \in \Psi$, so that $r_h : \mathcal{P} \longrightarrow \mathcal{P}$ gives the right action of Ψ on $\mathcal{P}$, and the corresponding action of Ψ on $T\mathcal{P}$ is $(r_h)_* : T\mathcal{P} \longrightarrow T\mathcal{P}$. Note that the corresponding action of Ψ on TS is then $(l_{h^{-1}})_* : TS \longrightarrow TS$. Now,

$$\begin{aligned} d^{\mathcal{H}} f \circ (r_h)_* &= f_* \circ \mathrm{proj}_{\mathcal{H}} \circ (r_h)_* = f_* \circ (r_h)_* \circ \mathrm{proj}_{\mathcal{H}} \\ &= (f \circ r_h)_* \circ \mathrm{proj}_{\mathcal{H}} = (l_{h^{-1}} \circ f)_* \circ \mathrm{proj}_{\mathcal{H}} \\ &= (l_{h^{-1}})_* \circ d^{\mathcal{H}} f \end{aligned}$$

where we have used the equivariance of both f and $\mathrm{proj}_{\mathcal{H}}$. So indeed, $d^{\mathcal{H}} f$ is equivariant. Now consider the quotient map $q' : \mathcal{P} \times TS \longrightarrow \mathcal{P} \times_{\Psi} TS$, where Ψ acts via r_h on $\mathcal{P}$ and $(l_{h^{-1}})_*$ on TS. This is a partial differential of the map $q : \mathcal{P} \times S \longrightarrow E$. Since $d^{\mathcal{H}} f$ is horizontal, it vanishes on the kernel of $\pi_* : T\mathcal{P} \longrightarrow TM$. Given $\tilde{f}$, the section of the associated bundle $\mathcal{P} \times_{\Psi} S$ that corresponds to f, we can use $d^{\mathcal{H}} f$ to define the unique map

$$d^{\mathcal{H}} \tilde{f} : TM \longrightarrow \mathcal{P} \times_{\Psi} TS \tag{4.12}$$

such that

$$d^{\mathcal{H}} \tilde{f} \circ \pi_* = \left(\pi_{T\mathcal{P}}, d^{\mathcal{H}} f\right) \circ q'$$

where $\pi_{T\mathcal{P}} : T\mathcal{P} \longrightarrow \mathcal{P}$ is the bundle projection for $T\mathcal{P}$. Moreover, $d^{\mathcal{H}} \tilde{f}$ covers the identity map on M, and hence is a section of the fiber product $TM \times_M (\mathcal{P} \times_{\Psi} TS)$. This construction is summarized in the commutative diagram in Fig. 3.

Of course, if S is a vector space, then this reduces to the usual definition of the exterior covariant derivative of a vector bundle-valued function and $d^{\mathcal{H}} f$ is a vector-bundle-valued 1-form.

Given the above correspondence between equivariant maps from $\mathcal{P}$ and sections of associated bundles, for convenience, we will work with equivariant maps rather than sections. This will allow us to use the properties of $\mathbb{L}$ from the previous section more directly.

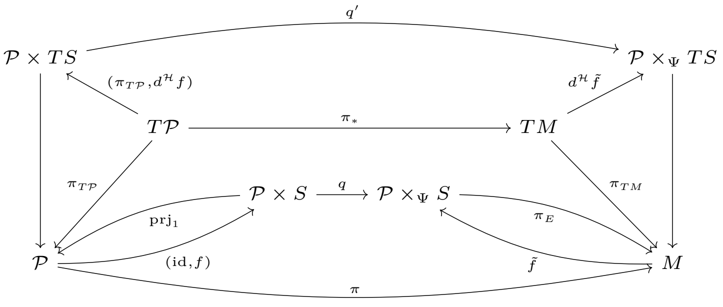

Fig. 3. Covariant derivative on maps and sections.

Given a $\mathfrak{p}$-valued connection 1-form ω on $\mathcal{P}$, we can concretely work out $d^{\mathcal{H}}f$. Suppose $X \in \Gamma(T\mathcal{P})$ is a vector field on $\mathcal{P}$, then using the definition (4.10), we have

$$\begin{aligned}\left(d^{\mathcal{H}}f\right)(X) =& df\left(\mathrm{proj}_{\mathcal{H}}(X)\right)\\ =& df\left(X - v(X)\right)\\ =& df(X) - df\left(\sigma\left(\pi_{TP}(X), \omega(X)\right)\right)\end{aligned}$$

where from (4.9), for $p \in \mathcal{P}$,

$$\sigma\left(\pi_{TP}(X), \omega(X)\right)_p = \frac{d}{dt}\left(p \exp\left(t\omega\left(X_p\right)\right)\right)\Big|_{t=0}.$$

Now, let $\gamma(t) = \exp\left(t\omega\left(X_p\right)\right) \in \Psi$, and note that $\gamma(t)^{-1} = \gamma(-t)$, so that

$$\begin{aligned}df\left(\sigma\left(\pi_{TP}(X), \omega(X)\right)\right)\big|_p =& \frac{d}{dt}\left(f\left(p\gamma(t)\right)\right)\Big|_{t=0}\\ =& -\frac{d}{dt}\left(\exp\left(t\omega\left(X_p\right)\right)f(p)\right)\Big|_{t=0}\\ =& -\omega\left(X_p\right)\cdot f(p)\end{aligned} \tag{4.13}$$

where we have used the equivariance of f and where, $\omega\left(X_p\right) \cdot f(p) \in T_{f(p)}S$ denotes the infinitesimal action of $\omega\left(X_p\right) \in \mathfrak{p}$ on S.

Lemma 4.6. *Let s be a Ψ-equivariant S-valued function on $\mathcal{P}$ and let ω be a $\mathfrak{p}$-valued connection 1-form on $\mathcal{P}$, then the covariant differential $d^{\mathcal{H}}s : T\mathcal{P} \longrightarrow TS$ is given by*

$$d^{\mathcal{H}}s = ds + \omega \cdot s \tag{4.14}$$

where $\omega \cdot s : T_p\mathcal{P} \longrightarrow T_{s(p)}S$ for each $p \in \mathcal{P}$ gives the infinitesimal action of ω on S.

Now, more concretely, given a principal connection ω on $\mathcal{P}$, consider the induced covariant derivatives on equivariant $\mathbb{L}$- and $\mathring{\mathbb{L}}$-valued maps. To avoid confusion, denote $d^{\mathcal{H}}$ acting on $\mathbb{L}$-valued maps by D and by $\mathring{D}$ when it is acting on $\mathring{\mathbb{L}}$-valued maps. Similarly, consider equivariant $\mathfrak{l}$-valued maps from $\mathcal{P}$. Given $\xi : \mathcal{P} \longrightarrow \mathfrak{l}$ such that $\xi_{ph} = \left(h^{-1}\right)'_* (\xi)$, define the covariant derivative $d^{\mathcal{H}}\xi$ via (4.14), so overall, given $X \in \Gamma(T\mathcal{P})$,

$$d^{\mathcal{H}}_X \xi = d_X \xi + \omega(X) \cdot \xi \tag{4.15}$$

where $\omega(X) \cdot \xi$ refers to the linear representation of the Lie algebra $\mathfrak{p}$ on $\mathfrak{l}$ given by (3.87).

We have the following useful relation between D and $\mathring{D}$.

Lemma 4.7. *Suppose $A : \mathcal{P} \longrightarrow \mathbb{L}$ and $s : \mathcal{P} \longrightarrow \mathring{\mathbb{L}}$ are equivariant, and let $p \in \mathcal{P}$. Then,*

$$\mathring{D}(As)\Big|_p = \left(R_{s_p}\right)_* DA|_p + \left(L_{A_p}\right)_* \mathring{D}s\Big|_p. \tag{4.16}$$

Note that $\mathring{D}(As)\Big|_p : T_p\mathcal{P} \longrightarrow T_{As}\mathring{\mathbb{L}}$.

Proof. Let $X_p \in T_p\mathcal{P}$ and let $p(t)$ be a curve on $\mathcal{P}$ with $p(0) = p$ and $\dot{p}(0) = \operatorname{proj}_{\mathcal{H}}(X_p) \in \mathcal{H}_p\mathcal{P}$. Consider

$$\mathring{D}(As)\Big|_p (X_p) = \frac{d}{dt}\left(A_{p(t)} s_{p(t)}\right)\Big|_{t=0} \tag{4.17}$$

However,

$$\begin{aligned} \frac{d}{dt}\left(A_{p(t)} s_{p(t)}\right)\Big|_{t=0} &= \frac{d}{dt}\left(A_{p(t)} s_p\right)\Big|_{t=0} + \frac{d}{dt}\left(A_p s_{p(t)}\right)\Big|_{t=0} \\ &= \left(R_{s_p}\right)_* (DA)_p (X_p) + \left(L_{A_p}\right)_* \left(\mathring{D}s\right)_p (X_p) \end{aligned} \tag{4.18}$$

and thus (4.16) holds. □

Suppose now $(\mathbb{L}, \Psi, \mathcal{P}, s)$ is a loop bundle structure, as in Definition 4.4, so that s is an $\mathring{\mathbb{L}}$-valued equivariant map. Then we have the following important definition.

Definition 4.8. The *torsion* $T^{(s,\omega)}$ of $(\mathbb{L}, \Psi, \mathcal{P}, s)$ with respect to ω is a horizontal $\mathfrak{l}$-valued 1-form on $\mathcal{P}$ given by

$$T^{(s,\omega)} = \theta_s \circ \operatorname{proj}_{\mathcal{H}} \tag{4.19}$$

where θ_s is the Darboux derivative of s. Equivalently, at $p \in \mathcal{P}$, we have

$$T^{(s,\omega)}\Big|_p = \left(R_{s_p}^{-1}\right)_* \mathring{D}s\Big|_p. \tag{4.20}$$

Thus, $T^{(s,\omega)}$ is the horizontal component of θ_s. We also easily see that it is Ψ-equivariant. Using the equivariance of s and $\mathring{D}s$, we have for $h \in \Psi$,

$$T_{ph}^{(s,\omega)} = (h'_*)^{-1} T_p^{(s,\omega)}. \tag{4.21}$$

Thus, $T^{(s,\omega)}$ is a *basic* (i.e. horizontal and equivariant) $\mathfrak{l}$-valued 1-form on $\mathcal{P}$, and thus defines a 1-form on M with values in the associated vector bundle $\mathcal{A} = \mathcal{P} \times_{\Psi'_*} \mathfrak{l}$. We also have the following key property of $T^{(s,\omega)}$.

Theorem 4.9. *Suppose $T^{(s,\omega)}$ is as given in Definition 4.8 and also let $\hat{\omega}^{(s)} \in \Omega^1(\mathcal{P}, \mathfrak{l})$ be given by*

$$\hat{\omega}^{(s)} = \varphi_s(\omega). \tag{4.22}$$

Then,

$$\theta_s = T^{(s,\omega)} - \hat{\omega}^{(s)}. \tag{4.23}$$

In particular, $T^{(s,\omega)}$ and the quantity $-\hat{\omega}^{(s)}$ are respectively the horizontal and vertical components of θ_s.

Proof. Let $p \in \mathcal{P}$. Then, from (4.14) we have

$$\begin{aligned} \left(R_{s_p}^{-1}\right)_* \mathring{D}s\Big|_p &= \left(R_{s_p}^{-1}\right)_* ds|_p + \left(R_{s_p}^{-1}\right)_* (\omega \cdot s_p) \\ &= \theta_s|_p + \frac{d}{dt}\left(\exp(t\omega_p)(s_p)\right)\Big/ s_p\Big|_{t=0} \\ &= \theta_s|_p + \varphi_{s_p}(\omega_p) \end{aligned} \tag{4.24}$$

where we have used the definition (3.74) of φ_s. Hence we get (4.23). □

Suppose $p(t)$ is a curve on $\mathcal{P}$ with $p(0) = p$ and with a horizontal initial velocity vector $\dot{p}(0) = X_p^{\mathcal{H}}$. Then, by definition,

$$\frac{d}{dt} s_{p(t)}\Big|_{t=0} = \mathring{D}_X s\Big|_p = \left(R_{s_p}\right)_* T_{X_p}^{(s,\omega)}\Big|_p, \tag{4.25}$$

where $T_X^{(s,\omega)} = T^{(s,\omega)}(X) \in \mathfrak{l}$. This observation will come in useful later on.

Remark 4.10. If $s_p \in \mathcal{C} \cong \ominus/H$ for all $p \in \mathcal{P}$, then as we know, the structure group of $\mathcal{P}$ is reduced to H. Moreover, the reduced holonomy group of ω is contained in H if, and only if, there exists such a map s with $d^{\mathcal{H}}s = 0$. This is equivalent to $T^{(s,\omega)} = 0$, so this is the motivation for calling this quantity the torsion. If s is not necessarily in $\mathcal{C}$, then we can still say something about the holonomy of ω in the case $d^{\mathcal{H}}s = 0$. Let $p \in \mathcal{P}$ and suppose $\Gamma(t)$ is the horizontal lift with respect to the connection ω of some closed curve based at $\pi(p)$. Then, the endpoint of Γ is $\Gamma(1) = ph$ for some $h \in \Psi$. The set of all such $h \in \Psi$ form the holonomy group $\mathrm{Hol}_p(\omega)$ of ω at p [25]. Now if we have an equivariant map $s : \mathcal{P} \longrightarrow \mathbb{L}$, then $s \circ \Gamma$ is a curve on $\mathbb{L}$ with $s(\Gamma(1)) = s_{ph} = h^{-1}s_p$. However, $\frac{d}{dt}(s \circ \Gamma(t)) = (d^{\mathcal{H}}s)_{s\circ\Gamma(t)}\dot{\Gamma}(t)$ since the velocity vectors of $\Gamma(t)$ are horizontal. Thus, if $d^{\mathcal{H}}s = 0$ everywhere, then the curve $s \circ \Gamma(t)$ is constant, and hence $h^{-1}s_p = s_p$. By (2.44), this means that $h \in \mathrm{Aut}(\mathbb{L}, \circ_{s_p})$. This is true for any horizontal lift Γ, hence we see that $\mathrm{Hol}_p(\omega) \subset \mathrm{Aut}(\mathbb{L}, \circ_{s_p})$.

The torsion also enters expressions for covariant derivatives of the loop product, loop algebra bracket, as well as the map φ_s.

Theorem 4.11. *Suppose $A, B : \mathcal{P} \longrightarrow \mathbb{L}$, and $s : \mathcal{P} \longrightarrow \mathring{\mathbb{L}}$ are equivariant, and let $p \in \mathcal{P}$. Then,*

$$D(A \circ_s B)|_p = \left(R_{B_p}^{(s_p)}\right)_* DA|_p + \left(L_{A_p}^{(s_p)}\right)_* DB|_p \\ + \left[A_p, B_p, T^{(s,\omega)}\Big|_p\right]^{(s_p)}. \tag{4.26}$$

If $\xi, \eta : \mathcal{P} \longrightarrow \mathfrak{l}$ are equivariant, then

$$d^{\mathcal{H}}[\xi, \eta]^{(s)} = \left[d^{\mathcal{H}}\xi, \eta\right]^{(s)} + \left[\xi, d^{\mathcal{H}}\eta\right]^{(s)} + \left[\xi, \eta, T^{(s,\omega)}\right]^{(s)} - \left[\eta, \xi, T^{(s,\omega)}\right]^{(s)}. \tag{4.27}$$

The $\mathfrak{l} \otimes \mathfrak{p}^*$-valued map $\varphi_s : \mathcal{P} \longrightarrow \mathfrak{l} \otimes \mathfrak{p}^*$ satisfies

$$d^{\mathcal{H}}\varphi_s = \mathrm{id}_{\mathfrak{p}} \cdot T^{(s,\omega)} - \left[\varphi_s, T^{(s,\omega)}\right]^{(s)}, \tag{4.28}$$

where $\mathrm{id}_{\mathfrak{p}}$ is the identity map of $\mathfrak{p}$ and $\cdot$ denotes the action of the Lie algebra $\mathfrak{p}$ on $\mathfrak{l}$ given by (3.87).

Proof. Let $X_p \in T_p\mathcal{P}$ and let $p(t)$ be a curve on $\mathcal{P}$ with $p(0) = p$ and $\dot{p}(0) = \mathrm{proj}_{\mathcal{H}}(X_p) \in \mathcal{H}_p\mathcal{P}$. To show (4.26), first note that

$$D(A \circ_s B)|_p(X_p) = \frac{d}{dt}\left(A_{p(t)} \circ_{s_{p(t)}} B_{p(t)}\right)\Big|_{t=0}. \tag{4.29}$$

However,

$$\frac{d}{dt}\left(A_{p(t)} \circ_{s_{p(t)}} B_{p(t)}\right)\Big|_{t=0} = \frac{d}{dt}\left(A_{p(t)} \circ_{s_p} B_p\right)\Big|_{t=0} + \frac{d}{dt}\left(A_p \circ_{s_p} B_{p(t)}\right)\Big|_{t=0} \\ + \frac{d}{dt}\left(A_p \circ_{s_{p(t)}} B_p\right)\Big|_{t=0}$$

$$= \left(R^{(s_p)}_{B_p}\right)_* DA|_p (X_p) + \left(L^{(s_p)}_{A_p}\right)_* DB|_p (X_p) + \frac{d}{dt}\left(A_p \circ_{s_{p(t)}} B_p\right)\Big|_{t=0} \tag{4.30}$$

and then, using Lemma A.1,

$$\begin{aligned} \frac{d}{dt}\left(A_p \circ_{s_{p(t)}} B_p\right)\Big|_{t=0} &= \frac{d}{dt}\left(\left(A_p \cdot B_p s_{p(t)}\right)/s_{p(t)}\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(\left(A_p \cdot B_p s_{p(t)}\right)/s_p\right)\Big|_{t=0} \\ &\quad + \frac{d}{dt}\left(\left(A_p \cdot B_p s_p\right)/s_p \cdot s_{p(t)}\right)/s_p\Big|_{t=0}. \end{aligned} \tag{4.31}$$

Looking at each term in (4.31), we have

$$\begin{aligned} \left(A_p \cdot B_p s_{p(t)}\right)/s_p &= \left(A_p \cdot B_p\left({}^{s_{p(t)}}\!/_{s_p} \cdot s_p\right)\right)/s_p \\ &= A_p \circ_{s_p}\left(B_p \circ_{s_p}\left({}^{s_{p(t)}}\!/_{s_p}\right)\right) \end{aligned}$$

and

$$\left(\left(A_p \cdot B_p s_p\right)/s_p \cdot s_{p(t)}\right)/s_p = \left(A_p \circ_{s_p} B_p\right) \circ_{s_p} \left({}^{s_{p(t)}}\!/_{s_p}\right).$$

Overall (4.30) becomes,

$$\frac{d}{dt}\left(A_p \circ_{s_{p(t)}} B_p\right)\Big|_{t=0} = \left(\left(L^{(s_p)}_{A_p} \circ L^{(s_p)}_{B_p}\right)_* - \left(L^{(s_p)}_{A_p \circ_{s_p} B_p}\right)_*\right)\left(R^{-1}_{s_p}\right)_* \mathring{D}s\Big|_p (X_p) \tag{4.32}$$

and hence we get (4.26) using the definitions of $T^{(s,\omega)}$ and the mixed associator (3.11).

To show (4.27), note that

$$\begin{aligned} d^{\mathcal{H}}_X\left([\xi,\eta]^{(s)}\right)\Big|_p &= \frac{d}{dt}\left[\xi_{p(t)},\eta_{p(t)}\right]^{(s_{p(t)})}\Big|_{t=0} \\ &= \left[d^{\mathcal{H}}_X\xi\big|_p, \eta_p\right]^{(s_p)} + \left[\xi_p, d^{\mathcal{H}}_X\eta\big|_p\right]^{(s_p)} \\ &\quad + \frac{d}{dt}\left[\xi_p,\eta_p\right]^{(s_{p(t)})}\Big|_{t=0}. \end{aligned}$$

However, using (3.39) and (4.25), the last term becomes

$$\frac{d}{dt}\left[\xi_p,\eta_p\right]^{(s_{p(t)})}\Big|_{t=0} = \left[\xi_p,\eta_p, T^{(s,\omega)}_X\Big|_p\right]^{(s_p)} - \left[\eta_p,\xi_p, T^{(s,\omega)}_X\Big|_p\right]^{(s_p)}$$

and hence we obtain (4.27).

Let us now show (4.28). From (3.81), given $\gamma \in \mathfrak{p}$, setting $\hat{\gamma}(r) = \varphi_r(\gamma)$ for each $r \in \mathbb{L}$, we have

$$d\hat{\gamma}|_r(\rho_r(\xi)) = \gamma \cdot \xi - [\hat{\gamma}(r), \xi]^{(r)} \tag{4.33}$$

for some $\xi \in \mathfrak{l}$. Now for a map $s : \mathcal{P} \longrightarrow \mathbb{L}$ and some vector field X on $\mathcal{P}$, we have at each $p \in \mathcal{P}$

$$\begin{aligned} d(\varphi_s(\gamma))|_p(X) &= d\hat{\gamma}|_{s_p} \circ ds|_p(X) \\ &= d\hat{\gamma}|_{s_p}\left(\rho_{s_p}(\theta_s(X_p))\right) \\ &= \gamma \cdot \theta_s(X_p) - \left[\varphi_{s_p}(\gamma), \theta_s(X_p)\right]^{(s_p)}. \end{aligned} \tag{4.34}$$

Therefore, $d\varphi_s$ is given by

$$d\varphi_s(\gamma) = \gamma \cdot \theta_s - [\varphi_s(\gamma), \theta_s]^{(s)}. \tag{4.35}$$

To obtain $d^{\mathcal{H}}\varphi_s$ we take the horizontal component, and hence using (4.23), we just replace θ_s in (4.35) by $T^{(s,\omega)}$, which gives (4.28). □

Remark 4.12. If $\mathbb{L}$ is associative, i.e. is a group, then certainly $A \circ_s B = AB$ and this is then an equivariant section, if A and B are such. In (4.26) the second term on the right vanishes, and thus D satisfies the product rule with respect to multiplication on $\mathbb{L}$.

We can rewrite (4.16) as

$$\begin{aligned} \mathring{D}(As) &= (DA)s + A\left(\left(\mathring{D}s\right)/s \cdot s\right) \\ &= (DA)s + \left(A \circ_s T^{(s,\omega)}\right)s. \end{aligned} \tag{4.36}$$

Using this, we can then define an adapted covariant derivative $D^{(s)}$ on equivariant $\mathbb{L}$-valued maps, given by

$$D^{(s)}A\Big|_p = \left(R_{s_p}^{-1}\right)_* \mathring{D}(As)\Big|_p = DA|_p + \left(L_{A_p}^{(s_p)}\right)_* T_p^{(s,\omega)} \tag{4.37}$$

with respect to which,

$$D^{(s)}(A \circ_s B)\Big|_p = \left(R_{B_p}^{(s_p)}\right)_* DA|_p + \left(L_{A_p}^{(s_p)}\right)_* D^{(s)}B\Big|_p. \tag{4.38}$$

This is the precise analog of the octonion covariant derivative from [15]. The derivative $D^{(s)}$ essentially converts an $\mathbb{L}$-valued map into an $\mathring{\mathbb{L}}$-valued one using s and then differentiates it using $\mathring{D}$ before converting back to $\mathbb{L}$. In particular, if we take $A = 1$,

$$D^{(s)}1 = T^{(s,\omega)}. \tag{4.39}$$

Remark 4.13. Up to the sign of T, (4.26) and (4.37) are precisely the expressions obtained in [15] for the covariant derivative with respect to the Levi-Civita connection of the product on the octonion bundle over a 7-manifold. In that case, T is precisely the torsion of the G_2-structure that defines the octonion bundle. This provides additional motivation for calling this quantity the torsion of s and ω. In the case of G_2-structures, usually one takes the torsion with respect to the preferred Levi-Civita connection, however in this more general setting, we don't have a preferred connection, thus $T^{(s,\omega)}$ should also be taken to depend on the connection.

Corollary 4.14. *Suppose $\mathbb{L}$ is an alternative loop, so that the associator is skew-symmetric. Suppose $\xi, \eta \longrightarrow \mathfrak{l}$ and $s : \mathcal{P} \longrightarrow \mathring{\mathbb{L}}$ are equivariant. Then, defining a modified exterior derivative $d^{(s)}$ on equivariant maps from $\mathcal{P}$ to $\mathfrak{l}$ via*

$$d^{(s)}\xi = d^{\mathcal{H}}\xi + \frac{1}{3}\left[\xi, T^{(s)}\right]^{(s)}, \tag{4.40}$$

it satisfies

$$d^{(s)}\left[\xi,\eta\right]^{(s)} = \left[d^{(s)}\xi,\eta\right]^{(s)} + \left[\xi, d^{(s)}\eta\right]^{(s)}. \tag{4.41}$$

Proof. If $\mathbb{L}$ is alternative, then the loop Jacobi identity (3.46) becomes

$$\left[\xi,\left[\eta,\gamma\right]^{(s)}\right]^{(s)} + \left[\eta,\left[\gamma,\xi\right]^{(s)}\right]^{(s)} + \left[\gamma,\left[\xi,\eta\right]^{(s)}\right]^{(s)} = 6\left[\xi,\eta,\gamma\right]^{(s)}. \tag{4.42}$$

On the other hand, (4.27) becomes

$$d^{\mathcal{H}}\left[\xi,\eta\right]^{(s)} = \left[d^{\mathcal{H}}\xi,\eta\right]^{(s)} + \left[\xi,d^{\mathcal{H}}\eta\right]^{(s)} + 2\left[\xi,\eta,T^{(s)}\right]^{(s)}. \tag{4.43}$$

Thus, using both (4.42) and (4.43), we obtain

$$\begin{aligned} d^{(s)}\left[\xi,\eta\right]^{(s)} =& d^{\mathcal{H}}\left[\xi,\eta\right]^{(s)} + \frac{1}{3}\left[\left[\xi,\eta\right]^{(s)}, T^{(s)}\right]^{(s)} \\ =& \left[d^{(s)}\xi,\eta\right]^{(s)} + \left[\xi,d^{(s)}\eta\right]^{(s)} \\ & - \frac{1}{3}\left[\left[\xi,T^{(s)}\right]^{(s)},\eta\right]^{(s)} - \frac{1}{3}\left[\xi,\left[\eta,T^{(s)}\right]^{(s)}\right]^{(s)} \\ & + \frac{1}{3}\left[\left[\xi,\eta\right]^{(s)},T^{(s)}\right]^{(s)} + 2\left[\xi,\eta,T^{(s)}\right]^{(s)} \\ =& \left[d^{(s)}\xi,\eta\right]^{(s)} + \left[\xi,d^{(s)}\eta\right]^{(s)}. \quad \square \end{aligned}$$

Remark 4.15. In the case of G_2-structures and octonions, the derivative (4.40) exactly replicates the modified covariant derivative that preserves the G_2-structure that was introduced in [10].

Example 4.16. The map φ_s is equivariant on $\mathcal{P}$ and hence defines a section of the associated bundle $\mathcal{A} \otimes \operatorname{ad}(\mathcal{P})^*$ over M. If $\mathbb{L}$ is the loop of unit octonions and $\mathfrak{l} \cong \operatorname{Im} \mathbb{O}$, and we have a G_2-structure on M, then φ_s corresponds to a section of $TM \otimes \Lambda^2 TM$, which up to a constant factor is a multiple of the corresponding G_2-structure 3-form φ with indices raised using the associated metric. The torsion T of φ with respect to the Levi-Civita connection on TM is then a section of $TM \otimes T^*M$. Noting that $\mathfrak{so}(7)$ acts on $\mathbb{R}^7$ by matrix multiplication, if we set $(\varphi_s)^{abc} = -\frac{1}{4}\varphi^{abc}$ in local coordinates, then (4.28) precisely recovers the well-known formula for $\nabla\varphi$ in terms of T. Indeed, suppose $\xi \in \Gamma\left(\Lambda^2 T^*M\right)$, then in a local basis $\{e_a\}$, for some fixed vector field X, we have

$$\begin{aligned}
(\nabla_X \varphi_s)(\xi) &= \xi \cdot T_X - [\varphi_s(\xi), T_X]^{(s)} \\
&= \left(\xi^a{}_b T_X^b + \frac{1}{2}\varphi^a{}_{bc}\varphi^{bde}\xi_{de}T_X^c\right) e_a \\
&= \left(\xi^a{}_b T_X^b - \frac{1}{2}\left(\psi^a{}_c{}^{de} + g^{ad}g_c{}^e - g^{ae}g_c{}^d\right)\xi_{de}T_X^c\right) e_a \\
&= \frac{1}{2}T_X^c \psi_c{}^{ade}\xi_{de}e_a,
\end{aligned}$$

where $\psi = *\varphi$. Hence, indeed,

$$\nabla_X \varphi = -2T_X \lrcorner \psi, \tag{4.44}$$

which is exactly as in [15], taking into account that the torsion here differs by a sign from [15]. Here we also used the convention that $[X, Y] = 2X \lrcorner Y \lrcorner \varphi$ and also contraction identities for φ [14,22]. This is also consistent with the expression (4.27) for the covariant derivative of the bracket. Indeed, in the case of an alternative loop, (4.43) shows that the covariant derivative of the bracket function b_s is given by

$$d^{\mathcal{H}} b_s = 2\left[\cdot, \cdot, T^{(s,\omega)}\right]^{(s)}. \tag{4.45}$$

Taking $b_s = 2\varphi$ and $[\cdot, \cdot, \cdot]^{(s)}$ given by $\left([X, Y, Z]^{(s)}\right)^a = 2\psi^a{}_{bcd}X^b Y^c Z^d$, as in [15], we again recover (4.44).

Example 4.17. Suppose $\mathcal{P}$ is a principal $U(n)$-bundle and $\mathbb{L} \cong U(1)$, the unit complex numbers, as in Example 3.28. Then, (4.28) shows that $d^{\mathcal{H}}\varphi_s = 0$. If V is an n-dimensional complex vector space with the standard action of $U(n)$ on it and $\mathcal{V} = \mathcal{P} \times_{U(n)} V$ is the associated vector bundle to $\mathcal{P}$ with fiber V, then φ_s defines a Kähler form on $\mathcal{V}$.

Example 4.18. Suppose $\mathcal{P}$ is a principal $Sp(n)\,Sp(1)$-bundle and $\mathbb{L} \cong Sp(1)$, the unit quaternions, as in Example 3.29. Then, (4.28) shows that $d^{\mathcal{H}}\varphi_s = -\left[\varphi_s, T^{(s,\omega)}\right]_{\operatorname{Im}\mathbb{H}}$. If V is an n-dimensional quaternionic vector space with the standard action of $Sp(n)\,Sp(1)$ on it and $\mathcal{V} = \mathcal{P}\times_{Sp(n)Sp(1)}V$ is the associated vector bundle to $\mathcal{P}$ with fiber V, then φ_s defines a 2-form on $\mathcal{V}$ with values in $\operatorname{Im}\mathbb{H}$ (since the bundle $\mathcal{A}$ is trivial). So this gives rise to 3 linearly independent 2-forms $\omega_1, \omega_2, \omega_3$. If $T^{(s,\omega)} = 0$, then this reduces to a HyperKähler structure on $\mathcal{V}$. It is an interesting question whether the case $T^{(s,\omega)} \neq 0$ is related to “HyperKähler with torsion” geometry [13,50].

4.2. Curvature

Recall that the curvature $F \in \Omega^2(\mathcal{P}, \mathfrak{p})$ of the connection ω on $\mathcal{P}$ is given by

$$F^{(\omega)} = d^{\mathcal{H}}\omega = d\omega \circ \operatorname{proj}_{\mathcal{H}}, \tag{4.46}$$

so that, for $X, Y \in \Gamma(T\mathcal{P})$,

$$F^{(\omega)}(X,Y) = d\omega\left(X^{\mathcal{H}}, Y^{\mathcal{H}}\right) = -\omega\left(\left[X^{\mathcal{H}}, Y^{\mathcal{H}}\right]\right), \tag{4.47}$$

where $X^{\mathcal{H}}, Y^{\mathcal{H}}$ are the projections of X, Y to $\mathcal{H}\mathcal{P}$.

Similarly as $\hat{\omega}$, define $\hat{F}^{(s,\omega)} \in \Omega^2(\mathcal{P}, \mathfrak{l})$ to be the projection of the curvature $F^{(\omega)}$ to $\mathfrak{l}$ with respect to s, such that for any $X_p, Y_p \in T_p\mathcal{P}$,

$$\begin{aligned}\hat{F}^{(s,\omega)}(X_p, Y_p) =& \varphi_s\left(F^{(\omega)}\right)(X_p, Y_p) \\ =& \frac{d}{dt}\left(\exp\left(tF^{(\omega)}(X_p, Y_p)\right)(s_p)\right)\Big/ s_p\Big|_{t=0}.\end{aligned} \tag{4.48}$$

We easily see that

$$d^{\mathcal{H}}\hat{\omega}^{(s)} = \hat{F}^{(s,\omega)}. \tag{4.49}$$

Indeed,

$$d^{\mathcal{H}}\hat{\omega}^{(s)} = d^{\mathcal{H}}\left(\varphi_s(\omega)\right) = d^{\mathcal{H}}\varphi_s \wedge \left(\omega \circ \operatorname{proj}_{\mathcal{H}}\right) + \varphi_s\left(d^{\mathcal{H}}\omega\right) = \hat{F}^{(s,\omega)},$$

where we have used the fact that ω is vertical.

We then have the following structure equations

Theorem 4.19. *$\hat{F}^{(s,\omega)}$ and $T^{(s,\omega)}$ satisfy the following structure equation*

$$\hat{F}^{(s,\omega)} = d^{\mathcal{H}}T^{(s,\omega)} - \frac{1}{2}\left[T^{(s,\omega)}, T^{(s,\omega)}\right]^{(s)}, \tag{4.50}$$

where a wedge product between the 1*-forms* $T^{(s,\omega)}$ *is implied. Equivalently, (4.50) can be written as*

$$d\hat{\omega}^{(s)} + \frac{1}{2}\left[\hat{\omega}^{(s)}, \hat{\omega}^{(s)}\right]^{(s)} = \hat{F}^{(s,\omega)} - d^{\mathcal{H}}\varphi_s \wedge \omega, \tag{4.51}$$

where $\left(d^{\mathcal{H}}\varphi_s \wedge \omega\right)(X, Y) = \left(d^{\mathcal{H}}_X \varphi_s\right)(\omega(Y)) - \left(d^{\mathcal{H}}_Y \varphi_s\right)(\omega(X))$ *for any vector fields* X *and* Y *on* $\mathcal{P}$*.*

Proof. Using (4.23), we have

$$\begin{aligned} d^{\mathcal{H}}T^{(s,\omega)} &= dT^{(s,\omega)} \circ \mathrm{proj}_{\mathcal{H}} \\ &= \left(d\theta_s + d\hat{\omega}^{(s)}\right) \circ \mathrm{proj}_{\mathcal{H}}. \end{aligned} \tag{4.52}$$

Now consider the first term. Let $X_p, Y_p \in T_p\mathcal{P}$, then

$$\begin{aligned} d\theta_s\left(X_p^{\mathcal{H}}, Y_p^{\mathcal{H}}\right) &= (d\theta)_{s_p}\left(s_* X_p^{\mathcal{H}}, s_* Y_p^{\mathcal{H}}\right) \\ &= (d\theta)_{s_p}\left(\mathring{D}_{X_p} s, \mathring{D}_{Y_P} s\right) \qquad (4.53) \\ &= \left[\theta\left(\mathring{D}_{X_p} s\right), \theta\left(\mathring{D}_{Y_P} s\right)\right]^{(s_p)} \\ &= \left[T^{(s,\omega)}(X_p), T^{(s,\omega)}(Y_p)\right]^{(s_p)}, \qquad (4.54) \end{aligned}$$

where we have used the Maurer-Cartan structural equation for loops (3.35). Using (4.49) for the second term, overall, we obtain (4.50).

From the Maurer-Cartan equation (3.35),

$$d\theta_s - \frac{1}{2}\left[\theta_s, \theta_s\right]^{(s)} = 0.$$

We also have from (4.23)

$$\left[\theta_s, \theta_s\right]^{(s)} = \left[T^{(s,\omega)}, T^{(s,\omega)}\right]^{(s)} - 2\left[\hat{\omega}^{(s)}, T^{(s,\omega)}\right]^{(s)} + \left[\hat{\omega}^{(s)}, \hat{\omega}^{(s)}\right]^{(s)}.$$

Hence

$$d\theta_s = dT^{(s,\omega)} - d\hat{\omega}^{(s)} = \frac{1}{2}\left[T^{(s,\omega)}, T^{(s,\omega)}\right]^{(s)} - \left[\hat{\omega}^{(s)}, T^{(s,\omega)}\right]^{(s)} + \frac{1}{2}\left[\hat{\omega}^{(s)}, \hat{\omega}^{(s)}\right]^{(s)}.$$

Noting that

$$dT^{(s,\omega)} = d^{\mathcal{H}}T^{(s,\omega)} - \omega \dot{\wedge} T^{(s,\omega)}$$

we find

$$d\hat{\omega}^{(s)} + \frac{1}{2}\left[\hat{\omega}^{(s)}, \hat{\omega}^{(s)}\right]^{(s)} = d^{\mathcal{H}} T^{(s,\omega)} - \omega \dot{\wedge} T^{(s,\omega)} - \frac{1}{2}\left[T^{(s,\omega)}, T^{(s,\omega)}\right]^{(s)} + \left[\hat{\omega}^{(s)}, T^{(s,\omega)}\right]^{(s)}$$

and then using (4.50) and (4.28) we obtain (4.51). □

Corollary 4.20 *(Bianchi identity). The quantity $\hat{F}^{(s,\omega)}$ satisfies the equation*

$$\begin{aligned} d^{\mathcal{H}} \hat{F}^{(s,\omega)} &= d^{\mathcal{H}} \varphi_s \wedge F \\ &= F \dot{\wedge} T^{(s,\omega)} - \left[\hat{F}^{(s,\omega)}, T^{(s,\omega)}\right]^{(s)} \end{aligned} \tag{4.55}$$

where $\dot{\wedge}$ denotes the linear action of $\mathfrak{p}$ on $\mathfrak{l}$ combined with a wedge product.

Proof. Using the definition (4.48) of $\hat{F}^{(s,\omega)}$, we have

$$d^{\mathcal{H}} \hat{F}^{(s,\omega)} = d^{\mathcal{H}}\left(\varphi_s\left(F\right)\right) = d^{\mathcal{H}} \varphi_s \wedge F + \varphi_s\left(d^{\mathcal{H}} F\right),$$

however using the standard Bianchi identity, $d^{\mathcal{H}} F = 0$, and (4.28), we obtain (4.55). □

Remark 4.21. Since the Bianchi identity (4.55) is not a standard one, one may wonder if differentiating it leads to additional identities. It is however a straightforward exercise using previously established identities for $\hat{F}$ and T, as well as the Akivis identity (3.46), to show that the identity (4.55) does not lead to any other additional identities.

Example 4.22. The equation (4.50) is the precise analog of what is known as the "G_2-structure Bianchi identity" [15,23] (not to be confused with the Bianchi identity (4.55)). In the case of G_2-structures, $\hat{F}$ corresponds precisely to the quantity $\frac{1}{4}\pi_7 \operatorname{Riem}$, which is the projection of the endomorphism part of Riem to the 7-dimensional representation of G_2. In local coordinates, it is given by $\frac{1}{4} \operatorname{Riem}_{abcd} \varphi^{cde}$.

Example 4.23. In the complex case, with $\mathbb{L} = U\mathbb{C}$ and $\mathcal{P}$ a principal $U(n)$-bundle, (4.50) shows that $\hat{F}^{(s,\omega)} = dT^{(s,\omega)}$. Here $d^{\mathcal{H}} = d$ on $\mathfrak{l}$-valued forms because the action of $\mathfrak{p}_n$ on $\mathfrak{l}$ is trivial (as in Example 3.28). If s is a global section, then this shows that $\hat{F}$ is an exact 2-form - and so the class $\left[\hat{F}\right] = 0$. This is consistent with a vanishing first Chern class which is a necessary condition for existence of a global s. On the other hand, if we suppose that s is only a local section, so that $T^{(s,\omega)}$ is a local 1-form, then we only get that $\hat{F}^{(s,\omega)}$ is closed, so in this case it may define a non-trivial first Chern class. If $\mathcal{P}$ is the unitary frame bundle over a complex manifold, it defines a Kähler metric, and then $\hat{F}$ precisely corresponds to the Ricci curvature, so that the Ricci-flat condition for reduction to a Calabi-Yau manifold is $\hat{F} = 0$.

The equation (4.51) is interesting because this is an analog of the structure equation for the connection 1-form ω on $\mathcal{P}$. However, in the case of ω, the quantity $d\omega - \frac{1}{2}[\omega, \omega]$ is horizontal. However, for $\hat{\omega}^{(s)}$, $\hat{F}^{(s,\omega)}$ gives the horizontal component, while the remaining terms give mixed vertical and horizontal components. The fully vertical components vanish. We also see that $\hat{\omega}^{(s)}$ satisfies the loop Maurer-Cartan equation if, and only if, $\hat{F}^{(s,\omega)} = 0$ and $d^{\mathcal{H}}\varphi_s = 0$. In the G_2 case, $\nabla\varphi = 0$ of course is equivalent to $T = 0$ and hence implies $\frac{1}{4}\pi_7 \operatorname{Riem} = 0$. More generally, this may not need to be the case.

Lemma 4.24. *Suppose $\mathbb{L}$ is a left-alternative loop and suppose $-\hat{\omega}^{(s)}$ satisfies the Maurer-Cartan equation*

$$d\hat{\omega}^{(s)} + \frac{1}{2}\left[\hat{\omega}^{(s)}, \hat{\omega}^{(s)}\right]^{(s)} = 0, \tag{4.56}$$

then for any $\alpha, \beta \in \mathfrak{q}^{(s_p)} \cong T_1\mathcal{C}^R\left(\mathbb{L}, \circ_{s_p}\right)$,

$$\left[\alpha, \beta, T_p^{(s,\omega)}\right]^{(s_p)} = 0. \tag{4.57}$$

Proof. Taking the exterior derivative of (4.56) and applying (3.166), we find $\hat{\omega}^{(s)}$ satisfies

$$0 = \left[\hat{\omega}^{(s)}, \hat{\omega}^{(s)}, \theta_s + \hat{\omega}^{(s)}\right]^{(s)} = \left[\hat{\omega}^{(s)}, \hat{\omega}^{(s)}, T^{(s,\omega)}\right]^{(s)}. \tag{4.58}$$

Since $\mathbb{L}$ is left-alternative, we know that the $\mathbb{L}$-algebra associator is skew in the first two entries, so if given vector fields X, Y, Z on $\mathcal{P}$, we have

$$\begin{aligned} 0 = &\left[\hat{\omega}^{(s)}(X), \hat{\omega}^{(s)}(Y), T^{(s,\omega)}(Z)\right]^{(s)} + \left[\hat{\omega}^{(s)}(Y), \hat{\omega}^{(s)}(Z), T^{(s,\omega)}(X)\right]^{(s)} \\ &+ \left[\hat{\omega}^{(s)}(Z), \hat{\omega}^{(s)}(X), T^{(s,\omega)}(Y)\right]^{(s)}. \end{aligned} \tag{4.59}$$

Let $\xi \in \mathfrak{p}$ and let $X = \sigma(\xi)$ be a vertical vector field on $\mathcal{P}$, then

$$\hat{\omega}^{(s)}(X) = \varphi_s(\omega(X)) = \varphi_s(\xi).$$

In (4.59), we take $X = \sigma(\xi)$ and $Y = \sigma(\eta)$ to be vertical vector fields and $Z = Z^h$ a horizontal vector field. Then since $\hat{\omega}^{(s)}$ is vertical and $T^{(s,\omega)}$ is horizontal, we find that for any $\xi, \eta \in \mathfrak{p}$,

$$\left[\varphi_s(\xi), \varphi_s(\eta), T^{(s,\omega)}(Z)\right]^{(s)} = 0.$$

We know that for each $p \in \mathcal{P}$, the map φ_{s_p} is surjective onto $\mathfrak{q}^{(s_p)} \subset \mathfrak{l}^{(s_p)}$ and thus (4.57) holds. $\square$

Theorem 4.25. *Suppose $\mathcal{P}$ is 1-connected and $\mathbb{L}$ a smooth loop such that*

1. *$\mathfrak{l}$ is a left-alternative algebra (i.e. the associator on $\mathfrak{l}$ is skew-symmetric in the first two entries),*
2. $\dim\left(\mathcal{N}^R(\mathbb{L})\right) = \dim\left(\mathcal{N}^R(\mathfrak{l})\right)$.

Moreover, suppose $s_p \in \mathcal{C}^R(\mathbb{L})$ for every $p \in \mathcal{P}$, then $\hat{\omega}^{(s)}$ satisfies the Maurer-Cartan equation (4.56) if, and only if, there exists a map $f : \mathcal{P} \longrightarrow \mathcal{N}^R(\mathbb{L})$ such that

$$T^{(s,\omega)} = -(\mathrm{Ad}_s)_* \theta_f. \tag{4.60}$$

Proof. Since s has values in $\mathcal{C}^R(\mathbb{L})$, using Lemma 4.24, we see that the conditions of Corollary 3.61 are satisfied, and hence there exists a map $f : \mathcal{P} \longrightarrow \mathcal{N}^R(\mathbb{L})$ such that

$$\begin{aligned} -\hat{\omega}^{(s)} &= \theta_{sf} \\ &= \theta_s + (\mathrm{Ad}_s)_* \theta_f. \end{aligned}$$

From (4.23),

$$T^{(s,\omega)} = \theta_s + \hat{\omega}^{(s)} = -(\mathrm{Ad}_s)_* \theta_f.$$

Conversely, suppose (4.60) holds for some right nucleus-valued map f. Then, clearly $\hat{\omega}^{(s)} = -\theta_{sf}$, and thus $-\hat{\omega}^{(s)}$ satisfies (4.56). □

Remark 4.26. Theorem 4.25 shows that if $\mathbb{L}$ has a sufficiently large nucleus, then $\hat{F}^{(s,\omega)} = 0$ and $d^{\mathcal{H}}\varphi_s = 0$ do not necessarily imply that $T^{(s,\omega)} = 0$. In the case of unit octonions, the nucleus is just $\{\pm 1\}$, so any nucleus-valued map is constant on connected components, hence in this case if $\hat{\omega}^{(s)}$ satisfies (4.56), then $T^{(s,\omega)} = 0$.

4.3. Deformations

The torsion of a loop structure is determined by the equivariant $\mathring{\mathbb{L}}$-valued map s and the connection ω on $\mathcal{P}$. There are several possible deformations of s and ω. In particular, s may be deformed by the action of Ψ or by left multiplication action of $\mathbb{L}$. The connection ω may be deformed by the affine action of $\Omega^1_{basic}(\mathcal{P}, \mathfrak{p})$ or by gauge transformations in Ψ. Moreover, of course, these deformations may be combined or considered infinitesimally. Since $T^{(s,\omega)}$ is the horizontal part of θ_s, when considering deformations of s it is sufficient to consider what happens to θ_s and then taking the horizontal component.

Recall that the space of connections on $\mathcal{P}$ is an affine space modeled on equivariant horizontal (i.e. basic) $\mathfrak{p}$-valued 1-forms on $\mathcal{P}$. Thus, any connection $\tilde{\omega} = \omega + A$ for some basic $\mathfrak{p}$-valued 1-form A. Then,

$$T^{(s,\tilde{\omega})} = \theta_s + \varphi_s(\tilde{\omega}) = T^{(s,\omega)} + \hat{A} \tag{4.61}$$

where $\hat{A} = \varphi_s(A)$. Thus, we can set $T^{(s,\tilde{\omega})} = 0$ by choosing A such that $\hat{A} = -T^{(s,\omega)}$ if, and only if, for each $p \in P$, $T_p^{(s,\omega)} \in \mathfrak{q}^{(s_p)} = \varphi_{s_p}(\mathfrak{p})$. Since $\hat{\omega}$ is always in the image of φ_s, we conclude there exists a connection $\tilde{\omega}$ for which $T^{(s,\tilde{\omega})} = 0$ if, and only if, $\theta_s|_p \in \mathfrak{q}^{(s_p)}$ for each p. In that case, $\theta_s = -\varphi_s(\tilde{\omega})$. From Theorem 3.53, we then see that $\tilde{\omega}$ has curvature with values in $\mathfrak{h}_s$.

Recall that if $\phi : \mathcal{P} \longrightarrow \mathcal{P}$ is a gauge transformation, then there exists an Ad_Ψ-equivariant map $u : \mathcal{P} \longrightarrow \Psi$ such that for each $p \in \mathcal{P}$, $\phi(p) = pu_p$. Each such map then corresponds to a section of the associated bundle $\mathrm{Ad}(\mathcal{P})$. The gauge-transformed connection 1-form is then $\omega^\phi = u^*\omega$, where

$$u^*\omega = (\mathrm{Ad}_{u^{-1}})_*\,\omega + u^*\theta_\Psi \tag{4.62}$$

where θ_Ψ is the *left*-invariant Maurer-Cartan form on Ψ. Then,

$$\begin{aligned} d^{u^*\mathcal{H}}s &= \left(l_u^{-1}\right)_* d^{\mathcal{H}}(l_u s) \\ &= d^{\mathcal{H}}s + (u^*\theta_\Psi)^{\mathcal{H}} \cdot s_p \end{aligned} \tag{4.63}$$

where at each $p \in \mathcal{P}$.

$$(u^*\theta_\Psi)^{\mathcal{H}}\Big|_p = \left(l_{u_p}\right)_*^{-1} \circ \left(d^{\mathcal{H}}u\right)_p .$$

Hence,

$$T^{(s,u^*\omega)} = \left(R_s^{-1}\right)_* d^{u^*\mathcal{H}}s = T^{(s,\omega)} + \varphi_s\left((u^*\theta_\Psi)^{\mathcal{H}}\right). \tag{4.64}$$

Consider the curvature $F^{u^*\omega}$ of the connection $u^*\omega$. It is well-known that it is given by

$$F^{u^*\omega} = (\mathrm{Ad}_{u^{-1}})_*\,F. \tag{4.65}$$

From Theorem 3.25, we then have

$$\hat{F}^{(s,u^*\omega)} = \varphi_s\left((\mathrm{Ad}_{u^{-1}})_* F\right) = \left(u^{-1}\right)'_* \hat{F}^{(u(s),\omega)}. \tag{4.66}$$

On the other hand, using (4.63) and (4.37) we have

$$\begin{aligned} T^{(s,u^*\omega)} &= \left(R_s^{-1}\right)_* \left(u^*\mathring{D}\right)(s) \\ &= \left(R_s^{-1}\right)_* \left(u^{-1}\right)_* \mathring{D}(u(s)) \\ &= \left(u^{-1}\right)'_* \left(R_{u(s)}^{-1}\right)_* \mathring{D}(u(s)) \\ &= \left(u^{-1}\right)'_* T^{(u(s),\omega)}. \end{aligned}$$

Summarizing, we have the following.

Theorem 4.27. *Suppose $s : \mathcal{P} \longrightarrow \mathring{\mathbb{L}}$ and $u : \mathcal{P} \longrightarrow \Psi$ are equivariant smooth maps. Then,*

$$T^{(s,u^*\omega)} = T^{(s,\omega)} + \varphi_s\left(\left(u^*\theta_\Psi\right)^{\mathcal{H}}\right) \tag{4.67a}$$

$$= \left(u^{-1}\right)'_* T^{(u(s),\omega)}$$

$$\hat{F}^{(s,u^*\omega)} = \left(u^{-1}\right)'_* \hat{F}^{(u(s),\omega)}. \tag{4.67b}$$

In particular,

$$T^{\left(u^{-1}(s),u^*\omega\right)} = \left(u'\right)_*^{-1} T^{(s,\omega)} \quad \textit{and } \hat{F}^{\left(u^{-1}(s),u^*\omega\right)} = \left(u^{-1}\right)'_* \hat{F}^{(s,\omega)}. \tag{4.68}$$

This shows that both T and $\hat{F}$ transform equivariantly with respect to a simultaneous transformation of s and ω. In particular, if we have a Riemannian metric on the base manifold M and a Ψ-covariant metric on $\mathfrak{l}$, then with respect to the induced metric on $T^*\mathcal{P} \otimes \mathfrak{l}$, the quantities $|T|^2$ and $|F|^2$ are invariant with respect to the transformation $(s,\omega) \mapsto \left(u^{-1}(s), u^*\omega\right)$. In the case of G_2-structure, the key question is regarding the holonomy of the Levi-Civita connection, so in this general setting, if we are interested in the holonomy of ω, it makes sense to consider individual transformations $s \mapsto As$ for some equivariant $A \in C^\infty(\mathcal{P}, \mathbb{L})$ and $\omega \mapsto u^*\omega$ because each of these transformations leaves the holonomy group unchanged. We also see that every transformation $s \mapsto u(s)$ for some equivariant $u \in C^\infty(\mathcal{P}, \Psi)$ corresponds to a transformation $s \mapsto As$, where $A = h(s)/s$. From (2.40), this is precisely the companion of the corresponding map $u_s \in \Psi(\mathbb{L}, \circ_s)$. Moreover, this correspondence is one-to-one if, and only if, $\mathbb{L}$ is a G-loop. It is easy to see that A is then an equivariant $\mathbb{L}$-valued map. Thus, considering transformations $s \mapsto As$ is more general in some situations.

Theorem 4.28. *Suppose $A : \mathcal{P} \longrightarrow \mathbb{L}$ and $s : \mathcal{P} \longrightarrow \mathring{\mathbb{L}}$. Then,*

$$T^{(As,\omega)} = \left(R_A^{(s)}\right)_*^{-1} DA + \left(\mathrm{Ad}_A^{(s)}\right)_* T^{(s,\omega)} = \left(R_A^{(s)}\right)_*^{-1} D^{(s)}A \tag{4.69a}$$

$$\hat{F}^{(As,\omega)} = \left(R_A^{(s)}\right)_*^{-1} \left(F' \cdot A\right) + \left(\mathrm{Ad}_A^{(s)}\right)_* \hat{F}^{(s,\omega)}, \tag{4.69b}$$

where $F' \cdot A$ denotes the infinitesimal action of $\mathfrak{p}$ on $\mathbb{L}$.

Proof. Recall from (3.158), that

$$\theta_{As} = \theta_A^{(s)} + \left(\mathrm{Ad}_A^{(s)}\right)_* \theta_s. \tag{4.70}$$

Now, $T^{(s,\omega)}$ is just the horizontal part of θ_s, so taking the horizontal projection in (4.70), we immediately get (4.69a). To obtain (4.69b), from (3.77) we have

$$\hat{F}^{(As,\omega)} = \varphi_{As}(F) = \left(R_A^{(s)}\right)_*^{-1}(F' \cdot A) + \left(\mathrm{Ad}_A^{(s)}\right)_* \varphi_s(F), \tag{4.71}$$

and hence we obtain (4.69b). □

Remark 4.29. The expression (4.69a) precisely replicates the formula for the transformation of torsion of a G_2-structure within a fixed metric class, as derived in [15].

Now suppose s_t is a 1-parameter family of equivariant $\mathring{\mathbb{L}}$-valued maps that satisfy

$$\frac{\partial s_t}{\partial t} = (R_{s_t})_* \xi_t \tag{4.72}$$

where ξ_t is a 1-parameter family of $\mathfrak{l}$-valued maps. In particular, if $\xi(t)$ is independent of t, then $s(t) = \exp_{s_0}(t\xi) s_0$. Then let us work out the evolution of $T^{(s(t),\omega)}$ and $\hat{F}^{(s(t),\omega)}$. First consider the evolution of $\theta_{s(t)}$ and $\varphi_{s(t)}$.

Lemma 4.30. *Suppose $s(t)$ satisfies (4.72), then*

$$\frac{\partial \theta_{s(t)}}{\partial t} = d\xi(t) - \left[\theta_{s(t)}, \xi(t)\right]^{(s(t))} \tag{4.73a}$$

$$\frac{\partial \varphi_{s(t)}}{\partial t} = \mathrm{id}_{\mathfrak{p}} \cdot \xi(t) - \left[\varphi_{s(t)}, \xi(t)\right]^{(s(t))}. \tag{4.73b}$$

Proof. For $\theta_{s(t)}$, suppressing pushforwards, we have

$$\begin{aligned}
\frac{\partial \theta_{s(t)}}{\partial t} &= \frac{\partial}{\partial t}\left((ds(t))/s(t)\right) \\
&= (d\dot{s})/s - ((ds)/s \cdot \dot{s})/s \\
&= d(\xi s)/s - ((ds)/s \cdot (\xi s))/s \\
&= d\xi - \left[\theta_{s(t)}, \xi\right]^{(s(t))}.
\end{aligned} \tag{4.74}$$

Similarly, for $\varphi_{s(t)}$, let $\eta \in \mathfrak{p}$, then

$$\begin{aligned}
\frac{\partial \varphi_{s(t)}(\eta)}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{d}{d\tau}\exp(\tau\eta)(s)/s\Big|_{\tau=0}\right) \\
&= \frac{d}{d\tau}\exp(\tau\eta)((\xi s)/s)\Big|_{\tau=0} - \frac{d}{d\tau}(\exp(\tau\eta)((s)/s)\cdot(\xi s))/s\Big|_{\tau=0} \\
&= \frac{d}{d\tau}\exp(\tau\eta)'(\xi)\Big|_{\tau=0} + \frac{d}{d\tau}(\xi\exp(\tau\eta)(s))/s\Big|_{\tau=0} \\
&\quad - \frac{d}{d\tau}(\exp(\tau\eta)((s)/s)\cdot(\xi s))/s\Big|_{\tau=0} \\
&= \eta \cdot \xi(t) - \left[\varphi_{s(t)}(\eta), \xi(t)\right]^{(s(t))}. \quad \square
\end{aligned} \tag{4.75}$$

To obtain the evolution of $T^{(s(t),\omega)}$ and $\hat{F}^{(s(t),\omega)}$, we just take the horizontal component of (4.73b) and substitute F into (4.73b).

Corollary 4.31. *Suppose $s(t)$ satisfies (4.72), then*

$$\frac{\partial T^{(s(t),\omega)}}{\partial t} = d^{\mathcal{H}}\xi(t) - \left[T^{(s(t),\omega)}, \xi(t)\right]^{(s(t))} \tag{4.76a}$$

$$\frac{\partial \hat{F}^{(s(t),\omega)}}{\partial t} = F\cdot\xi(t) - \left[\hat{F}^{(s(t),\omega)}, \xi(t)\right]^{(s(t))}. \tag{4.76b}$$

The expression (4.76a) is the analog of a similar expression for the evolution of the torsion of a G_2-structure, as given in [17,23].

Remark 4.32. Suppose u_t is a 1-parameter family of equivariant Ψ-valued maps that satisfy

$$\frac{\partial u_t}{\partial t} = (l_{u_t})_* \gamma_t \tag{4.77}$$

for a 1-parameter family γ_t of equivariant $\mathfrak{p}$-valued maps. Then, each u_t defines a gauge transformation of the connection ω. Define

$$\omega_t = u_t^*\omega. \tag{4.78}$$

Then, it is easy to see that

$$\frac{\partial \omega_t}{\partial t} = d\gamma_t + [\omega_t, \gamma_t]_{\mathfrak{p}} = d^{\mathcal{H}_t}\gamma_t, \tag{4.79}$$

where $d^{\mathcal{H}_t}$ is the covariant derivative corresponding to ω_t. Similarly, the corresponding curvature F_t evolves via the equation

$$\frac{\partial F_t}{\partial t} = [F_t, \gamma_t]_{\mathfrak{p}}. \tag{4.80}$$

Using (4.79) together with (4.76a) gives

$$\frac{\partial T^{(s_t,\omega_t)}}{\partial t} = d^{\mathcal{H}_t}\xi_t - \left[T^{(s_t,\omega_t)}, \xi_t\right]^{(s_t)} + \varphi_{s_t}\left(d^{\mathcal{H}_t}\gamma_t\right). \tag{4.81}$$

However,

$$\begin{aligned}\varphi_{s_t}\left(d^{\mathcal{H}_t}\gamma_t\right) &= d^{\mathcal{H}_t}\hat{\gamma}_t^{(s_t)} - \left(d^{\mathcal{H}_t}\varphi_{s_t}\right)(\gamma_t)\\ &= d^{\mathcal{H}_t}\hat{\gamma}_t^{(s_t)} - \gamma_t\cdot T^{(s_t,\omega_t)} - \left[T^{(s_t,\omega_t)}, \hat{\gamma}_t^{(s_t)}\right]^{(s_t)}\end{aligned}$$

and thus (4.81) becomes

$$\frac{\partial T^{(s_t,\omega_t)}}{\partial t} = -\gamma_t \cdot T^{(s_t,\omega_t)} + d^{\mathcal{H}_t}\left(\xi_t + \hat{\gamma}_t^{(s_t)}\right) - \left[T^{(s_t,\omega_t)}, \xi_t + \hat{\gamma}_t^{(s_t)}\right]^{(s_t)}. \tag{4.82}$$

For the curvature, using (4.80) together with (4.76b) gives

$$\frac{\partial \hat{F}^{(s_t,\omega_t)}}{\partial t} = F_t \cdot \xi_t - \left[\hat{F}^{(s_t,\omega_t)}, \xi_t\right]^{(s_t)} + \varphi_{s_t}\left([F_t, \gamma_t]_{\mathfrak{p}}\right). \tag{4.83}$$

Using (3.96), we then get

$$\frac{\partial \hat{F}^{(s_t,\omega_t)}}{\partial t} = -\gamma_t \cdot \hat{F}_t + F_t \cdot \left(\xi_t + \hat{\gamma}_t^{(s_t)}\right) - \left[\hat{F}^{(s_t,\omega_t)}, \xi_t + \hat{\gamma}_t^{(s_t)}\right]^{(s_t)}. \tag{4.84}$$

Taking $\xi_t = -\hat{\gamma}_t^{(s_t)}$ in (4.82) and (4.84), we obtain the infinitesimal versions of (4.68).

5. Non-associative gauge theory

In general we have seen that the loop bundle structure is given by $\mathring{\mathbb{L}}$-valued equivariant map s as well as a connection ω on $\mathcal{P}$. We call the pair (s, ω) the configuration of the loop bundle structure. Each point in the configuration space gives rise to the corresponding torsion $T^{(s,\omega)}$ and curvature $\hat{F}^{(s,\omega)}$. Previously we considered T and $\hat{F}$ as horizontal equivariant forms on $\mathcal{P}$, but of course we can equivalently consider them as bundle-valued differential forms on the base manifold M. The loop bundle framework will allow us to generalize various aspects of gauge theory to the nonassociative setting.

To be able to define functionals on M, let us suppose M has a Riemannian metric and moreover, $\mathbb{L}$ has the following properties:

1. For each $s \in \mathring{\mathbb{L}}$, the Killing form $K^{(s)}$ is nondegenerate and invariant with respect to $\mathrm{ad}^{(s)}$ and the action of $\mathfrak{p}$.
2. $\mathbb{L}$ is a G-loop, so that in particular, for each $s \in \mathring{\mathbb{L}}$, $\mathfrak{l}^{(s)} = \mathfrak{q}_s$.
3. For each $s \in \mathring{\mathbb{L}}$, the space $\mathfrak{q}_s$ is an irreducible representation of the Lie algebra $\mathfrak{h}_s$.

These properties may not be strictly necessary, but they will simplify arguments. Moreover, these are the properties satisfied by the loop of unit octonions, which is the key example. The non-degeneracy of $K^{(s)}$ means we can define the map φ_s^t, and then the second and third properties together make sure that there exists a constant λ such that for any $s \in \mathring{\mathbb{L}}$, $\varphi_s \varphi_s^t = \lambda \,\mathrm{id}_{\mathfrak{l}}$ and $\varphi_s^t \varphi_s = \lambda \pi_{\mathfrak{h}_s^{\perp}}$, as per Lemma 3.43. If $\mathfrak{q}_s$ is a reducible representation, then each irreducible component may have its own constant. Moreover, the first and second properties together imply that $K^{(s)}$ is independent of the choice of s, and when extended as an inner product on sections, it is covariantly constant with respect to a principal connection on $\mathcal{P}$.

For a fixed $s \in \mathring{\mathbb{L}}$, let us consider what happens to the torsion T and the curvature $\hat{F}$ with respect to deformations of the connection.

Lemma 5.1. *Let $s \in \mathring{\mathbb{L}}$ be fixed. Suppose we have a path of connections on $\mathcal{P}$ given by $\tilde{\omega}(t) = \omega + tA$ for some basic $\mathfrak{p}$-valued 1-form A and a fixed principal connection ω. Then, defining $T(t) = T^{(s,\tilde{\omega}(t))}$ and $\hat{F}(t) = \hat{F}^{(s,\hat{\omega}(t))}$, we have*

$$\left.\frac{d}{dt}T(t)\right|_{t=0} = \hat{A} \tag{5.1a}$$

$$\left.\frac{d}{dt}\hat{F}(t)\right|_{t=0} = d^{\mathcal{H}}\hat{A} + A \cdot T - \left[\hat{A}, T\right]^{(s)}, \tag{5.1b}$$

where $T = T(0) = T^{(s,\omega)}$.

Proof. Using (4.61), we have

$$T(t) = T^{(s,\tilde{\omega}(t))} = \theta_s + \varphi_s(\tilde{\omega}(t)) = T^{(s,\omega)} + t\hat{A},$$

and hence we get (5.1a). Also, using (4.50),

$$\begin{aligned}\hat{F}(t) =& \hat{F}^{(s,\hat{\omega}(t))} = \varphi_s\left(F^{\tilde{\omega}(t)}\right) = \hat{F}^{(s,\omega)} + t\varphi_s\left(d^{\mathcal{H}}A\right) \\ &+ \frac{1}{2}t^2\varphi_s\left([A,A]_{\mathfrak{p}}\right),\end{aligned} \tag{5.2}$$

and then using (4.28),

$$\begin{aligned}\left.\frac{d}{dt}\hat{F}(t)\right|_{t=0} &= \varphi_s\left(d^{\mathcal{H}}A\right) = d^{\mathcal{H}}\hat{A} - \left(d^{\mathcal{H}}\varphi_s\right) \wedge A \\ &= d^{\mathcal{H}}\hat{A} + A \cdot T - \left[\hat{A}, T\right]^{(s)}. \quad \square\end{aligned} \tag{5.3}$$

From (5.1a), we see that if for each $p \in \mathcal{P}$, $A_p \in \mathfrak{h}_{s_p}$, then $\hat{A} = 0$, and thus the torsion is unaffected, so these deformations are not relevant for the loop bundle structure. Therefore, let us assume that $A_p \in \mathfrak{h}_{s_p}^{\perp}$ for each $p \in \mathcal{P}$. Equivalently, this means that $A \in \varphi_s^t(\mathfrak{l})$. So now suppose $\xi \in \Omega^1_{basic}(\mathcal{P}, \mathfrak{l})$ is a basic $\mathfrak{l}$-valued 1-form on $\mathcal{P}$ such that $A = \frac{1}{\lambda}\varphi_s^t(\xi)$, and thus, $\hat{A} = \xi$. Then the deformations of T and $\hat{F}$ become the following.

Corollary 5.2. *Suppose $\xi \in \Omega^1_{basic}(\mathcal{P}, \mathfrak{l})$ is a basic $\mathfrak{l}$-valued 1-form on $\mathcal{P}$ such that $A = \frac{1}{\lambda}\varphi_s^t(\xi)$, and thus, $\hat{A} = \xi$. Then,*

$$\left.\frac{d}{dt}T(t)\right|_{t=0} = \xi \tag{5.4a}$$

$$\left.\frac{d}{dt}\hat{F}(t)\right|_{t=0} = d^{\mathcal{H}}\xi + \frac{1}{2\lambda^2}[\xi, T]_{\varphi_s} - \frac{1}{2}[\xi, T]^{(s)}. \tag{5.4b}$$

Proof. The first equation follows immediately from (5.1a). For the deformation of $\hat{F}$, from (3.122), we see that

$$A \cdot T = \frac{1}{\lambda}\varphi_s^t(\xi) \cdot T = \frac{1}{2\lambda^2}[\xi, T]_{\varphi_s} + \frac{1}{2}[\xi, T]^{(s)}, \tag{5.5}$$

where the bracket $[\cdot,\cdot]_{\varphi_s}$ on $\mathfrak{l}$ is given by

$$[\xi, \eta]_{\varphi_s} = \varphi_s\left(\left[\varphi_s^t(\xi), \varphi_s^t(\eta)\right]_{\mathfrak{p}}\right), \tag{5.6}$$

as defined in (3.123). Substituting this and $\hat{A} = \xi$ into (5.1b), we this obtain (5.4b). □

5.1. Loop Chern-Simons functional

Using the above technical results we can now generalize some aspects of Chern-Simons theory to loops.

Definition 5.3. Suppose M is a 3-dimensional compact manifold. For a fixed section $s \in \mathring{\mathcal{Q}}$, consider now the *loop Chern-Simons functional* $\mathcal{F}^{(s)}$ on the space of connections on $\mathcal{P}$ modulo $\mathfrak{h}_s$, given by

$$\mathcal{F}^{(s)}(\omega) = \int_M \left\langle T, \hat{F}\right\rangle^{(s)} - \frac{1}{6\lambda^2}\left\langle T, [T, T]_{\varphi_s}\right\rangle^{(s)}, \tag{5.7}$$

where wedge products between forms are implicit.

From the properties of $T, \hat{F}$, $[\cdot,\cdot]_{\varphi_s}$, and $\langle\cdot,\cdot\rangle^{(s)}$ that were obtained in Section 4.3, we see that this is invariant under simultaneous gauge transformation $(s, \omega) \mapsto \left(u^{-1}(s), u^*\omega\right)$. This shows that this is an appropriate invariant functional to use.

Theorem 5.4. *The critical points of the functional $\mathcal{F}^{(s)}$ are connections for which $\hat{F} = 0$.*

Proof. Using (5.4) consider deformations of each piece of (5.7). For the first term, using (5.4), we obtain

$$\begin{aligned} \frac{d}{dt}\int_M \left\langle T(t), \hat{F}(t)\right\rangle^{(s)}\Bigg|_{t=0} &= \int_M \left\langle \xi, \hat{F}\right\rangle^{(s)} \\ &\quad + \int_M \left\langle T, d^{\mathcal{H}}\xi + \frac{1}{2\lambda^2}[\xi, T]_{\varphi_s} - \frac{1}{2}[\xi, T]^{(s)}\right\rangle^{(s)} \\ &= \int_M \left\langle \xi, \hat{F} + d^{\mathcal{H}}T + \frac{1}{2\lambda^2}[T, T]_{\varphi_s} - \frac{1}{2}[T, T]^{(s)}\right\rangle^{(s)} \end{aligned}$$

$$= \int_M \left\langle \xi, 2\hat{F} + \frac{1}{2\lambda^2} [T,T]_{\varphi_s} \right\rangle^{(s)}. \tag{5.8}$$

For the second term in (5.7), using Lemma 3.50, we obtain

$$-\frac{1}{6\lambda^2} \frac{d}{dt} \int_M \left\langle T, [T,T]_{\varphi_s} \right\rangle^{(s)} \Big|_{t=0} = -\frac{1}{2\lambda^2} \int_M \left\langle \xi, [T,T]_{\varphi_s} \right\rangle^{(s)}. \tag{5.9}$$

Combining (5.8) and (5.9), we obtain

$$\frac{d}{dt} \mathcal{F}^{(s)} (\tilde{\omega}(t)) \Big|_{t=0} = 2 \int_M \left\langle \xi, \hat{F} \right\rangle^{(s)}. \tag{5.10}$$

Therefore, we see that the critical points of $\mathcal{F}^{(s)}$ are precisely the connections for which $\hat{F} = 0$. □

From the loop Bianchi identity (4.55), we also obtain an integrability condition for $\hat{F} = 0$:

$$F \dot{\wedge} T = 0, \tag{5.11}$$

where as before, $\dot{\wedge}$ denotes the linear action of $\mathfrak{p}$ on $\mathfrak{l}$ combined with a wedge product. Differentiating (5.11) however, we do not get any additional conditions, due to the standard Bianchi identity for F and the relations (3.90) and (4.50).

Remark 5.5. Theorem 5.4 shows that the condition $\hat{F} = 0$ is the loop generalization of the flat curvature condition that corresponds to the critical points of the standard Chern-Simons functional. The condition $\hat{F} = 0$ means that each point, the curvature $F^{(\omega)}$ lies in $\mathfrak{h}_s$. This is a restriction on the Lie algebra part of the curvature. The flat curvature condition is of course is a very special case, in which the curvature is restricted to the trivial Lie subalgebra. It may be tempting to regard $\hat{F} = 0$ as some kind of instanton, however instantons have restrictions on the 2-form part of the curvature, rather than the Lie algebra part. So what we have here is a different kind of condition to an instanton, and there is term for this, coined by Spiro Karigiannis - an *extanton*. As we see from Example 4.23, on a Kähler manifold, this just corresponds to the Ricci-flat condition.

The above construction on 3-manifolds can be extended to an n-dimensional manifold M if we have a closed $(n-3)$-dimensional form ψ. This idea was first introduced in [9] and then developed further in [40]. In this setting, let us define the *generalized loop Chern-Simons functional* as

$$\mathcal{G}^{(s)}(\omega) = \int_{M^n} \left(\left\langle T, \hat{F} \right\rangle^{(s)} - \frac{1}{6\lambda^2} \left\langle T, [T,T]_{\varphi_s} \right\rangle^{(s)} \right) \wedge \psi. \tag{5.12}$$

It is then easy to see the following.

Theorem 5.6. *The critical points of the functional $\mathcal{G}^{(s)}$ are connections for which*

$$\hat{F} \wedge \psi = 0. \tag{5.13}$$

This also implies that

$$F \dot{\wedge} T \wedge \psi = 0. \tag{5.14}$$

For example if M is a 7-dimensional manifold with a *co-closed* G_2-structure, i.e. $\psi = *\varphi$ is closed, then (5.13) shows that as a 2-form, $\hat{F}$ has a vanishing component in the 7-dimensional representation of G_2. In contrast, G_2-instantons (also known as Donaldson-Thomas connections) [9,40] satisfy $F \wedge \psi = 0$. If $F = \mathrm{Riem}$, is the Riemann curvature on the frame bundle, then equation (5.13) shows that, in local coordinates,

$$\mathrm{Riem}_{ijkl}\, \varphi^{ij}{}_{\alpha} \varphi^{kl}{}_{\beta} = 0. \tag{5.15}$$

The quantity on the left-hand side of (5.15), is sometimes denoted as Ric^* [7,8,18]. The traceless part of Ric^* corresponds to a component of the Riemann curvature that lies in a 27-dimensional representation of G_2, with another 27-dimensional component given by the traceless Ricci tensor Ric. The condition (5.14) is then given by

$$\mathrm{Riem}_{ijkl}\, T_m{}^l \varphi^{ijm} = 0. \tag{5.16}$$

Remark 5.7. In the spirit of Remark 5.5, we may refer to connections on bundles over compact 7-manifolds with co-closed G_2-structures that satisfy (5.13) as *G_2-extantons*. This is a generalization of the G_2-instanton condition.

The torsion T and the curvature $\hat{F}$ of a configuration (s, ω) of a loop bundle structure depend on both s and ω. So far, we have considered variations of the corresponding Chern-Simons functionals with respect to changes of ω. Suppose M is again a compact 3-dimensional manifold and let us consider (5.7) as functional on sections of $\mathring{Q}$ for a fixed connection ω, so that now we vary s. Thus, we now have the functional $\mathcal{F}^{(\omega)}$.

$$\mathcal{F}^{(\omega)}(s) = \int_M \left\langle T, \hat{F} \right\rangle^{(s)} - \frac{1}{6\lambda^2} \left\langle T, [T, T]_{\varphi_s} \right\rangle^{(s)}. \tag{5.17}$$

Let us now make additional assumptions:

1. $[\cdot,\cdot]_{\varphi_s} = k\,[\cdot,\cdot]^{(s)}$, for some scalar k.
2. $\mathbb{L}$ is alternative.

The last assumption implies in particular, that the associator is skew-symmetric, and moreover, for any $\alpha, \beta, \xi, \eta \in \mathfrak{l}^{(s)}$,

$$\langle a_s(\alpha, \beta, \xi), \eta\rangle^{(s)} = \langle \xi, a_s(\alpha, \beta, \eta)\rangle^{(s)}. \tag{5.18}$$

Thus, we can rewrite $\mathcal{F}^{(\omega)}$ as

$$\mathcal{F}^{(\omega)}(s) = \int_M \left\langle T, \hat{F}\right\rangle^{(s)} - \frac{k}{6\lambda^2}\left\langle T, [T,T]^{(s)}\right\rangle^{(s)}. \tag{5.19}$$

Theorem 5.8. *The critical points of the functional $\mathcal{F}^{(\omega)}$ satisfy*

$$F \cdot T - \left(\frac{1}{2} + \frac{k}{2\lambda^2}\right)\left[\hat{F}, T\right]^{(s)} + \frac{2k}{3\lambda^2}[T,T,T]^{(s)} = 0. \tag{5.20}$$

Proof. As in (4.72), suppose $s(t)$ is a path of defining sections of $\mathring{\mathcal{Q}}$ that satisfy

$$\frac{\partial s(t)}{\partial t} = (R_s)_* \eta(t) \tag{5.21}$$

where $\eta(t)$ is a 1-parameter family of sections of $\mathcal{A}$ (i.e. correspond to equivariant $\mathfrak{l}$-valued maps). From (4.76),

$$\frac{\partial T^{(s(t),\omega)}}{\partial t} = d^{\mathcal{H}}\eta(t) - \left[T^{(s(t),\omega)}, \eta(t)\right]^{(s(t))} \tag{5.22a}$$

$$\frac{\partial \hat{F}^{(s(t),\omega)}}{\partial t} = F \cdot \eta(t) - \left[\hat{F}^{(s(t),\omega)}, \eta(t)\right]^{(s(t))}. \tag{5.22b}$$

From this, we find that the derivative of $\mathcal{F}^{(\omega)}(s)$ is

$$\begin{aligned}\frac{d}{dt}\mathcal{F}^{(\omega)}(s(t))\bigg|_{t=0} &= \int_M \left\langle d^{\mathcal{H}}\eta - [T,\eta]^{(s)}, \hat{F}\right\rangle^{(s)} + \int_M \left\langle T, F\cdot\eta - \left[\hat{F},\eta\right]^{(s)}\right\rangle^{(s)} \\ &\quad - \frac{k}{2\lambda^2}\int_M \left\langle d^{\mathcal{H}}\eta - [T,\eta]^{(s)}, [T,T]^{(s)}\right\rangle^{(s)} \\ &\quad - \frac{k}{6\lambda^2}\int_M \langle T, a_s(T,T,\eta)\rangle^{(s)},\end{aligned} \tag{5.23}$$

where we have used (3.39) for the derivative of the bracket $[\cdot,\cdot]^{(s)}$, as well as the assumption that the Killing form $\langle\cdot,\cdot\rangle^{(s)}$ is invariant. Consider the first two terms in (5.23).

$$\int_M \left\langle d^{\mathcal{H}}\eta - [T,\eta]^{(s)}, \hat{F} \right\rangle^{(s)} = \int_M \left\langle \eta, -d^{\mathcal{H}}\hat{F} - \left[\hat{F}, T\right]^{(s)} \right\rangle^{(s)} \tag{5.24a}$$

$$\int_M \left\langle T, F\cdot\eta - \left[\hat{F},\eta\right]^{(s)} \right\rangle^{(s)} = \int_M \left\langle \eta, \left[\hat{F}, T\right]^{(s)} - F\cdot T \right\rangle^{(s)}. \tag{5.24b}$$

The third term in (5.23) becomes

$$\begin{aligned}\int_M \left\langle d^{\mathcal{H}}\eta - [T,\eta]^{(s)}, [T,T]^{(s)} \right\rangle^{(s)} &= \int_M \left\langle \eta, -d^{\mathcal{H}}[T,T]^{(s)} + \left[T,[T,T]^{(s)}\right]^{(s)} \right\rangle^{(s)} \\ &= \int_M \left\langle \eta, -2\left[\hat{F},T\right]^{(s)} + a_s(T,T,T) \right\rangle^{(s)},\end{aligned}$$

where we used the Akivis identity (3.45) to get $\left[T,[T,T]^{(s)}\right]^{(s)} = a_s(T,T,T)$. Using, alternativity, the last term in (5.23) is

$$\int_M \langle T, a_s(T,T,\eta)\rangle^{(s)} = \int_M \langle \eta, a_s(T,T,T)\rangle^{(s)}.$$

Therefore

$$\begin{aligned}\left.\frac{d}{dt}\mathcal{F}^{(\omega)}(s(t))\right|_{t=0} &= -\int_M \left\langle \eta, d^{\mathcal{H}}\hat{F} + F\cdot T - \frac{k}{\lambda^2}\left[\hat{F},T\right]^{(s)} \right\rangle^{(s)} \\ &\quad - \int_M \left\langle \eta, \frac{2k}{3\lambda^2} a_s(T,T,T) \right\rangle^{(s)}. \quad \square\end{aligned} \tag{5.25}$$

From the Bianchi identity (4.55),

$$d^{\mathcal{H}}\hat{F} = F\cdot T - \left[\hat{F},T\right]^{(s)},$$

and by definition of a_s,

$$a_s(T,T,T) = 2[T,T,T]^{(s)}.$$

Hence,

$$\left.\frac{d}{dt}\mathcal{F}^{(\omega)}(s(t))\right|_{t=0} = -\int_M \left\langle \eta, 2F\cdot T - \left(1+\frac{k}{\lambda^2}\right)\left[\hat{F},T\right]^{(s)} \right\rangle^{(s)} \tag{5.26}$$

$$-\int_M \left\langle \eta, \frac{4k}{3\lambda^2}\left[T,T,T\right]^{(s)}\right\rangle^{(s)}. \tag{5.27}$$

Thus, the critical points with respect to deformations of s satisfy

$$F\cdot T-\left(\frac{1}{2}+\frac{k}{2\lambda^2}\right)\left[\hat{F},T\right]^{(s)}+\frac{2k}{3\lambda^2}\left[T,T,T\right]^{(s)}=0. \tag{5.28}$$

Example 5.9. In the case when $\mathbb{L}$ is a Lie group, the associator vanishes, and $k=\lambda=1$, so we just obtain $d^{\mathcal{H}}\hat{F}=0$, which is of course the standard Bianchi identity. This shows that we just have a reduction from a $\Psi^R(\mathbb{L})$-connection to an $\mathbb{L}$-connection. In the case of $\mathbb{L}$ being the loop of unit octonions, it is easy to verify that $\lambda=\frac{3}{8}$ and $k=3\lambda^3=\frac{81}{512}$ so (5.20) becomes

$$F\cdot T-\frac{17}{16}\left[\hat{F},T\right]^{(s)}+\frac{3}{4}\left[T,T,T\right]^{(s)}=0. \tag{5.29}$$

The significance of this condition is not immediately clear.

We have considered separately the critical points of the functional $\mathcal{F}$ with respect to deformations of the connection ω and the defining section s. Combining the two variations of $\mathcal{F}$, we immediately find the following.

Corollary 5.10. *Consider the functional*

$$\mathcal{F}(s,\omega)=\int_M \left\langle T,\hat{F}\right\rangle^{(s)}-\frac{1}{6\lambda^2}\left\langle T,\left[T,T\right]_{\varphi_s}\right\rangle^{(s)}. \tag{5.30}$$

The critical points (s,ω) of the functional $\mathcal{F}$ satisfy

$$\begin{cases} \hat{F}=0 \\ \left[T,T,T\right]^{(s)}=0 \end{cases}. \tag{5.31}$$

In the spirit of (5.12), given an n-dimensional manifold and an $(n-3)$-form ψ, we could also define a generalized functional

$$\mathcal{G}(s,\omega)=\int_{M^n}\left(\left\langle T,\hat{F}\right\rangle^{(s)}-\frac{1}{6\lambda^2}\left\langle T,\left[T,T\right]_{\varphi_s}\right\rangle^{(s)}\right)\wedge\psi. \tag{5.32}$$

If ψ is assumed to be independent of s, the critical points of $\mathcal{G}$ would then satisfy

$$\begin{cases} \hat{F}\wedge\psi=0 \\ \left[T,T,T\right]^{(s)}\wedge\psi=0 \end{cases}. \tag{5.33}$$

If M is a 7-dimensional manifold with a G_2-structure, and $\psi = *\varphi$ is the G_2-structure 4-form, then the second condition in (5.33) says that as a 3-form, $[T,T,T]^{(s)}$ lies in the 7 and 27 dimensional representations of G_2, i.e. the 1-dimensional component vanishes.

Remark 5.11. We have defined the Chern-Simons functional in 3 dimensions and for higher dimensions followed the ideas from standard higher dimensional gauge. However, in the non-associative case, since the Jacobi does not hold, the $\mathfrak{l}$-valued 3-form $\left[T,[T,T]^{(s)}\right]^{(s)}$ is non-trivial. Moreover, the brackets may be iterated to obtain higher rank forms, and thus there are additional ways in which to define similar higher-dimensional functionals. It will be the subject of further work to understand the significance of such Chern-Simons type functionals. The functional $\mathcal{F}$ is invariant under simultaneous gauge transformations of (s,ω), but not the individual ones. The standard Chern-Simons functional in 3 dimensions not gauge invariant, which causes it to be multi-valued, and only the exponentiated action functional is truly gauge-invariant. It will be interesting to see if there are any analogous properties in the non-associative case.

5.2. Loop Yang-Mills functional

Using the quantity $\hat{F}$, we may also define a loop Yang-Mills functional. Indeed, on a compact n-dimensional Riemannian manifold (M,g), define

$$\mathcal{Y}^{(s)}(\omega) = \int_M \left\langle \hat{F}, *\hat{F} \right\rangle^{(s)}, \tag{5.34}$$

where as before, a wedge product is assumed. We have the following result regarding critical points.

Theorem 5.12. *The critical points of $\mathcal{Y}^{(s)}$ are connections that satisfy*

$$d^{\mathcal{H}} * \hat{F} = (-1)^n \left(*\pi_{\mathfrak{h}_s^{\perp}} F \cdot T - \left[*\hat{F}, T \right]^{(s)} \right). \tag{5.35}$$

Proof. Using (5.4b), we have

$$\begin{aligned} & \frac{d}{dt} \int_M \left\langle \hat{F}(t), *\hat{F}(t) \right\rangle^{(s)} \Big|_{t=0} \\ =& 2\int_M \left\langle d^{\mathcal{H}}\xi + \frac{1}{2\lambda^2}[\xi,T]_{\varphi_s} - \frac{1}{2}[\xi,T]^{(s)}, *\hat{F} \right\rangle \\ =& 2\int_M \left\langle \xi, d^{\mathcal{H}} * \hat{F} - \frac{(-1)^n}{2\lambda^2}\left[*\hat{F},T\right]_{\varphi_s} + \frac{(-1)^n}{2}\left[*\hat{F},T\right]^{(s)} \right\rangle . \end{aligned}$$

Thus, critical points satisfy

$$d^{\mathcal{H}} * \hat{F} = (-1)^n \left(\frac{1}{2\lambda^2} \left[* \hat{F}, T \right]_{\varphi_s} - \frac{1}{2} \left[* \hat{F}, T \right]^{(s)} \right). \tag{5.36}$$

However, from (3.122), we have

$$* \frac{1}{\lambda_s} \varphi_s^t \left(\hat{F} \right) \cdot T = \frac{1}{2\lambda_s^2} \left[* \hat{F}, T \right]_{\varphi_s} + \frac{1}{2} \left[* \hat{F}, T \right]^{(s)},$$

and from (3.113),

$$\varphi_s^t \left(\hat{F} \right) = \varphi_s^t \varphi_s \left(F \right) = \lambda_s \pi_{\mathfrak{h}_s^\perp} F.$$

Hence

$$\frac{1}{2\lambda^2} \left[* \hat{F}, T \right]_{\varphi_s} - \frac{1}{2} \left[* \hat{F}, T \right]^{(s)} = * \pi_{\mathfrak{h}_s^\perp} F \cdot T - \left[* \hat{F}, T \right]^{(s)},$$

and we obtain (5.35). □

Example 5.13. Since we have $* \pi_{\mathfrak{h}_s^\perp} F \cdot T = \frac{1}{\lambda_s} \varphi_s^t \left(\hat{F} \right)$, then as in Example 3.47, if $\mathbb{L} = U\mathbb{O}$,

$$* \pi_{\mathfrak{h}_s^\perp} F \cdot T = \frac{1}{3} \left[* \hat{F}, T \right]^{(s)}$$

and critical points of $\mathcal{Y}^{(s)}$ satisfy

$$d^{\mathcal{H}} * \hat{F} = (-1)^{n+1} \frac{2}{3} \left[* \hat{F}, T \right]^{(s)}.$$

Similarly, following Example 3.48, if $\mathbb{L}$ is a Lie group, then $* \pi_{\mathfrak{h}_s^\perp} F \cdot T = \left[* \hat{F}, T \right]^{(s)}$, and hence we recover the standard Yang-Mills condition for $\hat{F}$, which is the restriction of F to $\mathfrak{h}^\perp \cong \mathfrak{l}$. In this case, we just have standard gauge theory with gauge group $\mathbb{L}$. This justifies considering (5.34) as the *loop* Yang-Mills functional.

If $n = 4$, then we may decompose the 2-form $\hat{F}$ into self-dual and anti-self-dual parts.

Lemma 5.14. *Suppose M is a compact 4-dimensional manifold. If $\hat{F}^{(s,\omega)}$ is self-dual or anti-self-dual, then ω is a critical point of $\mathcal{Y}^{(s)}$ if and only if*

$$\pi_{\mathfrak{h}_s} F \cdot T = 0. \tag{5.37}$$

Proof. Suppose $\hat{F}$ is self-dual, so that $*\hat{F} = \hat{F}$, then from the loop Bianchi identity (4.55),

$$d^{\mathcal{H}} * \hat{F} = F \cdot T - \left[*\hat{F}, T\right]^{(s)}.$$

Comparing with (5.35), we see that ω with a self-dual $\hat{F}$ is a critical point of (5.34) if and only if

$$*\pi_{\mathfrak{h}_s^{\perp}} F \cdot T = F \cdot T.$$

However,

$$\begin{aligned} *\pi_{\mathfrak{h}_s^{\perp}} F =& \frac{1}{\lambda_s} \varphi_s^t \left(*\hat{F}\right) \\ =& \frac{1}{\lambda_s} \varphi_s^t \left(\hat{F}\right) \\ =& \pi_{\mathfrak{h}_s^{\perp}} F. \end{aligned}$$

Thus we see that in this case, ω is a critical point of (5.34) if and only if

$$\pi_{\mathfrak{h}_s} F \cdot T = 0. \tag{5.38}$$

Similarly, suppose $\hat{F}$ is anti-self-dual, so that $*\hat{F} = -\hat{F}$, then the loop Bianchi identity gives

$$d^{\mathcal{H}} * \hat{F} = -F \cdot T - \left[*\hat{F}, T\right]^{(s)}.$$

Now however,

$$*\pi_{\mathfrak{h}_s^{\perp}} F \cdot T = -\pi_{\mathfrak{h}_s^{\perp}} F$$

and we see that ω with a anti-self-dual $\hat{F}$ is also critical point of (5.34) if and only if (5.38) holds. □

Remark 5.15. From Lemma 5.14, we may define the notion of a *loop instanton* on a 4-manifold: a connection for which $\hat{F}$ is self-dual or anti-self-dual, and the $\mathfrak{h}$-component of F satisfies $\pi_{\mathfrak{h}_s} F \cdot T = 0$.

Similarly, as for the loop Chern-Simons functionals, we may also consider the variations of $\mathcal{Y}$ with respect to deformations of s. So if we fix ω, and instead define

$$\mathcal{Y}^{(\omega)}(s) = \int_M \left\langle \hat{F}, *\hat{F} \right\rangle^{(s)}, \tag{5.39}$$

where as before we assume a wedge product. We then have the following.

Theorem 5.16. *The critical points of the functional* $\mathcal{Y}^{(\omega)}$ *satisfy*

$$F\cdot\left(*\hat{F}\right)=0. \tag{5.40}$$

Proof. As before, suppose $s\left(t\right)$ is a path of defining sections of $\mathring{\mathcal{Q}}$ that satisfy

$$\frac{\partial s\left(t\right)}{\partial t}=\left(R_{s}\right)_{*}\eta\left(t\right) \tag{5.41}$$

where $\eta\left(t\right)$ is a 1-parameter family of sections of $\mathcal{A}$. Also, from (5.22), we have

$$\frac{\partial\hat{F}^{(s(t),\omega)}}{\partial t}=F\cdot\eta\left(t\right)-\left[\hat{F}^{(s(t),\omega)},\eta\left(t\right)\right]^{(s(t))}. \tag{5.42}$$

From this, we find that the derivative of $\mathcal{Y}^{(\omega)}\left(s\right)$ is

$$\begin{aligned}\left.\frac{d}{dt}\mathcal{Y}^{(\omega)}\left(s\left(t\right)\right)\right|_{t=0}&=\int_{M}\left\langle F\cdot\eta-\left[\hat{F},\eta\right]^{(s)},*\hat{F}\right\rangle^{(s)}\\&=\int_{M}\left\langle\eta,\left[\hat{F},*\hat{F}\right]^{(s)}-F\cdot*\hat{F}\right\rangle.\end{aligned}$$

However, due to symmetry considerations, $\left[\hat{F},*\hat{F}\right]^{(s)}=0$, and thus we obtain that critical points satisfy (5.40). □

Remark 5.17. From (3.122), we see that

$$\begin{aligned}F_{\mathfrak{h}_{s}^{\perp}}\cdot\left(*\hat{F}\right)&=\frac{1}{\lambda}\varphi_{s}^{t}\left(\hat{F}\right)\cdot\left(*\hat{F}\right)\\&=\frac{1}{2\lambda^{2}}\left[\hat{F},*\hat{F}\right]_{\varphi_{s}}+\frac{1}{2}\left[\hat{F},*\hat{F}\right]^{(s)}\\&=0,\end{aligned}$$

and thus $*\hat{F}$ is always invariant with respect to the action of the $\mathfrak{h}_{s}^{\perp}$-component of F, and thus the condition (5.40) is actually equivalent to saying that

$$F_{\mathfrak{h}_{s}}\cdot\left(*\hat{F}\right)=0. \tag{5.43}$$

However, φ_{s} is invariant under the action of $\mathfrak{h}_{s}$, so

$$\begin{aligned} F_{\mathfrak{h}_s} \cdot \left(*\hat{F} \right) &= F_{\mathfrak{h}_s} \cdot \left(\varphi_s \left(*F \right) \right) \\ &= \varphi_s \left(\left[F_{\mathfrak{h}_s}, *F \right]_{\mathfrak{p}} \right) \\ &= \varphi_s \left(\left[F_{\mathfrak{h}_s}, *F_{\mathfrak{h}_s^\perp} \right]_{\mathfrak{p}} \right). \end{aligned}$$

This shows that (5.43) is equivalent to

$$\left[\left\langle F_{\mathfrak{h}_s}, F_{\mathfrak{h}_s^\perp} \right\rangle_M \right]_{\mathfrak{p}} \in \mathfrak{h}_s$$

at every point. Here $\langle \cdot, \cdot \rangle_M$ is the inner product of 2-forms on M.

5.3. Energy functional

In the context of G_2-structures, another functional has been considered in several papers [5,10,15,17,31], namely the L_2-norm of the torsion, considered as functional on the space of isometric G_2-structures, i.e. G_2-structures that correspond to the same metric. In the context of loop structures we may define a similar functional. Given a compact Riemannian manifold (M, g) and a fixed connection ω on $\mathcal{P}$, for any section $s \in \Gamma\left(\mathring{\mathcal{Q}}\right)$ let $T^{(s)}$ be the torsion of s with respect to ω. Then define the energy functional on $\Gamma\left(\mathring{\mathcal{Q}}\right)$ given by:

$$\mathcal{E}(s) = \int_M \left\langle T^{(s)}, *T^{(s)} \right\rangle^{(s)}, \tag{5.44}$$

where the wedge product is assumed. We then have the following.

Theorem 5.18. *The critical points of the functional $\mathcal{E}$ satisfy*

$$\left(d^{\mathcal{H}}\right)^* T^{(s)} = 0. \tag{5.45}$$

Proof. With respect to deformations of s given by (4.72) and the corresponding deformation of T given by (5.22) we have

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(s(t)) \Big|_{t=0} &= 2 \int_M \left\langle d^{\mathcal{H}} \eta - \left[T^{(s)}, \eta \right]^{(s)}, *T^{(s)} \right\rangle^{(s)} \\ &= -2 \int_M \left\langle \eta, d^{\mathcal{H}} * T^{(s)} - \left[T^{(s)}, *T^{(s)} \right]^{(s)} \right\rangle^{(s)} \\ &= -2 \int_M \left\langle \eta, d^{\mathcal{H}} * T^{(s)} \right\rangle^{(s)}, \end{aligned} \tag{5.46}$$

where $\left[T^{(s)}, *T^{(s)}\right]^{(s)} = 0$ due to symmetry considerations. Hence we obtain (5.45). $\square$

Thus the critical points of $\mathcal{E}$ satisfy which is precisely the analog of the "divergence-free torsion" condition in [5,10,15,17,31]. Also, similarly as in [31], if we assume $\mathcal{P}$ is compact, the functional $\mathcal{E}$ may be related to the equivariant Dirichlet energy functional for maps from $\mathcal{P}$ to $\mathring{\mathbb{L}}$. Given a metric $\langle\cdot,\cdot\rangle^{(s)}$ on $\mathfrak{l}$, we may extend it to a metric on all of $\mathbb{L}$ via right translations: $\langle\cdot,\cdot\rangle_p^{(s)} = \left\langle (R_p)_*^{-1}\cdot, (R_p)_*^{-1}\cdot\right\rangle^{(s)}$. Then, the Dirichlet energy functional on *equivariant* maps from $\mathcal{P}$ to $\mathring{\mathbb{L}}$ is given by

$$\mathcal{D}(s) = \int_{\mathcal{P}} |ds|^2 = \int_{\mathcal{P}} |\theta_s|^2, \tag{5.47}$$

where we endow $T\mathcal{P}$ with a metric such that the decomposition $T\mathcal{P} = \mathcal{HP} \oplus \mathcal{VP}$ is orthogonal with respect to it, and moreover such that it is compatible with the metrics on M and Ψ. Then, using (4.23)

$$\mathcal{D}(s) = \int_{\mathcal{P}} \left|T^{(s)}\right|^2 + \int_{\mathcal{P}} \left|\hat{\omega}^{(s)}\right|^2 \tag{5.48}$$

Note that given an orthogonal basis $\{X_i\}$ on $\mathfrak{p}$, $\left|\hat{\omega}^{(s)}\right|^2 = \left|\hat{\omega}^{(s)}(\sigma(X_i))\right|^2 = \left|\hat{X}_i\right|^2 = \lambda_s \dim \mathfrak{l}$. With our previous assumptions, $\lambda_s = \lambda$, and thus does not depend on s, so we have

$$\mathcal{D}(s) = a\mathcal{E}(s) + b$$

where $a = \operatorname{Vol}(\Psi)$ and $b = \lambda(\dim \mathbb{L}) \operatorname{Vol}(\mathcal{P})$. Hence, the critical points of $\mathcal{E}(s)$ are precisely the critical points of $\mathcal{D}(s)$ with respect to deformations through equivariant maps, i.e. equivariant harmonic maps. So indeed, to understand the properties of these critical points, a rigorous equivariant harmonic map theory is required, as initiated in [31].

6. Concluding remarks

Given a smooth loop $\mathbb{L}$ with tangent algebra $\mathfrak{l}$ and a group Ψ that acts smoothly on $\mathbb{L}$ via pseudoautomorphism pairs, we have defined the concept of a loop bundle structure $(\mathbb{L}, \Psi, \mathcal{P}, s)$ for a principal Ψ-bundle and a corresponding equivariant $\mathring{\mathbb{L}}$-valued map s, that also defines a section of the corresponding associated bundle. If we moreover have a connection ω on $\mathcal{P}$, then horizontal component of the Darboux derivative of s defines an $\mathfrak{l}$-valued 1-form $T^{(s,\omega)}$, which we called the torsion. This object $T^{(s,\omega)}$ then satisfies a structural equation based on the loop Maurer-Cartan equation and gives rise to an $\mathfrak{l}$-valued component of the curvature $\hat{F}^{(s,\omega)}$. Overall, there are several possible directions to further this non-associative theory.

1. From a more algebraic perspective it would be interesting to construct additional examples of smooth loops, in particular those that are not Moufang and possibly are not even G-loops in order to more concretely study the corresponding bundles in those situations. In fact, it may not even be necessary to have a full loop structure - it may be sufficient to just have a right loop structure, so that division is possible only on the right. Left division was used rarely, and it may be possible to build up a full theory without needing it. New examples of loops may give rise to new geometric structures.
2. In Lie theory, the Maurer-Cartan equation plays a central role. As we've seen there is an analog in smooth loop theory as well. A better understanding of this equation is needed. The standard Maurer-Cartan equation is closely related to the concept of integrability, but it is not clear how to interpret the non-associative version.
3. In defining the loop bundle structure, we generally have assumed that the map s is globally defined. However, this may place strict topological restrictions. It may be reasonable to allow s to be defined only locally. This would give more flexibility, but it would need to be checked carefully whether other related quantities are well-defined.
4. We have defined a functional of Chern-Simons type in Section 5.1. There are further properties that need to be investigated. For example, is it possible to use the associator to define reasonable functionals on higher-dimensional manifolds? If the section s is defined only locally, are these functionals well-defined? Finally, do these functionals have any topological meaning?
5. In G_2-geometry, significant progress has been made in [5,10,15,17,31] regarding the existence of critical points of the energy functional (5.44) via a heat flow approach. However, it is likely that a more direct approach, similar to Uhlenbeck's existence result for the Coulomb gauge [49], could also be used. This would give existence of a preferred section s for a given connection or conversely, a preferred connection in a gauge class for a fixed section s.

Overall, the framework presented in this paper may give an impetus to the development of a larger theory of "nonassociative geometry".

Appendix A

Lemma A.1. *Suppose $A(t)$ and $B(t)$ are smooth curves in $\mathbb{L}$ with $A(0) = A_0$ and $B(0) = B_0$, then*

$$\left.\frac{d}{dt}A(t)/B(t)\right|_{t=0} = \left.\frac{d}{dt}A(t)/B_0\right|_{t=0} - \left.\frac{d}{dt}\left(A_0/B_0 \cdot B(t)\right)/B_0\right|_{t=0} \tag{A.1a}$$

$$\left.\frac{d}{dt}B(t)\backslash A(t)\right|_{t=0} = \left.\frac{d}{dt}B_0\backslash A(t)\right|_{t=0} - \left.\frac{d}{dt}B_0\backslash\left(B(t) \cdot B_0\backslash A_0\right)\right|_{t=0} \tag{A.1b}$$

Proof. First note that

$$\begin{aligned}\frac{d}{dt}A(t)\Big|_{t=0} &= \frac{d}{dt}\left(A(t)/B(t)\cdot B(t)\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(A(t)/B(t)\right)\cdot B_0\Big|_{t=0} + \frac{d}{dt}\left(A_0/B_0\cdot B(t)\right)\Big|_{t=0} \\ &= \left(R_{B_0}\right)_* \frac{d}{dt}A(t)/B(t)\Big|_{t=0} + \frac{d}{dt}\left(A_0/B_0\cdot B(t)\right)\Big|_{t=0}\end{aligned}$$

Hence, applying $\left(R_{B_0}^{-1}\right)_*$ to both sides, we obtain (A.1a). Similarly,

$$\begin{aligned}\frac{d}{dt}A(t)\Big|_{t=0} &= \frac{d}{dt}\left(B(t)\cdot B(t)\backslash A(t)\right)\Big|_{t=0} \\ &= \left(L_{B_0}\right)_* \frac{d}{dt}\left(B(t)\backslash A(t)\right)\Big|_{t=0} + \frac{d}{dt}\left(B(t)\cdot B_0\backslash A_0\right)\Big|_{t=0}\end{aligned}$$

and applying $\left(L_{B_0}^{-1}\right)_*$ to both sides gives (A.1b). □

Lemma A.2 *(Lemma 3.13). For fixed $\eta,\gamma\in\mathfrak{l}$,*

$$db|_p(\eta,\gamma) = [\eta,\gamma,\theta_p]^{(p)} - [\gamma,\eta,\theta_p]^{(p)}, \tag{A.2}$$

where $[\cdot,\cdot,\cdot]^{(p)}$ is the $\mathbb{L}$-algebra associator *on $\mathfrak{l}^{(p)}$ given by*

$$\begin{aligned}[\eta,\gamma,\xi]^{(p)} &= \frac{d^3}{dtd\tau d\tau'}\exp(\tau\eta)\circ_p\left(\exp(\tau'\gamma)\circ_p\exp(t\xi)\right)\Big|_{t,\tau,\tau'=0} \\ &\quad - \frac{d^3}{dtd\tau d\tau'}\left(\exp(\tau\eta)\circ_p\exp(\tau'\gamma)\right)\circ_p\exp(t\xi)\Big|_{t,\tau,\tau'=0}.\end{aligned} \tag{A.3}$$

Moreover,

$$[\eta,\gamma,\xi]^{(p)} = \frac{d^3}{dtd\tau d\tau'}\left[\exp(\tau\eta),\exp(\tau'\gamma),\exp(t\xi)\right]^{(\mathbb{L},\circ_p)}\Big|_{t,\tau,\tau'=0} \tag{A.4}$$

where $[\cdot,\cdot,\cdot]^{(\mathbb{L},\circ_p)}$ is the loop associator on $(\mathbb{L},\circ_p)$ as defined by (2.34).

Proof. Let $X=\rho(\xi)$ and $x(t)=\exp_p(t\xi)\,p$, then consider

$$\begin{aligned}X\left(b(\eta,\gamma)\right)_p &= \frac{d}{dt}\left([\eta,\gamma]^{x(t)}\right)\Big|_{t=0} \\ &= \frac{d^3}{dtd\tau d\tau'}\left(\exp(\tau\eta)\circ_{x(t)}\exp(\tau'\gamma)\right)\Big|_{t,\tau,\tau'=0}\end{aligned} \tag{A.5}$$

$$-\frac{d^3}{dtd\tau d\tau'}\left(\exp\left(\tau'\gamma\right)\circ_{x(t)}\exp\left(\tau\eta\right)\right)\Big|_{t,\tau,\tau'=0}$$

where we have used (3.29). Then,

$$\exp\left(\tau\eta\right)\circ_{x(t)}\exp\left(\tau'\gamma\right)=\left(\exp\left(\tau\eta\right)\left(\exp\left(\tau'\gamma\right)x\left(t\right)\right)\right)/x\left(t\right). \tag{A.6}$$

For brevity let us Ξ for exp, and d_0^3 for $\frac{d^3}{dtd\tau d\tau'}\Big|_{t,\tau,\tau'=0}$, so that, using Lemma A.1, we get

$$\begin{aligned}d_0^3\left(\Xi\left(\tau\eta\right)\circ_{x(t)}\Xi\left(\tau'\gamma\right)\right)=&d_0^3\left(\Xi\left(\tau\eta\right)\left(\Xi\left(\tau'\gamma\right)x\left(t\right)\right)/p\right)\\&-d_0^3\left(\left(\Xi\left(\tau\eta\right)\left(\Xi\left(\tau'\gamma\right)p\right)/p\right)\cdot x\left(t\right)/p\right)\end{aligned} \tag{A.7}$$

However,

$$\begin{aligned}\left(\Xi\left(\tau\eta\right)\left(\Xi\left(\tau'\gamma\right)x\left(t\right)\right)\right)/p&=\left(\Xi\left(\tau\eta\right)\left(\Xi\left(\tau'\gamma\right)\left(x\left(t\right)/p\cdot p\right)\right)\right)/p\\&=\left(\Xi\left(\tau\eta\right)\left(\left(\Xi\left(\tau'\gamma\right)\circ_p\left(x\left(t\right)/p\right)\right)p\right)\right)/p\\&=\Xi\left(\tau\eta\right)\circ_p\left(\Xi\left(\tau'\gamma\right)\circ_p\Xi_p\left(t\xi\right)\right)\end{aligned} \tag{A.8}$$

and similarly,

$$\left(\Xi\left(\tau\eta\right)\left(\Xi\left(\tau'\gamma\right)p\right)/p\cdot x\left(t\right)\right)/p=\left(\Xi\left(\tau\eta\right)\circ_p\Xi\left(\tau'\gamma\right)\right)\circ_p\Xi_p\left(t\xi\right). \tag{A.9}$$

The derivatives of $\Xi_p\left(t\xi\right)$ and $\Xi\left(t\xi\right)$ with respect to t at $t=0$ are equal, thus, from (A.7), we find

$$\frac{d^3}{dtd\tau d\tau'}\left(\Xi\left(\tau\eta\right)\circ_{x(t)}\Xi\left(\tau'\gamma\right)\right)\Big|_{t,\tau,\tau'=0}=\left[\eta,\gamma,\xi\right]^{(p)} \tag{A.10}$$

and hence, from (A.5),

$$X\left(b\left(\eta,\gamma\right)\right)_p=\left[\eta,\gamma,\xi\right]^{(p)}-\left[\gamma,\eta,\xi\right]^{(p)}. \tag{A.11}$$

For the last part, using (2.34) and Lemma A.1, we get

$$\begin{aligned}d_0^3\left[\Xi\left(\tau\eta\right),\Xi\left(\tau'\gamma\right),\Xi\left(t\xi\right)\right]^{(\mathbb{L},\circ_p)}=&d_0^3\left(\Xi\left(\tau\eta\right)\circ_{\Xi_p(t\xi)p}\Xi\left(\tau'\gamma\right)\right)/_p\left(\Xi\left(\tau\eta\right)\circ_p\Xi\left(\tau'\gamma\right)\right)\\=&d_0^3\left(\Xi\left(\tau\eta\right)\circ_{\Xi_p(t\xi)p}\Xi\left(\tau'\gamma\right)\right)/_p\Xi\left(\tau'\gamma\right)\\&-d_0^3\left(\Xi\left(\tau\eta\right)\circ_p\Xi\left(\tau'\gamma\right)\right)/\Xi\left(\tau\eta\right)\\=&d_0^3\left(\Xi\left(\tau\eta\right)\circ_{\Xi_p(t\xi)p}\Xi\left(\tau'\gamma\right)\right)\end{aligned}$$

and hence from (A.10) we see that indeed (3.41) holds. □

Lemma A.3. *Suppose $s(t)$ and $f(t)$ are smooth curves in $\mathbb{L}$ with $s(0) = s$, $f(0) = f$, $\dot{s}(0) = \dot{s}$, $\dot{f}(0) = \dot{f}$. Also, let $\xi \in \mathfrak{l}$, then*

$$\frac{d}{dt}\left(\mathrm{Ad}_{f(t)}^{(s(t))}\right)_*\xi\Big|_{t,\tau=0} = \left[\left(R_f^{(s)}\right)_*^{-1}\dot{f}, \left(\mathrm{Ad}_f^{(s)}\right)_*\xi\right]^{(fs)} - \left(R_f^{(s)}\right)_*^{-1}\left[\left(R_f^{(s)}\right)_*^{-1}\dot{f}, f, \xi\right]^{(s)} + \left(R_f^{(s)}\right)_*^{-1}\left[f, \xi, (R_s)_*^{-1}\dot{s}\right]^{(s)} - \left(R_f^{(s)}\right)_*^{-1}\left[\left(\mathrm{Ad}_f^{(s)}\right)_*\xi, f, (R_s)_*^{-1}\dot{s}\right]^{(s)}. \tag{A.12}$$

Proof. Let $\xi \in \mathfrak{l}$, and consider $s_t = s(t)$, $f_t = f(t)$, then, for brevity suppressing pushforwards, we have

$$\begin{aligned}
\frac{d}{dt}\left(\mathrm{Ad}_{f_t}^{(s_t)}\right)_*\xi\Big|_{t=0} &= \frac{d}{dt}\left(f_t \circ_{s_t} \xi\right)/_{s_t} f_t\Big|_{t=0} \\
&= \frac{d}{dt}\left(f \circ_{s_t} \xi\right)/_{s_t} f\Big|_{t=0} + \frac{d}{dt}\left(f_t \circ_s \xi\right)/_s f_t\Big|_{t=0} \\
&= \frac{d}{dt}\left(f \cdot \xi s_t\right)/\left(f s_t\right)\Big|_{t=0} + \frac{d}{dt}\left(f_t \circ_s \xi\right)/_s f\Big|_{t=0} \\
&\quad - \frac{d}{dt}\left(\left(f \circ_s \xi\right)/_s f \circ_s f_t\right)/_s f\Big|_{t=0} \\
&= \frac{d}{dt}\left(f \cdot \xi s_t\right)/\left(fs\right)\Big|_{t=0} - \frac{d}{dt}\left(\left(f \cdot \xi s\right)/\left(fs\right) \cdot f s_t\right)/fs\Big|_{t=0} \\
&\quad + \frac{d}{dt}\left(f_t \circ_s \xi\right)/_s f\Big|_{t=0} - \frac{d}{dt}\left(\mathrm{Ad}_f^{(s)}\xi \circ_s f_t\right)/_s f\Big|_{t=0}
\end{aligned} \tag{A.13}$$

Now consider the first two terms (suppressing the derivatives for clarity):

$$\begin{aligned}
\left(f \cdot \xi s_t\right)/\left(fs\right) &= \left(f \circ_s \left(\xi \circ_s \left(s_t/s\right)\right)\right)/_s f \\
\left(\left(f \cdot \xi s\right)/\left(fs\right) \cdot f s_t\right)/fs &= \left(\left(\left(f \circ_s \xi\right)/_s f\right) \circ_s \left(f \circ_s s_t/s\right)\right)/_s f \\
&= \left(\left(f \circ_s \xi\right) \circ_s s_t/s\right)/_s f + \left[\mathrm{Ad}_f^{(s)}\xi, f, s_t/s\right]^{(s)}/_s f
\end{aligned}$$

Thus,

$$\left(f \cdot \xi s_t\right)/\left(fs\right) - \left(\left(f \cdot \xi s\right)/\left(fs\right) \cdot f s_t\right)/fs = \left[f, \xi, s_t/s\right]^{(s)}/_s f \tag{A.14}$$

$$- \left[\mathrm{Ad}_f^{(s)} \xi, f, s_t/s \right]^{(s)} /_s f.$$

The next two terms in (A.13) become

$$\begin{aligned} (f_t \circ_s \xi) /_s f &= ((f_t/_s f \circ_s f) \circ_s \xi) /_s f \\ &= (f_t/_s f \circ_s (f \circ_s \xi)) /_s f - [f_t/_s f, f, \xi]^{(s)} /_s f \\ &= (f_t/_s f) \circ_{fs} \mathrm{Ad}_f^{(s)} \xi - [f_t/_s f, f, \xi]^{(s)} /_s f \\ \left(\mathrm{Ad}_f^{(s)} \xi \circ_s f_t \right) /_s f &= \mathrm{Ad}_f^{(s)} \xi \circ_{fs} (f_t/_s f) \end{aligned}$$

Thus,

$$(f_t \circ_s \xi) /_s f - \left(\mathrm{Ad}_f^{(s)} \xi \circ_s f_t \right) /_s f = \left[f_t/_s f, \mathrm{Ad}_f^{(s)} \xi \right]^{(fs)} - [f_t/_s f, f, \xi]^{(s)} /_s f \tag{A.15}$$

Overall, combining (A.14) and (A.15) and now using proper notation, we obtain (A.12). □

Theorem A.4 *(Theorem 3.40). The bilinear form $K^{(s)}$ (3.106) on $\mathfrak{l}$ has the following properties.*

1. *Let $h \in \Psi^R(\mathbb{L})$, then for any $\xi, \eta \in \mathfrak{l}$,*

$$K^{(h(s))} (h'_* \xi, h'_* \eta) = K^{(s)} (\xi, \eta). \tag{A.16}$$

2. *Suppose also $\gamma \in \mathfrak{l}$, then*

$$\begin{aligned} K^{(s)} \left(\mathrm{ad}_\gamma^{(s)} \eta, \xi \right) &= - K^{(s)} \left(\eta, \mathrm{ad}_\gamma^{(s)} \xi \right) + \mathrm{Tr} \left(\mathrm{Jac}_{\xi,\gamma}^{(s)} \circ \mathrm{ad}_\eta^{(s)} \right) \\ &\quad + \mathrm{Tr} \left(\mathrm{Jac}_{\eta,\gamma}^{(s)} \circ \mathrm{ad}_\xi^{(s)} \right), \end{aligned} \tag{A.17}$$

   where $\mathrm{Jac}_{\gamma,\xi}^{(s)} : \mathfrak{l} \longrightarrow \mathfrak{l}$ is given by $\mathrm{Jac}_{\eta,\gamma}^{(s)}(\xi) = \mathrm{Jac}^{(s)}(\xi, \eta, \gamma)$.

3. *Let $\alpha \in \mathfrak{p}$, then*

$$\begin{aligned} K^{(s)} (\alpha \cdot \xi, \eta) &= - K^{(s)} (\xi, \alpha \cdot \eta) + \mathrm{Tr} \left(a_{\eta,\hat{\alpha}}^{(s)} \circ \mathrm{ad}_\xi^{(s)} \right) \\ &\quad + \mathrm{Tr} \left(a_{\xi,\hat{\alpha}}^{(s)} \circ \mathrm{ad}_\eta^{(s)} \right), \end{aligned} \tag{A.18}$$

   where $a_{\xi,\eta}^{(s)} : \mathfrak{l} \longrightarrow \mathfrak{l}$ is given by $a_{\xi,\eta}^{(s)}(\gamma) = [\gamma, \xi, \eta]^{(s)} - [\xi, \gamma, \eta]^{(s)}$ and $\hat{\alpha} = \varphi_s(\alpha)$.

Proof. 1. Let $h \in \Psi$, and then using the cyclic property of trace as well as (3.57), we have

$$
\begin{aligned}
K^{(h(s))}\left(h'_*\xi, h'_*\eta\right) &= \operatorname{Tr}\left(\operatorname{ad}^{(h(s))}_{h'_*\xi} \circ \operatorname{ad}^{(h(s))}_{h'_*\eta}\right) \\
&= \operatorname{Tr}\left(\left[h'_*\xi, \left[h'_*\eta, \cdot\right]^{(h(s))}\right]^{(h(s))}\right) \\
&= \operatorname{Tr}\left(\left[h'_*\xi, \left[h'_*\eta, h'_*\left(h'_*\right)^{-1}\cdot\right]^{(h(s))}\right]^{(s)}\right) \\
&= \operatorname{Tr}\left(\left[h'_*\xi, h'_*\left[\eta, \left(h'_*\right)^{-1}\cdot\right]^{(s)}\right]^{(s)}\right) \\
&= \operatorname{Tr}\left(h'_*\left[\xi, \left[\eta, \left(h'_*\right)^{-1}\cdot\right]^{(s)}\right]^{(s)}\right) \\
&= \operatorname{Tr}\left(h'_* \circ \left(\operatorname{ad}^{(s)}_{\xi} \circ \operatorname{ad}^{(s)}_{\eta}\right) \circ \left(h'_*\right)^{-1}\right) \\
&= \operatorname{Tr}\left(\operatorname{ad}^{(s)}_{\xi} \circ \operatorname{ad}^{(s)}_{\eta}\right) \\
&= K^{(s)}\left(\xi, \eta\right).
\end{aligned}
$$

2. From (3.38), we see that

$$
\begin{aligned}
\operatorname{ad}^{(s)}_{[\eta,\gamma]^{(s)}} &= -\left[\cdot, [\eta,\gamma]^{(s)}\right]^{(s)} \\
&= \left[\eta, [\gamma, \cdot]^{(s)}\right]^{(s)} - \left[\gamma, [\eta, \cdot]^{(s)}\right]^{(s)} - \operatorname{Jac}^{(s)}_{\eta,\gamma} \\
&= \operatorname{ad}^{(s)}_{\eta} \circ \operatorname{ad}^{(s)}_{\gamma} - \operatorname{ad}^{(s)}_{\gamma} \circ \operatorname{ad}^{(s)}_{\eta} - \operatorname{Jac}^{(s)}_{\eta,\gamma}
\end{aligned} \tag{A.19}
$$

Hence,

$$
\begin{aligned}
\operatorname{ad}^{(s)}_{[\eta,\gamma]^{(s)}} \circ \operatorname{ad}^{(s)}_{\xi} = &\operatorname{ad}^{(s)}_{\eta} \circ \operatorname{ad}^{(s)}_{\gamma} \circ \operatorname{ad}^{(s)}_{\xi} - \operatorname{ad}^{(s)}_{\gamma} \circ \operatorname{ad}^{(s)}_{\eta} \circ \operatorname{ad}^{(s)}_{\xi} \\
&- \operatorname{Jac}^{(s)}_{\eta,\gamma} \circ \operatorname{ad}^{(s)}_{\xi}
\end{aligned}
$$

and so using the cycling symmetry of trace, we have

$$
\begin{aligned}
K^{(s)}\left([\eta,\gamma]^{(s)}, \xi\right) &= \operatorname{Tr}\left(\operatorname{ad}^{(s)}_{\eta} \circ \left(\operatorname{ad}^{(s)}_{\gamma} \circ \operatorname{ad}^{(s)}_{\xi} - \operatorname{ad}^{(s)}_{\xi} \circ \operatorname{ad}^{(s)}_{\gamma}\right)\right) \\
&\quad - \operatorname{Tr}\left(\operatorname{Jac}^{(s)}_{\eta,\gamma} \circ \operatorname{ad}^{(s)}_{\xi}\right) \\
&= \operatorname{Tr}\left(\operatorname{ad}^{(s)}_{\eta} \circ \operatorname{ad}^{(s)}_{[\gamma,\xi]^{(s)}}\right) + \operatorname{Tr}\left(\operatorname{ad}^{(s)}_{\eta} \circ \operatorname{Jac}^{(s)}_{\gamma,\xi}\right) \\
&\quad - \operatorname{Tr}\left(\operatorname{Jac}^{(s)}_{\eta,\gamma} \circ \operatorname{ad}^{(s)}_{\xi}\right) \\
&= K^{(s)}\left(\eta, [\gamma,\xi]^{(s)}\right) + \operatorname{Tr}\left(\operatorname{Jac}^{(s)}_{\gamma,\xi} \circ \operatorname{ad}^{(s)}_{\eta}\right)
\end{aligned}
$$

$$+\operatorname{Tr}\left(\mathrm{Jac}_{\gamma,\eta}^{(s)}\circ\mathrm{ad}_{\xi}^{(s)}\right).$$

This then gives (A.17).

3. Now let $\alpha\in\mathfrak{p}$ and consider

$$K^{(s)}\left(\alpha\cdot\xi,\eta\right)=\operatorname{Tr}\left(\mathrm{ad}_{\alpha\cdot\xi}^{(s)}\circ\mathrm{ad}_{\eta}^{(s)}\right).$$

Denote by $l_{\alpha}:\mathfrak{l}\longrightarrow\mathfrak{l}$ the left action of $\mathfrak{p}$ on $\mathfrak{l}$. From (3.90), we then have

$$\mathrm{ad}_{\alpha\cdot\xi}^{(s)}=l_{\alpha}\circ\mathrm{ad}_{\xi}^{(s)}-\mathrm{ad}_{\xi}^{(s)}\circ l_{\alpha}+a_{\xi,\hat{\alpha}}^{(s)} \tag{A.20}$$

So now,

$$\begin{aligned}K^{(s)}\left(\alpha\cdot\xi,\eta\right)=&\operatorname{Tr}\left(l_{\alpha}\circ\mathrm{ad}_{\xi}^{(s)}\circ\mathrm{ad}_{\eta}^{(s)}-\mathrm{ad}_{\xi}^{(s)}\circ l_{\alpha}\circ\mathrm{ad}_{\eta}^{(s)}\right)\\&+\operatorname{Tr}\left(a_{\xi,\hat{\alpha}}^{(s)}\circ\mathrm{ad}_{\eta}^{(s)}\right)\\=&\operatorname{Tr}\left(\mathrm{ad}_{\xi}^{(s)}\circ\left(\mathrm{ad}_{\eta}^{(s)}\circ l_{\alpha}-l_{\alpha}\circ\mathrm{ad}_{\eta}^{(s)}\right)\right)+\operatorname{Tr}\left(a_{\xi,\hat{\alpha}}^{(s)}\circ\mathrm{ad}_{\eta}^{(s)}\right)\\=&-\operatorname{Tr}\left(\mathrm{ad}_{\xi}^{(s)}\circ\mathrm{ad}_{\alpha\cdot\eta}^{(s)}\right)+\operatorname{Tr}\left(\mathrm{ad}_{\xi}^{(s)}\circ a_{\eta,\hat{\alpha}}^{(s)}\right)+\operatorname{Tr}\left(a_{\xi,\hat{\alpha}}^{(s)}\circ\mathrm{ad}_{\eta}^{(s)}\right)\\=&-K^{(s)}\left(\xi,\alpha\cdot\eta\right)+\operatorname{Tr}\left(a_{\eta,\hat{\alpha}}^{(s)}\circ\mathrm{ad}_{\xi}^{(s)}\right)+\operatorname{Tr}\left(a_{\xi,\hat{\alpha}}^{(s)}\circ\mathrm{ad}_{\eta}^{(s)}\right).\quad\square\end{aligned}$$

References

[1] M.A. Akivis, The local algebras of a multidimensional three-web, Sib. Mat. Zh. 17 (1) (1976) 5–11, 237.

[2] A. Anastasiou, L. Borsten, M.J. Duff, L.J. Hughes, S. Nagy, An octonionic formulation of the M-theory algebra, J. High Energy Phys. 11 (2014) 022, https://doi.org/10.1007/JHEP11(2014)022, front matter+9.

[3] J. Baez, The Octonions, Bull. Am. Math. Soc. (N.S.) 39 (2002) 145–205, https://doi.org/10.1090/S0273-0979-01-00934-X.

[4] J.C. Baez, J. Huerta, Division algebras and supersymmetry I, in: Superstrings, Geometry, Topology, and C^*-Algebras, in: Proc. Sympos. Pure Math., vol. 81, Amer. Math. Soc., Providence, RI, 2010, pp. 65–80.

[5] L. Bagaglini, The energy functional of G_2-structures compatible with a background metric, J. Geom. Anal. 31 (2021) 346–365, https://doi.org/10.1007/s12220-019-00264-6.

[6] M. Berger, Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes, Bull. Soc. Math. Fr. 83 (1955), https://doi.org/10.24033/bsmf.1464.

[7] R. Cleyton, S. Ivanov, On the geometry of closed G_2-structures, Commun. Math. Phys. 270 (1) (2007) 53–67, https://doi.org/10.1007/s00220-006-0145-7, arXiv:math/0306362.

[8] R. Cleyton, S. Ivanov, Curvature decomposition of G_2-manifolds, J. Geom. Phys. 58 (10) (2008) 1429–1449, https://doi.org/10.1016/j.geomphys.2008.06.002.

[9] S.K. Donaldson, R.P. Thomas, Gauge theory in higher dimensions, in: The Geometric Universe, Oxford, 1996, Oxford Univ. Press, Oxford, 1998, pp. 31–47.

[10] S. Dwivedi, P. Gianniotis, S. Karigiannis, A gradient flow of isometric G_2 structures, J. Geom. Anal. 31 (2021) 1855–1933, https://doi.org/10.1007/s12220-019-00327-8, arXiv:1904.10068.

[11] A. Figula, Bol loops as sections in semi-simple Lie groups of small dimension, Manuscr. Math. 121 (3) (2006) 367–384, https://doi.org/10.1007/s00229-006-0042-6.

[12] A. Floer, An instanton-invariant for 3-manifolds, Commun. Math. Phys. 118 (2) (1988) 215–240, http://projecteuclid.org/euclid.cmp/1104161987.

[13] G. Grantcharov, Y.S. Poon, Geometry of hyper-Kähler connections with torsion, Commun. Math. Phys. 213 (1) (2000) 19–37, https://doi.org/10.1007/s002200000231.
[14] S. Grigorian, Deformations of G_2-structures with torsion, Asian J. Math. 20 (1) (2016) 123–155, https://doi.org/10.4310/AJM.2016.v20.n1.a6, arXiv:1108.2465.
[15] S. Grigorian, G_2-structures and octonion bundles, Adv. Math. 308 (2017) 142–207, https://doi.org/10.1016/j.aim.2016.12.003, arXiv:1510.04226.
[16] S. Grigorian, G_2-structures for $N = 1$ supersymmetric AdS_4 solutions of M-theory, Class. Quantum Gravity 35 (8) (2018) 085012, https://doi.org/10.1088/1361-6382/aaad4a, arXiv:1711.10619.
[17] S. Grigorian, Estimates and monotonicity for a heat flow of isometric G_2-structures, Calc. Var. Partial Differ. Equ. 58 (5) (2019) 175, https://doi.org/10.1007/s00526-019-1630-0, 37.
[18] S. Grigorian, Flows of co-closed G_2-structures, in: Lectures and Surveys on G2 Manifolds and Related Topics, in: Fields Institute Communications, Springer, New York, 2019, https://doi.org/10.1007/978-1-0716-0577-6_12, arXiv:1811.10505.
[19] S. Grigorian, Isometric flows of G_2-structures, arXiv:2008.06593, 2020.
[20] F.R. Harvey, Spinors and Calibrations, Perspectives in Mathematics, vol. 9, Academic Press, Inc., Boston, MA, 1990.
[21] K.H. Hofmann, K. Strambach, Topological and analytic loops, in: Quasigroups and Loops: Theory and Applications, in: Sigma Ser. Pure Math., vol. 8, Heldermann, Berlin, 1990, pp. 205–262.
[22] S. Karigiannis, Deformations of G_2 and $Spin(7)$ structures on manifolds, Can. J. Math. 57 (2005) 1012, https://doi.org/10.4153/CJM-2005-039-x, arXiv:math/0301218.
[23] S. Karigiannis, Flows of G_2-structures, I, Q. J. Math. 60 (4) (2009) 487–522, https://doi.org/10.1093/qmath/han020, arXiv:math/0702077.
[24] H. Kiechle, Theory of K-Loops, Lecture Notes in Mathematics, vol. 1778, Springer-Verlag, Berlin, 2002.
[25] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry. Vol. I, Wiley Classics Library, John Wiley & Sons Inc., New York, 1996. Reprint of the 1963 original, a Wiley-Interscience Publication.
[26] K. Kunen, G-loops and permutation groups, J. Algebra 220 (2) (1999) 694–708, https://doi.org/10.1006/jabr.1999.8006.
[27] E.N. Kuzmin, Malcev algebras and their representations, Algebra Log. 7 (4) (1968) 48–69.
[28] E.N. Kuzmin, The connection between Malcev algebras and analytic Moufang loops, Algebra Log. 10 (1971) 3–22.
[29] N.C. Leung, Riemannian geometry over different normed division algebras, J. Differ. Geom. 61 (2) (2002) 289–333, https://doi.org/10.4310/jdg/1090351387.
[30] O. Loos, Über eine Beziehung zwischen Malcev-Algebren und Lietripelsystemen, Pac. J. Math. 18 (1966) 553–562, http://projecteuclid.org/euclid.pjm/1102994136.
[31] E. Loubeau, H.N. Sá Earp, Harmonic flow of geometric structures, arXiv:1907.06072, 2019.
[32] A.I. Malcev, Analytic loops, Mat. Sb. (N.S.) 36 (78) (1955) 569–576.
[33] J.W. Milnor, J.D. Stasheff, Characteristic Classes, Annals of Mathematics Studies, vol. 76, Princeton University Press/University of Tokyo Press, Princeton, N.J./Tokyo, 1974.
[34] G.P. Nagy, Some remarks on simple Bol loops, Comment. Math. Univ. Carol. 49 (2) (2008) 259–270.
[35] P.T. Nagy, K. Strambach, Loops in Group Theory and Lie Theory, De Gruyter Expositions in Mathematics, vol. 35, Walter de Gruyter & Co., Berlin, 2002.
[36] A.L. Onishchik (Ed.), Lie Groups and Lie Algebras. I, Encyclopaedia of Mathematical Sciences., vol. 20, Springer-Verlag, Berlin, 1993.
[37] H.O. Pflugfelder, Historical notes on loop theory, in: Loops'99 (Prague), Vol. 41, 2000, pp. 359–370, https://dml.cz/handle/10338.dmlcz/119169.
[38] D.A. Robinson, Bol loops, Trans. Am. Math. Soc. 123 (1966) 341–354, https://doi.org/10.2307/1994661.
[39] D.A. Robinson, A Bol loop isomorphic to all loop isotopes, Proc. Am. Math. Soc. 19 (1968) 671–672, https://doi.org/10.2307/2035860.
[40] H.N. Sá Earp, Generalised Chern-Simons theory and G_2-instantons over associative fibrations, SIGMA 10 (2014) 083, https://doi.org/10.3842/SIGMA.2014.083, 11.
[41] L.V. Sabinin, Smooth Quasigroups and Loops, Mathematics and Its Applications., vol. 492, Kluwer Academic Publishers, Dordrecht, 1999.
[42] L.V. Sabinin, P.O. Mikheev, The differential geometry of Bol loops, Dokl. Akad. Nauk SSSR 281 (5) (1985) 1055–1057.
[43] A.A. Sagle, Malcev algebras, Trans. Am. Math. Soc. 101 (1961) 426–458, https://doi.org/10.2307/1993472.

[44] S. Salamon, Riemannian Geometry and Holonomy Groups, Pitman Research Notes in Mathematics Series, vol. 201, Longman Scientific & Technical, Harlow, 1989.

[45] R.W. Sharpe, Differential Geometry, Graduate Texts in Mathematics, vol. 166, Springer-Verlag, New York, 1997.

[46] I.P. Shestakov, Every Akivis algebra is linear, Geom. Dedic. 77 (2) (1999) 215–223, https://doi.org/10.1023/A:1005157524168.

[47] I.P. Shestakov, U.U. Umirbaev, Free Akivis algebras, primitive elements, and hyperalgebras, J. Algebra 250 (2) (2002) 533–548, https://doi.org/10.1006/jabr.2001.9123.

[48] J.D.H. Smith, An Introduction to Quasigroups and Their Representations, Studies in Advanced Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2007.

[49] K.K. Uhlenbeck, Connections with L^p bounds on curvature, Commun. Math. Phys. 83 (1) (1982) 31–42, http://projecteuclid.org/euclid.cmp/1103920743.

[50] M. Verbitsky, HyperKähler manifolds with torsion, supersymmetry and Hodge theory, Asian J. Math. 6 (4) (2002) 679–712, https://doi.org/10.4310/AJM.2002.v6.n4.a5.

[51] F.W. Warner, Foundations of Differentiable Manifolds and Lie Groups, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York-Berlin, 1983.

[52] S.-T. Yau, On the Ricci curvature of a compact Kaehler manifold and the complex Monge-Ampère equation. I, Commun. Pure Appl. Math. 31 (1978) 339–411, https://doi.org/10.1002/cpa.3160310304.