Flows of co-closed G_2-structures

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Abstract

We survey recent progress in the study of G_2-structure Laplacian coflows, that is, heat flows of co-closed G_2-structures. We introduce the properties of the original Laplacian coflow of G_2-structures as well as the modified coflow, reviewing short-time existence and uniqueness results for the modified coflow and well as recent Shi-type estimates that apply to a more general class of G_2-structure flows.

1 Introduction

One of the most successful techniques in geometric analysis has been the application of geometric flows to various problems in geometry and topology, most notably the use of the Ricci flow [20, 30] to solve the Poincaré Conjecture [31]. The Ricci flow is a non-linear weakly parabolic partial differential equation for the Riemannian metric g

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}_g \tag{1.1}$$

so that the evolution of the metric is given by the Ricci curvature defined by the metric. This can further be interpreted as a heat equation for the metric. In G_2-geometry, there have been a number of proposals for geometric flows of G_2-structures. The general idea is that given an initial G_2-structure with weaker assumptions than vanishing torsion, the flow should eventually seek out a torsion-free G_2-structure, if one exists on the given manifold. A G_2-structure is defined by a positive 3-form φ, which in turn defines the metric g, and the corresponding Hodge dual 4-form $*\varphi =: \psi$. Therefore, a natural equation to consider is the analog of the heat equation for the 3-form φ

$$\frac{\partial \varphi}{\partial t} = \Delta_\varphi \varphi. \tag{1.2}$$

This Laplacian flow of the 3-form φ is now nonlinear in φ, because the metric and hence the Laplacian depend on φ itself. A particular case of this flow has been first studied by Robert Bryant [5], where he restricted it to closed G_2-structures, that is ones where $d\varphi = 0$. For a closed G_2-structure, $\Delta\varphi = dd^*\varphi$, so in this case, the 3-form φ stays closed under the flow

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(1.2), and in fact remains within the same cohomology class since $\Delta\varphi$ is exact. Short-time existence and uniqueness of solutions to (1.2) was proved in [6]. Moreover, on a compact manifold M, this flow can be interpreted as the gradient flow of the Hitchin functional V given by

$$V(\varphi) = \frac{1}{7}\int_M \varphi \wedge *_\varphi \varphi. \tag{1.3}$$

The functional V is then the volume of the manifold M. It was shown by Nigel Hitchin in [21] that if φ is closed, then the critical points of the functional V within the cohomology class $[\varphi]$ correspond precisely to torsion-free G_2-structures, and in particular, these critical points are maxima in the directions transverse to diffeomorphisms. Under the flow (1.2), V increases monotonically, so if the growth of V is bounded, then $\varphi(t)$ would be expected to approach a torsion-free G_2-structure as $t \longrightarrow \infty$. The stability and analyticity of this flow has recently been proved by Lotay and Wei [26, 27, 28]

Alternatively, a G_2-structure and the corresponding metric may also be defined by the 4-form ψ (up to a choice of orientation). Therefore, instead of deforming φ, we may deform ψ. Using this idea, Karigiannis, McKay, and Tsui, introduced the *Laplacian coflow* for the 4-form ψ in [25]. Instead of considering the heat flow equation for φ, they instead considered the flow:

$$\frac{\partial \psi}{\partial t} = \Delta_\psi \psi. \tag{1.4}$$

If restricted to *co-closed* G_2-structures (that is, ones with $d\psi = 0$ and equivalently, those with a symmetric torsion tensor T) this flow preserves the co-closed condition and in fact preserves the cohomology class of ψ. In [14], it was shown that this flow has similar characteristics to the original Laplacian flow for closed G_2-structures. In fact, (1.4) can also be regarded as a gradient flow of the Hitchin functional (but now reformulated via 4-forms). However, a major difference compared with the Laplacian flow of closed G_2-structures (1.2) is that (1.4) is not even a weakly parabolic equation. In fact, the symbol of the linearized equation is indefinite. In order to have any hope of proving the existence of solutions, a *modified Laplacian coflow* of co-closed G_2-structures was introduced in [14]:

$$\frac{d\psi}{dt} = \Delta_\psi \psi + 2d((A - \operatorname{Tr} T)\varphi) \tag{1.5}$$

where $\operatorname{Tr} T$ is the trace of the full torsion tensor T of the G_2-structure defined by ψ, and A is a positive constant. This flow is now weakly parabolic in the direction of closed forms and hence it is possible to relate it to a strictly parabolic flow using an application of DeTurck's trick. Recently, the methods of Lotay and Wei for Shi-type estimates for the flow (1.2) have been extended by Gao Chen [7] to cover a more general class of G_2-structure flows that includes (1.5) as well. We will first survey the properties of G_2-structures and the Laplacian $\Delta_\varphi \varphi$ in sections 2 and 3. Then, in section 4 we will focus on Laplacian coflows.

Despite the apparent similarity between closed and co-closed G_2-structures, there are also important differences. As shown in [10], co-closed G_2-structures always satisfy the h-principle (on both open and closed manifolds) and hence always exist whenever a manifold admits G_2-structures. This is in contrast to closed G_2-structures for which the h-principle only holds on open manifolds. Therefore, co-closed G_2-structures are in some sense more generic than closed ones. This is both good and bad - it's good because they always exist, but bad because one cannot expect their flows to always behave nicely. This is also in part shown by the non-parabolicity of the original coflow (1.4).

In this survey we will focus on analytic properties of flows on general 7-manifolds, however another approach to understand the specific behavior of geometric flows and obtain explicit solutions has been to consider manifolds with some symmetry, in which case the number of degrees of freedom in the PDE will be reduced. Both the original Laplacian coflow (1.4) and the modified Laplacian coflow (1.5) have been studied on a variety of such manifolds with symmetry. Note that while in these situations mostly the original coflow (1.4) with the negative sign has been studied, results for the coflow with the positive sign (1.4) would be similar because equations reduce to ODEs. In [25] and [16], the coflow and the modified coflow, respectively, have been studied on warped product manifolds of the form $N^6 \times L$ where N^6 is a 6-dimensional manifold with $SU(3)$-structure such as a Calabi-Yau or nearly Kähler manifold and L is either $\mathbb{R}$ or S^1. In particular, soliton solutions in both cases have been obtained. In [1], Bagaglini, Fernandez, and Fino, also studied both the coflows on the 7-dimensional Heisenberg group. In particular, they have shown that the long-term existence properties of the flow (1.5) depend on the constant A. Similarly, in [2], Bagaglini and Fino studied the Laplacian coflow on 7-dimensional almost-abelian Lie groups and showed long-term existence properties and constructed soliton solutions. In [29], Manero, Otal, and Villacampa studied both the Laplacian flow (4.1) and the coflow (1.4) on solvmanifolds, but instead of restricting to closed or co-closed G_2-structures, they instead restricted to *locally conformally parallel* G_2-structures, which are the ones where only the 7-dimensional τ_1 component of the torsion may be nonvanishing.

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2 Laplacian of a G_2-structure

Suppose M is a smooth 7-dimensional manifold with a G_2-structure φ. Then we know φ uniquely defines a compatible Riemannian metric g_φ, the volume form vol_φ, Hodge star $*_\varphi$, and the dual 4-form $\psi = *_\varphi \varphi$. There is arbitrary choice of orientation, which affects the relative sign of ψ. We use the same convention as [4] and [13, 14, 15, 16, 18], which is opposite from the convention used in [23, 24]. For further properties of φ and ψ, as well as different identities that they satisfy, we refer the reader to the above references. We will also use the following notation. The symbol $\lrcorner$ will denote contraction of a vector with the differential form:

$$(u \lrcorner \varphi)_{mn} = u^a \varphi_{amn}. \tag{2.1}$$

Note that we will also use this symbol for contractions of differential forms using the metric, for example $(T \lrcorner \varphi)_a = T^{mn} \varphi_{mna}$. Given a symmetric 2-tensor h on M, we define the map $\mathrm{i}_\varphi : \Gamma(\mathrm{Sym}(T^*M)) \longrightarrow \Lambda^3_1 \oplus \Lambda^3_{27}$ as

$$\mathrm{i}_\varphi(h)_{abc} = h^d_{[a} \varphi_{bc]d}.$$

We will define the operators π_1, π_7, π_{14} and π_{27} to be the projections of differential forms onto the corresponding representations. Sometimes we will also use $\pi_{1\oplus 27}$ to denote the projection of 3-forms or 4-forms into $\Lambda^3_1 \oplus \Lambda^3_{27}$ or $\Lambda^4_1 \oplus \Lambda^4_{27}$ respectively. For convenience, when writing out projections of forms, we will sometimes just give the vector that defines the 7-dimensional component, the function that defines the 1-dimensional component or the symmetric 2-tensor that defines the $1 \oplus 27$ component whenever there is no ambiguity. For instance,

$$\begin{array}{lll} \pi_1(f\varphi) = f & \pi_1(f\psi) = f & \\ \pi_7(X \lrcorner \varphi)^a = X^a & \pi_7(X \lrcorner \psi)^a = X^a & \pi_7(X \wedge \varphi)^a = X^a \\ \pi_{1\oplus 27}(\mathrm{i}_\varphi(h))_{ab} = h_{ab} & \pi_{1\oplus 27}(*\mathrm{i}_\varphi(h))_{ab} = h_{ab} & \end{array} \tag{2.2}$$

The above-mentioned references give more information regarding the properties of decomposition of differential forms with respect to G_2 representations.

The *intrinsic torsion* of a G_2-structure is defined by $\nabla\varphi$, where ∇ is the Levi-Civita connection for the metric g that is defined by φ. Following [24], we have

$$\nabla_a \varphi_{bcd} = T_a{}^e \psi_{ebcd} \tag{2.3a}$$

$$\nabla_a \psi_{bcde} = -4T_{a[b}\varphi_{cde]} \tag{2.3b}$$

where T_{ab} is the *full torsion tensor*. In general we can split T_{ab} according to representations of G_2 into *torsion components*:

$$T = \frac{1}{4}\tau_0 g - \tau_1 \lrcorner \varphi + \frac{1}{2}\tau_2 - \frac{1}{3}\tau_3 \tag{2.4}$$

where τ_0 is a function, and gives the **1** component of T. We also have τ_1, which is a 1-form and hence gives the **7** component, and, $\tau_2 \in \Lambda^2_{14}$ gives the **14** component and τ_3 is traceless symmetric, giving the **27** component. As shown by Karigiannis in [24], the torsion components τ_i relate directly to the expression for $d\varphi$ and $d\psi$. In fact, in our notation,

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *\mathrm{i}_\varphi(\tau_3) \tag{2.5a}$$

$$d\psi = 4\tau_1 \wedge \psi + *\tau_2. \tag{2.5b}$$

Note that in [14, 15, 16, 18] a different convention is used: τ_1 in that convention corresponds to $\frac{1}{4}\tau_0$ here, τ_7 corresponds to $-\tau_1$ here, $\mathrm{i}_\varphi(\tau_{27})$ corresponds to $-\frac{1}{3}\tau_3$, and τ_{14} corresponds to $\frac{1}{2}\tau_2$. The notation used here is widely used elsewhere in the literature.

An important special case is when the G_2-structure is said to be torsion-free, that is, $T = 0$. This is equivalent to $\nabla\varphi = 0$ and also equivalent, by Fernández and Gray [12], to $d\varphi = d\psi = 0$. Moreover, a G_2-structure is torsion-free if and only if the holonomy of the corresponding metric is contained in G_2 [22]. On a compact manifold, the holonomy group is then precisely equal to G_2 if and only if the fundamental group π_1 is finite. If $d\varphi = 0$, then we say φ defines a *closed* G_2-structure. In that case, $\tau_0 = \tau_1 = \tau_3 = 0$ and only τ_2 is in general non-zero. In this case, $T = -\frac{1}{2}\tau_2$ and is hence skew-symmetric. If instead, $d\psi = 0$, then we say that we have a *co-closed* G_2-structure. In this case, τ_1 and τ_2 vanish in (2.5b) and we are left with τ_0 and τ_3 components. In particular, the torsion tensor T_{ab} is now symmetric.

We will be using the following notation, as in [14]. Given a tensor ω, the rough Laplacian is defined by

$$\Delta\omega = g^{ab}\nabla_a\nabla_b\omega = -\nabla^*\nabla\omega. \tag{2.6}$$

whereas the Hodge Laplacian defined by φ or ψ will be denoted by Δ_φ or Δ_ψ, respectively. For a vector field X, define the *divergence* of X as

$$\operatorname{div} X = \nabla_a X^a. \tag{2.7}$$

This operator can be extended to a 2-tensor β:

$$(\operatorname{div}\beta)_b = \nabla^a \beta_{ab}. \tag{2.8}$$

Also, for a vector X, we can use the G_2-structure 3-form φ to define a "curl" operator, similar to the standard one on $\mathbb{R}^3$:

$$(\operatorname{curl} X)^a = (\nabla_b X_c)\varphi^{abc}. \tag{2.9}$$

This curl operator can then also be extended to 2-tensor β:

$$(\operatorname{curl}\beta)_{ab} = (\nabla_m \beta_{na})\varphi_b{}^{mn}. \tag{2.10}$$

Note that when β_{ab} is symmetric, $\operatorname{curl}\beta$ is traceless. It is also not difficult to see that schematically,

$$\operatorname{curl}\big((\operatorname{curl}\beta)^t\big) = -\Delta\beta^t + \nabla(\operatorname{div}\beta) + \operatorname{Riem}\circledast\beta + T\circledast\nabla\beta + (\nabla T)\circledast\beta + T\circledast T\circledast\beta \tag{2.11}$$

where ^{t} denotes transpose and $\circledast$ is some multilinear operator involving g, φ, ψ. From the context it will be clear whether the curl operator is applied to a vector or a 2-tensor.

As in [14], we can also use the G_2-structure 3-form to define a commutative product $\alpha\circ\beta$ of two 2-tensors α and β

$$(\alpha\circ\beta)_{ab} = \varphi_{amn}\varphi_{bpq}\alpha^{mp}\beta^{nq} \tag{2.12}$$

Note that $(\alpha\circ\beta)^t = \big(\alpha^t\circ\beta^t\big)$. If α and β are both symmetric or both skew-symmetric, then $\alpha\circ\beta$ is a symmetric 2-tensor. Also, for a 2-tensor we have the standard matrix product $(\alpha\beta)_{ab} = \alpha_a{}^k\beta_{kb}$.

From [8, 15, 24] we know that the torsion of a G_2-structure satisfies the following integrability condition:

$$\frac{1}{2}\operatorname{Riem}_{ij}{}^{\beta\gamma}\varphi^\alpha{}_{\beta\gamma} = \nabla_i T_j{}^\alpha - \nabla_j T_i{}^\alpha + T_i{}^\beta T_j{}^\gamma \varphi^\alpha{}_{\beta\gamma}. \tag{2.13}$$

Taking projections of (2.13) to different representations of G_2, we obtain the following expressions:

Lemma 2.1 *The torsion tensor T satisfies the following identities*

$$(\nabla T)\lrcorner\psi = -(T\lrcorner\varphi)\lrcorner T + T^2\lrcorner\varphi + (\operatorname{Tr}T)(T\lrcorner\varphi) \tag{2.14a}$$

$$0 = d(\operatorname{Tr}T) - \operatorname{div}\big(T^t\big) - (T\lrcorner\varphi)\lrcorner T^t \tag{2.14b}$$

$$\operatorname{Ric} = -\operatorname{Sym}\big(\operatorname{curl}T^t - \nabla(T\lrcorner\varphi) + T^2 - \operatorname{Tr}(T)T\big) \tag{2.14c}$$

$$\frac{1}{4}\operatorname{Ric}^* = \operatorname{curl}T + \frac{1}{2}T\circ T \tag{2.14d}$$

$$\mathrm{R} = 2\operatorname{Tr}(\operatorname{curl}T) - \psi(T,T) - \operatorname{Tr}\big(T^2\big) + (\operatorname{Tr}T)^2 \tag{2.14e}$$

where $(\operatorname{Ric}^*)_{ab} = \operatorname{Riem}_{mnpq}\varphi^{mn}{}_a\varphi^{pq}{}_b$ *and* $\psi(T,T) = \psi_{abcd}T^{ab}T^{cd}$. *Note that from (2.4),* $\operatorname{Tr}T = \frac{7}{4}\tau_0$ *and* $T\lrcorner\varphi = -6\tau_1$.

The symmetric 2-tensor Ric^* has been defined and studied by Cleyton and Ivanov in [8, 9]. Note that $\mathrm{Tr}(\mathrm{Ric}^*) = 2\mathrm{R}$, where R is the scalar curvature. Thus the tensors Ric and Ric^* span the components of Riem that lie in $1 \oplus 27 \oplus 27$ representations of G_2. It is know that Riem has no components in the 7 or 14 dimensional representations of G_2. The identities (2.14a), (2.14b), as well as the projection of (2.14d) to Λ^2_{14} are a consequence of this. In fact, taking the skew-symmetric part of (2.14d) and using the fact that Ric^* is by definition symmetric, gives us

$$\mathrm{Skew}(\mathrm{curl}\, T) = -\frac{1}{2}\,\mathrm{Skew}(T \circ T). \tag{2.15}$$

In particular, this shows that $\mathrm{curl}\, T$ is symmetric whenever T is skew-symmetric or symmetric, and in particular, if φ is closed or co-closed.

Let us now look at the properties of $\Delta_\varphi \varphi = dd^*\varphi + d^*d\varphi$.

Proposition 2.2 ([14]) *Suppose φ defines a G_2-structure. Then $\Delta_\varphi \varphi = X \lrcorner \psi + 3\mathrm{i}_\varphi(h)$ with*

$$X = -\,\mathrm{div}\, T \tag{2.16a}$$

$$\begin{aligned} h = {} & -\frac{1}{4}\,\mathrm{Ric}^* + \frac{1}{6}\Big(\mathrm{R} + 2|T|^2\Big)g - T^tT - \frac{1}{2}(T\lrcorner\varphi)(T\lrcorner\varphi) \\ & + \frac{1}{4}T \circ T + \frac{1}{4}T^t \circ T^t - \frac{1}{2}T \circ T^t + \mathrm{Sym}\big((T)(T\lrcorner\psi) - \big(T^t\big)(T\lrcorner\psi)\big). \end{aligned} \tag{2.16b}$$

In particular,

$$\mathrm{Tr}\, h = \frac{2}{3}\mathrm{R} + \frac{4}{3}|T|^2. \tag{2.17}$$

The leading order terms in $\Delta_\varphi \varphi$ are those that contain second derivatives of φ, and hence first derivatives of T. Thus, $\mathrm{div}\, T$ fully defines the Λ^3_7 component of $\Delta_\varphi \varphi$ and the leading order terms in $\Lambda^3_{1\oplus 27}$ are given by

$$-\frac{1}{4}\,\mathrm{Ric}^* + \frac{1}{6}\,\mathrm{R}\, g \sim -\,\mathrm{curl}\, T + \frac{1}{3}\,\mathrm{Tr}(\mathrm{curl}\, T) g. \tag{2.18}$$

3 Flows of G_2-structures

Suppose $\varphi(t)$ is a one-parameter family of G_2-structures on a manifold M that satisfies

$$\frac{\partial \varphi(t)}{\partial t} = X(t)\lrcorner\psi(t) + 3\mathrm{i}_{\varphi(t)}(h(t)). \tag{3.1}$$

As shown by Karigiannis in [24], the associated quantities $g(t), \mathrm{vol}_t, \psi(t),\ T(t)$ satisfy the following evolution equations:

Lemma 3.1 ([24]) *If $\varphi(t)$ satisfies the equation (3.1), then we also have the following equa-*

tions:

$$\frac{\partial g}{\partial t} = 2h \tag{3.2a}$$

$$\frac{\partial \operatorname{vol}}{\partial t} = \operatorname{Tr}(h)\operatorname{vol} \tag{3.2b}$$

$$\frac{\partial \psi}{\partial t} = 4\mathrm{i}_\psi(h) - X\wedge\varphi \tag{3.2c}$$

$$\frac{\partial T}{\partial t} = \nabla X - \operatorname{curl} h + Th - (T)(X\lrcorner\varphi) \tag{3.2d}$$

where $\mathrm{i}_\psi(h)_{abcd} = -h^e_{[a}\psi_{bcd]e}$ *and equivalently,* $4\mathrm{i}_\psi(h) = -3*\mathrm{i}_\varphi(h) + (\operatorname{Tr} h)\psi$.

Similarly, as in [14], we can consider flows of ψ, given by

$$\frac{\partial\psi(t)}{\partial t} = *(X(t)\lrcorner\psi(t)) + 3*\mathrm{i}_{\varphi(t)}(s(t)) \tag{3.3}$$

for some symmetric 2-tensor s. Since $3*\mathrm{i}_\varphi(s) = 4\mathrm{i}_\psi\left(\frac{1}{4}(\operatorname{Tr} s)g - s\right)$, comparing (3.3) with (3.2c) give us corresponding evolution equations for $\varphi(t)$, $g(t), \operatorname{vol}_t, T(t)$ from (3.1) and (3.2) by taking $h = \frac{1}{4}(\operatorname{Tr} s)g - s$.

When constructing geometric flows, there are two main considerations: 1) the flow's stationary points should correspond to geometrically interesting objects; and 2) the flow should be parabolic in some sense. The first property is the main motivation for studying a flow, since we ideally want the flow to deform a geometric structure to one that has nicer or more constrained properties and the second property is a minimal requirement to at least guarantee short-time existence and uniqueness of solutions. In [7], Chen defined a class of *reasonable* flows (3.1) of G_2-structures that satisfy the following 4 general conditions:

1. The metric should evolve by the Ricci flow to leading order, and be no more than quadratic in the torsion, that is

$$\frac{\partial g}{\partial t} = 2h = -2\operatorname{Ric} + Cg + L(T) + T\circledast T \tag{3.4}$$

   where C is a constant and L is some linear operator involving g, φ, ψ.

2. The vector field X is at most linear in ∇T and at most quadratic in T:

$$X = L(\nabla T) + L(T) + L(\operatorname{Riem}) + T\circledast T + C. \tag{3.5}$$

3. The torsion tensor should evolve by ΔT to leading order, and be at most linear in Riem and ∇T, and at most cubic in T:

$$\begin{aligned}\frac{\partial T}{\partial t} = {} & \Delta T + L(\nabla T) + L(\operatorname{Riem}) + \operatorname{Riem}\circledast T + \nabla T\circledast T \\ & + L(T) + T\circledast T + T\circledast T\circledast T.\end{aligned} \tag{3.6}$$

4. The flow (3.1) has short-time existence and uniqueness.

As one of the key properties of *reasonable* flows defined above is that the flow of the metric is the Ricci flow to leading order, we will instead refer to flows that satisfy properties 1.-4. as *Ricci-like flows.* This is appropriate because a variety of techniques that originated from the study of the Ricci flow have been applied to these flows. In particular, under the Ricci flow, invariants of the metric Riem, Ric, R, all satisfy heat-like equations. Therefore it is appropriate that for a Ricci-like flow of a G_2-structure, the torsion, which an invariant of the G_2-structure also satisfies a heat-like equation (3.6). This is important because then $\nabla^k T$ and $|T|^2$ also satisfy heat-like equations and this is necessary to be able to obtain estimates using the maximum principle.

Using techniques developed by Shi in [32] for the Ricci flow and their adaptation to G_2-structures by Lotay and Wei [26], Chen then showed that a reasonable flow satisfies the following Shi-type estimate.

Theorem 3.2 ([7, Theorem 2.1]) *Suppose (3.1) is a Ricci-like flow of G_2-structures, such that the coefficients in equations (3.1), (3.4), (3.5), and (3.6) are bounded by a constant Λ. Let $B_r(p)$ be a ball of radius r with respect to the initial metric $g(0)$. If*

$$|\mathrm{Riem}(x,t)|_{g(t)} + |T(x,t)|^2_{g(t)} + |\nabla T(x,t)|_{g(t)} < \Lambda \tag{3.7}$$

for any $(x,t) \in B_r(p) \times [0,t_0]$, then

$$\left|\nabla^k \mathrm{Riem}(x,t)\right|_{g(t)} + \left|\nabla^{k+1} T(x,t)\right|_{g(t)} < C(k,r,\Lambda,t) \tag{3.8}$$

for any $(x,t) \in B_{r/2}(p) \times \left[\frac{t_0}{2}, t_0\right]$ for all $k = 1,2,3,...$

It should be noted that in [26], the condition analogous to (3.7) does not include a $|T|^2$ term. This is because in the case of a closed G_2-structure, $|T|^2 = -\mathrm{R} \leq C|\mathrm{Riem}|$. Therefore, the norm of the torsion can always be bounded in terms of the norm of Riem . For other torsion classes, and in particular, co-closed G_2-structures, this is no longer true, therefore $|T|^2$ needs to be included in (3.7).

Using the estimates from Theorem 3.2, Chen then derived an estimate for the blow-up rate on a compact manifold.

Theorem 3.3 ([7, Theorem 5.1]) *If $\varphi(t)$ is a solution to a Ricci-like flow of G_2-structures on a compact manifold in a finite maximal time interval $[0,t_0)$, then*

$$\sup_M \Big(|\mathrm{Riem}(x,t)|^2_{g(t)} + |T(x,t)|^4_{g(t)} + |\nabla T(x,t)|^2_{g(t)}\Big)^{\frac{1}{2}} \geq \frac{C}{t_0 - t} \tag{3.9}$$

for some positive constant C.

The estimate (3.9) shows that a solution will exist as long the quantity of the left-hand side of (3.9) remains bounded.

A classic example of a Ricci-like flow of G_2-structures is the Laplacian flow of G_2-structures that was introduced by Bryant in [5]:

$$\frac{\partial \varphi}{\partial t} = \Delta_\varphi \varphi. \tag{3.10}$$

If the initial G_2-structure is closed, then this property is preserved along the flow. It is then natural to think of (3.10) as a flow of closed G_2-structures. In this case, since $T^t = -T$, from (2.14), $\mathrm{Ric}^* = 4\,\mathrm{Ric} + T \circledast T$ and $\mathrm{R} = 2\,\mathrm{Tr}(\mathrm{curl}\, T) - \psi(T,T) - \mathrm{Tr}\big(T^2\big) = -|T|^2$; and thus, from (2.16b), $h = -\mathrm{Ric} + T \circledast T$, and so from (3.2a), we do find that (3.4) holds. Moreover, from (2.14b), we see that $\mathrm{div}\, T = 0$ in this case, and hence $X = 0$. The expression (3.6) comes from (3.2d) and using $h = -\,\mathrm{curl}\, T + T \circledast T$

$$\frac{\partial T}{\partial t} = \mathrm{curl}(\mathrm{curl}\, T) + \nabla T \circledast T + T \circledast T \circledast T. \tag{3.11}$$

Using (2.11) to expand $\mathrm{curl}(\mathrm{curl}\, T)$ together the facts that $\mathrm{curl}\, T$ is symmetric, T is skew-symmetric, and $\mathrm{div}\, T = 0$, allows to express the right-hand side of (3.11) as $\Delta T + \mathrm{Riem} \circledast T + \nabla T \circledast T + T \circledast T \circledast T$. Finally, short-term existence and uniqueness of the flow (3.10) has been first proved by Bryant and Xu in [6]. For more on the properties of this flow, as well as the details of the above calculations, the reader is referred to the series of papers by Lotay and Wei [26, 27, 28]. The results in Theorems 3.2 and 3.3 are extensions of similar results for the Laplacian flow of closed G_2-structures in [26].

4 Laplacian coflow

In [25], Karigiannis, McKay, and Tsui introduced an alternative flow of G_2-structures, called the Laplacian *coflow*:

$$\frac{\partial \psi}{\partial t} = -\Delta_\psi \psi. \tag{4.1}$$

If the initial G_2-structure is co-closed, i.e. $d\psi = 0$, then this property is preserved along the flow. Therefore, the coflow may be regarded as a natural flow of co-closed G_2-structures. In order to understand flows of co-closed G_2-structures, we need to understand better the properties of T and the Hodge Laplacian in this case. Rewriting Lemma 2.1 and Proposition 2.2 in the case of symmetric T, we find the following.

Proposition 4.1 *Suppose φ is a co-closed G_2-structure, then the torsion tensor T satisfies the following identities*

$$\begin{aligned}
\mathrm{div}\, T &= d(\mathrm{Tr}\, T) && \text{(4.2a)}\\
\mathrm{curl}\, T &= (\mathrm{curl}\, T)^t && \text{(4.2b)}\\
\mathrm{Ric} &= \mathrm{curl}\, T - T^2 + \mathrm{Tr}(T)T && \text{(4.2c)}\\
\frac{1}{4}\mathrm{Ric}^* &= \mathrm{curl}\, T + \frac{1}{2} T \circ T = \mathrm{Ric} + \frac{1}{2} T \circ T + T^2 - \mathrm{Tr}(T)T && \text{(4.2d)}\\
\mathrm{R} &= (\mathrm{Tr}\, T)^2 - |T|^2. && \text{(4.2e)}
\end{aligned}$$

The Hodge Laplacian is given by $\Delta_\varphi \varphi = X \lrcorner \psi + 3\mathrm{i}_\varphi(s)$ *with*

$$\begin{aligned} X &= -\operatorname{div} T & (4.3a)\\ s &= -\operatorname{Ric} + \frac{1}{6}\Big(\mathrm{R} + 2|T|^2\Big) g + \operatorname{Tr}(T)T - 2T^2 - \frac{1}{2} T \circ T & (4.3b)\\ &= -\operatorname{curl} T + \frac{1}{6}\Big((\operatorname{Tr} T)^2 + |T|^2\Big) g - T^2 - \frac{1}{2} T \circ T & (4.3c)\\ \operatorname{Tr} s &= \frac{2}{3}\mathrm{R} + \frac{4}{3}|T|^2 = \frac{2}{3}\Big((\operatorname{Tr} T)^2 + |T|^2\Big). & (4.3d) \end{aligned}$$

Comparing (4.1) with (3.3) and using (4.3), we see that to leading order the evolution of the metric is given by 2 Ric, that is the opposite of the Ricci flow. Thus, in order for the flow to be Ricci-like and to have any hope of existence and uniqueness, the sign in (4.1) needs to be reversed. Therefore, let us redefine the Laplacian coflow as

$$\frac{d\psi}{dt} = \Delta_\psi \psi. \tag{4.4}$$

We then find that

$$\frac{\partial g}{\partial t} = -2\operatorname{Ric} + T \circ T + 2(\operatorname{Tr} T)T \tag{4.5}$$

which now satisfies (3.4). Also, $X = -\operatorname{div} T$, which satisfies (3.5). To obtain the general form of the evolution of the torsion, note that to leading order, $h = -s = \operatorname{curl} T$, so from (3.2d),

$$\frac{\partial T}{\partial t} = -\nabla(\operatorname{div} T) - \operatorname{curl}(\operatorname{curl} T) + \nabla T \circledast T$$

however, since both T and $\operatorname{curl} T$ are symmetric,

$$\operatorname{curl}(\operatorname{curl} T) = -\Delta T + \nabla(\operatorname{div} T) + \operatorname{Riem} \circledast T + (\nabla T) \circledast T + T \circledast T \circledast T$$

Hence, overall,

$$\frac{\partial T}{\partial t} = \Delta T - 2\nabla(\operatorname{div} T) + \operatorname{Riem} \circledast T + (\nabla T) \circledast T + T \circledast T \circledast T. \tag{4.6}$$

Notice that this does not satisfy (3.6). In fact, we can see that the presence of the $\nabla(\operatorname{div} T)$ term in (4.6) is due to the negative sign of $\operatorname{div} T$ in (4.3a). As it was shown in [14], the sign of $\operatorname{div} T$ also causes problems at a much more fundamental level: it prevents the flow (4.4) from being parabolic even along closed 4-forms. Proposition 4.2 below gives the linearization of Δ_ψ. It is then easy to see that for closed 4-forms, the symbol will be negative in the Λ^4_7 direction, but non-negative in Λ^4_{27}.

Proposition 4.2 ([14, Prop. 4.7]) *The linearization of* Δ_ψ *at* ψ *is given by*

$$\begin{aligned} \pi_7(D_\psi \Delta_\psi)(\chi) &= d(\operatorname{div} X) \wedge \varphi + l.o.t. & (4.7a)\\ \pi_{1\oplus 27}(D_\psi \Delta_\psi)(\chi) &= \frac{3}{2} * \mathrm{i}_\varphi\Big(\Delta h + \frac{1}{4}\operatorname{Hess}(\operatorname{Tr} h) - \frac{1}{2}(\Delta \operatorname{Tr} h) g & (4.7b)\\ &\quad - \operatorname{Sym}\big(\nabla \operatorname{div} h + \operatorname{curl}(\nabla X)^t\big) + l.o.t.\Big) \end{aligned}$$

where $\chi = *(X \lrcorner \psi + 3\mathrm{i}_\varphi(h))$*. Moreover, if* χ *is closed, we can write* $D_\psi\Delta_\psi$ *as*

$$D_\psi\Delta_\psi(\chi) = -\Delta_\psi\chi - \mathcal{L}_{V(\chi)}\psi + 2d((\operatorname{div} X)\varphi) + dF(\chi) \tag{4.8}$$

where

$$V(\chi) = \frac{3}{4}\nabla \operatorname{Tr} h - 2\operatorname{curl} X \tag{4.9}$$

and $F(\chi)$ *is a* 3*-form-valued algebraic function of* χ*.*

Looking closer at the leading terms in the linearization (4.8) evaluated at closed forms, we see that the term $2d((\operatorname{div} X)\varphi)$ appears for exactly the same reason as the term $-2\nabla(\operatorname{div} T)$ in (4.6) - namely the "wrong" sign of the π_7 component of $\Delta_\psi\psi$. To fix this problem, in [14], a *modified Laplacian coflow* has been proposed:

$$\frac{\partial\psi}{\partial t} = \Delta_\psi\psi + 2d((A - \operatorname{Tr} T)\varphi) \tag{4.10}$$

where A is some constant. Since for co-closed G_2-structures, $\operatorname{Tr} T = \operatorname{div} T$, the leading term in the modification precisely reverses the sign of the Λ^4_7 component of the original flow (1.4). However, because we want the right hand side of the flow to be an exact 4-form for co-closed G_2-structures, there are some additional lower order terms. The constant A could be set to zero, however adding it may allow for more flexibility. The linearization of the modified coflow at a closed 4-form is now given by

$$\frac{\partial\chi}{\partial t} = -\Delta_\psi\chi - \mathcal{L}_{V(\chi)}\psi + d\hat{F}(\chi) \tag{4.11}$$

where $V(\chi)$ is as in (4.9) and $\hat{F}(\chi)$ involves no derivatives of χ. Hence, in the direction of closed forms, this flow is now weakly parabolic. Moreover, the undesired term is removed from the evolution equation for T and its evolution is now given by (3.6).

The additional term in (4.10) now also allows to prove short-time existence and uniqueness, hence completing the requirements for (4.10) to be a Ricci-like flow. The proof, as given in [14], follows a procedure similar to the approach taken by Bryant and Xu [6] for the proof of short-time existence and uniqueness for the Laplacian flow (3.10), which is in turn based on DeTurck's [11] and Hamilton's [19] approaches to the proof of short-time existence and uniqueness of the Ricci flow. Let $\psi(t) = \psi_0 + \chi(t)$ where $\chi(t)$ is an exact 4-form with $\chi(0) = 0$. Then, given this initial condition, the flow (4.10) can be rewritten as an initial value problem for $\chi(t)$. From the linearization (4.11) we see that by adding the term $\mathcal{L}_{V(\chi(t))}\psi(t)$ we obtain a strictly parabolic flow in the direction of closed forms, which is related to the original flow by diffeomorphism:

$$\frac{\partial\chi}{\partial t} = \Delta_\psi\psi + 2d((A - \operatorname{Tr} T_\psi) *_\psi \psi) + \mathcal{L}_{V(\chi)}\psi. \tag{4.12}$$

This is the essence of what is known as "DeTurck's trick" - turning a weakly parabolic flow into a strictly parabolic one. In the case of Ricci flow this is enough to obtain short-time existence and uniqueness, however in this case, the parabolicity is only along closed forms, hence we cannot apply the standard parabolic theory right away, and more steps are needed.

Let us also define the spaces of time-dependent and time-independent exact 4-forms $\mathcal{F}$ and $\mathcal{G}$, respectively. Moreover, since we know that $\psi(t)$ always defines a G_2-structure and is thus a positive 4-form, χ will always lie in an open subset $\mathcal{U} \subset \mathcal{F}$ defined by

$$\mathcal{U} = \{\chi \in \mathcal{F} : \psi_0 + \chi \text{ is a positive 4-form}\}. \tag{4.13}$$

Moreover, let us now define a map $F : \mathcal{U} \longrightarrow \mathcal{F} \times \mathcal{G}$ given by

$$\chi \longrightarrow \left(\frac{\partial \chi}{\partial t} - \Delta_\psi \psi - 2d((A - \operatorname{Tr} T_\psi) *_\psi \psi) - \mathcal{L}_{V(\chi)}\psi, \, \chi|_{t=0} \right). \tag{4.14}$$

Adapting the results in [6], it is easy to see $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H} := \mathcal{F} \times \mathcal{G}$ are *graded tame Fréchet spaces*. Moreover, it was then shown in [14] that F is smooth *tame* map of Fréchet spaces, such that its derivative $DF(\chi) : \mathcal{F} \longrightarrow \mathcal{H}$ is an isomorphism for all $\chi \in \mathcal{U}$ and the inverse $(DF)^{-1} : \mathcal{U} \times \mathcal{H} \longrightarrow \mathcal{F}$ is smooth tame. The significance of these facts are that in the category of Fréchet spaces there exists an inverse function theorem - the Nash-Moser Inverse Function [19], which tells us that the map F is locally invertible. From this it follows that the flow (4.12) has short-time existence and uniqueness.

To prove short-time existence and uniqueness for the flow (4.10) we need to relate (4.10) and (4.12) via diffeomorphisms. Suppose $\bar{\chi}(t)$ is the unique short-time solution to (4.12), and $\bar{\psi} = \psi_0 + \bar{\chi}$. Consider the following ODE for a family of diffeomorphisms ϕ_t:

$$\begin{cases} \frac{\partial \phi_t}{\partial t} = -V(\bar{\chi}(t)) \\ \phi_0 = \mathrm{id} \end{cases} \tag{4.15}$$

This has a unique solution ϕ_t. Now let $\psi(t) = (\phi_t)^* \bar{\psi}(t)$, then $\psi(0) = \psi_0$, and since diffeomorphisms commute with d, $\psi(t)$ is closed for all t. Moreover, as shown in [14, Theorem 6.9], $\psi(t)$ now satisfies (4.10). Uniqueness is obtained similarly using the uniqueness of solutions of (4.15). Hence, overall, we obtain a unique short-time solution for the modified Laplacian coflow (4.10) and can now conclude that it is a Ricci-like flow.

Theorem 4.3 *The Laplacian coflow (4.10) of co-closed G_2-structures is a Ricci-like flow.*

5 Further directions

There are several important unanswered questions regarding flows of co-closed G_2-structures. An intriguing question is whether it is possible to obtain at least short-time existence and uniqueness of the unmodified Laplacian coflow (1.4). To leading order the only difference with the modified coflow is the sign of the Λ^4_7 component which is given by $\operatorname{div} T$. So in particular, if $\operatorname{div} T$ vanishes, then the two flows agree. It is also known [23] that deformations in the Λ^4_7 directions keep the metric unchanged. Moreover, in [17], the torsion T has been shown to play a role of an octonionic connection on the bundle of G_2-structures that correspond to the same metric, which can be given the structure of an octonion bundle. In this interpretation, on a compact manifold, the condition $\operatorname{div} T = 0$ corresponds to critical points of the functional $\int |T|^2 \operatorname{vol}$, and is hence the analog of a Coulomb gauge. It is therefore tempting to think that to relate the flows (1.4) and (1.5), a gauge-fixing condition such as $\operatorname{div} T = 0$ needs to be introduced.

There are also multiple questions relating to the modified coflow itself. As it is a Ricci-like flow, Shi-type estimates apply to it, so it is likely that in addition to Chen's results in [7], more properties such as real analyticity and stability could be proved using techniques similar to the ones used by Lotay and Wei in [26, 27, 28]. Indeed, as this article was being finalized, the author was made aware that Bedulli and Vezzoni [3] have generalized the proof of stability from [28] to a wider class of geometric flows that also includes the modified Laplacian coflow with $A = 0$.

Apart from the Laplacian flow and the coflows, there could be more interesting flows of G_2-structures. For co-closed G_2-structures, it is an open question whether the flow $\frac{\partial \varphi}{\partial t} = d^* d\varphi$ satisfies the co-closed condition. More generally, the conditions for a flow to be Ricci-like is a good set of conditions that flows should satisfy. In particular, one could try to construct flows using the first 3 conditions, but then also making sure that short-time existence and uniqueness is satisfied.

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