A Feasibility Governor for Enlarging the Region of Attraction of Linear Model Predictive Controllers

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***Abstract*—This paper proposes a method for enlarging the region of attraction of Linear Model Predictive Controllers (MPC) when tracking piecewise-constant references in the presence of pointwise-in-time constraints. It consists of an add-on unit, the Feasibility Governor (FG), that manipulates the reference command so as to ensure that the optimal control problem that underlies the MPC feedback law remains feasible. Offline polyhedral projection algorithms based on multi-objective linear programming are employed to compute the set of feasible states and reference commands. Online, the action of the FG is computed by solving a convex quadratic program. The closed-loop system is shown to satisfy constraints, be asymptotically stable, exhibit zero-offset tracking, and display finite-time convergence of the reference.**

I. Introduction

Model Predictive Control [1], [2] (MPC) defines a feedback policy as the solution of a receding horizon optimal control problem (OCP). MPC is widely used in applications, it enables high-performance control while systematically enforcing state and control constraints and is supported by a robust theoretical literature. Stability guarantees are typically obtained by incorporating "terminal ingredients" into the OCP. For example, adding a terminal penalty and an invariant set based terminal constraint is sufficient to guarantee asymptotic stability and constraint satisfaction [3]; the closed-loop region of attraction (ROA) is then the set of all states from which it is possible to reach the terminal set within the prediction horizon.

Many applications of MPC require the capability to track piecewise constant references and to safely transition between them. However, if the change in the reference is large the system may not be able to reach the new terminal set within the prediction horizon, resulting in infeasibility and failure of the MPC controller. The obvious strategy for avoiding infeasibility is increasing the size of the ROA by either enlarging the terminal set or increasing the prediction horizon. Unfortunately, the maximum size of the terminal set is fixed by the constraints [4], and increasing the prediction horizon increases the computational footprint of the controller.

Another strategy is to treat aspects of the terminal set, e.g., size, location, or shape, as optimization variables and use these additional degrees of freedom to enlarge the feasible set. This approach has been applied to economic operation of nonlinear systems with terminal state constraints [5] and regulation of linear systems using terminal set constraints [6]. It has also been applied to reference tracking problems for linear systems [7], [8] using various parameterizations of the terminal sets. Computing a contractive sequence of terminal sets offline which are incorporated into the OCP to enlarge the ROA is proposed in [9]. This approach is done in a more general nonlinear MPC setting and the computed sets are not necessarily invariant. The major disadvantage of these approaches is that they require redesigning the OCP and increase the computational complexity of the controller.

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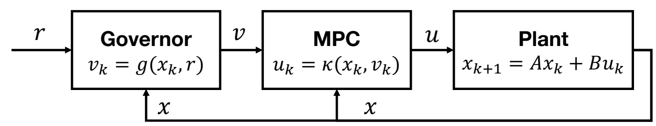

Fig. 1. A block diagram of the control architecture. Given a reference r, the feasibility governor manipulates the auxiliary reference v to ensure that the primary MPC controller is able to produce a valid control input u.

In this paper, we propose the Feasibility Governor (FG) (illustrated in Figure 1), an add-on unit in the tradition of reference/command governors [10], [11], that modifies the reference signal to ensure that the terminal set remains reachable within the prediction horizon. The FG does not require any modifications to the primary MPC controller, exhibits finite time convergence to the desired reference, and expands the ROA of the MPC controller to all states that can reach the terminal set of any steady-state admissible reference. It also takes advantage of offline polyhedral set manipulation tools [12], [13] to limit online complexity and minimize conservatism.

There is existing literature on avoiding infeasibility in MPC using reference manipulation. The dual-mode controller in [14] features a recovery mode that simultaneously computes a modified reference and control input. This approach converges in finite-time but is invasive and may reduce performance. An FG like algorithm is proposed in [15] and is used as an intermediate design stage in the construction of a piecewise-affine control law that combines a governor and explicit MPC controller into a single unit. This approach suffers from the well known complexity limitations of explicit MPC [16] as the dimension of the state, prediction horizon, and number of constraints increases.

This paper shows that the FG can be scaled to larger systems/longer horizons, and provides a more detailed treatment of both the theoretical properties of the governor, including using under-approximation of the feasible set, and the computation of the terminal and feasible sets. A governor-like algorithm using ellipsoidal terminal sets is proposed in [17] and can be considered a special case of the FG that uses a specific reference parameterization and conservative inner approximation of the feasible set. In [18] the authors propose a suboptimal continuous-time analog of the governor in [17]. In [19], the authors use an MPC to govern the reference of a closed-loop system, thus recovering recursive feasibility. This method is extended in [20], where the inner-loop is closed using an MPC, leading to a bi-level optimization problem. The bi-level formulation aligns the cost functions of the MPC and RG, leading to good performance, but is computationally expensive. Finally, a spatial governor is proposed in [21]. It is specific to precision machining applications and adjusts the velocity profile passed to a path tracking MPC controller to ensure recursive feasibility of constraints representing manufacturing error tolerances.

This paper significantly extend the conference version [22]. Specifically, it addresses the case where the desired reference under/over determines the steady state of the system, removes the requirement that the terminal controller be the linear-quadratic regulator (LQR), includes additional numerical examples for both simulations and feasible set computations, and relaxes the forward invariance requirement on the feasible set to strong returnability. The final extension is both technically challenging and practically significant as computation of the feasible set is computationally expensive, relaxing the forward invariance requirement allows the use of cheaper-to-compute underapproximations of the feasible set.

Notation: For vectors a and b, $(a,b) = [a^T \quad b^T]^T$. The identity and zero matrices are denoted $I_N \in \mathbb{R}^{N\times N}$ and $0_{N\times M} \in \mathbb{R}^{N\times M}$, respectively with the subscripts absent whenever the dimensions are clear from context. Given $M \in \mathbb{R}^{m\times n}$ and $\mathcal{U} \subseteq \mathbb{R}^n$, Ker $M = \{x \mid Mx = 0\}$, $M\mathcal{U} = \{Mx \mid x \in \mathcal{U}\}$, $M^{-1}\mathcal{U} = \{x \mid Mx \in \mathcal{U}\}$, and Int$\mathcal{U}$ denotes the interior of $\mathcal{U}$. Set addition/subtraction is defined as $\mathcal{U}\pm\mathcal{V} = \{u\pm v \mid (u,v) \in \mathcal{U}\times\mathcal{V}\}$ and for $\lambda \in \mathbb{R}$, $\lambda\mathcal{U} = \{\lambda u \mid u \in \mathcal{U}\}$. Positive (semi) definiteness of a matrix $P \in \mathbb{R}^{n\times n}$ is denoted by $(P \succeq 0)$ $P \succ 0$; and $\|x\|_P = \sqrt{x^T P x}$ for $x \in \mathbb{R}^n$. Consider $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and a set $\Gamma \subseteq \mathbb{R}^{n+m}$, the projection of Γ onto x is the image $\Pi_x\Gamma$ where $\Pi_x = [I_n \ 0_{n\times m}]$, i.e., $x = \Pi_x(x,y)$. The slice (or cross-section) operation is $S_y(\Gamma, x) = \{y \mid (x,y) \in \Gamma\}$. For $x \in \mathbb{R}^n$, and $\delta \geq 0$, $\mathcal{B}_\delta(x) = \{y \mid \|y - x\| \leq \delta\}$. For a sequence $\{x_k\} \subseteq \mathbb{R}^n$ and a set $\Gamma \subseteq \mathbb{R}^n$ we write that $x_k \to \Gamma$ as $k \to \infty$, if and only if $\lim_{k\to\infty} \ \inf_{y\in\Gamma} \|y - x_k\| = 0$. Our use of comparison functions, i.e., class $\mathcal{K}, \mathcal{K}_\infty$ and $\mathcal{KL}$ functions follows [23].

II. Problem Setting

Consider the linear time invariant (LTI) system

$$x_{k+1} = Ax_k + Bu_k \tag{1a}$$
$$y_k = Cx_k + Du_k \tag{1b}$$
$$z_k = Ex_k + Fu_k, \tag{1c}$$

where $k \in \mathbb{N}$ is the discrete-time index and $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$, $y_k \in \mathbb{R}^{n_y}$, and $z_k \in \mathbb{R}^{n_z}$ are the states, control inputs, constrained outputs, and tracking outputs, respectively.

Assumption 1. *The pair (A, B) is stabilizable.*

The system (1) is subject to pointwise-in-time constraints

$$\forall k \in \mathbb{N} \quad y_k \in \mathcal{Y}, \tag{2}$$

where $\mathcal{Y} \subseteq \mathbb{R}^{n_y}$ is a specified constraint set.

Assumption 2. *The set $\mathcal{Y}$ is a compact polyhedron with representation $\mathcal{Y} = \{y \mid Yy \leq h\}$ and satisfies $0 \in \text{Int } \mathcal{Y}$.*

As detailed in [7], Assumption 1 implies that the matrix

$$Z = \begin{bmatrix} A - I & B & 0 \\ E & F & -I \end{bmatrix} \tag{3}$$

has a non-trivial kernel. As a result, is possible to introduce an auxiliary reference $v \in \mathbb{R}^{n_v}$ that parameterizes the equilibrium manifold, i.e., every solution to $Z\,[x^T, u^T, z^T]^T = 0$, as

$$\bar{x}_v = G_x v, \quad \bar{u}_v = G_u v, \text{ and } \quad \bar{z}_v = G_z v \tag{4}$$

where $G^T \equiv \begin{bmatrix} G_x^T & G_u^T & G_z^T \end{bmatrix}$ is a basis for Ker (Z). Using v instead of r allows us to handle under/over parameterization of the equilibrium manifold. If G_z is square and invertible, then we can pick $v = r$. The following assumption excludes pathological cases, e.g., $G_z = 0$, that are indicative of an ill-posed problem.

Assumption 3. *The matrix G_z is full rank.*

Next, we introduce a design parameter $\epsilon \in (0, 1)$ and the corresponding set of strictly steady-state admissible auxiliary references

$$\mathcal{V}_\epsilon \equiv G_y^{-1}(1-\epsilon)\mathcal{Y} = \{v \mid G_y v \in (1-\epsilon)\mathcal{Y}\}, \tag{5}$$

where $G_y \equiv CG_x + DG_u$, and strictly admissible references

$$\mathcal{R}_\epsilon \equiv G_z\mathcal{V}_\epsilon = \{G_z v \mid v \in \mathcal{V}_\epsilon\}. \tag{6}$$

The parameter ϵ is needed because MPC controllers cannot stabilize points on the boundary of the feasible set.

We are now ready to formally state our control objectives.

Control Objectives: Given the LTI system (1), let $\mathcal{Y} \subseteq \mathbb{R}^{n_y}$ be given, and let $r \in \mathbb{R}^{n_z}$ be a target reference. The goal of this paper is to design a state feedback law that achieves the following:

- *Safety:* Ensure $y_k \in \mathcal{Y} \quad \forall k \geq 0$;
- *Convergence:* $\lim_{k\to\infty} z_k = r^*$, where $r^* = \underset{s\in\mathcal{R}_\epsilon}{\arg\min} \ \|s - r\|$.
- *Asymptotic Stability:* $\lim_{k\to\infty}(x_k, v_k) = (x_r^*, v_r^*)$ where $(x_r^*, v_r^*) = (G_x v_r^*, v_r^*)$ is stable and satisfies $r^* = G_z v_r^*$.

III. Primary Controller Design

Due to the constraints, we approach the control objectives using a typical MPC formulation where the feedback policy is defined using the solution to the following optimal control problem (OCP)

$$J(x,v) = \min_{\mu} ||\xi_N - \bar{x}_v||_P^2 + \sum_{i=0}^{N-1} ||\xi_i - \bar{x}_v||_Q^2 + ||\mu_i - \bar{u}_v||_R^2 \tag{7a}$$
$$\text{s.t.} \quad \xi_0 = x, \quad (\xi_N, v) \in \mathcal{T} \tag{7b}$$
$$\xi_{i+1} = A\xi_i + B\mu_i, \quad i \in \mathbb{N}_{[0,N-1]}, \tag{7c}$$
$$C\xi_i + D\mu_i \in \mathcal{Y}, \qquad i \in \mathbb{N}_{[0,N-1]}. \tag{7d}$$

where $N \in \mathbb{N}_{>0}$ is the prediction horizon, $\mu = (\mu_0, \ldots \mu_{N-1})$ are the decision variables, $P \in \mathbb{R}^{n_x\times n_x}$, $Q \in \mathbb{R}^{n_x\times n_x}$, and $R \in \mathbb{R}^{n_u\times n_u}$ are weighting matrices, and $\mathcal{T} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_v}$ is the terminal set, which is assumed to be polyhedral, i.e.,

$$\mathcal{T} = \{(x,v) \mid T_x x + T_v v \leq c\}, \tag{8}$$

and $\bar{x}_v, \bar{u}_v$ are defined in (4) and will be manipulated.

The following assumptions ensure that (7) is well-posed and can be used to construct a stabilizing feedback law.

Assumption 4. *The stage cost matrices satisfy $Q = Q^T \succeq 0$, with (A, Q) observable, and $R = R^T \succ 0$.*

Once the stage weights are defined, the terminal penalty P and the terminal set $\mathcal{T}$ can be obtained using a gain $K \in \mathbb{R}^{n_u\times n_x}$ and a fictitious terminal control law

$$\kappa_N(x,v) \equiv -K(x - \bar{x}_v) + \bar{u}_v. \tag{9}$$

Assumption 5. *The terminal set $\mathcal{T}$ is invariant and constraint admissible under* (9)*. Moreover, given $(x,v) \in \mathcal{T}$ and the terminal control law* (9)*, the terminal cost matrix satisfies $P = P^T \succeq 0$ and $\|(A-BK)\delta x\|_P^2 - \|\delta x\|_P^2 \leq -\|\delta x\|_{(Q+K^TRK)}^2$ where $\delta x \equiv x - \bar{x}_v$.*

The terminal control law (9) is not used online but is needed to synthesize P and $\mathcal{T}$. A conservative choice is $K = 0$, $P = 0$, and $\mathcal{T} = \{(\bar{x}_v, v)\}$. Alternatively, for any K such that $A - BK$ is Schur, P can be obtained by solving the discrete algebraic Riccati equation.

Methods for computing polyhedral approximations of the largest possible set $\mathcal{T}$ are detailed in Appendix A.

It is only possible to compute a control action if (7) admits a solution. The set of all parameters for which this is possible, i.e., the feasible set, is

$$\Gamma_N \equiv \{(x,v) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_v} \mid \exists\, \mu : \text{ (7b)} - \text{(7d)}\}, \tag{10}$$

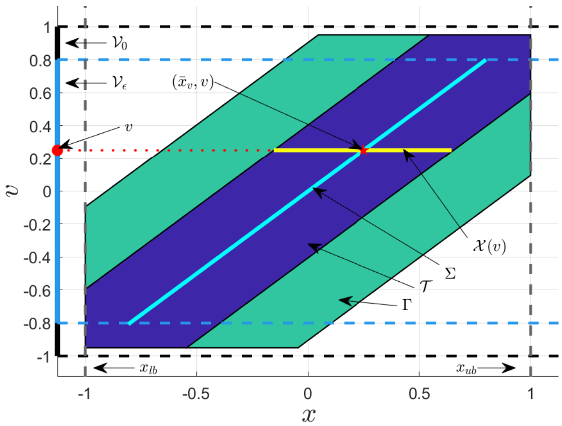

Fig. 2. The sets used in the paper for the integrator $x_{k+1} = x_k + u_k$ subject to $|x_k| \leq 1$, $|u_k| \leq 0.25$ and with $\epsilon = 0.2$, $\mathcal{T} = \tilde{O}_\infty^{0.05}$ and $N = 2$.

which is the N-step backwards reachable set of $\mathcal{T}$. The set of strictly steady-state admissible equilibria is

$$\Sigma \equiv \{(x, v) \mid x = G_x v,\ v \in \mathcal{V}_\epsilon\}. \tag{11}$$

If $\mathcal{T}$ is polyhedral, then Γ_N is polyhedral as well and can be computed offline, see Section IV-C. Figure 2 illustrates the various sets defined in this section. The following technical assumption guarantees convergence and always holds when $\mathcal{T}$ is synthesized using the procedure in Appendix A. Some conditions under which it holds are given in Lemma 1 which is proven in Appendix C.

Assumption 6. $\Sigma \subset \text{Int}\ \Gamma_N$

Lemma 1. *Given Assumption 1 either (i)* $\Sigma \subset \text{Int}\ \mathcal{T}$*, or (ii)* (A, B) *is controllable,* $\Sigma \subseteq \mathcal{T}$*, and* $N \geq \nu$*, where the controllability index* ν *is the smallest positive integer such that* $\begin{bmatrix} B & AB & \cdots & A^{\nu-1}B \end{bmatrix}$ *is full rank is sufficient for Assumption 6 to hold.*

The MPC feedback policy $\kappa : \Gamma_N \to \mathbb{R}^{n_u}$ is

$$\kappa(x, v) \equiv \mu_0^\star(x, v) \tag{12}$$

where $\mu^\star(x, v) = [\mu_0^{\star T}, \mu_1^{\star T}, \ldots, \mu_{N-1}^{\star T}]^T$ is the minimizer of (7). The following theorem summarizes the properties of the closed-loop system for a constant auxiliary reference. The idea behind the proof is to show that the optimal cost J in (7) is a Lyapunov function for the closed-loop system.

Theorem 1. *([2, Theorem 4.4.2]) Let Assumptions 1–5 hold and let* $\phi(\ell, x, v)$ *denote the solution at timestep* $\ell \geq 0$ *of the closed-loop dynamics* $x_{\ell+1} = Ax_\ell + B\kappa(x_\ell, v)$ *with initial condition* $x_0 = x$*. Then for all* $(x, v) \in \Gamma_N$ $(\phi(\ell, x, v), v) \in \Gamma_N$, $\forall \ell \geq 0$, $y_\ell \in \mathcal{Y}$, $\forall \ell \geq 0$, *and* $\lim_{\ell \to \infty} \phi(\ell, x, v) = \bar{x}_v$*. If, in addition,* $v \in \text{Int}\ \mathcal{V}_0$ *then* $\bar{x}_v$ *is asymptotically stable.*

Theorem 1 achieves the control objectives assuming there exists a v_0 such that $G_z v_0 = r$ and x_0 satisfying $(x_0, v_0) \in \Gamma_N$. Its main limitation, however, lies in the fact that the OCP (7) is infeasible if x_0 cannot be steered to $\mathcal{X}(v_0) = \{x \mid (x, v_0) \in \mathcal{T}\}$ within N steps. Although increasing N may seem like a suitable workaround, this solution may be inapplicable in practice since the computational time required to solve (7) scales unfavorably with N.

In the next section, we describe an *add-on* unit that expands the closed-loop region of attraction without extending the prediction horizon or modifying the MPC formulation.

IV. The Feasibility Governor

The MPC feedback policy (12) is stabilizing only if the terminal set associated with the target equilibrium is N-step reachable from the current state. Intuitively, if the target can be manipulated, this limitation can be overcome by selecting a sequence of intermediate targets that are pair-wise reachable. This paper formalizes this idea by redefining the auxiliary reference v as a time-varying signal v_k to ensure $(x_k, v_k) \in \Gamma_N,\ \forall k \in \mathbb{N}$ and $G_z v_k = r$ for sufficiently large $k \in \mathbb{N}$. The resulting control architecture is displayed in Figure 1.

A. Governor Design

The idea behind the FG is to modify the reference so that the MPC problem remains feasible. This follows the minimal interference philosophy of command governors (CGs) where we assume the inner-loop controller is well-designed and we do not wish to modify it. Drawing inspiration from the CG literature [11], the action of the FG can be computed via the following optimization problem.

$$\min_{v \in \mathcal{V}_\epsilon} \quad \|G_z v - r\|_2^2 \tag{13a}$$

$$\text{s.t.} \quad (x, v) \in \Gamma_N. \tag{13b}$$

The FG operates on the same principle as the CG: manipulate the auxiliary reference to remain within a safe invariant set associated with an underlying primary controller. In the case of the CG, the invariant sets are typically slices of O_∞, the maximum constraint admissible set [4] associated with a linear feedback law such as (9). In contrast, the FG uses slices of Γ_N which are invariant under the nonlinear MPC feedback (12). Assuming the common choice $\mathcal{T} = O_\infty$, the set Γ_N is a superset of O_∞ and grows larger as N increases [22]. The use of a more permissive constraint set leads to better performance, as the MPC controller is "aware" of the constraints, something which is not possible using linear feedback.

Unfortunately, if G_z does not have full column rank then (13) will not have a unique minimizer. This is problematic from a stability/robustness perspective; a mechanism for resolving degeneracies is needed. As such, we extend (13) and define the FG feedback as

$$g(x, r) \equiv \underset{v \in \mathcal{V}_\epsilon}{\arg\min} \quad \{\psi(v, r) \mid (x, v) \in \Gamma_N\} \tag{14}$$

where

$$\psi(v, r) \equiv \begin{cases} \|G_z v - r\|_2^2 & \text{if } G_z \text{ is injective} \\ \|v - v_r^*\|_2^2 & \text{otherwise,} \end{cases} \tag{15}$$

and the designer can select any v_r^* satisfying

$$v_r^* \in \mathcal{V}_r^* \equiv \underset{v \in \mathcal{V}_\epsilon}{\arg\min} \ \|G_z v - r\|_2^2. \tag{16}$$

Since ψ is strongly convex, (14) has a unique solution and can be solved reliably online. The combined action of the FG-MPC is shown in Algorithm 1.

Algorithm 1 FG-MPC

Measure: x_k, r_k

1: $v_k = g(x_k, r_k)$ where the FG action g is defined in (14).

2: $u_k = \kappa(x_k, v_k)$ where the feedback law κ is defined in (12).

B. Properties

When combined with (12) and placed in closed-loop with (1), the combined FG-MPC feedback policy ensures constraint satisfaction, renders the point $(x_r^*, v_r^*) = (G_x v_r^*, v_r^*)$ asymptotically stable, and exhibits finite time convergence of $v_k \to v_r^*$. These results are rigorously formulated and proven in Section V.

Moreover, the addition of the FG expands the region of attraction of the closed-loop system from $\mathcal{D}_{MPC} = S_x(\Gamma_N, v_r^*)$, the set of states from which it is possible to reach $\mathcal{X}(v_r^*) = S_x(\mathcal{T}, v_r^*)$ in N-steps, to

$$\mathcal{D}_{FG} = \bigcup_{v \in \mathcal{V}_\epsilon} S_x(\Gamma_N, v), \tag{17}$$

the set of states from which it is possible to reach $\mathcal{X}(v)$ for *any* $v \in \mathcal{V}_\epsilon$ in N-steps. In particular, the addition of the FG guarantees safe transitions between any $r_1, r_2 \in \mathcal{R}_\epsilon$.

Remark 1. *The FG can be applied to systems with disturbances by noting that the essential property required by the FG is that the feasible set Γ_N of the model predictive controller is forward invariant for any constant auxiliary reference. We can readily replace the MPC formulation* (7) *with any alternative OCP with a forward invariant feasible set. Some examples include the tube MPC [24, Algorithm 3.1], or robust MPC [25], which both render the terminal set positively disturbance invariant [24, Theorem 3.2].*

C. Implementation

In our problem setting, Γ_N and $\mathcal{V}_\epsilon$ are polyhedral and thus (14) is a strongly convex quadratic program (QP). For example, the lateral vehicle example in Section VI-A has 1 variable and around 6000 inequality constraints. Dual active-set methods [26] can solve the FG problems efficiently and reliably; they start from the unconstrained optimum and only consider a limited number of active constraints at a given time.

Implementation of the FG also requires a half-space representation of the feasible set. Two methods for obtaining one via polyhedral calculus are described below.

1) Block Method: The MPC OCP (7) is a QP and can be written in the condensed form[1]

$$\min_{\mu} \quad \frac{1}{2}\mu^T H \mu + \mu^T W \theta \tag{18a}$$
$$\text{s.t.} \quad M\mu + L\theta \le b, \tag{18b}$$

with parameter $\theta = (x, v)$. The feasible set (10) can therefore be expressed as $\Gamma_N = \Pi_\theta\{(\mu,\theta) \mid M\mu + L\theta \le b\}$.

2) Recursive Method: The feasible set Γ_N is the N-step backwards reachable set of $\mathcal{T}$ and can be computed recursively. Define the matrices

$$A_e = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} \quad B_e = \begin{bmatrix} B \\ 0 \end{bmatrix}, \text{ and } M_e = \begin{bmatrix} A_e & B_e \end{bmatrix}, \tag{19}$$

and the set $\mathcal{W} = \{(x,v,u) \mid Cx + Du \in \mathcal{Y}\}$. Then Γ_N can be computed via the recursion $\Gamma_{i+1} = \Pi_\theta\left(M_e^{-1}\Gamma_i \cap \mathcal{W}\right)$ starting from the initial condition $\Gamma_0 = \mathcal{T}$.

For both the recursive and block methods, the complexity of computing Γ_N is dominated by the projection operation. The projection is performed offline but can quickly become intractable as all known projection algorithms suffer from the curse of dimensionality [27].

There are several toolboxes available for performing polyhedral calculus (e.g., projections, images etc.). We tested both MPT3 [12] and `bensolve tools` [13] packages, and observed that `bensolve tools` was more effective when computing Γ_N. Figure 3 illustrates Γ_N for several values of N for the lateral vehicle model (Section VI-A). The recursive method is marginally faster than the block method, we recorded the wall-clock time on a 2019 Macbook Pro (2.8 GHz i9, 32GB RAM) running MATLAB 2019b. Both methods display exponential scaling in the horizon length, as expected for polyhedral projection methods.

[1]Expressions for the matrices in (18) are provided in Appendix B.

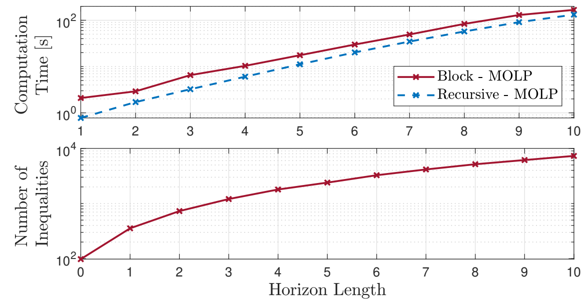

Fig. 3. The computational cost of computing Γ_N. The recursive method marginally outperforms the block method.

Our investigations confirm that projection based methods for computing Γ_N are tractable only for moderately sized systems. One strategy for applying the FG to larger systems is to replace Γ_N with an easier to compute approximation.

D. Under-approximating the Feasible Set

In some scenarios, it may be advantageous (or necessary) to use an approximation of the feasible set. Luckily, with some minor modifications, a set $\mathcal{F} \subseteq \Gamma_N$ can be used in place of Γ_N. In this case, the FG is re-defined as follows

$$g(x,v,r) \equiv \begin{cases} \bar{g}(x,r) & \text{if } (x,v) \in \mathcal{F} \\ v & \text{else} \end{cases} \tag{20a}$$

where

$$\bar{g}(x,r) \equiv \underset{v \in \mathcal{V}_\epsilon}{\arg\min} \ \{\psi(v,r) \mid (x,v) \in \mathcal{F}\}. \tag{20b}$$

At time k the auxiliary reference is then computed as $v_k = g(x_k, v_{k-1}, r)$. The set $\mathcal{F}$ and must satisfy the following:

Assumption 7. *The set $\mathcal{F} \subseteq \Gamma_N$ is closed, convex, and satisfies $\Sigma \subset \text{Int } \mathcal{F}$.*

Note that, as detailed in [28], the set does not need to be polyhedral. The idea behind (20) is that, while the slices of $\mathcal{F}$ are not invariant like those of Γ_N, they are strongly returnable [29] under Assumption 7. That is, if v remains constant, the state trajectories are guaranteed to eventually return to $\mathcal{F}$ due to the properties of the MPC feedback, so the FG simply holds v constant in the meantime. This approach preserves the qualitative theoretical properties (convergence, safety etc.) of the closed-loop system but, unsurprisingly, results in the smaller region of attraction

$$\bar{\mathcal{D}}_{FG} = \bigcup_{v \in \mathcal{V}_\epsilon} S_x(\mathcal{F}, v) \subseteq \mathcal{D}_{FG}. \tag{21}$$

An obvious way to generate the under-approximation is to pick $\mathcal{F} = \Gamma_i$ for $0 \le i < N$ with the limit case $\mathcal{F} = \Gamma_0 = \mathcal{T}$. The ability to use under-approximations also provides the flexibility to design $\mathcal{F}$ so as to limit the number of inequalities, for example by picking $\mathcal{F}$ as a box within Γ_N or as the convex hull of a pre-specified number of points sampled from the boundary of Γ_N. Finally, obtaining under-approximations of Γ_N through parallelizable approaches, such as sampling based algorithms, is likely key for enabling the application of the FG to higher dimensional systems and an important direction for future work.

V. Theoretical Analysis

This section analyzes the properties of the closed-loop system under the combined FG and MPC feedback policy. We consider the case from Section IV-D where the under-approximation $\mathcal{F}$ is used in place of Γ_N. The reference r is assumed constant throughout this section, we suppress any dependencies on r to simplify the notation.

The feasible and invariant sets of the FG are

$$\Phi \equiv \mathcal{F} \cap (\mathbb{R}^{n_x} \times \mathcal{V}_\epsilon), \text{ and } \Lambda \equiv \Gamma_N \cap (\mathbb{R}^{n_x} \times \mathcal{V}_\epsilon). \tag{22}$$

Using these sets, the action of the FG can be expressed as

$$g(x,v) = \begin{cases} \bar{g}(x) & (x,v) \in \Phi \\ v & (x,v) \in \Lambda \setminus \Phi \end{cases} \tag{23a}$$

$$\bar{g}(x) = \underset{v \in S_v(\Phi,x)}{\arg\min} \quad \psi(v,r), \tag{23b}$$

where ψ is defined in (15). Then the closed-loop dynamics of (1) under the combined FG and MPC feedback law are

$$v_k = g(x_k, v_{k-1}) \tag{24a}$$

$$x_{k+1} = f(x_k, v_k), \tag{24b}$$

$$y_k = h(x_k, v_k) \tag{24c}$$

where $f(x,v) \equiv Ax + B\kappa(x,v)$, $h(x,v) \equiv Cx + D\kappa(x,v)$, and κ is the MPC feedback law. The update equations can then be written compactly as

$$(x_{k+1}, v_{k+1}) = T(x_k, v_k), \tag{25}$$

where $T(x,v) \equiv (f(x,v), g(f(x,v),v))$.

The continuity properties of (24) are as follows.

Lemma 2. *([16, Theorem 4]) Given Assumptions 1–5, the functions* $f : \Gamma_N \to \mathbb{R}^{n_x}$ *in* (24b) *and* $\bar{g} : \Phi \to \mathcal{V}_\epsilon$ *are Lipschitz continuous.*

Proof. The MPC feedback policy κ and $\bar{g}$ are solution mappings of the strongly convex multi-parametric quadratic programs (18) and (23b) and are therefore Lipschitz continuous. Lipschitz continuity of f follows immediately. □

A. Safety and Recursive Feasibility

The following theorem provides sufficient conditions under which the (FG) achieves the *Safety* objective and proves that the set Λ is forward invariant.

Theorem 2 (Safety & Invariance)**.** *Given Assumptions 1–5 and 7, consider the closed-loop dynamics* (24)*. Suppose* $x_0 \in \Pi_x\Phi$*, then the sequence* $\{(x_k, v_k)\}_{k=0}^{\infty} \subseteq \Lambda$ *is well defined and* $y_k \in \mathcal{Y}$ *for all* $k \in \mathbb{N}$.

Proof. The proof is by induction. At time $k = 0$ if $x_0 \in \Pi_x\Phi$ then (23b) is feasible and $(x_0, v_0) \in \Phi \subseteq \Lambda$. Next, assume $(x_k, v_k) \in \Lambda$, the functions f and g are both defined on Λ and thus (x_{k+1}, v_{k+1}) is well defined. If $(x_{k+1}, v_k) \in \Phi$ then $S_v(\Phi, x_{k+1}) \neq \emptyset$, i.e., (23b) is feasible, and $v_{k+1} = g(x_{k+1}, v_k) = \bar{g}(x_{k+1}) \in S_v(\Phi, x_{k+1})$, and thus $(x_{k+1}, v_{k+1}) \in \Phi \subseteq \Lambda$. Otherwise, if $(x_{k+1}, v_k) \notin \Phi$, (23) yields $v_{k+1} = v_k$, thus $(x_{k+1}, v_{k+1}) = (x_{k+1}, v_k) \in \Lambda$ (by Theorem 1). Therefore, by induction, $(x_k, v_k) \in \Lambda \subset \Gamma_N$ for all $k \in \mathbb{N}$ which implies that $\forall k \in \mathbb{N}$ $y_k \in \mathcal{Y}$ (by Theorem 1) and that the sequence $\{(x_k, v_k)\}_{k=0}^{\infty}$ is well-defined. □

B. Convergence and Stability

Having established safety, we now consider convergence and stability. We begin by introducing the Lyapunov function candidate

$$V(v) \equiv \psi(v,r) \geq 0, \tag{26}$$

with ψ defined in (15) and the notation $V_k = V(v_k)$ and $V^* = V(v^*)$ where v^* is defined in (16). The following Lemma addresses how V evolves along solutions of (24).

Lemma 3. *Given Assumptions 1–5, define the increment*

$$\Delta V(x,v) \equiv V(g(f(x,v),v)) - V(v), \tag{27}$$

then for all $(x,v) \in \Lambda$*, there exists* $\eta > 0$ *such that*

$$\Delta V(x,v) \leq -\eta \|g(f(x,v),v) - v\|^2. \tag{28}$$

Proof. Partition the set Λ into $\Lambda = \Lambda_1 \cup \Lambda_2$ where $\Lambda_1 = \{(x,v) \mid (f(x,v),v) \in \Lambda \backslash \Phi\}$ and $\Lambda_2 = \{(x,v) \mid (f(x,v),v) \in \Phi\}$. If $(x,v) \in \Lambda_1$ then $g(f(x,v),v) = v$ by (23) and (28) clearly holds.

Next the case $(x,v) \in \Lambda_2$. Recall that V is a strongly convex quadratic function. Thus, there exists $\eta > 0$ such that

$$V(v) \geq V(v') + \nabla V(v')^T (v - v') + \eta \|v - v'\|^2 \tag{29}$$

for all $v', v \in \mathbb{R}^{n_v}$. Letting $x^+ = f(x,v)$, we have that by (23), $g(x^+, v) = \bar{g}(x^+)$ for all $(x,v) \in \Lambda_2$. Moreover, recall that optimality conditions associated with $\bar{g}(x^+) = \arg\min_{s \in S_v(\Phi, x^+)} V(s)$ are [30]

$$\nabla V(\bar{g}(x^+))^T (v - \bar{g}(x^+)) \geq 0, \quad \forall v \in S_v(\Phi, x^+). \tag{30}$$

Substituting $v' = \bar{g}(x^+)$ and (30) into (29), and rearranging, we obtain that, for all $(x,v) \in \Lambda_2$

$$V(\bar{g}(f(x,v))) - V(v) \leq -\eta \|\bar{g}(f(x,v)) - v\|^2 \leq 0.$$

Since $\Lambda = \Lambda_1 \cup \Lambda_2$ this completes the proof. □

Corollary 1. *Consider* (24)*, under Assumptions 1–7, if* $x_0 \in \Pi_x\Phi$ *then* $V(v_{k+1}) - V(v_k) \leq 0$.

The next Lemma provides a sufficient condition under which the auxiliary reference changes.

Lemma 4. *Given Assumptions 1–7, define*

$$\mathcal{B}_\delta(\Sigma) \equiv \{(x,v) \mid v \in \mathcal{V}_\epsilon, \ \|x - G_x v\| \leq \delta\}, \tag{31}$$

where $\Sigma = \mathcal{B}_0(\Sigma) = \{(x,v) \mid x = G_x v, \ v \in \mathcal{V}_\epsilon\}$*. Then, there exists* $\delta^\star > 0$ *such that* $\mathcal{B}_{\delta^\star}(\Sigma) \subset \text{Int } \mathcal{F}$*. Moreover,* $\delta \in [0, \delta^\star]$*,* $(x,v) \in \mathcal{B}_\delta(\Sigma)$*, and* $v \neq v^*$ *implies that* $g(x,v) \neq v$.

Proof. To show that $(x,v) \in \mathcal{B}_\delta(\Sigma) \wedge v \neq v^* \implies g(x,v) \neq v$ we will construct a point $v' \in S_v(\Phi, x)$ such that $V(v') < V(v)$. By Assumption 7, $\Sigma \subset \text{Int } \mathcal{F}$ and thus there exists $\delta^\star > 0$ such that $\mathcal{B}_\delta(\Sigma) \subset \text{Int } \mathcal{F}$ for all $\delta \in [0, \delta^*]$. Moreover, because $\mathcal{B}_\delta(\Sigma) \subset \text{Int } \mathcal{F}$, for any $(x,v) \in \mathcal{B}_\delta(\Sigma)$, there exists $\alpha = \alpha(\delta) > 0$ such that $\mathcal{B}_\alpha(v) \subseteq S_v(\mathcal{F}, x)$.

Fix any $\delta \in [0, \delta^*]$ and the corresponding $\alpha = \alpha(\delta)$. Then, define the set $\mathcal{C}_\alpha = \mathcal{V}_\epsilon \cap \mathcal{B}_\alpha(v)$, the ray $v'(t) = v + t(v^* - v) \quad t \in [0,1]$ and assume $v \neq v^*$. The first step is to show that $t \in [0, \gamma] \implies v'(t) \in \mathcal{C}_\alpha$ where $\gamma = \min\left(1, \frac{\alpha}{\|v - v^*\|}\right) \in (0,1]$. To prove this, recall that $\mathcal{V}_\epsilon$ is convex and $v, v^* \in \mathcal{V}_\epsilon$ thus $v'(t) \in \mathcal{V}_\epsilon$ for $t \in [0,1]$. Moreover, $\|v'(\gamma) - v\| \leq \|v'(\alpha / \|v - v^*\|) - v)\| = \alpha$ and therefore $\forall t \in [0, \gamma], \ v'(t) \in \mathcal{C}_\alpha$.

To establish that V decreases along $v'(t)$, recall that V is convex and therefore

$$V(v'(t)) = V((1-t)v + tv^*) \leq V(v) - t[V(v) - V^*]$$

for all $v \in \mathcal{V}_\epsilon \setminus v^*$ and $t \in [0,1]$. Further, using that $V^* < V(v)$ for all $v \in \mathcal{V}_\epsilon \setminus v^*$ and that $\gamma \in (0,1]$ we conclude that $V(v'(\gamma)) < V(v)$. Thus we have constructed a point $v'(\gamma) \in \mathcal{C}_\alpha \subseteq S_v(\Phi, x)$ satisfying $V(v'(\gamma)) < V(v)$, this implies that

$$V(g(x,v)) = \min_{s \in S_v(\Phi,x)} V(s) \leq V(v'(\gamma)) < V(v). \tag{32}$$

Finally, strong convexity of V combined with $V(g(x,v)) < V(v)$ implies that $g(x,v) \neq v$ as claimed. □

The next lemma extends Theorem 1 for varying v.

Lemma 5. *Given Assumptions 1–5, and the system $x_{k+1} = f(x_k, v_k)$, the error signal $e_k = x_k - G_x v_k$ is input-to-state stable (ISS) [31] with respect to the input $\Delta v_k = v_{k+1} - v_k$, i.e., there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that*

$$\|x_k - G_x v_k\|_Q \leq \beta(k, \|x_0 - G_x v_0\|) + \gamma \left(\sup_{j \geq 0} \|\Delta v_j\| \right).$$

Proof. Under Assumptions 1–5, it is well known, see e.g., [3], that J the optimal cost function of the MPC feedback law in (7), is a Lyapunov function for the closed-loop system, i.e., there exist $\alpha, \alpha_l, \alpha_u \in \mathcal{K}_\infty$ such that

$$J(f(x, v), v) - J(x, v) \leq -\alpha(\|x - G_x v\|_Q), \tag{33}$$

$$\alpha_l(\|x - G_x v\|_Q) \leq J(x, v) \leq \alpha_u(\|x - G_x v\|_Q) \tag{34}$$

for all $(x, v) \in \Gamma_N$. Moreover, under our assumptions, J is uniformly continuous [25, Prop 1] and thus there exists $\sigma_x, \sigma_v \in \mathcal{K}$ such that $|J(x', v') - J(x, v)| \leq \sigma_x(\|x' - x\|) + \sigma_v(\|v' - v\|)$. Hence, for any $(x, v) \in \Lambda$, $x^+ = f(x, v)$ and $v^+ \in S_v(\Lambda, x^+)$, let $\Delta J = J(x^+, v^+) - J(x, v)$ then

$$\Delta J = J(x^+, v) - J(x, v) + J(x^+, v^+) - J(x^+, v) \tag{35}$$

$$\leq -\alpha(\|x - G_x v\|_Q) + |J(x^+, v^+) - J(x^+, v)| \tag{36}$$

$$\leq -\alpha(\|x - G_x v\|_Q) + \sigma_v(\|v^+ - v\|) \tag{37}$$

which demonstrates ISS [31, Lemma 3.5]. □

Corollary 2. *Let Assumptions 1–5 hold, and let $T : \Lambda \to \Lambda$ be the operator defined in* (25). *Then $\tilde{T} : \Phi \to \Lambda$, the restriction of T to Φ, is continuous and can be expressed explicitly as $\tilde{T}(x, v) \equiv (f(x, v), \bar{g}(f(x, v)))$.*

Proof. Recall that, by (23), for all $(x, v) \in \Phi$ $T(x, v) = (f(x, v), \bar{g}(f(x, v))) = \tilde{T}(x, v)$. Since f and $\bar{g}$ are continuous by virtue of Lemma 2, $\tilde{T}$ is also continuous. □

Having assembled the required components, we are ready to show asymptotic stability.

Theorem 3 (Asymptotic Stability)**.** *Let Assumptions 1–7 hold. The point (x^*, v^*), where $x^* = G_x v^*$, is an asymptotically stable equilibrium of* (24) *and $x_0 \in \Pi_x \Phi \implies \lim_{k \to \infty} (x_k, v_k) = (x^*, v^*)$.*

Proof. First, note that, by Theorem 2, $x_0 \in \Pi_x \Phi$ guarantees that the sequence $\{(x_k, v_k)\}_{k=0}^\infty \subseteq \Lambda$ is well defined. Moreover, the sequence $\{V_k\}_{k=0}^\infty$ is non-increasing (Corollary 1) and bounded from below, hence converging. By virtue of Lemma 3, we have that there exists $\eta > 0$ such that $\|v_{k+1} - v_k\|^2 \leq \eta^{-1}|V_{k+1} - V_k| \to 0$ as $k \to \infty$ and thus $\lim_{k \to \infty} ||\Delta v_k|| = 0$. Moreover, combining Lemma 5 and [31, Lemma 3.8], there exists $\gamma \in \mathcal{K}$ such that

$$\limsup_{k \to \infty} \|x_k - G_x v_k\|_Q \leq \gamma \left(\limsup_{k \to \infty} \|\Delta v_k\| \right), \tag{38}$$

together with the observability of (A, Q), this implies that

$$\lim_{k \to \infty} \|x_k - G_x v_k\| = 0. \tag{39}$$

Therefore, there exists $t \geq 0$ such that $\|x_k - G_x v_k\| \leq \delta^\star$ for all $k \geq t$ and thus $(x_k, v_k) \in \mathcal{B}_{\delta^\star}(\Sigma)$ for all $t \geq k$, where $\delta^\star$ and $\mathcal{B}_\delta(\Sigma)$ are defined in Lemma 4.

By virtue of Lemma 4, $\mathcal{B}_{\delta^*}(\Sigma) \subset \text{Int } \mathcal{F}$ implying that $\mathcal{B}_{\delta^*}(\Sigma) \cap (\mathbb{R}^n \times \mathcal{V}_\epsilon) \subset \text{Int } \mathcal{F} \cap (\mathbb{R}^n \times \mathcal{V}_\epsilon) \subsetneq \Phi$, and thus $\{(x_k, v_k)\}_{k=t}^\infty \subseteq \mathcal{B}_{\delta^*}(\Sigma) \subset \Phi$. Hence, for all $k \geq t$, $(x_{k+1}, v_{k+1}) = \tilde{T}(x_k, v_k)$, where $\tilde{T}$ is defined in Corollary 2. As $\tilde{T}$ is continuous (Corollary 2), and V is non-increasing along solutions of (24) (Corollary 1), the invariance principle [32, Theorem 6.3] implies that $(x_k, v_k) \to \mathcal{M}$ as $k \to \infty$ where $\mathcal{M} \subset \Phi$ denotes the largest invariant subset of

$$\Omega = \{(x, v) \in \Phi \mid V(\bar{g}(f(x, v)) - V(v) = 0\}. \tag{40}$$

Moreover, (39) implies that $(x_k, v_k) \to \Sigma$ as $k \to \infty$ and thus $(x_k, v_k) \to \mathcal{M} \cap \Sigma$ as $k \to \infty$.

We claim that $\mathcal{M} \cap \Sigma = \{(x^*, v^*)\}$; evidently, $(x^*, v^*) \in \mathcal{M}$ and $(x^*, v^*) \in \Sigma$. Recall that $\mathcal{M} \subset \Omega \subset \Phi$, thus by Lemma 3, $(x, v) \in \mathcal{M} \implies \bar{g}(f(x, v)) = v$. Furthermore, by virtue of Theorem 1, $(x, v) \in \Sigma \implies x = f(x, v)$ and thus

$$(x, v) \in \mathcal{M} \cap \Sigma \implies \bar{g}(f(x, v)) = \bar{g}(x) = v. \tag{41}$$

Moreover, by Lemma 4, for all $(x, v) \in \Sigma = \mathcal{B}_0(\Sigma)$ we have that $v \neq v^* \implies g(x) \neq v$ and thus

$$(x, v) \in \Sigma \text{ and } \bar{g}(x) = v \implies v = v^*. \tag{42}$$

Taking the logical conjunction of right-hand sides of (41) and (42) immediately yields the implication $(x, v) \in \mathcal{M} \cap \Sigma \implies v = v^*$ and thus $\mathcal{M} \cap \Sigma = \{(x, v) \mid x = G_x v,\ v = v^*\} = \{(x^*, v^*)\}$ as claimed.

Lyapunov stability of (x^*, v^*) follows from Corollary 1 and the ISS stability of the tracking error. Therefore, the sequence $\{(x_k, v_k)\}_{k=0}^\infty \subseteq \Lambda$ is well defined, $x_0 \in \Pi_x \Phi$ implies that $(x_k, v_k) \to (x^*, v^*)$ as $k \to \infty$ and (x^*, v^*) is a Lyapunov stable equilibrium point of (24). □

Theorem 4 (Finite-time Convergence)**.** *Let Assumptions 1–7 hold and consider the closed-loop system* (24). *Then, for all $x_0 \in \Pi_x \Phi$, there exists $t \geq 0$ such that $v_k = v^*$ for all $k \geq t$.*

Proof. Thanks to Lemma 1 we know that $(x^*, v^*) \in \Sigma \subset \text{Int } \mathcal{F}$ and thus $x^* \in \text{Int } S_x(\mathcal{F}, v^*)$. In addition, the definition of Φ implies that $S_x(\Phi, v) = S_x(\mathcal{F}, v)$ for all $v \in \mathcal{V}_\epsilon$ and thus $x^* \in \text{Int } S_x(\Phi, v^*)$.

Since $x_k \to x^* \in \text{Int } S_x(\Phi, v^*)$ as $k \to \infty$ (Theorem 3) there exists a finite $t \geq 0$ such that $x_t \in S_x(\Phi, v^*)$. From the definition (23), it is evident that $g(x, v) = \bar{g}(x) = v^*$ for all $x \in S_x(\Phi, v^*)$ and thus $v_t = g(x_t, v_{t-1}) = v^*$. Finally, thanks to Theorem 1, $x_k \in S_x(\Lambda, v^*)$ implies that $x_{k+1} = f(x_k, v_k) \in S_x(\Lambda, v^*)$ and thus we can consider two cases, corresponding to the partition $\Lambda = \Phi \cup (\Lambda \setminus \Phi)$. If $x_k \in S_x(\Phi, v^*)$ then $v_k = g(x_k, v^*) = \bar{g}(x_k) = v^*$, and if $x_k \in S_x(\Lambda \setminus \Phi, v^*)$ then $v_k = g(x_k, v^*) = v^*$ and thus $v_k = v^*$ for all $k \geq t$. □

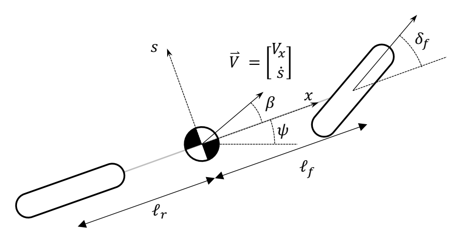

Fig. 4. The bicycle model of the lateral vehicle dynamics.

VI. Numerical Examples

A. Lateral Vehicle Dynamics

This section applies the FG to the lateral dynamics of a car moving forward at a constant speed of $V_x = 30m/s$. The model is based on the one in [33] and roughly represents a 2017 BMW 740i sedan.

A diagram of the bicycle model is displayed in Figure 4. The state of the system is $x^T = [s \ \ \psi \ \ \beta \ \ \omega]$ where s is the lateral position of the vehicle, ψ is the yaw angle, $\beta = \dot{s}/V_x$ is the sideslip angle, and

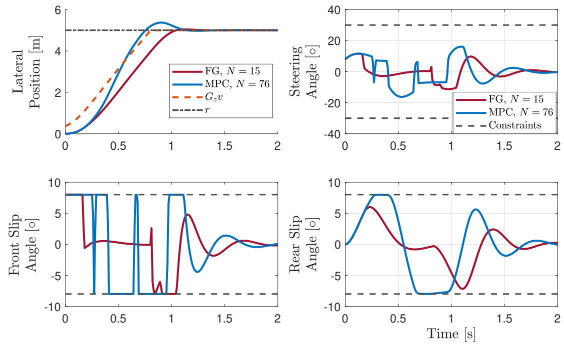

Fig. 5. Closed-loop lateral vehicle dynamics responses for the FG with $N = 15$ vs. an ungoverned MPC controller with $N = 76$, the shortest N such that the initial problem is feasible. The performance (rise-time) of the FG-MPC combination is only marginally slower than the ungoverned MPC controller which needs a significantly longer horizon to ensure feasibility.

TABLE I
EXECUTION TIME DATA FOR THE LATERAL VEHICLE DYNAMICS EXAMPLE.

	FG($N=15$)	MPC($N=15$)	MPC($N=75$)
TAVE [ms]	1.4	0.22	11.7
TMAX [ms]	2.7	0.53	54.5

$\omega = \dot{\psi}$ is the yaw rate. The control input is the front steering angle $u = \delta_f$ and the system is subject to constraints on $y^T = [\alpha_f \;\; \alpha_r \;\; \delta_f]$ where α_f and α_r are the front and rear slip angles. The tracking output is $z = s$. The continuous-time system matrices are

$$A = \begin{bmatrix} 0 & V_x & V_x & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{2C_\alpha}{mV_x} & \frac{C_\alpha(\ell_r - \ell_f)}{mV_x^2} - 1 \\ 0 & 0 & \frac{C_\alpha(\ell_r - \ell_f)}{I_{zz}} & -\frac{C_\alpha(\ell_r^2 + \ell_f^2)}{I_{zz}V_x} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \frac{C_\alpha}{mV_x} \\ \frac{C_\alpha \ell_f}{I_{zz}} \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & -1 & -\frac{\ell_f}{V_x} \\ 0 & 0 & -1 & \frac{\ell_r}{V_x} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$E = [1 \; 0 \; 0 \; 0]$, and $F = 0$, where $m = 2041 \; kg$ is the mass of the vehicle, $I_{zz} = 4964 \; kg \cdot m^2$ is the moment of inertia about the yaw axis, $\ell_f = 1.56 \; m$ and $\ell_r = 1.64 \; m$ are the moment arms of the front and rear wheels relative to the center of mass, and $C_\alpha = 246994 \; N/rad$ is the tire stiffness. The continuous time system matrices are converted to discrete-time using a zero-order hold with a sampling time of $t_s = 0.01$ seconds. The constraint set is $\mathcal{Y} = [-8°, \; 8°] \times [-8°, \; 8°] \times [-30°, \; 30°]$ which represents limits on the front and rear slip angles (to prevent tire slip and drifting) and a mechanical limit on the steering angle. The initial condition is $x_0 = 0$, the target position is $r = 5 \; m$, and the weighting matrices are $Q = E^T E$ and $R = 0.1$. The terminal penalty and gain are computed using the linear quadratic regulator and the terminal set is $\mathcal{T} = \tilde{O}_\infty^{0.01}$, computed using the procedure in Appendix A.

Figure 5 compares the combined FG-MPC feedback law for $N = 15$ with an ungoverned MPC controller with $N = N^* = 76$ where $N^* = N^*(x_0, r, \mathcal{T}) = \inf_i \; \{i \mid (x_0, G_z^{-1} r) \in \Gamma_i\}$. The rise and settling times of the combined feedback law is comparable with that of the ungoverned MPC controller despite a 94% reduction in worst-case computation time, see Table I.

Figure 6 compares the response of the closed-loop system for several values of N and with the CG + LQR. As expected, the FG-MPC solution provides a faster response than the CG + LQR solution and the system response becomes faster as N increases. As $N \to N^*$ the filtering effect diminishes until the response of the pure MPC controller is recovered.

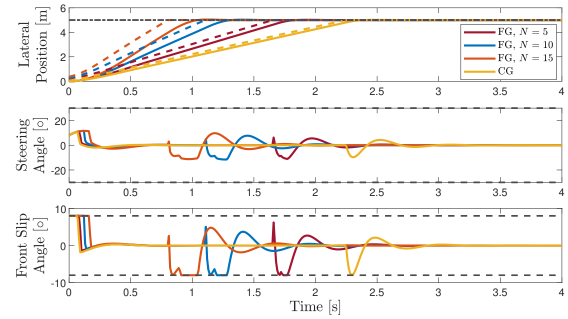

Fig. 6. Closed-loop lateral vehicle dynamics responses for varying horizon lengths. The FG outperforms the CG and the system responds more quickly as N increases.

VII. Conclusions

This paper proposed the Feasibility Governor (FG), an add-on unit that expands the region of attraction of linear-quadratic model predictive controllers by minimally manipulating the reference input passed to the controller. It was shown that the FG is safe, converges in finite time, and extends the region of attraction of MPC controllers at a fraction of the computation cost associated with increasing the prediction horizon. Future work includes extending the FG to nonlinear settings, extending the FG beyond piecewise constant reference signals, considering inner-loop controllers that achieve offset-free tracking in the presence of disturbances, and exploring parallelizeable methods for synthesizing inner-approximation of the feasible set to enable to application of the FG to large scale systems.

Appendix

A. Computing the Terminal Set

Substituting the terminal control law (9) into the open-loop dynamics (1) and using that $x_v = G_x v$ and $u_v = G_u v$ yields

$$x_{k+1} = \bar{A} x_k + \bar{B} v \quad (43)$$

$$y_k = \bar{C} x_k + \bar{D} v \in \mathcal{Y} \quad (44)$$

where $\bar{A} = A - BK$, $\bar{B} = B\,(KG_x + G_u)$, $\bar{C} = C - DK$, and $\bar{D} = D\,(KG_x + G_u)$. This is a standard form in the reference governor literature, see e.g., [4], [10], [34], which makes use of the maximal constraint admissible set,

$$O_\infty = \{(x, v) \mid \bar{C}\bar{A}^k x + \bar{C}\left(I - \bar{A}\right)^{-1}\left(I - \bar{A}^k\right)\bar{B}v + \bar{D}v \in \mathcal{Y}, \; \forall k \geq 0\}. \quad (45)$$

Since O_∞ is maximal, invariant, and constraint admissible under constant v [34, Theorem 1.1] $\mathcal{T} = O_\infty$ is the largest possible terminal set (for a given terminal feedback law). However, if O_∞ can not be finitely determined, we replace it with $\tilde{O}_\infty^\epsilon = O_\infty \cap O^\epsilon$ where $O^\epsilon = \{(x, v) \mid (\bar{D} + \bar{C}(I - \bar{A})^{-1}\bar{B})v \in (1 - \epsilon)\mathcal{Y}\}$. The set $\tilde{O}_\infty^\epsilon$ is guaranteed to be representable by a finite number of linear inequalities and is still forward invariant and constraint admissible. Algorithms for computing $\tilde{O}_\infty^\epsilon$ are well established and can be found in [4], [35]; they yield matrices $[T_x \;\; T_v]$ and a vector c, such that

$$\tilde{O}_\infty^\epsilon = \{(x, v) \mid T_x x + T_v v \leq c\}. \quad (46)$$

B. Condensed Matrix Definitions

Let $\otimes$ denote the Kronecker product and define

$$\hat{A} = \begin{bmatrix} I \\ A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}, \hat{B} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ B & 0 & \cdots & 0 \\ AB & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ A^{N-1}B & \cdots & AB & B \end{bmatrix}, \hat{C} = \begin{bmatrix} I_N \otimes YC & 0 \\ 0 & T_x \end{bmatrix},$$

$$\hat{D} = \begin{bmatrix} I_N \otimes YD \\ 0 \end{bmatrix}, \hat{H} = \begin{bmatrix} I_N \otimes Q & 0 \\ 0 & P \end{bmatrix}, \text{ and } \hat{T}_v = \begin{bmatrix} 0 \\ T_v \end{bmatrix}.$$

Then the matrices in (18) are

$$H = \hat{B}^T \hat{H} \hat{B} + I_N \otimes R,\ W_x = \hat{B}^T \hat{H} \hat{A},\ W = \begin{bmatrix} W_x & W_v \end{bmatrix}$$
$$W_v = -\left(W_x G_x + H\left(1_N \otimes G_u\right)\right),$$
$$M = \hat{C}\hat{B} + \hat{D},\ L = \begin{bmatrix} \hat{C}\hat{A} & \hat{T}_v \end{bmatrix}, \text{ and } b = \begin{bmatrix} (1_N \otimes h)^T & c^T \end{bmatrix}^T,$$

where 1_N is a column of N ones.

C. Proof of Lemma 1

Depending on which condition of Assumption 6 is satisfied, one of the following holds: 1) Following from Assumption 5, the terminal control law (9) ensures constraint satisfaction $\forall (x, v) \in \mathcal{T}$. Therefore, it follows from (10) that $\mathcal{T} \subseteq \Gamma_N$. The statement is then proven by noting $\Sigma \subset \text{Int } \mathcal{T} \subseteq \text{Int } \Gamma_N$. 2) Since (A, B) is controllable, there exists a deadbeat gain matrix L such that $(A - BL)^\nu = 0$ [36]. Thus, given the control law $u_k = \bar{u}_v - L(x_k - \bar{x}_v)$, the closed-loop dynamics of (1) satisfy $x_k = \bar{x}_v,\ \forall k \geq \nu$. Let O_∞ denote the maximum constraint admissible set [4] associated with the deadbeat dynamics. It follows by definition that $(x, v) \in O_\infty$ ensures $y_k \in \mathcal{Y}$, which implies $O_\infty \subseteq \Gamma_N$ due to (10). Since $(A - BL)$ is Schur [36, Property 2] and $v \in \mathcal{V}_\epsilon$, it follows from [4, Theorem 2.1] that $\Sigma \subset \text{Int } O_\infty \subseteq \text{Int } \Gamma_N$.

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