Drinfeld's Lemma for Perfectoid Spaces and Overconvergence of Multivariate (φ, Γ)-Modules

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Abstract. Let p be a prime, let K be a finite extension of $\mathbb{Q}_p$, and let n be a positive integer. We construct equivalences of categories between continuous p-adic representations of the n-fold product of the absolute Galois group G_K and (φ, Γ)-modules over one of several rings of n-variable power series. The case $n = 1$ recovers the original construction of Fontaine and the subsequent refinement by Cherbonnier–Colmez; for general n, the case $K = \mathbb{Q}_p$ had been previously treated by the third author. To handle general K uniformly, we use a form of Drinfeld's lemma on the profinite fundamental groups of products of spaces in characteristic p, but for perfectoid spaces instead of schemes. We also construct the multivariate analogue of the Herr complex to compute Galois cohomology; the case $K = \mathbb{Q}_p$ had been previously treated by Pal and the third author, and we reduce to this case using a form of Shapiro's lemma.

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1 Introduction

1.1 Overview

Throughout this paper, fix a prime number p. The theory of (φ, Γ)-*modules* was introduced by Fontaine [16] as a tool for describing and classifying continuous representations of the Galois group of a finite extension of $\mathbb{Q}_p$ on a finite-dimensional $\mathbb{Q}_p$-vector space. Thanks to subsequent refinements, notably the work of Cherbonnier–Colmez [10] and Berger [3, 5], it has become clear that essentially all of p-adic Hodge theory can be formulated in terms of (φ, Γ)-modules; moreover, this formulation has driven much recent progress in the subject and powered some notable applications in arithmetic geometry. See [23] for a quick introduction to this circle of ideas or [37] for a more in-depth treatment.

The goal of this paper is to initiate a systematic development of *multivariate* (φ, Γ)-modules, founded upon the theory of perfectoid spaces, as a tool for studying representations of *products* of Galois groups of p-adic fields. The relevance of such representations may not be immediately clear from general considerations of arithmetic geometry; however, products of Galois groups occur naturally in the approach to geometric Langlands developed for GL_2 by Drinfeld [13] and extended to GL_n by L. Lafforgue [31] and to other reductive groups by V. Lafforgue [32, 33].

The relationship between multivariate (φ, Γ)-modules and Galois representations was previously explored by the third author [36, 47] from a slightly different point of view: this line of inquiry emerged as part of a program to extend Colmez's construction of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ [11] by exhibiting analogues of (φ, Γ)-modules obtained from higher-rank groups [38, 48].

One motivation for consolidating the theory of multivariate (φ, Γ)-modules is to prepare for an eventual unification of this program with the work of V. Lafforgue described above; however, such a unification lies far beyond the scope of the present work. Another motivation is to flesh out the point of view suggested in the last paragraph of the introduction of [47], which relates multivariate (φ, Γ)-modules to a form of Drinfeld's lemma for perfectoid spaces (more on which below).

1.2 Main results

Before diving into the weeds of perfectoid spaces, we give a brief indication of our main results (and recall that the case $K = \mathbb{Q}_p$ was treated in [36, 47]). Let K denote a finite extension of $\mathbb{Q}_p$ with absolute Galois group G_K. Let $G_{K,\Delta}$ be the Cartesian power of G_K indexed by the finite set Δ. (One can also consider products $G_{K_1} \times \cdots \times G_{K_n}$ where $K_1, \dots, K_n$ are possibly distinct finite extensions of $\mathbb{Q}_p$; to keep notation under control, we suppress this level of generality until the end of the paper.)

Theorem 1.1 (see Theorem 6.15). The category of continuous representations of $G_{K,\Delta}$ on finite free $\mathbb{Z}_p$-modules is canonically equivalent to the category of projective étale $(\varphi_\Delta, \Gamma_\Delta)$-modules over each of the rings

$$\mathcal{O}_{\mathcal{E}_\Delta(K)}, \quad \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}, \quad \mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}, \quad \widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)},$$

described below. A similar statement also holds for representations of $G_{K,\Delta}$ on finite-dimensional $\mathbb{Q}_p$-vector spaces; see Theorem 6.16.

Theorem 1.2 (see Theorem 7.10). In Theorem 1.1, the Galois cohomology of a representation of $G_{K,\Delta}$ is canonically isomorphic to the continuous $(\varphi_\Delta, \Gamma_\Delta)$-cohomology of the corresponding $(\varphi_\Delta, \Gamma_\Delta)$-module (i.e., the cohomology of the complex of continuous $(\varphi_\Delta, \Gamma_\Delta)$-cochains valued in the module).

To unpack this, let us start with the case where Δ is a singleton set. In this case, $\mathcal{O}_{\mathcal{E}_\Delta(K)}$ is none other than Fontaine's ring $\mathcal{O}_{\mathcal{E}}$ which serves as his original base ring for (φ, Γ)-modules [16]; it is the p-adic completion of a Laurent series ring in a variable ϖ with coefficients in a certain finite étale extension of $\mathbb{Z}_p$, carrying actions of a Frobenius lift φ and a profinite group Γ_K (the Galois group of the maximal cyclotomic extension of K), which commute with each other. The other rings are built from $\mathcal{O}_{\mathcal{E}_\Delta(K)}$ in such a way as to also carry actions of φ and Γ_K: the tilde indicates an enlargement that makes the action of φ bijective (*perfection*), while the dagger indicates passage to a subring satisfying a growth condition (*overconvergence*). A *projective étale* (φ, Γ)*-module* over one of these rings is then a finite projective (hence free) module M equipped with commuting semilinear actions of φ and Γ_K, for which the linearization $\varphi^* M \to M$ of the φ-action is an isomorphism. The equivalence of categories stated in Theorem 1.1 then incorporates Fontaine's original equivalence, together with its refinement by Cherbonnier–Colmez [10] using the overconvergent subring. The description of Galois cohomology stated in Theorem 1.2 is due to Herr [19, 20].

For general Δ, the rings in question are certain topological Cartesian powers of the rings arising in the singleton case. In particular, for each $\alpha \in \Delta$, there will be an element ϖ_α arising from the factor of the product indexed by α; moreover, there will be a partial Frobenius lift φ_α and a profinite group $\Gamma_{K,\alpha}$ which act on ϖ_α but not on the other variables. (The use of the symbols Δ and α is meant to suggest roots of a Lie algebra; the relevance of this will not be apparent herein, but can be seen more directly in earlier work on the subject, especially [48].)

1.3 Perfectoid spaces and Drinfeld's lemma

We next explain what perfectoid spaces and Drinfeld's lemma have to do with each, and with the aforementioned results.

The theory of *perfectoid spaces*, while having notched diverse achievements since its promulgation in the early 2010s, is at its heart a geometric reinterpretation of the core ideas of p-adic Hodge theory (as in [9, 27, 28, 39]). It begins with a vast generalization of the "field of norms" isomorphism between the absolute Galois groups of $\mathbb{Q}_p(\mu_{p^\infty})$ and $\mathbb{F}_p(\!(t)\!)$ introduced by Fontaine–Wintenberger [17, 18] and underpinning the classical theory of (φ, Γ)-modules. In this generalization, every field K of characteristic 0 which is complete for a nonarchimedean absolute value and "sufficiently large" (i.e., *perfectoid*) has associated with it a corresponding field $K^\flat$ of characteristic p (the *tilt* of K) with a canonically isomorphic Galois group. Following the model of the *almost* *purity theorem* of Faltings, one then spreads out this correspondence to obtain a similar correspondence of spaces that matches up étale topoi; however, this takes place in the realm of analytic rather than algebraic geometry, specifically in Huber's category of *adic spaces*.

The term *Drinfeld's lemma* refers collectively to a statement used by Drinfeld [13, Theorem 2.1], [14, Proposition 6.1] in his study of the Langlands correspondence for GL_2 over global function fields of characteristic p, together with subsequent generalizations [31, IV.2, Théorème 4], [34, Lemma 8.1.2], [42, Theorem 17.2.4], [24, Theorem 4.2.12]. These results address the behavior of (profinite) étale fundamental groups of schemes under formation of products, and in particular the significant discrepancy between this behavior in characteristic 0 and in characteristic p. In characteristic 0, étale fundamental groups are the profinite completions of topological fundamental groups, and so their formation commutes with taking fiber products over an algebraically closed field. By contrast, it is easy to construct examples in characteristic p where this commutativity fails; however, one obtains a similar statement by taking fiber products over $\mathbb{F}_p$, but with all of the objects divided by Frobenius (in the natural stack-theoretic sense).

It was first observed by Scholze [42, Lecture 17] that Drinfeld's lemma might be related to the geometric simple connectivity of Fargues–Fontaine curves. These appear in [15] as geometric objects whose vector bundles are closely related to p-adic Galois representations and (φ, Γ)-modules. The geometric simple connectivity property of the "basic" Fargues–Fontaine curve (the one associated to a completed algebraic closure of $\mathbb{Q}_p$) was established independently by Fargues–Fontaine [15, Chapter 8] and Weinstein [45]; this has subsequently been extended to the curves associated to arbitrary algebraically closed perfectoid fields by the second author [26]. This result may be interpreted as the analogue of Drinfeld's lemma for the product of two geometric points; by emulating some of the steps in the case of schemes, one obtains a full analogue of Drinfeld's lemma for perfectoid spaces [24, Theorem 4.3.14]. (It is natural to state the latter result in Scholze's language of *diamonds* [42]; we do so here, but diamonds are not essential for our present work.)

Using Drinfeld's lemma for perfectoid spaces, it is almost but not entirely straightforward to recover the multivariate analogue of Fontaine's original construction of (φ, Γ)-modules. The one difficulty is that taking a product of perfectoid fields in the sense of Drinfeld's lemma does not quite yield the adic space associated to the base ring of multivariate (φ, Γ)-modules, but rather a large open subspace thereof. To bridge the gap, we need an argument about bounded functions on certain non-quasicompact perfectoid spaces, which can be viewed as an application of the *perfectoid Riemann extension theorem (Hebbarkeitssatz)* appearing in the work of Scholze [40] on torsion Galois representations associated to automorphic forms, and in the proofs of the direct summand conjecture by André [1, 2] and Bhatt [8]. With this in place, we can then exhibit the analogue of the Cherbonnier–Colmez refinement; for this, we prefer the simplified approach of [23] which avoids any use of Tate–Sen formalism.

As in [36, 47] for the case $K = \mathbb{Q}_p$, one can extend various results about univariate (φ, Γ)-modules to the multivariate case, such as Herr's description of Galois cohomology [19, 20]. While these could be derived from scratch, we instead follow the approach of Liu [35] of reduction to the case $K = \mathbb{Q}_p$ using a form of Shapiro's lemma for (φ, Γ)-modules.

1.4 Followup questions

At the end of the paper, we discuss a number of followup questions that emerge naturally from this line of investigation. One of these is to extend the correspondence between Galois representations and (φ, Γ)-modules to products of étale fundamental groups (in the sense of de Jong [12]) of rigid analytic spaces and perfectoid spaces, in the style of [27]. Another is to modify the construction to reproduce some other examples of multivariate (φ, Γ)-modules in the literature, such as those exhibited by Berger [6, 7] using Lubin–Tate towers; in particular, as suggested in [25], it may be possible to establish an analogue of Cherbonnier–Colmez via this approach.

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2 Notation

Fix a finite extension $K/\mathbb{Q}_p$, an algebraic closure K^{alg} of K, and a finite set Δ with n elements. We are interested in continuous $\mathbb{Z}_p$-representations of the group

$$G_{K,\Delta} := \prod_{\alpha \in \Delta} \mathrm{Gal}(K^{\mathrm{alg}}/K).$$

Along the way, we will also encounter the groups

$$H_{K,\Delta} := \prod_{\alpha \in \Delta} \mathrm{Gal}(K^{\mathrm{alg}}/K(\mu_{p^\infty}))$$
$$\Gamma_{K,\Delta} := \prod_{\alpha \in \Delta} \mathrm{Gal}(K(\mu_{p^\infty})/K).$$

As mentioned in the introduction, one can also handle products of Galois groups of possibly distinct finite extensions of $\mathbb{Q}_p$, but to simplify notation we postpone discussion of this case until Section 8.

2.1 The univariate case

We begin by defining the suite of rings used to describe continuous $\mathbb{Z}_p$-representations of $G_K := \mathrm{Gal}(K^{\mathrm{alg}}/K)$. This will amount to treating the case where Δ is a singleton set; we will then go back and define various product constructions to handle general Δ. See [4] for a more detailed treatment.

Notation 2.1. Let $\mathcal{O}_K$ be the valuation ring of K and k its residue field. Let $K_0 := \operatorname{Frac} W(k)$, identified with a subfield of K. Notice that the completion $\widehat{K_0(\mu_{p^\infty})}$ is a perfectoid field; let E_0 be its *tilt* in the sense of the general theory of perfectoid rings (see Subsection 3.1). For the moment, we recall that E_0 is set-theoretically the inverse limit of $\widehat{K_0(\mu_{p^\infty})}$ under the p-power map, so we may choose $\epsilon := (1, \zeta_p, \zeta_{p^2}, \ldots) \in E_0$ where ζ_{p^n} denotes a primitive p^n-th root of unity; we then have $E_0 \simeq \widehat{k(\!(\bar{\varpi})\!)^{\mathrm{perf}}}$ for $\bar{\varpi} := \epsilon - 1$.
Let $\varpi := [\epsilon] - 1 \in W(E_0)$. Let $\mathcal{O}_{\mathcal{E}_0}$ be the p-adic completion of $\mathcal{O}_{K_0}(\!(\varpi)\!) \subseteq W(E_0)$, i.e.

$$\mathcal{O}_{\mathcal{E}_0} = \left\{ \sum_{n=-\infty}^{\infty} a_n \varpi^n \,\middle|\, a_n \in \mathcal{O}_{K_0},\ a_n \to 0 \text{ as } n \to -\infty \right\}.$$

The ring $\mathcal{O}_{\mathcal{E}_0}$ is a complete discrete valuation ring, with maximal ideal generated by p and residue field $k(\!(\bar{\varpi})\!)$. Let $\mathcal{E}_0 := \operatorname{Frac} \mathcal{O}_{\mathcal{E}_0} = \mathcal{O}_{\mathcal{E}_0}[p^{-1}]$. Denote by φ the unique ring homomorphism on $\mathcal{O}_{K_0}$ lifting the absolute Frobenius on k. The rings $\mathcal{E}_0$ and $\mathcal{O}_{\mathcal{E}_0}$ have commuting, $\mathcal{O}_{K_0}$-semilinear actions of the map φ and the group Γ_{K_0}, defined by

$$\varphi(\varpi) = (1+\varpi)^p - 1 \qquad \text{and} \qquad \gamma(\varpi) = (1+\varpi)^\gamma - 1,$$

where we identify an element $\gamma \in \Gamma_{K_0}$ with an element of $\mathbb{Z}_p^\times$ via the cyclotomic character. (The action of φ on coefficients is via the Witt vector Frobenius; the action of Γ_{K_0} on coefficients is trivial.)
Let $\widetilde{\mathcal{O}}_{\mathcal{E}_0} := W(E_0)$; we may represent elements of $\widetilde{\mathcal{O}}_{\mathcal{E}_0}$ as series

$$\sum_{n \in \mathbb{Z}[1/p]} a_n [\bar{\varpi}]^n$$

with coefficients $a_n \in \mathcal{O}_{K_0}$ such that $a_n \to 0$ as $n \to -\infty$ and, for each $c > 0$ and $r > 0$, there are at most finitely many coefficients a_n with $|a_n|_p \geq c$ and $n \leq r$, since the ring of all such series satisfies the universal property for Witt vectors: it is p-adically complete and separated, and its residue field consists of elements

$$\sum_{n \in \mathbb{Z}[1/p]} \bar{a}_n \bar{\varpi}^n$$

such that for each $r > 0$ there are at most finitely many nonzero $\bar{a}_n$ with $n < r$; such elements constitute the ring E_0. The point is that we have a natural inclusion $\mathcal{O}_{\mathcal{E}_0} \subseteq \widetilde{\mathcal{O}}_{\mathcal{E}_0}$ (in fact, $\widetilde{\mathcal{O}}_{\mathcal{E}_0}$ is the completion of $\mathcal{O}_{\mathcal{E}_0}$ with respect to the *weak topology*, defined below), but note that $\varpi \neq [\bar{\varpi}]$. Let $\widetilde{\mathcal{E}}_0 := \operatorname{Frac} \widetilde{\mathcal{O}}_{\mathcal{E}_0} = \widetilde{\mathcal{O}}_{\mathcal{E}_0}[p^{-1}]$. (Note the typographical convention whereby we write $\widetilde{\mathcal{O}}_{\mathcal{E}}$ instead of the more logical $\mathcal{O}_{\widetilde{\mathcal{E}}}$.)

So far, everything we have constructed depends only on K_0. We now introduce corresponding constructions depending on K, for which the previous constructions amount to the special case $K_0 = K$.

Notation 2.2. By the properties of the tilting construction, we have canonical isomorphisms

$$H_{K_0} \simeq \operatorname{Gal}(E_0^{\mathrm{sep}}/E_0) \simeq \operatorname{Gal}(\mathcal{E}_0^{\mathrm{nr}}/\mathcal{E}_0) \simeq \operatorname{Gal}(\widetilde{\mathcal{E}}_0^{\mathrm{nr}}/\widetilde{\mathcal{E}}_0),$$

where $\mathcal{E}_0^{\mathrm{nr}}$ and $\widetilde{\mathcal{E}}_0^{\mathrm{nr}}$ denote the maximal unramified extensions of $\mathcal{E}_0$ and $\widetilde{\mathcal{E}}_0$, respectively. Define

$$E := (E_0^{\mathrm{sep}})^{H_K}, \quad \mathcal{O}_{\mathcal{E}} := (\mathcal{O}_{\mathcal{E}_0^{\mathrm{nr}}})^{H_K}, \quad \mathcal{E} := (\mathcal{E}_0^{\mathrm{nr}})^{H_K} = \mathcal{O}_{\mathcal{E}}[p^{-1}],$$
$$\widetilde{\mathcal{O}}_{\mathcal{E}} := (\widetilde{\mathcal{O}}_{\mathcal{E}_0^{\mathrm{nr}}})^{H_K}, \quad \widetilde{\mathcal{E}} := (\widetilde{\mathcal{E}}_0^{\mathrm{nr}})^{H_K} = \widetilde{\mathcal{O}}_{\mathcal{E}}[p^{-1}],$$

so that

$$H_K \simeq \operatorname{Gal}(E^{\mathrm{sep}}/E) \simeq \operatorname{Gal}(\mathcal{E}^{\mathrm{nr}}/\mathcal{E}).$$

The rings $\mathcal{E}$, $\mathcal{O}_{\mathcal{E}}$, $\widetilde{\mathcal{E}}$, and $\widetilde{\mathcal{O}}_{\mathcal{E}}$ are stable under the actions of φ and Γ_K.
The ring $\mathcal{O}_{\mathcal{E}}$ is again a complete discrete valuation ring with maximal ideal generated by p. Its residue field is a finite separable extension of $k(\!(\bar{\varpi})\!)$, which can itself be (noncanonically) identified with $k'(\!(\bar{\varpi}_K)\!)$ where k' is the residue field of $K(\mu_{p^\infty})$. The completed perfect closure of the residue field of $\mathcal{O}_{\mathcal{E}}$ is canonically isomorphic to E, the residue field of $\widetilde{\mathcal{O}}_{\mathcal{E}}$.

Definition 2.3. A *series parameter* in $\mathcal{O}_{\mathcal{E}}$ is an element ϖ_K which maps to $\bar{\varpi}_K$ under some isomorphism $\mathcal{O}_{\mathcal{E}}/p\mathcal{O}_{\mathcal{E}} \simeq k'(\!(\bar{\varpi}_K)\!)$. (By the Cohen structure theorem, this just means that ϖ_K maps to a uniformizer of the complete discretely valued field $\mathcal{O}_{\mathcal{E}}/p\mathcal{O}_{\mathcal{E}}$.) For any such element, we may write

$$\mathcal{O}_{\mathcal{E}} = \left\{ \sum_{n=-\infty}^{\infty} a_n \varpi_K^n \,\middle|\, a_n \in W(k'),\ a_n \to 0 \text{ as } n \to -\infty \right\}.$$

An *overconvergent series parameter* in $\mathcal{O}_{\mathcal{E}}$ is a series parameter ϖ_K satisfying the following additional condition: there exists a positive integer c such that for each positive integer n, the element $[\epsilon - 1]^{cn}\varpi_K \in W(E)$ is congruent modulo p^n to some element of $W(\mathcal{O}_E)$. The existence of such a series parameter follows from the discussion in [23, Definition 2.1.4].

2.2 Product constructions

We now adapt the preceding constructions to products over the finite set Δ. In the notation, we suppress the field K for visual clarity; when it is necessary to specify K explicitly, we will write $\mathcal{E}_\Delta(K)$ in place of $\mathcal{E}_\Delta$ in all of the notations. It will also be convenient to label the elements of Δ as $\alpha_1, \dots, \alpha_n$, but nothing will depend in an essential way on this ordering.

Notation 2.4. For each $\alpha \in \Delta$, let $\widetilde{\mathcal{O}}_{\mathcal{E}_\alpha}$ be a copy of $\widetilde{\mathcal{O}}_{\mathcal{E}}$, let $\mathcal{O}_{\mathcal{E}_\alpha}$ denote the corresponding copy of $\mathcal{O}_{\mathcal{E}}$ inside $\widetilde{\mathcal{O}}_{\mathcal{E}_\alpha}$, and let ϖ_α be an overconvergent series parameter in $\mathcal{O}_{\mathcal{E}_\alpha}$. The choice of ϖ_α is needed in order to articulate the subsequent definitions, but again nothing will depend in an essential way on this choice. Further, let $\bar{\varpi}_\alpha$ denote the image of ϖ_α in $R_\alpha := \widetilde{\mathcal{O}}_{\mathcal{E}_\alpha}/p\widetilde{\mathcal{O}}_{\mathcal{E}_\alpha}$.
Let

$$R_\Delta = R_\Delta(K) := (R_{\alpha_1} \widehat{\otimes}_{\mathbb{F}_p} \cdots \widehat{\otimes}_{\mathbb{F}_p} R_{\alpha_n})[\bar{\varpi}_{\alpha_1}^{-1}, \dots, \bar{\varpi}_{\alpha_n}^{-1}],$$

where we identify $\bar{\varpi}_{\alpha_1}$ with $\bar{\varpi}_{\alpha_1} \otimes 1 \otimes \cdots \otimes 1$, etc., and the hats in the tensor product denote completion with respect to the $(\bar{\varpi}_{\alpha_1}, \dots, \bar{\varpi}_{\alpha_n})$-adic topology. Let $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta} := W(R_\Delta)$. Then

$$\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta} = \varprojlim_m (\widetilde{\mathcal{O}}_{\mathcal{E}_{\alpha_1}}/p^m \widetilde{\mathcal{O}}_{\mathcal{E}_{\alpha_1}}) \widehat{\otimes}_{\mathbb{Z}_p} \cdots \widehat{\otimes}_{\mathbb{Z}_p} (\widetilde{\mathcal{O}}_{\mathcal{E}_{\alpha_n}}/p^m \widetilde{\mathcal{O}}_{\mathcal{E}_{\alpha_n}})$$

where similarly the hats denote completion with respect to the $(\varpi_{\alpha_1}, \dots, \varpi_{\alpha_n})$-adic topology. Define

$$\mathcal{O}_{\mathcal{E}_\Delta} := \varprojlim_m (\mathcal{O}_{\mathcal{E}_{\alpha_1}}/p^m \mathcal{O}_{\mathcal{E}_{\alpha_1}}) \widehat{\otimes}_{\mathbb{Z}_p} \cdots \widehat{\otimes}_{\mathbb{Z}_p} (\mathcal{O}_{\mathcal{E}_{\alpha_n}}/p^m \mathcal{O}_{\mathcal{E}_{\alpha_n}}),$$

viewed as a subring of $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta}$.
For any of the above rings, let φ_α and $\Gamma_{K,\alpha}$ denote the actions of φ and Γ_K on the factor indexed by α in the product, fixing the other factors. Denote by φ_Δ the monoid generated by the φ_α for $\alpha \in \Delta$.

Remark 2.5. The ring $\mathcal{O}_{\mathcal{E}_\Delta}$ is noetherian, but the ring $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta}$ is not (because R_Δ is not).

We define a family of "Gauss norms" on $\mathcal{O}_{\mathcal{E}_\Delta}$ as follows.

Notation 2.6. Let e be the ramification index of $K(\mu_{p^\infty})$ over $K_0(\mu_{p^\infty})$ (or equivalently, of E over E_0). For $j = 1, \dots, n$ and real number $r > 0$, define the submultiplicative norm $|\ |_{j,r}$ on $\mathcal{O}_{\mathcal{E}_\Delta}$ by

$$\left| \sum_{i_1,\dots,i_n} (a_{i_1} \varpi_{\alpha_1}^{i_1}) \otimes \cdots \otimes (a_{i_n} \varpi_{\alpha_n}^{i_n}) \right|_{j,r} = \sup_{i_1,\dots,i_n} \{ p^{-r i_j p/(e(p-1))} |a_{i_1} \cdots a_{i_n}|_p \}.$$

For $r > 0$, define the submultiplicative norm $|\ |_r$ on $\mathcal{O}_{\mathcal{E}_\Delta}$ by

$$\left| \sum_{i_1,\dots,i_n} (a_{i_1} \varpi_{\alpha_1}^{i_1}) \otimes \cdots \otimes (a_{i_n} \varpi_{\alpha_n}^{i_n}) \right|_r = \sup_{i_1,\dots,i_n} \{ p^{-rp \min\{i_j\}/(e(p-1))} |a_{i_1} \cdots a_{i_n}|_p \}.$$

Let $\mathcal{O}_{\mathcal{E}_\Delta}^{j,r-}$ denote those $Y \in \mathcal{O}_{\mathcal{E}_\Delta}$ with $|Y|_{j,r}$ finite, and let $\mathcal{O}_{\mathcal{E}_\Delta}^{r-}$ denote those $Y \in \mathcal{O}_{\mathcal{E}_\Delta}$ with $|Y|_r$ finite. Notice that $|Y|_r = \max_j |Y|_{j,r}$; in particular, $Y \in \mathcal{O}_{\mathcal{E}_\Delta}^{r-}$ if and only if $|Y|_{j,r}$ is finite for each j.

For $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta}$, the following related construction is more useful.

Notation 2.7. For $j = 1, \dots, n$, define the submultiplicative norm $|\ |'_j$ on R_Δ by

$$\left| \sum_{i_1,\dots,i_n} (\bar{a}_{i_1} \bar{\varpi}_{\alpha_1}^{i_1}) \otimes \cdots \otimes (\bar{a}_{i_n} \bar{\varpi}_{\alpha_n}^{i_n}) \right|'_j = (p^{-p/(e(p-1))})^{\min\{i_j \,|\, \bar{a}_{i_1} \cdots \bar{a}_{i_n} \neq 0\}}$$

and define $|\ |' := \max_j \{ |\ |'_j \}$. For $r > 0$, define submultiplicative norms $|\ |_{j,r}$ and $|\ |_r$ on $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta}$ as follows: for $x = \sum_{m=0}^\infty p^m [\bar{x}_m] \in \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta}$, set

$$|x|_{j,r} = \sup_m \{ p^{-m} |\bar{x}_m|_j^{\prime r} \}, \quad |x|_r = \sup_m \{ p^{-m} |\bar{x}_m|^{\prime r} \}.$$

In the sequel we equip R_Δ with the topology induced by $|\ |'$ which we call the *perfectoid topology* (see 4.3). Note that R_Δ is complete with respect to $|\ |'$, but the induced topology is coarser than the $(\varpi_{\alpha_1}, \dots, \varpi_{\alpha_n})$-adic topology.

Similarly, the rings $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta}$ and $\mathcal{O}_{\mathcal{E}_\Delta}$ are complete with respect to each of the following topologies:

- the p-adic topology;
- the *weak topology*: the inverse limit topology induced by the $(\varpi_{\alpha_1}, \dots, \varpi_{\alpha_n})$-adic topology modulo each power of p;
- the *perfectoid topology*: the inverse limit topology induced by the perfectoid topology modulo each power of p.

Proposition 2.8. For $Y \in \bigcup_{r>0} \mathcal{O}_{\mathcal{E}_\Delta}^{r-}$, let $\bar{Y}$ denote the reduction of Y modulo p. If $\bar{Y} \neq 0$, we have

$$\lim_{r \to 0^+} |Y|_r = 1.$$

If $\bar{Y} = 0$, then

$$\limsup_{r \to 0^+} |Y|_r \leq p^{-1}.$$

These statements also hold with $|\ |_r$ replaced with $|\ |_{j,r}$ and/or with $\mathcal{O}$ replaced by $\widetilde{\mathcal{O}}$.

Proof. See [23, Remark 1.7.3]. □

Proposition 2.9. For $j = 1, \dots, n$, for r sufficiently small (depending on the choice of ϖ_{α_j}), the definitions of $|\,|_{j,r}$ on $\mathcal{O}_{\mathcal{E}_\Delta}$ and $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta}$ agree. Consequently, for r sufficiently small (depending on the choices of all of the ϖ_α) the definitions of $|\,|_r$ on $\mathcal{O}_{\mathcal{E}_\Delta}$ and $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta}$ agree.

Proof. See [23, Remark 2.2.8]. □

Notation 2.10. We may now define the following rings:

$$\mathcal{O}^{j,\dagger}_{\mathcal{E}_\Delta} := \bigcup_{r>0} \mathcal{O}^{j,r-}_{\mathcal{E}_\Delta} \qquad \mathcal{O}^{\dagger}_{\mathcal{E}_\Delta} := \bigcup_{r>0} \mathcal{O}^{r-}_{\mathcal{E}_\Delta}$$
$$\widetilde{\mathcal{O}}^{j,\dagger}_{\mathcal{E}_\Delta} := \bigcup_{r>0} \widetilde{\mathcal{O}}^{j,r-}_{\mathcal{E}_\Delta} \qquad \widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E}_\Delta} := \bigcup_{r>0} \widetilde{\mathcal{O}}^{r-}_{\mathcal{E}_\Delta}.$$

We also define $\mathcal{E}^\dagger_\Delta := \mathcal{O}^\dagger_{\mathcal{E}_\Delta}[p^{-1}]$ and $\widetilde{\mathcal{E}}^\dagger_\Delta := \widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta}[p^{-1}]$. (Note again the typographical choice to write $\mathcal{O}^\dagger_{\mathcal{E}}$ instead of the more logical $\mathcal{O}_{\mathcal{E}^\dagger}$, and so on.) The rings $\mathcal{O}^\dagger_{\mathcal{E}_\Delta}$, $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta}$, $\mathcal{E}^\dagger_\Delta$, and $\widetilde{\mathcal{E}}^\dagger_\Delta$ are preserved by the actions of φ_Δ and $\Gamma_{K,\Delta}$. By Proposition 2.9, within $\widetilde{\mathcal{E}}_\Delta$ we have the equalities

$$\mathcal{O}_{\mathcal{E}_\Delta} \cap \widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta} = \mathcal{O}^\dagger_{\mathcal{E}_\Delta}, \quad \mathcal{E}_\Delta \cap \widetilde{\mathcal{E}}^\dagger_\Delta = \mathcal{E}^\dagger_\Delta.$$

2.3 (φ, Γ)-modules

We now give the definition of (φ, Γ)-modules over the various rings we have constructed; this is formally similar to the univariate case.

Definition 2.11. Let $\mathcal{O}$ be a ring with commuting actions of φ_Δ and $\Gamma_{K,\Delta}$ (such as $\mathcal{O}_{\mathcal{E}_\Delta}$ or $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta}$). A *$\varphi_\Delta$-module* over $\mathcal{O}$ is a finitely presented $\mathcal{O}$-module with commuting, semilinear actions of the φ_α. A *$(\varphi_\Delta, \Gamma_{K,\Delta})$-module* over $\mathcal{O}$ is a finitely presented $\mathcal{O}$-module M with commuting semilinear actions of the φ_α and the $\Gamma_{K,\alpha}$. We apply additional ring-theoretic modifiers (such as “torsion” or “projective”) by passing them through to the underlying $\mathcal{O}$-module.

Definition 2.12. Let $\mathcal{O}$ be one of the rings $\mathcal{O}_{\mathcal{E}_\Delta}$, $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta}$, $\mathcal{O}^\dagger_{\mathcal{E}_\Delta}$, or $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta}$. A φ_Δ-module M over $\mathcal{O}$ is *étale* if the induced maps

$$\varphi^*_\alpha M \to M, \qquad a \otimes x \mapsto a\varphi_\alpha(x)$$

are isomorphisms for all $\alpha \in \Delta$; here $\varphi^*_\alpha M$ denotes the module $\mathcal{O} \otimes_{\varphi,\mathcal{O}} M$, in which $a \otimes bx = a\varphi_\alpha(b) \otimes x$ for $a, b \in \mathcal{O}$, $x \in M$. In the case when M is a free module, this condition holds if and only if for each α, φ_α maps some basis of M to another basis of M; it then maps every basis of M to another basis. A $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over $\mathcal{O}$ is *étale* if its underlying φ_Δ-module is étale.

DEFINITION 2.13. Let $\mathcal{O}$ be a ring of the form $\mathcal{O}_0[p^{-1}]$ where $\mathcal{O}_0$ is one of the rings $\mathcal{O}_{\mathcal{E}_\Delta}$, $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta}$, $\mathcal{O}^\dagger_{\mathcal{E}_\Delta}$, or $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta}$; that is, $\mathcal{O}$ is one of the rings $\mathcal{E}_\Delta$, $\widetilde{\mathcal{E}}_\Delta$, $\mathcal{E}^\dagger_\Delta$, or $\widetilde{\mathcal{E}}^\dagger_\Delta$. A φ_Δ-module or $(\varphi_\Delta,\Gamma_{K,\Delta})$-module M over $\mathcal{O}$ is *étale* if it has the form $M_0[p^{-1}]$ for some projective étale φ_Δ-module or $(\varphi_\Delta,\Gamma_{K,\Delta})$-module M_0 over $\mathcal{O}_0$; in particular, in this setup we are insisting that M be projective.

REMARK 2.14. We point out two subtleties in the previous definitions which do not have much impact on our work here, but may become relevant when comparing with other literature. On the one hand, in Definition 2.13, our definition of étale $(\varphi_\Delta,\Gamma_{K,\Delta})$-modules imposes the étale condition not just on the action of φ_Δ, but also on $\Gamma_{K,\Delta}$. In some settings, it might be preferable to have a definition in which the étale condition is described solely in terms of φ_Δ.
On the other hand, to avoid imposing the étale condition on $\Gamma_{K,\Delta}$, one probably has to replace it with a condition asserting that the action map $\Gamma_{K,\Delta}\times M\to M$ is continuous for some topology on M. That topology should be induced by some topology on the base ring $\mathcal{O}$ for which the action of $\Gamma_{K,\Delta}$ is itself continuous (e.g., the p-adic topology or the weak topology); since M appears in both the source and target of the action map, the continuity conditions for different topologies on $\mathcal{O}$ are not immediately comparable even if the topologies themselves are comparable.
In our setup, for étale $(\varphi_\Delta,\Gamma_{K,\Delta})$-modules, any continuity condition of this form will follow *a posteriori* from the comparison between these objects and representations of $G_{K,\Delta}$, and so no such condition needs to be included in either definitions or theorem statements. For an example of the tradeoff when we try to weaken the étale condition, see Theorem 6.19.

REMARK 2.15. Suppose M and N are étale φ_Δ-modules (or étale $(\varphi_\Delta,\Gamma_{K,\Delta})$-modules) over $\mathcal{O}$. Then provided that $\mathrm{Hom}_{\mathcal{O}}(M,N)$ is finitely presented, we may view it as an étale φ_Δ-module (or étale $(\varphi_\Delta,\Gamma_{K,\Delta})$-module) by requiring, as appropriate,

$$\varphi_\alpha(f)(\varphi_\alpha(\mathbf{e}))=\varphi_\alpha(f(\mathbf{e}))$$
$$\gamma_\alpha(f)(\gamma_\alpha(\mathbf{e}))=\gamma_\alpha(f(\mathbf{e}))$$

for $\alpha\in\Delta$. The morphisms $M\to N$ of φ_Δ-modules (or of $(\varphi_\Delta,\Gamma_{K,\Delta})$-modules) are exactly those $\mathcal{O}$-module homomorphisms fixed by φ_Δ (or by φ_Δ and $\Gamma_{K,\Delta}$). We point out two key cases in which the finite presentation condition on $\mathrm{Hom}_{\mathcal{O}}(M,N)$ is always satisfied: the case where M is projective, and the case where $\mathcal{O}$ is noetherian. As noted above, the latter holds for $\mathcal{O}=\mathcal{O}_{\mathcal{E}_\Delta}$; it also holds for $\mathcal{O}^\dagger_{\mathcal{E}_\Delta}$, but we will not need this fact.

REMARK 2.16. In connection with the previous remark, we note that by [27, Proposition 3.2.13], a φ_Δ-module over $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta}$ or $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta}$ which is flat over $\mathbb{Z}/p^n\mathbb{Z}$ is a finite projective $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta}/p^n\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta}$-module.

3 Drinfeld's lemma for diamonds

In this section, we give a very brief summary of Drinfeld's lemma for perfectoid spaces and diamonds, following [24, Lecture 4].

3.1 Adic spaces, perfectoid spaces, and diamonds

We begin by recalling some terminology and results about adic spaces, perfectoid spaces, and diamonds. Good introductions to this material can be found in [21], [46], and [24]; see also [42].

Definition 3.1. We say that (A, A^+) is a *Huber pair* if

1. A is a *Huber ring*, i.e. it is a topological ring which contains an open subring A_0 whose topology is the I-adic topology for some finitely generated ideal I of A_0 (the ring A_0 is called a *ring of definition* and the ideal I an *ideal of definition*);
2. A^+ is a *ring of integral elements*, i.e. A^+ is an open, integrally closed subring contained in the subring A° of power-bounded elements of A.

Definition 3.2. A Huber ring A (or a Huber pair (A, A^+)) is *uniform* if the subring A° is bounded. It is *analytic* if the topologically nilpotent elements of A generate the unit ideal. It is *Tate* if A contains a *pseudo-uniformizer*, i.e. a topologically nilpotent unit.

Remark 3.3. Beware that the condition that a Huber pair (A, A^+) be uniform is only a condition on A. While it does imply that (A, A°) is also a Huber pair, it does not force the inclusion $A^+ \subseteq A^\circ$ to be an equality. On the other hand, the difference between the two is not large: the topologically nilpotent elements of A° form an ideal of A°, and this ideal is itself contained in A^+ (because A^+ is open and integrally closed).

Definition 3.4. The *adic spectrum* of a Huber pair (A, A^+) is the set $\mathrm{Spa}(A, A^+)$ of equivalence classes of continuous valuations v on A such that $v(f) \leq 1$ for all $f \in A^+$.

Definition 3.5 ([24, Definition 1.2.1]). Given a Huber pair (A, A^+), a *rational subspace* of $X = \mathrm{Spa}(A, A^+)$ is a set of the form

$$X\left(\frac{f_1, \dots, f_n}{g}\right) := \{v \in X \,|\, v(f_i) \leq v(g) \neq 0 \;\forall\, i\}$$

for some collection of elements $f_1, \dots, f_n, g$ which generate an open ideal in A. Rational subspaces provide a basis for a topology on X. The *rational localization* corresponding to a rational subspace Y of X is a morphism $(A, A^+) \to (B, B^+)$ of complete Huber pairs which is initial among morphisms for which $\mathrm{Spa}(B, B^+)$ maps into Y. This morphism induces a homeomorphism

$\mathrm{Spa}(B, B^+) \simeq Y$ which moreover identifies rational subspaces of $\mathrm{Spa}(B, B^+)$ with rational subspaces of X contained in Y.
In the case that $\{f_1, \dots, f_n, g\}$ generates the unit ideal, B is canonically isomorphic to the quotient of $A\langle T_1, \dots, T_n\rangle$ by the closure of the ideal generated by $\{gT_1 - f_1, \dots, gT_n - f_n\}$. This is always the case when A is analytic. (Otherwise, one must also invert g.)

Definition 3.6. Let (A, A^+) be a Huber pair. We define the *structure presheaf* of $X := \mathrm{Spa}(A, A^+)$ as follows: If $U \subseteq X$ is an open subspace, let

$$\mathcal{O}(U) := \varprojlim B$$
$$\mathcal{O}^+(U) := \varprojlim B^+$$

where the limits run over rational localizations $(A, A^+) \to (B, B^+)$ such that $\mathrm{Spa}(B, B^+) \subseteq U$. In particular, if $U = \mathrm{Spa}(B, B^+)$, then $(\mathcal{O}(U), \mathcal{O}^+(U)) = (B, B^+)$. The structure presheaf has the property that

$$\mathcal{O}^+(U) = \{f \in \mathcal{O}(U) \,|\, v(f) \le 1 \ \forall \ v \in U\}.$$

A Huber pair (A, A^+) is *sheafy* if $\mathcal{O}$ is a sheaf; in this case $\mathcal{O}^+$ is also.
In general, the sheafiness condition in a Huber pair is highly nontrivial, which causes severe complications in setting up the theory of adic spaces. Fortunately, for perfectoid spaces this complication disappears; see Remark 3.10.

Definition 3.7. An *adic space* is a topological space X together with a sheaf of topological rings $\mathcal{O}_X$ and a continuous valuation on the stalk $\mathcal{O}_{X,x}$ for each $x \in X$, such that X can be covered by open subsets of the form $\mathrm{Spa}(A, A^+)$ where each (A, A^+) is a sheafy Huber pair. In particular, for any identification of an open subset of X with a space $\mathrm{Spa}(A, A^+)$, and a corresponding identification of $\mathcal{O}_X$ with the structure sheaf on $\mathrm{Spa}(A, A^+)$, the valuation on $\mathcal{O}_{X,x}$ is the one whose valuation ring is the stalk of $\mathcal{O}_X^+$ at x.

Definition 3.8. A *perfectoid ring* is a uniform, analytic Huber ring A which contains a ring A^+ of integral elements (which is then a ring of definition) and an ideal of definition $I \subseteq A^+$ such that $p \in I^p$ and the p-th power map $A^+/I \to A^+/I^p$ is surjective. A *perfectoid pair* is a Huber pair (A, A^+) with A perfectoid.

Remark 3.9. An important special case is when A is an analytic Huber ring of characteristic p which is perfect (that is, its Frobenius endomorphism is bijective). In this case, A is automatically uniform (see [24, Example 2.1.2]) and hence a perfectoid ring.

Remark 3.10 ([46, Theorem 3.1.3], [24, Theorem 2.5.3]). Let (A, A^+) be a perfectoid pair. Then

1. (A, A^+) is sheafy;

2. $X := \mathrm{Spa}(A, A^+)$ is an adic space;
3. $\mathcal{O}_X(U)$ is a perfectoid ring for all rational subsets $U \subseteq X$.

Theorem 3.11 (Tilting Correspondence, [24, Theorem 2.3.9]). There is an equivalence of categories

$$(A, A^+) \mapsto (A^\flat, A^{\flat+}, I) = \left(\varprojlim_{x^p \leftarrow x} A, \varprojlim_{x^p \leftarrow x} A^+, \ker\left(W(A^{\flat+}) \to A^+\right)\right)$$
$$\left(W^b(R)/IW^b(R), W(R^+)/I\right) \leftarrow (R, R^+, I)$$

between the category of perfectoid pairs (A, A^+) and the category of characteristic p perfectoid pairs (R, R^+) together with a primitive ideal I of $W(R^+)$. Here $W(R^+)$ denotes the ring of p-typical Witt vectors over R^+, and $W^b(R)$ denotes the ring of p-typical Witt vectors

$$\sum_{n=0}^{\infty} p^n[\bar{x}_n]$$

such that the set $\{\bar{x}_n \mid n \in \mathbb{N}\}$ is bounded in R. An ideal of $W(R^+)$ is *primitive* if it is principal on some generator $z = \sum_{n=0}^{\infty} p^n[\bar{z}_n]$ for which $\bar{z}_0$ is topologically nilpotent and $\bar{z}_1$ is a unit in R^+.

Theorem 3.12 ([23, Theorem 1.5.6]). If L is a perfectoid field, then the absolute Galois groups of L and $L^\flat$ are isomorphic as topological groups.

Theorem 3.13 ([24, Theorem 2.5.1]). Given a perfectoid pair (A, A^+), there is a homeomorphism

$$\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(A^\flat, A^{\flat+})$$
$$v \mapsto v^\flat,$$

where $v^\flat((f_n)_n) = v(f_0)$. If $f = (f_n)_n \in A^\flat$, we define $f^\sharp := f_0 \in A$; then we have $v^\flat(f) = v(f^\sharp)$.

Definition 3.14. A *perfectoid space* is an adic space which is covered by open subspaces of the form $\mathrm{Spa}(A, A^+)$ with A perfectoid. Any such subspace is called an *affinoid perfectoid space*.

Definition 3.15. A morphism $(R, R^+) \to (S, S^+)$ of Huber pairs is *finite étale* if S is a finite étale R-algebra with the induced topology and S^+ is the integral closure of R^+ in S.
A morphism $f\colon X \to Y$ of adic spaces is *finite étale* if there is a cover of Y by open affinoids $V \subseteq Y$ such that the pre-image $U = f^{-1}(V)$ is affinoid and the associated morphism of Huber pairs

$$(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$$

is finite étale.

A morphism $f\colon X \to Y$ of adic spaces is *étale* if for all points $x \in X$, there exist open neighborhoods U of x and V of $f(x)$ and a commutative diagram

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "f|_U"'] & W \arrow[d, "p"] \\ & V \end{tikzcd}$$

with j an open embedding and p finite étale.

A morphism $f\colon X \to Y$ of perfectoid spaces is *pro-étale* if locally on X it is of the form $\mathrm{Spa}(A_\infty, A_\infty^+) \to \mathrm{Spa}(A, A^+)$, where A and A_∞ are perfectoid rings, and

$$(A_\infty, A_\infty^+) = \left[\varinjlim (A_i, A_i^+)\right]^\wedge$$

is a filtered colimit of pairs (A_i, A_i^+) such that $\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A, A^+)$ is étale.

Definition 3.16. Let **Perf** denote the category of perfectoid spaces of characteristic p. We hereafter view **Perf** as a site using the *pro-étale topology*, whose coverings (*pro-étale coverings*) are collections of morphisms $\{f_i\colon X_i \to X \mid i \in I\}$ such that each f_i is pro-étale and for all quasi-compact open $U \subseteq X$, there exists a collection of quasi-compact open sets $U_i \subseteq X_i$ indexed by a finite subset $I_U \subseteq I$ such that

$$U = \bigcup_{i \in I_U} f_i(U_i).$$

The pro-étale topology is subcanonical; that is, for $X \in \mathbf{Perf}$, the functor h_X on **Perf** represented by X is a sheaf (see Theorem 3.20).

Definition 3.17. A morphism of sheaves $\mathcal{F} \to \mathcal{G}$ on **Perf** is *pro-étale* if for all perfectoid spaces X and maps $h_X \to \mathcal{G}$, the pullback $h_X \times_{\mathcal{G}} \mathcal{F}$ is representable by a perfectoid space Y, and the morphism $Y \to X$ corresponding to the map $h_Y = h_X \times_{\mathcal{G}} \mathcal{F} \to h_X$ is pro-étale.

Let $\mathcal{F}$ be a sheaf on **Perf**. A *pro-étale equivalence relation* is a monomorphism $\mathcal{R} \to \mathcal{F} \times \mathcal{F}$ in the category of sheaves on **Perf** such that each projection $\mathcal{R} \to \mathcal{F}$ is pro-étale, and such that for all objects S of **Perf**, the image of the map $\mathcal{R}(S) \to \mathcal{F}(S) \times \mathcal{F}(S)$ is an equivalence relation on $\mathcal{F}(S)$.

Definition 3.18. A *diamond* is a sheaf $\mathcal{F}$ on **Perf** which is the quotient of a perfectoid space by a pro-étale equivalence relation. More precisely, there exist a perfectoid space X and a representable equivalence relation $\mathcal{R} \to h_X \times h_X$ such that the two projections $\mathcal{R} \to h_X$ are pro-étale. (Compare [41, Definition 11.1].)

Definition 3.19. 1. Given a perfectoid space X, we denote by $X^\diamond$ the representable sheaf $h_{X^\flat}$.

2. The *diamond spectrum* of a perfectoid ring A is the sheaf $\operatorname{Spd} A = (\operatorname{Spa}(A, A^\circ))^\diamond$.

Theorem 3.20 ([46, Theorem 3.5.2], [24, Theorem 3.8.2]). If X is a perfectoid space, then $X^\diamond$ is a diamond. Moreover, for any perfectoid space S, the functor $X \mapsto X^\diamond$ from perfectoid spaces over S to diamonds over $S^\diamond$ is fully faithful.

3.2 Finite étale covers and profinite fundamental groups

Notation 3.21. Let X be a scheme, a perfectoid space, or a diamond. We denote by $\mathbf{FEt}(X)$ the category of finite étale coverings of X. (For X a diamond, a finite étale covering of X is a representable morphism $Y \to X$ whose pullback to any perfectoid space is a finite étale covering.)
We use a similar notation for formal quotients by group actions: if we take some X and formally quotient by a group Γ of automorphisms of X, then $\mathbf{FEt}(X/\Gamma)$ is the category of objects of $\mathbf{FEt}(X)$ equipped with an action of Γ for which the structure morphism is Γ-equivariant.

Notation 3.22. Let $X_1, \dots, X_n$ be diamonds, and let $X = X_1 \times \cdots \times X_n$. Let φ_i denote the absolute Frobenius of X_i (induced by the p-th power map on rings), and let $\varphi = \varphi_1 \times \cdots \times \varphi_n$ be the absolute Frobenius of X. As per [24, Remark 4.2.14], let X/Φ be the functor from perfectoid spaces to sets taking Y to the set of tuples $(f, \beta_1, \dots, \beta_n)$ where $f : Y \to X$ is a morphism and $\beta_i : Y \to \varphi_i^* Y$ are morphisms which "commute and compose to φ" in the sense of [24, Definition 4.2.10]. That is, for any i and j, the composition

$$Y \xrightarrow{\beta_i} \varphi_i^* Y \xrightarrow{\beta_j} \varphi_i^* \varphi_j^* Y = (\varphi_i \circ \varphi_j)^* Y = (\varphi_j \circ \varphi_i)^* Y$$

is the same as the corresponding composition with i and j reversed; and the composition

$$Y \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_n} (\varphi_1 \circ \cdots \circ \varphi_n)^* Y = \varphi^* Y$$

is relative Frobenius for Y/X.
A more concrete, but less symmetric, description of X/Φ can be given by picking an index $j \in \{1, \dots, n\}$; one then has a canonical isomorphism

$$X/\Phi \cong X/\langle \varphi_1, \dots, \widehat{\varphi_j}, \dots, \varphi_n \rangle$$

given by discarding β_j and instead recovering it from the other data. In other words, X/Φ is the formal quotient of X by the group Φ generated by $\langle \varphi_1, \dots, \widehat{\varphi_j}, \dots, \varphi_n \rangle$; we will refer to the group Φ on its own in various contexts where the choice of j does not matter.

Definition 3.23. Let X and Φ be as above. Given a geometric point $\bar{x}$ of X, the *profinite fundamental group* $\pi_1^{\mathrm{prof}}(X/\Phi, \bar{x})$ is the group of natural isomorphisms of the functor $\mathbf{FEt}(X/\Phi) \to \mathbf{Set}$ taking a covering Y to the underlying set of $Y \times_X \bar{x}$.

Theorem 3.24 ([24, Remark 4.1.4]). Suppose that F is a perfectoid field of characteristic p, and let F^{alg} be an algebraic closure of F. Then

$$G_F = \mathrm{Gal}(F^{\mathrm{alg}}/F) \simeq \pi_1^{\mathrm{prof}}(X, \bar{x})$$

for $X := \mathrm{Spd}(F)$ and $\bar{x} := \mathrm{Spd}(F^{\mathrm{alg}})$.

Theorem 3.25 (Drinfeld's lemma, [24, Theorem 4.3.14]). Let $X_1, \ldots, X_n$ be connected spatial (in the sense of [42, Definition 17.3.1]) diamonds and put $X := X_1 \times \cdots \times X_n$. Then X/Φ is a connected (that is, X admits no Φ-invariant disconnection) spatial diamond, and for any geometric point $\bar{x}$ of X, the map

$$\pi_1^{\mathrm{prof}}(X/\Phi, \bar{x}) \to \prod_{i=1}^{n} \pi_1^{\mathrm{prof}}(X_i, \bar{x})$$

is an isomorphism of profinite groups.

Remark 3.26. The formulation of Theorem 3.25 above uses the language of diamonds, but this is not strictly necessary for our purposes: we will be interested exclusively in the case where $X_1, \ldots, X_n$ are perfectoid spaces, in which case so are X (compare 4.23) and X/Φ (because Φ acts properly discontinuously on X; compare [24, Corollary 4.3.16]). We will see an explicit example of this phenomenon in Subsection 4.3.

4 (φ, Γ)-modules and representations

In this section, we use Drinfeld's lemma for diamonds to make an initial link between multivariate (φ, Γ)-modules and representations of $G_{K,\Delta}$, culminating in the following result.

Theorem 4.1 (see Theorem 4.29, Theorem 4.39). The category of continuous $\mathbb{Z}_p$-representations of $G_{K,\Delta}$ is equivalent to the category of étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules over either $\mathcal{O}_{\mathcal{E}_\Delta(K)}$ or $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$. Moreover, these equivalences are exact.

This will then be refined in later sections to include the rings $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$ and $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$.

4.1 Product constructions for Huber rings

We start by adapting the construction of the ring R_Δ (Notation 2.4) to form topological products of arbitrary perfectoid rings.

Definition 4.2. For (R, R^+) a perfect Huber pair of characteristic p, an R^+-module is said to be *almost zero* if it is annihilated by every topologically nilpotent element of R^+. For any fixed pseudo-uniformizer ϖ of R, it is sufficient to require the module to be annihilated by $\varpi^{p^{-n}}$ for all nonnegative integers n.
For example, since R is perfectoid (Remark 3.9), (R, R°) is also a Huber pair (Remark 3.3), but the inclusion $R^+ \subseteq R^\circ$ need not be an equality. However, the quotient R°/R^+ is almost zero.

Notation 4.3. For $i = 1, \dots, n$, let (R_i, R_i^+) be a perfect Huber pair of characteristic p. For $i = 1, \dots, n$, choose a pseudo-uniformizer ϖ_i of R_i and define

$$\begin{aligned} \varpi &:= \varpi_1 \cdots \varpi_n \\ R_0 &:= (R_1^+ \otimes_{\mathbb{F}_p} \cdots \otimes_{\mathbb{F}_p} R_n^+)^\wedge_{(\varpi_1, \dots, \varpi_n)} \\ R &:= R_0[\varpi^{-1}]. \end{aligned}$$

Note that R_0 is perfect; it is also complete for both the $(\varpi_1, \dots, \varpi_n)$-adic topology and for the finer $(\varpi_1 \cdots \varpi_n)$-adic topology. By equipping R_0 with the latter topology, we may give R the structure of a perfect Huber ring containing R_0 as an open subring; by Remark 3.9, R is also uniform and hence perfectoid. The choice of the ϖ_i has no ultimate effect on either R_0 or R; moreover, R does not depend (either algebraically or topologically) on the choice of the R_i^+ within R_i. We refer to R as the *completed tensor product* of $R_1, \dots, R_n$. As for ordinary tensor products, any continuous endomorphism of R_i extends naturally to R so as to fix R_j for $j \neq i$.
In the setting of Notation 2.4, R_Δ with the perfectoid topology coincides with the completed tensor product of $R_{\alpha_1}, \dots, R_{\alpha_n}$.

In Notation 4.3, it is not immediately clear that R_0 is an integrally closed subring of R, which would then ensure that (R, R_0) is a perfectoid Huber pair. For our purposes, it will be sufficient to check something slightly weaker.

Lemma 4.4. In Notation 4.3, the quotient R°/R_0 is killed by $(\varpi_1 \cdots \varpi_n)^{p^{-m}}$ for every nonnegative integer m.

Proof. If $x \in R^\circ$, then by definition the sequence $\{x^{p^m}\}_m$ is topologically bounded. Since R_0 is open, there exists a single nonnegative integer h such that $(\varpi_1 \cdots \varpi_n)^h x^{p^m} \in R_0$ for all $m \geq 0$. But since R_0 is perfect, this implies that $(\varpi_1 \cdots \varpi_n)^{hp^{-m}} x \in R_0$ for all $m \geq 0$. Since $hp^{-m} \to 0$ as $m \to \infty$, this implies the claim. $\square$

4.2 Mod-p representations: full faithfulness

As in the usual theory of (φ, Γ)-modules, the desired statement about $\mathbb{Z}_p$-representations will be deduced from a corresponding statement about torsion representations, and most of the key ideas appear already in the study of $\mathbb{F}_p$-representations. We correspondingly start by proving the following theorem, which will occupy us for the entirety of this and the next two subsections.

Theorem 4.5. The category of continuous $\mathbb{F}_p$-representations of $G_{K,\Delta}$ is equivalent to the category of étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules over $R_\Delta = \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}/p\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$.

In this subsection, we formulate a more general version of this statement (Theorem 4.6), then in that context produce a fully faithful functor from Galois representations to (φ, Γ)-modules (Proposition 4.11). Essential surjectivity of this functor will be established in Subsection 4.3, modulo a technical result about perfectoid spaces which we defer to Subsection 4.4.
We make the following observations:

1. The completion of the field $K(\mu_{p^\infty})$ is perfectoid with tilt E (see Notation 2.2).

2. The absolute Galois groups of $K(\mu_{p^\infty})$, its completion, and E are isomorphic.

3. The action of $G_{K(\mu_{p^\infty})} \simeq G_E$ on E^{alg} extends to an action of G_K by functoriality of tilting. This leads to an action of Γ_K on E.

Thus it suffices to establish an equivalence between the category of representations of $H_{K,\Delta}$ and the category of étale φ_Δ-modules over R_Δ, for then the action of $\Gamma_{K,\Delta}$ allows us to recover the categories of $\mathbb{F}_p$-representations of $G_{K,\Delta}$ and étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules over R_Δ. In fact we prove a slightly more general statement; the following is [24, Corollary 4.3.16], but here we fill in many details of the proof.

Theorem 4.6. Let $F_1, \dots, F_n$ be perfectoid fields of characteristic p. Let φ_{α_i} act on F_i via the absolute Frobenius. Let R be the completed tensor product of $F_1, \dots, F_n$ as per Notation 4.3. Then the functor D defined in Definition 4.9 below yields an equivalence of categories between the category of continuous $\mathbb{F}_p$-representations of

$$G_{F_\Delta} := G_{F_1} \times \cdots \times G_{F_n}$$

and the category of étale φ_Δ-modules over R (that is, finite projective R-modules M having commuting semilinear bijective actions of $\varphi_{\alpha_1}, \dots, \varphi_{\alpha_n}$).

Definition 4.7. For each i, fix a completed algebraic closure $\bar{F}_i$ of F_i and identify G_{F_i} with $\mathrm{Gal}(\bar{F}_i/F_i)$. Let $\bar{R}$ be the completed tensor product of $\bar{F}_1, \dots, \bar{F}_n$

as per Notation 4.3. The ring $\bar{R}$ has a natural action of G_{F_Δ}. Moreover, whenever $F_1 = F_2 = \cdots = F_n$ is the tilt of $K(\mu_{p^\infty})$ for some finite extension $K/\mathbb{Q}_p$, then the action of $G_{F_\Delta} \simeq H_{K,\Delta}$ extends to an action of $G_{K,\Delta}$.

Lemma 4.8. We have $\bar{R}^{G_{F_\Delta}} = R$.

Proof. Note that $\mathrm{Spa}(\bar{R}, \bar{R}^+) \to \mathrm{Spa}(R, R^+)$ is a pro-étale covering; the claim thus follows from the fact that the structure sheaf on $\mathrm{Spa}(R, R^+)$ is a sheaf also for the pro-étale topology [28, Theorem 3.5.5]. $\square$

Definition 4.9. We define a functor D which maps an $\mathbb{F}_p$-representation V of G_{F_Δ} to the R-module

$$D(V) := (V \otimes_{\mathbb{F}_p} \bar{R})^{G_{F_\Delta}},$$

where G_{F_Δ} acts diagonally on the tensor product. For each i, we define an action of φ_{α_i} on $D(V)$ by

$$\varphi_{\alpha_i}(x \otimes a) = x \otimes \varphi_{\alpha_i}(a).$$

This defines commuting semilinear bijective actions of the φ_{α_i} on $D(V)$.

Proposition 4.10. For V an $\mathbb{F}_p$-representation of G_{F_Δ}, the module $D(V)$ is a finite, projective R-module.

Proof. Fix a particular choice of representation V. Since V is a finite-dimensional $\mathbb{F}_p$-vector space, and thus a finite set, the kernel of the action of G_{F_Δ} must be an open subgroup. Thus there exist finite, Galois extensions E_i/F_i in $\bar{F}_i$ such that, for S the completed tensor product of $E_1, \dots, E_n$,

$$D(V) = (V \otimes_{\mathbb{F}_p} S)^{\mathrm{Gal}(E_1/F_1) \times \cdots \times \mathrm{Gal}(E_n/F_n)}.$$

By the Normal Basis Theorem, $E_i \otimes_{F_i} E_i$ has an F_i-basis of elements $\sigma_{ij}(e_i) \otimes \sigma_{ik}(e_i)$ for $E_i \otimes_{F_i} E_i$, where $\mathrm{Gal}(E_i/F_i) = \{\sigma_{i1}, \dots, \sigma_{in}\}$. Since the maps σ_{ij} are F_i-linear, we can assume that the element e_i has norm at most 1. By taking $e_{ij} = \sigma_{ij}(e_i)$, it follows that the elements $e_{ij} \otimes \sigma_{ik}(e_{ij})$ form a basis of $E_i \otimes_{F_i} E_i$. Letting $s_j = e_{1j} \otimes \cdots \otimes e_{nj}$, we conclude that the elements $s_j \otimes \sigma(s_j)$ form an R-basis of $S \otimes_R S$, where σ runs over the elements of

$$\mathrm{Gal}(E_\Delta/F_\Delta) = \mathrm{Gal}(E_1/F_1) \times \cdots \times \mathrm{Gal}(E_n/F_n).$$

The map given by

$$v \otimes s \otimes \sigma(s) \mapsto s \otimes \sigma(v) \otimes \sigma(s)$$

with respect to this basis is a descent datum

$$(V \otimes_{\mathbb{F}_p} S) \otimes_R S \to S \otimes_R (V \otimes_{\mathbb{F}_p} S)$$

with respect to the faithfully flat map $R \to S$ (faithfully flat since S is a finite, free R-module). It follows that

$$D(V) \otimes_R S \simeq V \otimes_{\mathbb{F}_p} S,$$

and thus that $D(V)$ is a finite, projective R-module. $\square$

Proposition 4.11. The functor D is fully faithful.

Proof. Recall that for $\mathbb{F}_p$-representations V and W, the set $\mathrm{Hom}_{\mathbb{F}_p}(V, W)$ is itself an $\mathbb{F}_p$-representation of G_{F_Δ}, and by Remark 2.15, $\mathrm{Hom}_R(D(V), D(W))$ is a φ_Δ-module over R. The morphisms $V \to W$ are those elements of $\mathrm{Hom}_{\mathbb{F}_p}(V, W)$ which are fixed by G_{F_Δ}, while the morphisms $D(V) \to D(W)$ are those elements of $\mathrm{Hom}_R(D(V), D(W))$ that are fixed by φ_Δ. Using the fact that

$$D(\mathrm{Hom}_{\mathbb{F}_p}(V, W)) \simeq \mathrm{Hom}_R(D(V), D(W)),$$

we reduce the problem of showing that D is fully faithful to showing that

$$V^{G_{F_\Delta}} \simeq D(V)^{\varphi_\Delta}$$

for an arbitrary $\mathbb{F}_p$-representation V playing the role of $\mathrm{Hom}_{\mathbb{F}_p}(V, W)$. We have

$$\begin{aligned} V^{G_{F_\Delta}} &= (V \otimes_{\mathbb{F}_p} \bar{R})^{\varphi_\Delta, G_{F_\Delta}} \\ &\simeq (D(V) \otimes_R \bar{R})^{G_{F_\Delta}, \varphi_\Delta} \\ &= D(V)^{\varphi_\Delta} \end{aligned}$$

by the proof of Lemma 4.8, as desired. □

4.3 Mod-p representations: essential surjectivity

Continuing with our study of mod-p representations, we next address essential surjectivity of the functor in Theorem 4.6. Let D be a finitely presented (and hence projective by Remark 2.16) R-module with commuting semilinear bijective actions of $\varphi_{\alpha_1}, \dots, \varphi_{\alpha_n}$. Let $\varphi := \varphi_{\alpha_1} \circ \cdots \circ \varphi_{\alpha_n}$.

Remark 4.12. We have seen in the proof of Proposition 4.11 that if $D \simeq D(V)$ for some $\mathbb{F}_p$-representation V of G_{F_Δ}, then at the level of étale sheaves we have $V \simeq D^{\varphi_\Delta}$; it is thus natural to try to show that the expression on the right gives rise to a representable functor on $\mathbf{FEt}(R)$. As a first step, we consider φ-invariants instead of φ_Δ-invariants in Proposition 4.13 below.

Proposition 4.13. The functor that maps a finite étale R-algebra T to the $\mathbb{F}_p$-vector space $(D \otimes_R T)^\varphi$ is represented by a finite étale R-algebra S—that is, there is a natural isomorphism

$$(D \otimes_R T)^\varphi \simeq \mathrm{Hom}_R(S, T) \tag{4.1}$$

for all finite étale R-algebras T. Moreover, S carries natural actions of $\varphi_{\alpha_1}, \dots, \varphi_{\alpha_n}$.

Proof. We show this in analogy with [27, Lemma 3.2.6] (but correcting some errors therein). Supposing first that D is a free R-module, let $e_1, \dots, e_r$ be a basis for D. Let A be the matrix of the action of φ on D with respect to this basis; since D is étale the matrix A is invertible. We observe that each element of $D \otimes_R T$, where T is any finite étale R-algebra, can be written in the form $t_1 e_1 + \cdots + t_r e_r$ for some elements $t_i \in T$, and such an element belongs to $(D \otimes_R T)^\varphi$ if and only if

$$t_i = \sum_{j=1}^{r} t_j^p a_{ij}$$

for all i, that is, if

$$\begin{pmatrix} t_1 \\ \vdots \\ t_r \end{pmatrix} = A \begin{pmatrix} t_1^p \\ \vdots \\ t_r^p \end{pmatrix}.$$

Let $S := R[T_1, \dots, T_r]/((T_i^p)_i - A^{-1}(T_i)_i)$. Then S is a finite R-algebra of degree p^r, and the Jacobian matrix $(\partial f_i/\partial T_j)_{i,j}$ has full rank, where the f_i are the polynomials defining the ideal $((T_i^p)_i - A^{-1}(T_i)_i)$. Thus by the Jacobian criterion, S is étale over R. Since this construction is independent of the choice of basis, we can glue to obtain S in the general case.
Now each partial Frobenius φ_{α_i} induces an R-module isomorphism

$$D \otimes_{R,\varphi_{\alpha_i}} R \to D,$$

so it follows that

$$\mathrm{Hom}_R(S,T) \simeq (D \otimes_{R,\varphi_{\alpha_i}} R \otimes_R T)^\varphi \simeq (D \otimes_R (\varphi_{\alpha_i}^* T))^\varphi. \tag{4.2}$$

Combining (4.1) and (4.2), we can conclude that

$$\mathrm{Hom}_R(S, \varphi_{\alpha_i}^* T) \simeq \mathrm{Hom}_R(S,T)$$

for all étale R-algebras T, and in particular $\mathrm{Hom}_R(S, \varphi_{\alpha_i}^* S) \simeq \mathrm{Hom}_R(S,S)$. Let $f_i \colon S \to \varphi_{\alpha_i}^* S$ be the map corresponding to id_S under this correspondence. As the composition

$$S \to \varphi_{\alpha_1}^* S \to \varphi_{\alpha_2}^* \varphi_{\alpha_1}^* S \to \cdots \to \varphi_{\alpha_n}^* \cdots \varphi_{\alpha_1}^* S$$

is just the isomorphism $S \to \varphi^* S$, it follows that each f_i is an isomorphism. The inverse of f_i thus gives us a semilinear action of φ_{α_i} on S, for which the composition $\varphi_{\alpha_1} \circ \cdots \circ \varphi_{\alpha_n}$ is the absolute Frobenius. □

We now tie this construction to Drinfeld's lemma, discovering a hitch along the way whose resolution we postpone to Subsection 4.4.

Definition 4.14. Let $X_i := \mathrm{Spd}(F_i)$ and $\bar{x}_i := \mathrm{Spd}(\bar{F}_i)$. Let $X := X_1 \times \cdots \times X_n$, and let $\bar{x}$ be a geometric point of X lying over each $\bar{x}_i$. By Drinfeld's Lemma (3.25), we have

$$\pi_1^{\mathrm{prof}}(X/\Phi, \bar{x}) \simeq \prod_{i=1}^{n} \pi_1^{\mathrm{prof}}(X_i, \bar{x})$$

as profinite groups, and by Theorem 3.24, the right-hand side is isomorphic to G_{F_Δ}.

If it were the case that we could identify X with the diamond associated to $\mathrm{Spa}(R,R^\circ)$, we could then identify the finite étale R-algebra S with a finite étale cover of X carrying natural actions of $\varphi_{\alpha_1},\dots,\varphi_{\alpha_n}$, then use Drinfeld's lemma to see that such a cover must be dominated by a cover formed by taking a product of finite extensions of the F_i. Unfortunately, the situation turns out to be a bit more subtle than this.

Proposition 4.15. We have a natural identification $X = Y^\diamond$ for

$$Y := \{|\ | \in \mathrm{Spa}(R,R^\circ) \,|\, |\bar{\varpi}_i|^m \to 0, m \to \infty\ \forall\ i\}$$

viewed as an open subspace of $\mathrm{Spa}(R,R^\circ)$ by writing it as the increasing union of the affinoid subspaces

$$U_m := \{|\ | \in \mathrm{Spa}(R,R^\circ) \,|\, |\bar{\varpi}_i|^m \le |\bar{\varpi}_j|\ \forall\ i,j\}. \tag{4.3}$$

Proof. We show that $X = Y^\diamond$ by showing that every morphism $\mathrm{Spa}(T,T^+) \to \mathrm{Spa}(R,R^\circ)$ with image in Y (for $\mathrm{Spa}(T,T^+)$ an affinoid adic space) corresponds to a tuple of morphisms $\mathrm{Spa}(T,T^+) \to \mathrm{Spa}(F_i,F_i^\circ)$. That is to say, every morphism to $\mathrm{Spa}(R,R^\circ)$ with image in Y factors through $\mathrm{Spa}(F_1,F_1^\circ)\times\cdots\times\mathrm{Spa}(F_n,F_n^\circ)$, identifying $Y^\diamond$ with $\mathrm{Spd}(F_1)\times\cdots\times\mathrm{Spd}(F_n) = X$.

Supoose that $f\colon \mathrm{Spa}(T,T^+) \to \mathrm{Spa}(R,R^\circ)$ has image in Y. Then for each i, f maps $\bar{\varpi}_i$ to a topologically nilpotent element of T and so the composition $F_i \to R \to T$ is continuous (even though $F_i \to R$ is not). This gives us a family of morphisms $f_i\colon \mathrm{Spa}(T,T^+) \to \mathrm{Spa}(F_i,F_i^\circ)$.

Conversely, suppose that we have maps $f_i\colon \mathrm{Spa}(T,T^+) \to \mathrm{Spa}(F_i,F_i^\circ)$ for all i. Because the elements $f_i(\bar{\varpi}_i)$ are all topologically nilpotent, the induced map $F_1^\circ \otimes_{\mathbb{F}_p} \cdots \otimes_{\mathbb{F}_p} F_n^\circ \to T$ extends to R_0. We thus recover a continuous map $R \to T$ and thus a map $f\colon \mathrm{Spa}(T,T^+) \to \mathrm{Spa}(R,R^\circ)$. □

Remark 4.16. Proposition 4.15 implies that the complement of Y in $\mathrm{Spa}(R,R^\circ)$ is substantial; it is a sort of "boundary" consisting of points at which the power-bounded elements $\bar{\varpi}_1,\dots,\bar{\varpi}_n$ are not all individually topologically nilpotent, but at least one of them is (ensuring that their product is also). For example, when $n=2$, the boundary disconnects into two pieces depending on whether $\bar{\varpi}_1$ or $\bar{\varpi}_2$ is topologically nilpotent.

The identification of Y with a subspace of $\mathrm{Spa}(R,R^\circ)$ induces a morphism $R \to H^0(Y,\mathcal{O})$ which is injective but not surjective. For example,

$$\sum_{n=1}^{\infty} \bar{\varpi}_1^{p^n} \bar{\varpi}_2^{-n}$$

is not contained in the image of R.

This has the following effect on the proof of essential surjectivity in Theorem 4.6. Given the finite étale R-algebra S from Subsection 4.13, we can pull it back to a finite étale cover Y of X/Φ, then apply Drinfeld's lemma to choose finite extensions E_i of F_i with the property that for T the completed product of the E_i, we can pull back the morphism $\mathrm{Spa}(T, T^\circ) \to \mathrm{Spa}(R, R^\circ)$ to X/Φ to obtain a finite étale cover that dominates Y. This is enough to produce a candidate representation V, but not enough to construct an isomorphism $D(V) \simeq D$; for this, we need to show that $\mathrm{Spa}(T, T^\circ)$ dominates $\mathrm{Spa}(S, S^\circ)$.

Our workaround for the preceding issue is given by the following statement, whose proof we defer to Subsection 4.4.

Proposition 4.17 (see Proposition 4.27). The natural map $R^\circ \to H^0(Y, \mathcal{O}^+)$ is injective and almost surjective (that is, its cokernel is killed by $(\bar{\varpi}_1 \cdots \bar{\varpi}_n)^{p^{-m}}$ for all nonnegative integers m).

Corollary 4.18. We have

$$R^{\varphi_\Delta} = \mathbb{F}_p. \tag{4.4}$$

Proof. Any element of R^{φ_Δ} maps to a φ_Δ-invariant locally constant function $|Y| \to \mathbb{F}_p$. By Proposition 4.15, if such an element were not in $\mathbb{F}_p$, then it would imply the existence of a disconnection of X/Φ, which would contradict Theorem 3.25. $\square$

Let Z be the diamond corresponding to the adic space $Y \times_{\mathrm{Spa}(R,R^\circ)} \mathrm{Spa}(S, S^\circ)$. The set $V := Z \times_X \bar{x}$ carries an action of $\pi_1^{\mathrm{prof}}(X/\Phi, \bar{x}) \simeq G_{F_\Delta}$, by definition, and in fact we can say more:

Proposition 4.19. The set V has the structure of an $\mathbb{F}_p$-representation of G_{F_Δ}.

Proof. Let $e \in (D \otimes_R S)^\varphi \simeq \mathrm{Hom}_R(S, S)$ be the element corresponding to the identity map, and let ι_1, ι_2 be the two natural inclusions of $(D \otimes_R S)^\varphi$ into $(D \otimes_R S \otimes_R S)^\varphi$. Then the element

$$\iota_1(e) + \iota_2(e) \in (D \otimes_R S \otimes_R S)^\varphi \simeq \mathrm{Hom}_R(S, S \otimes_R S)$$

induces an addition law

$$\mathrm{Spa}(S, S^\circ) \times_{\mathrm{Spa}(R,R^\circ)} \mathrm{Spa}(S, S^\circ) = \mathrm{Spa}(S \otimes_R S, (S \otimes_R S)^+) \to \mathrm{Spa}(S, S^\circ)$$

(where $(S \otimes_R S)^+$ denotes the integral closure of $S^\circ \otimes_{R^\circ} S^\circ$ in $S \otimes_R S$). This in turn induces an addition law on $\mathrm{Spd}(S \otimes_R \bar{k})$, where $\bar{k}$ is an algebraically closed field such that $\bar{x} = \mathrm{Spec}(\bar{k})$. It remains to observe that $Z \times_X \bar{x} \simeq \mathrm{Spd}(S \otimes_R \bar{k})$ and that this addition law induces an $\mathbb{F}_p$-vector space structure, which by virtue of its naturality commutes with the action of G_{F_Δ}. (Note that p-fold addition corresponds to the map $S \to S$ obtained by post-composing the p-fold coaddition law $S \to S \otimes_R \cdots \otimes_R S$ with the multiplication map $S \otimes_R \cdots \otimes_R S \to S$; this composition corresponds to the element $pe = 0$ in $(D \otimes_R S)^\varphi$.) $\square$

As described in Remark 4.16, it remains to confirm that $D(V) \simeq D$. The crucial case is when V is a trivial representation of G_{F_Δ}.

Proposition 4.20. If V is a trivial representation of G_{F_Δ}, then D is a trivial φ_Δ-module of rank $\dim_{\mathbb{F}_p} V$, and in particular is isomorphic to $D(V)$.

Proof. In this case, we have

$$Y \times_{\mathrm{Spa}(R,R^\circ)} \mathrm{Spa}(S, S^\circ) \simeq \coprod Y.$$

That is, the map $\mathrm{Spa}(S, S^\circ) \to \mathrm{Spa}(R, R^\circ)$ splits completely after pullback to Y.

With notation as in Proposition (4.3), let $R_m := H^0(U_m, \mathcal{O})$ and $R_m^+ := H^0(U_m, \mathcal{O}^+)$; then by Proposition 4.27, $R^\circ \to \varprojlim_m R_m^+$ is an almost isomorphism and $R \to \varprojlim_m R_m$ is injective. Let $\widetilde{S}$ be the pushforward of the structure sheaf on $\mathrm{Spa}(S, S^\circ)$ to $\mathrm{Spa}(R, R^+)$, and let $S_m := H^0(U_m, \widetilde{S}) = S \otimes_R R_m$; since S is a finite, étale R-algebra, S_m is a finite, étale R_m-algebra. Consider the Φ-invariant idempotents corresponding to the decomposition

$$\mathcal{O}(\mathrm{Spa}(S, S^\circ)) \times_{\mathcal{O}(\mathrm{Spa}(R,R^\circ))} \mathcal{O}(Y) \simeq \bigoplus \mathcal{O}(Y).$$

The restrictions from Y to U_m induce isomorphisms

$$S_m \simeq S \otimes_R R_m \simeq \bigoplus R_m.$$

The given idempotents induce compatible systems of idempotents in S_m, and thus the idempotents in question belong to $\varprojlim_m S_m$.

Fix a presentation of S as a direct summand of a finite, free R-module. This gives a choice of coordinates in R for each element in S, which in turn gives a choice of coordinates in $\varprojlim_m R_m$ for each element of $\varprojlim_m S_m$. We remark that an element in $\varprojlim_m S_m$ belongs to S if an only if each of its coordinates belongs to R. Now, let U be a rational subspace of $\mathrm{Spa}(R, R^\circ)$ in Y containing a fundamental domain for the action of Φ. As U is quasicompact, the restriction of each idempotent to $H^0(U, \widetilde{S})$ must be an element with bounded coordinates, so there exists an m such that the coordinates belong to $\bar{\varpi}_n^{-m} H^0(U, \mathcal{O}^+)$. (Note that as there is no preferred choice of valuation on $H^0(U, \mathcal{O})$, we measure elements against powers of the single element $\bar{\varpi}_n$, which is topologically nilpotent on U.) On the other hand, since $\varphi = \varphi_{\alpha_1} \circ \cdots \circ \varphi_{\alpha_n}$ acts trivially on Y, the group Φ acts on Y via its quotient modulo φ, which can be generated by the classes of $\varphi_{\alpha_1}, \dots, \varphi_{\alpha_{n-1}}$; in particular, the coordinates of our idempotents when restricted to $\psi(U)$ will still belong to $\bar{\varpi}_n^{-m} H^0(\psi(U), \mathcal{O}^+)$ for any $\psi \in \Phi$. Thus after glueing we see that the coordinates of our idempotents belong to $\bar{\varpi}_n^{-m} (\bar{\varpi}_1 \cdots \bar{\varpi}_n)^{-1} R^\circ \subseteq R$. It follows that our Φ-invariant idempotents belong to S. This shows that $S \simeq \bigoplus R$; by (4.4), this is a decomposition of S into Φ-connected components.

The components in this decomposition correspond to various φ_Δ-equivariant homomorphisms $S \to R$. Via the natural isomorphism (4.1) (with $T = R$), these in turn correspond to elements of $D^{\varphi_\Delta} \subseteq D^\varphi$. That is, we have a natural injective morphism $V \to D^{\varphi_\Delta}$ of $\mathbb{F}_p$-vector spaces.

Let r be the rank of D. By the naturality of the previous construction, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p^{\oplus r}, V) \times \mathrm{Hom}_{\mathbb{F}_p}(V, \mathbb{F}_p^{\oplus r}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p^{\oplus r}, \mathbb{F}_p^{\oplus r}) \arrow[d] \\ \mathrm{Hom}_R(R^{\oplus r}, D)^{\varphi_\Delta} \times \mathrm{Hom}_R(D, R^{\oplus r})^{\varphi_\Delta} \arrow[r] & \mathrm{Hom}_R(R^{\oplus r}, R^{\oplus r})^{\varphi_\Delta}, \end{tikzcd}$$

in which the vertical arrows are injective and the horizontal arrows denote composition: $(f, g) \mapsto g \circ f$. Consequently, if we choose any isomorphism $f : \mathbb{F}_p^{\oplus r} \simeq V$ of $\mathbb{F}_p$-vector spaces and take g to be its inverse, applying the left vertical arrow to (f, g) yields a pair of Φ-equivariant morphisms between $R^{\oplus r}$ and D whose composition is the identity on $R^{\oplus r}$. Since both $R^{\oplus r}$ and D are finite projective R-modules of rank r, this implies that the two morphisms are indeed inverses on both sides, so D is a trivial φ_Δ-module as claimed. □

We finish by returning to the case when V is not necessarily a trivial $\mathbb{F}_p$-representation of G_{F_Δ}.

Proposition 4.21. The module D arises from the representation V in general.

Proof. Let E_i/F_i be finite extensions such that V is a trivial representation of G_{E_Δ}. Let R_E be the completed tensor product of $E_1, \ldots, E_n$. By Proposition 4.11 and Proposition 4.20, there is a canonical φ_Δ-equivariant and G_{E_Δ}-equivariant isomorphism

$$D \otimes_R R_E \simeq V \otimes_{\mathbb{F}_p} R_E;$$

that is, $D \otimes_R R_E \simeq D(V|_{E_\Delta})$. By canonicality, this isomorphism is also G_{F_Δ}-equivariant for the diagonal action on both sides; by taking G_{F_Δ}-invariants, we obtain an isomorphism $D \simeq D(V)$. □

This completes the proof of Theorem 4.6, and thus of Theorem 4.5, modulo the comparison between R° and $H^0(Y, \mathcal{O}^+)$; we give this next.

4.4 $H^0(Y, \mathcal{O}^+)$: a geometric interlude

In this subsection, we complete the comparison between R° and $H^0(Y, \mathcal{O}^+)$ alluded to above (Proposition 4.27); this will in turn complete the proof of Theorem 4.6, and will appear at a corresponding point in the study of mod-p^n representations (Proposition 4.37). For this calculation, it is convenient to work more generally, by taking products not of fields but of arbitrary perfect, analytic Huber rings. We note in passing that much of the previous section can be carried over to this level of generality (as in the comparison of φ-modules and $\mathbb{Z}_p$-local systems given in [27, §8]), but that will not be necessary for our present purposes.

We start by describing a special case in detail, in order to illustrate why such an assertion is reasonable to expect. This example was suggested in [24, Remark 4.3.18].

Example 4.22. Let $\mathcal{O}_{F_1}$ be the T_1-adic completion of $\mathbb{F}_p[\![T_1]\!][T_1^{p^{-\infty}}]$, which we denote by $\mathbb{F}_p[\![T_1^{p^{-\infty}}]\!]$. Let $F_1 := \mathcal{O}_{F_1}[T_1^{-1}]$. Let

$$\begin{aligned} A^+ &:= \mathcal{O}_{F_1}\langle T_2^{p^{-\infty}}\rangle \\ &= \Bigg\{ \sum_{j\in\mathbb{Z}[1/p]_{\geq 0}} \Bigg(\sum_{i\in\mathbb{Z}[1/p]_{\geq 0}} a_{ij}T_1^i \Bigg) T_2^j \,\Bigg|\, \forall\ k,\ \exists \text{ finitely many nonzero} \\ &\qquad a_{ij}\in\mathbb{F}_p \text{ with } i+j\leq k;\ \Bigg| \sum_{i\in\mathbb{Z}[1/p]_{\geq 0}} a_{ij}T_1^i \Bigg|_{T_1} \to 0 \text{ as } j\to\infty \Bigg\} \\ A &:= F_1\langle T_2^{p^{-\infty}}\rangle \\ &= \Bigg\{ \sum_{j\in\mathbb{Z}[1/p]_{\geq 0}} \Bigg(\sum_{i\in\mathbb{Z}[1/p]_{\geq m_0}} a_{ij}T_1^i \Bigg) T_2^j \,\Bigg|\, m_0\in\mathbb{Z};\ \forall\ k,\ \exists \text{ finitely many} \\ &\qquad \text{nonzero } a_{ij} \text{ with } i+j\leq k;\ \Bigg| \sum_{i\in\mathbb{Z}[1/p]_{\geq m_0}} a_{ij}T_1^i \Bigg|_{T_1} \to 0 \text{ as } j\to\infty \Bigg\} \end{aligned}$$

Consider

$$Y := \{|\ | \in \operatorname{Spa}(A, A^+) \,|\, 0 < |T_2| < 1\}$$

as an adic space by identifying it with the union of the rational open subspaces

$$\begin{aligned} U_m &:= \{|\ | \in \operatorname{Spa}(A, A^+) \,|\, |T_1^m| \leq |T_2| \leq |T_1^{1/m}|\} \\ &= U(T_1^m/T_2, T_2/T_1^{1/m}) \end{aligned}$$

as m runs over the powers of p. Let $(\mathcal{O}, \mathcal{O}^+)$ denote the structure sheaves of $\operatorname{Spa}(A, A^+)$. Then $\mathcal{O}^+(U_m)$ is the completion of the integral closure of

$A^+[T_1^m/T_2, T_2/T_1^{1/m}]$. That is,

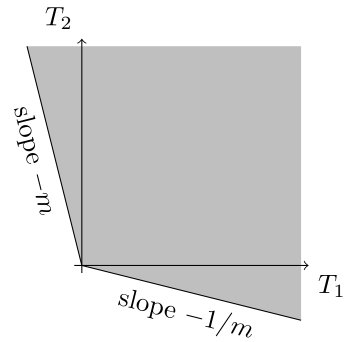
$$\begin{aligned}\mathcal{O}^+(U_m) &= A\langle T_1^m/T_2, T_2/T_1^{1/m}\rangle^+ \\ &= \Bigg\{ \sum_{r,s\in\mathbb{Z}[1/p]_{\geq 0}} \left(\sum_{i,j\in\mathbb{Z}[1/p]_{\geq 0}} a_{ijrs}T_1^iT_2^j \right) \left(\frac{T_1^m}{T_2}\right)^r \left(\frac{T_2}{T_1^{1/m}}\right)^s \Bigg| \\ &\quad \forall\ k,\ \exists \text{ finitely many nonzero } a_{ijrs}\in\mathbb{F}_p \text{ with } i+j+r(m-1)+ \\ &\quad s(m-1)\leq k; \left| \sum_{i\in\mathbb{Z}[1/p]_{\geq 0}} a_{ijrs}T_1^i \right|_{T_1} \to 0 \text{ as } j\to\infty; \\ &\quad \left| \sum_{i,j\in\mathbb{Z}[1/p]_{\geq 0}} a_{ijrs}T_1^iT_2^j \right|_{T_1} \to 0 \text{ as } \max\{r,s\}\to\infty \Bigg\}.\end{aligned}$$

Now, for any choices of coefficients a_{ij} such that for each k, at most finitely many a_{ij} with $i+j\leq k$ are nonzero, we have

$$\begin{aligned}\sum_{i,j\in\mathbb{Z}[1/p]_{\geq 0}} a_{ij}T_1^iT_2^j &= \sum_{i,j\in\mathbb{Z}[1/p]_{\geq 0}} a_{ij}T_1^{i+\lfloor j\rfloor/m}T_2^{j-\lfloor j\rfloor}(T_2/T_1^{1/m})^{\lfloor j\rfloor} \\ &= \sum_{s\in\mathbb{Z}_{\geq 0}} \left(\sum_{u,v\in\mathbb{Z}[1/p]_{\geq 0}, v<1} a_{(u-s/m)v}T_1^uT_2^v \right) \left(\frac{T_2}{T_1^{1/m}}\right)^s \\ &\in \mathcal{O}^+(U_m),\end{aligned}$$

so

$$\begin{aligned}&\mathbb{F}_p[\![T_1^{p^{-\infty}}, T_2^{p^{-\infty}}]\!] \subseteq \mathcal{O}^+(U_m) \\ &\quad\subseteq \Bigg\{ \sum_{i,j\in\mathbb{Z}[1/p]} a_{ij}T_1^iT_2^j \ \Bigg|\ i+mj\geq 0, mi+j\geq 0,\ \forall\ k\ \exists \text{ finitely many} \\ &\qquad \text{nonzero } a_{ij} \text{ with } i+j\leq k \Bigg\}.\end{aligned}$$



It follows that

$$\begin{aligned}\mathcal{O}^+(Y) &= \mathcal{O}^+\left(\bigcup_{m=p^k} U_m\right) \\ &= \bigcap_{m=p^k} \mathcal{O}^+(U_m) \\ &= \mathbb{F}_p[\![T_1^{p^{-\infty}}, T_2^{p^{-\infty}}]\!].\end{aligned}$$

In particular, in this case the map $R^\circ \to H^0(Y, \mathcal{O}^+)$ is a genuine isomorphism, not just an almost isomorphism. $\lozenge$

We now set notation so as to work with a more general completed tensor product of perfectoid rings.

Notation 4.23. With notation as in Notation 4.3, for $i = 1, \dots, n$, let $\varphi_i \colon R \to R$ be the ring homomorphism obtained by tensoring the Frobenius morphism on R_i with the identity morphism on R_j for $j \neq i$. Put

$$Y := \{v \in \mathrm{Spa}(R, R^\circ) \,|\, v(\varpi_1)^m, \dots, v(\varpi_n)^m \to 0, m \to \infty\},$$

viewed as an open subspace of $\mathrm{Spa}(R, R^+)$ as in Proposition 4.15; by the same proof, we have $\mathrm{Spd}(R_1) \times \cdots \times \mathrm{Spd}(R_n) = Y^\diamond$.

In order to simulate arguments using ordinary power series, we recall the following construction.

Definition 4.24. Let k be a ring and let Γ be a totally ordered abelian group (written multiplicatively). The ring $k(\!|\Gamma|\!)$ of *Hahn–Mal'cev–Neumann generalized power series* is the set of functions $\Gamma \to k$ whose support is a well-ordered subset of Γ. (In view of the multiplicative notation for Γ, a well-ordered subset here must be taken to be one containing no infinite *increasing* sequence.) We represent the function $\gamma \mapsto c_\gamma$ as a formal sum $\sum_{\gamma \in \Gamma} c_\gamma[\gamma]$; then $k(\!|\Gamma|\!)$ forms a ring with respect to the operations

$$\sum_\gamma c_\gamma[\gamma] + \sum_\gamma d_\gamma[\gamma] = \sum_\gamma (c_\gamma + d_\gamma)[\gamma]$$

$$\sum_\gamma c_\gamma[\gamma] \times \sum_\gamma d_\gamma[\gamma] = \sum_\gamma \left(\sum_{\gamma'\gamma''=\gamma} c_{\gamma'} d_{\gamma''} \right) [\gamma].$$

The condition on well-ordered supports is needed to establish that multiplication is well-defined; one must first check that the sum over γ', γ'' is always finite, and then that the support of the resulting sum is again well-ordered. See for example [22, §4].

From its construction, the ring $k(\!(\Gamma)\!)$ comes equipped with a natural valuation: the function assigning to every nonzero formal sum the maximal element of its support (which exists by the well-ordered condition).

Lemma 4.25 (Kaplansky). Let F be a nonarchimedean field of equal characteristics, with value group Γ and residue field k, which is algebraically closed and *maximally complete*. (The latter condition means that there is no nontrivial extension of F with the same value group and residue field as F.) Then there exists an isomorphism $F \simeq k(\!(\Gamma)\!)$ of fields with valuation.

Proof. See [22, Theorem 7]. □

Lemma 4.26. Let $\mathbb{R}^+$ denote the multiplicative group of positive real numbers. Suppose that $n = 2$ and there exists an isomorphism $R_1 \simeq k_1(\!(\mathbb{R}^+)\!)$ for some perfect ring k_1 (which need not be a field). Then the map $R^\circ \to H^0(Y, \mathcal{O}^+)$ is an isomorphism.

Proof. Fix a power-multiplicative norm $|\,|$ defining the topology on R_2. We extend this norm to $R_2 \otimes_{\mathbb{F}_p} k_1$ as follows: choose a basis of k_1 as an $\mathbb{F}_p$-vector space, use it to view $R_2 \otimes_{\mathbb{F}_p} k_1$ as a direct sum of copies of R_2, and take the supremum norm of the coordinates. This does not depend on the choice of the basis of k_1.

We may explicitly describe R as a certain set of formal sums $\sum_{\gamma\in\mathbb{R}^+} c_\gamma[\gamma]$ with coefficients in $R_2 \otimes_{\mathbb{F}_p} k_1$. These sums must satisfy the following conditions.

(i) For any neighborhood U of 0 in R_2, the set of $\gamma \in \mathbb{R}^+$ such that $c_\gamma \notin U$ is well-ordered; and for each $\gamma_0 \in \mathbb{R}^+$, there exists a finitely generated $\mathbb{F}_p$-submodule M of R_2 such that the quantities c_γ for $\gamma \geq \gamma_0$ all belong to $(U + M) \otimes_{\mathbb{F}_p} k_1$.

(ii) The set $\{\gamma \in \mathbb{R}^+ \mid c_\gamma \neq 0\}$ is bounded above in $\mathbb{R}^+$ (but not necessarily well-ordered), and the set $\{c_\gamma \mid \gamma \in \mathbb{R}^+\}$ is bounded in $R_2 \otimes_{\mathbb{F}_p} k_1$.

In terms of these formal sums, for each $r > 0$ we may define a norm v_r on R by the formula

$$v_r \left(\sum_\gamma c_\gamma[\gamma] \right) = \max\{\gamma^r |c_\gamma| \mid \gamma \in \mathbb{R}^+\}.$$

Now observe that the following two families of seminorms are mutually cofinal; that is, any member of one family is eventually dominated by all members of the other family.

(a) The functions $\max\{v_r, v_{1/r}\}$ as $r \to 0^+$.

(b) The supremum norms over the subspaces

$$\{|\ | \in Y \mid |[\gamma]| \leq |\varpi_2| \leq |[\gamma^{-1}]|\}$$

as $\gamma \to 0^+$.

In (b), the subspaces in question have union Y; therefore, taking the inverse limit of the completions of R with respect to these seminorms yields $H^0(Y, \mathcal{O})$. By the cofinal property, the same is true of (a); consequently, $H^0(Y, \mathcal{O})$ may be identified with the set of formal sums $\sum_{\gamma \in \mathbb{R}^+} c_\gamma[\gamma]$ with coefficients in $R_2 \otimes_{\mathbb{F}_p} k_1$ subject to the same condition (i) and a slightly weaker version of (ii):

(ii)$'$ For any $r > 0$, the set $\{\gamma^r |c_\gamma| \mid \gamma \in \mathbb{R}^+\}$ is bounded above in $\mathbb{R}$.

As for $H^0(Y, \mathcal{O}^+)$, we may characterize it as the set of $s \in H^0(Y, \mathcal{O})$ for which each of the supremum norms in (b) is at most 1; by cofinality, this is equivalent to requiring that $v_r(s) \leq 1$ for all $r > 0$.
From these descriptions, it is immediate that $R^\circ \to H^0(Y, \mathcal{O}^+)$ is injective. To check surjectivity, note that if $s = \sum_\gamma c_\gamma[\gamma] \in H^0(Y, \mathcal{O}^+)$, then for each $\gamma \in \mathbb{R}^+$ for which $c_\gamma \neq 0$, we must have $\gamma^r |c_\gamma| \leq 1$. In particular, if $\gamma > 1$ then $c_\gamma = 0$, while if $\gamma \leq 1$ then $c_\gamma \in R_2^\circ$. From this it is apparent that $s \in R^\circ$, so $R^\circ \to H^0(Y, \mathcal{O}^+)$ is surjective. (See [26, Lemma 5.1] for a related argument.) □

Proposition 4.27. With notation as in Notation 4.23, the map $R^\circ \to H^0(Y, \mathcal{O}^+)$ is injective and almost surjective.

Proof. Since R_i is allowed to be a perfect ring, not necessarily a field, we may deduce the general case by repeatedly applying the case $n = 2$; we thus assume $n = 2$ hereafter. Since R_1 is uniform, it embeds as a closed subring of a product of perfectoid fields; by Lemma 4.25, each of these fields can be embedded into a ring of the form $k(\!(\mathbb{R}^+)\!)$ for some perfect field k. Let R_1' be the set of bounded elements (for the supremum norm) in the product of these rings and put $R_1'' = R_1' \widehat{\otimes}_{R_1} R_1'$. Let R' (resp. R'') be the completed tensor product of R_1' (resp. R_1'') with R_2. Let Y' (resp. Y'') be the subspace of the spectrum of R' (resp. R'') defined as in Notation 4.23. In the commutative diagram

$$\begin{tikzcd}
R^\circ \arrow[r] \arrow[d] & R_1'^\circ \widehat{\otimes}_{\mathbb{F}_p} R_2^\circ \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & R_1''^\circ \widehat{\otimes}_{\mathbb{F}_p} R_2^\circ \arrow[d] \\
R^\circ \arrow[r] \arrow[d] & R'^\circ \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & R''^\circ \arrow[d] \\
H^0(Y, \mathcal{O}^+) \arrow[r] & H^0(Y', \mathcal{O}^+) \arrow[r, shift left] \arrow[r, shift right] & H^0(Y'', \mathcal{O}^+)
\end{tikzcd}$$

the vertical arrows between the first and second rows are almost isomorphisms by Lemma 4.4; the top and bottom rows are almost equalizer diagrams by the fact that $\mathcal{O}^+$ induces an almost acyclic sheaf for the *v-topology* [24, Definition 3.8.5] on any affinoid perfectoid space [28, Theorem 3.5.5]; and the lower vertical arrow in the second column is injective and almost surjective by Lemma 4.26 (applied to each factor of R_1').

From this, we may first deduce that the lower vertical arrow in the first column is injective. By similar logic, the lower vertical arrow in the third column is injective; we may thus deduce that the lower vertical arrow in the first column is almost surjective. □

Remark 4.28. The preceding proposition is conceptually related to the *perfectoid Riemann extension theorem*. The first such statement is due to Scholze [40]; similar statements are used in the proofs of the direct summand conjecture by André [1, 2] and Bhatt [8].

To make this link explicit, we describe an alternate proof of Proposition 4.27 in the case $n = 2$ (which again suffices for the general case), in which we appeal directly to the extension theorem as formulated by Bhatt. Consider the covering of $X := \mathrm{Spa}(R, R^\circ)$ by the two rational subspaces

$$U_1 := \{|\ | \in X \,|\, |\varpi_1| \leq |\varpi_2|\}, \qquad U_2 := \{|\ | \in X \,|\, |\varpi_2| \leq |\varpi_1|\}.$$

Since R is perfectoid, it is sheafy, so it will suffice to check that for $i = 1, 2$ the map

$$H^0(U_i, \mathcal{O}^+) \to H^0(U_i \cap Y, \mathcal{O}^+)$$

is injective and an almost isomorphism. By symmetry, we may restrict to the case $i = 1$. In this case, we define $R' := H^0(U_1, \mathcal{O}) = R\langle T\rangle/(\varpi_2 T - \varpi_1)$, so that R'° is almost isomorphic to $H^0(U_1, \mathcal{O}^+)$. We may then make the identification

$$U_1 \cap Y \simeq \{|\ | \in \mathrm{Spa}(R', R'^\circ) \,|\, |\varpi_2| > 0\}$$

and apply [8, Theorem 4.2] to deduce that $R'^\circ \to H^0(U_i \cap Y, \mathcal{O}^+)$ is injective and its cokernel is almost zero. (As an aside, we note that the proof of [8, Theorem 4.2] can be conceptualized rather well in terms of the pictures from Example 4.22.)

4.5 Mod-p^m representations

We return to the notation of Subsection 4.2 and prove the following.

Theorem 4.29. The category of continuous $\mathbb{Z}_p$-representations of $G_{K,\Delta}$ is equivalent to the category of étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules over $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$.

As in the proof of Theorem 4.5, this reduces to the following statement.

Theorem 4.30. With notation as in Theorem 4.6, there exists an equivalence of categories between the category of continuous $\mathbb{Z}_p$-representations of G_{F_Δ} and the category of étale φ_Δ-modules over $W(R)$.

Moreover, it suffices to check that the analogous equivalence holds modulo p^m on both sides for each m, since we can then obtain the desired result by taking limits. That is, we need to prove the following.

Theorem 4.31. With notation as in Theorem 4.6, for each positive integer m, the functor D defined in Definition 4.32 below yields an equivalence of categories between the category of continuous representations of G_{F_Δ} on finite free $\mathbb{Z}/p^m\mathbb{Z}$-modules and the category of étale φ_Δ-modules over $W(R)/p^m W(R)$.

Our method of proof is similar to that of the case $m = 1$; consequently, we summarize a few points that do not differ substantially from the previous case.

Definition 4.32. Define a functor D from the category of continuous representations of G_{F_Δ} on finite projective $\mathbb{Z}/p^m\mathbb{Z}$-modules to the category of $W(R)/p^m W(R)$-modules with commuting semilinear bijective actions of $\varphi_{\alpha_1}, \dots, \varphi_{\alpha_n}$ by

$$D(V) := (V \otimes_{\mathbb{Z}_p} W(\bar{R}))^{G_{F_\Delta}},$$

where $\bar{R}$ is as in Definition 4.7.

Proposition 4.33. The module $D(V)$ is a finite, projective $W(R)/p^m W(R)$-module.

Proof. As in the proof of 4.5, for any fixed V we can find finite, Galois extensions E_i/F_i such that the action of G_{F_Δ} on V factors through $\mathrm{Gal}(E_\Delta/F_\Delta)$ and, for S the completed tensor product of $E_1, \dots, E_n$,

$$D(V/p^m V) = (V/p^m V \otimes_{\mathbb{Z}_p} W(S))^{\mathrm{Gal}(E_\Delta/F_\Delta)},$$

As S is free over R, it is in particular faithfully finite flat over R, and in fact it is faithfully finite étale, so by [27, Proposition 5.5.4], the extension $W(R) \to W(S)$ is faithfully finite étale. It follows that the extension $W(R)/p^m W(R) \to W(S)/p^m W(S)$ is faithfully finite étale, and in particular faithfully flat. The map

$$\begin{gathered}(V \otimes_{\mathbb{Z}_p} S) \otimes_R S \to S \otimes_R (V \otimes_{\mathbb{Z}_p} S) \\ v \otimes s \otimes \sigma(s) \mapsto s \otimes \sigma(v) \otimes \sigma(s)\end{gathered}$$

induces a map

$$(V/p^m V \otimes_{\mathbb{Z}_p} W(S)) \otimes_{W(R)} W(S) \to W(S) \otimes_{W(R)} (V/p^m V \otimes_{\mathbb{Z}_p} W(S)),$$

which is then a descent datum with respect to the faithfully flat extension $W(R)/p^m W(R) \to W(S)/p^m W(S)$. It follows by faithfully flat descent for modules that

$$D(V) \otimes_{W(R)} W(S) \simeq V \otimes_{\mathbb{Z}_p} W(S),$$

and in particular $D(V)$ is a finite, projective $W(R)/p^m W(R)$-module for each m. $\square$

Proposition 4.34. The functor D is fully faithful.

Proof. The proof is analogous to the proof of Proposition refD-fully-faithful.□

We now turn to essential surjectivity. Let D be a finitely presented (hence projective by Remark 2.16) module over $W(R)/p^m W(R)$ carrying commuting, semilinear actions of $\varphi_{\alpha_1}, \dots, \varphi_{\alpha_n}$, and let $\varphi := \varphi_{\alpha_1} \circ \cdots \circ \varphi_{\alpha_n}$.

Proposition 4.35. There exist a finite étale R-algebra T_m such that $D \otimes_{W(R)} W(T_m)$ admits a φ-invariant basis over $W(T_m)/pW(T_m)$. (Note that we are not currently looking for a φ_Δ-invariant basis.)

Proof. We proceed by induction on m, the case $m = 1$ being included in Theorem 4.5. Suppose that the induction hypothesis holds with m replaced by $m - 1$; then there exists a finite étale R-algebra T_{m-1} such that $D \otimes_{W(R)} W(T_{m-1})/p^{m-1}W(T_{m-1})$ admits a φ-invariant basis $e_1, \dots, e_r$. Choose lifts of these elements to $D \otimes_{W(R)} W(T_{m-1})$; by Nakayama's lemma, they still form a basis over $W(T_{m-1})/p^m W(T_{m-1})$. Let

$$F := (f_{ij})_{i,j} \in M_r(W(T_{m-1})/p^m W(T_{m-1}))$$

be the matrix of the action of φ on $D \otimes_{W(R)} W(T_{m-1})$ with respect to this basis, i.e. $\varphi(e_j) = \sum_{i=1}^r f_{ij} e_i$ for each j. As this basis is φ-invariant modulo p^{m-1}, we have $F = I + p^{m-1}A$ for some matrix A uniquely determined modulo p. We want to find a matrix B (with coefficients potentially in a larger ring) which conjugates the basis $(e_i)_i$ to a φ-invariant basis; such a matrix would satisfy $B^{-1}F\varphi(B) = I$, in other words $(I + p^{m-1}A)\varphi(B) = B$. Since this is satisfied modulo p^{m-1} for $B = I$, we can look for a matrix of the form $I + p^{m-1}C$, where the entries of $C = ([\bar{c}_{ij}])_{i,j}$ are Teichmüller lifts. Thus we are looking for elements $\bar{c}_{ij}$ in some finite étale T_{m-1}-algebra T_m with the property that

$$(I + p^{m-1}A)(I + p^{m-1}([\bar{c}_{ij}^p])_{i,j}) = I + p^{m-1}([\bar{c}_{ij}])_{i,j}$$

in $M_r(W(T_m)/p^m W(T_m))$. Expanding the left-hand side, and noting that $p^{2(m-1)} = 0$ in this ring, we reduce to finding elements $\bar{c}_{ij} \in T_m$ such that

$$A + ([\bar{c}_{ij}^p])_{i,j} = ([\bar{c}_{ij}])_{i,j}.$$

In other words, we need only adjoin to T_{m-1} roots of the equations $x^p - x - \bar{a}_{ij}$ for each entry a_{ij} of A. The resulting ring T_m is then finite étale over T_{m-1}, and consequently over R. □

Proposition 4.36. There exist a finite étale R-algebra S_m which represents the functor F_m defined as follows:

$$F_m(T) := (D \otimes_{W(R)} W(T))^\varphi.$$

Proof. Consider T_m as in Proposition 4.35 and define $D_m := D \otimes_{W(R)} W(T_m)$. By construction, D_m admits a φ-invariant basis $e_1, \dots, e_r$ over $W(T_m)/p^m W(T_m)$. We first show that there exists a finite étale R-algebra Q_m which represents the functor that maps a finite étale T_m-algebra T to the $\mathbb{Z}/p^m\mathbb{Z}$-module $(D_m \otimes_{W(T_m)} W(T))^{\varphi}$. We can write

$$D_m = (W(T_m)/p^m W(T_m))e_1 + \cdots + (W(T_m)/p^m W(T_m))e_r.$$

Let T be a finite étale T_m-algebra. Then

$$\begin{aligned}&(D_m \otimes_{W(T_m)} W(T)/p^m W(T))^{\varphi} \\ &\qquad\qquad = (W(T)/p^m W(T))^{\varphi} e_1 + \cdots + (W(T)/p^m W(T))^{\varphi} e_r;\end{aligned}$$

the ring $(W(T)/p^m W(T))^{\varphi}$ consists of one copy of $\mathbb{Z}/p^m\mathbb{Z}$ per connected component of T.
Let $Q_m := T_m[\bar{x}_{ij} \mid i = 1, \dots r, \quad j = 1, \dots, m]/(\cdots)$, where $\cdots$ are the relations necessary to ensure

$$[\bar{x}_{i1}] + p[\bar{x}_{i2}] + p^2[\bar{x}_{i3}] + \cdots + p^{m-1}[\bar{x}_{im}]$$

is a φ-invariant element in $W(Q_m)/p^m W(Q_m)$ for each i. Now

$$t_1 e_1 + \cdots + t_r e_r \in (D_m \otimes_{W(T_m)} W(T))^{\varphi}$$

if and only if each element $t_i = \sum_{j=0}^{m-1} p^j [\bar{t}_{ij}]$ is fixed by φ, so that such an element corresponds to the mapping

$$W(Q_m)/p^m W(Q_m) \to W(T)/p^m W(T)$$

sending the element

$$[\bar{x}_{i1}] + p[\bar{x}_{i2}] + p^2[\bar{x}_{i3}] + \cdots + p^{m-1}[\bar{x}_{im}]$$

to t_i for each i. Thus we have

$$\begin{aligned}(D_m \otimes_{W(T_m)} W(T))^{\varphi_R} &\simeq \operatorname{Hom}_{T_m}(Q_m, T) \\ &\simeq \operatorname{Hom}_{W(T_m)/p^m W(T_m)}(W(Q_m)/p^m W(Q_m), W(T)/p^m W(T)),\end{aligned}$$

as desired.
To summarize, so far we have a finite étale T_m-algebra Q_m which represents the functor

$$T \mapsto (D \otimes_{W(T_m)} W(T)/p^m W(T))^{\varphi}$$

on finite étale T_m-algebras. We want to show that Q_m has the form $S_m \otimes_r T_m$, where S_m is a finite étale R-algebra that represents the functor F_m on finite étale R-algebras. On the same category, define the functors

$$\begin{aligned}G_1(T) &:= \operatorname{Hom}_{T_m}(Q_m \otimes_{T_m,1} (T_m \otimes_R T_m), T) \\ &\simeq \operatorname{Hom}_{T_m}(Q_m, T) \times \operatorname{Hom}_{T_m,1}(T_m \otimes_R T_m, T) \\ &\simeq F_{T_m}(T) \times \operatorname{Hom}_{T_m,1}(T_m \otimes_R T_m, T),\end{aligned}$$

and similarly define

$$\begin{aligned} G_2(T) &:= \mathrm{Hom}_{T_m}(Q_n \otimes_{T_m,2} (T_m \otimes_R T_m), T) \\ &\simeq F_{T_m}(T) \times \mathrm{Hom}_{T_m,2}(T_m \otimes_R T_m, T), \end{aligned}$$

where the 1 and 2 indicate whether $T_m \otimes_R T_m$ is considered as a T_m-algebra via its first or second component. In the case when D is a trivial φ-module, the functor which maps a finite étale R-module T to $(D \otimes_{W(R)} W(T))^\varphi$ is represented by $R^{p^{rm}}$, where r is the rank of D, and so we have a canonical identification $Q_m \simeq R^{p^{rm}} \otimes_R T_m$. In this case there is a canonical descent datum

$$(R^{p^{rm}} \otimes_R T_m) \otimes_R T_m \to T_m \otimes_R (R^{p^{rm}} \otimes_R T_m).$$

This induces a natural isomorphism $G_2 \to G_1$ (by identifying $Q_m \otimes_{T_m,1} (T_m \otimes_R T_m)$ with $Q_m \otimes_R T_m$ and $Q_m \otimes_{T_m,2} (T_m \otimes_R T_m)$ with $T_m \otimes_R Q_m$), which in turn induces natural isomorphisms $F_{T_m} \to F_{T_m}$ and $\mathrm{Hom}_{T_m,2}(T_m \otimes_R T_m, *) \to \mathrm{Hom}_{T_m,1}(T_m \otimes_R T_m, *)$. But these two functors do not depend on Q_m; thus in the case when D need not be trivial, these isomorphisms may be used to construct a natural isomorphism $G_2 \to G_1$, which by Yoneda's lemma arises from an isomorphism

$$Q_m \otimes_{T_m,1} (T_m \otimes_R T_m) \to Q_m \otimes_{T_m,2} (T_m \otimes_R T_m).$$

This isomorphism provides a descent datum to which we may apply faithfully flat descent to deduce that $Q_m = S_m \otimes_R T_m$ for some finite étale R-algebra S_m. We show that S_m represents the functor F_m. Consider the equalizer diagram

$$T \to T \otimes_R T_m \rightrightarrows T \otimes_R T_m \otimes_R T_m.$$

This induces equalizer diagrams

$$\mathrm{Hom}_{T_m}(Q_m, T) \to \mathrm{Hom}_{T_m}(Q_m, T \otimes_R T_m) \rightrightarrows \mathrm{Hom}_{T_m}(Q_m, T \otimes_R T_m \otimes_R T_m)$$

and

$$\begin{aligned} (D \otimes_{W(R)} W(T))^\varphi &\to (D \otimes_{W(R)} W(T) \otimes_{W(R)} W(T_m))^\varphi \\ &\rightrightarrows (D \otimes_{W(R)} W(T) \otimes_{W(R)} W(T_m) \otimes_{W(R)} W(T_m))^\varphi. \end{aligned}$$

Since Q_m represents the functor F_{T_m}, the right two objects in each equalizer diagram are isomorphic. It follows that the left objects are also isomorphic. But

$$\mathrm{Hom}_{T_m}(Q_m, T) \simeq \mathrm{Hom}_{T_m}(S_m \otimes_R T_m, T) \simeq \mathrm{Hom}_R(S_m, T),$$

as desired. □

Define X and $\bar{x}$ as in Definition 4.14, so that $\pi_1^{\mathrm{prof}}(X/\Phi, \bar{x}) \simeq G_{F_\Delta}$. Define the perfectoid space Y as in Proposition 4.15, so that $X \simeq Y^\diamond$. With notation as in the proof of Proposition 4.20, let Z be the diamond corresponding to the adic space $Y \times_{\mathrm{Spa}(R,R^\circ)} \mathrm{Spa}(S_m, S_m^+)$. As in Proposition 4.19, the set $V := Z \times_X \bar{x}$ has an action of the group $\pi_1^{\mathrm{prof}}(X/\Phi, \bar{x}) \simeq G_{F_\Delta}$ by definition, and starting from the element

$$\begin{aligned} e \in &(D \otimes_{W(R)} W(T_m))^\varphi \\ &\simeq \mathrm{Hom}_{W(R)/p^m W(R)}(W(S_m)/p^m W(S_m), \\ &\quad W(S_m)/p^m W(S_m) \otimes_{W(R)} W(T_m)/p^m W(T_m)) \end{aligned}$$

we can construct an addition law on $\mathrm{Spd}(S_m \otimes_R \bar{k}) \simeq V$, under which the p^m-fold sum of e corresponds to the element $p^m e = 0$ in $(D \otimes_{W(R)} W(T_m))^\varphi$. Thus V carries the structure of a $\mathbb{Z}/p^m\mathbb{Z}$-module with an action of G_{F_Δ}.

Proposition 4.37. If V is a trivial $\mathbb{Z}/p^m\mathbb{Z}$-representation of G_{F_Δ}, then D is a trivial φ_Δ-module, and in particular is isomorphic to $D(V)$.

Proof. In this case, we have

$$Y \times_{\mathrm{Spa}(R,R^+)} \mathrm{Spa}(S_m, S_m^+) \simeq \coprod Y.$$

As in the proof of Proposition 4.20, we have $S_m = \bigoplus R$; by (4.4), this is a decomposition of S_m into Φ-connected components. Again as in the proof of Proposition 4.20, we have a natural injective morphism $V \to D^{\varphi_\Delta}$ of $\mathbb{Z}/p^m\mathbb{Z}$-modules; using naturality of the construction, we may deduce that D is a trivial φ_Δ-module. □

Proposition 4.38. The φ-module D arises from V in general.

Proof. The proof is analogous to the proof of Proposition 4.21. □

This completes the proof of Theorem 4.31: the functor D is fully faithful by Proposition 4.34 and essentially surjective by Proposition 4.38. This in turn implies Theorem 4.30 and Theorem 4.29.

4.6 Descent from $\widetilde{\mathcal{O}}$ to $\mathcal{O}$

Returning to the case where $F_1 = F_2 = \cdots = F_n$ is the tilt of $K(\mu_{p^\infty})$ for some finite extension $K/\mathbb{Q}_p$, denote by $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta^{\mathrm{nr}}}$ the ring $W(\bar{R})$. In the case $K = \mathbb{Q}_p$, the following result was proved previously (by another method) by the third author [47].

Theorem 4.39. The category of continuous $\mathbb{Z}_p$-representations of $G_{K,\Delta}$ is equivalent to the category of étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules over $\mathcal{O}_{\mathcal{E}_\Delta(K)}$. In particular, by Theorem 4.29, base extension of $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules from $\mathcal{O}_{\mathcal{E}_\Delta(K)}$ to $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ is an equivalence of categories.

Proof. By Theorem 4.29, the functor

$$\widetilde{M}(V) := (V \otimes_{\mathbb{Z}_p} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta^{\mathrm{nr}}})^{H_{K,\Delta}}$$

is an equivalence of categories from the category of continuous $\mathbb{Z}_p$-representations of $G_{K,\Delta}$ to the category of étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules over $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$. This functor is the composition of the two functors

$$M(V) := (V \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{E}_\Delta^{\mathrm{nr}}})^{H_{K,\Delta}}$$
$$B(M) := M \otimes_{\mathcal{O}_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)},$$

the first taking a continuous $\mathbb{Z}_p$-representation of $G_{K,\Delta}$ to an étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over $\mathcal{O}_{\mathcal{E}_\Delta(K)}$, and the second base-changing to $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$. From the fact that $\widetilde{M}$ is an equivalence of categories, it follows that the functor B is essentially surjective; furthermore B is fully faithful, as we assume all $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules M are finite projective. Thus B is an equivalence of categories, and it follows that the category of continuous $\mathbb{Z}_p$-representations of $G_{K,\Delta}$ is equivalent to the category of étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules over $\mathcal{O}_{\mathcal{E}_\Delta(K)}$. □

5 Descent for overconvergent Witt vectors

In this section we show that base extension gives an equivalence between the categories of étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules over $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ and over $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$. Our arguments follow closely those in [23, Section 2.4] with a twist. In that setting, one first descends the action of φ from $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ to $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$ and then shows that this automatically causes the action of Γ_K to descend. Here, we use a similar method to show that once the action of φ is descended, this automatically causes the actions of both φ_Δ and $\Gamma_{K,\Delta}$ to descend.

Theorem 5.1 (see Theorem 5.10). Base extension of étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules from $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$ to $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ is an equivalence of categories. Consequently, both categories are equivalent to the category of continuous representations of $G_{K,\Delta}$ on finitely generated $\mathbb{Z}_p$-modules.

Remark 5.2. Note that the rings $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$ and $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ have the same p^n-torsion quotients for every positive integer n. Consequently, the main content in the equivalence between étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules over $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$ and $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ is at the level of p-torsion-free modules.

Remark 5.3. With Theorem 4.29 proven, perfectoid spaces will play no further role in the remainder of the paper. That said, the arguments given here also yield a corresponding refinement of Theorem 4.31 in terms of the ring $W^\dagger(R)$ of *overconvergent Witt vectors*: base extension of étale φ_Δ-modules from $W^\dagger(R)$ to $W(R)$ is an equivalence of categories. We omit further details here.

5.1 Full faithfulness

We start by showing full faithfulness of the base extension from $\widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E}_\Delta(K)}$ to $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$. We first justify the introduction of a convenient additional hypothesis.

Remark 5.4. Let M be an étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over either $\widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E}_\Delta(K)}$ or $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$. By Theorem 4.5, M/pM corresponds to a continuous $\mathbb{F}_p$-representation of $G_{K,\Delta}$, and we can find a finite extension L of K such that this representation restricts trivially to $G_{L,\Delta}$. This means that after base extension from K to L, M acquires a basis which modulo p is fixed by φ_Δ and $\Gamma_{L,\Delta}$. Our approach will be to prove everything under the assumption of the existence of such a basis, then return to the general case at the end via faithfully flat descent.

Lemma 5.5. Let M be an étale φ-module over $\widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E}_\Delta(K)}$ such that the action of φ is trivial modulo p. Then $M^{\varphi} = (M \otimes_{\widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)})^{\varphi}$.

Proof. The proof is analogous to the proof of [23, Lemma 2.4.2]. □

Corollary 5.6. Let M be an étale φ_Δ-module over $\widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E}_\Delta(K)}$ such that the action of φ_Δ is trivial modulo p. Then $M^{\varphi_\Delta} = (M \otimes_{\widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)})^{\varphi_\Delta}$.

Proof. We note that an element is fixed by φ_Δ if and only if it is fixed by φ_α for all $\alpha \in \Delta$. The result then follows immediately from Lemma 5.5. □

Corollary 5.7. Base extension of étale φ_Δ-modules which are trivial modulo p from $\widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E}_\Delta(K)}$ to $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ is fully faithful.

Proof. This follows directly from Corollary 5.6, by letting $\mathrm{Hom}_{\widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E}_\Delta(K)}}(M, N)$ play the role of M, as in the proof of Proposition 4.11, since by Remark 2.15 homomorphisms of φ_Δ-modules are exactly those elements of $\mathrm{Hom}_{\widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E}_\Delta(K)}}(M, N)$ that are fixed by φ_Δ. □

5.2 Compatibility of descent with extra structures

If a φ-module descends from $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ to $\widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E}_\Delta(K)}$, the following lemma allows us to conclude that the action of some commuting operator (say φ_α or $\gamma \in \Gamma_\alpha$ for $\alpha \in \Delta$) also descends, assuming that the actions of both φ and the commuting operator are trivial modulo p.

Lemma 5.8. Let ν be an endomorphism of $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ which commutes with φ and sends $\widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E}_\Delta(K)}$ into itself. Let M be an étale φ-module over $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ with a commuting semilinear action of ν. Suppose that the action of φ on M is trivial modulo p, and that M has a basis $e_1, \dots, e_d$ with respect to which the matrix F of the action of φ has entries in $\widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E}_\Delta(K)}$. Then the matrix of the action of ν with respect to $e_1, \dots, e_d$ also has entries in $\widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E}_\Delta(K)}$.

Proof. By changing the basis $e_1, \dots, e_d$ via a suitable invertible matrix over $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$, we can ensure that it is congruent mod p to a basis on which φ acts trivially mod p (whose existence was hypothesized). That is, we may assume that F is congruent to the identity matrix modulo p.
Define $\widetilde{\varphi}\colon M_d(\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}) \to M_d(\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)})$ by

$$\widetilde{\varphi}(B) = F\varphi(B)\nu(F)^{-1},$$

where φ and ν are applied to the matrices componentwise. The operator $\widetilde{\varphi}$ is φ-semilinear and thus induces the structure of a φ-module on the $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$-module $M_d(\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)})$. As F and $\nu(F)^{-1}$ have entries in $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$, the matrix of $\widetilde{\varphi}$ on the standard basis of $M_d(\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)})$ does also. Thus $\widetilde{\varphi}$ induces a φ-module structure on the $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$-module $M_d(\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)})$.
Let N be the matrix of the action of ν on the basis of the e_i. As ν commutes with φ, we have

$$F\varphi(N) = N\nu(F),$$

and thus $\widetilde{\varphi}(N) = N$, which is to say $N \in M_d(\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)})^{\widetilde{\varphi}}$. Since F is congruent to the identity matrix modulo p, the action of $\widetilde{\varphi}$ is trivial modulo p on the standard basis of $M_d(\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)})$, so by Lemma 5.5,

$$M_d(\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)})^{\widetilde{\varphi}} = (M_d(\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}) \otimes_{\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)})^{\widetilde{\varphi}} = M_d(\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)})^{\widetilde{\varphi}},$$

and thus N has entries in $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$. □

5.3 Completion of the proof

To complete the proof, we must first prove essential surjectivity of base extension from $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$ to $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ for étale φ_Δ-modules which are trivial modulo p (full faithfulness having been proved in Corollary 5.7), then perform a descent to eliminate the extra hypothesis.

Proposition 5.9. Let M be an étale φ_Δ-module over $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ with a basis $e_1, \dots, e_d$ that is fixed by φ modulo p. Then there exists a basis $e'_1, \dots, e'_d$ of M on which φ_Δ acts via invertible matrices over $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$.

Proof. Let $\varphi := \varphi_{\alpha_1} \circ \dots \circ \varphi_{\alpha_n}$. Let $F \in \mathrm{GL}_d(\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)})$ be the matrix of the action of φ on the e_i, i.e. $\varphi(e_j) = \sum_{i=1}^d F_{ij} e_i$. It follows from (the proof of) [27, Theorem 8.5.3] that there exists a basis $e'_1, \dots, e'_n$ of M on which φ acts via a matrix with entries in $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$. Since each of the maps φ_α commutes with φ, and the action of φ is trivial modulo p, it follows from Lemma 5.8 that the matrix of the action of each φ_α on the e'_i also has entries in $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$. □

Theorem 5.10. Base extension of étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules from $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$ to $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ is an equivalence of categories. Consequently, both categories are equivalent to the category of continuous representations of $G_{K,\Delta}$ on finitely generated $\mathbb{Z}_p$-modules.

Proof. To show that the base extension functor is fully faithful, let M be an étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$ (playing the role of $\mathrm{Hom}_{\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}}(M,N)$ as in the proof of Corollary 5.7). We need to show that

$$M^{\varphi_\Delta, \Gamma_{K,\Delta}} = (M \otimes_{\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)})^{\varphi_\Delta, \Gamma_{K,\Delta}}.$$

The space $M \otimes_{\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ is an étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$, and thus corresponds to a continuous representation of $G_{K,\Delta}$ on a finitely generated $\mathbb{Z}_p$-module T. The action of each $G_{K,\alpha}$ on T/pT factors through $\mathrm{Gal}(L/K)$ for each $\alpha \in \Delta$ for some finite extension L/K. Since G_L acts trivially on T/pT, the action of $G_{L,\Delta}$ on T corresponds to an étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(L)}$ which is trivial modulo p. Now since

$$(\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(L)})^{\varphi_\Delta, \Gamma_{K,\Delta}} = \mathbb{Z}_p \subseteq (\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(L)})^{\varphi_\Delta, \Gamma_{K,\Delta}},$$

we have

$$\begin{aligned}(M \otimes_{\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)})^{\varphi_\Delta, \Gamma_{K,\Delta}} &\subseteq (M \otimes_{\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(L)})^{\varphi_\Delta, \Gamma_{K,\Delta}} \\ &= (M \otimes_{\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(L)})^{\varphi_\Delta, \Gamma_{K,\Delta}}.\end{aligned}$$

But as

$$\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)} \cap \widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(L)} = \widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)},$$

we have

$$(M \otimes_{\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)})^{\varphi_\Delta, \Gamma_{K,\Delta}} \subseteq M^{\varphi_\Delta, \Gamma_{K,\Delta}},$$

as desired.

To show that the functor is essentially surjective, let M be an étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$. We want to show M descends to an étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$.

Suppose first that M is killed by p^m for some positive integer m. As noted in Remark 5.2, the rings $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}/p^m\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$ and $\widetilde{\mathcal{O}}\mathcal{E}_\Delta(K)/p^m\widetilde{\mathcal{O}}\mathcal{E}_\Delta(K)$ coincide, so M may be viewed equally well as a module over both of them.

Suppose next that M is flat over $\mathbb{Z}_p$; by applying Remark 2.16 to the quotients M/p^mM, we see that M is projective over $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$. As above, we can find a finite extension L/K for which $M \otimes_{\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(L)}$ admits a basis that is fixed by φ modulo p. By Proposition 5.9, $M \otimes_{\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(L)}$ descends to an étale

φ_Δ-module M' over $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(L)}$; by Lemma 5.8, the action of $\Gamma_{K,\Delta}$ also descends. Now $\mathrm{Gal}(L/K)$ acts on $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(L)}$, thus on $M \otimes_{\mathcal{E}_\Delta(K)} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(L)} = M' \otimes_{\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(L)}} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(L)}$, and therefore on M', since $\mathrm{Gal}(L/K)$ preserves $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(L)}$. By Galois descent, M' descends to the $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$-module $M^\dagger = (M')^{\mathrm{Gal}(L/K)}$.

Suppose finally that M is general. By Theorem 4.29, M corresponds to a continuous representation of $G_{K,\Delta}$ on a finitely generated $\mathbb{Z}_p$-module T. Let T_0 be the torsion submodule of T and put $T_1 = T/T_0$; we then have a short exact sequence

$$0 \to T_0 \to T \to T_1 \to 0$$

of continuous $\mathbb{Z}_p$-representations of $G_{K,\Delta}$ in which T_0 is killed by p^m for some positive integer m and T_1 is projective. Let

$$0 \to M_0 \to M \to M_1 \to 0 \tag{5.1}$$

be the corresponding exact sequence of étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules over $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$. By the previous paragraph, M_1 descends uniquely to an étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module $M_1^\dagger$ over $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$. Moreover, the splittings $M_1 \to M$ of (5.1) in the category of $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$-modules form a torsor for the group

$$\begin{aligned}
\mathrm{Hom}_{\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}}(M_1, M_0) &= \mathrm{Hom}_{\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}/p^m \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}}(M_1/p^m M_1, M_0) \\
&= \mathrm{Hom}_{\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}/p^m \widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}}(M_1^\dagger/p^m M_1^\dagger, M_0) \\
&= \mathrm{Hom}_{\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}}(M_1^\dagger, M_0).
\end{aligned}$$

It follows that (5.1) descends uniquely to an exact sequence

$$0 \to M_0 \to M^\dagger \to M_1^\dagger \to 0$$

of étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules over $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$. Thus M descends to $M^\dagger$ as desired. □

6 Descent for overconvergent power series

In this section we complete the proof of Theorem 1.1 by establishing the following.

Theorem 6.1 (see Theorem 6.15). Base extension of étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules from $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$ to $\mathcal{O}_{\mathcal{E}_\Delta(K)}$ is an equivalence of categories. Consequently, both categories are equivalent to the category of continuous representations of $G_{K,\Delta}$ on finitely generated $\mathbb{Z}_p$-modules.

We then deduce the corresponding equivalence for $\mathbb{Q}_p$-representations (see Theorem 6.16).

Our arguments follow closely those in [23, Sections 2.5–2.6]. As in the previous section, we first prove everything for modules which are trivial modulo p, and then reduce to this case by faithfully flat descent. For one $\alpha \in \Delta$ at a time, we use the action of $\Gamma_{K,\alpha}$ to eliminate fractional powers of ϖ_α.

Remark 6.2. By contrast with Remark 5.3, the arguments of this section do not have a direct analogue for more general perfectoid fields. The extent to which the arguments can be generalized is in fact far from clear; see [25] for a detailed discussion.

6.1 $\mathbb{Z}_p$-representations

In this subsection, we establish Theorem 6.15. As noted above, for most of the proof we will consider only modules that are trivial modulo p; keeping in mind the final paragraph of the proof of Theorem 5.10, it will also be harmless to further restrict to projective modules.
Under these conditions, we first establish full faithfulness by applying what we already know about other base extensions.

Lemma 6.3. Let M be a projective étale φ-module over $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$ such that the action of φ is trivial modulo p. Then $M^\varphi = (M \otimes_{\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}} \mathcal{O}_{\mathcal{E}_\Delta(K)})^\varphi$.

Proof. By Lemma 5.5, we have

$$(M \otimes_{\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}} \mathcal{O}_{\mathcal{E}_\Delta(K)})^\varphi \subseteq (M \otimes_{\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)})^\varphi = (M \otimes_{\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)})^\varphi.$$

Since M is a projective module, we have

$$(M \otimes_{\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}} \mathcal{O}_{\mathcal{E}_\Delta(K)}) \cap (M \otimes_{\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}) = M.$$

The result follows by taking fixed points on both sides. □

Corollary 6.4. Base extension of projective étale φ_Δ-modules which are trivial modulo p from $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$ to $\mathcal{O}_{\mathcal{E}_\Delta(K)}$ is fully faithful.

Proof. This follows directly from Lemma 6.3, by letting $\mathrm{Hom}_{\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}}(M, N)$ play the role of M, as in the proofs of Proposition 4.11 and Corollary 5.7, since by Remark 2.15 homomorphisms of φ_Δ-modules are exactly those elements of $\mathrm{Hom}_{\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}}(M, N)$ that are fixed by φ_Δ. □

We now begin the real work of the proof, to establish essential surjectivity under our harmless extra restrictions. We start by setting up an important decomposition of the ring $R_\Delta(K)$.

Notation 6.5. For $\alpha \in \Delta$, let $X_\alpha \in \mathcal{O}_{\mathcal{E}_\Delta(K)}$ be the copy of $[\epsilon] - 1$ in the copy of $\mathcal{O}_{\mathcal{E}}$ indexed by α. Let $\bar{X}_\alpha$ be the reduction of X_α modulo p.

Proposition 6.6. For $i = 1, \dots, n$, let $\bar{T}_i \subseteq R_\Delta(K)$ be the closure of the subgroup generated by

$$(1 + \bar{X}_{\alpha_1})^{e_1} \cdots (1 + \bar{X}_{\alpha_i})^{e_i} \mathcal{O}_{\mathcal{E}_\Delta(K)}/p\mathcal{O}_{\mathcal{E}_\Delta(K)}$$

for $e_1, \dots, e_i \in \mathbb{Z}[p^{-1}] \cap [0, 1)$ with $e_i \neq 0$. The natural map

$$\mathcal{O}_{\mathcal{E}_\Delta(K)}/p\mathcal{O}_{\mathcal{E}_\Delta(K)} \oplus \bar{T}_1 \oplus \cdots \oplus \bar{T}_n \to R_\Delta(K)$$

is an isomorphism of Banach spaces over $\mathcal{O}_{\mathcal{E}_\Delta(K)}/p\mathcal{O}_{\mathcal{E}_\Delta(K)}$.

Proof. The case when $n = 1$ was proved in [23, Lemma 2.5.3 and Corollary 2.5.4], and an analogous proof establishes the general case. □

Remark 6.7. Let us spell out what Proposition 6.6 is saying when $K = K_0$. In this case, we can write elements of $R_\Delta(K)$ as power series in fractional powers of the variables $\bar{\varpi}_{\alpha_i}$ for $i = 1, \dots, n$. The summand $\bar{T}_i$ corresponds to those power series in which each monomial includes a nonintegral power of $\bar{\varpi}_{\alpha_i}$, but only integral powers of $\bar{\varpi}_{\alpha_j}$ for any $j > i$. The complement of these corresponds to power series in the $\bar{\varpi}_{\alpha_i}$ themselves, which constitute precisely $\mathcal{O}_{\mathcal{E}_\Delta(K)}/p\mathcal{O}_{\mathcal{E}_\Delta(K)}$.

We next show that in a strong sense, the action of Γ_{K,α_i} has no fixed points on $\bar{T}_i$.

Proposition 6.8. For every $\gamma \in \Gamma_{K,\alpha_i}$ of infinite order, with $\alpha_i \in \Delta$, there exists a $c > 0$ such that every $\bar{Y} \in R_\Delta(K)$ can be written uniquely in the form

$$\bar{U} + (\gamma - 1)(\bar{V}_i) + \sum_{j \neq i} \bar{V}_j$$

with $\bar{U} \in \mathcal{O}_{\mathcal{E}_\Delta(K)}/p\mathcal{O}_{\mathcal{E}_\Delta(K)}$, all $\bar{V}_j \in \bar{T}_j$, and

$$\max\{|\bar{U}|', |\bar{V}_1|', \dots, |\bar{V}_n|'\} \leq c|\bar{Y}|'.$$

Proof. In analogy with [23, Lemma 2.5.5 and Corollary 2.5.6], which proved this result in the case $n = 1$, we can show that the map $(\gamma - 1)\colon \bar{T}_i \to \bar{T}_i$ is bijective with bounded inverse. The desired result then follows from Proposition 6.6. □

We next lift the preceding definitions and propositions out of characteristic p.

Proposition 6.9. For $i = 1, \dots, n$, let $T_i \subseteq \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ be the closure, with respect to the weak topology, of the subgroup generated by

$$(1 + X_{\alpha_1})^{e_1} \cdots (1 + X_{\alpha_i})^{e_i} \mathcal{O}_{\mathcal{E}_\Delta(K)}$$

for $e_1, \dots, e_i \in \mathbb{Z}[p^{-1}] \cap [0,1)$ with $e_i \neq 0$. Then for all $\gamma \in \Gamma_{K,\alpha_i}$ of infinite order, there exist $c, r_0 > 0$ such that every $Y \in \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ can be written uniquely as

$$U + (\gamma - 1)(V_i) + \sum_{j \neq i} V_j$$

with $U \in \mathcal{O}_{\mathcal{E}_\Delta(K)}$, all $V_j \in T_j$, and

$$\max\{|U|_r, |V_1|_r, \dots, |V_n|_r\} \leq c^r |Y|_r$$

for $r \in (0, r_0]$.

Proof. The case when $n = 1$ was proved in [23, Corollary 2.5.7], and an analogous proof establishes the general case. □

Notation 6.10. For $i = 0, \dots, n$, let $\mathcal{O}^{(i)}_{\mathcal{E}_\Delta(K)}$ be the subring $\mathcal{O}_{\mathcal{E}_\Delta(K)} \oplus T_1 \oplus \cdots \oplus T_i$ of $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$; note that $\mathcal{O}^{(0)}_{\mathcal{E}_\Delta(K)} = \mathcal{O}_{\mathcal{E}_\Delta(K)}$ and $\mathcal{O}^{(n)}_{\mathcal{E}_\Delta(K)} = \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$. Also let $\mathcal{O}^{(i),\dagger}_{\mathcal{E}_\Delta(K)} := \mathcal{O}^{(i)}_{\mathcal{E}_\Delta(K)} \cap \widetilde{\mathcal{O}}^{\dagger}_{\mathcal{E}_\Delta(K)}$. These rings are stable under φ_Δ and $\Gamma_{K,\Delta}$.

For these subrings of $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$, we have the following analogue of Lemma 5.5.

Lemma 6.11. For $i \in \{0, \dots, n\}$, let M be an étale φ-module over $\mathcal{O}^{(i)}_{\mathcal{E}_\Delta(K)}$. Then $M^\varphi = (M \otimes_{\mathcal{O}^{(i)}_{\mathcal{E}_\Delta(K)}} \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)})^\varphi$.

Proof. This amounts to the full faithfulness of base extension of étale φ-modules from $\mathcal{O}^{(i)}_{\mathcal{E}_\Delta(K)}$ to $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$. In fact this functor is an equivalence of categories, as may be shown following the proof of [28, Corollary 5.4.6]. □

This in turn yields an analogue of Lemma 5.8.

Lemma 6.12. Choose $i \in \{0, \dots, n\}$. Let ν be an endomorphism of $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ which commutes with φ and sends $\mathcal{O}^{(i)}_{\mathcal{E}_\Delta(K)}$ into itself. Let M be an étale φ-module over $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ with a commuting semilinear action of ν. Suppose that the action of φ on M is trivial modulo p, and that M has a basis $e_1, \dots, e_d$ with respect to which the matrix F of the action of φ has entries in $\mathcal{O}^{(i)}_{\mathcal{E}_\Delta(K)}$. Then the matrix of the action of ν with respect to $e_1, \dots, e_d$ also has entries in $\mathcal{O}^{(i)}_{\mathcal{E}_\Delta(K)}$.

Proof. Proceed as in the proof of Lemma 5.8, using Lemma 6.11 in place of Lemma 5.5. □

At last, we are ready for the crucial calculation, which enables us to eliminate fractional powers one variable at a time.

Lemma 6.13. Let M be a projective étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$ admitting a basis $e_1, \dots, e_d$ fixed modulo p by φ_Δ and $\Gamma_{K,\Delta}$. For $i = 1, \dots, n$, pick $\gamma_i \in \Gamma_{\alpha_i}$ with $\gamma_i \equiv 1 \mod p$ and $\gamma_i \neq 1$. Then for $i = 0, \dots, n$, there exists a basis $e_1^{(i)}, \dots, e_d^{(i)}$ of M congruent to $e_1, \dots, e_d$ modulo p, such that the matrices of the actions of φ and γ_i on this basis have entries in $\mathcal{O}^{(i),\dagger}_{\mathcal{E}_\Delta(K)}$.

Proof. We proceed by descending induction on i. For the base case $i = n$, we take $e_1^{(n)}, \dots, e_d^{(n)} = e_1, \dots, e_d$. Given the statement for some $i > 0$, let $G \in \mathrm{GL}_d(\mathcal{O}^{(i),\dagger}_{\mathcal{E}_\Delta(K)})$ be the matrix of action of γ_i on $e_1^{(i)}, \dots, e_d^{(i)}$, i.e.,

$$\gamma_i(e_k^{(i)}) = \sum_{j=1}^d G_{jk} e_j^{(i)}.$$

Since the $e_j^{(i)}$ are fixed by γ_i modulo p (by virtue of being congruent to the original e_j), we have $p \mid G - 1$. By Proposition 2.8, there exists an $r \in (0, r_0]$ such that

$$\epsilon := |G - 1|_r^{1/3} < \min\{c^{-r}, 1\}.$$

We now construct a sequence of matrices whose product will converge to a change of basis matrix converting $e_1^{(i)}, \dots, e_d^{(i)}$ into a new basis $e_1^{(i-1)}, \dots, e_d^{(i-1)}$. Let $U_0 := 1$. By Proposition 6.9, there exist matrices

$$Y_0 \in M_d(\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}) \qquad \text{and} \qquad Z_{0j} \in M_d(T_j)$$

for $j = 1, \dots, i$ such that

$$G = 1 + Y_0 + Z_{01} + \cdots + Z_{0(i-1)} + (\gamma_i - 1)(Z_{0i})$$

and

$$|Y_0|_r, |Z_{0j}|_r \leq c^r |G - 1|_r \leq \epsilon^2$$

for all i. Now suppose that we have constructed matrices $U_l \in \mathrm{GL}_d(\mathcal{O}^{(i),\dagger}_{\mathcal{E}_\Delta(K)})$, $Y_l \in M_d(\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)})$, and $Z_{lj} \in M_d(T_j)$ such that

$$U_l \equiv 1 \mod p,$$
$$|Y_l|_r, |Z_{l1}|_r, \dots, |Z_{l(i-1)}|_r \leq \epsilon^2,$$
$$|Z_{li}|_r \leq \epsilon^{l+2},$$

and

$$U_l^{-1} G \gamma_i(U_l) = 1 + Y_l + Z_{l1} + \cdots + Z_{l(i-1)} + (\gamma_i - 1)(Z_{li}).$$

Let $G_l := U_l^{-1} G \gamma_1(U_l)$. Let $U_{l+1} := U_l(1 - Z_{li})$. Then

$$\begin{aligned} G_{l+1} &= (1 - Z_{li})^{-1} U_l G \gamma_i(U_l) \gamma_i(1 - Z_{li}) \\ &= (1 - Z_{li})^{-1}(1 + Y_l + Z_{l1} + \cdots + Z_{l(i-1)} + (\gamma_i - 1)(Z_{li})(1 - \gamma_i(Z_{li})) \\ &= 1 + Y_l + Z_{l1} + \cdots + Z_{l(i-1)} + Z_{li} Y_l - Y_l \gamma_i(Z_{li}) + E_l \end{aligned}$$

for some $E_l \in M_d(\mathcal{O}^{(i)}_{\mathcal{E}_\Delta(K)})$ with $|E_l|_r \leq \epsilon^{2l+4}$. We have $Z_{li}Y_l - Y_l\gamma_i(Z_i) \in M_d(\mathcal{O}^{(i)}_{\mathcal{E}_\Delta(K)})$ and

$$|Z_{li}Y_l - Y_l\gamma_i(Z_{li})|_r \leq \epsilon^{l+4}.$$

Write $Z_{li}Y_l - Y_l\gamma_i(Z_{li})$ as

$$A_l + B_{l1} + \cdots + B_{l(i-1)} + (\gamma_i - 1)(Z_{li})$$

with $A_l \in M_d(\mathcal{O}^{\dagger}_{\mathcal{E}_\Delta(K)})$, $B_{lj} \in M_d(T_j)$ for all j, and

$$|A_l|_r, |B_{lj}|_r \leq c^r \epsilon^{l+4} \leq \epsilon^{l+3}.$$

Let $Y_{l+1} := Y_l + A_l$, $Z_{(l+1)j} := Z_{lj} + B_{lj}$ for $j = 1, \ldots, i-1$, and $Z_{(l+1)i} := B_{li}$. We then have

$$|Y_{l+1}|_r, |Z_{(l+1)1}|_r, \ldots, |Z_{(l+1)(i-1)}|_r \leq \epsilon^2 \qquad \text{and} \qquad |Z_{(l+1)i}|_r \leq \epsilon^{(l+1)+2}.$$

If $l > 1$, it follows from the fact that $|B_{li}|_r \leq \epsilon^{l+3}$ that $U_{l+1} \equiv 1 \mod p$. If $l = 1$, observe that since $p \mid G - 1$, we have $p \mid (\gamma_i - 1)(Z_{0i})$; it then follows that $p \mid Z_{0i}$ as a consequence of the fact that $\gamma_i - 1$ is bijective on $\bar{T}_i$, so $U_{l+1} \equiv 1 \mod p$ in this case also.

The product $U_1U_2\cdots$ converges to a matrix $U \in \mathrm{GL}_d(\mathcal{O}^{(i),\dagger}_{\mathcal{E}_\Delta(K)})$; we thus obtain a new basis $e_1^{(i-1)}, \ldots, e_d^{(i-1)}$ of M by setting

$$e_k^{(i-1)} := \sum_{j=1}^d U_{jk} e_j^{(i)}$$

for $k = 1, \ldots, d$. Let A and H be the matrices of action of φ and γ_i, respectively, on the $e_j^{(i-1)}$:

$$\varphi(e_k^{(i-1)}) = \sum_{j=1}^d A_{jk} e_j^{(i-1)}$$
$$\gamma_i(e_k^{(i-1)}) = \sum_{j=1}^d H_{jk} e_j^{(i-1)}.$$

Then $H = U^{-1}G\gamma_i(U) \in M_d(\mathcal{O}^{(i-1),\dagger}_{\mathcal{E}_\Delta(K)})$ with $H \equiv 1 \mod p$. Since φ and γ_i commute, we have $A\varphi(H) = H\gamma_i(A)$. Write $A = B + C_1 + \cdots + C_i$ with $B \in M_d(\mathcal{O}^{\dagger}_{\mathcal{E}_\Delta(K)})$ and $C_j \in M_d(T_j)$ for all j. Then

$$H^{-1}C_i\varphi(H) - C_i = (\gamma_i - 1)(C_i).$$

If $C_i \neq 0$, then let m be the largest integer such that $p^m \mid C_i$. Since $H \equiv 1 \mod p$, we have $p^{m+1} \mid H^{-1}C_i\varphi(H) - C_i$, but, referring again to the fact that

$\gamma_i - 1$ is bijective on $\bar{T}_i$, we have $p^{m+1} \nmid (\gamma_i - 1)(C_1)$. By contradiction, $C_i = 0$ and thus the matrix A has entries in $\mathcal{O}^{(i),\dagger}_{\mathcal{E}_\Delta(K)}$. By Lemma 5.8 and Lemma 6.12, the matrix of the action of γ_{i-1} on the basis of the $e_j^{(i-1)}$ also has entries in $\mathcal{O}^{(i-1),\dagger}_{\mathcal{E}_\Delta(K)}$; this completes the induction. $\square$

We finally tie everything together and eliminate our auxiliary hypotheses.

Theorem 6.14. Base extension of étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules from $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$ to $\mathcal{O}_{\mathcal{E}_\Delta(K)}$ is an equivalence of categories.

Proof. As in the proof of Theorem 5.10, there is nothing to check for modules killed by a power of p, and given the statement for projective modules we may easily obtain the general case. We thus consider hereafter only projective étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules.

By Theorem 4.39, base extension from $\mathcal{O}_{\mathcal{E}_\Delta(K)}$ to $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ is an equivalence; by Theorem 5.10, base extension from $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$ to $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ is an equivalence. It suffices to show that base extension from $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$ to $\mathcal{O}_{\mathcal{E}_\Delta(K)}$ is fully faithful and base extension from $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$ to $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$ is essentially surjective.

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{E}_\Delta(K)} \arrow[r] & \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)} \\
\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)} \arrow[u] \arrow[r] & \widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)} \arrow[u]
\end{tikzcd}$$

The fact that base extension from $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$ to $\mathcal{O}_{\mathcal{E}_\Delta(K)}$ is fully faithful follows from Corollary 6.4, just as in the proof of Theorem 5.10.

For the essential surjectivity of base extension from $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$ to $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$, as in the proof of Theorem 5.10 it suffices to consider an $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$-module with a basis fixed modulo p by φ_Δ and $\Gamma_{K,\Delta}$: let M be such a module. For $i = 1, \dots, n$, pick $\gamma_i \in \Gamma_{\alpha_i}$ with $\gamma_i \equiv 1 \mod p$. By Lemma 6.13 (applied with $i = 0$), there exists a basis $e_1^{(0)}, \dots, e_r^{(0)}$ of M on which the matrix of action of φ has entries in $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$ and is congruent to the identity matrix modulo p. Let $M^\dagger$ be the $\mathcal{O}^\dagger_{\mathcal{E}_\Delta}$-span of $e_1^{(0)}, \dots, e_r^{(0)}$. This is an étale φ-module over $\mathcal{O}^\dagger_{\mathcal{E}_\Delta}$ such that $M^\dagger \otimes_{\mathcal{O}^\dagger_{\mathcal{E}_\Delta}} \widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta} \simeq M$. As the matrix of the action of φ has entries in $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$, by Lemma 5.8 and Lemma 6.12 the matrices of each φ_α and of each $\gamma \in \Gamma_{K,\alpha}$ also have entries in $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$ for each $\alpha \in \Delta$. Hence $M^\dagger$ is indeed an étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over $\mathcal{O}^\dagger_{\mathcal{E}_{\Delta(K)}}$, completing the proof of essential surjectivity. $\square$

For convenience, we summarize everything we have established in a single theorem statement.

THEOREM 6.15. The category of continuous representations of $G_{K,\Delta}$ on finite free $\mathbb{Z}_p$-modules is equivalent to the category of projective étale $(\varphi_\Delta, \Gamma_\Delta)$-modules over each of the rings $\mathcal{O}_{\mathcal{E}_\Delta(K)}$, $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$, $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$, and $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$.

Proof. The equivalence between representations and $(\varphi_\Delta, \Gamma_\Delta)$-modules over $\mathcal{O}_{\mathcal{E}_\Delta(K)}$ and $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(K)}$ is given by Theorem 4.29. We add $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$ to the equivalence using Theorem 5.10, and $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(K)}$ using Theorem 6.14. □

6.2 $\mathbb{Q}_p$-REPRESENTATIONS

We next formulate a corresponding statement for $\mathbb{Q}_p$-representations. It is easy to obtain a single statement of the right form, but there are some subtleties around finding the strongest possible formulation; we touch briefly on these issues, but do not give a definitive resolution.

THEOREM 6.16. The category of continuous representations of $G_{K,\Delta}$ on finite dimensional $\mathbb{Q}_p$-vector spaces is equivalent to the category of projective étale $(\varphi_\Delta, \Gamma_\Delta)$-modules over each of the rings $\mathcal{E}_\Delta(K)$, $\widetilde{\mathcal{E}}_\Delta(K)$, $\mathcal{E}^\dagger_\Delta(K)$, and $\widetilde{\mathcal{E}}^\dagger_\Delta(K)$.

Proof. Since $G_{K,\Delta}$ is a profinite topological group, it is compact; consequently, any finite–dimensional $\mathbb{Q}_p$-vector space with a continuous $G_{K,\Delta}$-action admits a stable $\mathbb{Z}_p$-lattice. (For example, if one starts with any $\mathbb{Z}_p$-lattice, taking the sum of its images under all elements of $G_{K,\Delta}$ yields a stable lattice.) Consequently, Theorem 6.15 defines fully faithful functors from the category of continuous representations of $G_{K,\Delta}$ on finite–dimensional $\mathbb{Q}_p$-vector spaces to the various categories of projective étale $(\varphi_\Delta, \Gamma_\Delta)$-modules. In view of our definition of the étale condition in this setting, essential surjectivity of these functors also reduces at once to Theorem 6.15. □

While Theorem 6.16 will be sufficient for our present purposes, as noted in Remark 2.14 it should be possible to formally weaken the definition of a $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over a ring in which p is invertible. We give some limited evidence in this direction.

LEMMA 6.17. Let M be an étale φ_Δ-module over $\mathcal{O}_{\mathcal{E}_\Delta}/p\mathcal{O}_{\mathcal{E}_\Delta}$. Then the underlying module of M is projective over $\mathcal{O}_{\mathcal{E}_\Delta}/p\mathcal{O}_{\mathcal{E}_\Delta}$.

Proof. Since $\mathcal{O}_{\mathcal{E}_\Delta}$ is noetherian, M is finitely presented. Consequently, $\widetilde{M} := M \otimes_{\mathcal{O}_{\mathcal{E}_\Delta}/p\mathcal{O}_{\mathcal{E}_\Delta}} R_\Delta$ is a finitely presented étale φ_Δ-module over R_Δ. By Remark 2.16, $\widetilde{M}$ is projective over R_Δ. Now note that the morphism $\mathcal{O}_{\mathcal{E}_\Delta}/p\mathcal{O}_{\mathcal{E}_\Delta} \to R_\Delta$ is split in the category of $\mathcal{O}_{\mathcal{E}_\Delta}/p\mathcal{O}_{\mathcal{E}_\Delta}$-modules by the morphism

$$\sum_{i_1,\dots,i_n \in \mathbb{Z}[p^{-1}]} (\bar{a}_{i_1} \bar{\varpi}_{\alpha_1}^{i_1}) \otimes \cdots \otimes (\bar{a}_{i_n} \bar{\varpi}_{\alpha_n}^{i_n}) \mapsto \sum_{i_1,\dots,i_n \in \mathbb{Z}} (\bar{a}_{i_1} \bar{\varpi}_{\alpha_1}^{i_1}) \otimes \cdots \otimes (\bar{a}_{i_n} \bar{\varpi}_{\alpha_n}^{i_n})$$

(that is, discarding non-integral powers of the series parameters). This means that $\mathcal{O}_{\mathcal{E}_\Delta}/p\mathcal{O}_{\mathcal{E}_\Delta} \to R_\Delta$ is a *pure* morphism of rings, so the fact that $\widetilde{M}$ is finite projective over R_Δ implies that M is finite projective over $\mathcal{O}_{\mathcal{E}_\Delta}/p\mathcal{O}_{\mathcal{E}_\Delta}$ [44, Tag 08XD], as desired. □

Corollary 6.18. Let M be an étale φ_Δ-module over $\mathcal{O}_{\mathcal{E}_\Delta}$. If the underlying module of M is p-torsion-free, then it is projective over $\mathcal{O}_{\mathcal{E}_\Delta}$.

Proof. By 6.17, M/pM is projective of some finite rank r over $\mathcal{O}_{\mathcal{E}_\Delta}/p\mathcal{O}_{\mathcal{E}_\Delta}$. We next check that for each positive integer m, M/p^mM is projective of rank r over $\mathcal{O}_{\mathcal{E}_\Delta}/p^m\mathcal{O}_{\mathcal{E}_\Delta}$. Since M is p-torsion-free, it is flat over $\mathbb{Z}_p$; consequently, M/p^mM is flat over $\mathbb{Z}/p^m\mathbb{Z}$. Since M/pM is flat over $\mathcal{O}_{\mathcal{E}_\Delta}/p\mathcal{O}_{\mathcal{E}_\Delta}$, we may apply [44, Tag 06A5] to deduce that M/p^mM is flat over $\mathcal{O}_{\mathcal{E}_\Delta}/p^m\mathcal{O}_{\mathcal{E}_\Delta}$; we may then apply [44, Tag 05CG] to deduce that M/p^mM is projective over $\mathcal{O}_{\mathcal{E}_\Delta}/p^m\mathcal{O}_{\mathcal{E}_\Delta}$. The rank-$r$ condition is then enforced by Nakayama's lemma.
Now let $\mathrm{Fitt}_i(M)$ denote the sequence of Fitting ideals of M (see for example [44, Tag 07Z6]). For each positive integer m, the fact that M/p^mM is projective of rank r over $\mathcal{O}_{\mathcal{E}_\Delta}/p^m\mathcal{O}_{\mathcal{E}_\Delta}$ implies that $\mathrm{Fitt}_i(M)$ is contained in $p^m\mathcal{O}_{\mathcal{E}_\Delta}$ for all $i < r$ and is equal to $\mathcal{O}_{\mathcal{E}_\Delta}$ for all $i \geq r$. Running over all m, we deduce that $\mathrm{Fitt}_i(M) = 0$ for $i < r$ and is equal to $\mathcal{O}_{\mathcal{E}_\Delta}$ for all $i \geq r$; hence M is projective of rank r. □

Theorem 6.19. Let M be a $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over $\mathcal{E}_\Delta$ satisfying the following conditions.

(i) The underlying φ_Δ-module of M is the base extension of an étale φ_Δ-module over $\mathcal{O}_{\mathcal{E}_\Delta}$.

(ii) The action of $\Gamma_{K,\Delta}$ is *bounded*: for some (and hence any) finitely generated $\mathcal{O}_{\mathcal{E}_\Delta}$-submodule M_0 of M which generates M over $\mathcal{O}_{\mathcal{E}_\Delta}$, the action of $\Gamma_{K,\Delta}$ carries M_0 into $p^{-m}M_0$ for some nonnegative integer m.

Then M is an étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over $\mathcal{E}_\Delta$ in the sense of Definition 2.13; in particular, by Theorem 6.16 it corresponds to a continuous $\mathbb{Q}_p$-representation of $G_{K,\Delta}$.

Proof. In condition (i), there is no loss of generality in assuming that there exists a finitely generated φ_Δ-stable $\mathcal{O}_{\mathcal{E}_\Delta}$-submodule M_0 of M which generates M over $\mathcal{O}_{\mathcal{E}_\Delta}$, such that M_0 is an étale φ_Δ-module over $\mathcal{O}_{\mathcal{E}_\Delta}$. Let M_1 be the $\mathcal{O}_{\mathcal{E}_\Delta}$-submodule of M generated by $\gamma(M_0)$ for all $\gamma \in \Gamma_{K,\Delta}$; by condition (ii), M_1 is again a finitely generated $\mathcal{O}_{\mathcal{E}_\Delta}$-module. In particular, M_1 is an étale φ_Δ-module; by Corollary 6.18, M_1 is projective over $\mathcal{O}_{\mathcal{E}_\Delta}$. Hence M_1 has the structure of a projective étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over $\mathcal{O}_{\mathcal{E}_\Delta}$, and so M is an étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over $\mathcal{E}_\Delta$ in the sense of 2.13. □

Remark 6.20. One may extend Definition 6.19 to $(\varphi_\Delta, \Gamma_{K,\Delta})$-modules over $\mathcal{E}^\dagger_\Delta$ by similar arguments. We do not know how to extend it to $\widetilde{\mathcal{E}}_\Delta$ or $\widetilde{\mathcal{E}}^\dagger_\Delta$.

We also do not know whether in condition (i) of Theorem 6.19, one may consider the underlying φ-module instead of the underlying φ_Δ-module; in particular, it is unclear whether an analogue of the bounded condition must be applied to the partial Frobenius actions. To decide this, one may need some of the slope theory for modules over relative Robba rings developed in [27].

7 Galois cohomology

The goal of this section is to show that the group cohomology of $G_{K,\Delta}$ with values in a p-adic representation V is computed by the Herr complex of the multivariate $(\varphi_\Delta,\Gamma_{K,\Delta})$-module associated to V via Theorem 6.15 or Theorem 6.16. The case $K=\mathbb{Q}_p$ is proven in [36]; we deduce the general case by reducing to that case, using a generalization of Shapiro's Lemma for (φ,Γ)-modules [35, Theorem 2.2] to this context.
In the following discussion, we write $\mathbb{D}$ for the functor taking a $\mathbb{Z}_p$-representation (resp. a $\mathbb{Q}_p$-representation) to its corresponding étale $(\varphi_\Delta,\Gamma_{K,\Delta})$-module over $\mathcal{O}_{\mathcal{E}_\Delta(K)}$ (resp. over $\mathcal{E}_\Delta(K)$), and $\mathbb{D}^\dagger$ for the functor taking such a representation to its corresponding étale $(\varphi_\Delta,\Gamma_{K,\Delta})$-module over $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$ (resp. over $\mathcal{E}^\dagger_\Delta(K)$).

Definition 7.1. For any abelian group $D^?$ equipped with commuting operators φ_α ($\alpha\in\Delta$) we define the cochain complex

$$\Phi^\bullet(D^?)\colon 0\to D^?\to\bigoplus_{\alpha\in\Delta}D^?\to\cdots\to\bigoplus_{\{\alpha_1,\dots,\alpha_r\}\in\binom{\Delta}{r}}D^?\to\cdots\to D^?\to 0$$

where for all $0\le r\le|\Delta|-1$, the map $d^{\beta_1,\dots,\beta_{r+1}}_{\alpha_1,\dots,\alpha_r}\colon D^?\to D^?$ from the component in the rth term corresponding to $\{\alpha_1,\dots,\alpha_r\}\subseteq\Delta$ to the component corresponding to the $(r+1)$-tuple $\{\beta_1,\dots,\beta_{r+1}\}\subseteq\Delta$ is given by

$$d^{\beta_1,\dots,\beta_{r+1}}_{\alpha_1,\dots,\alpha_r}=\begin{cases}0 & \text{if } \{\alpha_1,\dots,\alpha_r\}\not\subseteq\{\beta_1,\dots,\beta_{r+1}\}\\ (-1)^\varepsilon(\mathrm{id}-\varphi_\beta) & \text{if } \{\beta_1,\dots,\beta_{r+1}\}=\{\alpha_1,\dots,\alpha_r\}\cup\{\beta\}\end{cases},$$

where $\varepsilon=\varepsilon(\alpha_1,\dots,\alpha_r,\beta)$ is the number of elements in the set $\{\alpha_1,\dots,\alpha_r\}$ smaller than β.

Let $K/\mathbb{Q}_p$ be a finite extension. We denote by $C_{K,\Delta}$ the torsion subgroup of $\Gamma_{K,\Delta}\simeq\prod_{\alpha\in\Delta}\mathrm{Gal}(K(\mu_{p^\infty})/K)$ and by $H^*_{K,\Delta}$ the kernel of the composite quotient map $G_{K,\Delta}\twoheadrightarrow\Gamma_{K,\Delta}\twoheadrightarrow\Gamma^*_{K,\Delta}:=\Gamma_{K,\Delta}/C_{K,\Delta}$. We choose topological generators $\gamma_{K,\alpha}\in\Gamma^*_{K,\alpha}:=\Gamma_{K,\alpha}/(\Gamma_\alpha\cap C_{K,\Delta})$ for each $\alpha\in\Delta$.

Definition 7.2. If A is an arbitrary (for now abstract) representation of the group $\Gamma^*_{K,\Delta}\simeq\prod_{\alpha\in\Delta}\mathbb{Z}_p$ on a $\mathbb{Z}_p$-module we denote by $\Gamma^\bullet_{K,\Delta}(A)$ the cochain complex

$$\Gamma^\bullet_{K,\Delta}(A)\colon 0\to A\to\bigoplus_{\alpha\in\Delta}A\to\cdots\to\bigoplus_{\{\alpha_1,\dots,\alpha_r\}\in\binom{\Delta}{r}}A\to\cdots\to A\to 0$$

where for all $0 \le r \le |\Delta| - 1$, the map $d^{\beta_1,\dots,\beta_{r+1}}_{\alpha_1,\dots,\alpha_r}\colon A \to A$ from the component in the rth term corresponding to $\{\alpha_1,\dots,\alpha_r\} \subseteq \Delta$ to the component corresponding to the $(r+1)$-tuple $\{\beta_1,\dots,\beta_{r+1}\} \subseteq \Delta$ is given by

$$d^{\beta_1,\dots,\beta_{r+1}}_{\alpha_1,\dots,\alpha_r} = \begin{cases} 0 & \text{if } \{\alpha_1,\dots,\alpha_r\} \not\subseteq \{\beta_1,\dots,\beta_{r+1}\} \\ (-1)^{\varepsilon}(\mathrm{id} - \gamma_{K,\beta}) & \text{if } \{\beta_1,\dots,\beta_{r+1}\} = \{\alpha_1,\dots,\alpha_r\} \cup \{\beta\} \end{cases},$$

where $\varepsilon = \varepsilon(\alpha_1,\dots,\alpha_r,\beta)$ is the number of elements in the set $\{\alpha_1,\dots,\alpha_r\}$ smaller than β.

Definition 7.3. Let D be an étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over any of the rings $\mathcal{O}_{\mathcal{E}_\Delta(K)}$, $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$, $\mathcal{E}_\Delta(K)$, or $\mathcal{E}^\dagger_\Delta(K)$. We define the cochain complex $\Phi\Gamma^\bullet_{K,\Delta}(D)$ as the total complex of the double complex $\Gamma^\bullet_{K,\Delta}(\Phi^\bullet(D^{C_{K,\Delta}}))$ and call it the *Herr complex* of D.

Definition 7.4. If $\mathbb{Q}_p \le F \le K$ are finite extensions and V is a continuous finite dimensional representation of $G_{K,\Delta}$ either over $\mathbb{Q}_p$ or $\mathbb{Z}_p$, then we denote by $\mathrm{Ind}_K^F V := \mathbb{Z}_p[G_{F,\Delta}] \otimes_{\mathbb{Z}_p[G_{K,\Delta}]} V$ the representation of $G_{F,\Delta}$ induced from the representation V of the finite index subgroup $G_{K,\Delta}$.

Remark 7.5. Since $G_{K,\Delta}$ has finite index in $G_{F,\Delta}$ the induced and coinduced representations are isomorphic, so we may use the latter.

By Shapiro's Lemma for continuous cohomology of profinite groups [43, Proposition I.2.5.10], we have a natural isomorphism of cohomological δ-functors $H^\bullet(G_{F,\Delta}, \mathrm{Ind}_K^F(\cdot)) \simeq H^\bullet(G_{K,\Delta}, \cdot)$.

Definition 7.6. Let D be an étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over any of the rings $\mathcal{O}_{\mathcal{E}_\Delta(K)}$, $\mathcal{E}_\Delta(K)$, $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$, or $\mathcal{E}^\dagger_\Delta(K)$. We define the induced $(\varphi_\Delta, \Gamma_{F,\Delta})$-module over the analogous ring with base F instead of K as

$$\mathrm{Ind}_K^F D := \mathbb{Z}_p[\Gamma_{F,\Delta}] \otimes_{\mathbb{Z}_p[\Gamma_{K,\Delta}]} D .$$

We equip $\mathrm{Ind}_K^F D$ with the obvious $\Gamma_{F,\Delta}$-action on the left factor, and we put $\varphi_\alpha(\gamma \otimes x) := \gamma \otimes (\varphi_\alpha(x))$ and $\lambda \cdot (\gamma \otimes x) := \gamma \otimes (\gamma^{-1}(\lambda) \cdot x)$ for $\gamma \in \Gamma_{F,\Delta}$, $\alpha \in \Delta$, $\lambda \in \mathcal{O}_{\mathcal{E}_\Delta(F)}$ (resp. $\mathcal{E}_\Delta(F)$, $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(F)}$, $\mathcal{E}^\dagger_\Delta(F)$). Here note that $\mathcal{O}_{\mathcal{E}_\Delta(F)}$ (resp. $\mathcal{E}_\Delta(F)$, $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(F)}$, $\mathcal{E}^\dagger_\Delta(F)$) is naturally a subring of $\mathcal{O}_{\mathcal{E}_\Delta(K)}$ (resp. of $\mathcal{E}_\Delta(K)$, $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$, $\mathcal{E}^\dagger_\Delta(K)$) and in the definition of $\mathrm{Ind}_K^F D$, we regard D as a module over this subring.

Proposition 7.7. Let $\mathbb{Q}_p \le F \le K$ be finite extensions and let V be a continuous representation of $G_{K,\Delta}$ on a finite dimensional $\mathbb{Q}_p$-vector space or finitely generated $\mathbb{Z}_p$-module. Then we have $\mathbb{D}(\mathrm{Ind}_K^F V) \simeq \mathrm{Ind}_K^F \mathbb{D}(V)$ and $\mathbb{D}^\dagger(\mathrm{Ind}_K^F V) \simeq \mathrm{Ind}_K^F \mathbb{D}^\dagger(V)$.

Proof. This is completely analogous to the proof of [35, Proposition 2.1]. Choose a set of representatives $\overline{U}$ of cosets of $\Gamma_{F,\Delta}/\Gamma_{K,\Delta}$ and lift it to a subset U of $G_{F,\Delta}$. Similarly, let $W \subset H_{F,\Delta}$ be a set of representatives of the cosets of $H_{F,\Delta}/H_{K,\Delta}$; then $UW = \{uw \mid u \in U, w \in W\}$ is a set of representatives of the cosets of $G_{F,\Delta}/G_{K,\Delta}$. We thus compute

$$\begin{aligned}\mathbb{D}(\mathrm{Ind}_K^F V) &= (\mathrm{Ind}_K^F V \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{E}_\Delta^{\mathrm{nr}}})^{H_{F,\Delta}} \\ &= \left((\mathbb{Z}_p[G_{F,\Delta}] \otimes_{\mathbb{Z}_p[G_{K,\Delta}]} V) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{E}_\Delta^{\mathrm{nr}}}\right)^{H_{F,\Delta}} \\ &= \left(\left(\bigoplus_{u\in U}\bigoplus_{w\in W} uw \otimes V\right) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{E}_\Delta^{\mathrm{nr}}}\right)^{H_{F,\Delta}} \\ &= \bigoplus_{u\in U}\left(\left(\bigoplus_{w\in W} uw \otimes V\right) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{E}_\Delta^{\mathrm{nr}}}\right)^{H_{F,\Delta}} .\end{aligned}$$

Now an element $x = \sum_i \sum_{w\in W} uw \otimes v_{i,w} \otimes \lambda_{i,w} \in (\bigoplus_{w\in W} uw \otimes V) \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{E}_\Delta^{\mathrm{nr}}}$ lies in the $H_{F,\Delta}$-invariant part if and only if

$$u^{-1}x = \sum_i \sum_{w\in W} w \otimes v_{i,w} \otimes u^{-1}(\lambda_{i,w})$$

does, since $H_{F,\Delta}$ is normalized by $u \in G_{F,\Delta}$. Using the invariance under the multiplication by $w'w^{-1} \in H_{F,\Delta}$ we deduce

$$x_u := \sum_i v_{i,w} \otimes w^{-1}u^{-1}(\lambda_{i,w}) = \sum_i v_{i,w'} \otimes w'^{-1}u^{-1}(\lambda_{i,w'})$$

for all $w, w' \in W$. Further, x_u must be $H_{K,\Delta}$-invariant, i.e. it belongs to $(V \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathcal{E}_\Delta^{\mathrm{nr}}})^{H_{K,\Delta}}$. So the isomorphism $\mathbb{D}(\mathrm{Ind}_K^F V) \to \mathrm{Ind}_K^F \mathbb{D}(V)$ is given by sending x to $\sum_{u\in U} \bar{u} \otimes x_u$ where $\bar{u}$ is the image of u under the quotient map $G_{F,\Delta} \to \Gamma_{F,\Delta}$. This is bijective since the ranks of the two sides are equal.
The statement on $\mathbb{D}^\dagger$ follows from Theorem 6.14 using the result on $\mathbb{D}$. □

Lemma 7.8. Let A be a $\mathbb{Z}_p[\Gamma^*_{K,\Delta}]$-module and assume the topological generators $\gamma_{K,\alpha}$ and $\gamma_{F,\alpha}$ are chosen so that $\gamma_{K,\alpha} = \gamma_{F,\alpha}^{p^r}$ for all $\alpha \in \Delta$ where $p^r := [\Gamma^*_{F,\alpha} : \Gamma^*_{K,\alpha}]$. Then the complex $\Gamma^\bullet_{K,\Delta}(A)$ is quasi-isomorphic to the complex $\Gamma^\bullet_{F,\Delta}(\mathbb{Z}_p[\Gamma^*_{F,\Delta}] \otimes_{\mathbb{Z}_p[\Gamma^*_{K,\Delta}]} A)$. The quasi-isomorphism is functorial in A.

Proof. We proceed by induction on $|\Delta|$. If $|\Delta| = 1$ then by our assumption on the topological generators the diagram

$$\begin{tikzcd} \Gamma^\bullet_K: & 0 \arrow[r] & A \arrow[r, "\gamma_K - \mathrm{id}"] \arrow[d, "(\sum_{j=0}^{p^r-1}\gamma_F^j)\otimes\mathrm{id}"'] & A \arrow[r] \arrow[d, "1\otimes\mathrm{id}"] & 0 \\ \Gamma^\bullet_F: & 0 \arrow[r] & \mathbb{Z}_p[\Gamma^*_{F,\Delta}] \otimes_{\mathbb{Z}_p[\Gamma^*_{K,\Delta}]} A \arrow[r, "\gamma_F - \mathrm{id}"] & \mathbb{Z}_p[\Gamma^*_{F,\Delta}] \otimes_{\mathbb{Z}_p[\Gamma^*_{K,\Delta}]} A \arrow[r] & 0 \end{tikzcd}$$

commutes. Since $(\sum_{j=0}^{p^r-1}\gamma_F^j)\otimes\mathrm{id}$ is injective, so is the induced map on h^0. Further, any element $x\in\mathbb{Z}_p[\Gamma^*_{F,\Delta}]\otimes_{\mathbb{Z}_p[\Gamma^*_{K,\Delta}]}A$ can uniquely be written as $x=\sum_{j=0}^{p^r-1}\gamma_F^j\otimes x_j$ with $x_j\in A$ and x is fixed by γ_F if and only if $x_0=x_1=\cdots=x_{p^r-1}$ is fixed by $\gamma_F^{p^r}=\gamma_K$. Hence we deduce $h^0\Gamma^\bullet_K(A)\simeq h^0\Gamma^\bullet_F(\mathbb{Z}_p[\Gamma^*_{F,\Delta}]\otimes_{\mathbb{Z}_p[\Gamma^*_{K,\Delta}]}A)$.
On the other hand if $1\otimes x=(\gamma_F-1)(\sum_{j=0}^{p^r-1}\gamma_F^j\otimes y_j)$ then $y_0=y_1=\cdots=y_{p^r-1}$ and x lies in the image of γ_K-1 since the γ_F^j-component of the right hand side has to vanish for $j\geq 1$. This shows the injectivity on h^1. Finally, $\sum_{j=0}^{p^r-1}\gamma_F^j\otimes x_j-1\otimes(\sum_{j=0}^{p^r-1}x_j)$ lies in the image of γ_F-1 showing that the induced map on h^1 is onto.
The induction step follows from the spectral sequences associated to the double complexes $\Gamma^\bullet_{K,\alpha}(\Gamma^\bullet_{K,\Delta\setminus\{\alpha\}}(A))$ and $\Gamma^\bullet_{F,\alpha}(\Gamma^\bullet_{F,\Delta\setminus\{\alpha\}}(\mathbb{Z}_p[\Gamma^*_{F,\Delta}]\otimes_{\mathbb{Z}_p[\Gamma^*_{K,\Delta}]}A))$. $\square$

Remark 7.9. Whenever the action of $\mathbb{Z}_p[\Gamma^*_{K,\Delta}]$ on A extends to the Iwasawa algebra $\mathbb{Z}_p[\![\Gamma^*_{K,\Delta}]\!]$, we may relax our assumption that $\gamma_{K,\alpha}=\gamma_{F,\alpha}^{p^r}$: for any other topological generator $\gamma'_{K,\alpha}$ of the group $\Gamma^*_{K,\alpha}$, the element $\frac{\gamma'_{K,\alpha}-1}{\gamma_{K,\alpha}-1}$ is a unit in the Iwasawa algebra, and therefore the complex defined using $\gamma'_{K,\alpha}$ instead of $\gamma_{K,\alpha}$ $(\alpha\in\Delta)$ is quasi-isomorphic to $\Gamma^\bullet_{K,\Delta}(A)$.

Theorem 7.10. We have a natural isomorphism of cohomological δ-functors

$$h^i\Phi\Gamma^\bullet_{F,\Delta}(\mathrm{Ind}_K^F(\cdot))\simeq h^i\Phi\Gamma^\bullet_{K,\Delta}(\cdot)$$

on the category of étale $(\varphi_\Delta,\Gamma_{K,\Delta})$-modules over $\mathcal{O}_{\mathcal{E}_\Delta(K)}$ (resp. over $\mathcal{E}_\Delta(K)$, $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$, and $\mathcal{E}^\dagger_\Delta(K)$).

Proof. Let D be an étale $(\varphi_\Delta,\Gamma_{K,\Delta})$-module over any of these rings. Since $\mathbb{Z}_p[\Gamma_{F,\Delta}]$ is free as a module over $\mathbb{Z}_p[\Gamma_{K,\Delta}]$, we have a natural identification

$$h^i\Phi^\bullet(\mathrm{Ind}_K^F D)\simeq\mathbb{Z}_p[\Gamma_{F,\Delta}]\otimes_{\mathbb{Z}_p[\Gamma_{K,\Delta}]}h^i\Phi^\bullet(D)$$

of cohomological δ-functors. Taking $C_{F,\Delta}$-invariants we obtain

$$h^i\Phi^\bullet((\mathrm{Ind}_K^F D)^{C_{F,\Delta}})\simeq\mathbb{Z}_p[\Gamma^*_{F,\Delta}]\otimes_{\mathbb{Z}_p[\Gamma^*_{K,\Delta}]}h^i\Phi^\bullet(D^{C_{K,\Delta}})\ .$$

In case $\gamma_{K,\alpha}=\gamma_{F,\alpha}^{p^r}$ for all $\alpha\in\Delta$ the result follows from Lemma 7.8 (with $A:=h^i\Phi^\bullet(D^{C_{K,\Delta}})$) using the spectral sequences associated to the double complexes $\Gamma^\bullet_{K,\Delta}(\Phi^\bullet(D^{C_{K,\Delta}}))$ and $\Gamma^\bullet_{F,\Delta}(\Phi^\bullet((\mathrm{Ind}_K^F D)^{C_{F,\Delta}}))$. Note that for $F=\mathbb{Q}_p$, the topological generator $\gamma_{\mathbb{Q}_p,\alpha}$ of $\Gamma^*_{\mathbb{Q}_p,\alpha}$ can be chosen arbitrarily by [36, Theorem 2.6.2, Corollary 3.5.10]. Hence by applying Remark 7.9 with $F=\mathbb{Q}_p$, we deduce that the Herr complex $\Phi\Gamma^\bullet_{K,\Delta}(D)$ does not depend on the choices of topological generators of $\Gamma^*_{K,\alpha}$ for $\alpha\in\Delta$ up to quasi-isomorphism. $\square$

Corollary 7.11. We have a natural isomorphism of cohomological δ-functors $H^i(G_{K,\Delta}, \cdot) \simeq h^i\Phi\Gamma^\bullet_{K,\Delta}(\mathbb{D}(\cdot)) \simeq h^i\Phi\Gamma^\bullet_{K,\Delta}(\mathbb{D}^\dagger(\cdot))$ on the categories of continuous $\mathbb{Z}_p$- or $\mathbb{Q}_p$-representations of $G_{K,\Delta}$.

Proof. This is a combination of Proposition 7.7, Theorem 7.10 (with $F := \mathbb{Q}_p$), and [36, Theorem 2.6.2]. The statement on $\mathbb{D}^\dagger$ follows in a similar fashion from [36, Corollary 3.5.10]. □

Now we turn to the discussion of the Iwasawa cohomology

$$H^i_{\mathrm{Iw}}(G_{K,\Delta}, V) := \varprojlim_{H_{K,\Delta} \leq H \leq G_{K,\Delta}} H^i(H, V)\ .$$

By Shapiro's Lemma we have $H^i_{\mathrm{Iw}}(G_{K,\Delta}, V) \simeq H^i(G_{K,\Delta}, \mathbb{Z}_p[\![\Gamma_{K,\Delta}]\!] \otimes_{\mathbb{Z}_p} V)$ where the right-hand side refers to continuous cochains via the diagonal action of $G_{K,\Delta}$ on the coefficients (see [36, Lemma 2.5.1]). In particular, $H^i_{\mathrm{Iw}}(G_{K,\Delta}, V)$ is a module over the Iwasawa algebra $\mathbb{Z}_p[\![\Gamma_{K,\Delta}]\!]$. On $\mathbb{Z}_p[\![\Gamma_{K,\Delta}]\!]$-modules, the functor $\mathbb{Z}_p[\![\Gamma_{F,\Delta}]\!] \otimes_{\mathbb{Z}_p[\![\Gamma_{K,\Delta}]\!]} \cdot$ is naturally isomorphic to the functor $\mathbb{Z}_p[\Gamma_{F,\Delta}] \otimes_{\mathbb{Z}_p[\Gamma_{K,\Delta}]} \cdot$.

Lemma 7.12. For any continuous finite dimensional representation V of $G_{K,\Delta}$ over $\mathbb{Z}_p$ or $\mathbb{Q}_p$, we have $\mathbb{Z}_p[\![\Gamma_{F,\Delta}]\!] \otimes_{\mathbb{Z}_p[\![\Gamma_{K,\Delta}]\!]} H^i_{\mathrm{Iw}}(G_{K,\Delta}, V) \simeq H^i_{\mathrm{Iw}}(G_{F,\Delta}, \mathrm{Ind}_K^F V)$ for all $i \geq 0$.

Proof. Let $H \leq G_{K,\Delta} H_{F,\Delta}$ be a subgroup containing $H_{F,\Delta}$ so that we have $G_{K,\Delta} H_{F,\Delta} = G_{K,\Delta} H$. Since the quotient $G_{F,\Delta}/H_{F,\Delta} \simeq \Gamma_{F,\Delta}$ is abelian, H is automatically normal both in $G_{F,\Delta}$ and $G_{K,\Delta} H_{F,\Delta}$. Therefore taking H-cohomologies commutes with $\mathrm{Ind}^{G_{F,\Delta}}_{G_{K,\Delta} H_{F,\Delta}}$. In particular, we compute

$$\begin{aligned} H^i(H, \mathrm{Ind}_K^F V) &\simeq H^i(H, \mathbb{Z}_p[G_{F,\Delta}] \otimes_{\mathbb{Z}_p[G_{K,\Delta} H_{F,\Delta}]} \mathbb{Z}_p[G_{K,\Delta} H_{F,\Delta}] \otimes_{\mathbb{Z}_p[G_{K,\Delta}]} V) \\ &\simeq \mathbb{Z}_p[G_{F,\Delta}] \otimes_{\mathbb{Z}_p[G_{K,\Delta} H_{F,\Delta}]} H^i(H, \mathbb{Z}_p[G_{K,\Delta} H] \otimes_{\mathbb{Z}_p[G_{K,\Delta}]} V). \end{aligned}$$

Now any set of representatives of $H/(H \cap G_{K,\Delta})$ is also a set of representatives of $G_{K,\Delta} H / G_{K,\Delta}$, whence we deduce

$$\mathbb{Z}_p[G_{K,\Delta} H] \otimes_{\mathbb{Z}_p[G_{K,\Delta}]} V \simeq \mathbb{Z}_p[H] \otimes_{\mathbb{Z}_p[H \cap G_{K,\Delta}]} V.$$

Using Shapiro's Lemma, we obtain

$$H^i(H, \mathrm{Ind}_K^F V) \simeq \mathbb{Z}_p[G_{F,\Delta}] \otimes_{\mathbb{Z}_p[G_{K,\Delta} H_{F,\Delta}]} H^i(H \cap G_{K,\Delta}, V)\ .$$

Taking the projective limit with respect to H, we deduce

$$\begin{aligned} \mathbb{Z}_p[\Gamma_{F,\Delta}] \otimes_{\mathbb{Z}_p[\Gamma_{K,\Delta}]} H^i_{\mathrm{Iw}}(G_{K,\Delta}, V) &\simeq \mathbb{Z}_p[G_{F,\Delta}] \otimes_{\mathbb{Z}_p[G_{K,\Delta} H_{F,\Delta}]} H^i_{\mathrm{Iw}}(G_{K,\Delta}, V) \\ &\simeq H^i_{\mathrm{Iw}}(G_{F,\Delta}, \mathrm{Ind}_K^F V) \end{aligned}$$

as $G_{K,\Delta}$ acts on $H^i_{\mathrm{Iw}}(G_{K,\Delta}, V)$ via its quotient $\Gamma_{K,\Delta}$ and $\varprojlim_H$ commutes with the functor $\mathbb{Z}_p[G_{F,\Delta}] \otimes_{\mathbb{Z}_p[G_{K,\Delta} H_{F,\Delta}]} \cdot$ since $\mathbb{Z}_p[G_{F,\Delta}]$ is finite free over $\mathbb{Z}_p[G_{K,\Delta} H_{F,\Delta}]$. □

Recall [36] that $G_{\mathbb{Q}_p,\Delta}$ is a Poincaré group at p of dimension $2|\Delta|$. The dualizing module is $I = \mu_{p^\infty,\Delta}$ which is by definition the $G_{\mathbb{Q}_p,\Delta}$-module isomorphic abstractly to μ_{p^∞} (i.e. to $\mathbb{Q}_p/\mathbb{Z}_p$) on which each component $G_{\mathbb{Q}_p,\alpha}$ ($\alpha \in \Delta$) acts as on μ_{p^∞} (i.e. via the cyclotomic character $\chi_\alpha \colon G_{\mathbb{Q}_p,\alpha} \twoheadrightarrow \Gamma_{\mathbb{Q}_p,\alpha} \to \mathbb{Z}_p^\times$). Let $\mathbb{Z}_p(\mathbf{1}_\Delta) := T_p(\mu_{p^\infty,\Delta}) = \varprojlim_n \mu_{p^n,\Delta}$ be the p-adic Tate module of $\mu_{p^\infty,\Delta}$. For a p-primary discrete $G_{\mathbb{Q}_p,\Delta}$-module A, we define the Tate twist $A(\mathbf{1}_\Delta) := A \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\mathbf{1}_\Delta)$ and the Cartier dual $\mathrm{Hom}(A, \mu_{p^\infty,\Delta}) = A^\vee(\mathbf{1}_\Delta)$.

Proposition 7.13. For any discrete p-primary $G_{K,\Delta}$-module A, the cup product pairing induces an isomorphism $H^i(G_{K,\Delta}, A) \simeq H^{2d-i}(G_{K,\Delta}, A^\vee(\mathbf{1}_\Delta))^\vee$ for every i, where $(\cdot)^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(\cdot, \mathbb{Q}_p/\mathbb{Z}_p)$ stands for the Pontryagin dual.

Proof. In case $K = \mathbb{Q}_p$ this is [36, Theorem 2.3.1]. For an arbitrary finite extension $K/\mathbb{Q}_p$, the statement follows from Shapiro's Lemma by inducing A from $G_{K,\Delta}$ to $G_{\mathbb{Q}_p,\Delta}$. □

By Proposition 7.13, we may further identify these cohomology groups using the Cartier dual $A^\vee(\mathbf{1}_\Delta)$ as follows:

$$\begin{aligned} H^i(G_{K,\Delta}, \mathbb{Z}_p[G_{K,\Delta}/H] \otimes_{\mathbb{Z}_p} A) &\simeq H^{2d-i}(G_{K,\Delta}, (\mathbb{Z}_p[G_{K,\Delta}/H] \otimes_{\mathbb{Z}_p} A)^\vee(\mathbf{1}_\Delta))^\vee \\ &\simeq H^{2d-i}(G_{K,\Delta}, \mathbb{Z}_p[G_{K,\Delta}/H] \otimes_{\mathbb{Z}_p} (A^\vee(\mathbf{1}_\Delta)))^\vee \\ &\simeq H^{2d-i}(H, A^\vee(\mathbf{1}_\Delta))^\vee \end{aligned}$$

since the index $|G_{K,\Delta} : H|$ is finite. The duals of the corestriction maps are the restriction maps, so we deduce

$$H^i_{\mathrm{Iw}}(G_{K,\Delta}, A) \simeq \left(\varinjlim_H H^{2d-i}(H, A^\vee(\mathbf{1}_\Delta)) \right)^\vee = H^{2d-i}(H_{K,\Delta}, A^\vee(\mathbf{1}_\Delta))^\vee .$$

Moreover, the complex $\Phi^\bullet(\mathbb{D}(A^\vee(\mathbf{1}_\Delta)))$ computes the $H_{K,\Delta}$-cohomology of $A^\vee(\mathbf{1}_\Delta)$ by [36, Proposition 2.1.4]. (That result is only stated for $K = \mathbb{Q}_p$, but the proof—including [47, Proposition 4.1]—goes over unchanged to the case of finite extensions $K/\mathbb{Q}_p$.) In particular, this shows

$$H^i_{\mathrm{Iw}}(G_{K,\Delta}, A) \simeq (h^{2d-i}\Phi^\bullet(\mathbb{D}(A^\vee(\mathbf{1}_\Delta)))^\vee) .$$

Recall [36] that in case $K = \mathbb{Q}_p$ and $D_{\mathbb{Q}_p}$ is an étale $(\varphi_\Delta, \Gamma_\Delta)$-module over $\mathcal{O}_{\mathcal{E}_\Delta}$ killed by a power of p, we have the residue pairing

$$\begin{aligned} \{\cdot,\cdot\} \colon D_{\mathbb{Q}_p} \times D^*_{\mathbb{Q}_p}(\mathbf{1}_\Delta) &\to \mathbb{Q}_p/\mathbb{Z}_p \\ (x, y) &\mapsto \{x, y\} := \mathrm{res}(y(x)) . \end{aligned} \tag{7.1}$$

Here $D^*_{\mathbb{Q}_p} := \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}_\Delta}}(D_{\mathbb{Q}_p}, \mathcal{E}_\Delta/\mathcal{O}_{\mathcal{E}_\Delta})$ is the dual $(\varphi_\Delta, \Gamma_\Delta)$-module. Further,

$$\mathrm{res} \colon \mathcal{E}_\Delta/\mathcal{O}_{\mathcal{E}_\Delta}(\mathbf{1}_\Delta) \to \mathbb{Q}_p/\mathbb{Z}_p$$

sends an element an element $F(X_\bullet)e$ in $\mathcal{E}_\Delta/\mathcal{O}_{\mathcal{E}_\Delta}(\mathbf{1}_\Delta)=\mathbb{Q}_p/\mathbb{Z}_p\otimes_{\mathbb{Z}_p}\mathcal{O}_{\mathcal{E}_\Delta}e$ (with $\varphi_\alpha(e)=e$, $\gamma_\alpha(e)=\chi_\alpha(\gamma_\alpha)e$ where $\chi_\alpha\colon\Gamma_\alpha\to\mathbb{Z}_p^\times$ is the cyclotomic character) to the coefficient $a_{-1_\bullet}\in\mathbb{Q}_p/\mathbb{Z}_p$ of $\frac{1}{X_\Delta}=\prod_{\alpha\in\Delta}X_\alpha^{-1}$ in the expansion of $\frac{F(X_\bullet)}{\prod_{\alpha\in\Delta}(1+X_\alpha)}$ as

$$\frac{F(X_\bullet)}{\prod_{\alpha\in\Delta}(1+X_\alpha)}=\sum_{i_\alpha\geq -N_F,\alpha\in\Delta}a_{i_\bullet}\prod_{\alpha\in\Delta}X_\alpha^{i_\alpha}$$

with $a_{i_\bullet}\in\mathbb{Q}_p/\mathbb{Z}_p$ for $i_\bullet=(i_\alpha)_{\alpha\in\Delta}\in\mathbb{Z}^\Delta$ and some integer $N_F\in\mathbb{Z}$ depending on F. Moreover, (7.1) is Γ_Δ- and $\prod_{\alpha\in\Delta}(1+X_\alpha)^{\mathbb{Z}_p}$-equivariant with respect to which the adjoint of φ_α is ψ_α $(\alpha\in\Delta)$.
Now if A is a continuous mod-p^n representation of $G_{K,\Delta}$, then the residue pairing (7.1) applies to $D_{\mathbb{Q}_p}:=\mathrm{Ind}_K^{\mathbb{Q}_p}\mathbb{D}(A)$.

LEMMA 7.14. We have $D_{\mathbb{Q}_p}^*\simeq\mathbb{Z}_p[\Gamma_{\mathbb{Q}_p,\Delta}]\otimes_{\mathbb{Z}_p[\Gamma_{K,\Delta}]}\mathbb{D}(A)^*$ where we put $\mathbb{D}(A)^*:=\mathrm{Hom}_{\mathcal{O}_{\mathcal{E}_\Delta(K)}}(\mathbb{D}(A),\mathcal{E}_\Delta(K)/\mathcal{O}_{\mathcal{E}_\Delta(K)})$. Moreover, under this identification $1\otimes\mathbb{D}(A)$ is orthogonal to $\gamma\otimes\mathbb{D}(A)^*(\mathbf{1}_\Delta)$ for all $\gamma\in\Gamma_{\mathbb{Q}_p,\Delta}$ not lying in $\Gamma_{K,\Delta}$. In particular the residue pairing (7.1) descends to a pairing

$$\{\cdot,\cdot\}\colon\mathbb{D}(A)\,(\simeq 1\otimes\mathbb{D}(A))\quad\times\quad(1\otimes\mathbb{D}(A)^*(\mathbf{1}_\Delta)\simeq)\,\mathbb{D}(A)^*(\mathbf{1}_\Delta)\to\mathbb{Q}_p/\mathbb{Z}_p$$

such that $\{\sum_{\gamma\in U}\gamma\otimes x_\gamma,\sum_{\gamma\in U}\gamma\otimes y_\gamma\}=\sum_{\gamma\in U}\{x_\gamma,y_\gamma\}$ for any choice $U\subset\Gamma_{\mathbb{Q}_p,\Delta}$ of coset representatives of $\Gamma_{\mathbb{Q}_p,\Delta}/\Gamma_{K,\Delta}$.

Proof. We define the map F as

$$\begin{array}{rcl}\mathbb{Z}_p[\Gamma_{\mathbb{Q}_p,\Delta}]\otimes_{\mathbb{Z}_p[\Gamma_{K,\Delta}]}\mathbb{D}(A)^* & \to & D_{\mathbb{Q}_p}^* \\ \sum_{\gamma\in U}\gamma\otimes f_\gamma & \mapsto & \left(\sum_{\gamma\in U}\gamma\otimes x_\gamma\mapsto\sum_{\gamma\in U}\mathrm{Tr}_{H_{\mathbb{Q}_p,\Delta}/H_{K,\Delta}}(f_\gamma(x_\gamma))\right)\end{array}$$

where $\mathrm{Tr}_{H_{\mathbb{Q}_p,\Delta}/H_{K,\Delta}}=\sum_{u\in H_{\mathbb{Q}_p,\Delta}/H_{K,\Delta}}u\colon\mathcal{E}_\Delta(K)/\mathcal{O}_{\mathcal{E}_\Delta(K)}\to\mathcal{E}_\Delta/\mathcal{O}_{\mathcal{E}_\Delta}$ is the trace map. The map F is $\mathcal{O}_{\mathcal{E}_\Delta}$-linear and $(\varphi_\Delta,\Gamma_{\mathbb{Q}_p,\Delta})$-equivariant by construction. For the bijectivity of F assume first that $pA=0$. Note that

$$\mathrm{Tr}_{H_{\mathbb{Q}_p,\Delta}/H_{K,\Delta}}\colon E_\Delta(K)\simeq p^{-1}\mathcal{O}_{\mathcal{E}_\Delta(K)}/\mathcal{O}_{\mathcal{E}_\Delta(K)}\to E_\Delta(\mathbb{Q}_p)\simeq p^{-1}\mathcal{O}_{\mathcal{E}_\Delta}/\mathcal{O}_{\mathcal{E}_\Delta}$$

is onto since it is the composite of the trace maps $\mathrm{Tr}_{H_{\mathbb{Q}_p,\alpha}/H_{K,\alpha}}\colon E_\alpha(K)\to E_\alpha(\mathbb{Q}_p)$ for all $\alpha\in\Delta$ that are each onto since $E_\alpha(K)/E_\alpha(\mathbb{Q}_p)$ is a Galois extension with Galois group $H_{\mathbb{Q}_p,\alpha}/H_{K,\alpha}$. In particular, $\mathrm{Ker}(\mathrm{Tr}_{H_{\mathbb{Q}_p,\alpha}/H_{K,\alpha}})$ does not contain any nonzero ideal of $E_\Delta(K)$ as its rank over $E_\Delta(\mathbb{Q}_p)$ equals $|H_{\mathbb{Q}_p,\Delta}/H_{K,\Delta}|-1<|H_{\mathbb{Q}_p,\Delta}/H_{K,\Delta}|$. Now if $\sum_{\gamma\in U}\gamma\otimes f_\gamma\neq 0$ then $f_\gamma\neq 0$ for at least one choice of $\gamma\in U$, so we may put $x_{\gamma'}:=0$ for all $\gamma\neq\gamma'\in U$ and choose x_γ so that $\mathrm{Tr}_{H_{\mathbb{Q}_p,\Delta}/H_{K,\Delta}}(f_\gamma(x_\gamma))=\mathrm{Tr}_{H_{\mathbb{Q}_p,\Delta}/H_{K,\Delta}}(x_\gamma f_\gamma(1))\neq 0$ (as we have just seen that $\mathrm{Tr}_{H_{\mathbb{Q}_p,\Delta}/H_{K,\Delta}}(E_\Delta(K)f_\gamma(1))\neq\{0\}$). Hence F is injective

if $pA = 0$. On the other hand, the ranks of the domain and codomain of F are equal, therefore it is an isomorphism by [47, Proposition 2.2]. The case of general A follows by devissage.
The second statement is deduced from the first one using the $\Gamma_{\mathbb{Q}_p,\Delta}$-invariance of the pairing $\{\cdot,\cdot\}$. $\square$

Definition 7.15. Let D be an étale $(\varphi_\Delta, \Gamma_{K,\Delta})$-module over any of the rings $\mathcal{O}_{\mathcal{E}_\Delta(K)}$, $\mathcal{E}_\Delta(K)$, $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(K)}$, or $\mathcal{E}^\dagger_\Delta(K)$. We define the cochain complex

$$\Psi^\bullet(D)\colon 0 \to D \to \bigoplus_{\alpha\in\Delta} D \to \cdots \to \bigoplus_{\{\alpha_1,\dots,\alpha_r\}\in\binom{\Delta}{r}} D \to \cdots \to D \to 0 \qquad (7.2)$$

where for all $0 \le r \le |\Delta| - 1$ the map $d^{\beta_1,\dots,\beta_{r+1}}_{\alpha_1,\dots,\alpha_r}\colon D \to D$ from the component in the rth term corresponding to $\{\alpha_1,\dots,\alpha_r\} \subseteq \Delta$ to the component corresponding to the $(r+1)$-tuple $\{\beta_1,\dots,\beta_{r+1}\} \subseteq \Delta$ is given by

$$d^{\beta_1,\dots,\beta_{r+1}}_{\alpha_1,\dots,\alpha_r} = \begin{cases} 0 & \text{if } \{\alpha_1,\dots,\alpha_r\} \not\subseteq \{\beta_1,\dots,\beta_{r+1}\} \\ (-1)^\eta(\mathrm{id} - \psi_\beta) & \text{if } \{\beta_1,\dots,\beta_{r+1}\} = \{\alpha_1,\dots,\alpha_r\} \cup \{\beta\} \end{cases},$$

where $\eta = \eta(\alpha_1,\dots,\alpha_r,\beta)$ is the number of elements in the set $\Delta \setminus \{\alpha_1,\dots,\alpha_r\}$ smaller than β, and ψ_β is the reduced trace of φ_β, ie. for an element $x = \sum_{j=0}^{p-1}(1+X_\beta)^j\varphi_\beta(x_j) \in D$ we put $\psi_\beta(x) = x_0$. (Note that the sign convention here is different from the one defining the complex $\Phi^\bullet(D)$. The reason for this is that with this choice of signs, the differentials are adjoint to each other under the residue pairing (7.1) defined above.)

Since the ψ-operators commute with the action of $\Gamma_{K,\Delta}$, we have $\Psi^\bullet(\mathrm{Ind}_K^F D) \simeq \mathbb{Z}_p[\Gamma_{F,\Delta}] \otimes_{\mathbb{Z}_p[\Gamma_{K,\Delta}]} \Psi^\bullet(D)$. Further, $\mathbb{Z}_p[\Gamma_{F,\Delta}]$ is finite free over $\mathbb{Z}_p[\Gamma_{K,\Delta}]$, so we deduce $h^i\Psi^\bullet(\mathrm{Ind}_K^F D) \simeq \mathbb{Z}_p[\Gamma_{F,\Delta}] \otimes_{\mathbb{Z}_p[\Gamma_{K,\Delta}]} h^i\Psi^\bullet(D)$ for all $i \ge 0$. Similarly, we have $\Phi^\bullet(\mathrm{Ind}_K^F D) \simeq \mathbb{Z}_p[\Gamma_{F,\Delta}] \otimes_{\mathbb{Z}_p[\Gamma_{K,\Delta}]} \Phi^\bullet(D)$ and $h^i\Phi^\bullet(\mathrm{Ind}_K^F D) \simeq \mathbb{Z}_p[\Gamma_{F,\Delta}] \otimes_{\mathbb{Z}_p[\Gamma_{K,\Delta}]} h^i\Phi^\bullet(D)$ for all $i \ge 0$.

Theorem 7.16. We have a natural isomorphism of cohomological δ-functors $H^i_{\mathrm{Iw}}(G_{K,\Delta},\cdot) \simeq h^{i-d}\Psi^\bullet(\mathbb{D}(\cdot)) \simeq h^{i-d}\Psi^\bullet(\mathbb{D}^\dagger(\cdot))$ on the categories of continuous $\mathbb{Z}_p$- or $\mathbb{Q}_p$-representations of $G_{K,\Delta}$.

Proof. The case $K = \mathbb{Q}_p$ is proven in [36, Corollary 3.5.10]. Let A be a p-power torsion representation of $G_{K,\Delta}$ and put $D := \mathbb{D}(A)$, $D_{\mathbb{Q}_p} := \mathrm{Ind}_K^{\mathbb{Q}_p} D$. The isomorphism is constructed via the residue pairing which induces a pairing between $\Psi^{d-r}(D_{\mathbb{Q}_p})$ and $\Phi^r(D^*_{\mathbb{Q}_p}(\mathbf{1}_\Delta))$, and hence between $h^{d-r}\Psi^\bullet(D_{\mathbb{Q}_p})$ and $h^r\Phi^\bullet(D^*_{\mathbb{Q}_p}(\mathbf{1}_\Delta))$ $(0 \le r \le d)$. By Lemma 7.14, the isomorphism

$$\eta_r(A)\colon h^{d-r}\Psi^\bullet(D_{\mathbb{Q}_p}) \to h^r\Phi^\bullet(D^*_{\mathbb{Q}_p}(\mathbf{1}_\Delta))^\vee \simeq H^r_{\mathrm{Iw}}(G_{\mathbb{Q}_p,\Delta}, \mathrm{Ind}_K^{\mathbb{Q}_p} A)$$

descends to a $\mathbb{Z}_p[\![\Gamma_{K,\Delta}]\!]$-linear map

$$\widetilde{\eta_r}(A) \colon h^{d-r}\Psi^\bullet(D) \to h^r\Phi^\bullet(D^*(\mathbf{1}_\Delta))^\vee \simeq H^r_{\mathrm{Iw}}(G_{K,\Delta}, A)$$

such that $\eta_r = \mathbb{Z}_p[\![\Gamma_{\mathbb{Q}_p,\Delta}]\!] \otimes_{\mathbb{Z}_p[\![\Gamma_{K,\Delta}]\!]} \widetilde{\eta_r}$. Since η_r is an isomorphism and the ring extension $\mathbb{Z}_p[\![\Gamma_{K,\Delta}]\!] \hookrightarrow \mathbb{Z}_p[\![\Gamma_{\mathbb{Q}_p,\Delta}]\!]$ is faithfully flat, we deduce that $\widetilde{\eta_r}$ is also an isomorphism. If T is an arbitrary continuous representation of $G_{K,\Delta}$ on a finitely generated $\mathbb{Z}_p$-module, then the result on $\mathbb{D}(T)$ follows by taking the projective limit of the isomorphisms $\widetilde{\eta_r}(T/p^nT)$.
Now if V is a continuous representation of $G_{K,\Delta}$ over $\mathbb{Q}_p$, then it contains a $G_{K,\Delta}$-invariant $\mathbb{Z}_p$-lattice $T \leq V$ by the compactness of $G_{K,\Delta}$ and the isomorphism for $\mathbb{D}(V)$ follows from that for $\mathbb{D}(T)$ by inverting p. Finally, the inclusion of $\mathbb{D}^\dagger(T) \hookrightarrow \mathbb{D}(T)$ (resp. $\mathbb{D}^\dagger(V) \hookrightarrow \mathbb{D}(V)$) induces a morphism $\Psi^\bullet(\mathbb{D}^\dagger(T)) \to \Psi^\bullet(\mathbb{D}(T))$ (resp. $\Psi^\bullet(\mathbb{D}^\dagger(V)) \to \Psi^\bullet(\mathbb{D}(V))$) and, by taking cohomologies, a $\mathbb{Z}_p[\![\Gamma_{K,\Delta}]\!]$-linear map $\iota_i \colon h^i\Psi^\bullet(\mathbb{D}^\dagger(T)) \to h^i\Psi^\bullet(\mathbb{D}(T))$ (resp. $\iota_i \colon h^i\Psi^\bullet(\mathbb{D}^\dagger(V)) \to h^i\Psi^\bullet(\mathbb{D}(V))$) for $i \geq 0$ which becomes an isomorphism after base change to $\mathbb{Z}_p[\![\Gamma_{\mathbb{Q}_p,\Delta}]\!]$ by [36, Corollary 3.5.10]. Hence ι_i is an isomorphism for all $i \geq 0$ using again the faithfully flat property of the ring extension $\mathbb{Z}_p[\![\Gamma_{K,\Delta}]\!] \hookrightarrow \mathbb{Z}_p[\![\Gamma_{\mathbb{Q}_p,\Delta}]\!]$. □

8 Distinct factors

We now formulate the corresponding results for products of Galois groups of distinct finite extensions of $\mathbb{Q}_p$.

Notation 8.1. Again let Δ be a finite set, but now let $\underline{K} = (K_\alpha : \alpha \in \Delta)$ be a tuple of finite extensions of $\mathbb{Q}_p$ and put

$$\begin{aligned} G_{\underline{K},\Delta} &:= \prod_{\alpha\in\Delta} \mathrm{Gal}(K_\alpha^{\mathrm{alg}}/K_\alpha) \\ H_{\underline{K},\Delta} &:= \prod_{\alpha\in\Delta} \mathrm{Gal}(K_\alpha^{\mathrm{alg}}/K_\alpha(\mu_{p^\infty})) \\ \Gamma_{\underline{K},\Delta} &:= \prod_{\alpha\in\Delta} \mathrm{Gal}(K_\alpha(\mu_{p^\infty})/K_\alpha). \end{aligned}$$

We then follow Notation 2.4, Notation 2.6, and Notation 2.10, but taking $\widetilde{\mathcal{O}}_{\mathcal{E}_\alpha}$ to be a copy of $\widetilde{\mathcal{O}}_{\mathcal{E}}$ for $\mathcal{E}$ associated to the field K_α. This yields rings

$$\begin{gathered} \mathcal{O}_{\mathcal{E}_\Delta(\underline{K})}, \quad \widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(\underline{K})}, \quad \mathcal{O}^\dagger_{\mathcal{E}_\Delta(\underline{K})}, \quad \widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(\underline{K})}, \\ \mathcal{E}_\Delta(\underline{K}), \quad \widetilde{\mathcal{E}}_\Delta(\underline{K}), \quad \mathcal{E}^\dagger_\Delta(\underline{K}), \quad \widetilde{\mathcal{E}}^\dagger_\Delta(\underline{K}). \end{gathered}$$

We then have the following extensions of Theorem 6.15 and Theorem 6.16.

Theorem 8.2. The category of continuous representations of $G_{\underline{K},\Delta}$ on finite free $\mathbb{Z}_p$-modules is equivalent to the category of projective étale $(\varphi_\Delta, \Gamma_\Delta)$-modules over each of the rings $\mathcal{O}_{\mathcal{E}_\Delta(\underline{K})}$, $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(\underline{K})}$, $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(\underline{K})}$, and $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta(\underline{K})}$.

Proof. Since Theorem 4.30 already includes the case where the fields F_i need not be equal, we may apply it to deduce the analogue of Theorem 4.29; this gives the equivalence between continuous $\mathbb{Z}_p$-representations of $G_{\underline{K},\Delta}$ and $(\varphi_\Delta, \Gamma_\Delta)$-modules over $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta(\underline{K})}$. To add the other rings, it suffices to do so after replacing each K_α with a single mutual extension K; we thus reduce to Theorem 6.15. □

Theorem 8.3. The category of continuous representations of $G_{\underline{K},\Delta}$ on finite dimensional $\mathbb{Q}_p$-vector spaces is equivalent to the category of projective étale $(\varphi_\Delta, \Gamma_\Delta)$-modules over each of the rings $\mathcal{E}_\Delta(\underline{K})$, $\widetilde{\mathcal{E}}_\Delta(\underline{K})$, $\mathcal{E}^\dagger_\Delta(\underline{K})$, and $\widetilde{\mathcal{E}}^\dagger_\Delta(\underline{K})$.

Proof. This follows from Theorem 8.2 just as Theorem 6.16 follows from Theorem 6.15. □

In order to extend the results on Galois cohomology to distinct finite extensions of $\mathbb{Q}_p$, note that Definition 7.1 of $\Phi^\bullet(\cdot)$ carries over to this situation. Further, putting $C_{\underline{K},\Delta}$ for the torsion subgroup of $\Gamma_{\underline{K},\Delta}$ and choosing generators $\gamma_\alpha \in \Gamma_{K_\alpha}/(\Gamma_{K_\alpha} \cap C_{\underline{K},\Delta})$ (where $\Gamma_{K_\alpha} := \mathrm{Gal}(K_\alpha(\mu_{p^\infty})/K_\alpha)$) we may also form the complex $\Phi\Gamma^\bullet_{\underline{K},\Delta}(D)$ as in Definition 7.2 and Definition 7.3 for a projective étale $(\varphi_\Delta, \Gamma_\Delta)$-module D over any of the rings $\mathcal{O}_{\mathcal{E}_\Delta(\underline{K})}$, $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(\underline{K})}$, $\mathcal{E}_\Delta(\underline{K})$, or $\mathcal{E}^\dagger_\Delta(\underline{K})$. Moreover, if $\underline{F} = (F_\alpha : \alpha \in \Delta)$ is another tuple of finite extensions of $\mathbb{Q}_p$ satisfying $F_\alpha \leq K_\alpha$ for all $\alpha \in \Delta$ then the induced $(\varphi_\Delta, \Gamma_\Delta)$-module $\mathrm{Ind}_{\underline{K}}^{\underline{F}} D := \mathbb{Z}_p[\Gamma_{\underline{F},\Delta}] \otimes_{\mathbb{Z}_p[\Gamma_{\underline{K},\Delta}]} D$ (as in Definition 7.6) also makes sense. So we have the following version of Shapiro's lemma.

Theorem 8.4. We have a natural isomorphism of cohomological δ-functors

$$h^i\Phi\Gamma^\bullet_{\underline{F},\Delta}(\mathrm{Ind}_{\underline{K}}^{\underline{F}}(\cdot)) \simeq h^i\Phi\Gamma^\bullet_{\underline{K},\Delta}(\cdot)$$

on the category of projective étale $(\varphi_\Delta, \Gamma_\Delta)$-modules over $\mathcal{O}_{\mathcal{E}_\Delta(\underline{K})}$ (resp. over $\mathcal{E}_\Delta(\underline{K})$, $\mathcal{O}^\dagger_{\mathcal{E}_\Delta(\underline{K})}$, and $\mathcal{E}^\dagger_\Delta(\underline{K})$).

Proof. The proof of Theorem 7.10 goes through unchanged. □

Applying this in case $F_\alpha = \mathbb{Q}_p$ for all $\alpha \in \Delta$ we deduce:

Corollary 8.5. We have a natural isomorphism of cohomological δ-functors $H^i(G_{\underline{K},\Delta}, \cdot) \simeq h^i\Phi\Gamma^\bullet_{\underline{K},\Delta}(\mathbb{D}(\cdot)) \simeq h^i\Phi\Gamma^\bullet_{\underline{K},\Delta}(\mathbb{D}^\dagger(\cdot))$ on the categories of continuous $\mathbb{Z}_p$- or $\mathbb{Q}_p$-representations of $G_{\underline{K},\Delta}$.

Finally, Definition 7.15 of $\Psi^\bullet(\cdot)$ also carries over to this case and we have:

Theorem 8.6. We have a natural isomorphism of cohomological δ-functors $H^i_{\mathrm{Iw}}(G_{\underline{K},\Delta}, \cdot) \simeq h^{i-d}\Psi^\bullet(\mathbb{D}(\cdot)) \simeq h^{i-d}\Psi^\bullet(\mathbb{D}^\dagger(\cdot))$ on the categories of continuous $\mathbb{Z}_p$- or $\mathbb{Q}_p$-representations of $G_{\underline{K},\Delta}$.

Proof. The proof of Theorem 7.16 goes through unchanged. □

9 Future directions

We end by recording some possible directions in which this work could be continued.

- Extend the comparison of Galois cohomology and the Herr complex to the rings $\widetilde{\mathcal{O}}_{\mathcal{E}_\Delta}$, $\widetilde{\mathcal{O}}^\dagger_{\mathcal{E}_\Delta}$, $\widetilde{\mathcal{E}}_\Delta$, and $\widetilde{\mathcal{E}}^\dagger_\Delta$, for which the maps φ_α are bijective and so the construction of the reduced trace ψ_α is not relevant.

- Construct the analogue of the Robba ring and relate $(\varphi_\Delta, \Gamma_\Delta)$-modules over it to vector bundles on a product of Fargues–Fontaine curves.

- Extend to representations with values in a coherent sheaf on a rigid analytic space over $\mathbb{Q}_p$, as in [30].

- Consider Iwasawa cohomology in terms of cyclotomic deformations from the point of view of [29]. This potentially allows for the use of towers other than the cyclotomic one.

- Extend to representations of a product of étale fundamental groups of rigid analytic spaces, as in [27, 28]. Note that Proposition 4.27 is already written at a suitable level of generality for this purpose.

- Apply Drinfeld's lemma for perfectoid spaces to other constructions of multivariate (φ, Γ)-modules, such as that of Berger [6, 7]. In that construction, one starts with a representation of a single copy of G_K, but it should be possible to interpret the resulting objects in terms of representations of a suitable power of G_K.

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