Backward Asymptotics in S-Unimodal Maps

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To Jim Yorke, on the occasion of his 80th birthday

While the forward trajectory of a point in a discrete dynamical system is always unique, in general, a point can have infinitely many backward trajectories. The union of the limit points of all backward trajectories through x was called by Hero the "special α-limit" ($s\alpha$-limit for short) of x. In this article, we show that there is a hierarchy of $s\alpha$-limits of points under iterations of a S-unimodal map: the size of the $s\alpha$-limit of a point increases monotonically as the point gets closer and closer to the attractor. The $s\alpha$-limit of any point of the attractor is the whole nonwandering set. This hierarchy reflects the structure of the graph of a S-unimodal map recently introduced jointly by Jim Yorke and the present author.

Keywords: S-unimodal map; backward dynamics; $s\alpha$-limit; graph of a dynamical system.

1. Introduction

Our interest in $s\alpha$-limits comes from their relation to the edges of the graph of a dynamical system, that we studied in a recent joint work with Yorke [De Leo & Yorke, 2021a] in case of S-unimodal maps. What we ultimately show in [De Leo & Yorke, 2021a] is that, in the one-dimensional real case, there is a hierarchy among repellors so that points arbitrarily close to a given repellor have trajectories that asymptote to all repellors below him in this hierarchy and to no other repellor. In [De Leo & Yorke, 2021b] we present numerical evidence that this phenomenon is common also in multidimensional real discrete dynamical systems.

Due to this hierarchy, backward limits are much more complicated than in the one-dimensional *complex* case, where there is, instead, a unique repellor, the Julia set, and this repellor is both backward and forward invariant. We believe that this is the main reason why the standard definition of α-limit was never questioned by the discrete complex dynamics community. In this article, we argue that the

concept of $s\alpha$-limit is a more suitable analogue of ω-limit and use our results in [De Leo & Yorke, 2021a] to find the $s\alpha$-limit of points of the interval under a S-unimodal map.

The idea of encoding the qualitative behavior of a dynamical system in a graph goes back to Smale that, in [Smale, 1967], proved that the nonwandering set Ω_f of an Axiom-A diffeomorphism f of a compact manifold decomposes in a unique way as the finite union of disjoint, closed, invariant indecomposable subsets Ω_i, on each of which f is topologically transitive. Hence one can associate to f a directed graph, describing its qualitative dynamics, whose nodes are these Ω_i and such that there is an edge from Ω_j to Ω_i if the stable manifold of Ω_i intersects the unstable manifold of Ω_j.

A decade later Conley, in [Conley, 1978], widely generalized this idea to continuous finite-dimensional dynamical systems replacing the nonwandering set with the chain-recurrent set. One of the main advantages of using the chain-recurrent set is that on this set there is a natural equivalence relation,

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that does not extend in general to the nonwandering set, whose equivalence classes turn out to be the exact analogue of the indecomposable subsets of Axiom-A diffeomorphisms. Conley's construction was later generalized to discrete dynamics by Norton [1995a] and to several other settings by other authors (see [De Leo & Yorke, 2021b] for a detailed bibliography on the subject).

The chain-recurrent set $\mathcal{R}_f$ of a dynamical system f on X is defined via ε-chains, namely finite sequences of points each of which is within ε from the image of the previous one under f (see the next section for more precise definitions about the concepts used in this introduction). A point $x \in X$ is *upstream* from a point $y \in X$ if there is an ε-chain that starts at x and ends at y for every $\varepsilon > 0$. We also say that y is *downstream* from x. A point is chain-recurrent if it is upstream (or, equivalently, downstream) from itself.

The relation

$$x \sim y \quad \text{if } x \text{ is both upstream and}$$
$$\text{downstream from } y$$

is an equivalence relation. We call nodes its equivalence classes and we associate with f the graph Γ_f having the nodes of $\mathcal{R}_f$ as its nodes and so that there is an edge from node A to node B if there is a bi-infinite trajectory (bitrajectory)

$$t = (\ldots, t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots),$$

whose limit points for $n \to -\infty$ lie in A and those for $n \to +\infty$ lie in B. The dynamical relevance of Γ_f comes from Conley's decomposition theorem: given a continuous map f of a compact metric space X into itself, a point $x \in X$ either belongs to a node of Γ_f, and so it has some kind or "recurrent" dynamics, or is "gradient-like", namely it defines an edge between two distinct nodes of Γ_f (see [Norton, 1995a], Thm. 4.10).

In [De Leo & Yorke, 2021a] we studied the structure of the graph of a S-unimodal map and proved that such graph is always a tower, namely that there is an edge between each pair of nodes. Since there cannot be loops in the graph of a dynamical system, this means that there is a linear hierarchy among all nodes: there is a first node N_0 that has edges towards each other node, a node N_1 that has edges towards each other node except N_0 and so on up to the attracting node N_p. Each node but one is repelling and every repelling node, which
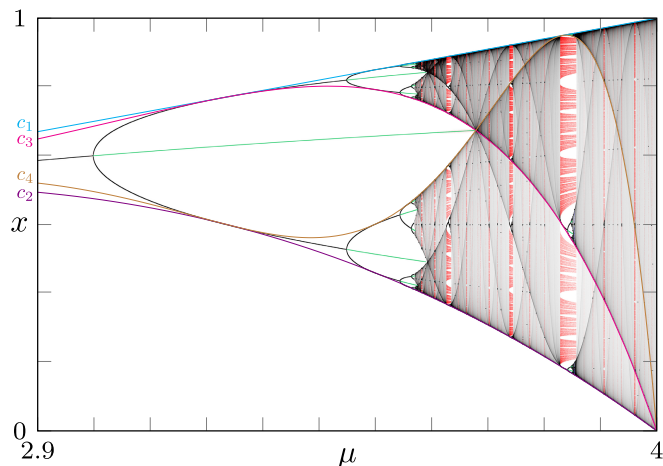

Fig. 1.   *Bifurcations diagram of the logistic map.* This picture shows the nodes of the logistic map $\ell_\mu(x) = \mu x (1-x)$ in the parameter range $[2.9, 4]$ that are visible at this resolution. Attractors are painted in shades of gray (depending on the density), repelling periodic orbits in green and repelling Cantor sets in red. The colored lines labeled by c_k are the lines $\ell_\mu^k(c)$, where $c = 0.5$ is the critical point of ℓ_μ. The black dots that are visible within the diagram are points belonging to low-period cycles, signaling the presence of bifurcation cascades within some window. The fact that some of them keep close to the line $x = c$ is a reflection of Singer's Theorem: when the attractor is a periodic orbit, c belongs to its immediate basin. Notice that, for each μ, all nodes but $N_0 = 0$ lie within the interval $[c_2, c_1]$ since points in $(c_1, 1]$ are outside the range of the map and, for $\mu \geq 2$, all points in $(0, c_2)$ eventually fall in $[c_2, c_1]$, which is forward invariant. See Sec. 3.2 for further comments of this figure.

can be either a cycle (for short we call *cycle* a periodic orbit) or a Cantor set, has an edge towards the attracting node.

Figure 1 shows the nodes of the logistic map $\ell_\mu(x) = \mu x (1 - x)$, $\mu \in [2.9, 4]$, that are visible at the picture's resolution. The attracting nodes are painted in shades of gray, representing the density of a generic orbit of the attractor. There are five kinds of attracting nodes: type A_1, a cycle; type A_2, a cycle of intervals; type A_3, an adding machine; type A_4, a 1-sided attracting cycle belonging to a repelling Cantor set (taking place at the beginning of each window); type A_5 (see Theorem G about this complicated case, taking place at the end of each window). The repelling nodes are painted either in green (if they are cycle nodes) or in red (if they are Cantor set nodes). The picture also shows the curves $c_k = \ell_\mu^k(c)$, $k = 1, \ldots, 4$, where $c = 1/2$ is the critical point of the logistic map. The points c_k play a fundamental role in the forward and backward dynamics of S-unimodal maps.

The main goal of the present article is to use our results in [De Leo & Yorke, 2021a] to study the structure of the $s\alpha$-limit sets under S-unimodal maps. After presenting, in Secs. 2 and 3, all definitions and main results we need from literature, we study in Sec. 4 the backward dynamics of points under a S-unimodal map $f : [0,1] \to [0,1]$, $f(0) = 0$. Except when N_p is of type A_5 (see Theorem 5 for a complete statement), our main result can be formulated as follows. When f has p nodes $N_0, \ldots, N_p$, $p \geq 1$, then $[0,1]$ can be written as the disjoint union of sets $U_{-1}, U_0, \ldots, U_p$ such that:

(1) each U_k, $k < p$, is the finite union of one or more intervals (these intervals can be open, half-open or closed);
(2) $U_{-1} = (c_2, 1]$ is the set of all points that have no bitrajectory passing through them;
(3) $U_0 = [0, c_2)$ is the set of all points with a single bitrajectory, which necessarily asymptotes backward to 0;
(4) $N_k \subset U_k$ for each $k = 0, \ldots, p$;
(5) $U_p = N_p$;
(6) $s\alpha_f(x) = \bigcup_{i=0}^{k} N_i$ for each $x \in U_k$, $k = 0, \ldots, p$.

Notice that proving this theorem requires different techniques for points that are not chain-recurrent and points that are so and, for chain-recurrent points, it requires different techniques for different types of nodes.

Figures 6–8 show examples of these U_k sets in several cases for the logistic map. For instance, let us describe U_1 in several cases when ℓ_μ has three nodes. When μ is close enough to 3 from the right, $N_1 = \{\bar{p}_\mu\}$ is the nonzero fixed point of ℓ_μ and $N_2 = \{p_1, p_2\}$, $p_1 < p_2$, a stable 2-cycle. In this case,

$$U_1 = [c_2, p_1) \cup (p_1, p_2) \cup (p_2, c_1].$$

When μ is a chaotic parameter with just three nodes and close enough from the left to $\mu_M \simeq 3.679$, which is the first parameter value at which c_3 and c_4 meet in Fig. 1, then $N_1 = \{\bar{p}_\mu\}$ is again the nonzero fixed point of ℓ_μ but this time $N_2 = [c_2, c_4] \cup [c_3, c_1]$. Hence, in this case,

$$U_1 = (c_4, c_3).$$

When μ is close enough from the right to $\mu_0 = 1 + 2\sqrt{2}$, which is the left endpoint of the period-3 window (see Figs. 1 and 5), then N_1 is a Cantor set (painted in red) and $N_2 = \{p_1, p_2, p_3\}$, $p_3 < p_1 < p_2$,

is a stable 3-cycle (painted in black). Hence

$$U_1 = [c_2, p_3) \cup (p_3, p_1) \cup (p_1, p_2) \cup (p_2, c_1].$$

Our main result above achieves also two sub-goals. First, it proves for S-unimodal maps Conjecture 1 (see Sec. 2) by Kolyada, Misiurewicz and Snoha, namely that $s\alpha$ limits are closed sets for all $x \in [0,1]$. Second, it supports several numerical observations by the author in planar discrete dynamical systems [De Leo, 2020] suggesting that, in case of real Newton maps on the plane, besides a nontrivial open set of points with backward trajectories that asymptote to (almost) the whole Julia set, just as it happens in the complex case (Theorem E), there are also nontrivial open sets from which one get smaller subsets of the nonwandering set or even the empty set.

2. Forward and Backward Limits

The study of (time-)discrete dynamical systems, namely iterations of a continuous map $f : X \to X$ on a metric space (X, d), goes back to the introduction of Poincaré maps [Poincaré, 1890, 1899] as a tool to study the qualitative behavior of the integral trajectories of a vector field. One of the main goals of the field is finding the asymptotics of points under a given map f: where a point $x \in X$ ultimately ends up (the "ω-limit" of x) and where it might come from (the "α-limit" of x).

2.1. *Forward asymptotics*

The asymptotics in the forward direction present no ambiguity: given a point $x \in X$, the set

$$\omega_f(x) = \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} \{f^m(x)\}}$$

consists in the accumulation points of the forward orbit of x, namely the sequence $\{f(x), f^2(x), f^3(x), \ldots\}$ where, as usual, we use the notation

$$f^n(x) = f(f^{n-1}(x)).$$

The setting where discrete dynamics has been possibly more successful to date is iterations of holomorphic (and therefore rational) maps on the Riemann sphere $\mathbb{CP}^1$. In this setting, F_f (Fatou set) denotes the largest open set over which the family of iterates $\{f^n\}$ is normal and J_f (Julia set) its complement and we have the fundamental full classification below.

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Definition 1. We use *cycle* as a synonim for periodic orbit. By *n*-cycle we mean a periodic orbit of period n (so, in particular, a n-cycle has n elements). We denote by $|\gamma|$ the period of a cycle γ.

Theorem A (ω-limits in $\mathbb{C}$ [Sullivan, 1985]). *Let f : $\mathbb{CP}^1 \to \mathbb{CP}^1$ be a rational function of degree larger than 1. Then*:

(1) *if $x \in J_f$, then $\omega_f(x)$ is either a repelling cycle or the whole J_f;*
(2) *if $x \in F_f$, then $\omega_f(x)$ is either a hyperbolic or parabolic attracting cycle or a Jordan curve.*

Recall that, in the complex setting, the duality *nonchaotic* versus *chaotic* dynamics coincides with the duality *attracting* versus *repelling* set. In the real case, except in particular cases such as Newton maps (e.g. see [De Leo, 2019]), the two dualities are independent (for instance, there are chaotic attractors). In regard to orbits asymptotics, the latter is the most relevant. An analogue of Theorem A for the real case is given by the following weaker result:

Theorem B (ω-limits in $\mathbb{R}$ [Martens *et al.*, 1992]). *Let f be either a generic noninvertible C^2 self-map of $\mathbb{RP}^1$ or a S-multimodal map on the interval. Then, for almost all x, $\omega_f(x)$ can be of the following three types*:

(1) *a cycle*;
(2) *a minimal Cantor set*;
(3) *a finite union of closed intervals containing a critical point and on which f acts transitively (see Definition 20).*

The last case is the one where the dynamics within the attractor is chaotic, in the sense of sensitivity to initial conditions (see [Kolyada, 2004] and [Robinson, 2008] for thorough discussions on alternate definitions of chaotic attractors).

2.2. *Backward asymptotics*

The asymptotics in the backward direction have been explored much less, even in the two settings mentioned above. Traditionally, the α-limit of a point is defined analogously to the ω-limit:

$$\alpha_f(x) = \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} \{f^{-m}(x)\}}.$$

Notice that, in this case, in general, $f^{-m}(x)$ consists of more than a single point and the growth of the number of counterimages with m

can be exponential. Very few theorems are known on $\alpha_f(x)$. We quote below the major ones known to us:

Theorem C (α-limits in $\mathbb{C}$ [Fatou, 1920]). *Let f : $\mathbb{CP}^1 \to \mathbb{CP}^1$ be a rational function of degree larger than 1. Then, for all $z \in \mathbb{CP}^1$, with at most two exceptions, $\alpha_f(z) \supset J_f$. Moreover, $\alpha_f(z) = J_f$ if and only if z belongs to either J_f or to the basin of attraction of some stable cycle of f (but not to the cycle itself).*

As in case of ω-limits, also the theorem above has a, much weaker, real analogue:

Theorem D (α-limits in $\mathbb{R}$ [De Melo & van Strien, 1993]). *Let f : $\mathbb{RP}^1 \to \mathbb{RP}^1$ be a generic analytic map and denote by Z_f its set of critical points. Then $\alpha_f(Z_f) = J_f$.*

More recently, Cui and Ding [2010] studied in detail the α-limit sets of S-unimodal maps.

2.3. *α-limits are not an optimal analogue of ω-limits*

While the definition of α-limit sets above has the appeal to be the precise analogue, replacing images with counterimages, of the definition of ω-limit sets, the example below suggests that it does not represent necessarily the best answer to the question of "where might x come from".

Definition 2. A *bitrajectory* of a map $f : X \to X$ based at $x \in X$ is a bi-infinite sequence $\{x_i\}_{i \in \mathbb{Z}}$ such that:

(1) $x_0 = x$;
(2) $f(x_i) = x_{i+1}$ for all $i \in \mathbb{Z}$.

Example 1. Consider the logistic map $\ell_\mu(x) = \mu x(1 - x)$ for some $\mu \in (1, 4)$. The point $x = 0$ is a repelling fixed point and $f(1) = 0$, so $\alpha_f(0) = \{0, 1\}$. For x_0 small enough, the equation $\ell_\mu(x) = x_0$ has two roots $x_1, \hat{x}_1$ such that:

(1) $x_1 \in (0, x_0)$;
(2) $\hat{x}_1 = 1 - x_1$ has no counterimage under ℓ_μ.

Hence $f^{-m}(x_0)$ always consists of just two points $x_m, \hat{x}_m$, with $x_m \to 0$ and $\hat{x}_m \to 1$. Ultimately, $\alpha_f(x_0) = \{0, 1\}$ for all x_0 small enough.

On the other side, every bitrajectory $\{t_i\}$ of ℓ_μ passing through x_0 satisfies $\lim_{n \to -\infty} t_n = 0$. Indeed, clearly the accumulation points of $\{t_{-i}\}_{i \in \mathbb{N}}$ belong to $\alpha_f(0)$ and, as long as $\mu < 4$, $\ell_\mu(0.5) < 1$ and so there is some neighborhood of 1 which is not

in $\ell_\mu([0,1])$. In particular, no point close enough to 1 can belong to a bitrajectory of ℓ_μ.

In fact, an equivalent definition for $\alpha_f(x)$ is given by the following [Hero, 1992]:

Definition 3. $\alpha_f(x)$ is the set of $y \in X$ for which there is a sequence of points $\{y_n\}$ and a sequence of monotonically increasing positive numbers k_n such that:

(1) $f^{k_n}(y_n) = x$;
(2) $\lim_{n \to \infty} y_n = y$.

Namely, arbitrarily close to each point in $\alpha_f(x)$, there are points whose forward trajectory passes through x.

The discussion above shows that, while formally there is a perfect symmetry between the definition of α-limit and the one of ω-limit, when a map is not invertible this symmetry does not extend to the sets obtained from those definitions: the ω-limit set of a point x contains only (and all) points that are asymptotically approached by x under f, while its α-limit contains also points that cannot be asymptotically approached, going backward, by a bitrajectory.

2.4. *sα-limits are a better analogue of ω-limits*

A definition that leads to backward limit sets which are symmetric, from the point of view highlighted above, to ω-limit sets can be achieved by slightly modifying the definition above:

Definition 4 [Hero, 1992]. Let $x \in X$. The *special α-limit* $s\alpha_f(x)$ is the set of all $y \in X$ for which there is a sequence of points $\{y_n\}$ and a sequence of monotonically increasing positive numbers k_n such that:

(1) $y_0 = x$;
(2) $f^{k_n}(y_n) = y_{n-1}$;
(3) $\lim_{n \to \infty} y_n = y$.

The symmetry between $s\alpha_f$ and ω_f becomes evident when we write them in terms of limit points of trajectories t.

Definition 5. Let t be a bitrajectory. We denote by $\omega(t)$ (resp., $\alpha(t)$) the set of all forward (resp., backward) limit points of t.

Proposition 1. *Let* $f : X \to X$ *and* $x \in X$. *Then*:

(1) $\omega_f(x) = \omega(t)$ *for any bitrajectory of* f *based at* x;
(2) $s\alpha_f(x) = \cup\alpha(t)$, *where the union is over all bitrajectories of* f *based at* x.

Not much is known on general properties of $s\alpha$-limit sets. The strongest claim related to them, in the author's knowledge, is relative to infinite backward trajectories of rational maps on the Riemann sphere:

Theorem E [Hawkins & Taylor, 2003]. *Let* f *be a rational map of degree larger than 1, z_0 a nonexceptional point and ν the equidistributed Bernoullli measure on the space of bitrajectories through z_0. Then, for ν-almost all bitrajectories t passing through z_0, we have that* $\alpha(t) = J_f$.

Only recently several authors started a thorough investigation of its general properties, in particular Balibrea *et al.* [2013], Mitchell [2020], Hantáková and Roth [2020], Kolyada *et al.* [2020]. In particular, the last three authors formulated the following conjecture:

Conjecture 1 [Kolyada *et al.*, 2020]. *For all continuous maps of the interval, all sα-limit sets of points are closed.*

3. Definitions and Graph of S-Unimodal Maps

A *discrete dynamical system* on a metric space (X, d) is given by the iterations of a continuous map $f : X \to X$. In this article, we are interested in the case where X is a closed interval and f is a S-unimodal map, as defined below:

Definition 6 [Singer, 1978]. A C^3 map $f : [a, b] \to [a, b]$ is *S-unimodal* if:

(1) either $f(a) = f(b) = a$ or $f(a) = f(b) = b$;
(2) f has a unique critical point $c \in (a, b)$;
(3) the Schwarzian derivative of f

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

is negative for all $x \in [a, b]$, $x \neq c$.

In all statements and examples throughout the article, we will assume that c is a maximum. Of course, the same proofs hold also when c is a minimum after trivial modifications that we leave to the reader.

Given any S-unimodal map f, the equation $f(x) = y$ has two distinct solutions for each $y < f(c)$. For each $p \neq c$ in the domain of f, we denote by $\hat{p}$ the solution different from p of the equation $f(x) = f(p)$.

Example 2. In case of the logistic map

$$\ell_\mu(x) = \mu x(1-x)$$

we have that $[a, b] = [0, 1]$, $f(0) = f(1) = 0$, $c = 0.5$ and $\hat{p} = 1 - p$. The Schwarzian derivative of ℓ_μ is trivially negative since $\ell_\mu''' \equiv 0$.

Recall that, as a consequence of Guckenheimer's results in [Guckenheimer, 1979], each S-unimodal map is topologically conjugated to a logistic map (see Theorem 6.4 and Remark 1 in [De Melo & van Strien, 1993]). In particular, each S-unimodal map has at most two fixed points.

The following several definitions are needed to define the graph of a dynamical system f.

Definition 7 [Bowen, 1975]. An *ε-chain* from $x \in X$ to $y \in X$ is a sequence of points $x_0 = x, x_1, \ldots,$ $x_n, x_{n+1} = y$ such that $d(f(x_i), x_{i+1}) < \varepsilon$ for all $i = 0, \ldots, n$. We say that x is *downstream* (resp., *upstream* from y) if, for every $\varepsilon > 0$, there is an ε-chain from y to x (resp., from x to y). A point $x \in X$ is *chain-recurrent* if it is downstream from itself.

3.1. *Nodes*

We denote the set of all chain-recurrent points of X under f by $\mathcal{R}_f$. The relation $x \sim y$ if and only if x is both upstream and downstream from y is an equivalent relation in $\mathcal{R}_f$ (e.g. see [Norton, 1995a]). The points of $\mathcal{R}_f/\sim$ will be the nodes of the graph and so we refer to them as *nodes*. Notice that every node is closed and invariant under f [Norton, 1995a]. The edges of the graph will be defined through the asymptotics of the system's bitrajectories, as explained below.

Proposition 2 [Norton, 1995a]. *Given any bitrajectory t, there exist nodes N_1, N_2 in $\mathcal{R}_f$, not necessarily distinct, such that $\alpha(t) \subset N_1$ and $\omega(t) \subset N_2$.*

Proof. If $x, y \in \omega(t)$ it means that, for every $\varepsilon > 0$, there are integers p, q, r such that $|x_i - x| < \varepsilon$ for $i = p, r$ and $|x_q - y| < \varepsilon$, namely x is both upstream and downstream from y and so they belong to the same node. The same argument can be repeated in the case of α-limit.   ∎

Notice that, when $\alpha(t)$ and $\omega(t)$ belong to the same node N, it means that all points of t actually belong to N.

Definition 8 [Milnor, 1985]. A closed invariant set A is an *attractor* if it satisfies the following conditions:

(1) the basin of attraction of A, namely the set of all $x \in X$ such that $\omega(x) \subset A$, has strictly positive measure;
(2) there is no strictly smaller invariant closed subset $A' \subset A$ whose basin differs from the basin of A by just a zero-measure set.

Definition 9. We call a node an *attracting node* if it contains an attractor, otherwise we call it a *repelling node*.

Definition 10. We say that a cycle γ is *critical*, *regular* or *flip* if, given any $p \in \gamma$, respectively $(f^n)'(p) = 0$, $(f^n)'(p) > 0$ or $(f^n)'(p) < 0$.

Notice that the definition is consistent because the derivative of f^n is constant on any n-cycle. Recall also that this derivative is zero, and so the cycle is superattracting, if and only if c belongs to the cycle.

Definition 11. We call period-k *trapping region* of f a collection $\mathcal{T}$ of k closed intervals $J_1, \ldots, J_k$ such that:

(1) c lies in the interior of J_1;
(2) the interiors of the J_i are pairwise disjoint;
(3) $f(J_i) \subset J_{i+1}$, $i = 1, \ldots, k$, where we use the notation $J_{k+1} = J_1$.

We say that $\mathcal{T}$ is *cyclic* if, moreover,

(4) $\partial J_1 = \{p_1, \hat{p}_1\}$ for some periodic point p_1;
(5) $f(\partial J_i) \subset \partial J_{i+1}$.

We denote by $|\mathcal{T}|$ the period of $\mathcal{T}$. When $\mathcal{T}$ is cyclic, we denote by $\Gamma(\mathcal{T})$ the periodic orbit passing through p_1. We say that $\Gamma(\mathcal{T})$ is the *minimal cycle* of $\mathcal{T}$ (see Theorem F). In general, we say that a cycle γ is minimal when $\gamma = \Gamma(\mathcal{T})$ for some trapping region $\mathcal{T}$. Sometimes, to emphasize, we denote by $J_i(\mathcal{T})$ the J_i interval of $\mathcal{T}$. We denote by $J^{\mathrm{int}} = J^{\mathrm{int}}(\mathcal{T})$ the union of the interiors of all the J_i. We say that a cyclic trapping region is a *flip trapping region* if any two J_i have an endpoint in common, otherwise we say it is a *regular trapping region*.

Proposition 3. *A cyclic trapping region $\mathcal{T}$ is flip* (*resp., regular*) *if and only if $\Gamma(\mathcal{T})$ is flip* (*resp., regular*)*. Moreover, $|\mathcal{T}| = |\Gamma(\mathcal{T})|$ if $\mathcal{T}$ is regular while $|\mathcal{T}| = 2|\Gamma(\mathcal{T})|$ if $\mathcal{T}$ is flip.*

Example 3. The interval $[a, b]$ is, trivially, a period-1 regular cyclic trapping region for every S-unimodal map.

Example 4. Let $\bar{p}_\mu$ be the fixed point of the logistic map ℓ_μ other than 0. The collection

$$\mathcal{T}_\mu = \{J_1 = [\ell_\mu^2(c), \bar{p}_\mu], J_2 = [\bar{p}_\mu, \ell_\mu(c)]\}$$

is a period-2 trapping region for all $\mu \in (1 + \sqrt{5}, 4]$. A direct calculation shows that $\mathcal{T}_\mu$ is *cyclic* only at

$$\mu_M = \frac{3[1 + (19 - 3\sqrt{33})^{1/3} + (19 + 3\sqrt{33})^{1/3}]}{2}$$

$$\simeq 3.67857,$$

which is exactly the point at which $\bar{p}_\mu$ plunges into a chaotic attractor and ceases to be a node in itself. At that point, the logistic map has a *crisis* [Grebogi *et al.*, 1982].

This example above can be generalized to S-unimodal maps, as shown below.

Definition 12. We call *core* of a S-unimodal map f the interval $[f^2(c), f(c)]$.

In order to simplify the notation, we often use the notation c_k in place of $f^k(c)$.

Proposition 4. *Let f be a S-unimodal map and $\bar{p}$ its internal fixed point. The core of f is forward-invariant if and only if $\bar{p} > c$.*

Proof. Notice first of all that, when $c_3 < c_2 < c$, we have that the restriction of f to $[c_3, c_2]$ is a orientation-preserving homeomorphism and so $f([c_3, c_2]) = [c_4, c_3]$. Hence, in general, $f([c_k, c_{k-1}]) = [c_{k+1}, c_k]$, namely the forward orbit of c is a monotonically decreasing sequence contained in $[a, c)$ and therefore it must converge to a fixed point at the left of c. In particular, when $c < \bar{p}$, this cannot happen because, when f has two fixed points, the boundary one $x = a$ is repelling (recall that f, being topologically conjugate to a logistic map, cannot have more than two fixed points). Now let us go over all possible cases:

Case 1. $\bar{p} \leq c$. Since f is monotonically increasing at the left of c and monotonically decreasing at its right, this can happen if and only if the graph of

f is not above the diagonal at $x = c$, namely we must have $c_1 \leq c$. The case $c_1 = c$ is degenerate: in that case $c = \bar{p}$ is fixed and $[c_2, c_1]$ is a single point. When $c_1 < c$, the restriction of f to $[c_1, c]$ is a orientation-preserving homeomorphism and we fall again in the case outlined above: $f([c_1, c]) = [c_2, c_1]$ and $f([c_2, c_1]) = [c_3, c_2]$, so in particular $[c_2, c_1]$ is not forward invariant.

Case 2. $c < \bar{p}$, $c \leq c_2$. In this case, the restriction of f to $[c_2, c_1]$ is a orientation-reversing homeomorphism and so $f([c_2, c_1]) = [c_2, c_3] \subset [c_2, c_1]$, showing that $c_2 < c_3$ and that $[c_2, c_1]$ is forward-invariant. It is not a trapping region, though, since $c \notin (c_2, c_1)$.

Case 3. $c_2 < c < \bar{p}$. In this case, we have that

$$f([c_2, c_1]) = f([c_2, c] \cup [c, c_1])$$

$$= [c_3, c_1] \cup [c_2, c_1] = [c_2, c_1]$$

since, by our argument above, the configuration $c_3 < c_2 < c < \bar{p}$ cannot take place and therefore we must have $c_2 \leq c_3$. Hence in this case $[c_2, c_1]$ is a period-1 trapping region.   ∎

Several examples of cyclic flipping and regular trapping regions are shown, respectively, in Figs. 2–5.

Theorem F (Repelling Nodes [De Leo & Yorke, 2021a]). *Let f be a S-unimodal map. Then the minimum distance between a repelling node N and the critical point of f is achieved at a periodic point p_1 of N. The closed interval with endpoints p_1 and $\hat{p}_1$ is the J_1 interval of a cyclic trapping region $\mathcal{T}(N)$ such that:*

(1) *$N \cap J^{\text{int}}(\mathcal{T}(N)) = \emptyset$;*
(2) *$p_1 \in \Gamma(\mathcal{T}(N))$ and each interval of $\mathcal{T}(N)$ has a point of $\Gamma(\mathcal{T}(N))$ at its boundary;*

Depending on the type of the corresponding trapping region, there are two types of repelling nodes:

(3) *A flip cycle. In this case, $\mathcal{T}(N)$ is a flip trapping region.*
(4) *A Cantor set with a dense orbit. In this case, $\mathcal{T}(N)$ is a regular trapping region.*

In order to keep notation light, we often write $\Gamma(N)$ to denote $\Gamma(\mathcal{T}(N))$ and so on.

Remark 1. A flip cycle node N is always inside the interior of the union of all $J_i(N)$ (see Fig. 4). On the contrary, a Cantor set node M has no point in common with the interior of the union of all $J_i(M)$ (see Fig. 5).
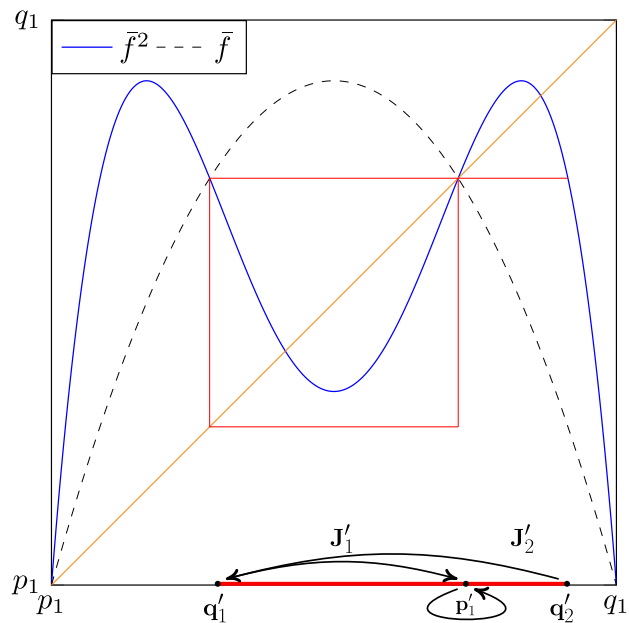
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Fig. 2. *Example of flip trapping region.* The interval $[p_1, q_1]$, where possibly $q_1 < p_1$, is the J_1 interval of some cyclic trapping region of some period r of a map f, so that J_1 is invariant under $\overline{f} = f^k$. Inside J_1 is shown a period-2 flip trapping region $\{J_1', J_2'\}$ of $\overline{f}$. The point p_1', which is fixed by $\overline{f}$, is an endpoint of both $J_{1,2}'$. Both the other endpoints q_1', q_2' eventually fall on p_1'. Notice that $\overline{f}(c) < q_2'$.
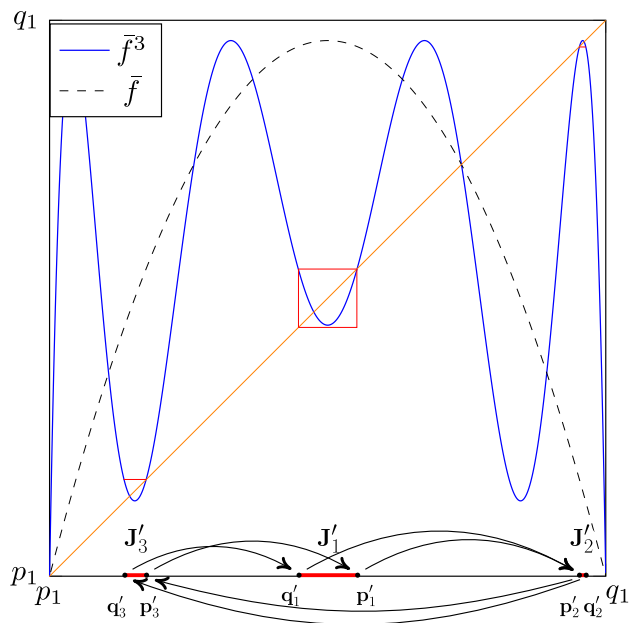

Fig. 3. *Example of regular trapping region.* The interval $[p_1, q_1]$, where possibly $q_1 < p_1$, is the J_1 interval of some cyclic trapping region of some period r of a map f, so that J_1 is invariant under $\overline{f} = f^k$. Inside J_1 is shown a period-3 regular trapping region $\{J_1', J_2', J_3'\}$ of $\overline{f}$. The arrows show how the endpoints map under $\overline{f}$: the endpoints p_1', p_2', p_3' form an unstable 3-cycle of $\overline{f}$ and all other endpoints q_1', q_2', q_3' eventually fall into this cycle.
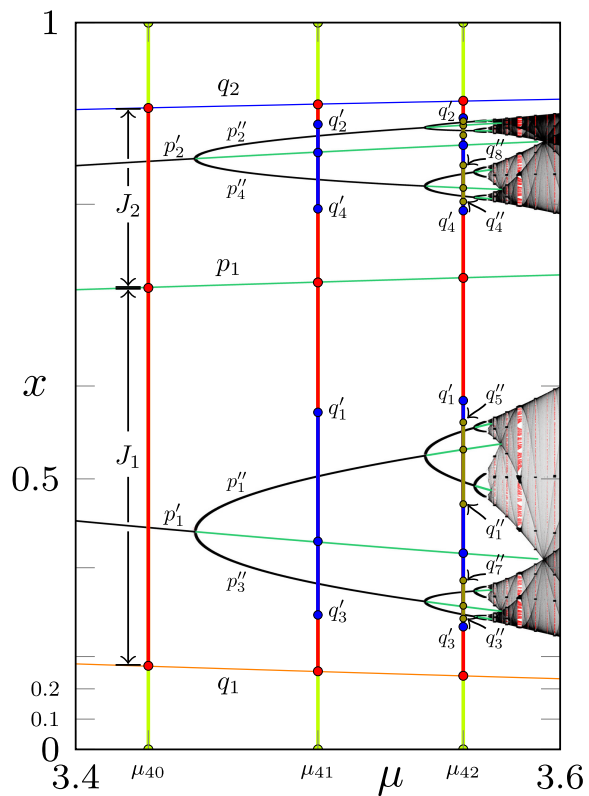

Fig. 4. *Examples of flipping trapping regions.* This picture is discussed in detail in Sec. 3.4.
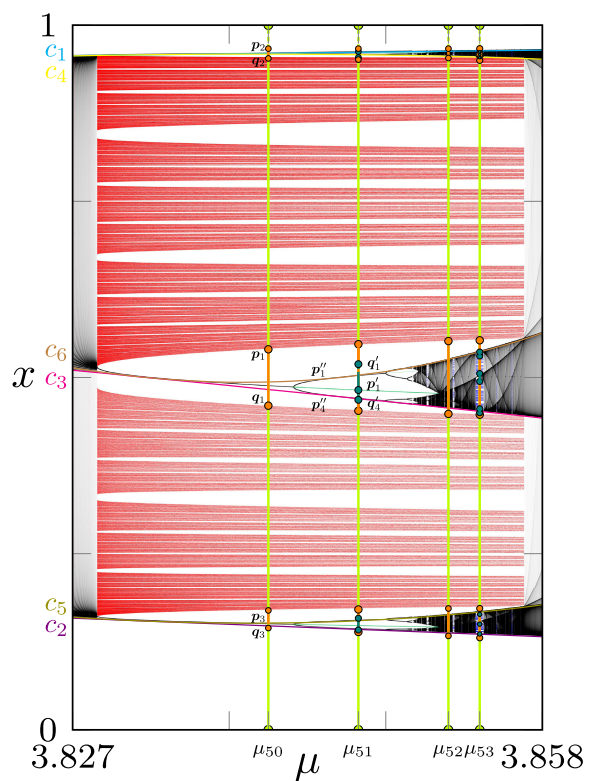

Fig. 5. *Examples of regular trapping regions.* This picture is discussed in detail in Sec. 3.4.

Theorem G (Attracting Nodes [De Leo & Yorke, 2021a]). *Let f be a S-unimodal map. Then its attracting nodes are of the following types*:

(A_1) *An attracting cycle.*

(A_2) *An attracting trapping region.*

(A_3) *An attracting minimal Cantor set* (*this is the case when there are infinitely many nodes*).

(A_4) *A repelling Cantor set containing a 1-sided attracting periodic orbit.*

(A_5) *A trapping region which strictly contains an attracting cyclic trapping region, a repelling Cantor set and part of the basin of attraction. The Cantor set and the attractor have a 1-sided attracting periodic orbit in common.*

Remark 2. Among the attracting and repelling nodes of S-unimodal maps, those of type A_5 are the only ones containing points that are not non-wandering points. In particular, since all points in $\omega(t)$ and $\alpha(t)$ are nonwandering points, this is the only node that cannot be equal to either the ω-limit or the α-limit of any bitrajectory.

3.2. *Examples of nodes*

Some of the nodes of the logistic map are shown in Fig. 1.

Repelling nodes of type R_1 (resp., R_2) are painted in green (resp., red). Each repelling node of either kind exists over some closed connected interval of parameter values and depends continuously on it. The parameters range over which a repelling Cantor set node N that arise is called a *window* $W(N)$. The period of $W(N)$ is the period of the periodic point $p_1(N)$. The largest window of the logistic map family, shown in detail in Fig. 5, is the period-3 window starting at $\mu = 1 + 2\sqrt{2} \simeq 3.828$ and ending at $\mu \simeq 3.857$ (see [Hasselblatt & Katok, 2003, p. 299], for an exact expression of this last number).

Attractors are shown in shades of gray. Attractors of type A_1 (isolated black lines) and of type A_2 are both visible. Attractors of types A_4 and A_5 are located, respectively, at the beginning and end of each window [De Leo & Yorke, 2021a]. Attractors of type A_3 are not visible because they arise only for a zero-measure set of parameter values [Lyubich, 2002]. The lines visible throughout the chaotic attractors are iterates of the critical point, about which the density of points under iterations is higher. In colors are highlighted the first four of these iterates.

3.3. *The graph*

Finally, we have all ingredients to introduce the graph of a discrete dynamical system:

Definition 13 [Conley, 1978]. We call graph Γ_f of the dynamical system f on X the directed graph having as nodes the elements of $\mathcal{R}_f/\sim$ and having an edge from node N_1 to node N_2 when there exist a bitrajectory t such that $\alpha(t) \subset N_1$ and $\omega(t) \subset N_2$.

The key result by Conley, obtained originally for continuous systems [Conley, 1978] and generalized later by Norton [1995b] to discrete ones, is that the dynamics outside of nodes is purely of gradient-like nature, as explained below.

Definition 14. A *Lyapunov function* [Wilson & Yorke, 1973] for $f : X \to X$ is a continuous function L such that:

(1) L is constant on each node;

(2) L assumes different values on different nodes;

(3) $L(f(x)) < L(x)$ if and only if x is not chain-recurrent.

Theorem H (Norton, 1995b). *Let f be a dynamical system on a compact metric space X. Then there exists a Lyapunov function for f. In other words, every point $x \in X$ either belongs to $\mathcal{R}_f$ or is gradient-like for f.*

Recall that a loop in a graph is a sequence of nodes $N_1, N_2, \ldots, N_k$ such that there is an edge from N_i to N_{i+1} for $i = 1, \ldots, n-1$ and then an edge from N_k back to N_1.

Corollary 1. *A graph Γ_f does not have loops.*

Definition 15. A graph Γ_f is a tower if there is an edge between each pair of nodes.

Notice that, because of the corollary above, if there is an edge from A to B then there cannot be an edge from B to A.

In [De Leo & Yorke, 2021a] we prove the following two fundamental results. Notice that the second theorem integrates some original result of ours together with other classic results by other authors (references are provided within the claim).

Theorem I [Tower Theorem]. *The graph of a S-unimodal map is a tower.*

Theorem J [Chain-Recurrent Spectral Theorem]. *Let f be a S-unimodal map with at least a repellor and denote by $N_0, N_1, \ldots, N_p$, where $p \geq 1$ is possibly infinite, the nodes of Γ_f sorted in the order*

determined by the edges (*in particular, N_p is the attracting node of f*). *Then*:

(1) *N_0 is the fixed endpoint of f.*
(2) *Each repelling node N_i, $0 \le i < p$, determines a cyclic trapping region $\mathcal{T}(N_i)$ for f such that $\Gamma(\mathcal{T}(N_i)) \subset N_i$ and each $J_k(\mathcal{T}(N_i))$ has a point of $\Gamma(\mathcal{T}(N_i))$ as an endpoint. There is no other point of N_i in $\mathcal{T}(N_i)$ [De Leo & Yorke, 2021a].*
(3) *The trapping regions $\mathcal{T}(N_i)$ are nested*:

$$\overline{J^{\text{int}}(\mathcal{T}_j)} \subset J^{\text{int}}(\mathcal{T}_{j-1}), \quad 1 \le j < p,$$

where, by convention, we take $\mathcal{T}(N_0) = \{[a,b]\}$ [De Leo & Yorke, 2021a].
(4) *Each N_j, $1 \le j < p$, is either of type R_1 or R_2 [De Leo & Yorke, 2021a]. The measure of each repelling node is zero and the action of f on it is topologically conjugated to a subshift of finite type [van Strien, 1981].*
(5) *Each N_j, $0 \le j < p$, is repelling and hyperbolic [van Strien, 1981].*
(6) *N_p is the unique attracting node of f [Guckenheimer, 1979] and it is one of the five types $A_1, \ldots, A_5$ [De Leo & Yorke, 2021a].*
(7) *N_p is hyperbolic unless it is of type A_4 or A_5 [van Strien, 1981].*
(8) *In each neighborhood of N_i, for each $j \ge i$, there are points falling eventually into N_j [De Leo & Yorke, 2021a].*
(9) *When $p = \infty$, the attracting node N_∞ is a Cantor set of zero Lebesgue measure on which f acts as an adding machine [van Strien, 1981].*

3.4. *Examples of trapping regions*

Several cyclic trapping regions are shown in Figs. 4 and 5 in case of the logistic map. In all cases, the top node is $N_0 = 0$ and its corresponding trapping region is the interval $J_1(N_0) = [0, 1]$.

In Fig. 4, we see the logistic map's bifurcation diagram for $\mu \in [3.4, 3.6]$. Attracting cycles are shown in black. In this range, most of the figure is occupied by the first bifurcation cascade of the diagram. Throughout this range, the first node N_0 is the unstable fixed point 0 and the second node N_1 is the other fixed point p_1 (painted in green), which is unstable too. We show respectively in orange and blue the points $q_1 = \hat{p}_1 = 1 - p_1$ and q_2, where q_2 is the root of $\ell_\mu^2(x) = p_1$ on the other side of q_1 with respect to p_1 (see Fig. 3). We paint in lime $\mathcal{T}(N_0)$.

At μ_{40}, the logistic map has a third and final node: the attracting 2-cycle $N_2 = \{p_1', p_2'\}$. The flip

trapping region $\mathcal{T}(N_1)$, painted in red, consists of the following two intervals: $J_1(N_1) = [q_1, p_1]$ and $J_2(N_1) = [p_1, q_2]$.

Between μ_{40} and μ_{41}, the 2-cycle N_2 bifurcates and becomes repelling and a new attracting 4-cycle node $N_3 = \{p_1'', \ldots, p_4''\}$ arises. We paint in blue the period-4 flip trapping region $\mathcal{T}(N_2)$, which consists of the following intervals: $J_1(N_2) = [p_1', q_1']$, $J_2(N_2) = [p_2', q_2']$, $J_3(N_2) = [q_3', p_1']$ and $J_4(N_2) = [q_4', p_2']$.

Similarly, at μ_{42} node N_3 is repelling and we have a new 8-cycle attractor and a new flip trapping region $\mathcal{T}(N_3)$ generated by $J_1(N_3) = [q_1'', p_1'']$ consisting of eight intervals, painted in olive.

Figure 5 shows a detail of the logistic map's period-3 window (see also Fig. 8 for a close-up of its middle cascade). Besides the nodes, we plot the supplementary lines c_k for $k = 1, \ldots, 6$. At the left endpoint of the window, at $\mu_0 = 1 + 2\sqrt{2}$, the graph has just two nodes: N_0 and an attracting node N_1 of type A_4. We paint in blue $\mathcal{T}(N_0)$.

For all parameter values in the interior of the window, node N_1 is the repelling Cantor set painted in red. The point $p_1 = p_1(N_1)$ belongs to the repelling 3-cycle p_1, p_2, p_3 created together with the other 3-cycle p_1', p_2', p_3', that is the attracting node for μ close enough to μ_0 from the right. The regular trapping region $\mathcal{T}(N_1)$ of the Cantor node, painted in orange, consists of the three intervals $J_1(N_1) = [q_1, p_1]$, $J_2(N_1) = [p_2, q_2]$ and $J_3(N_2) = [q_3, p_3]$. At the right endpoint, the Cantor set has a point in common with the attracting trapping region, forming an attracting node of type A_5.

At $\mu = \mu_{50}$ and $\mu = \mu_{52}$ the graph has just three nodes. In the first case, the attracting node N_2 is the 3-cycle p_1', p_2', p_3'; in the second case, it is the period-3 trapping region $[c_4, c_1], [c_2, c_5], [c_3, c_6]$.

At $\mu = \mu_{51}$, the node N_2 is repelling and the attracting node N_3 is a 6-cycle $\{p_i''\}_{i=1,\ldots,6}$. The flip trapping region $\mathcal{T}(N_2)$, painted in teal, consists of the six intervals $J_1(N_2) = [p_1', q_1']$, $J_2(N_2) = [p_2', q_2']$, $J_3(N_2) = [q_3', p_3']$, $J_4(N_2) = [q_4', p_1']$, $J_5(N_2) = [q_5', p_2']$, $J_6(N_2) = [p_3', q_6']$.

At $\mu = \mu_{53}$, node N_2 is a second Cantor set, painted in blue, namely this parameter value is within a window inside a window. Point $p_1(N_2)$ has period 9, so it is a period-9 window within a period-3 window. The attractor N_3 in this case is the period-9 orbit originating from the same bifurcation that created the repelling period-9 cycle that $p_1(N_2)$ belongs to.

4. Limit Sets of S-Unimodal Maps

We are now ready to study the $s\alpha$-limit sets of points under f. Throughout the rest of the paper, we will assume that f satisfies the following:

Assumption (A). f is a S-unimodal map with at least two nodes.

Under these conditions, the map is not surjective, i.e. $f(c) < b$, and the fixed point $x = a$ is a repelling node, i.e. $f'(a) > 1$ (note that $f'(a) > 0$ since we are assuming that the critical point of f is a maximum).

Example 5. In case of the logistic map $\ell_\mu : [0,1] \to [0,1]$, Assumption (A) amounts to $f'(0) > 1$ and $f(0.5) < 1$. These conditions are satisfied if and only if $\mu \in (1,4)$.

The nodes of f will be denoted by $N_0, N_1, \ldots, N_p$, where $p \geq 1$ is possibly infinite, and sorted in the order determined by the graph's edges (in particular, N_p is the attracting node of f). Recall that N_0, under Assumption (A), is always equal to the fixed endpoint of f.

Definition 16. For $k = 0, \ldots, p-1$, we denote by r_k the period of $\mathcal{T}(N_k)$ and we set

$$\overline{f}_i = f^{r_k}|_{J_i(N_k)} : J_i(N_k) \to J_i(N_k), \quad i = 1, \ldots, r_k.$$

We denote by $K(\overline{f}_i)$ the core of the map $\overline{f}_i$ and we call *core of* $\mathcal{T}(N_k)$ the collection of intervals

$$\mathcal{K}(N_k) = \{K(\overline{f}_1), \ldots, K(\overline{f}_{r_k})\}.$$

Finally, we denote by $K(N_k)$ the union of all intervals in $\mathcal{K}(N_k)$.

Example 6. $\mathcal{K}(N_0) = \{[c_2, c_1]\}$ since $r_0 = 1$ is the period of $\mathcal{T}(N_0) = \{[a,b]\}$. If the second fixed point $\bar{p}$ of f is a repelling node (as in Fig. 4), then $N_1 = \{\bar{p}\}$, $r_1 = 2$ and $\mathcal{K}(N_1) = \{[c_2, c_4], [c_3, c_1]\}$.

Suppose now that the second node N_1 is a Cantor set. Consider for instance the concrete case of the logistic map with $\mu = \mu_{50}$ in Fig. 5. Then $\mathcal{T}(N_1)$ has period equal to 3 and $\mathcal{K}(N_1) = \{[c_2, c_5], [c_3, c_6], [c_4, c_1]\}$.

Proposition 5. *The following properties hold for each $k = 0, 1, \ldots, p-1$:*

(1) $K(\overline{f}_{r_k - i}) = [c_{2r_k - i - 1}, c_{r_k - i - 1}]$;
(2) $K(N_k) \subset J^{\mathrm{int}}(N_k)$;
(3) *for $k \leq p - 2$, $\mathcal{K}(N_k)$ is a trapping region and $J^{\mathrm{int}}(N_{k+1}) \subset K(N_k)$.*

Proof. Recall that $J_1(N_k) = [p_1(N_k), \hat{p}_1(N_k)]$ is the interval of $\mathcal{T}(N_k)$ containing c, so that

$$K(\overline{f}_1) = [\overline{f}_1^2(c), \overline{f}_1(c)] = [c_{2r_k}, c_{r_k}].$$

Moreover, each map $f|_{J_i(N_k)} : J_i(N_k) \to J_{i+1}(N_k)$, $i = 2, \ldots, r_k$, where we set $J_{r_k+1}(N_k) = J_1(N_k)$, is a homeomorphism for $i > 1$, and so

$$K(\overline{f}_{r_k}) = f^{-1}(K(\overline{f}_1)) \cap J_{r_k}(N_k) = [c_{2r_k - 1}, c_{r_k - 1}]$$

and, ultimately,

$$K(\overline{f}_{r_k - i}) = [c_{2r_k - i - 1}, c_{r_k - i - 1}].$$

The assumption that N_k is not the last node implies the following two facts:

(1) $K(\overline{f}_1) \subset \mathrm{int}(J_1(N_k))$. Otherwise, we would have $\overline{f}_1(c) = p_1(N_k)$ or $\overline{f}_1(c) = \hat{p}_1(N_k)$ and, in both cases, N_k would be the attractor and there could be no other node besides it. Hence, in general, $K(\overline{f}_i) \subset \mathrm{int}(J_i(N_k))$ for $i = 1, \ldots, r_k$, which proves point (2).

(2) Since the cycle $\Gamma(N_k)$ is repelling and, for $k < p - 1$, N_{k+1} is repelling too, then by Proposition 4 we have that the core of each $\overline{f}_i$ is forward-invariant with respect to $\overline{f}_i$. Indeed, each $\overline{f}_i$ is a S-unimodal map on $J_i(N_k)$ whose nonboundary fixed point $\bar{p}_i$ must be at the right of the critical point, since otherwise $\bar{p}_i$ would be attractive and there would be no other node inside $J^{\mathrm{int}}(N_k)$. Hence

$$f(K(\overline{f}_1)) = f^2(K(\overline{f}_{r_k}))$$
$$= \cdots = f^{r_k}(K(\overline{f}_2)) \subset K(\overline{f}_2)$$

and, more generally, $f(K(\overline{f}_i)) \subset f(K(\overline{f}_{i+1}))$, $i = 1, \ldots, r_k$, where we put $\overline{f}_{r_k+1} = \overline{f}_1$.

Since the attractor is unique, it must lie inside $K(N_k)$ and so $J^{\mathrm{int}}(N_{k+1})$ must be completely contained inside $K(N_k)$ as well. In particular, $c \in J_1(N_{k+1}) \subset K(\overline{f}_1)$, so $\mathcal{K}(N_k)$ is a trapping region, proving point (3). ∎

Definition 17. We define sets $U_1, \ldots, U_p$ as follows:

- $U_p = N_p$;
- $U_k = K(N_{k-1}) \backslash K(N_k)$ for $1 \leq k < p - 1$;
- if $p < \infty$, $U_{p-1} = K(N_{p-2}) \backslash N_p$;

Finally, we set $U_0 = [a, c_2)$ and $U_{-1} = (c_1, b]$. We say, for short, that $x \in [a, b]$ is a *level-k point* if $x \in U_k$.

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Proposition 6. *The sets $U_{-1}, \ldots, U_p$ satisfy the following properties*:

(1) $[a, b] = \bigsqcup_{i=-1}^{p} U_i$;
(2) $N_k \subset U_k$ *for* $k = 0, \ldots, p$;
(3) $U_k \subset J^{\text{int}}(N_{k-1})$ *for* $k = 1, \ldots, p$.

Proof

(1) By construction, the U_i are all pairwise disjoint. Assume first that $p < \infty$. Then

$$\bigcup_{i=1}^{p} U_i = (K(N_0)\backslash K(N_1)) \cup (K(N_1)\backslash K(N_2))$$

$$\cup \cdots \cup (K(N_{p-2})\backslash N_p) \cup N_p$$

$$= K(N_0) = [c_2, c_1]$$

so that $\bigcup_{i=-1}^{p} U_i = [a, c_2) \cup [c_2, c_1] \cup (c_1, b] = [a, b]$. If $p = \infty$, then

$$N_p = \bigcap_{k \geq 1} \overline{J^{\text{int}}(N_k)} = \bigcap_{k \geq 1} K(N_k)$$

and so, even in this case, $\bigcup_{k \geq 1} U_k \cup U_\infty = [a, b]$.

(2), (3) Since $U_k \subset K(N_{k-1}) \subset J^{\text{int}}(N_{k-1})$, $U_k \cap N_i = \emptyset$ for $i \leq k - 1$. Since $U_{k+1} \subset K(N_k) \subset J^{\text{int}}(N_k)$, $U_{k+1} \cap N_k = \emptyset$. More generally, since $K(N_{k+1}) \subset K(N_k)$, $U_i \cap N_k = \emptyset$ for $i \geq k+1$. Since every point in $[a, b]$ belongs to some U_i, the only possibility is that $N_k \subset U_k$ for each $k = 0, \ldots, p-1$. ∎

Example 7. Consider the case, shown in Fig. 6, of the logistic map f with $\mu = \mu_{40}$. This map has three nodes: N_1 is the fixed point other than $x = 0$ and the attractor N_2 is a 2-cycle $\{p_1', p_2'\}$. In this case

$$U_2 = \{p_1', p_2'\}, \quad U_1 = [c_2, p_1') \cup (p_1', p_2') \cup (p_2', c_1].$$

Example 8. Consider the case, shown in Fig. 7, of the logistic map with $\mu = \mu_{53}$. This map has four nodes: N_1 is a Cantor set, painted in red; N_2 is a second Cantor set, painted in blue; the attractor N_3, three points of which are visible (in black) in Fig. 8, is a 9-cycle. In this case,

$$U_3 = N_3,$$

$$U_2 = [c_2, c_5] \cup [c_3, c_6] \cup [c_4, c_1]\backslash N_3,$$

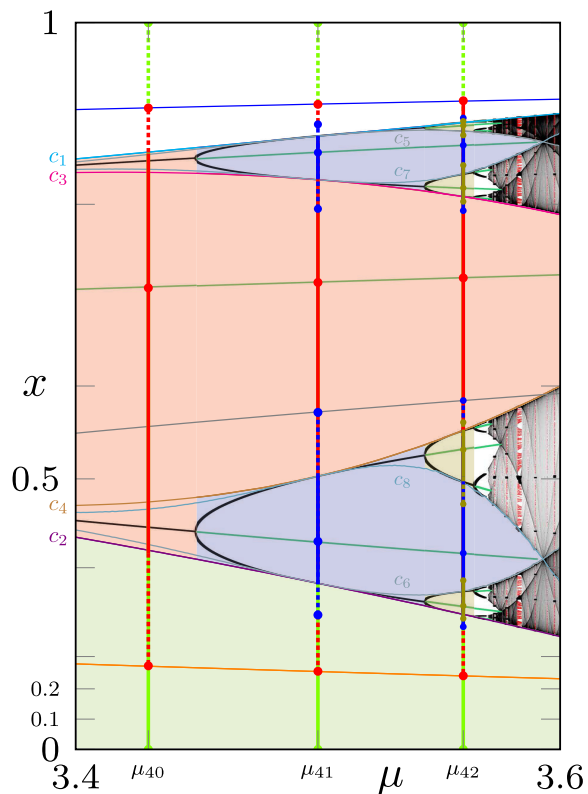$$U_1 = (c_5, c_3) \cup (c_6, c_5).$$

Fig. 6.   *Examples of U_k sets in case of flip trapping regions.* This picture is discussed in detail in Sec. 4.7.
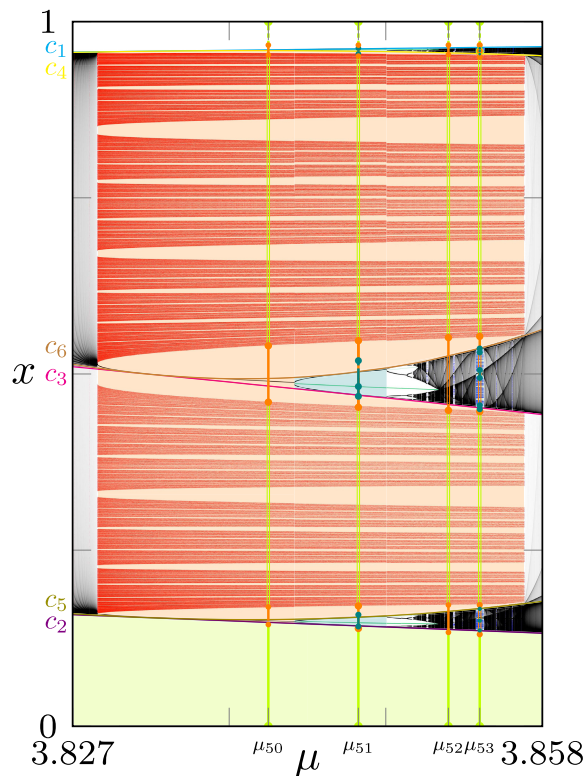

Fig. 7.   *Examples of U_k sets in case of regular trapping regions.* This picture is discussed in detail in Sec. 4.7.
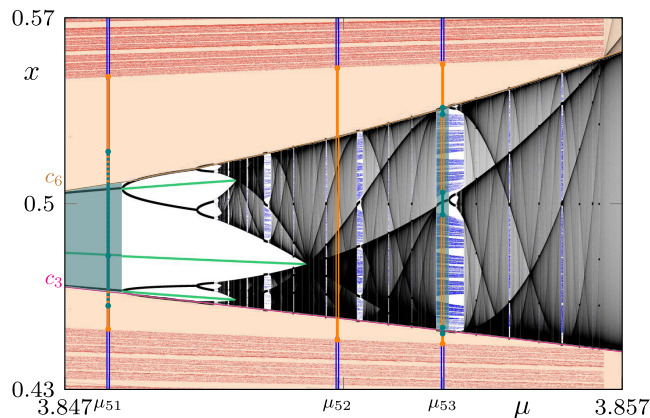
Fig. 8. *Close-up of the central cascade in Fig. 7.* This picture is discussed in detail in Sec. 4.7.

4.1. *Forward limits*

While the focus of this article is on backward limits, for completeness we include also the following result about the relation between the forward limit of a point and its position.

Theorem 1 [ω-limits of S-unimodal maps]. *Assume that f satisfies (A) and let $x \in U_k$. Then $\omega_f(x) \subset N_i$ for some unique $i \geq k$.*

Proof.   By the Chain-Recurrent Spectral Theorem, we know the following:

(1) Trapping regions are nested one into the other and are all forward invariant;
(2) $N_j \subset J^{\text{int}}(N_k)$ for all $j > k$;
(3) $N_j \cap J^{\text{int}}(N_k) = \emptyset$ for all $j \leq k$;
(4) $N_k \cap \overline{J^{\text{int}}(N_k)}$ is a single periodic orbit.

Hence, the ω-limit set of any x which lies into U_k must be contained in one of the nodes N_i for some $i = k, \ldots, p$.   ∎

In particular, this means that the set of possible outcomes for $\omega_f(x)$ is a locally constant monotonically decreasing function of the distance of x from the critical point c.

Remark 3. By Theorem B, $\omega(x)$ must be either a cycle or a Cantor set or an interval cycle. Moreover, for Lebesgue-almost all $x, \omega(x) = N_p$ [Guckenheimer, 1979; Guckenheimer, 1987]. Since all other nodes are repelling, when $\omega(x) \subset N_i$ for some $i \neq p$ the only possibility is that there is a forward trajectory starting at x that actually falls on N_i in a finite number of iterations. Equivalently, such x must belong to a backward trajectory from some point in N_i.

We turn now to the main goal of this article, which is studying the possible $s\alpha$-limits of a point depending on its level, namely depending on which U_k set it belongs to. Following Conley's observation, these points either belong to some node or are gradient-like, we discuss separately these two cases. All these partial results will be finally summarized in Theorem 5 (Sec. 4.6).

4.2. *Backward limits within repelling nodes*

As we recalled earlier, repelling nodes are either cycles (type R_1) or transitive (see Definition 20) Cantor sets (type R_2). The restriction of f to either case is topologically conjugated to a subshift of finite type, a fact that was proved first by van Strien in 1981 [van Strien, 1981] using kneading theory. For the sake of clarity and self-completeness, we sketch below a more explicit proof of this fact that will also lead us to establish that $s\alpha_f(x) \supset N$ for each point x belonging to a repelling node N.

The case of node N_1 in the period-3 window of ℓ_μ. Consider the concrete case of the period-3 window W of the logistic map $f = \ell_\mu$ (see Figs. 1 and 7). For every μ in the interior of W, the point $p_1(N_1)$ belongs to a repelling 3-cycle and $\mathcal{T}(N_1) = \{J_1, J_2, J_3\}$ is a regular period-3 trapping region. Since the J_i are disjoint, the set $C = [0,1] \backslash \bigcup_{k=0}^{\infty} f^{-k}(J_1)$ of all points that do not ever fall in J_1 is the complement of a dense countably infinite union of disjoint intervals and so is a Cantor set (see Prop. 3.21 in [De Leo & Yorke, 2021a] for more details).

Decomposing C. Recall that, when $\overline{p} > c$, each node different from N_0 lies inside $[f^2(c), f(c)]$ and that, since c belongs to either the basin of the attractor or to the attractor itself, each repelling node actually lies in $L_c = (f^2(c), f(c))$. Hence $N_1 \subset C \cap L_c$. Furthermore, by construction we have that

$$N_1 \subset C \cap (L_c \backslash J^{\text{int}}(N_1)).$$

The set $L_c \backslash J^{\text{int}}(N_1)$ is the disjoint union of the two closed intervals

$$A_0 = [p_3, 1 - p_1] \quad \text{and} \quad A_1 = [p_1, p_2]$$

and so, correspondingly, we can write

$$C \cap L_c = C_0 \sqcup C_1,$$

with $C_i = C \cap A_i, i = 0, 1$.

A map $s : C_0 \sqcup C_1 \to \Sigma_2$. Exploiting the decomposition above, one can associate to the trajectory $\{f^n(p)\}$ of each point $p \in C \cap L_c$ an element $s(p) = (s_0, s_1, \ldots)$ of the single-sided shift space Σ_2 in two symbols, say 0 and 1, so that $s_n = 0$ if $f^n(p) \in C_0$ and $s_n = 1$ otherwise.

For instance, for every $\mu \in W$, the map f has a fixed point $\bar{p}$ inside C_1, so

$$s(\bar{p}) = (111\ldots).$$

Since f^2 is a polynomial of degree 4 and both fixed points of f are also fixed points of f^2, for every $\mu \in W$ the map f has a single 2-cycle $\{\bar{q}_1, \bar{q}_2\}$, $\bar{q}_1 < \bar{q}_2$. Note that it is impossible that $\bar{q}_2 < c$ since $f(\bar{q}_1) = \bar{q}_2$ would imply $f(\bar{q}_2) > \bar{q}_2$. Moreover, since within W this 2-cycle is repelling, then necessarily $f^2(c) < c$ and so $c \in (\bar{q}_1, \bar{q}_2)$. Hence,

$$s(\bar{q}_1) = (0101\ldots),$$

$$s(f(\bar{q}_1)) = s(\bar{q}_2) = (1010\ldots).$$

Finally, in case of the 3-cycle $\Gamma(N_1)$, we have that

$$s(p_1(N_1)) = (110110\ldots),$$

$$s(f(p_1(N_1))) = s(p_2) = (101101\ldots),$$

$$s(f^2(p_1(N_1))) = s(p_3) = (011011\ldots).$$

Notice that, by construction,

$$s(f(p)) = \sigma(s(p)),$$

where $\sigma : \Sigma_2 \to \Sigma_2$ is the standard shift operator $\sigma((a_1 a_2 a_3 \ldots)) = (a_2 a_3 \ldots)$.

Each $s(p)$ misses the word "00"... While $(111\ldots)$ is in the range of s, $(000\ldots)$ is not, since the other fixed point of f, namely the node N_0, does not belong to N_1. This is an example of the following more general property: $\sigma(p)$ cannot contain the word "00". Indeed, since $[p_3, 1 - p_1] \subset [0, c]$,

$$f(A_0) = f([p_3, 1 - p_1])$$
$$= [f(p_3), f(1 - p_1)]$$
$$= [p_1, p_2] = A_1$$

while, since $[p_1, p_2] \subset [c, 1]$,

$$f(A_1) = f([p_1, p_2]) = [f(p_2), f(p_1)]$$
$$= [p_3, p_2] = A_0 \cup J_1(N_1) \cup A_1.$$

Since $C \cap L_c$ is invariant under f, this means ultimately that

$$f(C_0) \subset C_1, \quad f(C_1) \subset C_0 \sqcup C_1.$$

In particular, if the nth component of $s(p)$ is 0, then its $(n+1)$th component must be 1.

...but misses no other word. Denote by W_{00} the subset of Σ_2 of all elements not containing the word "00". Clearly W_{00} is a subshift of finite type, namely it is invariant under σ and is defined by a finite number of conditions (in fact, by just one condition). We claim that, for each element $w = W_{00}$, there is $p \in C \cap L_c$ such that $s(p) = w$.

Indeed, let $w = (w_1 w_2 w_3 \ldots)$ and define the following sets:

$$I_{w,1} = A_{w_1}$$
$$I_{w,2} = A_{w_1} \cap f^{-1}(A_{w_2})$$
$$I_{w,3} = A_{w_1} \cap f^{-1}(A_{w_2}) \cap f^{-2}(A_{w_3})$$
$$= A_{w_1} \cap f^{-1}(A_{w_2} \cap f^{-1}(A_{w_3}))$$
$$I_{w,4} = A_{w_1} \cap f^{-1}(A_{w_2}) \cap f^{-2}(A_{w_3}) \cap f^{-3}(A_{w_4})$$
$$= A_{w_1} \cap f^{-1}(A_{w_2} \cap f^{-1}(A_{w_3} \cap f^{-1}(A_{w_4})))$$
$$\vdots$$

By construction, $I_{w,k}$ is the open subset of C containing all points whose first k symbols coincide with the first k symbols of w. Set $I_w = \bigcap_{k=1}^{\infty} I_{w,k}$. Then $I_w \subset C_0 \sqcup C_1$ since the orbit of any point of I_w never leaves $A_0 \sqcup A_1$ and $s(p) = w$ for every $p \in I_w$.

In order to show that I_w is nonempty for each $w \in W_{00}$, we point out that, since f is unimodal, the counterimage of any interval not containing c is the union of two intervals lying on opposite side with respect to c. Given the action of f on A_0 and A_1, we furthermore have that the counterimage under $f|_{A_0 \sqcup A_1}$ of any interval in A_0 is a subinterval of A_1 while the counterimage of any interval in A_1 is a pair of intervals, one in A_0 and one in A_1. Hence, from the expression of $I_{w,k}$, it is clear that this set is empty if and only if, for some $r \leq k$, the interval $A_{w_r} \cap f^{-1}(\ldots)$ lies in A_0 (so $w_r = 0$) and, at the following step $A_{w_{r-1}} \cap f^{-1}(A_{w_r} \cap f^{-1}(\ldots))$, we have $A_{w_{r-1}} = A_0$ (namely $w_{r-1} = 0$). Ultimately, this argument shows that I_w is empty if and only if w contains the word "00".

s is injective. Since $I_w \subset C_0 \sqcup C_1$ is a closed interval and $C_0 \sqcup C_1$ is a Cantor set, the only possibility is that I_w is a single point, namely $s : C_0 \sqcup C_1 \to W_{00}$ is a bijection.

s is a homeomorphism. A basic neighborhood U_w of a point $w_0 \in W_{00}$ is the set of all $w \in W_{00}$ that coincide with w_0 up to their rth component for some $r > 0$. The set $s^{-1}(U_w)$ is the set we denoted above by $I_{w,r}$ and, as we pointed out above, this is always a neighborhood of $s^{-1}(w)$. Similarly, due to the continuity of f, given any $p \in C_0 \sqcup C_1$, for each $M > 0$ there is a neighborhood V_p of p such that $s(p)$ and $s(p')$ coincide up to their first M symbols for each $p' \in V_p$.

$f|_{N_1}$ is topologically conjugated to σ. This is an immediate consequence of the fact that, as pointed out above, $s \circ f = \sigma \circ s$ for all points of N_1 and that s is a homeomorphism.

$N_1 = C \cap L_c$. Because of the points above, it is enough to show that there is an ε-chain between every two points of $s(C \cap L_c)$. Given $x, y \in s(C \cap L_c)$ and $\varepsilon > 0$, there is an element z whose distance from x is smaller than ε so that all components z_i up to some finite index $n = n(\varepsilon) > 0$ coincide with the components x_i while $z_{n+i} = y_i$ for all $i > 0$. Then the sequence

$$x, z, \sigma(z), \ldots, \sigma^n(z) = y$$

is an ε-chain between x and y.

Hence, N_1 is the set of all points in $[f^2(c), f(c)]$ that do not fall into $J^{\text{int}}(N_1)$ under f.

The general case. Consider the general case of a Cantor set retractor N_k. Denote by r the period of the trapping region $\mathcal{T}(N_{k-1})$ and set $\overline{f}_i = f^r|_{J_i(N_{k-1})}$. Then each $\overline{f}_i : J_i(N_{k-1}) \to J_i(N_{k-1})$ is a S-unimodal map having first node $\overline{N}_{i,0} = \{p_i(N_{k-1})\}$ and second node the repelling Cantor set $\overline{N}_{i,1} = N_k \cap J_i(N_{k-1})$. Note that $|\mathcal{T}(\overline{N}_{i,1})| = |\mathcal{T}(N_k)|/|\mathcal{T}(N_{k-1})|$, $i = 1, \ldots, r$.

Similarly to the case above, $\overline{N}_{i,1} \subset [c_{2r-i-1}, c_{r-i-1}]$ and the set $(c_{2r-i-1}, c_{r-i-1}) \backslash J^{\text{int}}(\overline{N}_{i,1})$ is the disjoint union of $r-2$ closed intervals $A_{i,1}, \ldots, A_{i,r-2}$. Hence, like we did above, we can associate to each point $p \in (c_{2r-i-1}, c_{r-i-1}) \backslash \bigcup_{k=0}^{\infty} \times \overline{f}_i^{-k}(J_1(\overline{N}_{i,1}))$ an element $\sigma_i(p)$ belonging to the single-sided shift space Σ_{r-2}. Correspondingly to the action of $\overline{f}_i$ on the $A_{i,k}$, there will be some number of words that will not appear in any $\sigma_i(p)$. Since there is a finite number of $A_{i,k}$, there will be a finite number of forbidden words. Similarly to how we did above, one can prove that these maps σ_i are continuous and injective and all elements of Σ_{r-2} without those forbidden words in their sequence are the

image of some p. Hence, the image of each σ_i is a subshift of finite type.

Ultimately, we illustrated the following result:

Theorem K [van Strien, 1981]. *Let N be a repelling node of a S-unimodal map f. Then $f|_N$ is topologically conjugate to a subshift of finite type.*

4.3. *Backward dense orbits in subshifts of finite type*

Shifts have been thoroughly studied because they appear ubiquitously in every dynamics subfield. Nevertheless the following elementary result on backward dynamics, in the author's knowledge, did not appear explicitly so far in literature.

Theorem 2. *Let S be a 1-sided subshift of finite type with a dense trajectory. Then each point has a backward dense bitrajectory.*

Proof. Let F be the full 2-sided shift in the same symbols $\alpha_1, \ldots \alpha_k$ of S. A bitrajectory passing through an element $x = (x_1 x_2 x_3 \ldots) \in S$ is an element $X = (\ldots x_{-3} x_{-2} x_{-1} \cdot x_1 x_2 x_3 \ldots) \in F$.

Recall that, in a (1- or 2-sided) shift space X with a dense trajectory, for any two words $w, w' \in X$, there is a word $u \in X$ such that $wuw' \in X$ [Boyle, 2000]. Since in S there is an element $d = (d_1 d_2 d_3 \ldots)$ with a dense trajectory, the same is true about F: given any element $Y = (\ldots y_{-3} y_{-2} y_{-1} \cdot y_1 y_2 y_3 \ldots)$ there is some finite word $u_1 \ldots u_r$ such that

$$D = (\ldots y_{-3} y_{-2} y_{-1} u_1 \ldots u_r \cdot d_1 d_2 d_3 \ldots) \in F.$$

By Sec. 9.1 in [Lind & Marcus, 1995], every irreducible 2-sided shift (namely a 2-sided shift with a dense trajectory) has a doubly transitive point $T = (\ldots t_{-3} t_{-2} t_{-1} \cdot t_1 t_2 t_3 \ldots)$, namely a point whose orbit is dense both backward and forward.

Now, given any $x = (x_1 x_2 x_3 \ldots) \in S$, there is a finite word $u_1 \ldots u_s$ such that $X = (\ldots t_{-3} t_{-2} t_{-1} u_1 \ldots u_s \cdot x_1 x_2 x_3 \ldots) \in F$. This point represents precisely a bitrajectory passing through x whose backward orbit is dense by construction. ∎

As a corollary, we obtain our main result on retractors.

Theorem 3. *Every point of a repelling node N of f has a bitrajectory backward dense in N.*

4.4. *Backward limits within attracting nodes*

Of the five types of attracting nodes, only types A_2 and A_5, namely the two cases when the attractor is chaotic, are nontrivial. Indeed, cases A_1 (stable periodic orbit) and A_4 (single-sided stable periodic orbit belonging to a Cantor repellor) are covered by the results of the previous subsection. About case A_3, we recall that it is well-known that, in a minimal set, every point has a dense backward orbit [Kolyada & Snoha, 2009].

Backward orbits in a chaotic attractor. Let A be a chaotic attractor, namely an attracting trapping region. Recall that in A, as well as in every retracting node, there is a dense set of periodic points (but c does not belong to any of them, since any cycle containing c is superattracting). In each retractor node N_k we have a special cycle, the minimal cycle $\Gamma(N_k)$, which is the cycle passing through $p_1(N_k)$, the closest point of N_k to c. On the contrary, none of the cycles within A can be minimal, since A belongs to the attracting node and nodes are disjoint. Nevertheless, in analogy with minimal cycles, we give the following standard definition.

Definition 18. Given $p, q \in [a, b]$, we say that p is *closer to **c*** than q if $(p, \hat{p}) \subset (q, \hat{q})$, where as usual $\hat{p}$ is the root of $f(x) = f(p)$ other than p. Given a cycle γ, we denote by $p_1(\gamma)$ the point of γ that is the closest to c and we set $J_1(\gamma) = [p_1(\gamma), \hat{p}_1(\gamma)]$. We say that the cycle γ is closer to c than the cycle γ' when $p_1(\gamma)$ is closer to c than $p_1(\gamma')$.

Remark 4. The relation among cycles given by "being closer to c" is a linear order in the set of all cycles of f. In case of the logistic map, the distance between $p_1(\gamma)$ and c is the same as between $\hat{p}_1(\gamma) = 1 - p_1(\gamma)$ and c and so the "being closer" relation in this case is literal.

Minimal cycles are quite special: either there are only finitely many of them (when N_p is not of type A_3) or $p_1(\Gamma(N_k)) \to c$ (otherwise). On the contrary, the set of accumulation points of the set of all points $p_1(\gamma)$ of all cycles γ of a S-unimodal map f having at least one repelling Cantor set is large (easy to build examples in subshifts of finite type). Next definition highlights a type of cycle which represents a slight generalization minimal ones and is of central importance in the chaotic case.

Definition 19 [Guckenheimer, 1979]. Given a regular (resp., flip) n-cycle $\gamma \subset A$, we say that $p_1(\gamma)$ is *regular central* (resp., *flip central*) if f^n (resp., f^{2n}) is monotonic on $(p_1(\gamma), c)$, namely if $c \notin f^i(J_1(\gamma))$ for $i = 1, \ldots, n-1$ (resp., $i = 1, \ldots, 2n-1$). When $p_1(\gamma)$ is central, we also say that γ itself is central. Finally, we say that a closed interval H is a *homterval* for f if $c \notin f^k(H)$ for any integer $k \geq 0$, namely if all iterates f^k, $k = 0, 1, \ldots$, are strictly monotonic on H.

Example 9. Every minimal cycle $\Gamma(\mathcal{T})$ is central and its point $p_1(\mathcal{T})$ is a central point.

Example 10. Consider the unique 2-cycle $\gamma_2 = \{q_-, q_+\}$ of the logistic map. For each $\mu \in [1 + \sqrt{5}, 4]$, γ_2 is flip. For $\mu \in [1 + \sqrt{5}, \mu_{MF}]$, where μ_{MF} is the Myrberg–Feigenbaum point, γ_2 is a node. Hence, close enough from the right to μ_{MF}, γ_2 is a flip central point. On the other side, at $\mu = 4$ we have that $q_\pm = \frac{5 \pm \sqrt{5}}{8}$, so that $q_- < c < q_+$ and q_- is the closest to c. Set $J_1 = [q_-, 1 - q_-]$. Hence

$$f(J_1) = [q_+, 1],$$
$$f^2(J_1) = f([q_+, 1]) = [0, q_-],$$
$$f^3(J_1) = f([0, q_-]) = [0, q_+].$$

Since $c \in f^3(J_1)$, γ_2 is not central. For μ close enough to 4 from the left, c_1 is close to 1 and both c_2 and c_3 are close to 0, so that still $c \in f^3(J_1)$.

Proposition 7. *Let f be a S-unimodal map and $\bar{p}$ its internal fixed point. There is no flip cycle if $\bar{p} \leq c$. For $c < \bar{p}$, the lowest-period flip central point of f is $\bar{p}$. If f has a regular 3-cycle γ, then $p_1(\gamma)$ is the lowest-period regular central point of f.*

Proof. Set $\overline{J}_1 = [\hat{\bar{p}}, \bar{p}]$. Then $f(J_1) = [\bar{p}, c_1]$, which does not contain c, and so $\bar{p}$ is flip central. Consider now the case when f has a regular 3-cycle $\{p_1, p_2, p_3\}$. Set $J_1 = [\hat{p}_1, p_1]$. Then $f(J_1) = [p_2, c_1]$ and $f^2(J_1) = [c_2, p_3]$. Clearly c does not belong to either one of these two sets and so p_1 is central. ∎

The following facts are crucial for our results below:

Theorem L [Guckenheimer, 1979]. *Let f have a chaotic attractor $A = N_p$ and set $r = |\mathcal{T}(N_{p-1})|$. Then*:

(1) $c \in A$;
(2) $\alpha_f(c) = [a, b]$;

(3) *for any neighborhood U of the critical point, there is a central cycle γ with $J_1(\gamma) \subset U$;*
(4) *for each central s-cycle $\gamma \subset A$, with $s > r$ if γ is regular and $2s > r$ if it is flip, there is a central cycle $\gamma' \subset A$, closer to c than γ, and a $k > 0$ such that $f^k(J_1(\gamma')) \supset J_1(\gamma)$;*
(5) *f has no homtervals.*

The following proposition is implicitly used in a proof in [Guckenheimer, 1979] (see also Thm. II.7.9 in [Collet & Eckmann, 1980]).

Proposition 8. *The set C_f of all central points of a S-unimodal map f is finite if and only if N_p is of type A_1 or A_4. When C_f is infinite, the only point of accumulation of the set of all central points of f is the critical point.*

Proof. Suppose first that there are infinitely many central points. Consider a sequence q_k of central points of period n_k such that q_k is closer to c than q_{k-1} for all k and suppose that $q_k \to q_\infty \neq c$. Then there is some closed nontrivial interval $H \subset (q_\infty, 1 - q_\infty)$ not containing the critical point. Now, notice that $n_k \to \infty$ because there is a finite number of cycles of any given period. Hence $c \notin f^n(H)$ for any $n \geq 0$ since, for any given n, there are k for which $n_k > n$ and for such k we must have that $c \notin f^i(q_k, 1 - q_k)$ for $i = 1, \ldots, n_k - 1$. So H is a homterval, which contradicts point (5) of the theorem above.

By point (3) of the previous theorem, when the attractor is chaotic (namely N_p is of type A_2 or A_5) there are infinitely many central points. When N_p is of type A_3, there are infinitely many nodes N_k and each point $p_1(\mathcal{T}(N_k))$ is central. When the attractor is a cycle γ (namely N_p is of type A_1 or A_4), then γ is minimal and the interval $J^1(\gamma)$ is contained in the basin of attraction of γ and so cannot contain any point belonging to a cycle. ∎

The case $N_1 = A$. In this case there are only two nodes, the fixed endpoint N_0 and the chaotic attractor $N_1 = A$, which is of type A_2. The chaotic attractor must consist of a single interval, since the noninternal fixed point $\bar{p}$ of f must belong to it (or there would be a third node) and a trapping region with more than one component cannot contain any fixed point. Since $c \in A$, then $A = [c_2, c_1]$.

The general case. In case $p > 1$, set $r = |\mathcal{T}(N_{p-1})|$. Then the $J_i(N_{p-1})$ are invariant under

f^r and we set $\bar{f}_i = f^r|_{J_i(N_{p-1})}$. Then each $\bar{f}_i$ has just two nodes: the repelling fixed point $p_i(N_{p-1})$ and the chaotic attractor $N_p \cap J_i(N_{p-1}) = [c_{2r-i-1}, c_{r-i-1}]$. Hence, we reduce again the previous case.

In order to prove the existence of backward dense orbits, we use the two general results below about topological dynamical systems.

Definition 20 [Birkhoff, 1920]. A dynamical system $f : X \to X$ is *transitive* if it has a dense orbit.

Several properties equivalent to topological transitivity can be found in [Kolyada & Snoha, 1997].

Theorem M [Maličkỳ, 2012]. *Each transitive map f has a dense set of points with a backward dense orbit.*

Definition 21 [Parry, 1966]. A dynamical system $f : X \to X$ is *very strongly transitive* if, for every open set $U \subset X$, there is a $n \geq 1$ such that $\bigcup_{i=1}^n f^i(U) = X$.

Recently Akin, Auslander and Nagar gave the following characterization of very strongly transitive systems.

Theorem N [Akin *et al.*, 2016]. *A dynamical system $f : X \to X$ is very strongly transitive if and only if for any $x \in X$ and $\varepsilon > 0$ there is a $N > 0$ such that $A_N(x) = \bigcup_{i=1}^N f^{-i}(x)$ is ε-dense, namely $A_N(x)$ intersect all balls of radius ε in X.*

Proposition 9. *The restriction of f to $N_p = A$ is very strongly transitive.*

Proof. The general case argument above shows that it is enough to consider the case $p = 1$. Since c has a dense set of counterimages in A and there are central cycles arbitrarily close to c, we can assume without loss of generality that $U = \mathrm{int}(J_1(\gamma))$ for some central cycle γ.

As shown in Proposition 7, the fixed point $\bar{p}$ is a flip central point for f. Set $\overline{J}_1 = [\hat{\bar{p}}, \bar{p}]$. Then we have that $f(\overline{J}_1) = [\bar{p}, c_1]$ and $f^2(\overline{J}_1) = [c_2, \bar{p}]$, so that

$$A = f(\overline{J}_1) \cup f^2(\overline{J}_1), \quad f(\overline{J}_1) \cap f^2(\overline{J}_1) = \{\bar{p}\}.$$

Starting from the central cycle $\gamma_0 = \{\bar{p}\}$, we can build a sequence of central cycles γ_k so that, for each $k \geq 1$, $J_1(\gamma_k) \subset J_1(\gamma_{k-1})$ and there is a $n_k \geq 2$ such that $f^{n_k}(J_1(\gamma_k)) \supset J_1(\gamma_{k-1})$. In particular, for

each $k \geq 1$ there is a N_k such that $f^{N_k}(J_1(\gamma_k)) \supset J_1(\gamma_0)$. Since, by Proposition 8, central points can only accumulate at c, then $\lim_{k \to \infty} p_1(\gamma_k) = c$ and so there is some $r \geq 1$ such that γ_r is closer to c than γ and so $f^{n_r}(J_1(\gamma)) \supset J_1(\gamma_0)$. Hence $f^{n_r+1}(J_1(\gamma)) \supset [\bar{p}, c_1]$ and $f^{n_r+2}(J_1(\gamma)) \supset [c_2, \bar{p}]$, namely $\bigcup_{i=1}^{n_r+2} f^i(J_1(\gamma)) = A$. ∎

Now we are ready to prove our main result on chaotic attractors.

Theorem 4. *Every point of a chaotic attractor A of f has a bitrajectory backward dense in A. In particular, every point of an attracting node N of type A_2 has a backward orbit dense in N.*

Proof. By Theorem M applied to the restriction of f to A, there are points in A having a backward orbit dense in A. Let $b = (b_1, b_2, \ldots)$ be one of these backward dense orbits and let x_0 be any point of A. Let x_1 be a counterimage of x_0 of some order $n_1 \geq 0$ closer to b_1 than $1/1$. Such counterimage exists by Theorem N. Then let x_2 be a counterimage of x_1 of some order $n_2 \geq 0$ closer to b_2 than $1/2$ and so on. All these counterimages exist by Theorem N. The backward orbit $(x_0, x_1, x_2, \ldots)$ is easily seen to be dense as well. ∎

The case of nodes of type A_5. As we pointed out in Remark 2, a node of type A_5 contains, besides a chaotic attractor which is also a cyclic trapping region, the following two sets: (1) a repelling Cantor set, with which it shares a cycle, and (2) intervals of points that are not nonwandering. Since no backward trajectory can asymptote to points that are not nonwandering, in this case no point has a bitrajectory which is backward dense in the whole node.

4.5. *Backward limits of non-chain-recurrent points*

As pointed out initially by Conley in the continuous finite-dimensional case, and extended later by Norton to the discrete case [Norton, 1995b], all bitrajectories through non-chain-recurrent points have their backward and forward limits belonging to different nodes, so that they determine an edge of the graph. Below we present two examples that turn out to be typical.

Backward limits of level-0, 1 points when N_1 is a repelling cycle. This happens, for instance,

in case of the logistic map $f = \ell_\mu$ for $\mu \in (3, \mu_M)$. We already discussed in Example 5 the structure of the sets U_k.

Points in U_{-1} have no counterimage, so $s\alpha_f(x) = \emptyset$ if $x \in U_{-1}$.

Each $x \in U_0$ has two counterimages but one of them lies in U_{-1}, so there is a single bitrajectory passing through each such x. Since $x = 0$ is backward attracting, this bitrajectory asymptotes backward to 0. Hence, $s\alpha_f(x) = N_0$ if $x \in U_0$.

More generally, each point in $[0, c_1]$ has a backward trajectory asymptoting to 0 since any interval $[0, \varepsilon]$, $\varepsilon > 0$, will eventually cover the whole image of f under iterations, so $s\alpha_f(x) \supset N_0$ if $x \in [0, c_1]$. Indeed, recall that an interval not containing c such that none of its iterates contain c is called a *homterval* (see [De Leo & Yorke, 2021a] for more details on homtervals and on the argument below). As a consequence of a fundamental result of Guckenheimer [1979], the only configuration when a homterval can arise in a S-unimodal map is when the attractor N_p is a cycle and the node N_{p-1} is also a cycle. In that case, the interval between a point $x \in N_p$ and a point $y \in N_{p-1}$ such that no other node point lies in $[x, y]$ is a homterval. No other type of homterval can arise. This means that, in all possible cases, under iterations $[0, \varepsilon]$ will eventually cover c and so, in the following step, will cover all points in the image of the map.

Now consider the period-2 cyclic trapping region of N_1, namely the two intervals

$$J_1(N_1) = [q_1, p_1], \quad J_2(N_1) = [p_1, q_2]$$

where $q_1 = \hat{p}_1$ and $f(q_2) = q_1$. Notice that both J_1 and J_2 are forward invariant under $\overline{f} = f^2$ and let us set $g = f|_{J_2}$. Since $c \notin J_2$, the map $g : J_2 \to J_1$ is a diffeomorphism. For the same reasons above, within J_1 only points in $[\overline{f}(c), p_1] = [c_2, p_1]$ have a backward trajectory asymptoting to p_1. Hence, in J_2, only points in $[g^{-1}(c_2), g^{-1}(p_1)] = [p_1, c_1]$ have a backward trajectory asymptoting to p_1. This means ultimately that the only points with backward trajectories asymptoting to p_1 are the points in the interval $[c_2, c_1]$. Since $U_1 \subset [c_2, c_1]$, we have that $s\alpha_f(x) = N_0 \cup N_1$ if $x \in U_1$.

Backward limits of level-0, 1 points when N_1 is a repelling Cantor set. This happens, for instance, in case of the logistic map $f = \ell_\mu$ for $\mu \in (1 + 2\sqrt{2}, 3.868\ldots)$. We already discussed in Example 6 the structure of the sets U_k.

Just like the case above, points in U_{-1} have no bitrajectories passing through them and all points in $[0, c_1]$ have a backward trajectory asymptoting to 0.

As discussed in Sec. 4.2, the Cantor set N_1 is the set of all chain-recurrent points in $[c_2, c_1]$ that do not lie in $J^{\text{int}}(N_1)$. Each point $x \in [c_2, c_1]$ is either a point of N_1 or lies in a counterimage of $J_1(N_1)$. The case of points of N_1 has been discussed in Sec. 4.2. In the other case, since by definition $N_1 = [c_2, c_1] \backslash \bigcup_{k \geq 0} f^{-k}(J_1(N_1))$, every point in $[c_2, c_1] \backslash N_1$ belongs to a backward trajectory starting in the interior of $J_1(N_1)$ and asymptoting to N_1. In any case, then, $s\alpha_f(x) = N_0 \cup N_1$ if $x \in U_1$.

4.6. *Main theorem*

We are now ready to prove the main result of the article.

Theorem 5 [$s\alpha$-limits of S-unimodal maps]. *Assume that f satisfies (A), namely that f is a S-unimodal map with at least two nodes. If x is a level-k point, $k < p$, then $s\alpha_f(x) = \bigcup_{i=0}^{k} N_i$. If $k = p$ and N_p is not of type A_5, then $s\alpha_f(x) = \bigcup_{i=0}^{p} N_i = \Omega_f$ for all level-p points. If N_p is of type A_5, denote by C and A respectively the repelling Cantor set and the attractor contained in N_p. Then $s\alpha_f(x) = \Omega_f$ if $x \in A$ and $s\alpha_f(x) = \bigcup_{i=0}^{p-1} N_i \cup C = \Omega_f \backslash \text{int}(A)$ otherwise.*

Proof. We start by pointing out that, for $k = 0, \ldots, p-1$, $K(N_k)$ is the set of all points of $\overline{J^{\text{int}}(N_k)}$ that have a backward trajectory asymptoting to N_{k+1}. Indeed, let r be the period of $\mathcal{T}(N_k)$ and set $\overline{f} = f^r$. Then each $J_i(N_k)$ is forward-invariant under $\overline{f}$ and after restricting $\overline{f}$ to each of the $J_i(N_k)$ we reduce to either one of the two case examples in Sec. 4.5.

Since $K(N_k) \subset K(N_{k'})$ for $k' \leq k$, then every point of $K(N_k)$ has also a backward trajectory asymptoting to $N_{k'+1}$ for $k' \leq k$. Since $U_k \subset K(N_{k-1})$, then $s\alpha_f(x) \supset \bigcup_{i=0}^{k} N_i$ for every $x \in U_k$. Moreover, since $U_k \cap J^{\text{int}}(N_{k+1}) = \emptyset$, then no point of U_k can asymptote to nodes with $k' > k$. This shows that $s\alpha_f(x) = \bigcup_{i=0}^{k} N_i$ for every level-k point, $k < p$.

Consider now the case $k = p$. As showed in Sec. 4.4, each point of the attracting node N_p has a backward trajectory dense in N_p for all types of attracting node except A_5, so in all those cases $s\alpha_f(x) = \bigcup_{i=0}^{p} N_i = \Omega_f$ for every level-p point.

When N_p is of type A_5, the node contains points which are not nonwandering and these points cannot be obtained as limit of backward trajectories. Recall that, in this case, the attractor is a cyclic trapping region and C is the set obtained from N_p after removing from it all counterimages of the component of A containing the critical point c. Hence, if $x \in A$ then its $s\alpha$-limit contains both A, because each point in a chaotic attractor has a backward dense orbit, and C, because of the relation mentioned above between A and C, so that $s\alpha_f(x) = \Omega_f$. For the same reasons, if $x \in N_p \backslash A$, then $s\alpha_f(x) = \bigcup_{i=0}^{p-1} N_i \cup C$.   ∎

4.7. *Examples of U_k sets*

Several examples of U_k sets are shown in Figs. 6–8 in the concrete case of the logistic map ℓ_μ.

Figure 6 uses Fig. 4 as background, although we omit the points labels to improve the picture's readability. Over this background, we overlap the curves $c_k = \ell_\mu^k(c)$, $1 \leq k \leq 8$, and we shade points of the sets U_0 in lime and U_1 in red throughout the whole range of μ; we paint the sets U_2 in blue in the range when node N_2 is repelling and is equal to the 2-cycle $\{p_1', p_2'\}$; finally, we paint in olive the sets U_3 in the range when node N_3 is repelling and is equal to the 4-cycle $\{p_1'', \ldots, p_4''\}$ for μ close enough to the bifurcation of the 2-cycle attractor.

Note that we shade points in U_k with the same color as $\mathcal{T}(N_k)$. We leave U_{-1}, the set of points through which passes no bitrajectory, in white. Moreover, we dash in two colors the parts of a trapping region $\mathcal{T}(N_k)$ that are contained in a $U_{k'}$ with $k' < k$. For instance, for each point x of $\mathcal{T}(N_2)$ that is dashed in blue and red, we have $s\alpha(x) = N_0 \cup N_1$, while for each point y of $\mathcal{T}(N_2)$ that is painted in full blue and is not on the attractor we have that $s\alpha(x) = N_0 \cup N_1 \cup N_2$.

Close to the left boundary of the picture, the logistic map has three nodes: N_0 is the boundary fixed point, N_1 the internal fixed point and N_2 a 2-cycle attractor. For all $\mu \in (3.4, 3.6)$, all points of $[0, c_1]$ have a bitrajectory asymptoting backward to N_0 and all points in $K(N_0) = [c_2, c_1]$ have a bitrajectory asymptoting backward to N_1. The only points with bitrajectories asymptoting backward to the attractor N_2 are the points of the attractor itself.

At $\mu_0 = 1 + \sqrt{6} \in (\mu_{40}, \mu_{41})$, the 2-cycle undergoes a bifurcation, becomes repelling and a new

attracting 4-cycle N_3 arises. As soon as N_2 becomes repelling, all points in $K(N_1) = [c_2, c_4] \cup [c_3, c_1]$ (and no other point) have a bitrajectory asymptoting backward to it. This type of behavior repeats at each bifurcation. For instance, at $\mu = \mu_{42}$ we have an attracting 8-cycle N_4 and all points in $K(N_2) = [c_2, c_6] \cup [c_8, c_4] \cup [c_4, c_7] \cup [c_5, c_1]$ (and no other point) have a bitrajectory asymptoting backward to N_3.

Figure 7 uses Fig. 5 as background, although we omit the points labels to improve the picture's readability. Over this background, we overlap the curves $c_k = \ell_\mu^k(c)$, $1 \le k \le 6$, and we shade points of the sets U_0 in blue and of the sets U_1 in orange throughout the period-3 window. Finally, we paint in teal the points of the sets U_2 in two smaller ranges of parameters, one about the parameter value μ_{51} and one about μ_{53}. Note that we shade points in U_k with the same color as $\mathcal{T}(N_k)$. We leave U_{-1} in white.

As in case of Fig. 6, we dash in two colors the parts of a trapping region $\mathcal{T}(N_k)$ that are contained in a $U_{k'}$ with $k' < k$. Moreover, we set to transparent the interior of the line showing $\mathcal{T}(N_k)$ when its points belong to $U_{k'}$ with $k' > k$. For instance, in Fig. 8 at $\mu = \mu_{50}$, for all points $x \in [c_2, c_1] \subset J_1(N_0) = [0, 1]$ we have that $s\alpha_{\ell_\mu}(x) = N_0 \cup N_1$, where N_1 is the red Cantor set, so we paint $J_1(N_0)$ as a hollow blue line within that range.

Close to the left endpoint of the period-3 window, the logistic map has three nodes: N_0 is the boundary fixed point, N_1 the red Cantor set and N_2 is the attracting 3-cycle that arose together with the repelling 3-cycle at the boundary of the $J_i(N_1)$ (recall that the right endpoint of the period-3 window is the parameter value at which this repelling 3-cycle falls into a chaotic attractor). As above, all points in $[0, c_1]$ have a bitrajectory asymptoting backward to N_0 and all points in $K(N_0) = [c_2, c_1]$ have a bitrajectory asymptoting backward to N_1.

When the attracting 3-cycle bifurcates, we have the same kind of behavior just illustrated above for Fig. 6. For instance, after the first bifurcation (e.g. see $\mu = \mu_{51}$) all points in $K(N_1) = [c_2, c_5] \cup [c_3, c_6] \cup [c_4, c_1]$ have a bitrajectory asymptoting backward to the repelling 3-cycle N_2.

Case μ_{52} is completely analogous to case μ_{50}, the main difference being that at μ_{52} the attractor is not a cycle but the trapping region

$$N_2 = \{[c_2, c_5], [c_3, c_6], [c_4, c_1]\}.$$

When $\mu = \mu_{53}$, N_2 is the blue Cantor set and the attractor N_3 is a 9-cycle. Similarly to the case $\mu = \mu_{51}$, only points of $K(N_1)$ have a bitrajectory asymptoting backward to the blue Cantor set, while every point in $K(N_0)$ has a bitrajectory asymptoting backward to the red Cantor set. More details are visible in Fig. 8, where we show a detail of the central cascade.

4.8. *Two corollaries*

Theorem 5 can be used to prove the Conjecture by Kolyada, Misiurewicz and Snoha in the case of S-unimodal maps as follows.

Corollary 2. *Let $f : [a, b] \to [a, b]$ be a S-unimodal map. Then $s\alpha_f(x)$ is closed for any $x \in [a, b]$.*

Proof. The only nontrivial case is when the attractor is a Cantor set and x belongs to it. In this case, $p = \infty$ and $N_\infty = \bigcap_{i<\infty} \overline{J^{\text{int}}(N_i)}$.

To see this, we first show that $\bigcap_{i<\infty} \overline{J^{\text{int}}(N_i)}$ is a node. Indeed, let x and y be any two points in it. Given any $\varepsilon > 0$, there is some n_ε such that there is some component of U_k in both $(x - \varepsilon, x + \varepsilon)$ and $(y - \varepsilon, y + \varepsilon)$ for every $k > n_\varepsilon$. Hence, there is a point p' in the orbit of the periodic point $p_1(N_k)$ in the first neighborhood and one p'' in the second, so that the sequence $\{x, p', f(p'), \ldots, f^s(p') = p'', y\}$ is an ε-chain from x to y. For the same reason, $\{y, p'', f(p''), \ldots, f^{s'}(p'') = p', x\}$ is an ε-chain from x to y. Moreover, x and y are both chain-recurrent due to the same argument. Hence x and y belong to the same node, which can only be N_∞ since there is no other node. Since N_∞ is a subset of each $\overline{J^{\text{int}}(N_i)}$, these are all points of N_∞.

We showed in the previous theorem that, in this case, $s\alpha_f(x) = \bigcup_{i=0}^\infty N_i$, so it is enough to show that the limit points of the set of nodes coincides with the attractor. Given any converging sequence $x_k \in N_k$, $k = 1, 2, \ldots$, its limit point cannot be in any set U_i, with i finite, since, from some n on, every point of the sequence will be out of it. Hence it must belong to N_∞.   ∎

The corollary below is a "$s\alpha$" version of Theorem D:

Corollary 3. *Let $f : [a, b] \to [a, b]$ be a S-unimodal map. Then $s\alpha_f(c) = J_f \cap \Omega_f$ if and only if N_p is not a superattracting cycle.*

Proof. Notice, first of all, that all repelling nodes lie in J_f and that the attractor lies inside J_f if and only if N_p is of type A_2, A_3 or A_5 and that, in each of these three cases, $c \in N_p$. Hence, in this case, $s\alpha_f(c) = \bigcup_{k=0}^{p} N_k = J_f \cap \Omega_f$. Then notice that either $c \in N_p$ or $c \in U_{p-1}$. Indeed, when N_p is of type A_3, we have that $c \in N_p$ and, when N_p is not of type A_3, we have that $c \in J^{\text{int}}(N_{p-1}) \subset K(N_{p-2})$ and $U_{p-1} = K(N_{p-2}) \backslash N_p$. Hence, when N_p is of type A_1 or A_4 and the attracting cycle does not contain c, we must have that $c \in U_{p-1}$ and therefore $s\alpha_f(c) = \bigcup_{k=0}^{p-1} N_k = J_f \cap \Omega_f$. Finally, when N_p is superattracting, $s\alpha_f(c)$ contains the attractor but the attractor does not belong to J_f, so that $s\alpha_f(c)$ is strictly larger than $J_f \cap \Omega_f$. ∎

Acknowledgments

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