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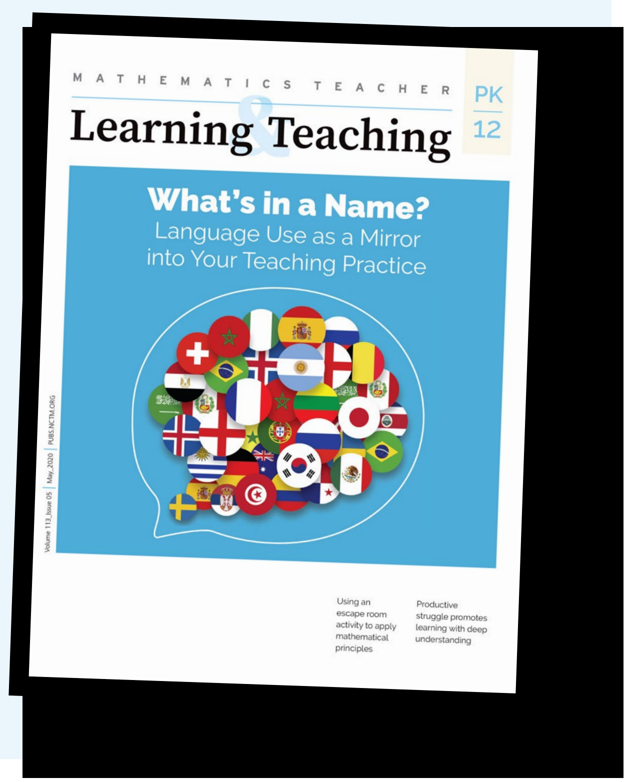


Promoting Generalizing in Algebra Class

Teachers can use a pattern task to promote and foster generalizing in the mathematics classroom, presenting opportunities to build on students' thinking and extending ideas to new contexts.

Allyson Hallman-Thrasher, Susanne Strachota, and Jennifer Thompson

Inherent in the Common Core's Standard for Mathematical Practice to "look for and express regularity in repeated reasoning" (SMP 8) is the idea that students engage in this practice by generalizing (NGA Center and CCSSO 2010, p. 8). In mathematics, generalizing involves "lifting" and communicating about ideas at a level where the focus is no longer on a particular instance, but rather on patterns and relationships between particular instances (Kaput 1999, p. 137). For example, a student generalizes when she graphs $y = 3x$, $y = 4x + 2$ and $y = (4/3)x - 3$ and then notices that equations of the form $y = mx + b$ are always a line. Research has shown that, at times, secondary school mathematics teachers find it challenging to respond to students' generalizations in mathematically productive ways (Demonty, Vlassis, and Annick Fagnant 2018). Knowing when students are generalizing and how to respond is important because generalizations are mathematically sophisticated ideas, and they present opportunities to build on students' thinking, extend ideas to new contexts, and move the mathematics forward.

IDENTIFYING GENERALIZING

When students generalize, they often use what we call general language, such as *always, every time, the pattern is, the rule is, any number, for all numbers,* and so on. Examples of generalizations that students may say using those words or phrases include the following:

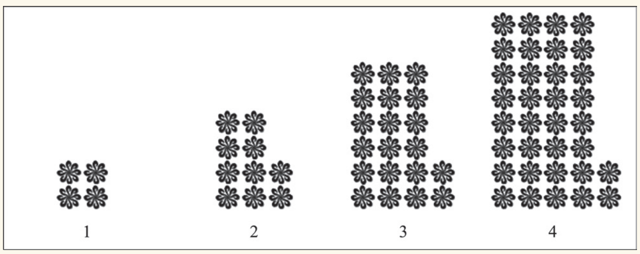
- "The sum of two even numbers is *always* an even number."
- "*Every time* we add a table, we can seat three more students."
- "*The pattern is* add three."
- "*The rule is* multiply the number of students by three."
- "When I add zero to *any number*, I get that number."
- "$a + b = b + a$ for all numbers."

Students' generalizations vary in sophistication. Imagine students are making observations about the pattern 2, 4, 6, 8, 10, . . . One student may say, "The pattern is add two," whereas another student may say, "$y = 2x + 2$." Both of these observations are generalizations because the student looked across the instances of the pattern and noticed something that was true across all the numbers, something general, and something about how these instances are related. Although "$y = 2x + 2$" is a more formal expected response in an algebra classroom, "the pattern is add two" is a generalization equally worthwhile to explore as a stepping stone to an explicit generalization. In trying to identify generalizations, teachers may consider the following questions:

- What are students saying that uses the "general" vocabulary and phrases noted above?
- What observations apply to (or could apply to) multiple instances of a phenomenon?
- What observations did students make that extend beyond the instances shown?
- What connections did students identify across multiple instances?
- Did students identify a pattern, rule, or relationship? What evidence did they provide to justify their pattern?

To frame our work on generalizing for preservice teachers (PSTs), we used a pattern task that we believed

Fig. 1



These are the first four instances of the pattern task that was used in Ms. Patton's 10th-grade algebra class. Adapted from Nguyen 2020 (visualpatterns.org)

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would present an accessible entrance into this work for novices. A typical pattern task (see figure 1 for an example) visually represents a series of objects in which the number of objects changes on the basis of a rule. The goal of the pattern task is to use the structure of the presented images (e.g., objects organized in arrays) to develop multiple ways of representing the rule. The change in the pattern across steps often demonstrates growth and can become the foundation for identifying a rule that can be generalized.

TOOLS TO SUPPORT THINKING ABOUT PATTERN TASKS AND GENERALIZING

To support our thinking about generalizing and the ways in which teachers (preservice and in-service) can promote generalizing, we adopted a framework that specifically addresses promoting generalizing. Tables 1 (Strachota 2020) and 2 (Ellis 2011) present definitions of the different types of priming and generalizing-promoting activities.

We applied this framework to design an activity for our PSTs that supported them in developing, planning, teaching, and reflecting on a pattern task to teach to their middle and high school students. We observed and video recorded three PSTs teach pattern tasks to grades 7–12 students. To support our PSTs in planning for their pattern task and to foster their vital reflection on their own teaching and our analysis of their teaching, we used several additional resources: the aforementioned framework for supporting students' generalizing (Ellis 2011; Strachota 2020), a framework for examining math-talk (Hallman-Thrasher 2017; Hufferd-Ackles, Fuson, and Sherin 2004), and a guide for recognizing question types (Boaler and Brodie 2004).

Our analysis was completed by coding the video data in 15-second segments using our three frameworks (Ellis 2011; Strachota 2020; Hallman-Thrasher 2017; Boaler and Brodie 2004; Hufferd-Ackles, Fuson, and Sherin 2004). The framework for promoting generalizing describes two important types of teacher actions: (1) priming actions (Strachota 2020) and (2) generalizing-promoting actions (Ellis 2011). Priming actions prepare students to build on an idea or refer to an idea later on, whereas generalizing-promoting actions prompt immediate activity. These definitions (see tables 1 and 2) illustrate the codes we used to capture activity related to generalizing (Ellis 2011; Strachota 2020). We also noted which of the nine types of questions were asked (Boaler and Brodie 2004). Table 3 describes the question types most prevalent in our study.

Table 1 Priming Activities

Activity	Example
Naming a phenomenon, clarifying vital terms, reviewing vital tools: Offering a common way to reference a phenomenon or emphasizing the meaning of a vital term or tool.	Referencing terms of the pattern as "the first term, the second term," and so on clarifies vital terms. Naming a phenomenon, such as particular strategies used by students (e.g., "breaking into squares"), gives everyone a common reference point for discussion. Encouraging students to make observations "without counting" introduces a vital tool for supporting students in noticing patterns.
Constructing or encouraging constructing searchable and relatable situations: Creating or identifying situations or objects that can be used for searching or relating. Situations that can be used for searching or relating involve particular instances or objects that students can identify as similar in some way. By presenting students with a pattern task, teachers construct searchable and relatable situations.	Recording an expression to represent each pattern term constructs a searchable and relatable situation because students can look across those expressions, search for similarities and differences, and in turn relate the terms.
Constructing extendable situations: Identifying situations or objects that can be used for extending. Extending involves applying a phenomenon to a larger range of cases than that from which it originated. By presenting students with a pattern task, teachers construct extendable situations because patterns are extendable.	Simply drawing an ellipsis or prompting students to consider their observations beyond the case at hand encourages students to extend a relationship.

To capture the quality of student-teacher interactions, we used a modified version (Hallman-Thrasher 2017) of the math-talk rubric (Hufferd-Ackles, Fuson, and Sherin 2004), which describes four levels (0–3) that range from a teacher-led to a student-centered classroom. Here, we focused specifically on the questioning and explaining mathematical thinking portions of that rubric to capture the specific back and forth of teacher–student interactions. The modified version also uses midlevels to capture the smaller scale changes needed in studying novice teachers (Hallman-Thrasher 2017).

The use of the three tools together gave a robust picture of each teaching episode. This fine-grained analysis, while not possible for classroom teachers to engage in on a regular basis, was important for our work to identify teacher pedagogical moves and classroom activity that supported students' generalizing. This careful analysis of PSTs' practice provides insight into productive strategies that support students in developing generalizations and can serve as a model for practicing teachers as they work to develop new strategies in their own classrooms.

A CLASSROOM EXAMPLE: MS. PATTON ENCOURAGES GENERALIZING

Here, we share data from one of our three PSTs, Ms. Patton, who was completing a year-long student teaching internship. We include excerpts of Ms. Patton's teaching and examples of student thinking collected from one lesson taught to two sections of her algebra 2 class, consisting of primarily 10th-grade students. This example illustrates teacher moves that supported students in articulating generalizations.

Table 2 Generalizing-Promoting Activities

Activity	Example
"*Encouraging relating and searching*[1]: Prompting the formation of an association between two or more entities; prompting the search for a pattern or stable relationship."	Asking students if they see and can articulate a relationship between an expression and an instance of the pattern.
"*Encouraging extending*: Prompting the expansion beyond the case at hand."	Once a relationship is identified, prompting students to apply it to a set of numbers (i.e., all real numbers or all natural numbers).
"*Encouraging reflection*: Prompting the creation of a verbal or algebraic description of a pattern, rule, or phenomenon."	Suggesting that a relationship is (or can be) represented using an equation (e.g., algebraically).
"*Encouraging justification*: Encouraging a student to reflect more deeply on a generalization or an idea by requesting an explanation or a justification. This may include asking students to clarify a generalization, describe its origins, or explain why it makes sense."	Questioning if a relationship holds true across cases in a way that prompts students to show or explain why the relationship applies to other cases.

[1]These categories and descriptions are from Ellis (2011, p. 316). However, we adapted encouraging relating and searching, following Strachota (2020), by combining these into one category because at times we were unable to differentiate relating and searching.

Table 3 Most Prevalent Question Types Observed

Question Types	
Exploring meanings or relationship	Seeks connections across representations or to concepts underlying ideas
Probing	Prompts for further clarification and justification
Generating discussion	Elicits ideas from students
Extending thinking	Extends beyond the cases at hand
Focusing	Directs student attention to important elements of the problem

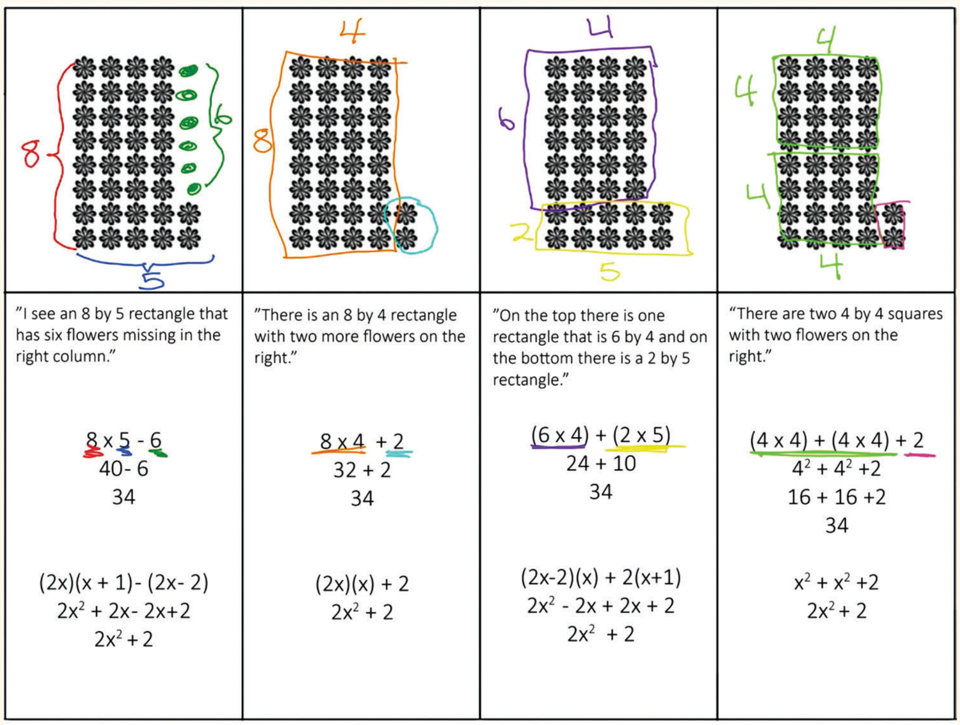
Students shared multiple methods of finding the number of flowers in step 4 of the pattern without counting. Figure 2 illustrates all the representations that Ms. Patton created on the basis of student contributions in different colors to show how students saw groupings of flowers. After multiple strategies were noted on the board, students were prompted to create numeric expressions for each visual representations. The final step was to construct a generalized expression for each.

To prepare students to represent a pattern algebraically, Ms. Patton first introduced one instance of the pattern (see step 4 in figure 1) and asked students to develop a numeric expression for the number of flowers "without counting." In doing so, she encouraged students to notice the structure and general properties of the pattern, which, in turn, shifted their focus away from numerical calculations. In response to Ms. Patton's prompts, one student developed the expression "8 × 4 + 2." When Ms. Patton explained why the student's equation "8 × 4 + 2" represented one instance of the pattern, she *encouraged relating* two instances—the equation and the pattern image (see figure 2 for all representations the class generated).

She then asked the class to consider the 100th step, which is an example of *encouraging extending*. In response, a student generalized when he asked, "Would it be, uh, two times the step, times that step, plus two? . . . Like for three, it would be two times three, times three, plus two." Following this student's contribution, Ms. Patton clarified that the student's idea of step was referring to the changing step number. To formalize the student's idea, she introduced "variable" as a tool for representing the unknown quantity in an equation, which is an example of *reviewing a vital tool*. In this situation, variables are a tool for representing a varying quantity.

Ms. Patton then said, "If you want to generalize the step number, let's use a variable. What variable do you want to use?" A student suggested *x*, so Ms. Patton replaced "the step" with *x* in the previous equation. She repeated the student's equation and asked, "Does that work for our first four patterns?" By asking this question, Ms. Patton encouraged students to *relate* particular instances and, in turn, *justify* the equation at hand ("two times *x*, times *x*, plus two"). Another student likely generalized when she shared that she had something similar, "two times the step number squared plus two."

Fig. 2



These are examples of students' representations of the fourth instance of the pattern.

Ms. Patton wrote her expression on the board, and together they realized that once the equations were simplified, they were equivalent. Again, Ms. Patton explicitly *related* the equations shared thus far with the new equation. She repeated the student's solution, "So, you said two times the step number squared plus two?" and asked, "Is that the same as that one (pointing to $2 \times 4 \times 4 + 2$ from the $2n^2 + 2$)?" Once students reached an agreement, the teacher confirmed, "Right, because we got two x, times x plus two, but it is the same as two x squared, plus two."

REFLECTION: HOW TEACHERS CAN PROMOTE GENERALIZATIONS

Although each of the three teachers we observed engaged in different types of priming activity and generalizing-promoting activity, in some teaching episodes, more instances of students generalizing occurred. Not surprisingly, the more frequently teachers engaged in both priming and generalizing-promoting activities, the more often students generalized. Additionally, we have rich examples of teachers engaging in generative cycles of priming activity, generalizing promoting activity, followed by another priming activity *before* a student shared a generalization. Rarely did priming activity alone lead to generalizing. We also noted instances of generalizing activity that could have been followed up by further priming activity, but these instances were typically not capitalized on by our novice teachers. Next, we share teaching moves that practicing teachers can use to effectively promote generalizing in their own classrooms.

PRIMING FIRST MATTERS

The priming activity that *purposefully* set the stage for students engaging in generalizing-promoting activity was more productive. For example, to prepare students to represent the pattern algebraically, Ms. Patton first used a priming action to introduce one instance of the pattern (see step 4 in figure 1) by asking how many flowers were in the pattern "without counting." She intentionally built on this priming by using a student representation of an 8×4 rectangle with two more flowers to engage in the generalizing-promoting activity of encouraging students to relate the expression that came from that priming activity ($8 \times 4 + 2$). The priming activity and the generalizing-promoting activity together justified the development of the expression. The act of justifying using the picture, in turn, supported the students' development of the generalization $2n^2 + 2$. This example illustrates the generative nature of priming, generalizing-promoting, and generalizing activities.

CONNECTIONS AMONG REPRESENTATIONS

Visual representations can support students in relating objects by establishing connections between how students see defining characteristics and how these characteristics translate to numeric and generalized expressions. When teachers encouraged students to relate instances of a pattern, it usually involved different representations of that pattern. For example, Ms. Patton showed her students a pictorial representation of a pattern. She then asked them to write a mathematical expression (using numbers) to represent that pattern. She recorded multiple student-generated expressions and encouraged students to verbalize (using words) their observations about the pattern too. In doing so, she attended to student thinking, elicited thinking, and pressed for clarification or ideas that supported students in justifying their claims (e.g., level 1.5 in the revised math-talk rubric). Additionally, by making student thinking public (e.g., displayed on the board) and orienting the class to one another's comments, she used students' thinking to promote student-to-student discourse (e.g., level 2) (Hallman-Thrasher 2017; Hufferd-Ackles, Fuson, and Sherin 2004).

The use of a variety of representations, especially the pictorial, numeric, and generalized representations in Ms. Patton's class, was a powerful way to support students' generalizing. By making available to students multiple representations of the pattern and multiple instances of each representation, Ms. Patton constructed relatable situations (priming activity). The different ways that Ms. Patton represented the pattern and the number of ways she represented the pattern supported setting up students for generalizing later in the lesson.

QUESTIONING TO SUPPORT EXPLAINING

Teacher questioning was central to supporting students in noticing and carefully articulating generalizations. Ms. Patton and other teachers who promoted student generalizations repeatedly responded to student ideas with questions, which is one way the teacher shifts the source of ideas from herself to the students (Hallman-Thrasher 2017; Hufferd-Ackles, Fuson, and Sherin 2004). When students suggested a numeric expression that involved subtracting the missing parts of the flower column on the right, Ms. Patton zeroed in on this student thinking to encourage them to generalize the idea by first clarifying: "What about that minus six? That one's a little tricky." Then, she revoiced the explanation to call the class's attention to the idea and posed further questions to prompt the students to generalize the six to other cases. Finally, she asked them to evaluate their generalizations: "Will that work . . . if you look at the third pattern?" This type of question supported students in extending their rule to other instances of the pattern, which is representative of extending the thinking question type (Boaler and Brodie 2004). Ms. Patton repeated the sequence of actions multiple times until the students developed a correct way to generalize the missing flowers in the final column ($2n - 2$). Ms. Patton's tendency to question in a series and vary the purpose of her questions across that series was more productive for promoting generalizations than posing only one-shot questions.

GUIDANCE FOR IMPLEMENTATION

From our observations in Ms. Patton's classroom and those of other PSTs, we isolated four practices that any classroom teacher can use to promote students articulating generalizations. First, teachers who used precise language influenced students' engagement with the priming and generalizing-promoting activity. This observation is not surprising given an extensive body of research around the importance of precise mathematical language (see Hill et al. 2008 for one example). Other PSTs' use of imprecise language seemed to hinder students' engagement with priming and generalizing-promoting activity. For example, another PST used the terms *equation* and *expression* interchangeably. This was confusing when she engaged in the generalizing-promoting activity of encouraging reflection because it was unclear whether she wanted students to write expressions or equations to represent an instance of the pattern. Another PST named different-colored parts of her pattern but inconsistently used their names and missed a chance to clarify a vital term. We learned the importance of precise language, specifically with terms that are relevant to generalizing, such as *expression*, *equation*, and *variable*.

Second, and related to precise language, a simple setup of the task was essential. Ms. Patton's entire setup was to introduce the picture of only one instance of the pattern and ask students to determine the number of flowers without counting one by one. Later, she was able to build on this straightforward task in complex ways by comparing students' responses and then extending those responses to showing more steps of the pattern. Other PSTs jumped immediately to introducing multiple steps of the pattern and asking students to generalize a priori any priming actions. In these examples, students were either unable to make the leap to generalizing or produced a rote equation without making connections among representations or among equivalent expressions. The pattern selected by one PST used different colors across steps that were intended to prime students to visualize the pattern in a variety of ways. However, without preparing students to use this vital tool, the intended support complicated the task for students.

Third, students were more productive when PSTs intentionally structured small-group explorations. Some PSTs allocated too much time for unstructured exploration or gave vague directions such as "work through these questions." Group work was more productive when it was shorter with a specific goal, such as "Find another expression different from this," and when the group work was interspersed with whole-class discussion. Additionally, when teachers checked in with groups or individuals, the interaction was most productive when it was specific to the work the group had produced. For example, targeted feedback, such as "Look at the three different methods and see how it compares to yours," meaningfully engaged a small group in a way that quickly checking in, such as "How are you doing on this?," did not. We learned the value of outlining clear expectations before individual or group work time, checking for student understanding of instructions before group work time began, and asking questions throughout that time on how students are meeting those expectations.

Fourth, when PSTs included multiple examples and complete representations in class discussion, students were better able to make connections. By representations of the pattern, we are referring to pictorial and numerical representations, expressions, equations, and verbal descriptions that depict that pattern or part of the pattern. Figure 2 shows four examples of different ways to see how many flowers are in the fourth instance of the pattern, and each of these four ways is represented with a picture, verbal description, numeric expression, and an algebraic expression. In some cases, PSTs provided only a few examples of representing the pattern (e.g., found only one expression for the pattern) or focused more on one type of representation, such as the numeric equation without pictorial or verbal representations. This was problematic because those examples are the material for engaging in relating. When PSTs engaged in the generalizing-promoting activity of encouraging relating, they linked or supported students in linking instances of the pattern. If limited instances of the pattern were represented, then fewer opportunities exist to relate, which is one way students identify relationships and, in turn, generalize. We learned the importance of teachers eliciting many examples of representations of instances of the pattern and recording these examples in an organized way that is easy to reference throughout the lesson to support students in relating and searching for the same relationship.

CONCLUSION

We have learned that it takes repeated intentional teacher interventions to effectively facilitate students' generalizing. We noticed that small changes in questioning and representing students' contributions can generate big changes in students' generalizing. Though our work focused on PSTs, the findings of our research apply to all mathematics teachers who aim to support their students in developing generalizations. When we first began working with PSTs on pattern tasks, we thought it was a straightforward lesson that would be easy for a novice to execute effectively. As we dug deeper into this topic, we learned that supporting students in generalizing is complex and messy work. We found ourselves discussing and unpacking this topic in an effort to understand the hard work of generalizing that we expect students to do. The frameworks used helped us to unpack this complex practice for our novice teachers, and we think they will support practicing teachers as well. In recognizing the complexities of teaching, we understand that our observations were of a small slice of a much bigger picture. Acknowledging that the work of generalizing is challenging for both teachers and learners, we hope that what we have shared will support all teachers as they test out these teaching strategies in their own classrooms.

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