Discrete Mathematics 345 (2022) 112836

Contents lists available at ScienceDirect

Discrete Mathematics

www.elsevier.com/locate/disc

Reconstructibility of a general DNA evolution model

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ARTICLE INFO

Article history:
Received 12 September 2021
Received in revised form 23 January 2022
Accepted 2 February 2022
Available online 9 February 2022

Keywords:
Classification
Markov random fields
Nonlinear dynamical system

ABSTRACT

In this paper, we analyze the tree reconstruction problem, to identify whether there is non-vanishing information of the root, as the level of the tree goes to infinity. Although it has been studied in numerous contexts, the existing literature with rigorous reconstruction thresholds established are very limited, and it becomes extremely challenging when the model under investigation has 4 states, one of whose interpretations is the four main bases found in Deoxyribonucleic acid (DNA) and Ribonucleic acid (RNA): guanine [G], cytosine [C], adenine [A], and thymine [T]. In this paper, we study a general DNA evolution model, which distinguishes between transitions and transversions, and allow transversions to occur at the same rate but that rate can be different from the rates for transitions. The sufficient condition for reconstruction is rigorously established.

1. Introduction

We firstly give the background and motivation in Section 1.1 and then state our contributions in Section 1.2.

1.1. Background and motivation

The tree reconstruction problem, as an interdisciplinary subject, has been studied in numerous contexts including statistical physics, information theory, and computational biology. The reconstructability plays a crucial role in phylogenetic reconstruction in evolutionary biology (see Mossel [25], Daskalakis et al. [7]), communication theory in the study of noisy computation (see Evans et al. [10]), analogous investigations in the realm of network tomography (see Bhamidi et al. [4]), reconstructability and distinguishability in the clustering problem of the stochastic block model (see Mossel et al. [28,29], Neeman and Netrapalli [30]), trace reconstruction problem (see Andoni et al. [1]), investigations of the random field models on sparse random graphs with replica symmetry breaking (see Lupo et al. [21]), etc. The reconstruction threshold, corresponds to the threshold for extremality of the infinite-volume Gibbs measure with free boundary conditions (see Georgii [13]), and is known to have a crucial determination effect on the efficiency of the Glauber dynamics on trees and random graphs (see Berger et al. [2], Martinelli et al. [22], Tetali et al. [34]).

The tree reconstruction model has two building blocks, with one being an irreducible aperiodic Markov chain on a finite characters set $\mathcal{C}$ and the other one being a rooted d-ary tree (every vertex having exactly d offspring). The tree is denoted as $\mathbb{T} = (\mathbb{V}, \mathbb{E}, \rho)$, where $\mathbb{V}$ stands for vertices, $\mathbb{E}$ stands for edges, and $\rho \in \mathbb{V}$ stands for the root. Denote σ_v as the state assigned to vertex v, and denote σ_ρ specially for the state of the root ρ that is chosen according to an initial distribution

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https://doi.org/10.1016/j.disc.2022.112836

π on $\mathcal{C}$. The root signal propagates in the tree according to a transition matrix $\mathbf{P}$ which is also called noisy channel, in a way that for each vertex v having u as its parent, the spin/configuration at v is assigned according to the probability $P_{ij} = \mathbf{P}(\sigma_v = j \mid \sigma_u = i)$ for $i, j \in \mathcal{C}$.

The reconstruction problem on an infinite tree is to analyze that given the configurations realized at the nth generation of the tree which is denoted as $\sigma(n)$, whether there exists non-vanishing information on the letter transmitted by the root, as n goes to infinity. Define $\sigma^i(n)$ as $\sigma(n)$ conditioned on $\sigma_\rho = i$. We are ready to give the mathematical definitions regarding reconstruction.

Definition 1.1. We say that a model is reconstructible on an infinite tree $\mathbb{T}$, if for some $i, j \in \mathcal{C}$

$$\limsup_{n\to\infty} d_{TV}(\sigma^i(n), \sigma^j(n)) > 0,$$

where d_{TV} is the total variation distance, i.e.

$$d_{TV}(\sigma^i(n), \sigma^j(n)) = \sup_A \left|\mathbf{P}(\sigma(n) = A \mid \sigma_\rho = i) - \mathbf{P}(\sigma(n) = A \mid \sigma_\rho = j)\right|.$$

When the lim sup is 0, we say that the model is non-reconstructible on $\mathbb{T}$.

The binary model with 2 states corresponds to the Ising model in statistical physics (see Giuliani and Mastropietro [14], Giuliani and Seiringer [15] and the references therein for more information), whose transition matrix is given by

$$\mathbf{P} = \frac{1}{2}\begin{pmatrix} 1+\theta & 1-\theta \\ 1-\theta & 1+\theta \end{pmatrix} + \frac{\Delta}{2}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad |\theta| + |\Delta| \leqslant 1,$$

where Δ is used to describe the deviation from the symmetric channel, i.e. when $\Delta \neq 0$ the channel is asymmetric. Whether the model is reconstructible is closely related to, the second largest eigenvalue by absolute value of the transition matrix $\mathbf{P}$, denoted as λ. For the binary symmetric channel, Bleher et al. [5] showed that the model is reconstructible if and only if $d\lambda^2 > 1$ (see also Evans et al. [10]), which is known as the Kesten-Stigum bound. For the binary asymmetric channel with sufficiently large asymmetry, Mossel [24,26] showed that the Kesten-Stigum bound is not the bound for reconstruction. When the asymmetry is sufficiently small, Borgs et al. [6] established the first tightness result of the Kesten-Stigum reconstruction bound in roughly a decade, and later Liu and Ning [19] gave a complete answer to the question on how small the asymmetry is necessary for the tightness of the reconstruction threshold.

For non-binary models, the simplest case is the q-state symmetric channel which corresponds to the Potts model in statistical physics (see Derrida et al. [8], Dhar [9] and the references therein for more information), with the following transition matrix

$$\mathbf{P} = \begin{pmatrix} p_0 & p_1 & \cdots & p_1 \\ p_1 & p_0 & \cdots & p_1 \\ \vdots & \vdots & \ddots & \vdots \\ p_1 & p_1 & \cdots & p_0 \end{pmatrix}_{q\times q}.$$

Sly [32] established the Kesten-Stigum bound for the 3-state Potts model on regular trees of large degree and showed that the Kesten-Stigum bound is not tight when $q \geqslant 5$. Liu et al. [20] proposed the following model to distinguish between transitions and transversions, whose transition matrix has two mutation classes with q states in each class

$$P_{ij} = \begin{cases} p_0 & \text{if } i = j, \\ p_1 & \text{if } i \neq j \text{ and } i, j \text{ are in the same category}, \\ p_2 & \text{if } i \neq j \text{ and } i, j \text{ are in different categories}. \end{cases}$$

When the number of states are more than or equal to 8, Liu et al. [20] showed that the Kesten-Stigum bound is not tight.

The 4-state cases give very important reconstruction on the tree models, especially for the applications in phylogenetic reconstruction since they correspond to some of the most basic phylogenetic evolutionary models (see the discussions in Section 2.5.1 of Mossel [27]). However, the 4-state case is much more challenging. The reason can be seen from equation (1) on page 1371 of Sly [32] where a key role is played by the sign of $q-4$; when $q \geqslant 5$, it is positive and this allows us to show that if $d\lambda^2$ is sufficiently close to 1 then the model is reconstructible beyond the Kesten–Stigum bound. Visualization of the challenge in handling 4-state cases can be seen from Figure 3 on page 042109 − 13 of Ricci-Tersenghi et al. [31]. Its reconstruction problem was open until very few new results established recently. For the symmetric model with 4 states, Ricci-Tersenghi et al. [31] showed that in the assortative (ferromagnetic) case the Kesten-Stigum bound is always tight, while in the disassortative (antiferromagnetic) case the Kesten-Stigum bound is tight in a large degree regime and not tight in a low degree regime. Later, Liu and Ning [18] investigated a 4-state asymmetric model based on the F81 model (Felsenstein [11]) and gave specific conditions under which the Kesten-Stigum bound is not tight.

DNA is a molecule composed of two chains coiling around each other to form a double helix, and these two DNA strands are composed of simpler monomeric units called nucleotides, each of which is further composed of one of four nitrogen-containing nucleobases (guanine [G], cytosine [C], adenine [A] or thymine [T]). Molecular phylogenetics is a branch of phylogeny to specially analyze the genetic or hereditary molecular differences in order to gain information on an organism's evolutionary relationships. Markov models of DNA sequence evolution were proposed and widely used in phylogenetic reconstruction. We refer interested readers to the classical book Felsenstein [12] for more information.

1.2. Our contributions

In this paper, we consider a general DNA evolution model, which follows the K80 model (Kimura [17]) and the TN93 model (Tamura and Nei [33]) in distinguishing between transitions (A↔ G, i.e. from purine to purine, or C ↔ T, i.e. from pyrimidine to pyrimidine) and transversions (from purine to pyrimidine or vice versa), and follows the TN93 model in designing that transversions occur at the same rate but that rate is allowed to be different from both of the rates for transitions. Specifically, in this paper, we focus on a 4-state model with the transition matrix of $\{A, G, T, C\}$, or the configuration set $\{1, 2, 3, 4\}$, of the form

$$\mathbf{P}=\left(\begin{array}{cc|cc} p_0 & p_1 & p_2 & p_2 \\ p_1 & p_0 & p_2 & p_2 \\ \hline p_2 & p_2 & \overline{p}_0 & \overline{p}_1 \\ p_2 & p_2 & \overline{p}_1 & \overline{p}_0 \end{array}\right). \tag{1}$$

Besides different out-block transition probabilities p_2, the model under investigation has different in-block transition probabilities: p_0 and p_1 in one block, $\overline{p}_0$ and $\overline{p}_1$ in the other block. It is easy to see that $\mathbf{P}$ has 4 eigenvalues: 1, $\lambda_1 = p_0 - p_1$, $\lambda_2 = p_0 + p_1 - 2p_2$, and $\lambda_3 = \overline{p}_0 - \overline{p}_1$. Let λ be the second largest eigenvalue by absolute value. Kesten and Stigum [16] showed that any model is reconstructible when $d\lambda^2 > 1$, so we only investigate $d\lambda^2 \leqslant 1$ in the following context. Since λ_1 and λ_3 play symmetric roles in this symmetric model (1), without loss of generality, we presume $|\lambda_1| > |\lambda_3|$ in the sequel.

In Section 2, we give detailed definitions and interpretations, conduct preliminary analyses, and then provide an equivalent condition for non-reconstruction:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \overline{x}_n = 0.$$

Here, x_n and $\overline{x}_n$ represent the probabilities of giving a correct guess of the root given the spins $\sigma(n)$ at distance n from the root minus the probability of guessing the root randomly which is $1/4$ in this case, for the root being in block 1 and block 2 respectively. Nonreconstruction means that the mutual information between the root and the spins at distance n goes to 0 as n tends to infinity, therefore one standard to classify reconstruction and nonreconstruction is to analyze the quantity x_n while in this paper we also need to consider the limiting behavior of $\overline{x}_n$.

In Section 3, after in-depth investigation of the recursive relationship, we develop a two dimensional dynamical system of the linear diagonal canonical form regarding quantities x_{n+1} and $\overline{z}_{n+1}$ through two new variables $\mathcal{X}_n = x_n + \overline{z}_n$ and $\mathcal{Z}_n = -\overline{z}_n$:

$$\begin{cases} \mathcal{X}_{n+1} = d\lambda_1^2\mathcal{X}_n + \frac{d(d-1)}{2}\left(-4\lambda_1^4\mathcal{X}_n^2 + 8\lambda_1^2\lambda_2^2\mathcal{X}_n\mathcal{Z}_n\right) + R_x + R_z + V_x \\ \mathcal{Z}_{n+1} = d\lambda_2^2\mathcal{Z}_n + \frac{d(d-1)}{2}\left[\lambda_1^4\mathcal{X}_n^2 - 8\lambda_2^4\mathcal{Z}_n^2 + \frac{1}{4}\lambda_3^4(\overline{x}_n - \overline{y}_n)^2\right] - R_z + V_z. \end{cases}$$

Here, $\overline{z}_n$ represents the opposite case of x_n as giving a wrong guess in another block. By symmetry, we can also obtain the dynamical system involving $\overline{x}_n$ simply through replacing λ_1 by λ_3. Our main result is the following theorem, whose rigorous proof is given in Section 5.

Main Theorem. *If $|\lambda_1| \neq |\lambda_3|$ and $0 < |\lambda_2| < \max\{|\lambda_1|, |\lambda_3|\}$, the model is reconstructible even if $d\lambda^2 < 1$ with λ being the second largest eigenvalue of the model.*

In Section 4, we show that R_x, R_z, V_x, and V_z are just small perturbations in the above dynamical system in order to study its stability, ensure that the decrease from x_n to x_{n+1} is never too large to lose construction, and establish crucial concentration results, by fully taking advantage of the Markov random field property and the symmetries in the probability transition matrix and the network structure.

In Section 5, by means of the method of reductio ad absurdum, we show that x_n and $\overline{x}_n$ can not simultaneously converge to zero as n goes to ∞, and then establish the nontightness of Kesten-Stigum bound in the Main Theorem of this paper.

2. Preliminary analysis

Let $u_1, \ldots, u_d$ be the children of the root ρ and $\mathbb{T}_v$ be the subtree of descendants of $v \in \mathbb{V}$. Denote the nth level of the tree by $L_n = \{v \in \mathbb{V} : d(\rho, v) = n\}$ with $d(\cdot,\cdot)$ being the graph distance on $\mathbb{T}$. Denote $\sigma(n)$ as the spins on L_n. Denote

$\sigma_j(n)$ as the spins on $L_n \cap \mathbb{T}_{u_j}$ where u_j is one of the children of the root ρ. For the notations involving $\sigma(n)$ in the sequel, we consistently use superscript to denote the conditional on a specific configuration of the root, and use the subscript to denote the conditional on a specific offspring of the root.

For a configuration A on the spins of L_n, define the posterior function by

$$f_n(i, A) = \mathbf{P}(\sigma_\rho = i \mid \sigma(n) = A) = \mathbf{P}(\sigma_{u_j} = i \mid \sigma_j(n+1) = A),$$

for $i = 1, 2, 3, 4$ and $j = 1, \cdots, d$, where the second equality holds by the recursive nature of the tree. Recalling that $\sigma^i(n)$ being $\sigma(n)$ conditioned on $\sigma_\rho = i$, by the definition of $f_n(i, A)$ given above, we have

$$f_n(i, \sigma^i(n)) = \mathbf{P}(\sigma_\rho = i \mid \sigma(n) = \sigma^i(n)).$$

Define $X_i(n)$ as the posterior probability that the root ρ is taking the configuration i given the random configuration $\sigma(n)$ on the spins in L_n, i.e.,

$$X_i(n) = f_n(i, \sigma(n)), \qquad i = 1, 2, 3, 4.$$

Apparently one has

$$X_1(n) + X_2(n) + X_3(n) + X_4(n) = 1.$$

By the block characteristic of the model, we know that regarding the first (resp. second) block, $X_1(n)$ and $X_2(n)$ (resp. $X_3(n)$ and $X_4(n)$) have the same distribution. Considering that the stationary distribution $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ of $\mathbf{P}$ is given by

$$\pi_1 = \pi_2 = \pi_3 = \pi_4 = \frac{1}{4},$$

we further have

$$\mathbf{E}(X_1(n)) = \mathbf{E}(X_2(n)) = \mathbf{E}(X_3(n)) = \mathbf{E}(X_4(n)) = \frac{1}{4}.$$

From the symmetry and the block characteristic of the model, we know that

$$f_n(i, \sigma^j(n)) = f_n(j, \sigma^i(n)), \qquad \text{for} \quad i \neq j, \quad i, j \in \{1, 2\} \text{ or } \{3, 4\},$$

and

$$f_n(1, \sigma^3(n)) = f_n(1, \sigma^4(n)).$$

Define $Y_{ij}(n)$ as the posterior probability that $\sigma_{u_j} = i$ given the random configuration $\sigma_j^1(n+1)$ on spins in $L(n+1) \cap \mathbb{T}_{u_j}$, i.e.,

$$Y_{ij}(n) = f_n(i, \sigma_j^1(n+1)), \qquad \text{for} \quad i = 1, 2, 3, 4, \quad j = 1, \cdots, d,$$

where the random variables $\{Y_{ij}(n)\}$ are independent and identically distributed and satisfy

$$Y_{1j}(n) + Y_{2j}(n) + Y_{3j}(n) + Y_{4j}(n) = 1.$$

We define the following moment variables to analyze the differences between different inferences of σ_ρ given the spins $\sigma(n)$ at distance n from the root ρ and the probability of guessing the root randomly:

$$x_n = \mathbf{E}\left(f_n(1, \sigma^1(n)) - \frac{1}{4}\right), \qquad y_n = \mathbf{E}\left(f_n(2, \sigma^1(n)) - \frac{1}{4}\right),$$
$$z_n = \mathbf{E}\left(f_n(1, \sigma^3(n)) - \frac{1}{4}\right), \qquad u_n = \mathbf{E}\left(f_n(1, \sigma^1(n)) - \frac{1}{4}\right)^2,$$
$$v_n = \mathbf{E}\left(f_n(2, \sigma^1(n)) - \frac{1}{4}\right)^2, \qquad w_n = \mathbf{E}\left(f_n(1, \sigma^3(n)) - \frac{1}{4}\right)^2,$$
$$\overline{x}_n = \mathbf{E}\left(f_n(3, \sigma^3(n)) - \frac{1}{4}\right), \qquad \overline{y}_n = \mathbf{E}\left(f_n(4, \sigma^3(n)) - \frac{1}{4}\right),$$
$$\overline{z}_n = \mathbf{E}\left(f_n(3, \sigma^1(n)) - \frac{1}{4}\right), \qquad \overline{u}_n = \mathbf{E}\left(f_n(3, \sigma^3(n)) - \frac{1}{4}\right)^2,$$
$$\overline{v}_n = \mathbf{E}\left(f_n(4, \sigma^3(n)) - \frac{1}{4}\right)^2, \qquad \overline{w}_n = \mathbf{E}\left(f_n(3, \sigma^1(n)) - \frac{1}{4}\right)^2.$$

We firstly establish some important lemmas which will be used frequently in the sequel.

Lemma 2.1. *For any $n \in \mathbb{N} \cup \{0\}$, we have*

(a) $x_n = 4\mathbf{E}\left(X_1(n) - \frac{1}{4}\right)^2 = u_n + v_n + 2w_n \geqslant 0.$
(b) $-\frac{x_n + y_n}{2} = z_n = \overline{z}_n = -\frac{\overline{x}_n + \overline{y}_n}{2} \leqslant 0.$
(c) $x_n + z_n \geqslant 0, \quad \overline{x}_n + z_n \geqslant 0.$

Proof. (a) By the law of total probability and Bayes' theorem, we have

$$\begin{aligned}\mathbf{E} f_n(1, \sigma^1(n)) &= \sum_A f_n(1, A)\mathbf{P}(\sigma(n) = A \mid \sigma_\rho = 1)\\ &= 4\sum_A f_n(1, A)\mathbf{P}(\sigma_\rho = 1 \mid \sigma(n) = A)\mathbf{P}(\sigma(n) = A)\\ &= 4\sum_A f_n^2(1, A)\mathbf{P}(\sigma(n) = A) = 4\mathbf{E}(X_1(n))^2.\end{aligned}$$

Recall that x_n is defined as $x_n = \mathbf{E}\left(f_n(1, \sigma^1(n)) - \frac{1}{4}\right)$, and then by the fact that $\mathbf{E}(X_1(n)) = \frac{1}{4}$ we have

$$x_n = 4\left(\mathbf{E}(X_1(n))^2 - \left(\frac{1}{4}\right)^2\right) = 4\mathbf{E}\left(X_1(n) - \frac{1}{4}\right)^2.$$

Furthermore, by the law of total expectation, we have

$$\begin{aligned}x_n =&4\mathbf{E}\left(X_1(n) - \frac{1}{4}\right)^2\\ =&4\sum_{i=1}^4 \mathbf{E}\left(\left(X_1(n) - \frac{1}{4}\right)^2 \middle| \sigma_\rho = i\right)\mathbf{P}(\sigma_\rho = i)\\ =&4\left[\mathbf{P}(\sigma_\rho = 1)\mathbf{E}\left(f_n(1, \sigma^1(n)) - \frac{1}{4}\right)^2 + \mathbf{P}(\sigma_\rho = 2)\mathbf{E}\left(f_n(1, \sigma^2(n)) - \frac{1}{4}\right)^2\right.\\ &\left. + \mathbf{P}(\sigma_\rho = 3)\mathbf{E}\left(f_n(1, \sigma^3(n)) - \frac{1}{4}\right)^2 + \mathbf{P}(\sigma_\rho = 4)\mathbf{E}\left(f_n(1, \sigma^4(n)) - \frac{1}{4}\right)^2\right]\\ =&u_n + v_n + 2w_n.\end{aligned}$$

(b) Similarly, we have

$$z_n = 4\mathbf{E}\left(X_1(n)X_3(n)\right) - \frac{1}{4} = \mathbf{E}\left(f_n(1, \sigma^3(n)) - \frac{1}{4}\right) = \overline{z}_n, \tag{2}$$

$$y_n + \frac{1}{4} = \sum_A f_n(2, A)\mathbf{P}(\sigma(n) = A \mid \sigma_\rho = 1) = 4\mathbf{E}\left(X_1(n)X_2(n)\right),$$

and then

$$y_n = 4\mathbf{E}\left(X_1(n) - \frac{1}{4}\right)\left(X_2(n) - \frac{1}{4}\right). \tag{3}$$

It follows from the Cauchy-Schwarz inequality that

$$\left[\mathbf{E}\left(X_1(n) - \frac{1}{4}\right)\left(X_2(n) - \frac{1}{4}\right)\right]^2 \leqslant \mathbf{E}\left(X_1(n) - \frac{1}{4}\right)^2 \mathbf{E}\left(X_2(n) - \frac{1}{4}\right)^2,$$

which implies

$$\left(\frac{1}{4}y_n\right)^2 \leqslant \left(\frac{1}{4}x_n\right)^2, \quad \text{i.e.} \quad -x_n \leqslant y_n \leqslant x_n. \tag{4}$$

By the definitions of x_n, y_n and z_n, we know that $z_n = -\frac{x_n + y_n}{2}$, and thus equation (4) implies $z_n \leqslant 0$.

(c) An analogous proof of

$$x_n + z_n = x_n - \frac{x_n + y_n}{2} = \frac{x_n - y_n}{2} \geqslant 0 \quad \text{and} \quad \overline{x}_n + z_n \geqslant 0$$

can be easily carried out.

Lemma 2.2. *For any $n \in \mathbb{N} \cup \{0\}$, we have*

(a) $\mathbf{E}\left(f_n(1,\sigma^1(n)) - \frac{1}{4}\right)\left(f_n(2,\sigma^1(n)) - \frac{1}{4}\right) = \frac{1}{4}y_n + \left(v_n - \frac{1}{4}x_n\right).$

(b) $$\mathbf{E}\left(f_n(1,\sigma^1(n)) - \frac{1}{4}\right)\left(f_n(3,\sigma^1(n)) - \frac{1}{4}\right) = \frac{1}{4}z_n - \frac{1}{2}\left(u_n - \frac{1}{4}x_n\right) - \frac{1}{2}\left(v_n - \frac{1}{4}x_n\right).$$

(c) $\mathbf{E}\left(f_n(2,\sigma^1(n)) - \frac{1}{4}\right)\left(f_n(3,\sigma^1(n)) - \frac{1}{4}\right) = \frac{1}{4}z_n - \left(v_n - \frac{1}{4}x_n\right).$

(d) $$\mathbf{E}\left(f_n(3,\sigma^1(n)) - \frac{1}{4}\right)\left(f_n(4,\sigma^1(n)) - \frac{1}{4}\right) = \frac{1}{4}\overline{y}_n + \frac{1}{2}\left(u_n - \frac{1}{4}x_n\right) + \frac{3}{2}\left(v_n - \frac{1}{4}x_n\right) - \left(\overline{w}_n - \frac{1}{4}\overline{x}_n\right).$$

(e) $\mathbf{E}\left(f_n(1,\sigma^3(n)) - \frac{1}{4}\right)\left(f_n(2,\sigma^3(n)) - \frac{1}{4}\right) = \frac{1}{4}y_n - \left(v_n - \frac{1}{4}x_n\right).$

Proof. We only prove (a) and (b) and the others can be shown analogously.

(a) By the law of total probability, one has

$$\begin{aligned}
&\mathbf{E}\Big(f_n(1,\sigma^1(n))f_n(2,\sigma^1(n))\Big)\\
=&\sum_A \mathbf{P}(\sigma_\rho = 1 \mid \sigma(n) = A)\mathbf{P}(\sigma_\rho = 2 \mid \sigma(n) = A)\mathbf{P}(\sigma(n) = A \mid \sigma_\rho = 1)\\
=&\sum_A \left[\mathbf{P}(\sigma_\rho = 2 \mid \sigma(n) = A)\right]^2 \mathbf{P}(\sigma(n) = A \mid \sigma_\rho = 1)\\
=&\mathbf{E}\left(f_n(2,\sigma^1(n))\right)^2,
\end{aligned}$$

therefore

$$\begin{aligned}
&\mathbf{E}\left(f_n(1,\sigma^1(n)) - \frac{1}{4}\right)\left(f_n(2,\sigma^1(n)) - \frac{1}{4}\right)\\
=&v_n + \frac{1}{4}(y_n - x_n) = \frac{1}{4}y_n + \left(v_n - \frac{1}{4}x_n\right).
\end{aligned}$$

(b) By the fact that $f_n(3,\sigma^1(n))$ and $f_n(4,\sigma^1(n))$ have the same distribution, and the equation that

$$f_n(1,\sigma^1(n)) + f_n(2,\sigma^1(n)) + f_n(3,\sigma^1(n)) + f_n(4,\sigma^1(n)) = 1,$$

plugging in the result of (a), we can obtain that

$$\begin{aligned}
&\mathbf{E}\left(f_n(1,\sigma^1(n)) - \frac{1}{4}\right)\left(f_n(3,\sigma^1(n)) - \frac{1}{4}\right)\\
=&\frac{1}{4}z_n - \frac{1}{2}\left(u_n - \frac{1}{4}x_n\right) - \frac{1}{2}\left(v_n - \frac{1}{4}x_n\right),
\end{aligned}$$

as desired.

Recall that $Y_{ij}(n)$ is defined as the posterior probability that $\sigma_{u_j} = i$ given the random configuration $\sigma_j^1(n+1)$ on spins in $L(n+1) \cap \mathbb{T}_{u_j}$, i.e., $Y_{ij}(n) = f_n(i, \sigma_j^1(n+1))$, for $i \in \{1,2,3,4\}$ and $j \in \{1,\cdots,d\}$. The random vectors $(Y_{ij}(n))_{i=1}^4$ are independent by the symmetry of the model, and its central moments are investigated in the following lemma.

Lemma 2.3. *For each $1 \leqslant j \leqslant d$, we have*

(a) $\mathbf{E}\left(Y_{1j}(n)-\frac{1}{4}\right)=\lambda_1 x_n+(\lambda_1-\lambda_2)z_n.$

(b) $\mathbf{E}\left(Y_{2j}(n)-\frac{1}{4}\right)=-\lambda_1 x_n-(\lambda_1+\lambda_2)z_n.$

(c) $\mathbf{E}\left(Y_{ij}(n)-\frac{1}{4}\right)=\lambda_2 z_n, \quad i=3,4.$

(d) $\mathbf{E}\left(Y_{1j}(n)-\frac{1}{4}\right)^2=\frac{1}{4}x_n+\lambda_1\left(u_n-\frac{1}{4}x_n\right)+(\lambda_1-\lambda_2)\left(w_n-\frac{1}{4}x_n\right).$

(e) $\mathbf{E}\left(Y_{2j}(n)-\frac{1}{4}\right)^2=\frac{1}{4}x_n-\lambda_1\left(u_n-\frac{1}{4}x_n\right)-(\lambda_1+\lambda_2)\left(w_n-\frac{1}{4}x_n\right).$

(f) $\mathbf{E}\left(Y_{ij}(n)-\frac{1}{4}\right)^2=\frac{1}{4}\overline{x}_n+\lambda_2\left(\overline{w}_n-\frac{1}{4}\overline{x}_n\right), \quad i=3,4.$

(g) $\mathbf{E}\left(Y_{1j}(n)-\frac{1}{4}\right)\left(Y_{2j}(n)-\frac{1}{4}\right)=\frac{1}{4}y_n+\lambda_2\left(v_n-\frac{1}{4}x_n\right).$

(h) $$\mathbf{E}\left(Y_{1j}(n)-\frac{1}{4}\right)\left(Y_{ij}(n)-\frac{1}{4}\right)=\frac{z_n}{4}+\frac{\lambda_1-\lambda_2}{2}\left(v_n-\frac{1}{4}x_n\right)+\frac{\lambda_1+\lambda_2}{2}\left(w_n-\frac{1}{4}x_n\right), \quad i=3,4.$$

(i) $$\mathbf{E}\left(Y_{2j}(n)-\frac{1}{4}\right)\left(Y_{ij}(n)-\frac{1}{4}\right)=\frac{z_n}{4}-\frac{\lambda_1+\lambda_2}{2}\left(v_n-\frac{1}{4}x_n\right)-\frac{\lambda_1-\lambda_2}{2}\left(w_n-\frac{1}{4}x_n\right), \quad i=3,4.$$

(j) $\mathbf{E}\left(Y_{3j}(n)-\frac{1}{4}\right)\left(Y_{4j}(n)-\frac{1}{4}\right)=\frac{1}{4}\overline{y}_n-\lambda_2\left(\overline{v}_n-\frac{1}{4}\overline{x}_n\right).$

Proof. We only prove (a), (b), and (c) and the others can be shown analogously.

(a) Conditioning on $\sigma_{u_j}=i$ for $i\in\{1,2,3,4\}$, we have

$$\begin{aligned}\mathbf{E}\left(Y_{1j}(n)-\frac{1}{4}\right)=&p_{11}\mathbf{E}\left(f_n(1,\sigma^1(n))-\frac{1}{4}\right)+p_{12}\mathbf{E}\left(f_n(1,\sigma^2(n))-\frac{1}{4}\right)\\&+p_{13}\mathbf{E}\left(f_n(1,\sigma^3(n))-\frac{1}{4}\right)+p_{14}\mathbf{E}\left(f_n(1,\sigma^4(n))-\frac{1}{4}\right)\\=&(p_0-p_1)x_n+2(p_2-p_1)z_n\\=&\lambda_1 x_n+(\lambda_1-\lambda_2)z_n.\end{aligned}$$

(b) Similar, we can obtain

$$\begin{aligned}\mathbf{E}\left(Y_{2j}(n)-\frac{1}{4}\right)=&(p_1-p_0)x_n+2(p_2-p_0)z_n\\=&-\lambda_1 x_n-(\lambda_1+\lambda_2)z_n.\end{aligned}$$

(c) It follows immediately from the identity $\sum_{i=1}^4 Y_{ij}(n)=1$ that, for $i=3,4$,

$$\mathbf{E}\left(Y_{ij}(n)-\frac{1}{4}\right)=-\frac{1}{2}\sum_{i=1}^{2}\mathbf{E}\left(Y_{ij}(n)-\frac{1}{4}\right)=\lambda_2 z_n.$$

If the model is reconstructible, $\sigma(n)$ contains significant information of the root variable. This can be expressed in several equivalent ways (see Mossel [24,26]). According to Definition 1.1, we have the following lemma (Proposition 2.1 on page 3 of Mossel [26]).

Lemma 2.4. *The model being non-reconstructible is equivalent to*

$$\lim_{n\to\infty} x_n=\lim_{n\to\infty}\overline{x}_n=0.$$

3. Recursive formulas

3.1. Distributional recursion

Consider A as a configuration on $L(n+1)$, and let $A_j (j = 1, \cdots, d)$ be its restriction to $\mathbb{T}_{u_j} \cap L(n+1)$ where u_j is the jth child of the root ρ. The following lemma provides a fundamental recursion formula in this section.

Lemma 3.1. *We have*

$$f_{n+1}(1, A) = \frac{N_1(n)}{N_1(n) + N_2(n) + N_3(n) + N_4(n)}, \tag{5}$$

where

$$N_k(n) = \prod_{j=1}^{d} \left[\sum_{i=1}^{4} p_{ki} \mathbf{P}(\sigma_j(n+1) = A_j \mid \sigma_{u_j} = i) \right], \quad k \in \{1, 2, 3, 4\}.$$

Proof. By the definition of $f_{n+1}(i, A)$ given in Section 2, we have

$$f_{n+1}(1, A) = \mathbf{P}(\sigma_\rho = 1 \mid \sigma(n+1) = A).$$

By Bayes' theorem we have

$$\begin{aligned} f_{n+1}(1, A) =& \frac{\mathbf{P}(\sigma(n+1) = A, \sigma_\rho = 1)}{\mathbf{P}(\sigma(n+1) = A)} \\ =& \frac{\prod_{j=1}^{d} \sum_{i=1}^{4} \mathbf{P}(\sigma_j(n+1) = A_j, \sigma_{u_j} = i, \sigma_\rho = 1)}{\prod_{j=1}^{d} \sum_{i,k=1}^{4} \mathbf{P}(\sigma_j(n+1) = A_j, \sigma_{u_j} = i, \sigma_\rho = k)} \\ =& \frac{\prod_{j=1}^{d} \sum_{i=1}^{4} \left[\mathbf{P}(\sigma_j(n+1) = A_j \mid \sigma_{u_j} = i, \sigma_\rho = 1) \mathbf{P}(\sigma_{u_j} = i \mid \sigma_\rho = 1) \mathbf{P}(\sigma_\rho = 1) \right]}{\prod_{j=1}^{d} \sum_{i,k=1}^{4} \left[\mathbf{P}(\sigma_j(n+1) = A_j \mid \sigma_{u_j} = i, \sigma_\rho = k) \mathbf{P}(\sigma_{u_j} = i \mid \sigma_\rho = k) \mathbf{P}(\sigma_\rho = k) \right]}. \end{aligned}$$

By the Markov property, we further have

$$f_{n+1}(1, A) = \frac{\prod_{j=1}^{d} \sum_{i=1}^{4} \left[\mathbf{P}(\sigma_j(n+1) = A_j \mid \sigma_{u_j} = i) \mathbf{P}(\sigma_{u_j} = i \mid \sigma_\rho = 1) \mathbf{P}(\sigma_\rho = 1) \right]}{\prod_{j=1}^{d} \sum_{i,k=1}^{4} \left[\mathbf{P}(\sigma_j(n+1) = A_j \mid \sigma_{u_j} = i) \mathbf{P}(\sigma_{u_j} = i \mid \sigma_\rho = k) \mathbf{P}(\sigma_\rho = k) \right]}.$$

Given that the configuration of root have equal chance to be $1, 2, 3, 4$, we have $\mathbf{P}(\sigma_\rho = k) = 1/4$, and then

$$\begin{aligned} f_{n+1}(1, A) =& \frac{\prod_{j=1}^{d} \sum_{i=1}^{4} \left[\mathbf{P}(\sigma_j(n+1) = A_j \mid \sigma_{u_j} = i) \mathbf{P}(\sigma_{u_j} = i \mid \sigma_\rho = 1) \right]}{\prod_{j=1}^{d} \sum_{i,k=1}^{4} \left[\mathbf{P}(\sigma_j(n+1) = A_j \mid \sigma_{u_j} = i) \mathbf{P}(\sigma_{u_j} = i \mid \sigma_\rho = k) \right]} \\ =& \frac{\prod_{j=1}^{d} \sum_{i=1}^{4} \left[p_{1i} \mathbf{P}(\sigma_j(n+1) = A_j \mid \sigma_{u_j} = i) \right]}{\prod_{j=1}^{d} \sum_{i,k=1}^{4} \left[p_{ki} \mathbf{P}(\sigma_j(n+1) = A_j \mid \sigma_{u_j} = i) \right]}, \end{aligned}$$

which completes the proof.

The following lemma elaborate the result of Lemma 3.1.

Lemma 3.2. *For any realization of $\sigma^1(n+1)$, denoted as $A = (A_1, \ldots, A_d)$, where A_j denoting the spins on $L_{n+1} \cap \mathbb{T}_{u_j}$, we have*

$$f_{n+1}(1, A) = \frac{Z_1(n)}{\sum_{k=1}^{4} Z_k(n)}$$

where, for $k = 1, 2, 3, 4$,

$$Z_k(n) = \prod_{j=1}^{d}\left[1 + \sum_{i=1}^{4} 4p_{ki}\left(f_n(i, A_j) - \frac{1}{4}\right)\right]$$
$$= 4\frac{\mathbf{P}(\sigma(n+1) = A)}{\prod_{j=1}^{d}\mathbf{P}(\sigma_j(n+1) = A_j)}\mathbf{P}(\sigma_\rho = k \mid \sigma(n+1) = A).$$

Proof. By Lemma 3.1, the definition of $f_{n+1}(i, A)$ given in Section 2, and Bayes' theorem, we have

$$f_{n+1}(1, A) = \frac{\prod_{j=1}^{d}\left[\sum_{i=1}^{4} p_{1i}\mathbf{P}(\sigma_j(n+1) = A_j \mid \sigma_{u_j} = i)\right]}{\sum_{k=1}^{4}\prod_{j=1}^{d}\left[\sum_{i=1}^{4} p_{ki}\mathbf{P}(\sigma_j(n+1) = A_j \mid \sigma_{u_j} = i)\right]}$$
$$= \frac{\prod_{j=1}^{d}\sum_{i=1}^{4} p_{1i} f_n(i, A_j)\frac{\mathbf{P}(\sigma_j(n+1)=A_j)}{\mathbf{P}(\sigma_{u_j}=i)}}{\sum_{k=1}^{4}\prod_{j=1}^{d}\sum_{i=1}^{4} p_{ki} f_n(i, A_j)\frac{\mathbf{P}(\sigma_j(n+1)=A_j)}{\mathbf{P}(\sigma_{u_j}=i)}}$$
$$= \frac{\prod_{j=1}^{d}\sum_{i=1}^{4} p_{1i} f_n(i, A_j)\mathbf{P}(\sigma_j(n+1) = A_j)}{\sum_{k=1}^{4}\prod_{j=1}^{d}\sum_{i=1}^{4} p_{ki} f_n(i, A_j)\mathbf{P}(\sigma_j(n+1) = A_j)},$$

where the last equality holds for the reason that the process started from uniform distribution and then $\mathbf{P}(\sigma_{u_j} = i) = \frac{1}{4}$ for $i = 1, 2, 3, 4$. Furthermore

$$f_{n+1}(1, A) = \frac{\left[\prod_{j=1}^{d}\sum_{i=1}^{4} p_{1i} f_n(i, A_j)\right] \times \left[\prod_{j=1}^{d}\mathbf{P}(\sigma_j(n+1) = A_j)\right]}{\left[\sum_{k=1}^{4}\prod_{j=1}^{d}\sum_{i=1}^{4} p_{ki} f_n(i, A_j)\right] \times \left[\prod_{j=1}^{d}\mathbf{P}(\sigma_j(n+1) = A_j)\right]}$$
$$= \frac{\prod_{j=1}^{d}\sum_{i=1}^{4} p_{1i} f_n(i, A_j)}{\sum_{k=1}^{4}\prod_{j=1}^{d}\sum_{i=1}^{4} p_{ki} f_n(i, A_j)}$$
$$= \frac{Z_1(n)}{\sum_{k=1}^{4} Z_k(n)},$$

where, since $\sum_{i=1}^{4} p_{ki} = 1$ for any $k = 1, 2, 3, 4$,

$$Z_k(n) = \prod_{j=1}^{d}\left[1 + \sum_{i=1}^{4} 4p_{ki}\left(f_n(i, A_j) - \frac{1}{4}\right)\right].$$

That is,

$$\frac{Z_k(n)}{4^d} = \prod_{j=1}^{d}\sum_{i=1}^{4} p_{ki} f_n(i, A_j).$$

Hence,

$$\frac{Z_k(n)}{4^d} \cdot \prod_{j=1}^{d}\mathbf{P}(\sigma_j(n+1) = A_j)$$
$$= \prod_{j=1}^{d}\sum_{i=1}^{4} p_{ki} f_n(i, A_j)\mathbf{P}(\sigma_j(n+1) = A_j)$$
$$= \prod_{j=1}^{d}\sum_{i=1}^{4} p_{ki}\mathbf{P}(\sigma_{u_j} = i, \sigma_j(n+1) = A_j)$$
$$= \prod_{j=1}^{d}\sum_{i=1}^{4} p_{ki}\mathbf{P}(\sigma_j(n+1) = A_j \mid \sigma_{u_j} = i)\mathbf{P}(\sigma_{u_j} = i)$$

$$
\begin{aligned}
&=\frac{1}{4^d}\prod_{j=1}^{d}\sum_{i=1}^{4}p_{ki}\mathbf{P}(\sigma_j(n+1)=A_j\mid\sigma_{u_j}=i)\\
&=\frac{1}{4^d}\prod_{j=1}^{d}\sum_{i=1}^{4}p_{ki}\mathbf{P}(\sigma_j(n+1)=A_j\mid\sigma_{u_j}=i,\sigma_\rho=k)\\
&=\frac{1}{4^d}\prod_{j=1}^{d}\sum_{i=1}^{4}\frac{\mathbf{P}(\sigma_{u_j}=i,\sigma_\rho=k)}{\mathbf{P}(\sigma_\rho=k)}\frac{\mathbf{P}(\sigma_j(n+1)=A_j,\sigma_{u_j}=i,\sigma_\rho=k)}{\mathbf{P}(\sigma_{u_j}=i,\sigma_\rho=k)}\\
&=\frac{4^d}{4^d}\prod_{j=1}^{d}\sum_{i=1}^{4}\mathbf{P}(\sigma_j(n+1)=A_j,\sigma_{u_j}=i,\sigma_\rho=k)\\
&=\prod_{j=1}^{d}\mathbf{P}(\sigma_j(n+1)=A_j,\sigma_\rho=k)\\
&=\prod_{j=1}^{d}\mathbf{P}(\sigma_j(n+1)=A_j\mid\sigma_\rho=k)\mathbf{P}(\sigma_\rho=k)\\
&=\frac{1}{4^d}\prod_{j=1}^{d}\mathbf{P}(\sigma_j(n+1)=A_j\mid\sigma_\rho=k)\\
&=\frac{1}{4^d}\mathbf{P}(\sigma(n+1)=A\mid\sigma_\rho=k)\\
&=\frac{4}{4^d}\mathbf{P}(\sigma(n+1)=A)\mathbf{P}(\sigma_\rho=k\mid\sigma(n+1)=A).
\end{aligned}
$$

Then we have

$$
Z_k(n)=4\frac{\mathbf{P}(\sigma(n+1)=A)}{\prod_{j=1}^{d}\mathbf{P}(\sigma_j(n+1)=A_j)}\mathbf{P}(\sigma_\rho=k\mid\sigma(n+1)=A),
$$

which completes the proof.

Given that Lemma 3.2 holds for any realization A of $\sigma^1(n+1)$, we are ready to extend the result of $f_{n+1}(1,A)$ to that of $f_{n+1}(1,\sigma^1(n+1))$ given in the following lemma.

Lemma 3.3. *We have*

$$
f_{n+1}(1,\sigma^1(n+1))=\frac{Z_1(n)}{Z_1(n)+Z_2(n)+Z_3(n)+Z_4(n)},\tag{6}
$$

where

$$
Z_i(n)=\begin{cases}
\prod_{j=1}^{d}\left[1+2(\lambda_1+\lambda_2)\left(Y_{1j}(n)-\frac{1}{4}\right)-2(\lambda_1-\lambda_2)\left(Y_{2j}(n)-\frac{1}{4}\right)\right], & i=1,\\
\prod_{j=1}^{d}\left[1-2(\lambda_1-\lambda_2)\left(Y_{1j}(n)-\frac{1}{4}\right)+2(\lambda_1+\lambda_2)\left(Y_{2j}(n)-\frac{1}{4}\right)\right], & i=2,\\
\prod_{j=1}^{d}\left[1+2(\lambda_2+\lambda_3)\left(Y_{3j}(n)-\frac{1}{4}\right)+2(\lambda_2-\lambda_3)\left(Y_{4j}(n)-\frac{1}{4}\right)\right], & i=3,\\
\prod_{j=1}^{d}\left[1+2(\lambda_2-\lambda_3)\left(Y_{3j}(n)-\frac{1}{4}\right)+2(\lambda_2+\lambda_3)\left(Y_{4j}(n)-\frac{1}{4}\right)\right], & i=4.
\end{cases}
$$

Proof. Given that

$$
f_n(1,A_j)+f_n(2,A_j)+f_n(3,A_j)+f_n(4,A_j)=1,
$$

plugging in the values of $\{p_{ki}\}$ from the probability transition matrix (1), by Lemma 3.2, we obtain

$$
\begin{aligned}
Z_1(n)=\prod_{j=1}^{d}\Bigg[1&+4p_0\left(f_n(1,A_j)-\frac{1}{4}\right)+4p_1\left(f_n(2,A_j)-\frac{1}{4}\right)\\
&+4p_2\left(f_n(3,A_j)-\frac{1}{4}\right)+4p_2\left(f_n(4,A_j)-\frac{1}{4}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{j=1}^{d} \Bigg[1 + 4p_0 \left(f_n(1, A_j) - \frac{1}{4} \right) + 4p_1 \left(f_n(2, A_j) - \frac{1}{4} \right) \\
&\qquad -4p_2 \left[\left(f_n(1, A_j) - \frac{1}{4} \right) + \left(f_n(2, A_j) - \frac{1}{4} \right) \right] \Bigg] \\
&= \prod_{j=1}^{d} \left[1 + 4(p_0 - p_2) \left(f_n(1, A_j) - \frac{1}{4} \right) + 4(p_1 - p_2) \left(f_n(2, A_j) - \frac{1}{4} \right) \right] \\
&= \prod_{j=1}^{d} \left[1 + 2(\lambda_1 + \lambda_2) \left(f_n(1, A_j) - \frac{1}{4} \right) - 2(\lambda_1 - \lambda_2) \left(f_n(2, A_j) - \frac{1}{4} \right) \right],
\end{aligned}
$$

where the last equality holds since $\lambda_1 = p_0 - p_1$ and $\lambda_2 = p_0 + p_1 - 2p_2$. Similarly, we have

$$
\begin{aligned}
Z_2(n) &= \prod_{j=1}^{d} \Bigg[1 + 4p_1 \left(f_n(1, A_j) - \frac{1}{4} \right) + 4p_0 \left(f_n(2, A_j) - \frac{1}{4} \right) \\
&\qquad +4p_2 \left(f_n(3, A_j) - \frac{1}{4} \right) + 4p_2 \left(f_n(4, A_j) - \frac{1}{4} \right) \Bigg] \\
&= \prod_{j=1}^{d} \Bigg[1 + 4p_1 \left(f_n(1, A_j) - \frac{1}{4} \right) + 4p_0 \left(f_n(2, A_j) - \frac{1}{4} \right) \\
&\qquad -4p_2 \left[\left(f_n(1, A_j) - \frac{1}{4} \right) + \left(f_n(2, A_j) - \frac{1}{4} \right) \right] \Bigg] \\
&= \prod_{j=1}^{d} \left[1 + 4(p_1 - p_2) \left(f_n(1, A_j) - \frac{1}{4} \right) + 4(p_0 - p_2) \left(f_n(2, A_j) - \frac{1}{4} \right) \right] \\
&= \prod_{j=1}^{d} \left[1 - 2(\lambda_1 - \lambda_2) \left(f_n(1, A_j) - \frac{1}{4} \right) + 2(\lambda_1 + \lambda_2) \left(f_n(2, A_j) - \frac{1}{4} \right) \right], \\
Z_3(n) &= \prod_{j=1}^{d} \Bigg[1 + 4p_2 \left(f_n(1, A_j) - \frac{1}{4} \right) + 4p_2 \left(f_n(2, A_j) - \frac{1}{4} \right) \\
&\qquad +4\bar{p}_0 \left(f_n(3, A_j) - \frac{1}{4} \right) + 4\bar{p}_1 \left(f_n(4, A_j) - \frac{1}{4} \right) \Bigg] \\
&= \prod_{j=1}^{d} \Bigg[1 - 4p_2 \left[\left(f_n(3, A_j) - \frac{1}{4} \right) + \left(f_n(4, A_j) - \frac{1}{4} \right) \right] \\
&\qquad +4\bar{p}_0 \left(f_n(3, A_j) - \frac{1}{4} \right) + 4\bar{p}_1 \left(f_n(4, A_j) - \frac{1}{4} \right) \Bigg] \\
&= \prod_{j=1}^{d} \left[1 + 4(\bar{p}_0 - p_2) \left(f_n(3, A_j) - \frac{1}{4} \right) + 4(\bar{p}_1 - p_2) \left(f_n(4, A_j) - \frac{1}{4} \right) \right] \\
&= \prod_{j=1}^{d} \left[1 + 2(\lambda_2 + \lambda_3) \left(f_n(3, A_j) - \frac{1}{4} \right) + 2(\lambda_2 - \lambda_3) \left(f_n(4, A_j) - \frac{1}{4} \right) \right],
\end{aligned}
$$

and

$$
\begin{aligned}
Z_4(n) &= \prod_{j=1}^{d} \Bigg[1 + 4p_2 \left(f_n(1, A_j) - \frac{1}{4} \right) + 4p_2 \left(f_n(2, A_j) - \frac{1}{4} \right) \\
&\qquad +4\bar{p}_1 \left(f_n(3, A_j) - \frac{1}{4} \right) + 4\bar{p}_0 \left(f_n(4, A_j) - \frac{1}{4} \right) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{j=1}^{d} \Bigg[1 - 4p_2 \left[\left(f_n(3, A_j) - \frac{1}{4} \right) + \left(f_n(4, A_j) - \frac{1}{4} \right) \right] \\
&\qquad + 4\bar{p}_1 \left(f_n(3, A_j) - \frac{1}{4} \right) + 4\bar{p}_0 \left(f_n(4, A_j) - \frac{1}{4} \right) \Bigg] \\
&= \prod_{j=1}^{d} \left[1 + 4(\bar{p}_1 - p_2) \left(f_n(3, A_j) - \frac{1}{4} \right) + 4(\bar{p}_0 - p_2) \left(f_n(4, A_j) - \frac{1}{4} \right) \right] \\
&= \prod_{j=1}^{d} \left[1 + 2(\lambda_2 - \lambda_3) \left(f_n(3, A_j) - \frac{1}{4} \right) + 2(\lambda_2 + \lambda_3) \left(f_n(4, A_j) - \frac{1}{4} \right) \right].
\end{aligned}
$$

Since the above results hold for any A as realization of $\sigma^1(n+1)$, recalling that $Y_{ij}(n) = f_n(i, \sigma_j^1(n+1))$, we complete the proof.

Lemma 3.4. *For any nonnegative $n \in \mathbb{Z}^+$, we have*

$$
\mathbf{E}\left(Z_1(n) Z_2(n)\right) = \mathbf{E} Z_2^2(n).
$$

Proof. For any configuration $A = (A_1, \ldots, A_d)$ with A_j denoting the spins on $L_{n+1} \cap \mathbb{T}_{u_j}$, by Lemma 3.2, we have

$$
Z_i(n) = 4 \frac{\mathbf{P}(\sigma(n+1) = A)}{\prod_{j=1}^{d} \mathbf{P}(\sigma_j(n+1) = A_j)} \mathbf{P}(\sigma_\rho = i \mid \sigma(n+1) = A), \quad \text{for } i = 1, 2.
$$

By the symmetry of the tree, we have

$$
\begin{aligned}
\mathbf{E}\left(Z_1(n) Z_2(n)\right) =& 16 \sum_{A} \left(\frac{\mathbf{P}(\sigma(n+1) = A)}{\prod_{j=1}^{d} \mathbf{P}(\sigma_j(n+1) = A_j)} \right)^2 \mathbf{P}(\sigma_\rho = 1 \mid \sigma(n+1) = A) \\
& \times \mathbf{P}(\sigma_\rho = 2 \mid \sigma(n+1) = A) \mathbf{P}(\sigma(n+1) = A \mid \sigma_\rho = 1) \\
=& 16 \sum_{A} \left(\frac{\mathbf{P}(\sigma(n+1) = A)}{\prod_{j=1}^{d} \mathbf{P}(\sigma_j(n+1) = A_j)} \right)^2 \mathbf{P}^2(\sigma_\rho = 2 \mid \sigma(n+1) = A) \\
& \times \mathbf{P}(\sigma(n+1) = A \mid \sigma_\rho = 1) \\
=& \mathbf{E} Z_2^2(n),
\end{aligned}
$$

as desired.

By Lemma 2.3, the means and variances of monomials of $Z_i(n)$ can be approximated as follows:

Lemma 3.5. *One has*

(i)
$$
\begin{aligned}
\mathbf{E} Z_1(n) =& 1 + d\lambda_1^2 4(x_n + z_n) - d\lambda_2^2 4z_n \\
& + \frac{d(d-1)}{2} \left[4\lambda_1^2(x_n + z_n) - 4\lambda_2^2 z_n \right]^2 + O(x_n^3).
\end{aligned}
$$

(ii)
$$
\begin{aligned}
\mathbf{E} Z_2(n) =& 1 - d\lambda_1^2 4(x_n + z_n) - d\lambda_2^2 4z_n \\
& + \frac{d(d-1)}{2} \left[4\lambda_1^2(x_n + z_n) + 4\lambda_2^2 z_n \right]^2 + O(x_n^3).
\end{aligned}
$$

(iii)
$$
\mathbf{E} Z_i(n) = 1 + d\lambda_2^2 4z_n + \frac{d(d-1)}{2} \left(4\lambda_2^2 z_n \right)^2 + O(x_n^3), \quad i = 3, 4.
$$

(iv) $\mathbf{E} Z_1^2(n) = 1 + d\Pi_1 + \frac{d(d-1)}{2} \Pi_1^2 + O(x_n^3)$, *where*

$$
\begin{aligned}
\Pi_1 =& \mathbf{E} \left[1 + 2(\lambda_1 + \lambda_2) \left(Y_{1j}(n) - \frac{1}{4} \right) - 2(\lambda_1 - \lambda_2) \left(Y_{2j}(n) - \frac{1}{4} \right) \right]^2 - 1 \\
=& 12\lambda_1^2(x_n + z_n) - 12\lambda_2^2 z_n + 16\lambda_1^2 \lambda_2 \left(u_n - \frac{1}{4} x_n \right) \\
& - 8(\lambda_1^2 - \lambda_2^2)\lambda_2 \left(v_n - \frac{1}{4} x_n \right) + 8(\lambda_1^2 - \lambda_2^2)\lambda_2 \left(w_n - \frac{1}{4} x_n \right).
\end{aligned}
$$

(v) $\mathbf{E}Z_2^2(n) = \mathbf{E}Z_1(n)Z_2(n) = 1 + d\Pi_2 + \frac{d(d-1)}{2}\Pi_2^2 + O(x_n^3)$, *where*

$$\begin{aligned}\Pi_2 =& \mathbf{E}\left[1 - 2(\lambda_1 - \lambda_2)\left(Y_{1j}(n) - \frac{1}{4}\right) + 2(\lambda_1 + \lambda_2)\left(Y_{2j}(n) - \frac{1}{4}\right)\right]^2 - 1 \\ =& -4\lambda_1^2(x_n + z_n) - 12\lambda_2^2 z_n - 16\lambda_1^2\lambda_2\left(u_n - \frac{1}{4}x_n\right) \\ & - 8(\lambda_1^2 - \lambda_2^2)\lambda_2\left(v_n - \frac{1}{4}x_n\right) - 8(3\lambda_1^2 + \lambda_2^2)\lambda_2\left(w_n - \frac{1}{4}x_n\right).\end{aligned}$$

(vi) $\mathbf{E}Z_i^2(n) = 1 + d\Pi_3 + \frac{d(d-1)}{2}\Pi_3^2 + O(x_n^3)$, *for* $i = 3, 4$, *where*

$$\begin{aligned}\Pi_3 =& \mathbf{E}\left[1 + 2(\lambda_2 + \lambda_3)\left(Y_{3j}(n) - \frac{1}{4}\right) + 2(\lambda_2 - \lambda_3)\left(Y_{4j}(n) - \frac{1}{4}\right)\right]^2 - 1 \\ =& 4\lambda_2^2 z_n + 2\lambda_3^2(\overline{x}_n - \overline{y}_n) - 8(\lambda_2^2 - \lambda_3^2)\lambda_2\left(\overline{v}_n - \frac{1}{4}\overline{x}_n\right) \\ & + 8(\lambda_2^2 + \lambda_3^2)\lambda_2\left(\overline{w}_n - \frac{1}{4}\overline{x}_n\right).\end{aligned}$$

(vii) $\mathbf{E}Z_1(n)Z_i(n) = 1 + d\Pi_4 + \frac{d(d-1)}{2}\Pi_4^2 + O(x_n^3)$, *for* $i = 3, 4$, *where*

$$\begin{aligned}\Pi_4 =& \mathbf{E}\left[1 + 2(\lambda_1 + \lambda_2)\left(Y_{1j}(n) - \frac{1}{4}\right) - 2(\lambda_1 - \lambda_2)\left(Y_{2j}(n) - \frac{1}{4}\right)\right] \\ & \times\left[1 + 2(\lambda_2 + \lambda_3)\left(Y_{3j}(n) - \frac{1}{4}\right) + 2(\lambda_2 - \lambda_3)\left(Y_{4j}(n) - \frac{1}{4}\right)\right] - 1 \\ =& 4\lambda_1^2(x_n + z_n) + 4\lambda_2^2 z_n + 8(\lambda_1^2 - \lambda_2^2)\lambda_2\left(v_n - \frac{1}{4}x_n\right) \\ & + 8(\lambda_1^2 + \lambda_2^2)\lambda_2\left(w_n - \frac{1}{4}x_n\right).\end{aligned}$$

(viii) $\mathbf{E}Z_2(n)Z_i(n) = 1 + d\Pi_5 + \frac{d(d-1)}{2}\Pi_5^2 + O(x_n^3)$, *for* $i = 3, 4$, *where*

$$\begin{aligned}\Pi_5 =& \mathbf{E}\left[1 - 2(\lambda_1 - \lambda_2)\left(Y_{1j}(n) - \frac{1}{4}\right) + 2(\lambda_1 + \lambda_2)\left(Y_{2j}(n) - \frac{1}{4}\right)\right] \\ & \times\left[1 + 2(\lambda_2 + \lambda_3)\left(Y_{3j}(n) - \frac{1}{4}\right) + 2(\lambda_2 - \lambda_3)\left(Y_{4j}(n) - \frac{1}{4}\right)\right] - 1 \\ =& -4\lambda_1^2(x_n + z_n) + 4\lambda_2^2 z_n - 8(\lambda_1^2 + \lambda_2^2)\lambda_2\left(v_n - \frac{1}{4}x_n\right) \\ & - 8(\lambda_1^2 - \lambda_2^2)\lambda_2\left(w_n - \frac{1}{4}x_n\right).\end{aligned}$$

(ix) $\mathbf{E}Z_3(n)Z_4(n) = 1 + d\Pi_6 + \frac{d(d-1)}{2}\Pi_6^2 + O(x_n^3)$, *where*

$$\begin{aligned}\Pi_6 =& \mathbf{E}\left[1 + 2(\lambda_2 + \lambda_3)\left(Y_{3j}(n) - \frac{1}{4}\right) + 2(\lambda_2 - \lambda_3)\left(Y_{4j}(n) - \frac{1}{4}\right)\right] \\ & \times\left[1 + 2(\lambda_2 - \lambda_3)\left(Y_{3j}(n) - \frac{1}{4}\right) + 2(\lambda_2 + \lambda_3)\left(Y_{4j}(n) - \frac{1}{4}\right)\right] - 1 \\ =& -4\lambda_3^2(\overline{x}_n + \overline{z}_n) + 4\lambda_2^2\overline{z}_n - 8(\lambda_2^2 + \lambda_3^2)\lambda_2\left(\overline{v}_n - \frac{1}{4}\overline{x}_n\right) \\ & + 8(\lambda_2^2 - \lambda_3^2)\lambda_2\left(\overline{w}_n - \frac{1}{4}\overline{x}_n\right).\end{aligned}$$

3.2. Main expansions of x_{n+1} and $\overline{z}_{n+1}$

In this section, we investigate the second order recursive relations associated with x_{n+1} and $\overline{z}_{n+1}$, with the assistance of the following identity

$$\frac{a}{s+r}=\frac{a}{s}-\frac{ar}{s^2}+\frac{r^2}{s^2}\frac{a}{s+r}. \tag{7}$$

Plugging $a=Z_1(n)$, $r=Z_1(n)+Z_2(n)+Z_3(n)+Z_4(n)-1$, and $s=1$ into equation (7), by the definition of x_n and equation (6), we have

$$\begin{aligned} & x_{n+1}+\frac{1}{4} \\ =&\mathbf{E}\frac{Z_1(n)}{Z_1(n)+Z_2(n)+Z_3(n)+Z_4(n)} \\ =&\mathbf{E}Z_1(n)-\mathbf{E}Z_1(n)\left(Z_1(n)+Z_2(n)+Z_3(n)+Z_4(n)-1\right) \\ &+\mathbf{E}\left(Z_1(n)+Z_2(n)+Z_3(n)+Z_4(n)-1\right)^2\frac{Z_1(n)}{Z_1(n)+Z_2(n)+Z_3(n)+Z_4(n)}. \end{aligned} \tag{8}$$

Next, plugging $a=Z_3(n)$, $r=Z_1(n)+Z_2(n)+Z_3(n)+Z_4(n)-1$, and $s=1$ in equation (7), by the definition of $\overline{z}_n$ and an analogous derivation as equation (6), we can obtain

$$\begin{aligned} & \overline{z}_{n+1}+\frac{1}{4} \\ =&\mathbf{E}Z_3(n)-\mathbf{E}Z_3(n)\left(Z_1(n)+Z_2(n)+Z_3(n)+Z_4(n)-1\right) \\ &+\mathbf{E}\left(Z_1(n)+Z_2(n)+Z_3(n)+Z_4(n)-1\right)^2\frac{Z_3(n)}{Z_1(n)+Z_2(n)+Z_3(n)+Z_4(n)}. \end{aligned} \tag{9}$$

Finally, plugging the results of Section 3.1 into equation (8) and equation (9), and then taking substitutions of

$$\mathcal{X}_n=x_n+\overline{z}_n \quad \text{and} \quad \mathcal{Z}_n=-\overline{z}_n,$$

we obtain a two-dimensional recursive formula of the linear diagonal canonical form:

$$\begin{cases} \mathcal{X}_{n+1}=d\lambda_1^2\mathcal{X}_n+\frac{d(d-1)}{2}\left(-4\lambda_1^4\mathcal{X}_n^2+8\lambda_1^2\lambda_2^2\mathcal{X}_n\mathcal{Z}_n\right)+R_x+R_z+V_x \\ \mathcal{Z}_{n+1}=d\lambda_2^2\mathcal{Z}_n+\frac{d(d-1)}{2}\left[\lambda_1^4\mathcal{X}_n^2-8\lambda_2^4\mathcal{Z}_n^2+\frac{1}{4}\lambda_3^4(\overline{x}_n-\overline{y}_n)^2\right]-R_z+V_z \end{cases} \tag{10}$$

where

$$\begin{aligned} R_x&=\mathbf{E}\left(\frac{Z_1(n)}{\sum_{i=1}^4 Z_i(n)}-\frac{1}{4}\right)\frac{\left(\sum_{i=1}^4 Z_i(n)-4\right)^2}{16}, \\ R_z&=\mathbf{E}\left(\frac{Z_3}{\sum_{i=1}^4 Z_i(n)}-\frac{1}{4}\right)\frac{\left(\sum_{i=1}^4 Z_i(n)-4\right)^2}{16}, \\ |V_x|,|V_z|&\leqslant C_V x_n^2\left(\left|\frac{u_n}{x_n}-\frac{1}{4}\right|+\left|\frac{w_n}{x_n}-\frac{1}{4}\right|+x_n\right)+C_V\overline{x}_n^2\left(\left|\frac{\overline{w}_n}{\overline{x}_n}-\frac{1}{4}\right|+\overline{x}_n\right) \end{aligned}$$

where C_V is an absolute constant.

4. Concentration analysis

In order to study the stability of the dynamical system (10), we show that R_x, R_z, V_x, and V_z are just small perturbations, in the following two lemmas. The proof of Lemma 4.1 resembles that of Lemma 9 in Liu and Ning [18] and is skipped for conciseness.

Lemma 4.1. *Assume $|\lambda_2|\geqslant\varrho>0$ and $|\lambda_1|/|\lambda_2|\geqslant\kappa$ for some $\kappa>1$. For any $\varepsilon>0$, there exist $N=N(\kappa,\varepsilon)$ and $\delta=\delta(\kappa,\varrho,\varepsilon)>0$, such that if $n\geqslant N$ and $\overline{x}_n\leqslant x_n\leqslant\delta$, then*

$$|R_x|,|R_z|\leqslant\varepsilon x_n^2.$$

The following lemma improves the result of Lemma 2.1 (c) by establishing the strict positivity of the sum of x_n and z_n.

Lemma 4.2. *Assume $\lambda_1 \neq 0$. For any nonnegative $n \in \mathbb{Z}$, we always have*

$$x_n + z_n > 0.$$

Proof. In Lemma 2.1 we proved that $x_n + z_n \geqslant 0$, so it suffices to exclude the equality. Now let us apply reductio ad absurdum and assume $x_n + z_n = 0$ for some $n \in \mathbb{N}$. Similar to the derivation in Lemma 2.1 (a) and (b), one can obtain that

$$\mathbf{E}(X_1(n) - X_2(n))^2 = 2\mathbf{E}(X_1(n))^2 - 2\mathbf{E}X_1(n)X_2(n) = x_n + z_n = 0.$$

For any configuration set A on the nth level, we always have

$$\mathbf{P}(\sigma_\rho = 1 \mid \sigma(n) = A) = \mathbf{P}(\sigma_\rho = 2 \mid \sigma(n) = A).$$

Denote the leftmost vertex on the nth level by $v_n(1)$, and it follows that

$$\mathbf{P}(\sigma_\rho = 1 \mid \sigma_{v_n(1)} = 1) = \mathbf{P}(\sigma_\rho = 2 \mid \sigma_{v_n(1)} = 1).$$

Define the transition matrices at distance s by $U_s = M^s_{1,1}$, $V_s = M^s_{1,2}$, and $W_s = M^s_{1,3}$. Then we have the following recursive system

$$\begin{cases} U_s = p_0 U_{s-1} + p_1 V_{s-1} + 2p_2 W_{s-1} \\ V_s = p_1 U_{s-1} + p_0 V_{s-1} + 2p_2 W_{s-1}. \end{cases}$$

The difference of the above two equations evolves as

$$U_s - V_s = \lambda_1 (U_{s-1} - V_{s-1}),$$

and then considering that $U_0 = 1$ and $V_0 = W_0 = 0$, we have

$$U_s - V_s = \lambda_1^s. \tag{11}$$

Finally, from the reversible property of the channel, we can conclude that

$$\lambda_1^n = U_n - V_n = \mathbf{P}(\sigma_\rho = 1 \mid \sigma_{v_n(1)} = 1) - \mathbf{P}(\sigma_\rho = 2 \mid \sigma_{v_n(1)} = 1) = 0,$$

i.e., $\lambda_1 = 0$, a contradiction to the assumption that $\lambda_1 \neq 0$.

The following lemma ensures that x_n does not drop too fast.

Lemma 4.3. *Suppose that there exists an integer $N > 0$, such that $x_n \geqslant \overline{x}_n$ when $n \geqslant N$. For any $\varrho > 0$, if $\min\{|\lambda_1|, |\lambda_2|\} \geqslant \varrho$, then there exists a constant $\gamma = \gamma(\varrho, N) > 0$ such that*

$$x_{n+1} \geqslant \gamma x_n.$$

Proof. Different to the definition of $Y_{ij}(n) = f_n(i, \sigma_j^1(n+1))$ which is the posterior probability that σ_{u_j} takes value i given the random configuration $\sigma_j^1(n+1)$ on spins in $\mathbb{T}_{u_j} \cap L(n+1)$, we consider a configuration set A on $\mathbb{T}_{u_1} \cap L(n+1)$ and define the posterior function $g_{n+1}(1, A)$ as

$$\begin{aligned} g_{n+1}(1, A) =& \mathbf{P}(\sigma_\rho = 1 \mid \sigma_1(n+1) = A) \\ =& \frac{1}{4} + p_0\left(f_n(1, A) - \frac{1}{4}\right) + p_1\left(f_n(2, A) - \frac{1}{4}\right) + p_2 \sum_{i=3,4}\left(f_n(i, A) - \frac{1}{4}\right) \\ =& \frac{1}{4} + \frac{\lambda_2 + \lambda_1}{2}\left(f_n(1, A) - \frac{1}{4}\right) + \frac{\lambda_2 - \lambda_1}{2}\left(f_n(2, A) - \frac{1}{4}\right). \end{aligned}$$

Setting $A = \sigma_1^1(n+1)$, by Lemma 2.3, we have

$$\begin{aligned} \mathbf{E}g_{n+1}(1, \sigma_1^1(n+1)) =& \frac{1}{4} + \frac{\lambda_2 + \lambda_1}{2}\mathbf{E}\left(Y_{11}(n) - \frac{1}{4}\right) + \frac{\lambda_2 - \lambda_1}{2}\mathbf{E}\left(Y_{21}(n) - \frac{1}{4}\right) \\ =& \frac{1}{4} + \lambda_1^2 x_n + (\lambda_1^2 - \lambda_2^2) z_n. \end{aligned}$$

Apparently, we have the following inequalities (see Mézard and Montanari [23]), regarding the estimator $g_{n+1}(1, \sigma_1^1(n+1))$ and the maximum-likelihood estimator:

$$\begin{aligned}\mathbf{EP}(\sigma_\rho = 1 \mid \sigma_1^1(n+1)) &\leqslant \mathbf{E}\max_{1\leqslant i\leqslant 4}\mathbf{P}(\sigma_\rho = i \mid \sigma(n+1)) = \mathbf{E}\max_{1\leqslant i\leqslant 4} X_i(n+1)\\ &\leqslant \frac{1}{4} + \left(\mathbf{E}\max_i\left(X_i(n+1)-\frac{1}{4}\right)^2\right)^{1/2}\\ &\leqslant \frac{1}{4} + \left(\mathbf{E}\sum_{i=1}^4\left(X_i(n+1)-\frac{1}{4}\right)^2\right)^{1/2}\\ &\leqslant \frac{1}{4} + x_{n+1}^{1/2},\end{aligned}$$

where the last inequality follows from the condition that $\bar{x}_{n+1} \leqslant x_{n+1}$. Therefore,

$$\frac{1}{4} + \lambda_1^2 x_n + (\lambda_1^2 - \lambda_2^2) z_n \leqslant \frac{1}{4} + x_{n+1}^{1/2}.$$

If $\lambda_1^2 \geqslant \lambda_2^2$, then it is concluded from $x_n \geqslant -z_n \geqslant 0$ in Lemma 2.1 that

$$\lambda_2^2 x_n \leqslant \lambda_2^2 x_n + (\lambda_1^2 - \lambda_2^2)(x_n + z_n) = \lambda_1^2 x_n + (\lambda_1^2 - \lambda_2^2) z_n \leqslant x_{n+1}^{1/2}.$$

If $\lambda_1^2 \leqslant \lambda_2^2$, then $\lambda_1^2 x_n \leqslant x_{n+1}^{1/2}$, since $z_n \leqslant 0$. To sum up, we always have

$$\min\{\lambda_1^2, \lambda_2^2\} x_n \leqslant x_{n+1}^{1/2}. \tag{12}$$

Under the condition that $x_{n+1} \geqslant \bar{x}_{n+1}$, it can be concluded from the dynamical system (10), Lemma 4.1, and the following inequalities achieved in Lemma 2.1

$$\left|\frac{u_n}{x_n} - \frac{1}{4}\right| \leqslant 1 \quad \text{and} \quad \left|\frac{w_n}{x_n} - \frac{1}{4}\right| \leqslant 1, \tag{13}$$

that there exists a $\delta = \delta(\varepsilon) > 0$ such that when $x_n < \delta$ one has

$$\mathcal{X}_{n+1} + \mathcal{Z}_{n+1} = x_{n+1} \geqslant (d\min\{\lambda_1^2, \lambda_2^2\} - \varepsilon) x_n.$$

Under the condition that $\min\{|\lambda_1|, |\lambda_2|\} \geqslant \varrho$ for any $\varrho > 0$, set $\varepsilon = \varrho^2$ and then we further obtain

$$(d\min\{\lambda_1^2, \lambda_2^2\} - \varepsilon) x_n \geqslant (d-1)\varrho^2 x_n \geqslant \varrho^2 x_n.$$

On the other hand, if $x_n \geqslant \delta$, by equation (12), one has

$$x_{n+1} \geqslant (\min\{\lambda_1^2, \lambda_2^2\} x_n)^2 \geqslant \varrho^4 \delta x_n.$$

Finally, by Lemma 4.2, it follows that $x_n \geqslant x_n + z_n > 0$, and thus $\frac{x_{n+1}}{x_n} > 0$ for all n. Therefore, taking

$$\gamma = \gamma(\varrho, N) = \min_{n=0,1,2,\ldots,N}\left\{\varrho^2, \varrho^4\delta, \frac{x_{n+1}}{x_n}\right\} > 0$$

completes the proof.

The following lemma provides the crucial concentration estimates of $u_n - \frac{x_n}{4}$ and $w_n - \frac{x_n}{4}$, when x_n is small.

Lemma 4.4. *Assume $|\lambda_2| \geqslant \varrho > 0$ and $|\lambda_1|/|\lambda_2| \geqslant \kappa$ for some $\kappa > 1$. For any $\varepsilon > 0$, there exist $N = N(\kappa, \varepsilon)$ and $\delta = \delta(\kappa, \varrho, \varepsilon) > 0$, such that if $n \geqslant N$ and $\bar{x}_n \leqslant x_n \leqslant \delta$, one has*

$$\left|\frac{u_n}{x_n} - \frac{1}{4}\right| < \varepsilon, \quad \left|\frac{w_n}{x_n} - \frac{1}{4}\right| < \varepsilon \quad \text{and} \quad \left|\frac{\overline{w}_n}{\bar{x}_n} - \frac{1}{4}\right| < \varepsilon.$$

As a result, we have the estimates

$$|V_x|, |V_z| \leqslant \varepsilon x_n^2.$$

Proof. It follows from Lemma 2.2 (d) and (e) that

$$\begin{aligned}&\mathbf{E}\left(f_n(3,\sigma^1(n))-\frac{1}{4}\right)\left(f_n(4,\sigma^1(n))-\frac{1}{4}\right)\\ =&\frac{1}{4}\overline{y}_n+\frac{1}{2}\left(u_n-\frac{1}{4}x_n\right)+\frac{3}{2}\left(v_n-\frac{1}{4}x_n\right)-\left(\overline{w}_n-\frac{1}{4}\overline{x}_n\right)\end{aligned}$$

and

$$\mathbf{E}\left(f_n(3,\sigma^1(n))-\frac{1}{4}\right)\left(f_n(4,\sigma^1(n))-\frac{1}{4}\right)=\frac{1}{4}\overline{y}_n-\left(\overline{v}_n-\frac{1}{4}\overline{x}_n\right).$$

Then by Lemma 2.1 (a) we have

$$\left(v_n-\frac{1}{4}x_n\right)-\left(w_n-\frac{1}{4}x_n\right)+\left(\overline{v}_n-\frac{1}{4}\overline{x}_n\right)-\left(\overline{w}_n-\frac{1}{4}\overline{x}_n\right)=0. \tag{14}$$

By the definitions of v_n, w_n, $\overline{v}_n$, and $\overline{w}_n$, and by symmetry, it follows that

$$\left(v_n-\frac{1}{4}x_n\right)-\left(w_n-\frac{1}{4}x_n\right)=0 \quad\text{and}\quad \left(\overline{v}_n-\frac{1}{4}\overline{x}_n\right)-\left(\overline{w}_n-\frac{1}{4}\overline{x}_n\right)=0. \tag{15}$$

Plugging $a=\left(Z_1(n)-\frac{1}{4}\sum_{i=1}^4 Z_i(n)\right)^2$, $r=\left(\left(\sum_{i=1}^4 Z_i(n)\right)^2-16\right)$, and $s=\frac{1}{16}$ into equation (7), we have

$$\begin{aligned}u_{n+1}=&\mathbf{E}\frac{\left(Z_1(n)-\frac{1}{4}\sum_{i=1}^4 Z_i(n)\right)^2}{\left(\sum_{i=1}^4 Z_i(n)\right)^2}\\ =&\frac{1}{16}\mathbf{E}\left(Z_1(n)-\frac{1}{4}\sum_{i=1}^4 Z_i(n)\right)^2\\ &-\frac{1}{256}\mathbf{E}\left(Z_1(n)-\frac{1}{4}\sum_{i=1}^4 Z_i(n)\right)^2\left(\left(\sum_{i=1}^4 Z_i(n)\right)^2-16\right)\\ &+\frac{1}{256}\mathbf{E}\frac{\left(Z_1(n)-\frac{1}{4}\sum_{i=1}^4 Z_i(n)\right)^2}{\left(\sum_{i=1}^4 Z_i(n)\right)^2}\left(\left(\sum_{i=1}^4 Z_i(n)\right)^2-16\right)^2.\end{aligned} \tag{16}$$

The first expectation of equation (16) will contribute to the major terms of the expansion:

$$\begin{aligned}&\mathbf{E}\left(Z_1(n)-\frac{1}{4}\sum_{i=1}^4 Z_i(n)\right)^2\\ =&\mathbf{E}(Z_1(n)-1)^2-\frac{1}{2}\mathbf{E}(Z_1(n)-1)\left(\sum_{i=1}^4 Z_i(n)-4\right)+\frac{1}{16}\mathbf{E}\left(\sum_{i=1}^4 Z_i(n)-4\right)^2\\ =&4d\lambda_1^2x_n+4d(\lambda_1^2-\lambda_2^2)z_n+16d\lambda_1^2\lambda_2\left(u_n-\frac{x_n}{4}\right)+O(x_n^2),\end{aligned}$$

where Lemma 3.5 is used in the last equity and the following derivations. Similarly, we can bound both the second and third terms of equation (16) by $O(x_n^2)$:

$$\mathbf{E}\left(Z_1(n)-\frac{1}{4}\sum_{i=1}^4 Z_i(n)\right)^2\left(\left(\sum_{i=1}^4 Z_i(n)\right)^2-16\right)=O(x_n^2),$$

and

$$\mathbf{E}\left(\left(\sum_{i=1}^4 Z_i(n)\right)^2-16\right)^2=O(x_n^2).$$

Considering that $\mathcal{X}_n = x_n + \bar{z}_n$ and $\mathcal{Z}_n = -\bar{z}_n$, the dynamical system (10) yields that

$$x_{n+1} = d\lambda_1^2 x_n + d(\lambda_1^2 - \lambda_2^2) z_n + O(x_n^2).$$

Equation (16) gives

$$u_{n+1} = \frac{x_{n+1}}{4} + d\lambda_1^2\lambda_2 \left(u_n - \frac{x_n}{4}\right) + O(x_n^2), \tag{17}$$

and then

$$\frac{u_{n+1}}{x_{n+1}} - \frac{1}{4} = d\lambda_1^2\lambda_2 \frac{x_n}{x_{n+1}} \left(\frac{u_n}{x_n} - \frac{1}{4}\right) + O\left(\frac{x_n^2}{x_{n+1}}\right). \tag{18}$$

Next display the discussion in the $\mathcal{X}O\mathcal{Z}$ plane. First consider the case that $|\lambda_1|/|\lambda_2| \geqslant \kappa$ for $\kappa > 1$. In a small neighborhood of $(0,0)$, since $d\lambda_2^2 < \kappa^2 d|\lambda_2^2| \leqslant d\lambda_1^2 < 1$ and $\mathcal{X}_n > 0$, the discrete trajectory approaches the origin point in a way that is "tangential" to the $\mathcal{X}$-axis, when x_n is small enough (see Bernussou and Abatut [3]). Furthermore, the conclusion of Lemma 4.2 excludes the possibility that the trajectory moves along the $\mathcal{Z}$-axis. Then for some $M > 1$, there exist constants $N_1 = N_1(\kappa, M)$ and $\delta_1 = \delta_1(\kappa, M)$, such that if $n \geqslant N_1$ and $x_n \leqslant \delta_1$, we have

$$\mathcal{X}_n \geqslant M\mathcal{Z}_n \quad \text{and} \quad \frac{1}{M(M+1)} d\lambda_1^2 x_n + O(x_n^2) > 0,$$

where the remainder term $O(x_n^2)$ comes from the expansion of x_{n+1}. Consequently, it follows

$$x_n + z_n = \mathcal{X}_n \geqslant \frac{M}{M+1}(\mathcal{X}_n + \mathcal{Z}_n) = \frac{M}{M+1} x_n,$$

and by the fact that $z_n \leqslant 0$ then

$$\begin{aligned} \frac{x_n}{x_{n+1}} &= \frac{x_n}{d\lambda_1^2 x_n + d(\lambda_1^2 - \lambda_2^2) z_n + O(x_n^2)} \leqslant \frac{x_n}{\frac{M}{M+1} d\lambda_1^2 x_n + O(x_n^2)} \\ &\leqslant \frac{x_n}{\left(1 - \frac{1}{M}\right) d\lambda_1^2 x_n} = \frac{M}{M-1} \frac{1}{d\lambda_1^2}. \end{aligned} \tag{19}$$

For fixed k, by the fact that $\frac{1}{4}\lambda_3^4(\bar{x}_n - \bar{y}_n)^2$ can be bounded by $O(x_n^2)$ for the reason that $|\bar{x}_n| > |\bar{y}_n|$ implied in Lemma 2.1 (b) and (c), it is known from the dynamical system (10) that

$$|x_{n+1} - (d\lambda_1^2 \mathcal{X}_n + d\lambda_2^2 \mathcal{Z}_n)| \leqslant C x_n^2.$$

Furthermore, one has

$$x_{n+1} \leqslant (d\lambda_1^2 \mathcal{X}_n + d\lambda_2^2 \mathcal{Z}_n) + C x_n^2 \leqslant (d\lambda_1^2 + C x_n) x_n,$$

and then there exists $\delta_2 = \delta_2(\kappa, M, k) < \delta_1$, such that if $x_n < \delta_2$ then for any $1 \leqslant \ell \leqslant k$ one has $x_{n+\ell} < 2\delta_2$. Therefore, for any positive integer k, equation (18) yields

$$\begin{aligned} \frac{u_{n+k}}{x_{n+k}} - \frac{1}{4} &= d\lambda_1^2\lambda_2 \frac{x_{n+k-1}}{x_{n+k}} \left(\frac{u_{n+k-1}}{x_{n+k-1}} - \frac{1}{4}\right) + O\left(x_{n+k-1} \frac{x_{n+k-1}}{x_{n+k}}\right) \\ &= (d\lambda_1^2\lambda_2)^k \left(\prod_{\ell=1}^{k} \frac{x_{n+\ell-1}}{x_{n+\ell}}\right) \left(\frac{u_n}{x_n} - \frac{1}{4}\right) + R, \end{aligned}$$

where, by equation (18) and with C denoting the O constant therein,

$$|R| \leqslant 2C\delta_2 \left(\sum_{i=1}^{k} \left(\frac{M}{M-1} \frac{1}{d\lambda_1^2}\right)^i (d\lambda_1^2 |\lambda_2|)^{i-1}\right) \leqslant \delta_2 \frac{1 - \left(\frac{M}{M-1}|\lambda_2|\right)^k}{1 - \left(\frac{M}{M-1}|\lambda_2|\right)} \frac{M}{M-1} \frac{1}{d\lambda_1^2},$$

and by equation (19)

$$(d\lambda_1^2\lambda_2)^k \left(\prod_{\ell=1}^{k} \frac{x_{n+\ell-1}}{x_{n+\ell}}\right) \leqslant (d\lambda_1^2|\lambda_2|)^k \left(\frac{M}{M-1} \frac{1}{d\lambda_1^2}\right)^k = \left(\frac{M}{M-1}|\lambda_2|\right)^k.$$

Firstly, from Lemma 2.1 (a) one has $0 \leqslant \frac{u_n}{x_n} \leqslant 1$, which implies that $\left|\frac{u_n}{x_n} - \frac{1}{4}\right| < 1$. Secondly, by the fact that $|\lambda_2| \leqslant |\lambda_1| \leqslant d^{-1/2} \leqslant 1/\sqrt{2}$, it is possible to achieve $\frac{M}{M-1}|\lambda_2| < 1$ by choosing $M = 4$. Therefore, we can conclude that it is feasible to take $k = k(\varepsilon)$ sufficiently large and $\delta_3 = \delta_3(\kappa, k, \varepsilon) = \delta_3(\kappa, \varepsilon) < \delta_2$ sufficiently small to guarantee that

$$\left|\frac{u_{n+k}}{x_{n+k}} - \frac{1}{4}\right| < \varepsilon.$$

Finally, under the condition that $|\lambda_2| \geqslant \varrho > 0$, by Lemma 4.3, we know that there exists $\gamma = \gamma(\varrho)$ such that $x_{n-k} \leqslant \gamma^{-k} x_n$. Thus, we can choose $N = N(\kappa, \varepsilon, k) = N(\kappa, \varepsilon) > N_1 + k$ and $\delta = \gamma^k \delta_3$, such that if $x_n \leqslant \delta$ and $n \geqslant N$ then

$$\left|\frac{u_n}{x_n} - \frac{1}{4}\right| < \varepsilon. \tag{20}$$

The second part of the lemma can be shown similarly as above.

5. Proof of main theorem

First, consider $\varrho \leqslant |\lambda_2| \leqslant |\lambda_1|$ for any fixed $\varrho > 0$. To investigate the non-tightness, it would be convenient to assume that $1 > d\lambda_1^2 \geqslant d\lambda_2^2 \geqslant \frac{1}{2}$, say, $|\lambda_1| \geqslant \frac{1}{\sqrt{2d}}$. We take $\varrho = \frac{1}{\sqrt{2d}}$ in the following context. Consider $|\lambda_2| > \varrho$ fixed and just λ_1 varying, and without loss of generality, assume $d\lambda_1^2 > \frac{1+d\lambda_2^2}{2}$. Consequently choose $\kappa = \kappa(d, \lambda_2) = \left(\frac{1+d\lambda_2^2}{2d\lambda_2^2}\right)^{1/2} > 1$ and thus $|\lambda_1|/|\lambda_2| \geqslant \kappa$.

By the definition of non-reconstruction in equation (2.4), it suffices to show that when $d\lambda_1^2$ is close enough to 1, $\mathcal{X}_n$ does not converge to 0 for the reason that it implies that x_n does not converge to 0 considering $0 \leqslant \mathcal{X}_n = x_n + z_n \leqslant x_n$. We apply reductio ad absurdum, by assuming that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \bar{x}_n = 0. \tag{21}$$

Therefore, there exists $\mathcal{N}_1 = \mathcal{N}_1(d)$, such that whenever $n > \mathcal{N}_1$, we have $x_n \leqslant \delta$. Next, recalling that $\mathcal{X}_n = x_n + \bar{z}_n$, we further define $\overline{\mathcal{X}}_n = \bar{x}_n + \bar{z}_n$. Then by the symmetry of the model, we can obtain the dynamical form for $\overline{\mathcal{X}}_n$ analogously as the dynamical form for $\mathcal{X}_n$ in equation (10):

$$\overline{\mathcal{X}}_{n+1} = d\lambda_3^2 \overline{\mathcal{X}}_n + \frac{d(d-1)}{2}\left(-4\lambda_3^4 \overline{\mathcal{X}}_n^2 + 8\lambda_3^2\lambda_2^2 \overline{\mathcal{X}}_n \mathcal{Z}_n\right) + R_{\bar{x}} + R_z + V_{\bar{x}}$$

where $R_{\bar{x}}$ and $V_{\bar{x}}$ are counterparts of R_x and V_x simply by replacing x by $\bar{x}$.

Then we display the discussion in the $\mathcal{X}O\overline{\mathcal{X}}$ plane. Since $|\lambda_1| > |\lambda_3|$ and $\mathcal{X}_n, \overline{\mathcal{X}}_n \to 0$ as $n \to \infty$ from equation (21), in a small neighborhood of $(0, 0)$, the discrete trajectory approaches the origin point in a way that is "tangential" to the $\mathcal{X}$-axis. Furthermore, the conclusion of Lemma 4.2 excludes the possibility that the trajectory moves along the $\overline{\mathcal{X}}$-axis. Therefore, it implies that there exists $\mathcal{N} = \mathcal{N}(d) > \mathcal{N}_1$, such that whenever $n > \mathcal{N}$,

$$\overline{\mathcal{X}}_n \leqslant \mathcal{X}_n, \quad \text{that is,} \quad \bar{x}_n \leqslant x_n. \tag{22}$$

From the proof of Lemma 4.4, we know that in the $\mathcal{X}O\mathcal{Z}$ plane there exist $N = N(\kappa, \varrho) > \mathcal{N}$ and $\delta = \delta(d, \kappa, \varrho) > 0$, such that if $n \geqslant N$ and $x_n \leqslant \delta$, then in the small neighborhood of $(0, 0)$, we have

$$\mathcal{X}_n \geqslant 4\mathcal{Z}_n \quad \text{that is,} \quad \mathcal{X}_n \geqslant \frac{4}{5} x_n \tag{23}$$

By equation (22), applying Lemma 4.1, and taking $\varepsilon = \frac{4}{25}\frac{d(d-1)}{4}\lambda_1^4$, one can obtain

$$|R_z| \leqslant \frac{4}{25}\frac{d(d-1)}{4}\lambda_1^4 x_n^2 \leqslant \frac{1}{4}\frac{d(d-1)}{4}\lambda_1^4 \mathcal{X}_n^2.$$

Next by the result of Lemma 4.4 that $\left|\frac{u_n}{x_n} - \frac{1}{4}\right| < \varepsilon'$ and $\left|\frac{w_n}{x_n} - \frac{1}{4}\right| < \varepsilon'$ for any $\varepsilon' > 0$, now we take $\varepsilon' = \frac{1}{12C_V}\frac{d(d-1)}{4}\lambda_1^4$. Therefore, by equation (10) and the condition that $\lambda_1 \geqslant \lambda_2$, we have

$$\begin{aligned}
\mathcal{Z}_{n+1} &= d\lambda_2^2 \mathcal{Z}_n + \frac{d(d-1)}{2}\left[\lambda_1^4 \mathcal{X}_n^2 - 8\lambda_2^4 \mathcal{Z}_n^2 + \frac{1}{4}\lambda_3^4(\bar{x}_n - \bar{y}_n)^2\right] - R_z + V_z \\
&\geqslant d\lambda_2^2 \mathcal{Z}_n + \frac{d(d-1)}{2}\left[\lambda_1^4 \mathcal{X}_n^2 - 8\lambda_2^4 \mathcal{Z}_n^2\right] - R_z + V_z \\
&\geqslant d\lambda_2^2 \mathcal{Z}_n + \frac{d(d-1)}{2}\left[\frac{1}{2}\lambda_1^4 \mathcal{X}_n^2 + \frac{1}{2}\lambda_1^4 16\mathcal{Z}_n^2 - 8\lambda_2^4 \mathcal{Z}_n^2\right] - R_z + V_z
\end{aligned} \tag{24}$$

$$\geqslant d\lambda_2^2 \mathcal{Z}_n + \frac{d(d-1)}{4}\lambda_1^4 \mathcal{X}_n^2 - |R_z| - C_V x_n^2 \left(\left| \frac{u_n}{x_n} - \frac{1}{4} \right| + \left| \frac{w_n}{x_n} - \frac{1}{4} \right| + x_n \right),$$

$$\geqslant d\lambda_2^2 \mathcal{Z}_n + \frac{1}{2}\frac{d(d-1)}{4}\lambda_1^4 \mathcal{X}_n^2,$$

$$\geqslant \mathcal{Z}_n \left[d\lambda_2^2 + \frac{d(d-1)}{2}\lambda_1^4 \mathcal{X}_n \right].$$

Note that the initial point $x_0 = 1 - \frac{1}{4} = \frac{3}{4} > 0$ and Lemma 4.3 implies that there exists $\gamma = \gamma(\varrho, \mathcal{N}) = \gamma(d)$ such that $x_n \geqslant x_0 \gamma^n$. Define $\varepsilon = \varepsilon(d) = \left(\frac{x_0 \gamma^N}{10}\right)^2 > 0$. Because ε is independent of λ_1, considering that $d\lambda_2^2$ sufficiently close to 1, we can choose $|\lambda_1| < d^{-1/2}$ such that

$$d\lambda_2^2 + \frac{d(d-1)}{2}\lambda_1^4 \varepsilon > 1. \tag{25}$$

Noting that $\frac{d(d-1)}{2}\lambda_1^4 \geqslant \left(\frac{d\lambda_1^2}{2}\right)^2 \geqslant \frac{1}{16}$, equation (24) implies that

$$\mathcal{Z}_{N+1} \geqslant \frac{1}{2}\frac{d(d-1)}{4}\lambda_1^4 \mathcal{X}_N^2 \geqslant \frac{1}{4}\frac{1}{16}\frac{16}{25}x_n^2 \geqslant \left(\frac{x_0 \gamma^N}{10}\right)^2 = \varepsilon.$$

Suppose $\mathcal{Z}_n \geqslant \varepsilon$ for some $n > N$, and it follows from equations (24) and (25) that

$$x_{n+1} \geqslant \mathcal{Z}_{n+1} \geqslant \mathcal{Z}_n \left[d\lambda_2^2 + \frac{d(d-1)}{2}\lambda_1^4 \varepsilon \right] > \mathcal{Z}_n \geqslant \varepsilon.$$

Therefore, by induction we have $x_n \geqslant \mathcal{Z}_n \geqslant \varepsilon$ for all $n > N$, which contradicts to the assumption imposed in equation (21). Thus, the proof is completed.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

We give special thanks to Joseph Felsenstein and Sébastien Roch for conversations on this subject. We also give special thanks the anonymous reviewers and the Editor for their constructive comments and efforts that greatly improved the quality of this paper. The research of Ning Ning was partially supported by NSF grant DMS-1761603.

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