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TOLERANCING FOR AN APPLE PIE

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ABSTRACT

Tolerancing began with the notion of limits imposed on the dimensions of realized parts both to maintain functional geometric dimensionality and to enable cost-effective part fabrication and inspection. Increasingly however, component fabrication depends on more than part geometry as many parts are fabricated as a result of a "recipe" rather than dimensional instructions for material addition or removal. Referred to as process tolerancing, this is the case, for example, with IC chips. In the case of tolerance optimization, a typical objective is cost minimization while achieving required functionality or "quality." This paper takes a different look at tolerances, suggesting that rather than ensuring merely that parts achieve a desired functionality at minimum cost, the underlying goal of product design is to make money, more is better and tolerances comprise additional design variables amenable to optimization in a decision theoretic framework. We further recognize that tolerances introduce additional product attributes that relate to product characteristics such as consistency, quality, reliability and durability. These important attributes complicate the computation of the expected utility of candidate designs, requiring additional computational steps for their determination. The resulting theory of tolerancing illuminates the assumptions and limitations inherent to Taguchi's loss function. We illustrate the theory using the example of tolerancing for an apple pie, which conveniently demands consideration of tolerances on both quantities and processes, and the interaction among these tolerances.

NOMENCLATURE

M A set of statements describing a particular design configuration
$\mathbf{x}$ A set of statements such as dimensions describing the measurable and differentiable variables that determine a basic design
$\mathbf{T}$ A set of real numbers describing tolerances on the variables $\mathbf{x}$
τ A set of attributes related to tolerances that affect demand for a product
$\mathbf{a}$ A set of attributes that determine the demand for a product
q Demand for a product
P Price at which a product is sold
R Revenue generated by the sale of a product
C Costs associated with the production of a product
$\mathbf{y}$ A set of statements that describe uncertainties on other variables
u Utility
$E\{u\}$ Expected utility
H Hessian matrix
v Eigenvalues of the Hessian matrix
$\mathbf{v}$ Eigenvectors of the Hessian matrix
V Net present value of profit
L Loss incurred because variables $\mathbf{x}$ do not achieve their target values

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r Discount rate
t Time

1 Introduction

It is likely that the modern concepts of tolerancing have their origins in the notion of interchangeability of parts [1,2]. Such concepts date back over half a millennium as Gutenberg's press (1450's) relied on interchangeable letters. Over the ensuing years, it became clear that making parts interchangeable is not as easy as one might expect. Nonetheless, the emergence of steam power in the 1780's demanded that parts be made with challenging accuracy. Benjamin Franklin reported, in 1785, of a French gunsmith making muskets with interchangeable parts. And, 100 years later with emergence of the Industrial Age, mass manufacturing on an assembly line required part interchangeability.

Parker [3,4], working at the Royal Torpedo Factory in Scotland, is credited by Liggett [5] with being the first to formally address "position tolerance theory." Since that time, tolerance theory has emerged as a major sub-discipline of engineering design and manufacturing. In the earlier years, tolerances were mainly associated with part geometry resulting in the discipline of geometric tolerancing. The need to properly interpret part specifications led to standards for dimensional tolerancing [6] and, with the emergence of computers, Requicha and Voelcker [7–10] developed a theory of geometric modeling that enabled computer-aided design (CAD).

A key problem in the setting of tolerances is referred to as the problem of "stack-up" [11]. This problem occurs when a series of parts must fit or work together within an overall tolerance. Problems of this sort led to the notion of optimizing the allocation of the individual part tolerances to achieve the overall desired tolerance at minimum cost [12–14].

A major contributor to a theory of tolerancing is Taguchi [15]. His philosophy may be summarized in four statements: "It is better to be precise and inaccurate than being accurate and imprecise; Quality should be designed into the product and not inspected; Quality is achieved by minimizing the deviation from the target; [and] The cost of quality should be measured as a function of the deviation from target." [16] This philosophy resulted in the concept of the *Taguchi loss function*.

More recently, it is noted that several products are described not so much by their dimensions as by a "recipe" according to which they are manufactured. This is the case of IC chips and food products such as an apple pie. In these cases, tolerances largely determine the quality, lifetime or reliability of the product. These are important attributes not often captured by product descriptions as they can significantly impact the proclivity of consumers to purchase a product. Again, recognizing that demanding narrower tolerances results in higher costs, several researchers have sought to meet a set of performance requirements at minimum cost [17,18].

The problem with minimizing manufacturing cost is that this objective results in the trivial solution of manufacturing none of the product. If the manufacturer manufactures no product, the manufacturing cost is $0.00. This, obviously, is not a helpful solution. To render the solution helpful, it is then necessary either to impose constraints on the optimization problem or to change the objective. Constraints typically take the form of a set of product requirements, whereas an alternative objective may seek minimum cost per item produced. Hazelrigg and Saari [19] note that constraints only remove alternatives from the allowable set of design choices and, if they remove the optimal point, that is, if the constraints are active, they always penalize performance. Thus, for optimal design, constraints should be avoided to the extent possible. A way to avoid constraints is to change the objective function to one that more accurately reflects the preference of the responsible decision maker. Noting that the underlying objective of a profit-making organization is to make money, Hazelrigg [20] presents a framework for product design optimization with this objective that also accounts for uncertainty.

The purpose of this paper is to show that the basic logic of Hazelrigg's framework, with minor modification, can be applied to the optimization of both geometric and process tolerances separately or concurrently with the product design, with the objective being the maximization of a measure of net revenue or profit. The medium used to illustrate this application is the tolerancing of an apple pie. Although the optimization framework is designed to an objective of profit maximization, it is conveniently adaptable to other valid preferences.

2 A General Framework for Tolerancing

The underlying tenet of this paper is that the purpose of tolerancing is to increase the value of a product to the producer of the product. This is a sensible tenet for a number of reasons. First, it is the producer of the product who decides what the tolerances should be and, for rationality, this choice must be based on a preference of the decision maker. Second, for a product that has multiple consumers, it is, in general, not possible to express a joint consumer preference that would enable rational choice of tolerances [21]. Third, for most products, the consumers are too far removed from the technical aspects of a product to care about tolerances or even understand them. Fourth, vendors or parts suppliers have conflicting interests with the producer and, for this and other reasons, cannot be left to select the tolerances on the parts they produce.

With this in mind, the framework that shall be used here is a modification of Hazelrigg's framework as shown in Fig. 1. There are three entry points in this framework: description of a baseline design, specification of a set of beliefs defined as "exogenous" variables that define the extant uncertainty, and the expression of a preference from which we will be able to determine a utility measure. A design configuration is given by a set of statements, M, that provide a detailed description of the product

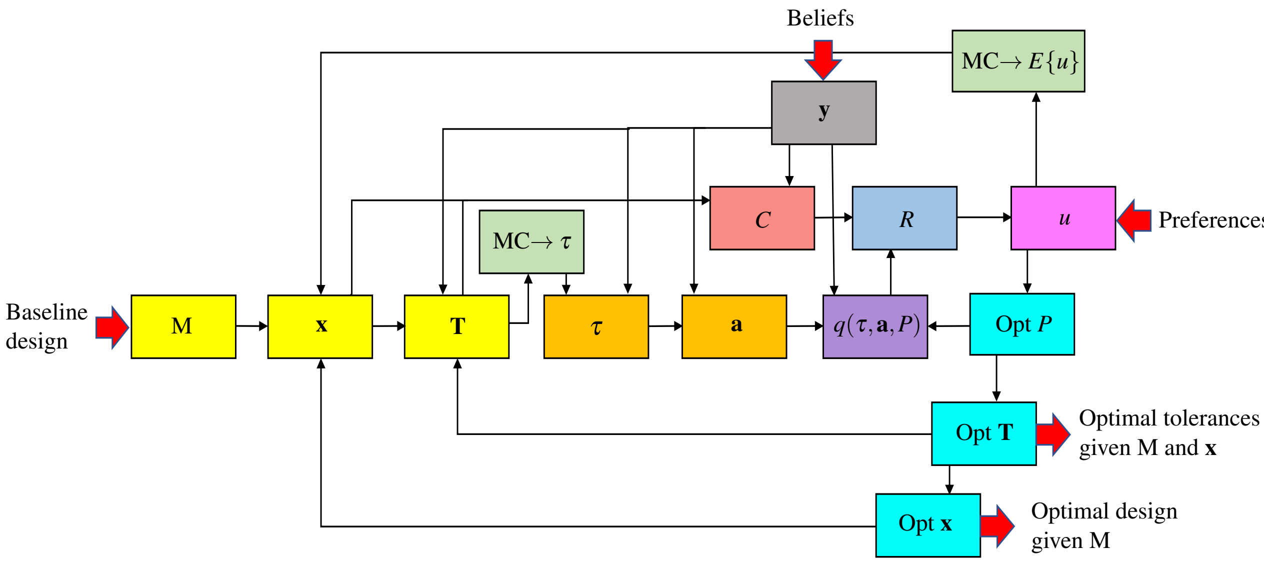

FIGURE 1. A framework for optimal tolerance design.

or system such as "the airplane has two engines." The variables $\mathbf{x}$ comprise a set of continuous and differentiable numerical values providing specifications such as dimensions, weights, volumes and other quantities. Together, M and $\mathbf{x}$ complete the specification of the basic design (including manufacture, distribution and sales, warranty, buy-back and disposal, etc.). Tolerances are typically applied to the continuous and differentiable variables $\mathbf{x}$, and not to the descriptive statements M. In order to analyze tolerances, we separate out their description, $\mathbf{T}$, and their resulting attributes, τ, from the basic design performance attributes, $\mathbf{a}$. Because the tolerances, $\mathbf{T}$, must be nonzero, the achieved values of $\mathbf{x}$ are random variables, deviating from their target values, $\mathbf{x}_0$. The complete specification of the design enables computation of the performance attributes of the product, and those attributes and price, P, which is set by the manufacturer, determine demand, q as a function of time, t. Then, demand and price determine revenue, R. The design specification also enables determination of all costs, C. Then revenue minus cost equals profit and thus, given a risk preference, utility, u. Expected utility, $E\{u\}$, namely the quantity that we wish to maximize, is determined via a Monte Carlo simulation. We now maximize expected utility with respect to price before comparing this design alternative to other alternatives. Next, we choose the numerical values of the design specification to maximize expected utility for the given configuration, and finally we can compare alternative configurations. With this simple overview, we'll now look at the elements of this framework in more detail.

Without denying the possibility of a producer having alternative preferences, the following is based on the notion that the underlying preference of a producer is to make money, and more is better. A full preference consists of three parts [22], the fundamental preference, taken here to be for money, a time preference and a risk preference. A time preference is generally expressed through discounting, and the risk preference is expressed through the curve of utility versus money. The net value derived from a product, corrected for its time value (discounted), is given by

$$V = \sum_{Today}^{\infty} \frac{R-C}{(1+r)^t} \tag{1}$$

where V is the net present value of profits, t is time per period and r is the discount rate per time interval ($r = 0$ infers that equal sums of money have equal value independent of when they are received). Revenues are generated by selling things, costs are generated by buying things. Normally, one sells the product produced, and the revenue generated at time t by the sale of the product is $R(t) = q_s(t)P_s(t)$, where $q_s(t)$ is the quantity sold at time t and $P_s(t)$ is the price at which it is sold. It is possible, however, that production of the product produces other salable items (things that may appear to be "waste"), and the revenue generated by their sale should be included.

Risk preferences derive from a decision-maker's willingness to wager on an uncertain return. Generally this is calculated as a function of utility and presented as an Arrow-Pratt [23,24] measure of absolute risk aversion (ARA).

$$\rho(V) = -\frac{d^2u(V)/dV^2}{du(V)/dV} \tag{2}$$

where $u(V)$ denotes the "utility" of V and $\rho(V)$ is a measure of

ARA. Thus, if $\rho(V)$ is positive the individual is risk averse, if it is negative the individual is risk proverse, and if it is zero the individual is risk neutral. Utility is a cardinal measure commonly determined via a decision-maker's response to a von Neumann-Morgenstern lottery [25]. Utility is typically a random variable, whereas expected utility is a deterministic ordinal variable. Thus, the use of expected utility as an objective for optimization converts a non-deterministic optimization function into a deterministic objective function as is mathematically required for its existence [26].

The quantity of product sold at each point in time depends on the demand for the product, which is a function of its attributes and its price. The attributes of the product are a result of its design and its tolerances. Variability in products is the result of nonzero tolerances. The more nearly identical that each individual product is to a nominal product, the more predictable it will be, and predictability of a product may itself be an attribute of concern to customers, frequently referred to as the product's "quality" [27–29]. For example, customers are often concerned about getting a "lemon," particularly in the purchase of a car, and they show this preference by paying more for cars that have good reliability reports.

3 The Mathematics of Tolerances

Referring again to Fig. 1, the elements of M describe the configuration of a product or system. These elements typically are not differentiable nor are tolerances applied to them. The variables, $\mathbf{x}$, on the other hand, are differentiable and typically are assigned tolerances. This differentiation between M and $\mathbf{x}$, together with the notion that variations in $\mathbf{x}$ because of nonzero tolerances will be small, enables us to write the expected utility of a design in the form of a Taylor series in a region near a reference design $\mathbf{x}_0$.

$$E\{u(\mathbf{x})\} - E\{u(\mathbf{x}_0)\} = \left.\frac{\partial E\{u\}}{\partial \mathbf{x}}\right|_{\mathbf{x}_0} \delta\mathbf{x} + \frac{1}{2}\delta\mathbf{x}^{\mathrm{T}} \left.\frac{\partial^2 E\{u\}}{\partial \mathbf{x}^2}\right|_{\mathbf{x}_0} \delta\mathbf{x} + \cdots \tag{3}$$

$\delta\mathbf{x} = (\mathbf{x} - \mathbf{x}_0)$, including deviations in $\mathbf{x}$ resulting from nonzero tolerances. Note that we write the Taylor series in terms of the performance variable $E\{u\}$. This differs from the work of Zhang et. al. [30] and Tarcolea [27], for example, who write the series in terms of the Taguchi loss function. While these authors then assume that the first order term is zero since the loss function achieves a minimum at the target value, when viewed as written in Eq. 3, this would not appear to be the case in general. In order that the first order term in Eq. 3 be zero, it is necessary that the expected utility of the design be maximized with respect to $\mathbf{x}$, as the condition of design optimality for the variables $\mathbf{x}$ is $\partial E\{u\}/\partial \mathbf{x} = 0$. This assures that the second order term, which is, in fact, a negative loss (that is, a benefit) term, dominates and that the loss is axisymmetric and quadratic about the optimum point $\mathbf{x}_0$. Thus, it would appear that, in order that tolerances be sensibly set, the design must first be optimized with respect to $\mathbf{x}$. If this were not the case, it could lead to solutions that encourage tolerances that essentially change the design, that is, that would result in very large values of $\delta\mathbf{x}$. That this phenomenon does not occur when writing the Taylor series in terms of the loss function is likely because this expansion misrepresents the true loss. We now write Eq. 3 in a slightly different form:

$$E\{u(\mathbf{x}_0)\} - E\{u(\mathbf{x})\} = -\left.\frac{\partial E\{u\}}{\partial \mathbf{x}}\right|_{\mathbf{x}_0} \delta\mathbf{x} - \frac{1}{2}\delta\mathbf{x}^{\mathrm{T}} \left.\frac{\partial^2 E\{u\}}{\partial \mathbf{x}^2}\right|_{\mathbf{x}_0} \delta\mathbf{x} - \cdots \tag{4}$$

where, if $\mathbf{x}_0$ is a maximizing solution for $E\{u\}$, then the loss function for small $\delta\mathbf{x}$ becomes

$$L \approx \frac{1}{2}\delta\mathbf{x}^{\mathrm{T}} \left\{ \left.\frac{\partial^2[-E\{u\}]}{\partial \mathbf{x}^2}\right|_{\mathbf{x}_0} \right\} \delta\mathbf{x} \tag{5}$$

where the $\delta\mathbf{x}$ are the result of non-zero tolerances. Note that $\partial^2[-E\{u\}]/\partial\mathbf{x}^2 = \mathrm{H}$ is a positive definite Hessian matrix. Therefore, a surface of constant loss forms an n-dimensional hyperellipsoid, with the value of the loss dependent on the eigenvalues and eigenvectors of the Hessian matrix. Furthermore, whereas typical formulations of loss functions in the case of tolerances on multiple elements of $\mathbf{x}$ tend to treat the losses as independent of each other, this formulation shows that, in general, they are not independent.

So far, we have recognized that non-zero tolerances, $\mathbf{T}$, impart losses to the value of a system or product. This alone would prompt a selection of $\mathbf{T} = \mathbf{0}$. However, countering this, tolerances come with a cost, and the smaller the tolerance, the higher the cost. Accordingly, the total loss is the sum of the loss function, Eq. 5, and the cost of the tolerance $\mathbf{C_T}(\mathbf{T})$,[1]

$$L_{tot} \approx \frac{1}{2}\delta\mathbf{x}^{\mathrm{T}} \left\{ \left.\frac{\partial^2[-E\{u\}]}{\partial \mathbf{x}^2}\right|_{\mathbf{x}_0} \right\} \delta\mathbf{x} + \mathbf{i}\,\mathbf{C_T}(\mathbf{T}) \tag{6}$$

where the row vector $\mathbf{i} = [1, 1, ..., 1]$. As noted in Section 5, it should be expected that the distribution of $\delta\mathbf{x}$ would be correlated with $\mathbf{C_T}(\mathbf{T})$. At this point, one might be inclined to minimize L_{tot}. But this would fail to account for the risk preference of the decision maker. Instead, one should maximize $E\{u\}$. This is a simple task only if the decision maker is risk neutral, that is, if utility equals profit, or if the variation in u because of the non-zero tolerances is sufficiently small that this is a reasonable approximation. Unfortunately, the latter is not always the case

[1] L_{tot} may contain additional terms relating to expenses such as warranty costs and liability costs.

as the quality attributes of a product have the potential to significantly affect demand. Hence, the selection of optimal tolerances is not as easy as the Taguchi method would have us believe.

Before proceeding further, it is appropriate to consider the terms that comprise the Hessian matrix in Eq. 6. This matrix provides an estimate of the revenue loss because of a degradation (real or perceived) in product quality, reflected as a shift in the demand curve. Taking the variation in demand to be a continuous, differentiable function of the quality attributes, we can write the demand function in the form of a Taylor series,

$$q(\tau) = q_0 + \frac{\partial q}{\partial \tau}\delta\tau - e\frac{q_0}{P_0}\delta P + \cdots \tag{7}$$

where $e = -(P/q)(\delta q/\delta P)$ is the price elasticity. To minimize the loss from reduced demand, depicted as the shaded area in Fig. 2, we must re-optimize the price.

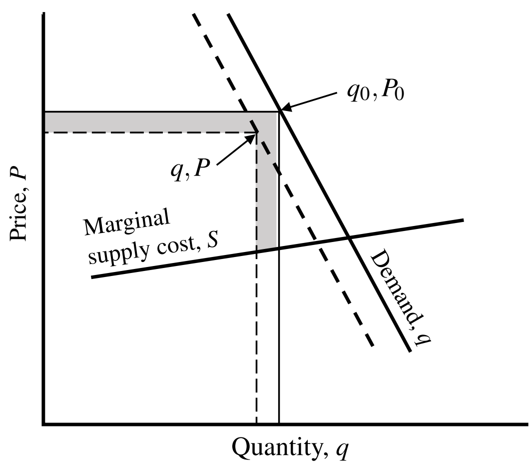

FIGURE 2. Profit loss resulting from a quality loss.

Let $\Delta q = (\partial q/\partial\tau)\delta\tau$ denote the shift in the demand curve resulting from a nonzero tolerance specification, α_S be the increase in marginal production cost per unit, that is, it is slope of the marginal supply cost curve, and S_0 be the marginal supply cost at a production rate of q_0. Then, the loss is given by

$$L = -q_0\delta P - (P_0 - S_0 + \delta P)\delta q + \frac{\alpha_S}{2}\delta q^2 \tag{8}$$

where

$$\delta q = -e\frac{q_0}{P_0}\delta P - \Delta q \tag{9}$$

Thus, L is a quadratic function of δP. Solving for the minimum loss, the optimum value of δP is given by

$$\delta P = -P_0\frac{\Delta q(P_0 + \alpha_S e q_0) - P_0 q_0(1-e) - e q_0 S_0}{e q_0(2P_0 + \alpha_S e q_0)} \tag{10}$$

This yields the interesting, albeit intuitive, result that, for high demand elasticity (somewhat greater than unity), increasing tolerances results in lower optimal product prices, whereas for inelastic demand, increasing tolerances results in higher optimal prices. Examples show that optimal adjustment of product price to match the selected tolerances can significantly improve profitability.

4 Computational Procedure

The computational framework follows the logic flow shown in Fig. 1, which outlines a procedure for the optimization of the product design, including tolerances, as a unified process. Unfortunately, for most products, this can lead to a highly complex and time-consuming set of computations. The complexity of the problem makes it desirable to resort to Monte Carlo methods, which sacrifice computational efficiency to achieve a more simplified and less-prone-to-error mathematical formulation. However, even this may leave the problem intractable. As a result, it is desirable to separate the dimensional optimization of the product from the tolerance optimization. The assumptions leading to Eq. 5 enable this separation. Thus, in practice, it is convenient to apply the framework in two steps, first the optimization of the "dimensions" (target values of $\mathbf{x}$) of the product and then, based upon these optimized values, the optimization of the tolerances placed on the target values.

We shall begin our outline of the computational procedure under the assumption that the basic product design has already been optimized. Keep in mind that the validity of Eq. 5 depends on this being the case. Tolerances place "constraints" on the variability of the outcomes of the decisions, $\mathbf{x}$, with a concomitant cost. The goal of the computational procedure is to enable a selection of these constraints such that they maximize the value, measured as the expected utility, of the product to the producer. Under the condition that the basic product design, assuming all $\mathbf{x}$ values achieve their nominal value, is optimized to achieve maximum expected utility, Eqs. 8, 9 and 10 afford some degree of independence from the basic design in the consideration of tolerances. Indeed, in the case that the decision maker is risk neutral, that is, for whom utility equals profit, minimizing the expected loss is a solution. However, minimization of the loss does not assure maximization of expected utility for decision makers who are not risk neutral. Because of this, we are forced to compute a utility difference in the context of the expected value of the basic design. This requires evaluation of the expected utility of the basic design and evaluation of the total loss function as a deviation from the expected utility of the basic design.

The first issue, which would appear to be overlooked in many applications of the Taguchi loss function, is the need to take product price into account as a variable of choice to the manufacturer that also must be optimized. The thing that makes the determination of the optimal price shift tricky is that the quality loss is not realized on a product-by-product basis one-by-one as products come off the production line, but rather on consumer perceptions based on a history of many products produced under the design variables and tolerances of the product. Thus, in order to simulate the demand shift, for each product outcome (achieved values of $\mathbf{x}$ on a product-by-product basis) we must compute the product loss function. This requires the inner Monte Carlo simulation shown in Fig. 1 between the selection of tolerances, $\mathbf{T}$, and their resulting quality measure, τ. This nesting of Monte Carlo loops can result in substantial increases in computational time. One approach to this problem is to assume that there is no significant variability in the outcome of $\mathbf{x}$ on a product-by-product basis, that is, a change large enough to alter the attributes, $\mathbf{a}$, and analyze only the impact of tolerance on one particular product instantiation. This provides an approximate result that can be later checked against a limited number of full simulations around the optimal tolerance design point.

The next problem we encounter is the appropriate expression for the cost of achieving a specific tolerance level. One way to achieve a given tolerance is to test to assure that all tolerances are met, and to discard any parts of or products that fail to meet the tolerance. This results in a cost of wastage. The wastage costs result from the costs of manufacturing unsalable product. It is obviously desirable to keep wastage small. But this means maintaining tolerances with a high per-product probability, and this often demands more sophisticated and concurrently more expensive manufacturing equipment. Ergo, as a tolerance is reduced, the manufacturer must consider the purchase of more expensive manufacturing equipment. The tolerance cost model must reflect these costs.

While tolerances denote the limits of acceptable outcomes of $\mathbf{x}$, they do not describe distribution of these outcomes. Yet, this distribution is needed in order to compute the loss function. While it might seem natural to describe the distributions of outcomes as Gaussian with a mean and standard deviation, the Gaussian distribution has the property that it extends infinitely in both positive and negative directions. This causes problems for two key reasons. First, actual parts don't have negative dimensions, and secondly, actual parts don't get infinitely large. One might think that, for a Gaussian distribution, these are extremely rare occurrences that can be neglected. The problem is, however, that a Monte Carlo analysis will interrogate the distribution hundreds of millions, perhaps even billions, of times, and rare events that will cause errors are bound to occur. Thus, we have chosen to represent tolerances for the analyses presented here as beta distributions, although other distributions can be used within the context of the theory presented here. Beta distributions have finite limits, can be skewed, and they can be shaped based on the distribution parameters α and β.

Lastly, we need to discuss the formulation and determination of the loss function. As noted above, the mathematics of the loss function confirm that it should take the form of an n-dimensional hyperellipsoid. Let the orthogonal unit vectors $\hat{\mathbf{x}}$ correspond to the elements of $\mathbf{x}$ to form a Cartesian coordinate system. Note that the critical point or center of a hyperellipsoid denoting a surface of constant loss is located at the design point $\mathbf{x}_0$, with its axes aligned with the orthogonal eigenvectors, $\mathbf{v}$, of the Hessian matrix H. Denote by $\tilde{\mathbf{x}}$ the coordinates of a point $\mathbf{x}$ in the rotated and translated coordinate system defined by $\mathbf{v}$. The critical point of the hyperellipsoid is at the center of this coordinate system. Then, the loss function corresponding to Eq. 5 is given by the equation of a hyperellipsiod in the rotated and translated coordinate system,

$$L = \gamma \sum_i v_i \tilde{x}_i^2 \tag{11}$$

where v_i are the eigenvalues of H and γ is a proportionality constant. The coordinates of $\mathbf{x}$ in the translated system are given by $\delta\mathbf{x} = (\mathbf{x} - \mathbf{x}_0)$. Rotating the axes to the eigenvector system gives the coordinates, $\tilde{\mathbf{x}}$, in the eigenvector system.

$$\tilde{\mathbf{x}} = \delta\mathbf{x}^T \hat{\mathbf{v}} \tag{12}$$

where

$$\hat{\mathbf{v}}_i = \frac{\mathbf{v}_i}{\sqrt{\mathbf{v}_i^T \mathbf{v}_i}} \tag{13}$$

are unit vectors comprising a coordinate system where the loss function hyperellipsoids are centered on the coordinate system with their axes aligned with the eigenvectors of the Hessian matrix.

5 Apple Pie

As an illustration of the decision theoretic formulation of the tolerancing problem, we have chosen the tolerancing of an apple pie. The detailed geometric tolerancing of an apple pie would be a formidable task and, in the end, rather futile. No two pies are exactly alike nor would anyone want that they would be. So, geometric tolerancing is not appropriate for a pie, save for, perhaps, the diameter of the pie as it has to fit in a box for marketing purposes. Instead of describing an apple pie by its detailed dimensions, which would involve volumes of numbers, we describe and tolerance an apple pie by its *recipe*. Accordingly, tolerances are placed on the measurable parameters of the recipe, including amounts of ingredients and processing parameters such as baking time and temperature. The apple pie recipe used in this example is given below.

APPLE PIE

Ingredients

½ cup sugar, more to taste
½ cup packed brown sugar
3 tablespoons all-purpose flour
1 teaspoon ground cinnamon
¼ teaspoon ground ginger
¼ teaspoon ground nutmeg
6-7 cups peeled & sliced tart apples
1 tablespoon lemon juice
dough for double-crust pie
1 tablespoon butter
1 large egg white

Process

Preheat oven, 375 deg.
Toss apples with lemon juice, add sugar, toss to coat
Combine sugars, flour and spices
Roll half of dough to ⅛-in.-thick circle,
  transfer to 9-in. pie plate,
  trim even with rim
Add filling, dot with butter
Roll remaining dough to ⅛-in.-thick circle
Place over filling, trim even with rim, seal and flute edge
Cut slits in top
Beat egg white until foamy, brush over crust
Sprinkle with sugar
Cover with foil, bake 25 minutes
Remove foil and bake another 25 minutes
Cool on wire rack

We assume that these ingredient amounts and process variables have been duly optimized and will now examine tolerances on the baking time and temperature. Note that these variables are measurable, continuous and differentiable. To begin, we construct an elliptical penalty function, taking into account the correlation between these variables. Obviously, if the oven temperature is a bit low, a longer cooking time will compensate at least partly for this deviation. Fig. 3 shows an elliptical loss function indicating that an increase in baking time of approximately 6 minutes will compensate optimally for a decrease in baking temperature of 10 degrees. The dashed-line box in Fig. 3 denotes example tolerance limits of ±5 degrees on temperature and ±5 minutes on baking time. If these tolerances were held, pies baked in conditions that exceed these tolerances would be discarded as a loss. However, we see that pies baked in conditions just outside the upper left and lower right corners of this box are classified as waste while they are considerably more acceptable than those being sold that are baked in conditions corresponding to the lower left and upper right corners. Intuitively, a multi-parameter tolerance criteria could enable the tolerances to be relaxed while reducing waste and maintaining or even improving quality. Multi-tolerance criteria can be easily implemented in the context of this tolerance-evaluation framework.

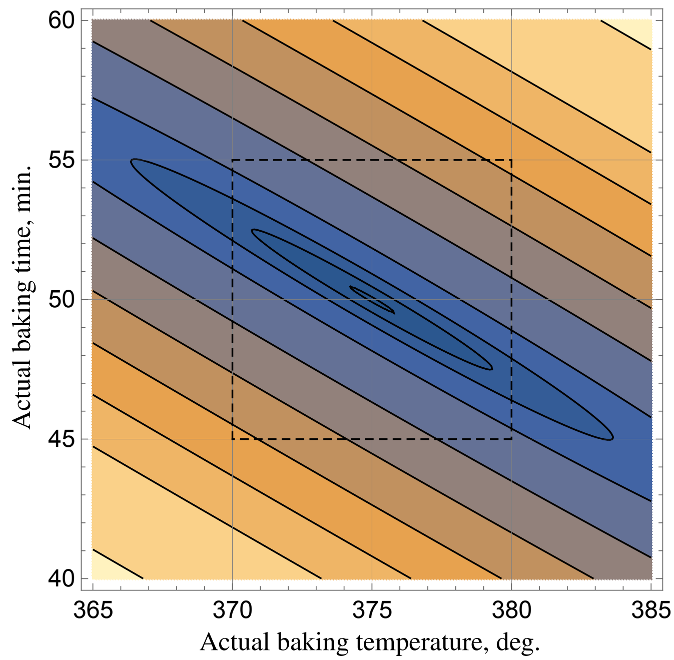

FIGURE 3. Loss as a function of actual baking temperature and time, with target values of 375 degrees and 50 minutes.

Fig 4. shows a commercially baked apple pie that failed to meet reasonable time and temperature baking conditions. The crust is rather burnt and bears the taste of burnt pastry. Clearly, were this the norm, demand for this producer's pie would be significantly reduced. One might be inclined to think that we have been a bit facetious in choosing to go to so much detail to analyze production tolerances on an apple pie. Be assured, however, that this is taken quite seriously in the apple pie baking industry [31–34]. Indeed, detailed studies have been conducted to identify the attributes of importance to apple pie customers, and to estimate how variations in these attributes might effect demand for the product. However, we did not choose variables for our example from the literature. Rather we chose them, while not entirely unreasonable, to emphasize aspects of tolerance optimization that one might encounter in typical manufacturing situations.

For our example, we assumed that the producer expected to be able to sell 10,000 pies per production period, with a demand elasticity of $e = 2.0$, a fixed investment cost (amortization of manufacturing equipment, rent, insurance, etc.) of \$5,000 per

FIGURE 4. A burnt apple pie. It did not taste very good.

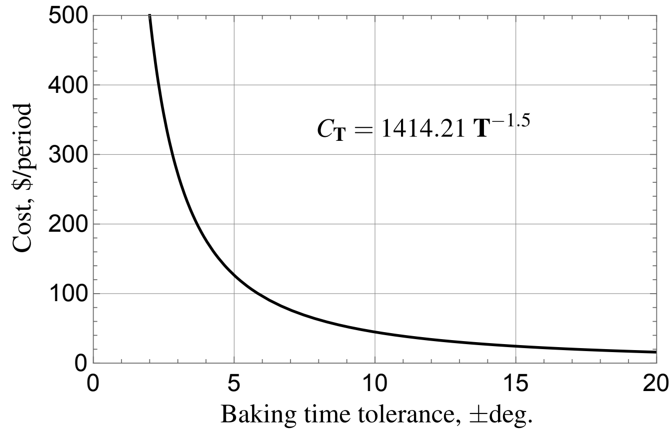

FIGURE 5. Per production period cost of maintaining a time tolerance.

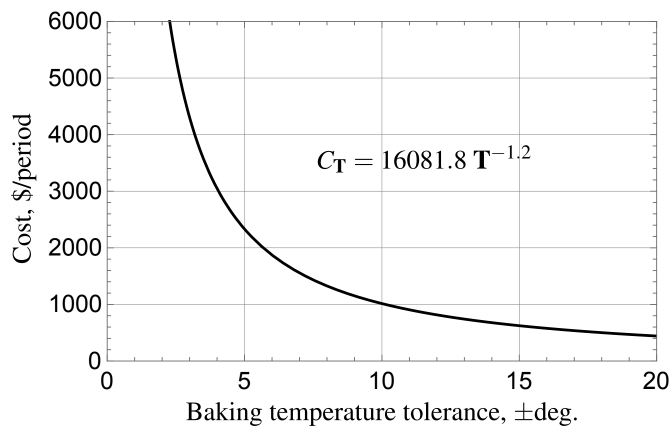

FIGURE 6. Per production period of maintaining a temperature tolerance.

production period, a marginal cost of production per pie of \$2.50 at a production rate of 10,000 pies per production period and increasing at a rate of \$0.001 per 1,000 pies. Tolerances around the target values of baking time and temperature were considered from near zero to ±7.8 minutes and ±29 degrees respectively. The distributions of times and temperatures were modeled as symmetrical beta distributions with parameters $\alpha = 4$ and $\beta = 4$, and with minimum and maximum limits of 25 percent below and above the tolerance limits. This leads to a rejection rate or wastage of about 12 pies per 1,000. These pies are discarded, incurring production costs but producing no revenue. Given this description of the tolerances and with time and temperature distributions dependent on the tolerance limits, it seemed reasonable to model a fixed, per-period cost of maintaining the specified tolerance as a hyperbolic function of the tolerance.

$$C_{\mathbf{T}} = \zeta \mathbf{T}^{\xi} \tag{14}$$

where $\mathbf{T}$ is the tolerance (±minutes or ±degrees), and ζ and ξ are constants. The functions used are shown in Figs. 5 and 6. As time is an easier tolerance to maintain than temperature, we modeled its cost as significantly lower than that for temperature, as the latter would likely demand more expensive ovens.

Finally, we took producer utility to be the log of the net revenue per production period. A simulation was coded that has the ability to take into account uncertainties in all major variables associated with the determination of performance (profit) as a function of tolerances. However, simulations of enough cases to map out expected utilities for even two tolerances, including all uncertainties, can be quite time consuming, as much as days of run time or longer. Thus, for the example provided here, we chose to assume that the demand, demand elasticity, and production costs are known deterministically. With these assumptions, Fig. 7 was produced by computing results for combinations of every combination of baking time and temperature corresponding to the tick marks of this plot. This comprised a total of 600 time-temperature tolerance cases, with 1 million simulations per case. The run time for this was about four hours.

Clearly, computer run times for cases that seek to optimize multiple tolerances with full consideration of uncertainties can be an impediment to application of this approach. Nonetheless, the approach can be used in a "deterministic" mode to locate the regions of optimal solutions, and these can be verified with limited computations in the vicinity of the optimal solutions taking uncertainties into account. The key factor driving high computing time is the need for the solution of a nested Monte Carlo simulation, which could require a total of a billion or more simulations to achieve adequate accuracy.

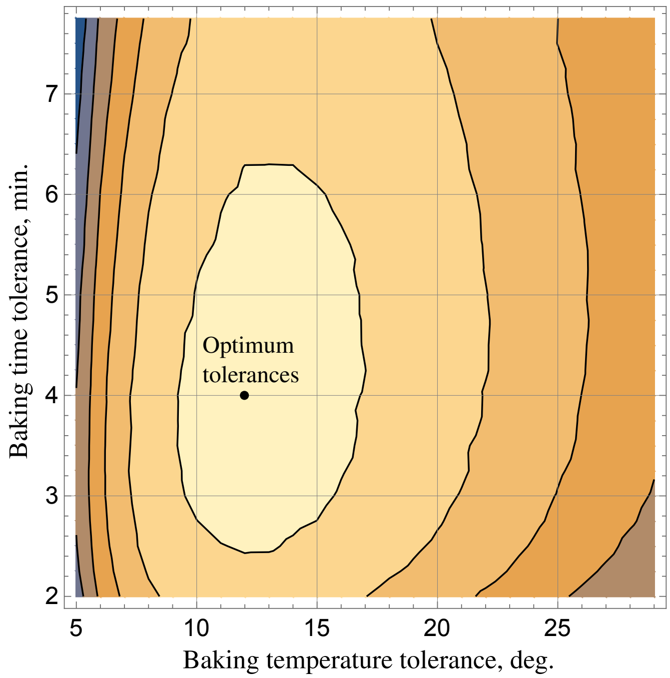

FIGURE 7. Expected utility as a function of baking temperature and time tolerances showing the optimum tolerances.

6 Conclusions

The objective of this research was to cast the problem of tolerancing in the framework of decision theory. It was found that Hazelrigg's design framework could provide a mathematically rigorous basis for a theory of tolerancing with modification to enable the analysis of so-called quality attributes emerging from product-to-product variability. The resulting analysis provides insights into the validity of the Taguchi loss function.

Taguchi defines a loss function that derives from a Taylor series around "target values" of design variables, with arguments that the first-order term of the series is zero because the loss is minimized at the target value. But this argument holds only if the design target values themselves are optimized with respect to an overall system or product value function, and only in the case that these variables are continuously differentiable in the vicinity of these optima, thus validating the Taylor series. Otherwise, the first-order terms do not vanish and, in fact, diminish the concept of a tolerance by allowing larger tolerances to have the potential to improve the product. Although Taguchi recognizes the need for an optimization criterion, it does not appear that this requirement is clearly recognized in applications of the loss function for tolerance optimization.

Secondly, the Taguchi method is most commonly applied assuming that the tolerances themselves are independent of each other. The decision theoretic formulation makes clear that this is not the case.

Third, while the Taguchi loss function treats the cost of tolerancing to the manufacturer and the loss of value to the customers, the decision theoretic formulation makes clear that the important factor is profit or net benefit to the designer/manufacturer. It is this entity that decides what the tolerances should be, reaps the benefits of production, and owns the loss. This entity would likely prefer to maintain a profitable level of demand for the product, whereas nonzero tolerances reduce demand. Through consideration of demand, the decision theoretic approach takes consumer preferences into account, without the need to assess a group preference [35].

Fourth, the loss attributed to diminished demand resulting from non-zero tolerances can be mitigated by re-optimization of the price at which the product is sold. Although this is required for the optimization of tolerances, we see no evidence that it has been considered in applications of the Taguchi loss function.

Fifth, the determination of tolerances in a decision-theoretic framework enables consideration of uncertainties affecting the optimal design of the entire product or system, and it accounts for the risk preference of the design decision maker. In this regard, it should be noted that, although the variation in the product resulting from non-zero tolerances may be small, it still has the potential to result in large losses in product value, thus invalidating the approximation of risk neutrality.

Lastly, we believe that the decision theoretic formulation of the tolerancing problem provides significant new insight into the mathematics of tolerancing and appears to encompass a range of tolerancing problems that span geometric dimensional tolerancing through process tolerancing.

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