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Hyperbolic Numerical Models for Unsteady Incompressible, Surcharged Stormwater Flows

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Abstract

The transition from open-channel to surcharged flow creates problems for numerical modeling of stormwater systems. Mathematically, problems arise through a discrete shock at the boundary between the hyperbolic Saint-Venant equations and the elliptic incompressible flow equations at the surcharge transition. Physically, problems arise through trapping of air pockets, creation of bubbly flows, and cavitation in rapid emptying and filling that are difficult to correctly capture in one-dimensional (1D) models. Discussed herein are three approaches for modeling surcharged flow with hyperbolic 1D equations: (i) Preissmann Slot (PS), (ii) Two-component Pressure Approach (TPA) and (iii) Artificial Compressibility (AC). Each provides approximating terms that are controlled by model coefficients to alter the pressure wave celerity through the surcharged system. Commonly, the implementation of these models involve slowing the pressure celerity below physical values, which allows the numerical solution to dissipate the transition shock between the free surface and surcharged flows without resorting to extraordinarily small time-steps. The different methods provide different capabilities and numerical implementations that affect their behavior and suitability for different problems.

Keywords: Pipe flow; Saint-Venant equations; Stormwater modeling; Pipe transient

1. INTRODUCTION

With a few notable exceptions, the broad goal of stormwater system design is to create a system that operates *mostly* with free-surface flow. However, the ability to safely operate in conditions that involve pressurization remains relevant, especially as climate change may cause urban areas to experience more intense rain events than historically considered in design criteria. The transition region between free-surface and surcharged (full pipe) conditions has been called ``mixed flow'' to distinguish from purely free-surface or purely surcharged flows (Song et al 1983). From the numerical modeling perspective, the possibility simultaneous free-surface and surcharged flows in a system creates a conundrum: numerical methods that are well-designed for solving the Saint-Venant equations for free-surface flow are typically inapplicable for incompressible surcharged pipe flow, and vice-versa. From a mathematical point of view, the problem stems from our choice of governing equations—the incompressible approximation applied in surcharged pipe leads to elliptic partial differential equations (i.e., a diagnostic problem driven solely by boundary conditions). By contrast, systems operating in free-surface flow are governed by hyperbolic partial differential equations (i.e., prognostic time-marching differential equations that form an initial-boundary value problem). In the free-surface flow the pressure celerity is the gravity wave speed, whereas the surcharged flow has near-instantaneous pressure transmission at an acoustic pressure wave celerity. To make matters worse, when the incompressibility approximation is used with rigid pipe walls and the hydrostatic approximation, the modeled surcharge pressure wave celerity becomes infinite. Thus, numerical models of mixed flows are faced with a pressure celerity shock across at the mixed-flow boundary that must be smoothly handled or it will destabilize the solution.

Fundamentally, it is a hopeless task to try to create a well-founded numerical model that smoothly solves a connection between discrete hyperbolic and elliptic equations—the boundary is mathematically ill-posed. With an implicit solution technique, the free-surface/surcharge shock creates a stiff problem that converges slowly (if at all). When explicit solvers are applied, the shock results in unphysical oscillations at the pressurization interface that lead to numerical instability (Vasconcelos et al. 2009). Formally, these problems are reduced if the unsteady surcharged flow is modeled using the slight compressibility of water and the elasticity of the pipe—i.e., introducing a hyperbolic component to the surcharged equations using an acoustic pressure celerity. Transient-resolving models (such as the method of characteristics) that are used for water hammer in distribution systems are arguably the most rigorous approach for such problems, but are computationally expensive due to their small time-step. These models generally require explicit tracking of pressurization interfaces through an expensive shock-fitting procedure (Cunge et al. 1981).

There is a long tradition of applying hyperbolic solvers to represent near-incompressible surcharged flow, including (1) Preissmann Slot, (2) Two-Component Pressure Approach, and (3) Artificial Compressibility. Underlying all three models is a concept of “transient storage”—i.e., the ``extra'' water that can be stored in a length of pipe of fixed nominal diameter. In the real world, the transient storage is composed of both compression of the water and expansion of the pipe. In the modeling realm, we can approximate the transient storage and its pressure celerity effects in a variety of ways. In the following sections we discuss each of these approximations and close with a comparison of their different interpretations.

2. PREISSMANN SLOT (PS)

The Preissmann Slot (PS) is the first mixed-flow model proposed in the literature, presented in detail by Cunge and Wegner (1964) apparently based on an idea of Preissmann. It is also arguably the simplest mixed-flow model, and is found in many established hydraulic models, including SWMM 5.1 and HEC-RAS. The concept behind the PS approach is simple to visualize, as illustrated in Fig. 1. A closed pipe is given an imaginary slot in the crown that runs down the length of the pipe. The slot is imagined as being bounded by walls of infinite height so the pipe can never actually pressurize—the fluid simply rises with a free surface in the slot.

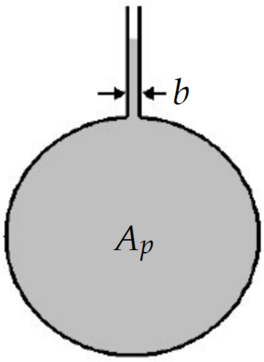

Figure 1. Schematic of the Preissmann Slot

In the PS approach the actual closed-pipe flow is modeled with a free-surface in the imaginary slot and can be represented by the standard Saint-Venant equations (omitted here for brevity). The difference between the height of water in the slot and the soffit (inside crown) of the pipe is the PS model representation of the surcharged head. The critical outcome of a PS model is that the selected pressure celerity (which is associated with transient storage) is given by:

$$c = \sqrt{\frac{gA_p}{b}} \qquad [1]$$

where g is gravitational acceleration and A_p is the pipe area (without the slot). As a consequence, the choice of b, the slot width, controls the celerity in the surcharged pipe and the shock that occurs at the transition from free-surface to surcharged flow. For illustration, consider a hypothetical condition where a pressure pulse travels in a pressurized pipe, raising the pressure head by h_s. Over a time interval of T seconds, the pressure front moves a distance of Tc, hence the “extra water volume” in the slot is given by:

$$\text{Volume} = Tcbh_s = Th_s\sqrt{gA_pb} \qquad [2]$$

which is the PS measure of the transient storage. Both the transient storage and the pressure celerity depend directly on the choice of the slot width (b).

Practical difficulties with the PS method typically stem from selecting too small of a slot width (relative to the pipe size) such that the resulting shock affects the stability of calculations. On the other hand, if the slot is chosen too wide the transient volume may be unrealistically large and the slow pressure celerity behavior will significantly diverge from reality. The traditional PS approach is also limited in that it neglects unique behaviors of the gas phase during rapid filling and rapid emptying events. Under rapid filling conditions, gas slugs and bubbles can become trapped in the liquid phase if the pipe is not adequately ventilated (Vasconcelos and Wright, 2006). The existence and compression of the gas phase effects the true transient storage (Wylie et al, 1993), which (to our knowledge) has not been directly represented in any PS model. Rapid emptying presents a different set of concerns that depend on pipe ventilation. Rapid emptying of an unventilated pipe can create sub-atmospheric closed pipe flow—i.e., a full pipe whose head is less than the piezometric pressure implied by the full pipe soffit. Standard PS models will simply regenerate a free-surface flow whenever sub-atmospheric pressures are present. However, Kerger et al. (2011) demonstrated a modified PS with a "negative slot" that can be used for unventilated sub-atmospheric flows, but this scheme is not yet commonly available in stormwater software. A further challenge in rapid emptying is that pressure drawdown can cause cavitation and degassing of dissolved gasses, forming bubbles and coalescing to slugs over longer times, which cannot be captured with PS methods. The popularity of two-equation SCL class of models to represent mixed flows, initially presented by Song et al (1983), stems from this important limitation; however, these more advanced approaches also require the application of shock-fitting algorithms that are computationally expensive. An alternative that addresses both rapid filling and rapid emptying problems problem is to split the pressure terms in two components, as is done in the Two-component Pressure Approach, discussed below.

3. TWO-COMPONENT PRESSURE APPROACH (TPA)

The Two-component Pressure Approach (TPA) was proposed to overcome the PS model limitations in representing sub-atmospheric transient flows during mixed-flow conditions (Vasconcelos et al. 2006). The strategy in TPA models is to separate the pressure component that results from the presence of the water in the conduit cross-section (i.e., hydrostatic pressure) from the pressure component that would be anticipated only in the case of pressurized flows (which is an analog to the depth of the water in the Preissmann Slot). The TPA method assumes that, due to the pipe wall elasticity, the cross-sectional area of the flow (A) can deviate from the nominal cross-sectional area of the pipe (A_p) by a small value ($A - A_p$) when flow becomes pressurized. Unlike the PS model, the pressure wave celerity is linked with the surcharge pressure by:

$$c = \sqrt{g h_s \left(\frac{A_p}{A - A_p} \right)} \qquad [3]$$

The surcharge pressure will create an increase of the cross-sectional area by $A - A_p$, which is governed by standard pipe elasticity equations and parameters (not presented for brevity). After a time interval T, the extra volume of water is:

$$\text{Volume} = c\,T\,(A - A_p) = T\sqrt{g A_p h_s (A - A_p)} \qquad [4]$$

which is the TPA measure of transient storage. Comparing to eq. [2], we see the PS transient storage scales on $h_s b^{1/2}$ where the slot width is the model parameter, whereas the TPA scales on $h_s^{1/2}(A - A_p)^{1/2}$ with the expansion determined by pipe elasticity as the model parameter. An important aspect of the TPA is that a sub-atmospheric (negative) surcharge pressure (h_s) will be accompanied by a negative transient storage, $A_p / (A - A_p)$, but the combination provides a positive pressure celerity.

Although the original TPA does not account for a gas phase fraction affected by filling and emptying, a variation of the model proposed by Vasconcelos and Marwell (2011) provides an approach that is analogous to the Discrete Gas Cavity Model (DGCM, Wylie et al 1993). Because the celerity is affected by a drop in pressure as the air fraction increases, the calculation procedure for this modified TPA approach is iterative, but this extra computational work enables to represent the low pressures observed during cavitation that are missing from the PS method and from Artificial Compressibility, discussed below.

4. ARTIFICIAL COMPRESSIBILITY (AC)

Instead of creating transient storage in an imaginary space, i.e., as in the PS, or expanding the pipe, as in TPA, the Artificial Compressibility (AC) method uses an artificial compression of the fluid—making the modeled water *more* compressible than real water. The approach creates an "imaginary time" over which transient storage is temporarily represented by compressing the fluid while retaining inelastic sidewalls. The goal is the AC method is to use the hyperbolic equations of compressible flow *without* the full equation of state for the fluid (Chorin, 1967). The motivation is quite simple: pressure pulses move at acoustic celerities that require a very small time step if we use the correct compressibility of water. However, if we treat water as a somewhat *more* compressible fluid, then the acoustic celerity is reduced, and a larger time step can be taken. Over a longer time-interval, both the correct compressibility and the artificial compressibility will arrive at the same quasi-steady conditions that are approximated by the incompressible flow equations. Note that the AC method is equally applicable in both surcharged and open channel flow.

The governing equations for the AC method in 1D flow are a somewhat modified version of the Saint-Venant equations, as derived in detail for mixed flow in Hodges (2020). The numerical method uses a dual-time-stepping scheme, where real time (t) is discretized with a backwards (implicit) stencil and pseudo-time (τ) is discretized with an explicit time march following Rogers et al. (1991) for multi-dimensional flow. Because this approach is unusual, it is useful to briefly present the conservation of mass and momentum. Neglecting lateral inflows these are:

$$\frac{\partial}{\partial \tau}(HA) = -D_h F^2 \left\{\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x}\right\} \quad [5]$$

$$\frac{\partial Q}{\partial \tau} = \left\{-\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + gA\frac{\partial \eta}{\partial x} - gAS_f\right\} \quad [6]$$

where A is the cross-sectional flow area, Q is the volumetric flow rate, η is the piezometric head, S_f is the friction slope, D_h is a modified hydraulic depth, and F is a Froude-like number related to the effective artificial celerity, γ, where $F = \gamma\ (gD_h)^{-1/2}$. In eqs. [5] and [6] the terms in the braces are the standard Saint-Venant equations and the other terms are the pseudo-time derivatives and parameters of the AC method. In solving this coupled equation set, the real time (t) derivatives are typically discretized with an implicit stencil and the pseudo-time (τ) derivatives are discretized with an explicit time march. In each real-time step, the explicit time march in pseudo-time is continued until the derivatives on the left-hand-side (LHS) of eqs. [5] and [6] vanish or reach some acceptably small residual. Note that in a surcharged pipe with inelastic sidewalls and without a Preissmann Slot, the dA/dt term must be exactly zero, but is retained in eq. [5] for use in the free-surface portion of mixed-flow conditions. The F parameter introduced by Hodges (2020) sets the response rate of the surcharge head to volume compression, which affects the pseudo-time step needed for stability. A fully-converged solution of the AC method (i.e., machine zero for LHS) for surcharged flow is a discrete solution that exactly satisfies momentum and enforces incompressible continuity.

Although the foundations of the AC method are almost as old as the PS method, its application to surcharged pipes and mixed flow is relatively recent (Hodges, 2020). As such, its characteristics and behaviors for these conditions are still under investigation. Unlike the PS and TPA methods, the AC method does *not* provide an obvious relationship between transient storage and celerity; i.e., we cannot (as yet) write an equation for the AC method that is equivalent to eqs. [2] and [4] relating celerity and transient volume. Indeed, in the limit as the LHS of eqs. [5] and [6] vanish, the AC method has zero transient storage and infinite pressure celerity in surcharged pipe; i.e., it converges to the incompressible-flow inelastic-pipe solution. However, experience to date has shown that this fully-converged solution is computationally impractical for mixed-flow conditions because of the celerity shock occurring at the interface between the free-surface and surcharged sections. That is, the AC method uses the hyperbolic equations so the solution procedure itself is smooth, but the converged condition is a celerity shock that can be difficult to capture without directly invoking shock-capturing schemes. As a practical matter, the pseudo-time march of the AC method will be stopped with some residual, providing a transient storage volume that depends on the integration of the fluxes over pseudo-time. For example, in a simple finite-volume formulation with Q_u and Q_d as upstream and downstream face fluxes, the transient storage volume would be:

$$\text{Volume} = \int_0^{\tau_f} Q_u d\tau - \int_0^{\tau_f} Q_d d\tau \quad [7]$$

where τ_f is the pseudo-time cutoff. Eq. [7] provides the net flux in/out of a finite volume during a single real-time step of the pseudo-time march. The relationship between the pseudo-time residual cutoff, the transient storage volume, and the effective celerity in real time will depend on the implicit stencils used for the real-time derivatives and the algorithm for the explicit pseudo-time march. As yet, these relationships are not well understood.

5. CONCLUSIONS

The Preissmann Slot, Two-component Pressure Approach, and Artificial Compressibility methods are three different approaches to a similar end: creating a transient storage term so that a hyperbolic equation can be used to model surcharged pipe flow and, more importantly, to smoothly capture the mixed flow transition from free-surface to surcharged flow. The general advantage of these approaches is that the mixed-flow boundary is represented as a boundary between two hyperbolic equations rather than coupling hyperbolic and elliptic equations (which is required when the incompressible approximation is directly used). The models differ in both their implementation and the parameters that affect the relationship between storage and surcharged head. The PS method needs but a single parameter: the slot width. In contrast, the TPA method uses an estimate of the pressure celerity to represent the gain in cross sectional area created by parameters controlling pipe elastic deformation. Finally, the AC method uses two parameters: an artificial celerity and a cutoff residual for pseudo-time iteration. Arguably, the PS method is the simplest to invoke and has the advantage of treating surcharged flow with the same equations as have traditionally been used for free-surface flow. However, the PS method has a reputation for being persnickety as few Saint-Venant solvers have been designed to handle the sharp celerity shock that occurs at the mixed flow transition—but this is perhaps an indictment of the Saint-Venant solvers rather than the PS method, per se. Both the TPA and AC methods introduce modified forms of the Saint-Venant equations that are applicable across both free-surface and surcharged flow, which helps them smoothly handle the celerity shock. Both the TPA and the modified PS (Kerger et al, 2011) can correctly handle unventilated sub-atmospheric flows, but this remains an uninvestigated area for AC methods. An important advantage of the TPA over both AC and PS is that the modified TPA (Vasconcelos and Marwell, 2011) provides more realistic behaviors for rapid filling and rapid emptying than has been demonstrated with the other methods. However, the advanced TPA implementation requires an iterative process to recompute the local celerity, which adds to the computational effort; it is not clear whether the TPA iterative effort is greater than or less than the additional pseudo-time iterations required in AC methods. The key disadvantages of the AC approach are (i) it has relatively limited flexibility in the discrete approach, (ii) the pseudo-time solution for the free-surface sections is typically slower than using a traditional real-time marching scheme, and (iii) the relationships between the artificial celerity, the time-marching residual and transient storage are not well understood. The key advantage of the AC method is that its ability to strictly control the transient storage by iterating in pseudo-time allows the method to be used to drive the solution to either the true transient storage or towards the incompressible ideal without sacrificing stability.

Computing mixed-flow conditions for stormwater systems is likely to remain a challenge for numerical models and modelers alike. The PS, TPA, and AC approaches can make the celerity shock at the mixed-flow boundary more tractable, but they are not a panacea: the pressure celerity shock can be reduced but cannot be eliminated without significantly distorting the underlying fluid mechanics. Thus, any numerical algorithm using one of these transient storage models for mixed-flow conditions must be able to handle the shock caused by the transition from a relatively slow pressure celerity in a free-surface pipe to the relatively fast celerity set by the selected transient storage model.

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