2021 60th IEEE Conference on Decision and Control (CDC)
December 13-15, 2021. Austin, Texas

Event-Triggered Safety-Critical Control for Systems with Unknown Dynamics

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Abstract— **This paper addresses the problem of safety-critical control for systems with unknown dynamics. It has been shown that stabilizing affine control systems to desired (sets of) states while optimizing quadratic costs subject to state and control constraints can be reduced to a sequence of quadratic programs (QPs) by using Control Barrier Functions (CBFs) and Control Lyapunov Functions (CLFs). Our recently proposed High Order CBFs (HOCBFs) can accommodate constraints of arbitrary relative degree. One of the main challenges in this approach is obtaining accurate system dynamics, which is especially difficult for systems that require online model identification given limited computational resources and system data. In order to approximate the real unmodeled system dynamics, we define adaptive affine control dynamics which are updated based on the error states obtained by real-time sensor measurements. We define an HOCBF for a safety requirement on the unmodeled system based on the adaptive dynamics and error states, and reformulate the safety-critical control problem as the above mentioned QP. Then, we determine the events required to solve the QP in order to guarantee safety, and derive a condition that guarantees the satisfaction of the HOCBF constraint between events. We illustrate the effectiveness of the proposed framework on adaptive cruise control and compare it with the classical time-driven approach.**

I. INTRODUCTION

Constrained optimal control problems with safety specifications are central to increasingly widespread safety critical autonomous and cyber physical systems. Control barrier functions enforcing safety have received increased attention in recent years [1] [2] [3].

Barrier functions (BFs) are Lyapunov-like functions [4], [5], whose use can be traced back to optimization problems [6]. More recently, they have been employed to prove set invariance [7], [8], [9] and for multi-objective control [10]. In [4], it was proved that if a BF for a given set satisfies Lyapunov-like conditions, then the set is forward invariant. A less restrictive form of a BF, which is allowed to grow when far away from the boundary of the set, was proposed in [1]. Another approach that allows a BF to be zero was proposed in [2], [11]. This simpler form has also been considered in time-varying cases and applied to enforce Signal Temporal Logic (STL) formulas as hard constraints [11].

Control BFs (CBFs) are extensions of BFs for control systems, and are used to map a constraint defined over system states to a constraint on the control input. The CBFs from [1] and [2] work for constraints that have relative degree one with respect to the system dynamics. A backstepping approach was introduced in [12] to address higher relative degree constraints, and it was shown to work for relative degree two. A CBF method for position-based constraints with relative degree two was also proposed in [13]. A more general form [14] for arbitrarily high relative degree constraints employs input-output linearization and finds a pole placement controller with negative poles. The high order CBF (HOCBF) proposed in [3] is simpler and more general than the exponential CBF [14].

Most works using CBFs to enforce safety are based on the assumption that the control system is affine in controls and the cost is quadratic in controls. Convergence to desired states is achieved by Control Lyapunov Functions (CLFs) [15]. The time domain is discretized, and the state is assumed to be constant over each time interval. The optimal control problem becomes a Quadratic Program (QP) in each time interval and the control is kept constant for the whole interval. Using this approach, the original optimal control problem is reduced to a (possibly large) sequence of quadratic programs (QP) - one for each interval [16]. One of the challenges in this QP-based approach is to determine the next time to solve the QP such that safety can still be guaranteed due to time discretization. The work in [17] proposed to find the next time to solve the QP by considering the system Lipschitz constants, and the work in [18] used a similar idea as the event-triggered control for Lyapunov functions [19]. All these approaches assume that the dynamics are accurately modeled, which is often not the case in reality.

In order to find accurate dynamics for systems with uncertainties, [20] proposed to use machine learning techniques; this, however, is computationally expensive and is not guaranteed to yield sufficiently accurate dynamics for the CBF method. The work in [21] proposed to use piecewise linear systems to estimate the system dynamics, which is also computationally expensive. All these works fail to work for systems (such as time-varying systems) that require online model identification.

In order to address the problem of safety-critical control for systems with unknown dynamics, especially for systems for which accurate modeling is hard and online identification is required, this paper contributes to define adaptive affine dynamics that are updated in a time-efficient way to approximate the actual unmodeled dynamics. The adaptive and real dynamics are related through the error states obtained by real-time sensor measurements. We define an HOCBF for a safety requirement on the actual system based on the adaptive

This work was supported in part by NSF under grants ECCS-1931600, DMS-1664644, CNS-1645681, IIS-1723995, and IIS-2024606, by ARPAE under grant DE-AR0001282 and by its NEXTCAR program under grant DE-AR0000796, by AFOSR under grant FA9550-19-1-0158, and by the MathWorks. The authors are with the Division of Systems Engineering and Center for Information and Systems Engineering, Boston University, Brookline, MA, 02446, USA {xiaowei, cbelta, cgc}@bu.edu

dynamics and error states, and reformulate the problem as the above mentioned QP. We determine the events required to solve the QP in order to guarantee safety and derive a condition that guarantees the satisfaction of the HOCBF constraint between events. The adaptive dynamics are updated at each event to accommodate the real dynamics according to the error states; this can reduce the number of events, thus improving the computational efficiency. Our framework can accommodate measurement uncertainties, guarantee safety for systems with unknown dynamics, and is time efficient. We illustrate our approach and compare with the classical time driven method on an ACC problem.

II. PRELIMINARIES

We assume the reader is familiar with the definitions of a class $\mathcal{K}$ function, relative degree of a (sufficiently many times) differentiable function or constraint, and forward invariance of a set with respect to given dynamics; otherwise, please refer to [22] [3] for details.

Consider an affine control system (assumed to be known in this section) of the form:

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} \tag{1}$$

where $\boldsymbol{x} \in X \subset \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times q}$ are Lipschitz continuous, and $\boldsymbol{u} \in U \subset \mathbb{R}^q$ is the control constraint set defined as ($\boldsymbol{u}_{min}, \boldsymbol{u}_{max} \in \mathbb{R}^q$):

$$U := \{\boldsymbol{u} \in \mathbb{R}^q : \boldsymbol{u}_{min} \le \boldsymbol{u} \le \boldsymbol{u}_{max}\}. \tag{2}$$

where the inequalities are interpreted elementwise.

For a constraint $b(\boldsymbol{x}) \ge 0$ with relative degree m, $b : \mathbb{R}^n \to \mathbb{R}$, and $\psi_0(\boldsymbol{x}) := b(\boldsymbol{x})$, we define a sequence of functions $\psi_i : \mathbb{R}^n \to \mathbb{R}, i \in \{1,\dots,m\}$:

$$\psi_i(\boldsymbol{x}) := \dot{\psi}_{i-1}(\boldsymbol{x}) + \alpha_i(\psi_{i-1}(\boldsymbol{x})), i \in \{1,\dots,m\}, \tag{3}$$

where $\alpha_i(\cdot), i \in \{1,\dots,m\}$ denotes a $(m-i)^{th}$ order differentiable class $\mathcal{K}$ function.

We further define a sequence of sets $C_i, i \in \{1,\dots,m\}$ associated with (3) in the form:

$$C_i := \{\boldsymbol{x} \in \mathbb{R}^n : \psi_{i-1}(\boldsymbol{x}) \ge 0\}, i \in \{1,\dots,m\}. \tag{4}$$

***Definition** 1:* (*High Order Control Barrier Function (HOCBF)* [3]) Let $C_1,\dots,C_m$ be defined by (4) and $\psi_1(\boldsymbol{x}),\dots,\psi_m(\boldsymbol{x})$ be defined by (3). A function $b : \mathbb{R}^n \to \mathbb{R}$ is a High Order Control Barrier Function (HOCBF) of relative degree m for system (1) if there exist $(m-i)^{th}$ order differentiable class $\mathcal{K}$ functions $\alpha_i, i \in \{1,\dots,m-1\}$ and a class $\mathcal{K}$ function α_m such that

$$\sup_{\boldsymbol{u}\in U}[L_f^m b(\boldsymbol{x}) + L_g L_f^{m-1} b(\boldsymbol{x})\boldsymbol{u} + R(b(\boldsymbol{x})) + \alpha_m(\psi_{m-1}(\boldsymbol{x}))] \ge 0, \tag{5}$$

for all $\boldsymbol{x} \in C_1\cap,\dots,\cap C_m$. In (5), L_f^m (L_g) denotes Lie derivatives along f (g) m (one) times, and $R(\cdot)$ denotes the remaining Lie derivatives along f with degree less than or equal to $m-1$ (omitted for simplicity, see [22]).

The HOCBF is a general form of the relative degree one CBF [1], [2], [11], i.e., setting $m = 1$ reduces the HOCBF to the common CBF form:

$$L_f b(\boldsymbol{x}) + L_g b(\boldsymbol{x})\boldsymbol{u} + \alpha_1(b(\boldsymbol{x})) \ge 0, \tag{6}$$

and it is also a general form of the exponential CBF [14].

Theorem 1: ([3]) Given an HOCBF $b(\boldsymbol{x})$ from Def. 1 with the associated sets $C_1,\dots,C_m$ defined by (4), if $\boldsymbol{x}(0) \in C_1\cap,\dots,\cap C_m$, then any Lipschitz continuous controller $\boldsymbol{u}(t)$ that satisfies (5), $\forall t \ge 0$ renders $C_1\cap,\dots,\cap C_m$ forward invariant for system (1).

***Definition** 2:* (*Control Lyapunov Function (CLF)* [15]) A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is an exponentially stabilizing control Lyapunov function (CLF) for system (1) if there exist constants $c_1 > 0, c_2 > 0, c_3 > 0$ such that for $\forall \boldsymbol{x} \in \mathbb{R}^n$, $c_1||\boldsymbol{x}||^2 \le V(\boldsymbol{x}) \le c_2||\boldsymbol{x}||^2$,

$$\inf_{\boldsymbol{u}\in U}[L_f V(\boldsymbol{x}) + L_g V(\boldsymbol{x})\boldsymbol{u} + c_3 V(\boldsymbol{x})] \le 0. \tag{7}$$

Many existing works [1], [14] combine CBFs and CLFs for systems with relative degree one with quadratic costs to form optimizations. Time is discretized, and these constraints are linear in control since the state value is fixed at the beginning of the interval. Therefore, each optimization is a quadratic program (QP). The optimal control obtained by solving each QP is applied at the current time step and held constant for the whole interval. The state is updated using dynamics (1), and the procedure is repeated. Replacing CBFs by HOCBFs allows us to handle constraints with arbitrary relative degree [22]. Throughout the paper, we will refer to this method as the *time driven* approach. The CBF method works if (1) is an accurate model for the system. However, this is often not the case in reality, especially for time-varying systems. In what follows, we show how we can find a safety-guaranteed controller for systems with unknown dynamics.

III. PROBLEM FORMULATION AND APPROACH

We consider a system (state $\boldsymbol{x} \in \mathbb{R}^n$ and control $\boldsymbol{u} \in U$) with unknown dynamics, as shown in Fig. 1. For the unknown dynamics, we make the following assumption:

Assumption 1: The relative degree of each component of $\boldsymbol{x}$ is known with repect to the real unknown dynamics.

For example, if the position of a vehicle (whose dynamics are unknown) is a component in $\boldsymbol{x}$ and the control is acceleration, then the relative degree of the position with respect to the unknown vehicle dynamics is two by Newton's law. We assume that we have sensors to monitor $\boldsymbol{x}$ and its derivatives with or without controls. Measuring derivatives of $\boldsymbol{x}$ is challenging, but accurate measurements may not be necessary: we can relax this requirement by limiting meansurement accuracy within some bound, as shown later.

Objective: (Minimizing cost) Consider an optimal control problem for the real unknown dynamics with the cost:

$$\min_{\boldsymbol{u}(t)} \int_0^T \mathcal{C}(||\boldsymbol{u}(t)||)dt + p_0||\boldsymbol{x}(T) - \boldsymbol{K}||^2 \tag{8}$$

where $T > 0, p_0 > 0, \boldsymbol{K} \in \mathbb{R}^n$, $||\cdot||$ denotes the 2-norm of a vector, $\mathcal{C}(\cdot)$ is a strictly increasing function of its argument.

Safety requirements: The real unknown dynamics should always satisfy a safety requirement:

$$b(\boldsymbol{x}(t)) \ge 0, \forall t \in [0,T]. \tag{9}$$

where $b : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and has relative degree $m \in \mathbb{N}$ with respect to the real system. The relative degree m is known by Assumption 1.

Control constraints: The control of the real system should always satisfy control bounds in the form of (2).

Problem 1: Find a control policy for the real unknown dynamics such that the cost (8) is minimized, and constraints (9) and (2) are satisfied.

Approach: Our approach to solve Problem 1 relies on the CBF-based QP method [1], and the solution is sub-optimal. In order to achieve solutions that are both safe and close to optimal, we may use desired planned trajectories to be optimally tracked as shown in [23]. There are four steps involved in the solution:

Step 1: Define adaptive affine dynamics. Under Assump. 1, we define affine dynamics that have the same relative degree for (9) as the real system to estimate the real dynamics:

$$\dot{\bar{\boldsymbol{x}}} = f_a(\bar{\boldsymbol{x}}) + g_a(\bar{\boldsymbol{x}})\boldsymbol{u} \tag{10}$$

where $f_a : \mathbb{R}^n \to \mathbb{R}, g_a : \mathbb{R}^n \to \mathbb{R}^{n\times q}$, and $\bar{\boldsymbol{x}} \in X \subset \mathbb{R}^n$ is the state vector corresponding to $\boldsymbol{x}$ in the unknown dynamics. Since $f_a(\cdot), g_a(\cdot)$ in (10) can be adaptively updated to accommodate the real unknown dynamics, as shown in the next section, we call (10) *adaptive affine dynamics*. The real unknown dynamics and (10) are related through the error states obtained from the real-time measurements of the system and the integration of (10). Clearly, we would like the adaptive dynamics (10) to "stay close" to the real dynamics. This notion will be formalized in the next section.

Step 2: Find an HOCBF that guarantees (9). Based on (10), the error state and its derivatives, we use an HOCBF to enforce (9). Details are shown in the next section.

Step 3: Formulate the CBF-based QP. We use a relaxed CLF to achieve a minimal value of the terminal state penalty in (8). If $\mathcal{C}(||\boldsymbol{u}(t)||) = ||\boldsymbol{u}(t)||^2$ in (8), then we can formulate Problem 1 using a CBF-CLF-QP approach [1], with a CBF replaced by an HOCBF [3] if $m > 1$.

Step 4: Determine the events required to solve the QP and the condition that guarantees the satisfaction of (9) between events. Since there is a difference between the adaptive dynamics (10) and the real unknown dynamics, in order to guarantee safety in the real system, we need to properly define events (dependent on the error state and the state of (10)) to solve the QP. In other words, we need to determine the times $t_k, k = 1, 2, \dots (t_1 = 0)$ at which the QP must be solved in order to guarantee (9) for the plant.

The proposed solution framework is shown in Fig. 1 where we note that we apply the same control from the QP to both the real unknown dynamics and (10).

IV. Event-Triggered Control

In this section, we provide the technical details involved in formulating the CBF-based QPs that guarantee the satisfaction of the safety constraint (9) for the real unknown system. We start with the case of a relative-degree-one safety constraint (9).

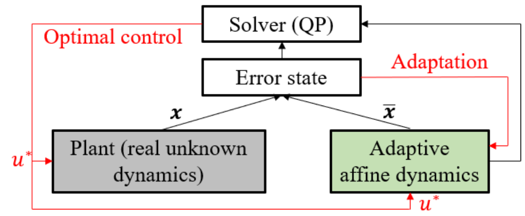


Fig. 1. The solution framework for Problem 1 and the connection between the real unknown dynamics and the adaptive affine dynamics (10). The state $\boldsymbol{x}$ is from the sensor measurements of the plant.

A. Relative-degree-one Constraints

Suppose the safety constraint in (9) has relative degree one with respect to both dynamics (10) and the actual dynamics.

Next, we show how to find a CBF that guarantees (9) for the real unknown dynamics. Let

$$\boldsymbol{e} := \boldsymbol{x} - \bar{\boldsymbol{x}}. \tag{11}$$

Note that $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ are state vectors from direct measurements and from the adaptive dynamics (10), respectively. Then,

$$b(\boldsymbol{x}) = b(\bar{\boldsymbol{x}} + \boldsymbol{e}). \tag{12}$$

Differentiating $b(\bar{\boldsymbol{x}} + \boldsymbol{e})$, we have

$$\frac{db(\bar{\boldsymbol{x}}+\boldsymbol{e})}{dt} = \frac{\partial b(\bar{\boldsymbol{x}}+\boldsymbol{e})}{\partial \bar{\boldsymbol{x}}}\dot{\bar{\boldsymbol{x}}} + \frac{\partial b(\bar{\boldsymbol{x}}+\boldsymbol{e})}{\partial \boldsymbol{e}}\dot{\boldsymbol{e}} \tag{13}$$

The CBF constraint that guarantees (9) for known dynamics (1) is as in (6), which is done by replacing $\dot{\boldsymbol{x}}$ with (1). However, for the unknown dynamics, the CBF constraint is: $\frac{db(\boldsymbol{x})}{dt} + \alpha_1(b(\boldsymbol{x})) \geq 0$. Equivalently, we have

$$\frac{db(\bar{\boldsymbol{x}}+\boldsymbol{e})}{dt} + \alpha_1(b(\bar{\boldsymbol{x}}+\boldsymbol{e})) \geq 0. \tag{14}$$

Combining (13), (14) and (10), we get the CBF constraint that guarantees (9):

$$\frac{\partial b(\boldsymbol{x})}{\partial \bar{\boldsymbol{x}}} f_a(\bar{\boldsymbol{x}}) + \frac{\partial b(\boldsymbol{x})}{\partial \bar{\boldsymbol{x}}} g_a(\bar{\boldsymbol{x}})\boldsymbol{u} + \frac{\partial b(\boldsymbol{x})}{\partial \boldsymbol{e}}\dot{\boldsymbol{e}} + \alpha_1(b(\boldsymbol{x})) \geq 0. \tag{15}$$

where $\dot{\boldsymbol{e}} = \dot{\boldsymbol{x}} - \dot{\bar{\boldsymbol{x}}}$ is evaluated online through $\dot{\boldsymbol{x}}$ (from direct measurements of the actual state derivative) and $\dot{\bar{\boldsymbol{x}}}$ as given through (10). Then, the satisfaction of (15) implies the satisfaction of $b(\bar{\boldsymbol{x}}+\boldsymbol{e}) \geq 0$ by Thm. 1 and (12), therefore, (9) is guaranteed to be satisfied for the real unknown dynamics.

Now, we can formulate an optimal control problem:

$$\min_{\boldsymbol{u}(t),\delta(t)} \int_0^T \left(||\boldsymbol{u}(t)||^2 + p\delta^2(t)\right) dt \tag{16}$$

subject to (15), (2), and the CLF constraint

$$L_{f_a}V(\bar{\boldsymbol{x}}) + L_{g_a}V(\bar{\boldsymbol{x}})\boldsymbol{u} + \epsilon V(\bar{\boldsymbol{x}}) \leq \delta(t), \tag{17}$$

where $V(\bar{\boldsymbol{x}}) = ||\bar{\boldsymbol{x}} - \boldsymbol{K}||^2$, $c_3 = \epsilon > 0$ in Def. 2, $p > 0$, $\delta(t)$ is a relaxation for the CLF constraint.

Following the approach introduced at the end of Sec. II, we solve the probem (16) at time $t_k, k = 1, 2 \dots$. However, at time t_k, the QP (16) does not generally know the error state $\boldsymbol{e}(t)$ and its derivative $\dot{\boldsymbol{e}}(t), \forall t > t_k$. Thus, it cannot guarantee that the CBF constraint (15) is satisfied in the time interval $(t_k, t_{k+1}]$, where t_{k+1} is the next time instant to solve the QP. In order to find a condition that guarantees the satisfaction of (15) $\forall t \in (t_k, t_{k+1}]$, we first let $\boldsymbol{e} = (e_1, \dots, e_n)$ and $\dot{\boldsymbol{e}} = (\dot{e}_1, \dots, \dot{e}_n)$ be bounded by $\boldsymbol{w} = (w_1, \dots, w_n) \in \mathbb{R}^n_{>0}$ and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n) \in \mathbb{R}^n_{>0}$:

$$|e_i| \leq w_i, \qquad |\dot{e}_i| \leq \nu_i, \qquad i \in \{1, \dots, n\}, \tag{18}$$

which can be rewritten as $|\boldsymbol{e}| \leq \boldsymbol{w}, |\dot{\boldsymbol{e}}| \leq \boldsymbol{\nu}$ for simplicity.

We now consider the state $\bar{\boldsymbol{x}}$ at time t_k, which satisfies:

$$\bar{\boldsymbol{x}}(t_k) - \boldsymbol{s} \leq \bar{\boldsymbol{x}}(t) \leq \bar{\boldsymbol{x}}(t_k) + \boldsymbol{s}, \tag{19}$$

where the inequalities are interpreted componentwise and $\boldsymbol{s} \in \mathbb{R}^n_{>0}$. The choice of $\boldsymbol{s}$ will be discussed later. We denote the set of states that satisfy (19) at time t_k by

$$S(t_k) = \{\boldsymbol{y} \in X : \bar{\boldsymbol{x}}(t_k) - \boldsymbol{s} \leq \boldsymbol{y} \leq \bar{\boldsymbol{x}}(t_k) + \boldsymbol{s}\}. \tag{20}$$

Now, with (18) and (19), we are ready to find a condition that guarantees the satisfaction of (15) in the time interval $(t_k, t_{k+1}]$. This is done by considering the minimum value of each component in (15), as shown next.

In (15), let $b_{f_a,min}(t_k) \in \mathbb{R}$ be the minimum value of $\frac{\partial b(\bar{\boldsymbol{x}}+\boldsymbol{e})}{\partial \bar{\boldsymbol{x}}} f_a(\bar{\boldsymbol{x}})$ for the preceding time interval that satisfies $\boldsymbol{y} \in S(t_k), |\boldsymbol{e}| \leq \boldsymbol{w}, \boldsymbol{y} + \boldsymbol{e} \in C_1$ starting at time t_k, i.e., let

$$b_{f_a,min}(t_k) = \min_{\boldsymbol{y} \in S(t_k), |\boldsymbol{e}| \leq \boldsymbol{w}, \boldsymbol{y}+\boldsymbol{e} \in C_1} \frac{\partial b(\boldsymbol{y}+\boldsymbol{e})}{\partial \boldsymbol{y}} f_a(\boldsymbol{y}) \tag{21}$$

Similarly, we can also find the minimum value $b_{\alpha_1,min}(t_k) \in \mathbb{R}$ and $b_{e,min}(t_k) \in \mathbb{R}$ of $\alpha_1(b(\boldsymbol{x}))$ and $\frac{\partial b(\boldsymbol{x})}{\partial \boldsymbol{e}} \dot{\boldsymbol{e}}$, respectively, for the preceding time interval that satisfies $\boldsymbol{y} \in S(t_k), |\boldsymbol{e}| \leq \boldsymbol{w}, \boldsymbol{y} + \boldsymbol{e} \in C_1, |\dot{\boldsymbol{e}}| \leq \boldsymbol{\nu}$ starting at time t_k, i.e., let

$$b_{\alpha_1,min}(t_k) = \min_{\boldsymbol{y} \in S(t_k), |\boldsymbol{e}| \leq \boldsymbol{w}, \boldsymbol{y}+\boldsymbol{e} \in C_1} \alpha_1(b(\boldsymbol{y}+\boldsymbol{e})) \tag{22}$$

$$b_{e,min}(t_k) = \min_{\boldsymbol{y} \in S(t_k), |\boldsymbol{e}| \leq \boldsymbol{w}, |\dot{\boldsymbol{e}}| \leq \boldsymbol{\nu}, \boldsymbol{y}+\boldsymbol{e} \in C_1} \frac{\partial b(\boldsymbol{y}+\boldsymbol{e})}{\partial \boldsymbol{e}} \dot{\boldsymbol{e}} \tag{23}$$

For the remaining term in (15), if $\frac{\partial b(\boldsymbol{x})}{\partial \bar{\boldsymbol{x}}} g_a(\bar{\boldsymbol{x}})$ is independent of $\bar{\boldsymbol{x}}$ and $\boldsymbol{e}$, then we do not need to find its limit value within the bound $\boldsymbol{y} \in S(t_k), |\boldsymbol{e}| \leq \boldsymbol{w}, \boldsymbol{y} + \boldsymbol{e} \in C_1$; otherwise, let $\bar{\boldsymbol{x}} = (\bar{x}_1, \dots, \bar{x}_n) \in \mathbb{R}^n$, $\boldsymbol{u} = (u_1, \dots, u_q) \in \mathbb{R}^q$ and $g_a = (g_1, \dots, g_q) \in \mathbb{R}^{n \times q}$. We assume each component of $\frac{\partial b(\boldsymbol{x})}{\partial \bar{\boldsymbol{x}}} g_a(\bar{\boldsymbol{x}})$ does not change sign $\forall \boldsymbol{x} \in X$; otherwise, we can define each sign change to be an update-triggering event (this is the subject of future work). The sign of $u_i(t_k), i \in \{1, \dots, q\}, k = 1, 2 \dots$ can be determined by solving the CBF-based QP (16) at t_k.

We can then determine the limit value $b_{g_i,lim}(t_k) \in \mathbb{R}, i \in \{1, \dots, q\}$ of $\frac{\partial b(\boldsymbol{x})}{\partial \bar{\boldsymbol{x}}} g_i(\bar{\boldsymbol{x}})$ by

$$b_{g_i,lim}(t_k) = \begin{cases} \min\limits_{\boldsymbol{y} \in S(t_k), |\boldsymbol{e}| \leq \boldsymbol{w}, \boldsymbol{y}+\boldsymbol{e} \in C_1} \frac{\partial b(\boldsymbol{y}+\boldsymbol{e})}{\partial \boldsymbol{y}} g_i(\boldsymbol{y}), & \text{if } u_i(t_k) \geq 0, \\ \max\limits_{\boldsymbol{y} \in S(t_k), |\boldsymbol{e}| \leq \boldsymbol{w}, \boldsymbol{y}+\boldsymbol{e} \in C_1} \frac{\partial b(\boldsymbol{y}+\boldsymbol{e})}{\partial \boldsymbol{y}} g_i(\boldsymbol{y}), & \text{otherwise} \end{cases} \tag{24}$$

Let $b_{g_a,lim}(t_k) = (b_{g_1,lim}(t_k), \dots, b_{g_q,lim}(t_k)) \in \mathbb{R}^q$, and we set $b_{g_a,lim}(t_k) = \frac{\partial b(\boldsymbol{x})}{\partial \bar{\boldsymbol{x}}} g_a(\bar{\boldsymbol{x}})$ if $\frac{\partial b(\boldsymbol{x})}{\partial \bar{\boldsymbol{x}}} g(\bar{\boldsymbol{x}})$ is independent of $\bar{\boldsymbol{x}}$ and $\boldsymbol{e}$ for notational simplicity.

The condition that guarantees the satisfaction of (15) in the time interval $(t_k, t_{k+1}]$ is then given by

$$b_{f_a,min}(t_k) + b_{g_a,lim}(t_k)\boldsymbol{u}(t_k) + b_{e,min}(t_k) + b_{\alpha_1,min}(t_k) \geq 0. \tag{25}$$

In order to apply the above condition to the QP (16), we just replace (15) by (25), i.e., we have

$$\min_{\boldsymbol{u}(t_k), \delta(t_k)} ||\boldsymbol{u}(t_k)||^2 + p\delta^2(t_k), \text{ s.t. } (25), (2), (17) \tag{26}$$

Based on the above, we define three events that determine the condition that triggers an instance of solving the QP (26):

- **Event 1:** $|\boldsymbol{e}| \leq \boldsymbol{w}$ is about to be violated.
- **Event 2:** $|\dot{\boldsymbol{e}}| \leq \boldsymbol{\nu}$ is about to be violated.
- **Event 3:** $\bar{\boldsymbol{x}}$ of (10) reaches the boundaries of $S(t_k)$.

In other words, the next time instant $t_{k+1}, k = 1, 2 \dots$ to solve the QP (26) is determined by:

$$\begin{aligned} t_{k+1} = \min\{t > t_k : |\boldsymbol{e}(t)| = \boldsymbol{w} \text{ or } |\dot{\boldsymbol{e}}(t)| = \boldsymbol{\nu} \\ \text{or } |\bar{\boldsymbol{x}}(t) - \bar{\boldsymbol{x}}(t_k)| = \boldsymbol{s}\}, \end{aligned} \tag{27}$$

where $t_1 = 0$. The first two events can be detected by direct sensor measurements after applying $\boldsymbol{u}(t_k)$, while Event 3 can be detected by monitoring the dynamics (10). The selected magnitude of each component of $\boldsymbol{s}$ is a tradeoff between the time complexity and the conservativeness of this approach. If the magnitude is large, then the number of events is small but the approach is conservative as we determine (25) through the minimum values as in (21)-(24).

Formally, we have the following theorem to show that the satisfaction of the safety constraint (9) is guaranteed for the plant with the condition (25) (proof is given in [24]):

Theorem 2: Given an HOCBF $b(\boldsymbol{x})$ with $m = 1$ as in Def. 1, let $t_{k+1}, k = 1, 2 \dots$ be determined by (27) with $t_1 = 0$, and (25) be determined by (21)-(24), respectively. Then, under Assumption 1, any control $\boldsymbol{u}(t_k)$ that satisfies (25) and updates the real unknown dynamics and the adaptive dynamics (10) within time interval $[t_k, t_{k+1})$ renders the set C_1 forward invariant for the real unknown dynamics.

Remark 1: We may also consider the minimum value of $\frac{\partial b(\boldsymbol{y}+\boldsymbol{e})}{\partial \boldsymbol{y}} f_a(\boldsymbol{y}) + \frac{\partial b(\boldsymbol{y}+\boldsymbol{e})}{\partial \boldsymbol{e}} \dot{\boldsymbol{e}} + \alpha_1(b(\boldsymbol{y} + \boldsymbol{e}))$ within $\boldsymbol{y} \in S(t_k), |\boldsymbol{e}| \leq \boldsymbol{w}, \boldsymbol{y} + \boldsymbol{e} \in C_1, |\dot{\boldsymbol{e}}| \leq \boldsymbol{\nu}$ instead of considering them separately as in (21)-(24). This will be less conservative as the constraint (25) is stronger compared with the CBF constraint (15), and we wish to find the largest possible value of the left-hand side of (15) that can support Thm. 2.

Events 1 and 2 will be frequently triggered if the modeling of the adaptive dynamics (10) has a large error with respect to the real dynamics. Therefore, we would like to model the adaptive dynamics (10) as accurately as possible in order to reduce the number of events required to solve the QP (26).

An additional important step is to synchronize the state of the real unknown dynamics and (10) such that we always have $\boldsymbol{e}(t_k) = 0$ and make $\dot{\boldsymbol{e}}(t_k)$ close to 0 by setting

$$\bar{\boldsymbol{x}}(t_k) = \boldsymbol{x}(t_k), \tag{28}$$

and by updating $f_a(\bar{\boldsymbol{x}}(t))$ of the adaptive dynamics (10) right after (t^+) an event occurs at t:

$$f_a(\bar{\boldsymbol{x}}(t^+)) = f_a(\bar{\boldsymbol{x}}(t^-)) + \sum_{i=0}^{k} \dot{\boldsymbol{e}}(t_i). \tag{29}$$

where t^+, t^- denote instants right after and before t. In this way, the dynamics (10) are adaptively updated at each event, i.e., at $t_k, k = 1, 2, \dots$. Note that we may also update $g_a(\cdot)$, which is harder than updating $f_a(\cdot)$ since $g_a(\cdot)$ is multiplied by $\boldsymbol{u}$ that is to be determined, i.e., the update of $g_a(\cdot)$ will depend on $\boldsymbol{u}$. This possibility is the subject of ongoing work.

Assuming the functions that define the real unknown dynamics and f_a, g_a in (10) are Lipschitz continuous, then there exists lower bounds for the occurance times of the three events [24]. As a summary, we get measurements from the plant at time $t_k, k = 1, 2, \dots$, and update (learn) the adaptive dynamics (10) by (28), (29). Then we solve the QP (26) at t_k, and apply the optimal control to the plant and (10). After t_k, we keep collecting measurements from the plant and evaluate the next event time t_{k+1} by (27) to solve the QP (26). This process is repeated until the final time. The algorithm for this event-triggered control can be found in [24].

Remark 2: (Measurement uncertainties) If the measurements $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ are subject to uncertainties, and the uncertainties are bounded, then we can employ some filters to the measurements and apply the bounds of $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ in evaluating t_{k+1} by (27) instead of $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ themselves. In other words, $\boldsymbol{e}(t)$ and $\dot{\boldsymbol{e}}(t)$ are determined by the bounds of $\boldsymbol{x}$, $\dot{\boldsymbol{x}}$ and the state of the adaptive system (10). This can also relax the earlier assumption that we can (exactly) measure $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$.

B. High-relative-degree Constraints

In this subsection, we consider the safety constraint (9) whose relative degree is larger than one with respect to the real unknown dynamics and (10). In other words, we consider the HOCBF constraint (5). The technique is similar to the last section, and thus, the details are skipped due to space limitation, but can be found in [24].

Similar to the last subsection, we find the error state $\boldsymbol{e}$ by (11), and have an alternative form of the HOCBF $b(\boldsymbol{x})$ as in (12). The difference is that we have $\boldsymbol{e}^{(i)} = \boldsymbol{x}^{(i)} - \bar{\boldsymbol{x}}^{(i)}, i \in \{1, \dots, m\}$ (the i_{th} derivative), and is evaluated online by $\boldsymbol{x}^{(i)}$ (from a sensor) of the real system and $\bar{\boldsymbol{x}}^{(i)}$ of (10).

In order to find a conditon that guarantees the satisfaction of the last equation in $[t_i, t_{i+1}), i = 1, 2, \dots$, we define bounds for $\boldsymbol{e}$ and $\boldsymbol{e}^{(i)}, i \in \{1, \dots, m\}$ as in (18). As in (21)-(24), we get a condition that guarantees the satisfaction of the safety constraint in the time interval $[t_k, t_{k+1})$ by

$$\begin{aligned} b_{f_a^m,min}(t_k) + b_{g_a,lim}(t_k)\boldsymbol{u}(t_k) + b_{e^m,min}(t_k) \\ + b_{\alpha_m,min}(t_k) + b_{R,min}(t_k) \geq 0. \end{aligned} \tag{30}$$

The parameters above are the min./lim. values corresponding to the error-state based HOCBF constraint as in (15).

The three events to solve the QP and the event time t_{k+1} are determined as in the last section. A result similar to Thm. 2 that shows the satisfaction of (9) can be found in [24].

V. CASE STUDIES

In this section, we consider the case study of an ACC problem. All the computations and simulations were conducted in MATLAB. We used quadprog to solve the quadratic programs and ode45 to integrate the dynamics.

The real vehicle dynamics are **unknown** to the controller:

$$\begin{bmatrix} \dot{v}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} \sigma_1(t) + \frac{\sigma_3(t)}{M}u(t) - \frac{1}{M}F_r(v(t)) \\ \sigma_2(t) + v_p - v(t) \end{bmatrix} \tag{31}$$

where $\boldsymbol{x} = (v, z)$ and $z(t)$ denotes the distance between the preceding and the ego vehicle, $v_p > 0, v(t)$ denote the velocities of the preceding and ego vehicles along the lane (the velocity of the preceding vehicle is assumed constant), respectively, and $u(t)$ is the control of the ego vehicle. $\sigma_1(t), \sigma_2(t), \sigma_3(t)$ denote three random processes whose pdf's have finite support. M denotes the mass of the ego vehicle and $F_r(v(t))$ denotes the resistance force, which is expressed [25] as: $F_r(v(t)) = f_0 sgn(v(t)) + f_1 v(t) + f_2 v^2(t)$, where $f_0 > 0$, $f_1 > 0$ and $f_2 > 0$ are unknown.

The adaptive dynamics will be automatically updated as shown in (29), and are in the form:

$$\underbrace{\begin{bmatrix} \dot{\bar{v}}(t) \\ \dot{\bar{z}}(t) \end{bmatrix}}_{\dot{\bar{\boldsymbol{x}}}(t)} = \underbrace{\begin{bmatrix} h_1(t) - \frac{1}{M}F_n(\bar{v}(t)) \\ h_2(t) + v_p - \bar{v}(t) \end{bmatrix}}_{f_a(\bar{\boldsymbol{x}}(t))} + \underbrace{\begin{bmatrix} \frac{1}{M} \\ 0 \end{bmatrix}}_{g_a(\bar{\boldsymbol{x}}(t))} u(t) \tag{32}$$

where $h_1(t) \in \mathbb{R}, h_2(t) \in \mathbb{R}$ denote the two adaptive terms in (29), $h_1(0) = 0, h_2(0) = 0$. $\bar{z}(t), \bar{v}(t)$ are corresponding to $z(t), v(t)$ in (31). $F_n(\bar{v}(t)) = g_0 sgn(\bar{v}(t)) + g_1 \bar{v}(t) + g_2 \bar{v}^2(t)$, which is *different* from F_r in (31), where $g_0 > 0$, $g_1 > 0$ and $g_2 > 0$ are empirically determined.

The control bound is defined as: $-c_d M g \leq u(t) \leq c_a M g$, where $c_a > 0$ and $c_d > 0$ are the maximum acceleration and deceleration coefficients, respectively, and g is the gravity constant. We require that the distance $z(t)$ between the ego vehicle (real dynamics) and its immediately preceding vehicle be greater than $l_p > 0$, i.e.,

$$z(t) \geq l_p, \quad \forall t \geq 0. \tag{33}$$

The objective is to minimize $\int_0^T ((u(t) - F_r(v(t)))/M)^2 dt$. The ego vehicle is also trying to achieve a desired speed $v_d > 0$, which is implemented by a CLF $V(\bar{\boldsymbol{x}}) = (\bar{v} - v_d)^2$ as in Def. 2. Since the relative degree of the constraint (33) is two, we define an HOCBF $b(\boldsymbol{x}) = z - l_p$ with $\alpha_1(b(\boldsymbol{x})) = b(\boldsymbol{x})$ and $\alpha_2(\psi_1(\boldsymbol{x})) = \psi_1(\boldsymbol{x})$ as in Def. 1 to implement the safety constraint. Then, the HOCBF constraint (5) which in this case is (with respect to the real dynamics (31)): $\ddot{b}(\boldsymbol{x}) + 2\dot{b}(\boldsymbol{x}) + b(\boldsymbol{x}) \geq 0$. Combining (11), (32) and this equation, we have an HOCBF constraint in the form: $-h_1(t) + \frac{F_n(\bar{v}(t))}{M} + \frac{-1}{M}u(t) + \ddot{e}_2(t) + 2(h_2(t) + v_p - \bar{v}(t) + \dot{e}_2(t)) + \bar{z}(t) + e_2(t) - l_p \geq 0$, where $\boldsymbol{e} = (e_1, e_2), e_1 = v - \bar{v}, e_2 = z - \bar{z}$.

Similar to (18), (19), we consider the state and bound the errors at step $t_k, k = 1, 2 \dots$ for the above HOCBF constraint in the form: $\bar{v}(t_k) - s_1 \leq \bar{v} \leq \bar{v}(t_k) + s_1, \bar{z}(t_k) - s_2 \leq \bar{z} \leq \bar{z}(t_k) + s_2, |e_2| \leq w_2, |\dot{e}_2| \leq \nu_{2,1}, |\ddot{e}_2| \leq \nu_{2,2}$, where $s_1 > 0, s_2 > 0, w_2 > 0, \nu_{2,1} > 0, \nu_{2,2} > 0$.

As in (20), (29), we also synchronize the state and update the adaptive dynamics (32) at step $t_k, k = 1, 2 \dots$ in the form: $\bar{v}(t_k) = v(t_k), \bar{z}(t_k) = z(t_k), h_1(t^+) = h_1(t^-) - \sum_{i=0}^k \ddot{e}_2(t_i), h_2(t^+) = h_2(t^-) + \sum_{i=0}^k \dot{e}_2(t_i)$, where $\dot{e}_2(t_k) = \dot{z}(t_k) - (h_2 + v_p - \bar{v}(t_k)), \ddot{e}_2(t_k) = \ddot{z}(t_k) - \frac{F_n(\bar{v}(t_k)) - u(t_k^-)}{M} + h_1(t_k), u(t_k^-) = u(t_{k-1})$ and $u(t_0) = 0$. $\dot{z}(t_k), \ddot{z}(t_k)$ are estimated by a sensor that measures the dynamics (31) at t_k.

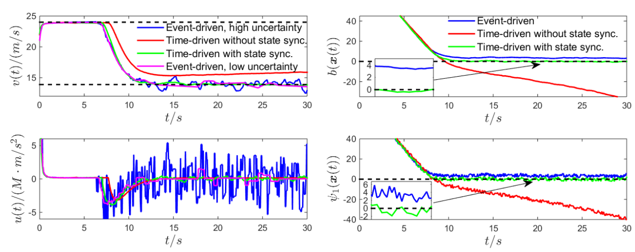
Then, we can find the limit values similar to (21)-(24), solve a QP similar to (26) at each time step $t_k, k = 1, 2 \dots$, and evaluate the next time step t_{k+1} as in (27) afterwards. In the evaluation of t_{k+1}, we have $e_2 = z - \bar{z}, \dot{e}_2 = \dot{z} -$

$(h_2 + v_p - \bar{v}), \ddot{e}_2 = \ddot{z} - \frac{F_n(\bar{v}) - u(t_k)}{M} + h_1$, where $z, \dot{z}, \ddot{z}$ are estimated by a sensor that measures the ego real dynamics (31), and $u(t_k)$ is already obtained by solving the QP and is held as a constant until we find t_{k+1}. The optimizations similar to (21)-(24) are either QPs or LPs. Each QP or LP can be solved with a computational time $< 0.01s$ in MATLAB (Intel(R) Core(TM) i7-8700 CPU @ 3.2GHz×2).

The simulation parameters can be found in [24]. The pdf's of $\sigma_1(t), \sigma_2(t), \sigma_3(t)$ are uniform over the intervals $[-0.2, 0.2]m/s^2, [-2, 2]m/s, [0.9, 1]$, respectively. The sensor sampling rate is 20Hz. We compare the proposed event driven framework with the time driven approach. The discretization time for the time driven approach is $\Delta t = 0.1$.

The simulation results are shown in Figs. 2(a) and 2(b). In the event-driven approach (blue lines), the control varies largely in order to be responsive to the random processes in the real dynamics. If we decrease the uncertainty levels by 10 times, the control is smoother (magenta lines). Thus, highly accurately modeled adaptive dynamics are desired.

It follows from Fig. 2(b) that the set $C_1 \cap C_2$ is forward invariant for the real vehicle dynamics (31), i.e., the safety constraint (33) is guaranteed with the proposed event driven approach. However, the safety is not guaranteed even with state synchronization under the time-driven approach.



(a) Speed and control profiles. (b) $b(\boldsymbol{x}(t))$ and $\psi_1(\boldsymbol{x}(t))$.

Fig. 2. Results for the proposed event driven framework and time driven with or without synchronization. $b(\boldsymbol{x}(t)) \geq 0$ and $\psi_1(\boldsymbol{x}(t)) \geq 0$ imply the forward invariance of $C_1 \cap C_2$ (for the proposed event driven framework, but not for the time driven case with or without synchronization).

In the event-driven approach, the number of QPs (events) within time $[0, T]$ is reduced by about 50% compared with the time-driven approach. If we multiply the bounds of the random processes $\sigma_1(t), \sigma_2(t)$ by 2, then the number of events increases by about 23% for both the 20Hz and 100Hz sensor sampling rate, which shows that accurate adaptive dynamics can reduce the number of events, and thus improves the computational efficient.

VI. CONCLUSION & FUTURE WORK

This paper proposes an event-triggered framework for safety-critical control of systems with unknown dynamics. This framework is based on defining adaptive affine dynamics to estimate the real system, an event-trigger mechanism for solving the problem and the finding of a condition that guarantees safety between events. The effectiveness has been demonstrated on adaptive cruise control. In the future, we will study the conservativeness of the proposed framework.

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