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High Order Control Lyapunov-Barrier Functions for Temporal Logic Specifications

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Abstract— **Recent work has shown that stabilizing an affine control system to a desired state while optimizing a quadratic cost subject to state and control constraints can be reduced to a sequence of Quadratic Programs (QPs) by using Control Barrier Functions (CBFs) and Control Lyapunov Functions (CLFs). In our own recent work, we defined High Order CBFs (HOCBFs) for systems and constraints with arbitrary relative degrees. In this paper, in order to accommodate initial states that do not satisfy the state constraints and constraints with arbitrary relative degree, we generalize HOCBFs to High Order Control Lyapunov-Barrier Functions (HOCLBFs). We also show that the proposed HOCLBFs can be used to guarantee the Boolean satisfaction of Signal Temporal Logic (STL) formulae over the state of the system. We illustrate our approach on a safety-critical optimal control problem (OCP) for a unicycle.**

I. Introduction

Barrier functions (BFs) are Lyapunov-like functions [17], whose use can be traced back to optimization problems [4]. More recently, they have been employed to prove set invariance [3], [14] and for the purpose of multi-objective control [13]. In [17], it was proved that if a BF for a given set satisfies Lyapunov-like conditions, then the set is forward invariant. A less restrictive form of a BF, which is allowed to grow when far away from the boundary of the set, was proposed in [1]. Another approach that allows a BF to take zero values was proposed in [6], [9]. Control BFs (CBFs) are extensions of BFs for control systems, and are used to map a constraint that is defined over system states to a constraint on the control input. Recently, it has been shown that, to stabilize an affine control system while optimizing a quadratic cost and satisfying state and control constraints, CBFs can be combined with control Lyapunov functions (CLFs) [2], [5] to form quadratic programs (QPs) [1], [6] that are solved in real time.

The CBFs from [1] and [6] work for constraints that have relative degree one with respect to the system dynamics. A CBF method for position-based constraints with relative degree two was proposed in [18]. A more general form, which works for arbitrarily high relative degree constraints, was proposed in [11]. The method in [11] employs input-output linearization and finds a pole placement controller with negative poles to stabilize the CBF to zero. In our recent work [19], we defined a High Order CBF (HOCBF) that can accommodate constraints with high relative degree and does not require linearization. In this paper, we propose an extension of the HOCBF from [19] that achieves two main objectives: (1) it works for states that are not initially in the safe set, and (2) it can guarantee the satisfaction of specifications given as Signal Temporal Logic (STL) formulae.

Recent works proposed the use of CBFs to enforce the satisfaction of temporal logic (TL) specifications. STL and Linear TL (LTL) were used as specification languages in [9] and [12], respectively, for systems and constraints with relative degree one. Many specifications and systems, however, lead to higher relative degrees. For example, a comfort requirement for an autonomous vehicle is usually expressed using jerk, which induces a high relative degree constraint. The authors of [9] defined time-varying functions to guarantee the satisfaction of a STL formula for systems with relative degree one. Extending time-varying functions to work for high relative degree constraints, even though possible, would be difficult, as it would require that the state of the system be in the intersection of a possibly large number of sets. TL specifications have also been considered in [16] by using finite-time convergence CBFs [8]. However, this approach is restricted to relative-degree-one constraints, and may lead to chattering behaviors that result from finite-time convergence, as will be shown in this paper. Barrier-Lyapunov functions, as proposed in [17], [15], could also be used, in principle, to implement STL specifications, as they combine (linear) state constraints with convergence.

In this paper, to accommodate STL specifications over nonlinear state constraints for high relative degree systems, we propose High Order Control Lyapunov-Barrier Functions (HOCLBF). The proposed HOCLBFs lead to controllers that stabilize a system inside a set within a specified time if the system state is initially outside this set, and ensure that the system remains in this set after it enters it. We also propose how to eliminate chattering behaviors with the HOCLBF method. We illustrate the usefulness of the proposed approach by applying it to a unicycle model.

II. Preliminaries

We assume the reader is familiar with the definitions of class $\mathcal{K}$ function, extended class $\mathcal{K}$ function, relative degree of a (sufficiently many times) differentiable function [7], and forward invariance of a set with respect to given dynamics. When a constraint is defined using a differentiable function, we will refer to the relative degree of the function as the relative degree of the constraint.

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Consider an affine control system of the form

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}) + g(\boldsymbol{x})\boldsymbol{u} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times q}$ are globally Lipschitz, and $\boldsymbol{u} \in U \subset \mathbb{R}^q$ (U denotes the control constraint set). Solutions $\boldsymbol{x}(t)$ of (1), starting at $\boldsymbol{x}(0)$, $t \geq 0$, are forward complete for all $\boldsymbol{u} \in U$.

Suppose the control bound U is defined as (the inequality is interpreted componentwise, $\boldsymbol{u}_{min}, \boldsymbol{u}_{max} \in \mathbb{R}^q$):

$$U := \{\boldsymbol{u} \in \mathbb{R}^q : \boldsymbol{u}_{min} \leq \boldsymbol{u} \leq \boldsymbol{u}_{max}\}. \tag{2}$$

(1) High Order Control Barrier Functions: For a constraint $b(\boldsymbol{x}) \geq 0$ with relative degree m, $b : \mathbb{R}^n \to \mathbb{R}$, and $\psi_0(\boldsymbol{x}) := b(\boldsymbol{x})$, we define a sequence of functions $\psi_i : \mathbb{R}^n \to \mathbb{R}, i \in \{1, \dots, m\}$:

$$\psi_i(\boldsymbol{x}) := \dot{\psi}_{i-1}(\boldsymbol{x}) + \alpha_i(\psi_{i-1}(\boldsymbol{x})), \quad i \in \{1, \dots, m\}, \tag{3}$$

where $\alpha_i(\cdot), i \in \{1, \dots, m\}$ denotes a $(m-i)^{th}$ order differentiable class $\mathcal{K}$ function. We further define a sequence of sets $C_i, i \in \{1, \dots, m\}$ associated with (3) in the form:

$$C_i := \{\boldsymbol{x} \in \mathbb{R}^n : \psi_{i-1}(\boldsymbol{x}) \geq 0\}, \quad i \in \{1, \dots, m\}. \tag{4}$$

Definition 1: (*High Order Control Barrier Function (HOCBF)* [19]) Let $C_1, \dots, C_m$ be defined by (4) and $\psi_1(\boldsymbol{x}), \dots, \psi_m(\boldsymbol{x})$ be defined by (3). A function $b : \mathbb{R}^n \to \mathbb{R}$ is a high order control barrier function (HOCBF) of relative degree m for system (1) if there exist $(m-i)^{th}$ order differentiable class $\mathcal{K}$ functions $\alpha_i, i \in \{1, \dots, m-1\}$ and a class $\mathcal{K}$ function α_m such that $\forall \boldsymbol{x} \in C_1 \cap, \dots, \cap C_m$,

$$\sup_{\boldsymbol{u}\in U}[L_f^m b(\boldsymbol{x}) + L_g L_f^{m-1} b(\boldsymbol{x})\boldsymbol{u} + S(b(\boldsymbol{x})) + \alpha_m(\psi_{m-1}(\boldsymbol{x}))] \geq 0. \tag{5}$$

In (5), L_f^m (L_g) denotes Lie derivatives along f (g) m (one) times, $S(\cdot)$ denotes the remaining Lie derivatives along f with degree $< m$ (omitted for simplicity, see [19]). Assume the number of $\boldsymbol{x}$ such that $L_g L_f^{m-1} b(\boldsymbol{x}) = 0$ is finite.

Theorem 1: ([19]) Given a HOCBF $b(\boldsymbol{x})$ from Def. 1 with the associated sets $C_1, \dots, C_m$ defined by (4), if $\boldsymbol{x}(0) \in C_1 \cap, \dots, \cap C_m$, then any Lipschitz continuous controller $\boldsymbol{u}(t)$ that satisfies (5), $\forall t \geq 0$, renders $C_1 \cap, \dots, \cap C_m$ forward invariant for system (1).

The HOCBF is a general form of the relative degree one CBF [1], [6], [9] (setting $m = 1$ reduces the HOCBF to the common CBF form in [1], [6], [9]). In order to accomodate initial conditions $\boldsymbol{x}(0)$ that are not in C_1, the extended class $\mathcal{K}$ functions are used in the definition of a relative degree one CBF [23], [1]. In this way, a system will be assymptotically stabilized to a safe set that is defined by a safety constraint if the system is initially outside this set, but this may not work for high relative degree constraints, as will be shown in the next section. The HOCBF is also a general form of the exponential CBF [11].

For system (1), consider the following cost:

$$J(\boldsymbol{u}(t)) = \int_0^T \mathcal{C}(||\boldsymbol{u}(t)||)dt \tag{6}$$

where $||\cdot||$ denotes the 2-norm of a vector, and $\mathcal{C}(\cdot)$ is a strictly increasing function.

***Problem** 1 (Optimal Control Problem (OCP)):* Given system (1) with initial condition $\boldsymbol{x}(0)$, find a control law that minimizes cost (6), while satisfying the control bounds (2) and a constraint $b(\boldsymbol{x}) \geq 0$, for all $t \in [0, T]$.

Under the assumption that the cost (6) is quadratic, a conservative solution of the OCP above is obtained through a sequence of QPs, by discretizing the time, keeping the state constant at its value at the beginning of each interval, and solving for a constant optimal control in each interval (note that constraint (5) is linear in control when the state is constant). Most existing approaches use a simpler form of (5), which corresponds to a constraint of relative degree 1 [1], [9], [11]. HOCBFs are used for arbitrary relative degree constraints in [19]. To guarantee the QP feasibility, we can use the analytical approach [22] or adaptive CBF methods [20].

(2) Signal Temporal Logic (STL): In this paper, we use the negation-free signal temporal logic (STL) to specify regions of interest to be reached by the states of system (1). Formal definitions for the syntax and semantics of STL can be found in [10]. Informally, the STL formulas that we use in this paper are predicates over the state $\mu := (b(\boldsymbol{x}) \geq 0)$ ($b : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function of relative degree m with respect to system (1)) connected using the usual Boolean operators (e.g., $\wedge$, $\vee$, $\Rightarrow$) and temporal operators such as $\mathcal{U}_I$ ("until"), $\mathcal{F}_I$ ("eventually"), and $\mathcal{G}_I$ ("always"), where $I = [t_a, t_b]$ is a time interval, with $t_b \geq t_a \geq 0$. We use $\boldsymbol{x} \models \varphi$ to denote that $\boldsymbol{x}$ satisfies φ.

Example: Consider a unicycle model:

$$\dot{x} = v\cos\theta, \qquad \dot{y} = v\sin\theta, \qquad \dot{\theta} = u, \tag{7}$$

where (x, y) denote the coordinates of the robot, $v > 0$ denotes its linear speed, θ is its heading angle, and u denotes its control (angular speed). Formula $\varphi_1 := \mathcal{G}_{[5,6]}(x^2(t) + y^2(t) \leq R^2)$, $R > 0$, requires the robot to satisfy the constraint $x^2(t) + y^2(t) \leq R^2$ for all times in $[5s, 6s]$. Formula $\varphi_2 := \mathcal{F}_{[5,6]}(x^2(t) + y^2(t) \leq R^2)$, $R > 0$, requires the robot to satisfy the constraint $x^2(t) + y^2(t) \leq R^2$ for at least a time instant in $[5s, 6s]$.

III. Problem Formulation and Approach

***Problem** 2 (OCP with STL constraints):* Given system (1) with initial state $\boldsymbol{x}(0)$, and given a STL formula φ over its state $\boldsymbol{x}$, find a control law that minimizes cost (6), while satisfying the control bounds (2) and formula φ.

Assume the STL formula φ can be satisfied for some controllers. In the case that it cannot be satisfied, we explore how to maximally satisfy it, i.e., to maximize the STL robustness. This will be further studied in future work.

Our approach to Problem 2 is based on two types of HOCLBF (class 1 and class 2, shown in the next section) and it can be summarized as follows. First, by exploiting the negation-free structure of formula φ, we break it down (assume it is tractable) into a set of atomic formulae of the type $\mathcal{G}_{[t_a,t_b]}(b(\boldsymbol{x}(t)) \geq 0)$ and $\mathcal{F}_{[t_a,t_b]}(b(\boldsymbol{x}(t)) \geq 0)$. Starting from time $t = 0$, we use a receding horizon $H > 0$ to determine the atomic formulae that we will consider at time t, i.e., we only consider the atomic formulae such that $[t, t+H] \cap [t_a, t_b] \neq \emptyset$ (the choice of H is discussed at the end of Sec. IV). For each predicate involved in these formulae, we define a HOCLBF (we discuss later how to address possible conflicts among these predicates). If the current state satisfies a predicate $b(\boldsymbol{x}(t)) \geq 0$ (the predicate most likely corresponds to a safety requirement), then we use a class 2 HOCLBF, which is a HOCBF as defined in our previous work [19], to derive a controller that makes sure the predicate stays true for all future times. If the current state does not satisfy the predicate (usually related to a state convergence requirement), we use a class 1 HOCLBF that makes sure the system satisfies the predicate before t_b for atomic formulae with $\mathcal{F}_{[t_a,t_b]}$, and before t_a for atomic formulae with $\mathcal{G}_{[t_a,t_b]}$. Once the predicate is satisfied, we switch to a class 2 HOCLBF. We show how the satisfaction of general STL formulae can be enforced with such class 1 and class 2 HOCLBFs.

IV. High Order Control Lyapunov-Barrier Functions

In this section, we define high order control Lyapunov-barrier functions (HOCLBFs) for system (1), and classify them into two classes to accommodate systems with arbitrary initial states. The proofs of all the results from this section are omitted and can be found in [21].

Example revisited: Consider the robot from the previous example and formula φ_1, which requires the satisfaction of constraint $x^2(t) + y^2(t) \leq R^2$ for all times in $[5s, 6s]$. This constraint has relative degree 2 for system (7). If this constraint is satisfied at time 0, then we can define a HOCBF $b(\boldsymbol{x}) := R^2 - x^2(t) - y^2(t)$ such that φ_1 is guaranteed to be satisfied if a controller u satisfies the corresponding HOCBF constraint (5). Otherwise, we cannot define a HOCBF for it since $b(\boldsymbol{x}(0)) < 0$ and the class $\mathcal{K}$ function $\alpha_1(\cdot)$ in (3) only allows for a non-negative argument. Thus, it is impossible to construct the corresponding sets C_1, C_2.

If $b(\boldsymbol{x}(0)) < 0$ and $\dot{b}(\boldsymbol{x}(0)) > 0$, we can then redefine $\psi_i(\boldsymbol{x})$ ($i \in \{1, 2\}$ in this case) in (3) as:

$$\begin{aligned} \psi_1(\boldsymbol{x}) &:= \dot{\psi}_0(\boldsymbol{x}) + p_1\beta_1(\psi_0(\boldsymbol{x})), \\ \psi_2(\boldsymbol{x}) &:= \dot{\psi}_1(\boldsymbol{x}) + \alpha_2(\psi_1(\boldsymbol{x})), \end{aligned} \tag{8}$$

where $\psi_0(\boldsymbol{x}) = b(\boldsymbol{x}), p_1 > 0$. $\beta_1(\cdot)$ and $\alpha_2(\cdot)$ are extended class $\mathcal{K}$ (e.g., $\beta_1(\psi_0(\boldsymbol{x})) = \psi_0^3(\boldsymbol{x})$) and class $\mathcal{K}$ (e.g., $\alpha_2(\psi_1(\boldsymbol{x})) = \psi_1^2(\boldsymbol{x})$) functions, respectively. Since $\dot{b}(\boldsymbol{x}(0)) > 0$ and $b(\boldsymbol{x}(0)) < 0$, we can always choose a small enough p_1 such that $\psi_1(\boldsymbol{x}(0)) \geq 0$ in (8). The HOCBF constraint (5) is the Lie derivative form of $\psi_2(\boldsymbol{x}) \geq 0$ in this case. It follows from Thm. 1 that $\psi_1(\boldsymbol{x}(t)) \geq 0, \forall t \geq 0$ if a controller satisfies the corresponding HOCBF constraint (5). Because $\beta_1(\cdot)$ is an extended class $\mathcal{K}$ function in (8), the robot will be asymptotically stabilized to the set $C_1 := \{\boldsymbol{x} : b(\boldsymbol{x}) \geq 0\}$, but it will never reach the set boundary in finite time, i.e., the STL specification φ_1 cannot be satisfied. If both $b(\boldsymbol{x}(0)) < 0$ and $\psi_1(\boldsymbol{x}(0)) < 0$, the HOCBF fails to work since $\psi_1(\boldsymbol{x}) \geq 0$ is not guaranteed to be satisfied in finite time. Since $\psi_1(\boldsymbol{x}) \geq 0$ is equivalent to $\dot{\psi}_0(\boldsymbol{x}) + \alpha_1(\psi_0(\boldsymbol{x})) \geq 0$ by (3), we have that the original constraint $b(\boldsymbol{x}) \geq 0$ is also not guaranteed to be satisfied. We explore how to solve this problem in the next section.

A. High Order Control Lyapunov-Barrier Function

We introduce HOCLBFs that stabilize a system to a set[1] defined by $b(\boldsymbol{x}) \geq 0$ whose relative degree is m w. r. t. system (1). Similar to (3), we define a sequence of functions:

$$\psi_i(\boldsymbol{x}) := \dot{\psi}_{i-1}(\boldsymbol{x}) + p_i\beta_i(\psi_{i-1}(\boldsymbol{x})), \;\; i \in \{1, \dots, m\}, \tag{9}$$

where $\psi_0(\boldsymbol{x}) := b(\boldsymbol{x})$ and $p_i \geq 0$. $\beta_i(\cdot), i \in \{1, \dots, m\}$ are extended class $\mathcal{K}$ functions.

We also define a sequence of sets as in (4). Note that $\boldsymbol{x}(0) \in C_1$ means that system (1) is initially in the set defined by the constraint $b(\boldsymbol{x}) \geq 0$. If $b(\boldsymbol{x}(0)) > 0$, we can always construct a non-empty set $C_1 \cap, \dots, \cap C_m$ at time 0 by choosing proper class $\mathcal{K}$ functions in the definition of a HOCBF. Otherwise, there are only some extreme cases (such as $b(\boldsymbol{x}(0)) = 0$ and $\dot{b}(\boldsymbol{x}(0)) > 0$) in which we can construct a non-empty set $C_1 \cap, \dots, \cap C_m$, as discussed in [19]. If we cannot construct such a non-empty set at time 0, we construct C_1 as in (4), and construct sets $C_i, i \in \{2, \dots, m\}$ by (9) and (4) such that $\boldsymbol{x}(0) \notin C_1 \cap, \dots, \cap C_m$. Then, we define a HOCLBF as follows:

Definition 2: (*High Order Control Lyapunov-barrier Function (HOCLBF)*) Let $C_1, \dots, C_m$ be defined by (4) and $\psi_1(\boldsymbol{x}), \dots, \psi_m(\boldsymbol{x})$ be defined by (9). A function $b : \mathbb{R}^n \to \mathbb{R}$ is a HOCLBF of relative degree m for system (1) if there exist $(m-i)^{th}$ order differentiable extended class $\mathcal{K}$ functions $\beta_i, i \in \{1, \dots, m-1\}$ and an extended class $\mathcal{K}$ function β_m such that

$$\sup_{\boldsymbol{u} \in U} [L_f^m b(\boldsymbol{x}) + L_g L_f^{m-1} b(\boldsymbol{x})\boldsymbol{u} + R(b(\boldsymbol{x})) + p_m\beta_m(\psi_{m-1}(\boldsymbol{x}))] \geq 0, \tag{10}$$

for all $\boldsymbol{x} \in \mathbb{R}^n$. In (10), $R(\cdot)$ denotes the remaining Lie derivatives along f with degree $< m$ (omitted for simplicity).

We make the following assumption, which is not true in some cases (such as asymptotically growing functions). However, we will relax it in the next subsection.

Assumption 1: If $\psi_{i-1}(\boldsymbol{x}(t)), i \in \{1, \dots, m\}$ is negative at time 0 and there exists a controller $\boldsymbol{u}(t) \in U$ that makes it strictly increasing $\forall t \geq 0$, then, under this controller, assume $\psi_{i-1}(\boldsymbol{x}(t))$ will become non-negative in finite time.

Theorem 2: Given a HOCLBF $b(\boldsymbol{x})$ from Def. 2 with the associated sets $C_1, \dots, C_m$ defined by (4), if $\boldsymbol{x}(0) \in C_1 \cap, \dots, \cap C_m$, then any Lipschitz continuous controller $\boldsymbol{u}(t)$ that satisfies (10), $\forall t \geq 0$ renders $C_1 \cap, \dots, \cap C_m$ forward invariant for system (1). Otherwise, any Lipschitz continuous controller $\boldsymbol{u}(t)$ that satisfies (10), $\forall t \geq 0$ stabilizes system (1) to the set $C_1 \cap, \dots, \cap C_m$.

B. Two Classes of HOCLBFs

In this subsection, we classify HOCLBFs into two classes: one that can achieve finite-time convergence (to a set defined by an arbitrary-relative-degree constraint) if a system is

[1]For simplicity, throughout the paper, we say that a system is *stabilized to a set* if, when initialized outside the set, it reaches the set in finite time and then it stays inside the set for all future times.

initially outside the set, which can help us relax Assumption 1 in Thm. 2, and another one that enforces set forward invariance if a system is initially inside the set.

Since power functions are often used for class $\mathcal{K}$ functions, we consider extended class $\mathcal{K}$ functions as power functions. If $q_i = k$ or $q_i = \frac{1}{k}$, where $k \geq 1$ is an odd number, we rewrite (9) in the form:

$$\psi_i(\boldsymbol{x}) = \dot{\psi}_{i-1}(\boldsymbol{x}) + p_i \psi_{i-1}^{q_i}(\boldsymbol{x}), \tag{11}$$

where $p_i > 0, i \in \{1, \dots, m\}$. Otherwise, the analysis is similar, and thus is omitted. If $q_i \geq 1$, the next lemma shows the asymptotic convergence property of $\psi_{i-1}(\boldsymbol{x})$ in a HOCLBF (we assume 0 is the initial time WLOG):

Lemma 1: Given a HOCLBF $b(\boldsymbol{x})$, if a controller $\boldsymbol{u}(t) \in U$ for (1) satisfies

$$\dot{\psi}_{i-1}(\boldsymbol{x}(t)) + p_i \psi_{i-1}^{q_i}(\boldsymbol{x}(t)) \geq 0, \forall t \geq 0, \tag{12}$$

with $p_i > 0, q_i \geq 1, i \in \{1, \dots, m\}$ and $\psi_{i-1}(\boldsymbol{x}(0)) = \psi_{i-1}^0 \neq 0$, then there exists a lower bound for $\psi_{i-1}(\boldsymbol{x}(t))$, and the lower bound asymptotically approaches 0 as $t \to \infty$.

Note that the extended class $\mathcal{K}$ function $p_i \psi_{i-1}^{q_i}(\boldsymbol{x})$ in (11) is not Lipschitz continuous when $\psi_{i-1}(\boldsymbol{x}) = 0$ if $0 < q_i < 1$. Then, we have the following lemma that demonstrates the finite-time convergence property of $\psi_{i-1}(\boldsymbol{x})$ in a HOCLBF:

Lemma 2: Given a HOCLBF $b(\boldsymbol{x})$, if a controller $\boldsymbol{u}(t) \in U$ for (1) satisfies (12) with $p_i > 0, q_i \in (0,1), i \in \{1, \dots, m\}$ and $\psi_{i-1}(\boldsymbol{x}(0)) = \psi_{i-1}^0 \neq 0$, then there exists a lower bound for $\psi_{i-1}(\boldsymbol{x})$, and the time at which this lower bound becomes 0 is $\frac{(\psi_{i-1}^0)^{1-q_i}}{p_i(1-q_i)}$.

Motivated by the properties from Lems. 1 and 2, we classify HOCLBFs into two classes:

- *Class 1*: if $\exists i \in \{1, \dots, m\}$, s. t. $0 < q_i < 1$ in (11),
- *Class 2*: $q_i \geq 1, \forall i \in \{1, \dots, m\}$ in (11).

Next, we continue to consider the *Class 1* HOCLBF to show its finite-time convergence property with the above lemmas. If $\psi_j(\boldsymbol{x}(t_i)) \geq 0, \forall j \in \{i, \dots, m\}$, where $i \in \{1, \dots, m\}, t_i \geq 0$, then we can define $\psi_i(\boldsymbol{x})$ as a HOCBF to guarantee that $\psi_j(\boldsymbol{x}(t)) \geq 0, \forall j \in \{i, \dots, m\}, \forall t \geq t_i$ [19]. Thus, we assume that a *Class 1* HOCLBF always defines $\psi_i(\boldsymbol{x})$ to be a HOCBF if $\psi_j(\boldsymbol{x}(t_i)) \geq 0, \forall j \in \{i, \dots, m\}$ as it better guarantees finite-time convergence.

Given a *Class 1* HOCLBF $b(\boldsymbol{x})$ with $b(\boldsymbol{x}(0)) < 0$ and $\psi_i(\boldsymbol{x}(0)) = \psi_i^0 \in \mathbb{R}, i \in \{1, \dots, m-1\}$, we define

$$m_0 = \begin{cases} \min\limits_{i \in \{1, \dots, m-1\}: \psi_i^0 > 0} i, & \text{if there exists } i \text{ s.t. } \psi_i^0 > 0 \\ m, & \text{otherwise.} \end{cases} \tag{13}$$

In summary, if $i \leq m_0$, we choose $q_i \in (0,1)$ in (11); otherwise, we choose $q_i \geq 1$ for a *Class 1* HOCLBF.

Let $t_i \geq 0, i \in \{1, \dots, m\}$ denote the starting time instant when $\psi_j(\boldsymbol{x}(t_i)) \geq 0, \forall j \in \{i, \dots, m\}$. Each t_i depends on $\boldsymbol{x}(0)$ and $\boldsymbol{u}(t), t \geq 0$. The following theorem provides the finite-time convergence property of a *Class 1* HOCLBF:

Theorem 3: Given a *Class 1* HOCLBF $b(\boldsymbol{x})$ with $b(\boldsymbol{x}(0)) < 0$, any controller $\boldsymbol{u}(t) \in U$ that satisfies (10) makes (1) converge to the set $C_1 \cap \dots \cap C_m$ within time

$$t_{up} = \sum_{i=1}^{m_0} \frac{(\psi_{i-1}(\boldsymbol{x}(t_i)))^{1-q_i}}{p_i(1-q_i)}. \tag{14}$$

Remark 1: (Chattering in *Class 1* HOCLBFs) By Lemma 2, we have that $\psi_{i-1}(\boldsymbol{x})$ will go to zero within time $t = \frac{(\psi_{i-1}(\boldsymbol{x}(t_i)))^{1-q_i}}{p_i(1-q_i)}$ when $\psi_{i-1}(\boldsymbol{x}(t_i))$ is negative. This could also be true when (12) becomes active if $\psi_{i-1}(\boldsymbol{x}(t_i))$ is positive, which is usually imposed by the state convergence requirement. After $\psi_{i-1}(\boldsymbol{x})$ becomes zero, it will become positive (negative) if it is initially negative (positive) due to the continuity of the dynamics (1). However, $\psi_{i-1}(\boldsymbol{x})$ may go to zero again after it becomes positive (negative), which is usually imposed by state convergence requirements. Recursively, this may cause a *chattering* behavior.

We can relax Assumption 1 by defining a *Class 1* HOCLBF when $\boldsymbol{x}(0) \notin C_1 \cap \dots \cap C_m$ since $\psi_{i-1}(x(t))$ will always cross the boundary $\psi_{i-1}(\boldsymbol{x}(t)) = 0$ in finite time when $\dot{\psi}_{i-1}(\boldsymbol{x}) > 0$, (a condition imposed by $\psi_i(\boldsymbol{x}(t)) \geq 0$ in Def. 2). After $\psi_{i-1}(\boldsymbol{x})$ becomes positive, we can re-define an extended power class $\mathcal{K}$ function with $q_i \geq 1$ for $\psi_i(\boldsymbol{x})$ in (11) in order to eliminate the chattering behavior. This **switching process** is formally shown in [21].

In a nutshell, we would like to define a *Class 1* HOCLBF when $b(\boldsymbol{x}(0)) \leq 0$ as the state of system (1) will converge to the set $C_1 \cap, \dots, \cap C_m$ without Assumption 1 in finite time, and define a *Class 2* HOCLBF when $b(\boldsymbol{x}(0)) > 0$ in which case we can always define $C_i, i \in \{1, \dots, m\}$ such that $\boldsymbol{x}(0) \in C_1 \cap, \dots, \cap C_m$, as shown in [19]. Then the set $C_1 \cap, \dots, \cap C_m$ is forward invariant, as shown in Thm. 2. If we want $\psi_{i-1}(\boldsymbol{x})$ to decrease to 0 slower, we can define a *Class 2* HOCLBF with large q_i value, as shown in [19].

C. HOCLBFs for STL Satisfaction

In this section, we show how we can use HOCLBFs to guarantee the satisfaction of a STL formula. A STL formula can be decomposed into atomic formulae composed of $\mathcal{G}, \mathcal{F}$ operators, and each atomic formula is mapped to a constraint over the state of (1). The receding horizon $H > 0$ of the STL is shown as in Sec. III. If the constraint is satisfied at the current state, we can define a *Class 2* HOCLBF to make sure the predicate always stays true. The implementation is the same as for HOCBF, and thus is omitted; otherwise, we can use *Class 1* HOCLBFs to guarantee it to be satisfied within specified time. Once this constraint is satisfied, we switch to a *Class 2* HOCLBF as shown next.

Always atomic formula $\mathcal{G}$: $\boldsymbol{x} \models \varphi$, where $\varphi := \mathcal{G}_{[t_a, t_b]}(||\boldsymbol{x}(t) - \boldsymbol{K}|| \leq \xi)$, $\boldsymbol{K} \in \mathbb{R}^n, 0 \leq t_a \leq t_b$, and $\xi > 0$, requires the trajectory $\boldsymbol{x}$ of system (1) to satisfy:

$$\forall t \in [t_a, t_b], \quad ||\boldsymbol{x}(t) - \boldsymbol{K}|| \leq \xi. \tag{15}$$

Let $b(\boldsymbol{x}) := \xi - ||\boldsymbol{x} - \boldsymbol{K}||$, where $b(\boldsymbol{x})$ has relative degree m for system (1) and $b(\boldsymbol{x}(0)) < 0$. If we define $b(\boldsymbol{x})$ to be a *Class 1* HOCLBF and choose $p_i > 0, q_i \in (0,1), i \in \{1, 2, \dots, m_0\}$ to satisfy

$$t_a \geq \sum_{i=1}^{m_0} \frac{(\psi_{i-1}(\boldsymbol{x}(t_i)))^{1-q_i}}{p_i(1-q_i)}, \tag{16}$$

then the constraint (15) is guaranteed to be satisfied at t_a following from Thm. 3 and is always satisfied after t_a when we define $b(\boldsymbol{x})$ to be a *Class 2* HOCLBF to avoid chattering. We remove the HOCLBF $b(\boldsymbol{x})$ after t_b. Thus, this atomic formula is guaranteed to be satisfied. Since $\psi_{i-1}(\boldsymbol{x}(t_i)), i \in \{2, \ldots, m\}$ depends on $p_j, q_j, \forall j \in [1, \ldots, i]$, choosing p_i, q_i to satisfy constraint (16) is difficult. However, this can be easily resolved if we define an Adaptive CBF (AdaCBF) [20] that makes p_i, q_i time-varying (adaptive). In this paper, we provide a simple approach to choose p_i, q_i, i.e., we redefine $\psi_i(\boldsymbol{x})$ in (11) as ($p_i > 0$):

$$\psi_i(\boldsymbol{x}) := \begin{cases} \dot{\psi}_{i-1}, \text{ if } i < m_0, \\ \dot{\psi}_{i-1}(\boldsymbol{x}) + p_i \psi_{i-1}^{q_i}, q_i \in (0,1), \text{ if } i = m_0, \\ \dot{\psi}_{i-1}(\boldsymbol{x}) + p_i \psi_{i-1}^{q_i}, q_i \geq 1, \text{ otherwise.} \end{cases} \tag{17}$$

Now, $\psi_{m_0}(\boldsymbol{x})$ in (17) excludes $p_i, q_i, \forall i \in \{1, \ldots, m_0 - 1\}$. We partition the time $[0, t_a]$ into m_0 intervals $\{t_1, \ldots, t_{m_0}\}$ such that $\sum_{i=1}^{m} t_i = t_a$. Each interval corresponds to the time necessary to drive $\psi_{i-1}(\boldsymbol{x}), i \in \{1, \ldots, m_0\}$ in (17) from negative to positive. We update $m_0 \leftarrow m_0 - 1$ whenever $\psi_{m_0-1}(\boldsymbol{x}) > 0$, and then design each pair of p_{m_0}, q_{m_0} according to Lem. 2 and the pre-partitioned time interval mentioned above. Each pair p_i, q_i is determined online (the algorithm is shown in [21]).

Example revisited. For the robot control problem in Sec. II, consider formula φ_1. Let $b(\boldsymbol{x}) = R^2 - x^2 - y^2$ be a *Class 1* HOCLBF. The initial condition of system (7) is given by $(0, -7.7, \frac{\pi}{4})$, $R = 4m, v = 1.732m/s$. We have $b(\boldsymbol{x}(0)) = -43.29$ and $\dot{b}(\boldsymbol{x}(0)) > 0$, and thus, $m_0 = 1$. If we choose $p_1 = 5, q_1 = \frac{1}{3}, t_1 = 4s$, then $\psi_1(\boldsymbol{x}(0)) = 1.3042 > 0$, and $t_1 > \frac{(b(\boldsymbol{x}(0)))^{1-q_1}}{p_1(1-q_1)}$ is satisfied. Thus, by Thm. 3, the formula φ_1 is guaranteed to be satisfied.

Eventually atomic formula $\mathcal{F}$: $\boldsymbol{x} \models \varphi$, where $\varphi := \mathcal{F}_{[t_a,t_b]}(||\boldsymbol{x}(t) - \boldsymbol{K}|| \leq \xi)$, $\boldsymbol{K} \in \mathbb{R}^n, 0 \leq t_a \leq t_b$, and $\xi > 0$, requires the trajectory $\boldsymbol{x}$ of system (1) to satisfy the quantified constraint:

$$\exists t \in [t_a, t_b], \quad ||\boldsymbol{x}(t) - \boldsymbol{K}|| \leq \xi. \tag{18}$$

Let $b(\boldsymbol{x}) := \xi - ||\boldsymbol{x} - \boldsymbol{K}||$, where $b(\boldsymbol{x})$ has relative degree m for system (1) and $b(\boldsymbol{x}(0)) < 0$. If we define $b(\boldsymbol{x})$ to be a *Class 1* HOCLBF and choose $p_i > 0, q_i \in (0,1), i \in \{1, \ldots, m_0\}$ to satisfy

$$t_b \geq \sum_{i=1}^{m_0} \frac{(\psi_{i-1}(\boldsymbol{x}(t_i)))^{1-q_i}}{p_i(1-q_i)}, \tag{19}$$

then constraint (18) is guaranteed to be satisfied before t_b following from Thm. 3. If the predicate $b(\boldsymbol{x}(t)) \geq 0$ is satisfied before t_a, then we will switch to a *Class 2* HOCLBF to make the predicate stay true. We remove the HOCLBF $b(\boldsymbol{x})$ once the constraint (18) is satisfied for any time instant in $[t_a, t_b]$. The approach to choose p_i, q_i is similar to the $\mathcal{G}$.

Disjunction, conjunction, and Until formulae: For conjunctions of atomic formulae, we consider the corresponding HOCLBFs at the same time. We also consider the corresponding HOCLBFs at the same time for the disjunctions of atomic formulae. However, we will relax the one whose barrier function value is smaller when any two of the atomic formulae conflict and remove all the HOCLBFs once any one of these HOCLBFs is non-negative. Note that an Until formula $\mathcal{U}$ is a conjunction of $\mathcal{G}$ and $\mathcal{F}$ atomic formulae [10].

Horizon H and conflict predicates: The horizon H (see the description of the approach in Sec. III) is chosen as large as possible given the available computation resources. While we define a HOCLBF for each atomic formula, it is likely that there will be conflict predicates among the predicates within H, which could make the problem infeasible. To address this, we relax the predicates in formulae with larger t_a, while minimizing the relaxation in the cost function.

V. Case Study

Consider the unicycle described by Eqn. (7). The objective is to minimize the control effort: $\min_{\boldsymbol{u}(t)} \int_0^T u^2(t)dt$. The STL specification is given by

$$\boldsymbol{x} \models (\varphi_1 \Rightarrow \varphi_2) \wedge (\varphi_0 \Rightarrow \varphi_3) \wedge \varphi_4 \wedge \varphi_5 \wedge \varphi_6 \wedge \varphi_7, \tag{20}$$

where $\varphi_0 := (b_1(\boldsymbol{x}(0)) < 0), \varphi_1 := (b_1(\boldsymbol{x}(0)) \geq 0)$, $\varphi_2 = G_{[0,t_b]}(b_1(\boldsymbol{x}) \geq 0)$, $\varphi_3 = G_{[t_a,t_b]}(b_1(\boldsymbol{x}) \geq 0), \varphi_4 = F_{[t_c,t_d]} b_2(\boldsymbol{x}) \geq 0, \varphi_5 = G_{[t_e,T]}(b_3(\boldsymbol{x}) \geq 0), \varphi_6 = G_{[0,T]}(b_4(\boldsymbol{x}) \geq 0), \varphi_7 = G_{[0,T]}(b_5(\boldsymbol{x}) \geq 0), 0 < t_a < t_b < t_c < t_d < t_e < T$, where

$$b_1(\boldsymbol{x}) := R_1^2 - x^2 - y^2 \geq 0, \tag{21}$$

$$b_2(\boldsymbol{x}) := \phi^2 - (\theta - \theta_d)^2 \geq 0, \tag{22}$$

$$b_3(\boldsymbol{x}) := R_2^2 - (x + A_x)^2 + (y + A_y)^2 \geq 0, \tag{23}$$

describe desired sets, with $R_1 > 0, R_2 > 0, \phi > 0, \theta_d \in \mathbb{R}, A_x \in \mathbb{R}, A_y \in \mathbb{R}$. Functions $b_4(\boldsymbol{x})$ and $b_5(\boldsymbol{x})$ describe two obstacles, i.e.,

$$b_4(\boldsymbol{x}) := (x + O_{x,1})^2 + (y + O_{y,1})^2 - R_3^2 \geq 0, \tag{24}$$

$$b_5(\boldsymbol{x}) := (x + O_{x,2})^2 + (y + O_{y,2})^2 - R_4^2 \geq 0, \tag{25}$$

where $R_3 > 0, R_4 > 0, (O_{x,1}, O_{y,1}) \in \mathbb{R}^2, (O_{x,2}, O_{y,2}) \in \mathbb{R}^2$.

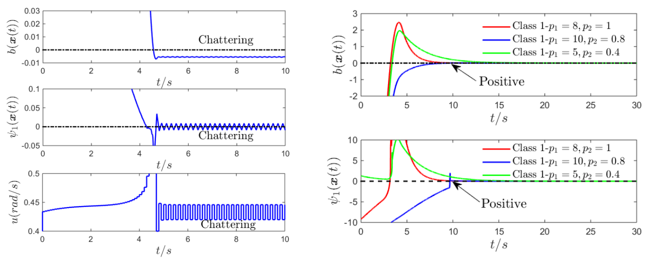
In plain English, the STL specification states that, if the robot is initially in the set defined by constraint (21), then it should stay there $\forall t \in [0, t_b]$. Otherwise, it should stay in this set $\forall t \in [t_a, t_b]$. The heading of the robot should be θ_d with error ϕ for at least a time instant in $[t_c, t_d]$, and the robot should stay in the set defined by (23) $\forall t \in [t_e, T]$. The robot should always avoid the obstacles defined by (25) (24).

The control limitation is defined as: $u_{min} \leq u \leq u_{max}$, where $u_{min} < 0, u_{max} > 0$. The relative degrees of all the constraints (21)-(25) with respect to (7) are 2. We solve the OCP with the approach introduced in Sec. II.

We implemented the proposed algorithms in MATLAB. We used Quadprog to solve the QPs and ODE45 to integrate the dynamics. Simulations for initially violated constraints to study *Class 1* and *Class 2* HOCLBFs can be found in [21]. We just show chattering and present the complete solution to the OCP with STL specifications in this paper.

(1) Chattering Behavior: We consider *Class 1* HOCLBFs to study chattering behaviors with the atomic formula φ_3. The robot starts inside the set $C_1 := \{\boldsymbol{x} : b_1(\boldsymbol{x}) \geq 0\}$

with $\boldsymbol{x}(0) = (0, -3.7, 0), v = 1.732m/s$. Other simulation parameters are $t_b = 30s, \Delta t = 0.1, u_{max} = -u_{min} = 0.6rad/s, R = 4m$. There would be chattering for the robot if we define a *Class 1* HOCLBF for the safety constraint (21), as the blue curves shown in Fig. 1(a). In order to avoid chattering, we switch a *Class 1* HOCLBF to a *Class 2* HOCLBF, as shown in Sec. IV-B. The results of three *Class 1* HOCLBFs with the switch method to avoid chattering are shown in Fig. 1(b).



(a) Chattering behaviors. (b) The switch method.

Fig. 1. Chattering behaviors ($p_1 = 6, p_2 = 0.14, q_1 = q_2 = \frac{1}{3}$) and the switch method for *Class 1* HOCLBFs.

(2) Complete Solution: The simulation parameters are $T = 32s, t_a = 4s, t_b = 5s, t_c = 7s, t_d = 9s, t_e = 21s, \Delta t = 0.1s, R_1 = 4m, R_2 = 4m, R_3 = 2m, R_4 = 3m, Ax = 10m, A_y = 10m, \phi = \frac{\pi}{12}, \theta_d = \frac{5\pi}{4}, O_{x,1} = 8m, O_{y,1} = 4m, O_{x,2} = 10m, O_{y,2} = 10m, u_{\max} = -u_{\min} = 0.9rad/s, v = 1.732m/s, H = 10s$. The robot initial state is $(0, -7.7, \frac{\pi}{4})$.

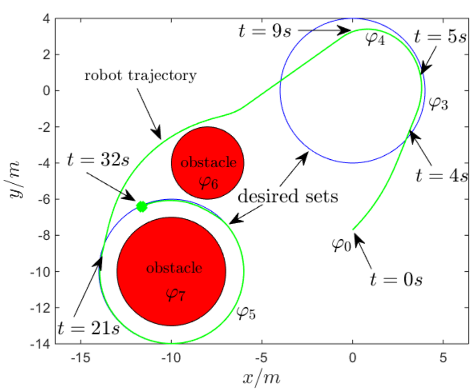


Fig. 2. A trajectory that satisfies the STL specification with HOCLBFs.

We choose $q_1 = q_2 = \frac{1}{3}$ for all *Class 1* HOCLBFs, and choose $q_1 = q_2 = 1$ for all *Class 2* HOCLBFs. Then we get (p_1, p_2) with the approach introduced in Sec. IV-C as $(5, 0.4), (0.8, N/A), (4.85, 0.4)$ for the atomic formulae $\varphi_3, \varphi_4, \varphi_5$, respectively. Note that the relative degree of (22) is one, so φ_4 only has p_1. The p_1, p_2 for φ_6, φ_7 are chosen according to the penalty method [19] such that the QP is feasible. When the *Class 1* HOCLBF (desired set) conflicts with the *Class 2* HOCLBF (safety), we relax the *Class 1* HOCLBF. The STL formula is satisfied, as shown in Fig. 2.

VI. Conclusion

We propose high order control Lyapunov-barrier functions (HOCLBF) that work for constraints with arbitrary relative degree and systems with arbitrary initial state. We show how the proposed HOCLBFs can be used to enforce the satisfaction of Signal Temporal Logic (STL) specifications. Simulation results on a unicycle model demonstrate the effectiveness of the proposed method. Future work will focus on the robust satisfaction of STL specifications.

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