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Exponential stability of impulsive stochastic differential equations with Markovian switching

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A B S T R A C T

This paper is devoted to moment exponential stability of a class of Markovian switching stochastic differential equations with impulsive perturbations. Taking into consideration the long time behavior of the switching device, we derive explicit criteria for stability and instability. The contribution of the Markovian switching and impulsive perturbations to the stability and instability is revealed.

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1. Introduction

This work focuses on moment exponential stability of a class of stochastic differential equations (SDEs) with Markovian switching and impulsive perturbations. The underlying process is a two-component process $(X(\cdot), \alpha(\cdot))$, where $X(\cdot)$ describes the diffusion behavior with impulsive perturbations while $\alpha(\cdot)$ is a Markov chain. Recently, the stability of such a class of stochastic processes has been studied in various settings for different domain of applications. To mention just a few, we refer to [1] for an intensive reference of impulsive differential equations and [2,3] for modeling and analysis of hybrid systems. Related works on asymptotic behaviors and stabilization of stochastic hybrid systems can be found in [3–8].

Let d, m be positive integers, $\mathcal{M} = \{1, 2, \dots, m\}$, $\mathbb{R}_+ = [0, \infty)$, and $\mathbb{N} = \{1, 2, \dots\}$. Consider the dynamic system in $\mathbb{R}^d \times \mathcal{M}$ given by

$$\begin{aligned} &dX(t) = b(X(t), t, \alpha(t))dt + \sigma(X(t), t, \alpha(t))dw(t), \quad t \ge 0,\\ &X(t_k) = I_k\big(X(t_k^-), \alpha(t_k^-)\big), \quad k \in \mathbb{N}, \end{aligned} \tag{1.1}$$

with initial condition $(X(0), \alpha(0)) = (x^0, i^0)$, where $b(\cdot)$, $\sigma(\cdot)$, and $I_k(\cdot)$ are suitable functions, $w(\cdot)$ is a Brownian motion, $\alpha(\cdot)$ is a finite state Markov chain, $\{t_k\}_{k\in\mathbb{N}}$ is a strictly increasing sequence of positive numbers satisfying $\lim_{k\to\infty} t_k = \infty$, $X(t_k^-) = \lim_{t\to t_k^-} X(t)$, and $\alpha(t_k^-) = \lim_{t\to t_k^-} \alpha(t)$. We defer the discussion of the precise formulation and conditions needed to the next section. It can be seen that there are impulsive jumps in the component $X(\cdot)$ at time t_k for $k \in \mathbb{N}$. If $I_k(x, i) = x$ for any $(x, i, k) \in \mathbb{R}^d \times \mathcal{M} \times \mathbb{N}$, then $\big(X(\cdot), \alpha(\cdot)\big)$ is simply a Markovian switching SDE studied in [2,6]. For SDEs under Markovian switching, it has been demonstrated that although they are similar to SDEs, they have some distinct features. With the impulsive perturbation taken into account, the distinctions are even more pronounced. However, the stability analysis for such hybrid systems is much more delicate than the impulsive-free case. In particular, one needs to treat both possible jumps in $\alpha(\cdot)$ and impulsive jumps in $X(\cdot)$.

As pointed out in [1,9,10], impulsive perturbations are observed in information science, electronics, automatic control systems, computer networking, artificial intelligence, robotics, telecommunications, population models, neural networks, and economics. Many sudden and sharp changes occur instantaneously, in the form of impulses, which cannot be well modeled by using purely continuous or purely discrete descriptions. Thus, it is important to study impulsive systems. In particular, it is critical to know whether the impulsive systems are stable; see [10, Chapter 4] for a detailed discussion concerning population models, neural networks, and economic models. Focusing on Eq. (1.1), we are interested in the notion of moment exponential stability, which is one of the main issues in stochastic stability. Under certain conditions, we show that the moment

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exponential stability implies almost sure exponential stability; see Theorem 3.13.

In recent years, there have been growing interests devoted to the study of moment exponential stability analysis of impulsive systems. To mention just a few, we refer to [11–14] for moment stability analysis of impulsive SDEs. The works [15–21] focus on moment exponential stability of various forms of impulsive stochastic functional differential systems. Moreover, recent results on Razumikhin-type theorems for stochastic functional differential equations with Lévy noise and Markov switching and for impulsive stochastic delay systems are obtained in [22,23]. Although the stability criteria in the aforementioned papers are useful, there are certain points that have not been fully investigated. First, we observe that most stability criteria for impulsive systems are given in terms of the existence of certain Lyapunov-type functions or the existence of certain matrices satisfying a set of conditions. In practice, it is not easy to construct such Lyapunov-type functions or matrices. It is important to construct verifiable criteria for stability and instability. Second, the contribution of $\alpha(\cdot)$ to the moment exponential stability has not been explicitly revealed for impulsive systems. In particular, can the switching process $\alpha(\cdot)$ make an impulsive system stable if an impulsive subsystem is stable while the other subsystems are unstable? This paper can also be seen as a further step of the work [14,21,24]. In [14], the authors studied moment stability of Eq. (1.1) while we focus on moment stability with an exponentially fast convergence. In [21], the authors considered moment stability and moment exponential stability for an extension of Eq. (1.1) with delays. However, the work [14] and [21] does not address the two aforementioned points. The paper [24] treats a Markovian switching SDEs with Poisson jumps. However, the results in [24] do not work for Eq. (1.1) because of impulsive perturbations in Eq. (1.1).

Focusing on SDEs with Markovian switching and impulsive perturbations, our objective is to address the aforementioned issues. The novelty of this work lies in the use of the martingale theory and large deviation techniques to establish new and explicit criteria for moment exponential stability and instability of switching impulsive systems. In contrast to the existing literature, our main contributions in this work can be summarized as follows.

(a) We construct general sufficient conditions and explicit criteria for moment exponential stability and instability. These criteria are verified based on the system coefficients $b(\cdot,\cdot)$, $\sigma(\cdot,\cdot)$, the impulsive functions $I_k(\cdot,\cdot)$, a function $\Lambda(\cdot)$ to be introduced, and the stationary distribution of $\alpha(\cdot)$. Such explicit criteria enables us to develop suitable controls for stabilization; see Examples 3.12 and 4.4. These criteria can also be used to investigate the stability of various impulsive systems in population dynamics, neural networks, and economic models to mention just a few; see Example 4.6, see also [10, Chapter 4] for more applications.

(b) We are successful to take into consideration the long time behavior of the switching device via a function $\Lambda(\cdot)$ and its stationary distribution. Thus, we reveal explicitly the impact of the switching process to the pth moment stability and instability of impulsive systems under consideration; see Remarks 3.8 and 3.9.

The rest of the work is organized as follows. Section 2 begins with the problem formulation. Section 3 proceeds with several general criteria of the moment exponential stability and instability. Section 4 furthers our investigation by establishing explicit criteria. Finally, the paper is concluded with a conclusion section.

2. Formulation

Assume throughout the work that both the Markov chain $\alpha(\cdot)$ and the d-dimensional standard Brownian motion $w(\cdot)$ are defined on a complete filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\})$ with the filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions (i.e., it is right-continuous and $\mathcal{F}_0$ contains all null sets). Suppose that $\alpha(\cdot)$ and $w(\cdot)$ are independent. Moreover, $\alpha(\cdot)$ takes values in $\mathcal{M}=\{1,2,\dots,m\}$ with the generator $Q=(q_{ij})\in\mathbb{R}^{m\times m}$. The evolution of $\alpha(\cdot)$ is described by a transition probability specification of the form

$$\mathbb{P}\{\alpha(t+\Delta t)=j|\alpha(t)=i,\alpha(s),s\le t\} = \begin{cases} q_{ij}\Delta t+o(\Delta t) & \text{if } i\neq j,\\ 1+q_{ii}\Delta t+o(\Delta t) & \text{if } i=j. \end{cases} \tag{2.1}$$

Note that $q_{ij}\ge 0$ if $i\neq j$ and $\sum_{j=1}^m q_{ij}=0$ for any $i\in\mathcal{M}$.

Let $\{t_k\}_{k\in\mathbb{N}}$ be a strictly increasing sequence of positive numbers satisfying $\lim_{k\to\infty} t_k=\infty$. For each $k\in\mathbb{N}$, the impulsive function at time t_k is given by $I_k:\mathbb{R}^d\times\mathcal{M}\to\mathbb{R}^d$. The component $X(\cdot)$ of the two-component process $\big(X(\cdot),\alpha(\cdot)\big)$ is given by the impulsive SDE

$$\begin{aligned} &dX(t)=b(X(t),t,\alpha(t))dt+\sigma(X(t),t,\alpha(t))dw(t), \quad t\ge 0,\\ &X(t_k)=I_k\big(X(t_k^-),\alpha(t_k^-)\big), \quad k\in\mathbb{N}. \end{aligned} \tag{2.2}$$

with initial condition $\big(X(0),\alpha(0)\big)=(x^0,i^0)\in\mathbb{R}^d\times\mathcal{M}$, $b:\mathbb{R}^d\times\mathbb{R}_+\times\mathcal{M}\to\mathbb{R}^d$, $\sigma:\mathbb{R}^d\times\mathbb{R}_+\times\mathcal{M}\to\mathbb{R}^{d\times d}$. For convenience, let $t_0=0$.

Notation. For two real numbers c_1,c_2, $c_1\wedge c_2$ denotes $\min\{c_1,c_2\}$. For a matrix $A\in\mathbb{R}^{d_1\times d_2}$, $A^\top$ denotes its transpose. For a matrix $A\in\mathbb{R}^{d\times d}$, its trace norm is given by $|A|=\sqrt{\operatorname{tr}(AA^\top)}$, while I_d denotes the $d\times d$ identity matrix. For $x=(x_1,\dots,x_d)^\top\in\mathbb{R}^d$, its Euclidean norm is denoted by $|x|=\big(\sum_{i=1}^d x_i^2\big)^{1/2}$. For any symmetric matrix $A\in\mathbb{R}^{d\times d}$, $\Lambda_{\max}(A)$ and $\Lambda_{\min}(A)$ denote its largest eigenvalue and smallest eigenvalue, respectively. Let $\rho(A)=\max\{|\Lambda_{\max}(A)|,|\Lambda_{\min}(A)|\}$.

The operator $\mathcal{G}$ associated with the process $(X(t),\alpha(t))$ is given as follows. Suppose $V:\mathbb{R}^d\times\mathbb{R}_+\times\mathcal{M}\to\mathbb{R}$ and $V(\cdot,\cdot,i)\in C^{2,1}(\mathbb{R}^d\times\mathbb{R}_+)$ for each $i\in\mathcal{M}$. Then

$$\begin{aligned}(\mathcal{G}V)(x,t,i)=&\,V_t(x,t,i)+b^\top(x,t,i)V_x(x,t,i)\\ &+\frac12\operatorname{tr}\big(\sigma^\top(x,t,i)V_{xx}(x,t,i)\sigma(x,t,i)\big)+(QV)(x,t,i),\end{aligned}$$

where $V_t(\cdot)=\partial V/\partial t$, $V_x(\cdot,t,i)$ and $V_{xx}(\cdot,t,i)$ denote the gradient and Hessian matrix of $V(\cdot,t,i)$, respectively, and $(QV)(x,t,i)=\sum_{j\in\mathcal{M}} q_{ij}V(x,t,j)$. Since Q is the generator of $\alpha(\cdot)$, $\sum_{j\in\mathcal{M}} q_{ij}=0$ for any $i\in\mathcal{M}$. It follows that

$$(QV)(x,t,i)=\sum_{j\in\mathcal{M},j\neq i} q_{ij}\big(V(x,t,j)-V(x,t,i)\big) \quad \text{for} \quad (x,t,i)\in\mathbb{R}^d\times\mathbb{R}_+\times\mathcal{M}.$$

The standing assumption is given below.

(A) (a) We have

$$b(0,t,i)=\sigma(0,t,i)=0 \quad \text{for all} \quad (t,i)\in\mathbb{R}_+\times\mathcal{M}.$$

(b) For any real number $T>0$ and $k\in\mathbb{N}$, there exists a positive number $K_{T,k}$ such that for all $t\in[0,T]$, $i\in\mathcal{M}$, and all $x,y\in\mathbb{R}^d$ with $\max\{|x|,|y|\}\le k$,

$$|b(x,t,i)-b(y,t,i)|^2+|\sigma(x,t,i)-\sigma(y,t,i)|^2\le K_{T,k}|x-y|^2.$$

Also there exists a constant $K_{0,0}>0$ such that

$$x^\top b(x,t,i)+|\sigma(x,t,i)|^2\le K_{0,0}(1+|x|^2) \quad \text{for all} \quad (x,t,i)\in\mathbb{R}^d\times\mathbb{R}_+\times\mathcal{M}.$$

(c) There exist positive constants γ_k and $\widetilde{\gamma}_k$ for $k \in \mathbb{N}$ such that

$$\gamma_k|x| \le \big|I_k(x,i)\big| \le \widetilde{\gamma}_k|x| \quad \text{for} \quad (x,i) \in \mathbb{R}^d \times \mathcal{M}, k \in \mathbb{N}.$$

(d) The Markov chain $\alpha(t)$ is irreducible; that is, the system of equations

$$\nu Q = 0, \quad \sum_{i\in\mathcal{M}} \nu_i = 1$$

has a unique solution $\nu = (\nu_1, \dots, \nu_m)$ satisfying $\nu_i > 0$ for each $i = 1, 2, \dots, m$.

Remark 2.1. Assumption (A)(a) indicates that the process $X(t) \equiv 0$ is a trivial solution of Eq. (2.2). Under assumption (A), with the same method as in [2,6], we can show that for each initial condition $(x^0, i^0) \in \mathbb{R}^d \times \mathcal{M}$, the system given by Eq. (2.1) and Eq. (2.2) has a unique strong solution $\big(X(\cdot), \alpha(\cdot)\big)$ satisfying $X(0) = x^0, \alpha(0) = i^0$. The sample paths of $\big(X(\cdot), \alpha(\cdot)\big)$ are right continuous and have left limits. Moreover, the solution $\big(X(\cdot), \alpha(\cdot)\big)$ is global; that is, it is defined for any $t \ge 0$. Then assumption (A)(c) guarantees the nonzero property; that is, if $x^0 \ne 0$, then

$$\mathbb{P}\Big(X(t) \ne 0 \quad \text{for all} \quad t \ge 0\Big) = 1.$$

It enables us to work with the functions that are twice continuously differentiable in $\mathbb{R}^d \setminus \{0\}$. Moreover, for any $T > 0$, $X(\cdot)$ satisfies

$$\mathbb{E}\Big[\sup_{0\le t\le T} |X(t)|^p\Big] < \infty \quad \text{for any} \quad p > 0;$$

see [2, Theorem 3.24] and [3, Proposition 2.3]. Assumption (A)(d) is related to the long time behavior of the switching device.

Now we state the definition of pth moment exponential stability and instability for switching SDEs with impulsive perturbations for a positive number p.

Definition 2.2.

(a) The trivial solution of Eq. (2.2) is said to be pth moment exponentially stable if there are positive constants β and K such that

$$\mathbb{E}\Big|X^{x^0,i^0}(t)\Big|^p \le Ke^{-\beta t}\Big|x^0\Big|^p \quad \text{for any} \quad t \ge 0, (x^0, i^0) \in \mathbb{R}^d \times \mathcal{M}.$$

(b) The trivial solution of Eq. (2.2) is said to be pth moment exponentially unstable if there are positive constants β and K such that

$$\mathbb{E}\Big|X^{x^0,i^0}(t)\Big|^p \ge Ke^{\beta t}\Big|x^0\Big|^p \quad \text{for any} \quad t \ge 0, (x^0, i^0) \in \mathbb{R}^d \times \mathcal{M}.$$

3. General criteria for moment exponential stability and instability

To prepare for the development in this section, we state the definition of $\Lambda : \mathbb{R}^m \to \mathbb{R}$; see [24, page 2598]. We refer to [24, Appendix A], [25, page 136–137], and [26, page 22] for details.

Definition 3.1. Let $\mathcal{P}$ be the set of all probability vectors on the state space $\mathcal{M}$. Denote

$$\mathcal{I}(\mathfrak{p}) = -\inf_{u_1>0,\dots,u_m>0} \sum_{i,j\in\mathcal{M}} \frac{\mathfrak{p}_i q_{ij} u_j}{u_i},$$

where $\mathfrak{p} = (\mathfrak{p}_1, \dots, \mathfrak{p}_m) \in \mathcal{P}$ is a probability vector. For $\eta = (\eta_1, \dots, \eta_m)^\top \in \mathbb{R}^m$, define

$$\Lambda(\eta) = \sup_{\mathfrak{p}\in\mathcal{P}}\Big(\sum_{i\in\mathcal{M}} \eta_i \mathfrak{p}_i - \mathcal{I}(\mathfrak{p})\Big).$$

Remark 3.2. Let $\eta = (\eta_1, \dots, \eta_m)^\top \in \mathbb{R}^m$. Throughout this paper, $\{\eta_{\alpha(t)}\}_{t\ge 0}$ is a process defined by $\eta_{\alpha(t)} = \eta_i$ if $\alpha(t) = i \in \mathcal{M}$. Thus, $\eta_{\alpha(t)}$ takes values in $\{\eta_1, \dots, \eta_m\}$. It is proved that

$$\Lambda(\eta) = \lim_{t\to\infty} \frac{1}{t} \ln\Big\{\mathbb{E}\Big[e^{\int_0^t \eta_{\alpha(s)}ds}\Big]\Big\};$$

see [27, Lemma A.3]. Moreover, $\sum_{i\in\mathcal{M}} \nu_i\eta_i \le \Lambda(\eta) \le \max_{i\in\mathcal{M}} \eta_i$. As recognized in [24,27], $\Lambda(\cdot)$ will play an important role in investigating the pth moment exponential stability of hybrid systems.

Theorem 3.3. *Assume* (A). *Let $V : \mathbb{R}^d \to \mathbb{R}$ be a twice continuously differentiable on $\mathbb{R}^d \setminus \{0\}$ satisfying $c_1|x|^p \le V(x) \le c_2|x|^p$ for all $x \in \mathbb{R}^d$, where c_1 and c_2 are two positive numbers. Suppose that there exist $\eta_p = (\eta_{p,1}, \eta_{p,2}, \dots, \eta_{p,m})^\top \in \mathbb{R}^m$ and a sequence of positive numbers $\{\delta_k\}_{k\in\mathbb{N}}$ such that*

$$(\mathcal{G}V)(x,t,i) \le \eta_{p,i}V(x) \quad \textit{for all} \quad (x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathcal{M}, x \ne 0 \tag{3.1}$$

and

$$V(I_k(x,i)) \le |\delta_k|^p V(x) \quad \textit{for all} \quad (x,i) \in \mathbb{R}^d \times \mathcal{M}, k \in \mathbb{N}. \tag{3.2}$$

Suppose that β is a positive number and

$$\Lambda(\eta_p) + \limsup_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\le t} p\ln|\delta_j|\Big] < -\beta. \tag{3.3}$$

Then the trivial solution of Eq. (2.2) *is pth moment exponentially stable. Moreover, there is a positive number K such that*

$$\mathbb{E}\Big|X^{x^0,i^0}(t)\Big|^p \le Ke^{-\beta t}\Big|x^0\Big|^p \quad \textit{for any} \quad t \ge 0, (x^0, i^0) \in \mathbb{R}^d \times \mathcal{M}.$$

Proof. We divide the proof into two steps.

Step 1. In this step, we work with a fixed value of $k \in \mathbb{N}$. Let $(x^0, i^0) \in \mathbb{R}^d \times \mathcal{M}$. Without loss of generality, we suppose $x^0 \ne 0$. For notational simplicity, we denote $X(t) = X^{x^0,i^0}(t)$ and $\alpha(t) = \alpha^{i^0}(t)$. Let $\{\tau_n\}_n$ be the sequence of stopping times defined by

$$\tau_n = \inf\{t \ge t_{k-1} : |X(t)| \ge n\} \wedge t_k \quad \text{for} \quad n \in \mathbb{N}.$$

Then $\tau_n \to t_k$ as $n \to \infty$ almost surely since the process $X(\cdot)$ is a global solution of Eq. (2.2). By the Itô formula, we obtain for $t \in [t_{k-1}, t_k)$ that

$$\begin{aligned} \exp\Big(-\int_{t_{k-1}}^{t\wedge\tau_n} \eta_{p,\alpha(s)}ds\Big)V\big(X(t\wedge\tau_n)\big) &= V\big(X(t_{k-1})\big) \\ &+ \int_{t_{k-1}}^{t\wedge\tau_n} \exp\Big(-\int_{t_{k-1}}^{s} \eta_{p,\alpha(u)}du\Big) \\ &\times\Big[-\eta_{p,\alpha(s)}V\big(X(s)\big) + (\mathcal{G}V)(X(s),s,\alpha(s))\Big]ds \\ &+M(t\wedge\tau_n), \end{aligned} \tag{3.4}$$

where

$$M(t) = \int_{t_{k-1}}^{t} \exp\Big(-\int_{t_{k-1}}^{s} \eta_{p,\alpha(u)}du\Big)V_x\big(X(s)\big)\sigma\big(X(s),s,\alpha(s)\big)dw(s), \quad t \in [t_{k-1}, t_k).$$

By (3.4) and assumption (3.1), we have

$$\exp\Big(-\textstyle\int_{t_{k-1}}^{t\wedge\tau_n} \eta_{p,\alpha(s)}ds\Big)V\big(X(t\wedge\tau_n)\big) \le V\big(X(t_{k-1})\big) + M(t\wedge\tau_n). \tag{3.5}$$

Let $\mathcal{D}_t$ be the σ-algebra generated by $\{\alpha(u), w(s) : 0 \le u, 0 \le s \le t\} \cup \mathcal{F}_0$. In other words, $\mathcal{D}_t$ is generated by the whole process $\{\alpha(s) : 0 \le s < \infty\}$, $\{w(s) : s \in [0,t]\}$, and $\mathcal{F}_0$. By

[28, Chapter 1, Thm. 31], the filtration $\{\mathcal{D}_t\}$ is right continuous and thus it satisfies the usual conditions. Moreover, $M(t)$ is a local martingale with respect to the filtration $\{\mathcal{D}_t\}$ for $t \in [t_{k-1}, t_k)$; see [29, Theorem 4.2.12]. Note also that $X(t_{k-1})$ is measurable with respect to $\mathcal{D}_{t_{k-1}}$. Thus,

$$\mathbb{E}\Big[V\big(X(t_{k-1})\big)\Big|\mathcal{D}_{t_{k-1}}\Big] = V\big(X(t_{k-1})\big),\quad \mathbb{E}\Big[M(t\wedge\tau_n)\Big|\mathcal{D}_{t_{k-1}}\Big] = 0 \text{ for } t \in [t_{k-1}, t_k).$$

This together with (3.5) implies

$$\mathbb{E}\Big[\exp\Big(-\int_{t_{k-1}}^{t\wedge\tau_n} \eta_{p,\alpha(s)}ds\Big)V\big(X(t\wedge\tau_n)\big)\Big|\mathcal{D}_{t_{k-1}}\Big] \le V\big(X(t_{k-1})\big).$$

Letting $n \to \infty$ yield

$$\mathbb{E}\Big[\exp\Big(-\int_{t_{k-1}}^{t} \eta_{p,\alpha(s)}ds\Big)V\big(X(t)\big)\Big|\mathcal{D}_{t_{k-1}}\Big] \le V\big(X(t_{k-1})\big). \tag{3.6}$$

Recall that $D_{t_{k-1}}$ is the σ-algebra generated by $\{\alpha(u), w(s) : 0 \le u, 0 \le s \le t_{k-1}\} \cup \mathcal{F}_0$. Hence, $\exp\Big(-\int_{t_{k-1}}^{t} \eta_{p,\alpha(s)}ds\Big)$ is measurable with respect to $\mathcal{D}_{t_{k-1}}$. As a result,

$$\mathbb{E}\Big[\exp\Big(-\int_{t_{k-1}}^{t} \eta_{p,\alpha(s)}ds\Big)V\big(X(t)\big)\Big|\mathcal{D}_{t_{k-1}}\Big] = \exp\Big(-\int_{t_{k-1}}^{t} \eta_{p,\alpha(s)}ds\Big)\mathbb{E}\Big[V\big(X(t)\big)\Big|\mathcal{D}_{t_{k-1}}\Big]. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\mathbb{E}\Big[V\big(X(t)\big)\Big|\mathcal{D}_{t_{k-1}}\Big] \le V\big(X(t_{k-1})\big)\exp\Big(\int_{t_{k-1}}^{t} \eta_{p,\alpha(s)}ds\Big) \quad \text{for} \quad t \in [t_{k-1}, t_k). \tag{3.8}$$

In view of (3.2), we obtain

$$\begin{aligned}\mathbb{E}\Big[V\big(X(t_k)\big)\Big|\mathcal{D}_{t_{k-1}}\Big] &\le |\delta_k|^p\mathbb{E}\Big[V\big(X(t_k^-)\big)\Big|\mathcal{D}_{t_{k-1}}\Big]\\ &\le |\delta_k|^p V\big(X(t_{k-1})\big)\exp\Big(\int_{t_{k-1}}^{t_k} \eta_{p,\alpha(s)}ds\Big).\end{aligned} \tag{3.9}$$

Step 2. To proceed, we note that (3.8) and (3.9) hold for any $k \in \mathbb{N}$. Letting $k = 1$ in (3.9) yield

$$\mathbb{E}\Big[V\big(X(t_1)\big)\Big|\mathcal{D}_0\Big] \le V(x^0)\exp\Big(\int_0^{t_1} \eta_{p,\alpha(s)}ds\Big)|\delta_1|^p. \tag{3.10}$$

Since $\mathcal{D}_0 \subset \mathcal{D}_{t_1}$, then

$$\mathbb{E}\Big[V\big(X(t_2)\big)\Big|\mathcal{D}_0\Big] = \mathbb{E}\Big[\mathbb{E}\Big[V\big(X(t_2)\big)\Big|\mathcal{D}_{t_1}\Big]\Big|\mathcal{D}_0\Big]. \tag{3.11}$$

Letting $k = 2$ in (3.9) yield

$$\mathbb{E}\Big[V\big(X(t_2)\big)\Big|\mathcal{D}_{t_1}\Big] \le V\big(X(t_1)\big)\exp\Big(\int_{t_1}^{t_2} \eta_{p,\alpha(s)}ds\Big)|\delta_2|^p. \tag{3.12}$$

It follows from (3.10), (3.11), and (3.12) that

$$\mathbb{E}\Big[V\big(X(t_2)\big)\Big|\mathcal{D}_0\Big] \le V(x^0)\exp\Big(\int_0^{t_2} \eta_{p,\alpha(s)}ds\Big)|\delta_2|^p|\delta_1|^p.$$

By induction, we arrive at

$$\mathbb{E}\Big[V\big(X(t_k)\big)\Big|\mathcal{D}_0\Big] \le V(x^0)\exp\Big(\int_0^{t_k} \eta_{p,\alpha(s)}ds\Big)\prod_{j=1}^k|\delta_j|^p,$$

which implies that

$$\mathbb{E}\Big[V\big(X(t_k)\big)\Big] \le V(x^0)\mathbb{E}\Big[\exp\Big(\int_0^{t_k} \eta_{p,\alpha(s)}ds\Big)\Big]\prod_{j=1}^k|\delta_j|^p \quad \text{for} \quad k \in \mathbb{N}. \tag{3.13}$$

In view of (3.8) and (3.13), we have

$$\mathbb{E}\Big[V\big(X(t)\big)\Big] \le V(x^0)\mathbb{E}\Big[\exp\Big(\int_0^{t} \eta_{p,\alpha(s)}ds\Big)\Big]\prod_{j=1}^k|\delta_j|^p \quad \text{for} \quad t \in [t_k, t_{k+1}).$$

Thus,

$$\mathbb{E}\Big[V\big(X(t)\big)\Big] \le V(x^0)\mathbb{E}\Big[\exp\Big(\sum_{j:t_j\le t} p\ln|\delta_j| + \int_0^{t} \eta_{p,\alpha(s)}ds\Big)\Big] \quad \text{for} \quad t \ge 0.$$

Since

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\ln\mathbb{E}\Big[\exp\Big(\sum_{j:t_j\le t} p\ln|\delta_j| + \int_0^{t} \eta_{p,\alpha(s)}ds\Big)\Big]\\ &= \Lambda(\eta_p) + \limsup_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\le t} p\ln|\delta_j|\Big] < -\beta,\end{aligned}$$

there is a positive number K such that

$$\mathbb{E}\Big[|X(t)|^p\Big] \le Ke^{-\beta t}|x^0|^p \quad \text{for all} \quad t \ge 0, (x^0, i^0) \in \mathbb{R}^d \times \mathcal{M}.$$

The conclusion follows. □

Remark 3.4. In the statement of Theorem 3.3, we state a list of several assumptions under which the trivial solution of Eq. (2.2) is pth moment exponentially stable. These assumptions are based on the existence of a Lyapunov function $V(\cdot)$ and the sequence $\{I_k(\cdot)\}$ satisfying (3.1), (3.2), and (3.3). In the next section, explicit criteria are established for common/practical classes of impulsive systems; see Theorems 4.1, 4.2, and 4.5.

Assume that $\sup_{k\in\mathbb{N}}(t_k - t_{k-1}) < \infty$. Then (3.3) is equivalent to

$$\Lambda(\eta_p) + \limsup_{k\to\infty}\frac{1}{t_k}\Big[\sum_{j=1}^k p\ln|\delta_j|\Big] < -\beta. \tag{3.14}$$

In order to verify (3.14), a sufficient condition is

$$p\ln|\delta_k| < -\big(\beta + \Lambda(\eta_p)\big)(t_k - t_{k-1}) \quad \text{for} \quad k \ge k_0,$$

where k_0 is a positive integer; that is,

$$|\delta_k| < \exp\Big(-\frac{\big(\beta + \Lambda(\eta_p)\big)(t_k - t_{k-1})}{p}\Big) \quad \text{for} \quad k \ge k_0.$$

Theorem 3.5. *Assume* (A). *Let $V : \mathbb{R}^d \to \mathbb{R}$ be a twice continuously differentiable on $\mathbb{R}^d \setminus \{0\}$ satisfying $c_1|x|^p \le V(x) \le c_2|x|^p$ for all $x \in \mathbb{R}^d$, where c_1 and c_2 are two positive numbers. Suppose that there exist $\eta_p = (\eta_{p,1}, \eta_{p,2}, \dots, \eta_{p,m})^\top \in \mathbb{R}^m$ and a sequence of real numbers $\{\delta_k\}_{k\in\mathbb{N}}$ such that*

$$(\mathcal{G}V)(x, t, i) \le \eta_{p,i}V(x) \quad \textit{for all} \quad (x, t, i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathcal{M}, x \ne 0 \tag{3.15}$$

and

$$V\big(I_k(x, i)\big) \le |\delta_k|^pV(x) \quad \textit{for all} \quad (x, i) \in \mathbb{R}^d \times \mathcal{M}, k \in \mathbb{N}. \tag{3.16}$$

Suppose that

$$\sum_{i\in\mathcal{M}}\nu_i\Big[\eta_{p,i} + \limsup_{t\to\infty}\Big(\frac{1}{t}\sum_{j:t_j\le t} p\ln|\delta_j|\Big)\Big] < 0. \tag{3.17}$$

Then the trivial solution of Eq. (2.2) *is $\widetilde{p}$th moment exponentially stable for some $\widetilde{p} > 0$.*

Proof. Let $\gamma\in(0,1)$ and consider the function $V_\gamma(x)=|V(x)|^\gamma$ for $x\in\mathbb{R}^d$. Then in view of (3.15), we have

$$\begin{aligned}(\mathcal{G}V_\gamma)(x,t,i)&=\gamma|V(x)|^{\gamma-1}(\mathcal{G}V)(x,t,i)-\frac{\gamma(1-\gamma)}{2}\\&\quad\times|V(x)|^{\gamma-2}\big|V_{xx}(x)\sigma(x,t,i)\big|^2\\&\le\gamma|V(x)|^{\gamma-1}\eta_{p,i}V(x)\\&=\gamma\eta_{p,i}V_\gamma(x)\quad\text{for}\quad(x,t,i)\in\mathbb{R}^d\times\mathbb{R}_+\times\mathcal{M},x\neq0.\end{aligned}\tag{3.18}$$

By virtue of [24, Lemma 3.5], we have

$$\lim_{\gamma\to0}\frac{\Lambda(\gamma\eta_p)}{\gamma}=\sum_{i\in\mathcal{M}}\nu_i\eta_{p,i}.\tag{3.19}$$

In view of (3.17) and (3.19), there exists $\gamma_0>0$ such that

$$\sum_{i\in\mathcal{M}}\nu_i\Big[\Lambda(\gamma_0\eta_p)+\limsup_{t\to\infty}\Big(\frac{1}{t}\sum_{j:t_j\le t}(\gamma_0p)\ln|\delta_j|\Big)\Big]<0.\tag{3.20}$$

Let $\widetilde{p}=\gamma_0p$. By virtue of Theorem 3.3, (3.18), and (3.20), the trivial solution of Eq. (2.2) is $\widetilde{p}$th moment exponentially stable. The conclusion follows. □

Theorem 3.6. *Suppose $t_k=Tk$ for some $T>0$. Let $V:\mathbb{R}^d\to\mathbb{R}$ be a twice continuously differentiable on $\mathbb{R}^d\setminus\{0\}$ satisfying $c_1|x|^p\le V(x)\le c_2|x|^p$ for any $x\in\mathbb{R}^d$, where c_1 and c_2 are two positive numbers. Suppose (3.1) is satisfied and there exist positive numbers $\delta_{p,i}$ ($i\in\mathcal{M}$) such that*

$$V(I_k(x,i))\le\delta_{p,i}V(x)\quad\text{for all}\quad(x,i)\in\mathbb{R}^d\times\mathcal{M},k\in\mathbb{N}.\tag{3.21}$$

If $\sum_{i\in\mathcal{M}}\left(\ln|\delta_{p,i}|+\eta_{p,i}T\right)\nu_i<0$, then the trivial solution of Eq. (2.2) is $\widetilde{p}$th moment exponentially stable for some $\widetilde{p}>0$.

Proof. Let $\mathcal{F}_2$ be the filtration generated by $\{\alpha(t),t\ge0\}$ and $\mathcal{F}_{2,t}$ the filtration generated by $\{\alpha(s),0\le t\le s\}$. From the proof of Theorem 3.3, we have

$$\mathbb{E}\big[V(X(t))\big|\mathcal{F}_2\big]\le V(x^0)\Big[\exp\Big(\sum_{k=1}^{[t/T]}\ln|\delta_{p,\alpha(Tk)}|+\int_0^t\eta_{p,\alpha(s)}ds\Big)\Big]\quad\text{for}\quad t\ge0.$$

Then, an application of Jensen's inequality yields

$$\mathbb{E}\big[V^\theta(X(t))\big|\mathcal{F}_2\big]\le V^\theta(x^0)\Big[\exp\Big(\theta\sum_{k=1}^{[t/T]}\ln|\delta_{p,\alpha(Tk)}|+\theta\int_0^t\eta_{p,\alpha(s)}ds\Big)\Big]\quad\text{for}\quad t\ge0,\theta\in(0,1).\tag{3.22}$$

Due to the ergodicity of the Markov chains $\{\alpha(t),t\ge0\}$ and $\{\alpha(kT),k\in\mathbb{N}\}$, we have

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}_i\left(\sum_{k=1}^n\ln|\delta_{p,\alpha(Tk)}|+\int_0^{nT}\eta_{p,\alpha(s)}ds\right)=\lambda<0,$$

where $\mathbb{E}_i$ indicates the initial value $\alpha(0)=i$. Then, there exists $n^*>0$ such that

$$\mathbb{E}_i\left(\sum_{k=1}^{n^*}\ln|\delta_{p,\alpha(Tk)}|+\int_0^{n^*T}\eta_{p,\alpha(s)}ds\right)\le\frac{\lambda n^*}{2}<0\quad\text{for any } i\in\mathcal{M}.$$

Then, as a result of a property of Laplace's transformation, see [7, Lemma 3.4], we have

$$\begin{aligned}&\mathbb{E}_i\exp\left\{\theta\left(\sum_{k=1}^{n^*}\ln|\delta_{p,\alpha(Tk)}|+\int_0^{n^*T}\eta_{p,\alpha(s)}ds\right)\right\}\\&\quad\le\rho:=\exp\left\{\frac{\lambda n^*}{4}\right\}<1\quad\text{for any } i\in\mathcal{M}.\end{aligned}$$

Because of the Markov property of $\{\alpha(t),t\ge0\}$ and $\{\alpha(kT),k\in\mathbb{N}\}$, we have

$$\begin{aligned}&\mathbb{E}_i\exp\left\{\theta\left(\sum_{k=1}^{2n^*}\ln|\delta_{p,\alpha(Tk)}|+\int_0^{2n^*T}\eta_{p,\alpha(s)}ds\right)\right\}\\&=\mathbb{E}_i\Bigg[\mathbb{E}_i\left(\exp\left\{\theta\left(\sum_{k=n^*+1}^{2n^*}\ln|\delta_{p,\alpha(Tk)}|+\int_{n^*T}^{2n^*T}\eta_{p,\alpha(s)}ds\right)\right\}\Big|\mathcal{F}_{2,n^*T}\right)\\&\qquad\times\exp\left\{\theta\left(\sum_{k=1}^{n^*}\ln|\delta_{p,\alpha(Tk)}|+\int_0^{n^*T}\eta_{p,\alpha(s)}ds\right)\right\}\Bigg]\\&=\mathbb{E}_i\Bigg[\mathbb{E}_{\alpha(n^*T)}\exp\left\{\theta\left(\sum_{k=1}^{n^*}\ln|\delta_{p,\alpha(Tk)}|+\int_0^{n^*T}\eta_{p,\alpha(s)}ds\right)\right\}\\&\qquad\times\exp\left\{\theta\left(\sum_{k=1}^{n^*}\ln|\delta_{p,\alpha(Tk)}|+\int_0^{n^*T}\eta_{p,\alpha(s)}ds\right)\right\}\Bigg]\\&\le\mathbb{E}_i\left[\rho\exp\left\{\theta\left(\sum_{k=1}^{n^*}\ln|\delta_{p,\alpha(Tk)}|+\int_0^{n^*T}\eta_{p,\alpha(s)}ds\right)\right\}\right]\\&\le\rho^2.\end{aligned}$$

Continuing this process, we obtain that

$$\mathbb{E}_i\exp\left\{\theta\left(\sum_{k=1}^{\ell n^*}\ln|\delta_{p,\alpha(Tk)}|+\int_0^{\ell n^*T}\eta_{p,\alpha(s)}ds\right)\right\}\le\rho^\ell,\quad\ell\in\mathbb{N}.\tag{3.23}$$

Because of the boundedness of $\ln|\delta_{p,i}|$ and $\eta_{p,i}$ for $i\in\mathcal{M}$, there exists a constant independent C_{n^*} of t such that if $\ell n^*T\le t<(\ell+1)n^*T$ for some $\ell\in\mathbb{N}$, we have

$$\begin{aligned}&\exp\left\{\theta\left(\sum_{k=1}^{[t/T]}\ln|\delta_{p,\alpha(Tk)}|+\int_0^t\eta_{p,\alpha(s)}\right)\right\}\\&\quad\le C_{n^*}\exp\left\{\theta\left(\sum_{k=1}^{\ell n^*}\ln|\delta_{p,\alpha(Tk)}|+\int_0^{\ell n^*T}\eta_{p,\alpha(s)}ds\right)\right\}.\end{aligned}\tag{3.24}$$

It follows from (3.22), (3.23), and (3.24) that

$$\mathbb{E}[V^\theta(X(t))]\le C_{n^*}V^\theta(x^0)\rho^{[t/(n^*T)]}.$$

Since $\rho<1$, we easily obtain the $\widetilde{p}$th moment exponential stability of the system, where $\widetilde{p}=\theta p$ for some small $\theta>0$. □

Remark 3.7. Assume that $\sup_{k\in\mathbb{N}}(t_k-t_{k-1})<\infty$. Then (3.17) is equivalent to

$$\sum_{i\in\mathcal{M}}\nu_i\zeta_i+\limsup_{k\to\infty}\frac{1}{t_k}\Big[\sum_{j=1}^k\ln|\delta_j|\Big]<0,\tag{3.25}$$

where $\zeta_i=\eta_{p,i}/p$ for $i\in\mathcal{M}$. In order to verify (3.25), a sufficient condition is

$$\ln|\delta_k|<-\Big(\sum_{i\in\mathcal{M}}\nu_i\zeta_i\Big)(t_k-t_{k-1})\quad\text{for}\quad k\ge k_0,$$

where k_0 is a positive integer; that is,

$$|\delta_k| < \exp\Big(-\Big(\sum_{i\in\mathcal{M}}\nu_i\zeta_i\Big)(t_k-t_{k-1})\Big) \quad \text{for} \quad k\geq k_0.$$

Remark 3.8. The associated switching SDE with no impulsive perturbations of $X(\cdot)$ is given by

$$d\widetilde{X}(t)=b(\widetilde{X}(t),t,\alpha(t))dt+\sigma(\widetilde{X}(t),t,\alpha(t))dw(t), \quad t\geq 0. \tag{3.26}$$

Associated with the switching SDE $\widetilde{X}(\cdot)$, there are m SDEs that interact and switch back and forth. These SDEs are denoted by $\widetilde{X}^{(1)}(\cdot),\widetilde{X}^{(2)}(\cdot),\dots,\widetilde{X}^{(m)}(\cdot)$ and for each $i\in\mathcal{M}$, $\widetilde{X}^{(i)}$ is given by the ith subsystem

$$d\widetilde{X}^{(i)}(t)=b(\widetilde{X}^{(i)}(t),t,i)dt+\sigma(\widetilde{X}^{(i)}(t),t,i)dw(t), \quad t\geq 0. \tag{3.27}$$

By virtue of Theorem 3.3, we have the following assertions on special cases of Eq. (2.2).

(a) If $\eta_{p,i}<0$, then the trivial solution of the ith subsystem given by (3.27) is pth moment exponentially stable.

(b) If $\Lambda(\eta_p)<0$, then the trivial solution of the switching SDE (3.26) is pth moment exponentially stable.

(c) If $\Lambda(\eta_p)+\limsup_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\leq t}p\ln|\delta_j|\Big]<0$, then the trivial solution of the switching SDE with impulsive perturbations Eq. (2.2) is pth moment exponentially stable.

Thus, Theorem 3.3 and in particular the sufficient condition (3.3) reveal the contribution to the pth moment exponential stability of the Markov chain $\alpha(\cdot)$ and the impulsive perturbations.

Remark 3.9. The impulsive switching SDE $X(\cdot)$ can also be viewed as a set of m impulsive SDEs that interact and switch back and forth. These impulsive SDEs are denoted by $X^{(1)}(\cdot),X^{(2)}(\cdot),\dots,X^{(m)}(\cdot)$ and for each $i\in\mathcal{M}$, $X^{(i)}$ is given by

$$\begin{aligned}&dX^{(i)}(t)=b(X^{(i)}(t),t,i)dt+\sigma(X^{(i)}(t),t,i)dw(t), \quad t\geq 0,\\&X^{(i)}(t_k)=I_k\big(X^{(i)}(t_k^-),i\big), \quad k\in\mathbb{N}.\end{aligned} \tag{3.28}$$

Note that $\sum_{i\in\mathcal{M}}\nu_i=1$ and $\Lambda(c\mathbb{1}_m+\eta_p)=c+\Lambda(\eta_p)$, where c is a real number and $\mathbb{1}_m=(1,1,\dots,1)^\top\in\mathbb{R}^m$. It follows that

$$\begin{aligned}&\Lambda(\eta_p)+\limsup_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\leq t}p\ln|\delta_j|\Big]\\&=\Lambda\bigg(\eta_p+\limsup_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\leq t}p\ln|\delta_j|\Big]\mathbb{1}_m\bigg).\end{aligned}$$

By virtue of Theorem 3.3, we have the following assertions on special cases of Eq. (2.2).

(a) If $\eta_{p,i}+\limsup_{t\to\infty}\frac{1}{t}\Big[\sum_{j:t_j\leq t}p\ln|\delta_j|\Big]<0$, then the trivial solution of the ith subsystem given by (3.28) is pth moment exponentially stable.

(b) Suppose that (3.1) and (3.2) are satisfied. Suppose that

$$\eta_{p,i}+\limsup_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\leq t}p\ln|\delta_j|\Big]<0$$

for some i; that is, some subsystem given by (3.28) is pth moment exponentially stable. Then we can choose a Markov chain $\alpha(\cdot)$ such that the switching SDE with impulsive perturbations (2.2) is pth moment exponentially stable. To illustrate, we look at the case $m=2$; that is, the state space of Markov chain $\alpha(\cdot)$ is $\mathcal{M}=\{1,2\}$. Let $\mathcal{P}$ be the set of all probability vectors on the state space $\mathcal{M}$. Let $\mathfrak{p}=(\mathfrak{p}_1,\mathfrak{p}_2)\in\mathcal{P}$. As in [27, page 1310], we have

$$\begin{aligned}\mathcal{I}(\mathfrak{p})&=-\inf_{u_1>0,u_2>0}\sum_{i,j\in\mathcal{M}}\frac{\mathfrak{p}_iq_{ij}u_j}{u_i}\\&=-\inf_{u_1>0,u_2>0}\Big[-\mathfrak{p}_1q_{12}-\mathfrak{p}_2q_{21}+\frac{\mathfrak{p}_1q_{12}u_2}{u_1}+\frac{\mathfrak{p}_2q_{21}u_1}{u_2}\Big]\\&=-(-\mathfrak{p}_1q_{12}-\mathfrak{p}_2q_{21}+2\sqrt{\mathfrak{p}_1q_{12}\mathfrak{p}_2q_{21}})\\&=\big(\sqrt{\mathfrak{p}_1q_{12}}-\sqrt{\mathfrak{p}_2q_{21}}\big)^2.\end{aligned}$$

Note that $\mathfrak{p}_2=1-\mathfrak{p}_1$. Thus, for $\eta_p=(\eta_{p,1},\eta_{p,2})^\top\in\mathbb{R}^2$,

$$\begin{aligned}\Lambda(\eta_p)&=\sup_{\mathfrak{p}\in\mathcal{P}}\big(\eta_{p,1}\mathfrak{p}_1+\eta_{p,2}\mathfrak{p}_2-\mathcal{I}(\mathfrak{p})\big)\\&=\sup_{\phi\in[0,1]}\Big[\eta_{p,1}\phi+\eta_{p,2}(1-\phi)-\big(\sqrt{\phi q_{12}}-\sqrt{(1-\phi)q_{21}}\big)^2\Big].\end{aligned}$$

Suppose $\eta_{p,1}<0$. Then we can choose a sufficiently large q_{21} and a sufficient small q_{12} such that

$$\Lambda(\eta_p)+\limsup_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\leq t}p\ln|\delta_j|\Big]<0,$$

as desired.

Next, we provide a sufficient condition for instability.

Theorem 3.10. *Assume* (A). *Let $V:\mathbb{R}^d\to\mathbb{R}$ be a twice continuously differentiable on $\mathbb{R}^d\setminus\{0\}$ satisfying $c_1|x|^p\leq V(x)\leq c_2|x|^p$ for all $x\in\mathbb{R}^d$, where c_1 and c_2 are two positive numbers. Suppose that there exist $\widetilde{\eta}_p=(\widetilde{\eta}_{p,1},\widetilde{\eta}_{p,2},\dots,\widetilde{\eta}_{p,m})^\top\in\mathbb{R}^m$ and a sequence of positive numbers $\{\widetilde{\delta}_k\}_{k\in\mathbb{N}}$ such that*

$$(\mathcal{G}V)(x,t,i)\geq\widetilde{\eta}_{p,i}V(x) \quad \textit{for all} \quad (x,t,i)\in\mathbb{R}^d\times\mathbb{R}_+\times\mathcal{M}, x\neq 0 \tag{3.29}$$

and

$$V\big(I_k(x,i)\big)\geq|\widetilde{\delta}_k|^pV(x) \quad \textit{for all} \quad (x,i)\in\mathbb{R}^d\times\mathcal{M}, k\in\mathbb{N}. \tag{3.30}$$

Suppose that

$$\Lambda(\widetilde{\eta}_p)+\liminf_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\leq t}p\ln|\widetilde{\delta}_j|\Big]>0. \tag{3.31}$$

Then the trivial solution of Eq. (2.2) *is pth moment exponentially unstable.*

Proof. The proof is a modification of that of Theorem 3.3. We will provide an outline and skip the details for brevity. We divide the proof into two steps.

Step 1. In this step, we still work with a fixed value of k. Let $(x^0,i^0)\in\mathbb{R}^d\times\mathcal{M}$. Let $\mathcal{D}_t$ be the σ-algebra generated by $\{\alpha(u),w(s):0\leq u,0\leq s\leq t\}\cup\mathcal{F}_0$. Without loss of generality, we suppose $x^0\neq 0$. With the similar arguments as in Step 1 of the proof of Theorem 3.3, we obtain

$$\mathbb{E}\Big[V\big(X(t)\big)\big|\mathcal{D}_{t_{k-1}}\Big]\geq V\big(X(t_{k-1})\big)\exp\Big(\int_{t_{k-1}}^t\widetilde{\eta}_{p,\alpha(s)}ds\Big) \quad \text{for} \quad t\in[t_{k-1},t_k). \tag{3.32}$$

It follows from (3.30) that

$$\begin{aligned}\mathbb{E}\Big[V\big(X(t_k)\big)\big|\mathcal{D}_{t_{k-1}}\Big]&\geq|\widetilde{\delta}_k|^p\mathbb{E}\Big[V\big(X(t_k^-)\big)\big|\mathcal{D}_{t_{k-1}}\Big]\\&\geq|\widetilde{\delta}_k|^pV\big(X(t_{k-1})\big)\exp\Big(\int_{t_{k-1}}^{t_k}\widetilde{\eta}_{p,\alpha(s)}ds\Big).\end{aligned} \tag{3.33}$$

Step 2. To proceed, we note that (3.32) and (3.33) hold for any $k\in\mathbb{N}$. With the similar arguments as in Step 2 of the proof of

Theorem 3.3, we obtain

$$\mathbb{E}\Big[V\big(X(t_k)\big)\Big] \geq V(x^0)\mathbb{E}\Big[\exp\Big(\int_0^{t_k} \widetilde{\eta}_{p,\alpha(s)}ds\Big)\Big]\prod_{j=1}^{k}|\widetilde{\delta}_j|^p \quad \text{for} \quad k \in \mathbb{N}. \tag{3.34}$$

In view of (3.32) and (3.34), we have

$$\mathbb{E}\Big[V\big(X(t)\big)\Big] \geq V(x^0)\mathbb{E}\Big[\exp\Big(\int_0^{t} \widetilde{\eta}_{p,\alpha(s)}ds\Big)\Big]\prod_{j=1}^{k}|\widetilde{\delta}_j|^p \quad \text{for} \quad t \in [t_k, t_{k+1}).$$

Thus,

$$\mathbb{E}\Big[V\big(X(t)\big)\Big] \geq V(x^0)\mathbb{E}\Big[\exp\Big(\sum_{j:t_j\leq t} p\ln|\widetilde{\delta}_j| + \int_0^{t} \widetilde{\eta}_{p,\alpha(s)}ds\Big)\Big] \quad \text{for all} \quad t \geq 0.$$

Since

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t}\ln\mathbb{E}\Big[\exp\Big(\sum_{j:t_j\leq t} p\ln|\widetilde{\delta}_j| + \int_0^{t} \widetilde{\eta}_{p,\alpha(s)}ds\Big)\Big]\\ &\quad = \Lambda(\widetilde{\eta}_p) + \liminf_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\leq t} p\ln|\widetilde{\delta}_j|\Big] > 0,\end{aligned}$$

there are positive numbers K and β such that

$$\mathbb{E}\Big[|X(t)|^p\Big] \geq Ke^{\beta t}|x^0|^p \quad \text{for all} \quad t\geq 0, (x^0, i^0) \in \mathbb{R}^d \times \mathcal{M}.$$

The conclusion follows. □

Remark 3.11. In view of Theorem 3.10 (see also Remark 3.4), we can construct impulsive perturbations so that the resulting switching SDE with impulsive perturbations is pth moment exponentially unstable even if the system with no impulsive perturbations is pth moment exponentially stable.

Example 3.12. We discuss the stabilization of switching SDEs by impulsive perturbations. Suppose the switching SDE (3.26) is pth moment exponentially unstable. We would like to design impulsive perturbations so that the resulting impulsive system (2.2) is pth moment exponentially stable. Moreover, we wish that the pth moment Lyapunov exponent is not greater than $-\beta$, where β is a given positive number.

Suppose we can perform impulsive perturbations at the sequence of impulsive times $\{t_k\}_{k\in\mathbb{N}}$ and $\sup_{k\in\mathbb{N}}(t_k - t_{k-1}) < \infty$. Theoretically, t_1 can be arbitrarily large. With $I_k(x, i) = \delta_k x$ for $k \in \mathbb{N}$, we proceed to determine the sequence $\{\delta_k\}_{k\in\mathbb{N}}$. By Remark 3.4, we can choose δ_k such that

$$|\delta_k| < \exp\Big(-\frac{(\beta + \Lambda(\eta_p))(t_k - t_{k-1})}{p}\Big) \quad \text{for} \quad k \in \mathbb{N}.$$

By virtue of Theorem 3.3, the impulsive switching SDE (2.2) is pth moment exponentially stable and the pth moment Lyapunov exponent is not greater than $-\beta$.

Theorem 3.13. *Assume* (A), $\inf_{k\in\mathbb{N}}(t_k - t_{k-1}) > 0$, *and there is a positive constant M such that*

$$|b(x, t, i)| + |\sigma(x, t, i)| \leq M|x| \quad \textit{for any} \quad (x, t, i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathcal{M}.$$

Suppose that the trivial solution of Eq. (2.2) *is pth moment exponentially stable; that is, there are positive constants β and K such that*

$$\mathbb{E}\big|X^{x^0,i^0}(t)\big|^p \leq Ke^{-\beta t}\big|x^0\big|^p \quad \textit{for any} \quad t\geq 0, (x^0, i^0) \in \mathbb{R}^d \times \mathcal{M}.$$

Then for any $(x^0, i^0) \in \mathbb{R}^d \times \mathcal{M}$,

$$\limsup_{t\to\infty}\frac{1}{t}\ln\Big(|X^{x^0,i^0}(t)|\Big) \leq -\frac{\beta}{p} \quad a.s.; \tag{3.35}$$

that is, the trivial solution of Eq. (2.2) *is almost surely exponentially stable.*

Proof. The proof is a modification of that of [6, Theorem 3.2]. We will provide an outline and skip the details for brevity. Let $(x^0, i^0) \in \mathbb{R}^d \times \mathcal{M}$. Let $\varepsilon \in (0, \beta/2)$ be arbitrary. Let $r > 0$ be sufficiently small for $(3M)^p\big(r^p + C_p r^{p/2}\big) < \frac{1}{2}$, where C_p is the constant given by the well-known Burkholder–Davis–Gundy inequality (see [2, Theorem 2.13]). Without loss of generality, we suppose that $\sup_{k\in\mathbb{N}}(t_k - t_{k-1}) \leq r$ (note that if $t > 0$ and $t \neq t_k$ for any $k \in \mathbb{N}$, we can treat t as an impulsive time with $X^{x^0,i^0}(t) = X^{x^0,i^0}(t^-)$). For a fixed $k \in \mathbb{N}$, we have

$$dX^{x^0,i^0}(t) = b(X^{x^0,i^0}(t), t, \alpha^{i^0}(t))dt + \sigma(X^{x^0,i^0}(t), t, \alpha^{i^0}(t))dw(t), \quad t \in [t_{k-1}, t_k).$$

By using the same arguments as those in the proof of [6, Theorem 3.2], we obtain

$$\mathbb{E}\Big[\sup_{t_{k-1}\leq t<t_k}|X^{x^0,i^0}(t)|^p\Big] \leq 2K|x^0|^p 3^p e^{-(\beta-\varepsilon)t_{k-1}}.$$

Hence,

$$\mathbb{P}\Big(\sup_{t_{k-1}\leq t<t_k}|X^{x^0,i^0}(t)| > e^{-(\beta-2\varepsilon)t_{k-1}/p}\Big) \leq 2K|x^0|^p 3^p e^{-\varepsilon t_{k-1}}.$$

Since $\inf_{k\in\mathbb{N}}(t_k - t_{k-1}) > 0$, $\sum_{k\in\mathbb{N}} e^{-\varepsilon t_{k-1}} < \infty$. By the Borel–Cantelli lemma, for almost all $\omega \in \Omega$,

$$\sup_{t_{k-1}\leq t<t_k}|X^{x^0,i^0}(t)| \leq e^{-(\beta-2\varepsilon)t_{k-1}/p} \tag{3.36}$$

holds for all but finitely many $k \in \mathbb{N}$. Thus, there is a function $\widetilde{k} : \Omega \to \mathbb{N}$ and an event $\widetilde{\Omega} \subset \Omega$ for which $\mathbb{P}(\Omega) = 1$ and (3.36) holds whenever $\omega \in \widetilde{\Omega}$ and $k \geq \widetilde{k}(\omega)$. Consequently, for $\omega \in \widetilde{\Omega}$, $k \geq \widetilde{k}(\omega)$, and $t \in [t_{k-1}, t_k)$,

$$\frac{1}{t}\ln\Big(|X^{x^0,i^0}(t)|\Big) \leq -\frac{(\beta-2\varepsilon)t_{k-1}}{pt} \leq -\frac{(\beta-2\varepsilon)(t-r)}{pt}.$$

Therefore,

$$\limsup_{t\to\infty}\frac{1}{t}\ln\Big(|X^{x^0,i^0}(t)|\Big) \leq -\frac{(\beta-2\varepsilon)}{p} \quad a.s.,$$

which gives us (3.35) by letting $\varepsilon \to 0$. □

4. Explicit criteria for pth moment exponential stability and instability

In what follows, we apply Theorems 3.3 and 3.5 to find explicit criteria for stability and instability.

Theorem 4.1. *Assume* (A) *and for each $i \in \mathcal{M}$, there exist real numbers $K_i^b, K_i^\sigma, K_i^d, L_i^d$ and a sequence of positive numbers $\{\delta_k\}_{k\in\mathbb{N}}$ such that*

$$\begin{aligned}&x^\top b(x, t, i) \leq K_i^b|x|^2, \qquad |\sigma(x, t, i)|^2 \leq K_i^\sigma|x|^2,\\ &L_i^d|x|^4 \leq \big|x^\top\sigma(x, t, i)\big|^2 \leq K_i^d|x|^4, \qquad \big|I_k(x, i)\big| \leq \delta_k|x|,\end{aligned}$$

for all $(x, t, i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathcal{M}$, and $k \in \mathbb{N}$. Denote for $i \in \mathcal{M}$,

$$\eta_{p,i} = \begin{cases} pK_i^b + \dfrac{p}{2}K_i^\sigma + \dfrac{p}{2}(p-2)L_i^d & \textit{for} \quad p \in (0, 2),\\ pK_i^b + \dfrac{p}{2}K_i^\sigma + \dfrac{p}{2}(p-2)K_i^\sigma & \textit{for} \quad p \in [2, \infty),\end{cases}$$

and $\eta_p = (\eta_{p,1}, \eta_{p,2}, \dots, \eta_{p,m})^\top \in \mathbb{R}^m$. *The following assertions hold.*

(a) *Suppose that*

$$\Lambda(\eta_p) + \limsup_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\le t} p\ln|\delta_j|\Big] < 0.$$

Then the trivial solution of Eq. (2.2) *is pth moment exponentially stable.*

(b) *Suppose that*

$$\sum_{i\in\mathcal{M}} \nu_i\Big(K_i^b + \frac{1}{2}K_i^\sigma - L_i^d + \limsup_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\le t}\ln|\delta_j|\Big]\Big) < 0.$$

Then the trivial solution of Eq. (2.2) *is* $\widetilde{p}$*th moment exponentially stable for some* $\widetilde{p} > 0$.

Proof. (a) Let $V(x) = |x|^p$ for $x \in \mathbb{R}^d$. In view of Theorem 3.3, it is sufficient to check that

$$(\mathcal{G}V)(x,t,i) \le \eta_{p,i}V(x) \quad \text{for all} \quad (x,t,i) \in \mathbb{R}^d\times\mathbb{R}_+\times\mathcal{M}, x\ne 0.$$

Indeed, we have

$$V_x(x) = p|x|^{p-2}x,$$
$$V_{xx}(x) = p\big[|x|^{p-2}I_d + (p-2)|x|^{p-4}xx^\top\big] \quad \text{for} \quad x\in\mathbb{R}^d, x\ne 0.$$

Thus,

$$\begin{aligned}(\mathcal{G}\mathcal{V})(x,t,i) &= p\big|x\big|^{p-2}b^\top(x,t,i)x + \frac{p}{2}\mathrm{tr}\Big[\big|x\big|^{p-2}\sigma(x,t,i)\sigma^\top(x,t,i)\Big]\\ &\quad + \frac{1}{2}p(p-2)\mathrm{tr}\Big[\big|x\big|^{p-4}xx^\top\sigma(x,t,i)\sigma^\top(x,t,i)\Big]\\ &\le |x|^p\Big(pK_i^b + \frac{p}{2}K_i^\sigma + \frac{p}{2}(p-2)\big|x\big|^{-4}\big|x^\top\sigma(x,t,i)\big|^2\Big).\end{aligned} \tag{4.1}$$

We consider 2 cases.
(i) If $p < 2$, then we have from (4.1) that

$$(\mathcal{G}\mathcal{V})(x,t,i) \le |x|^p\Big(pK_i^b + \frac{p}{2}K_i^\sigma + \frac{p}{2}(p-2)L_i^d\Big).$$

(ii) If $p \ge 2$, then we have from (4.1) that

$$(\mathcal{G}\mathcal{V})(x,t,i) \le |x|^p\Big(pK_i^b + \frac{p}{2}K_i^\sigma + \frac{p}{2}(p-2)K_i^\sigma\Big).$$

In any case, we have

$$(\mathcal{G}\mathcal{V})(x,t,i) \le \eta_{p,i}V(x) \quad \text{for all} \quad (x,t,i)\in\mathbb{R}^d\times\mathbb{R}_+\times\mathcal{M}, x\ne 0,$$

as desired. The conclusion follows.

(b) follows from Theorem 3.5 and the computations in part (a). □

Now we focus on linear regime-switching SDEs with impulsive perturbations. To proceed, recall that for any symmetric matrix $A \in \mathbb{R}^{d\times d}$, $\Lambda_{\max}(A)$ and $\Lambda_{\min}(A)$ denote its largest eigenvalue and smallest eigenvalue, respectively. Let $\rho(A) = \max\{\big|\Lambda_{\max}(A)\big|, \big|\Lambda_{\min}(A)\big|\}$.

Theorem 4.2. *Assume* (A) *and for each* $i\in\mathcal{M}$, *there exist matrices* $B(i), C_1(i), C_2(i),\dots, C_d(i) \in \mathbb{R}^{d\times d}$, *and a sequence of positive numbers* $\{\delta_k\}_{k\in\mathbb{N}}$ *such that*

$$b(x,t,i) = B(i)x, \quad \sigma(x,t,i) = \big(C_1(i)x, C_2(i)x,\dots,C_d(i)x\big),$$
$$\big|I_k(x,i)\big| \le \delta_k|x|$$

for all $(x,t,i)\in\mathbb{R}^d\times\mathbb{R}_+\times\mathcal{M}$, *and* $k\in\mathbb{N}$. *Denote for* $i\in\mathcal{M}$,

$$\eta_{p,i} = \begin{cases} \dfrac{p}{2}\Lambda_{\max}\Big(\displaystyle\sum_{k=1}^d C_k^\top(i)C_k(i) + B^\top(i) + B(i)\Big) & \text{for} \quad p\in(0,2),\\ \dfrac{p}{2}\Lambda_{\max}\Big(\displaystyle\sum_{k=1}^d C_k^\top(i)C_k(i) + B^\top(i) + B(i)\Big) & \\ \quad + \dfrac{p(p-2)}{8}\sum_{k=1}^d\Big[\rho\big(C_k^\top(i) + C_k(i)\big)\Big]^2 & \\ & \text{for} \quad p\in[2,\infty),\end{cases}$$

and $\eta_p = (\eta_{p,1},\eta_{p,2},\dots,\eta_{p,m})^\top \in \mathbb{R}^m$. *The following assertions hold.*

(a) *Suppose that*

$$\Lambda(\eta_p) + \limsup_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\le t} p\ln|\delta_j|\Big] < 0.$$

Then the trivial solution of Eq. (2.2) *is pth moment exponentially stable.*

(b) *Suppose that*

$$\sum_{i\in\mathcal{M}}\nu_i\Big(\frac{1}{2}\Lambda_{\max}\Big(\sum_{k=1}^d C_k^\top(i)C_k(i) + B^\top(i) + B(i)\Big) + \limsup_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\le t}\ln|\delta_j|\Big]\Big) < 0.$$

Then the trivial solution of Eq. (2.2) *is* $\widetilde{p}$*th moment exponentially stable for some* $\widetilde{p} > 0$.

Proof. (a) Let $V(x) = |x|^p$ for $x\in\mathbb{R}^d$. In view of Theorem 3.3, it is sufficient to check that

$$(\mathcal{G}V)(x,t,i) \le \eta_{p,i}(t)V(x) \quad \text{for all} \quad (x,t,i)\in\mathbb{R}^d\times\mathbb{R}_+\times\mathcal{M}, x\ne 0.$$

Indeed, we have

$$V_x(x) = p|x|^{p-2}x,$$
$$V_{xx}(x) = p\big[|x|^{p-2}I_d + (p-2)|x|^{p-4}xx^\top\big] \quad \text{for} \quad x\in\mathbb{R}^d.$$

Thus,

$$\begin{aligned}(\mathcal{G}V)(x,t,i) &= \frac{1}{2}\mathrm{tr}\Big(\sum_{k=1}^d x^\top C_k^\top(i)V_{xx}(x)C_k(i)x\Big) + \big(V_x(x)\big)^\top\big(B(i)x\big)\\ &\le p|x|^p\Big\{\frac{1}{2}\sum_{k=1}^d\Big(\frac{x^\top C_k^\top(i)C_k(i)x}{|x|^2} + (p-2)\frac{|x^\top C_k^\top(i)x|^2}{|x|^4}\Big)\\ &\quad + \frac{x^\top B(i)x}{|x|^2}\Big\}.\end{aligned} \tag{4.2}$$

We have

$$\begin{aligned}&\frac{1}{2}\sum_{k=1}^d\frac{x^\top C_k^\top(i)C_k(i)x}{|x|^2} + \frac{x^\top B(i)x}{|x|^2}\\ &= \frac{1}{2}\sum_{k=1}^d\frac{x^\top C_k^\top(i)C_k(i)x}{|x|^2} + \frac{x^\top\big(B^\top(i)+B(i)\big)x}{2|x|^2}\\ &\le \frac{1}{2|x|^2}x^\top\Big(\sum_{k=1}^d C_k^\top(i)C_k(i) + \big(B^\top(i)+B(i)\big)\Big)x\\ &= \frac{1}{2}\Lambda_{\max}\Big(\sum_{k=1}^d C_k^\top(i)C_k(i) + B^\top(i) + B(i)\Big).\end{aligned} \tag{4.3}$$

For a symmetric $A \in \mathbb{R}^{d\times d}$, we have $|x^\top Ax| \le \rho(A)|x|^2$ for any $x \in \mathbb{R}^d$. Thus,

$$0 \le \frac{|x^\top C_k^\top(i)x|^2}{|x|^4} = \frac{|x^\top\big(C_k^\top(i)+C_k(i)\big)x|^2}{4|x|^4} \le \frac{1}{4}\Big[\rho\Big(C_k^\top(i)+C_k(i)\Big)\Big]^2. \tag{4.4}$$

It follows from (4.2), (4.3), and (4.4) that

$$(\mathcal{G}V)(x,t,i) \le \eta_{p,i}V(x) \quad \text{for all} \quad (x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathcal{M}, x \ne 0,$$

as desired. The conclusion follows.

(b) follows from Theorem 3.5 and the computations in part (a). □

Using Theorem 3.5 and applying the same steps we did in Theorems 4.1 and 4.2, we can construct explicit criteria for exponentially instability. For linear regime-switching SDEs with impulsive perturbations, we have the following result. We skip the proof for brevity.

Theorem 4.3. *Assume* (A) *and for each $i \in \mathcal{M}$, there exist matrices $B(i), C_1(i), C_2(i), \dots, C_d(i) \in \mathbb{R}^{d\times d}$, and a sequence of positive numbers $\{\widetilde\delta_k\}_{k\in\mathbb{N}}$ such that*

$$b(x,t,i) = B(i)x,$$
$$\sigma(x,t,i) = \big(C_1(i)x, C_2(i)x, \dots, C_d(i)x\big), \quad \big|I_k(x,i)\big| \ge \widetilde\delta_k|x|$$

for all $(x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathcal{M}$, and $k \in \mathbb{N}$. Define

$$\widetilde\eta_{p,i} = \begin{cases} \dfrac{p}{2}\Lambda_{\min}\Big(\displaystyle\sum_{k=1}^d C_k^\top(i)C_k(i) + B^\top(i) + B(i)\Big) & \text{for} \quad p \in [2,\infty), \\ \dfrac{p}{2}\Lambda_{\min}\Big(\displaystyle\sum_{k=1}^d C_k^\top(i)C_k(i) + B^\top(i) + B(i)\Big) & \\ \quad + \dfrac{p(p-2)}{8}\sum_{k=1}^d\Big[\rho\Big(C_k^\top(i)+C_k(i)\Big)\Big]^2 & \\ & \text{for} \quad p \in (0,2), \end{cases}$$

and $\widetilde\eta_p = (\widetilde\eta_{p,1}, \widetilde\eta_{p,2}, \dots, \widetilde\eta_{p,m})^\top \in \mathbb{R}^m$. Suppose

$$\Lambda(\widetilde\eta_p) + \liminf_{t\to\infty}\frac{1}{t}\sum_{j:t_j\le t} p\ln|\widetilde\delta_j| > 0.$$

Then the trivial solution of Eq. (2.2) is pth moment exponentially unstable.

Example 4.4. Let $\alpha(\cdot)$ be a Markov chain with state space $\mathcal{M} = \{1,2,3\}$ and generator

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 3 & -4 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Consider a three-dimensional stochastic differential equation with Markovian switching of the form

$$dX(t) = B\big(\alpha(t)\big)X(t)dt + C\big(\alpha(t)\big)X(t)dw_1(t), \tag{4.5}$$

where

$$B(1) = \begin{pmatrix} 2 & 1 & 1.2 \\ -0.8 & 4.5 & -0.2 \\ 1 & 0.5 & 3 \end{pmatrix},$$
$$B(2) = \begin{pmatrix} 1.5 & 1 & 0.5 \\ 0.8 & 1.7 & 1 \\ -0.7 & -0.4 & 2.1 \end{pmatrix},$$
$$B(3) = \begin{pmatrix} 0.5 & -0.9 & -1 \\ 1 & 1.5 & -0.7 \\ 0.8 & 1 & 1.5 \end{pmatrix},$$
$$C(1) = \begin{pmatrix} 0.5 & 0.5 & 0 \\ -0.2 & 0.7 & 0.3 \\ -0.2 & -1 & 1 \end{pmatrix},$$
$$C(2) = \begin{pmatrix} -1 & 0.8 & -0.2 \\ -0.3 & 1 & 0.5 \\ 0.5 & -1 & 0.6 \end{pmatrix},$$
$$C(3) = \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ -1 & 0.1 & 0.4 \\ 0.7 & -0.5 & 0.2 \end{pmatrix}.$$

Detailed computation gives us the stationary distribution $\nu = (7/15, 3/15, 5/15)$. Using Theorem 4.3, we have $\widetilde\eta_{p,1} \ge 0.5p$, $\widetilde\eta_{p,2} \ge 0.01p$, $\widetilde\eta_{p,3} \ge 0.48p$. Then $\sum_{i=1}^3 \nu_i\widetilde\eta_{p,i} \ge 0.39p > 0$ for $p \in (0,2)$. Thus, by virtue of Theorem 4.3, the trivial solution of Eq. (4.5) is not pth moment exponentially stable for any $p > 0$.

Next, we introduce impulsive perturbations so that the trivial solution of the resulting system

$$\begin{aligned} dX(t) &= B\big(\alpha(t)\big)X(t)dt + C\big(\alpha(t)\big)X(t)dw_1(t), \quad t \ge 0, \\ X(t_k) &= I_k\big(X(t_k^-), \alpha(t_k^-)\big), \quad k \in \mathbb{N}. \end{aligned} \tag{4.6}$$

is pth moment exponentially stable for some $p > 0$. Let $t_k = k$ for $k \in \mathbb{N}$, $I_k(x,i) = \delta_k x$ where $\{\delta_k\}$ is a sequence of real numbers. We will apply Theorem 4.2. By virtue of Theorem 4.2 and Remark 3.7, it is sufficient to have

$$|\delta_k| < \exp\Big(-\sum_{i\in\mathcal{M}}\nu_i\zeta_i\Big) \quad \text{for} \quad k \in \mathbb{N},$$

where

$$\zeta_i = \frac{1}{2}\Lambda_{\max}\Big(C^\top(i)C(i) + B^\top(i) + B(i)\Big) \quad \text{for} \quad i = 1,2,3.$$

Detailed computations give us $\zeta_1 \approx 4.745957$, $\zeta_2 \approx 2.892707$, $\zeta_3 \approx 1.572366$. Thus, a sufficient condition for moment exponential stability is $|\delta_k| < \exp(-3.317443) \approx 0.03624539$ for $k \in \mathbb{N}$. We apply the Euler–Maruyama method to obtain numerical solutions of Eqs. (4.5) and (4.6). Fig. 1(a) provides trajectories of Eq. (4.5) with $X(0) = (-2,0,2)^\top$ and $\alpha(0) = 1$. Fig. 1(b) presents trajectories of Eq. (4.6) with $X(0) = (-2,0,2)^\top$, $\alpha(0) = 1$, $\delta_k = 0.03$ for any $k \in \mathbb{N}$. The figures illustrate the stabilization effects of impulsive perturbations.

The following result concerns the linear scalar switching diffusion with impulsive perturbations.

Theorem 4.5. *Assume* (A) *and for each $i \in \mathcal{M}$, there are real numbers μ_i, λ_i, and a sequence of positive numbers $\{\delta_k\}_{k\in\mathbb{N}}$ such that*

$$\begin{aligned} dX(t) &= \mu_{\alpha(t)}X(t)dt + \lambda_{\alpha(t)}X(t)dw(t), \quad t \ge 0, \\ X(t_k) &= \delta_k X(t_k^-), \quad k \in \mathbb{N}. \end{aligned} \tag{4.7}$$

Denote

$$\eta_{p,i} = p\mu_i + \frac{p(p-1)}{2}\lambda_i^2$$
$$\text{for} \quad i \in \mathcal{M}, \quad \eta_p = (\eta_{p,1}, \eta_{p,2}, \dots, \eta_{p,m})^\top \in \mathbb{R}^m.$$

Then the following statements hold.

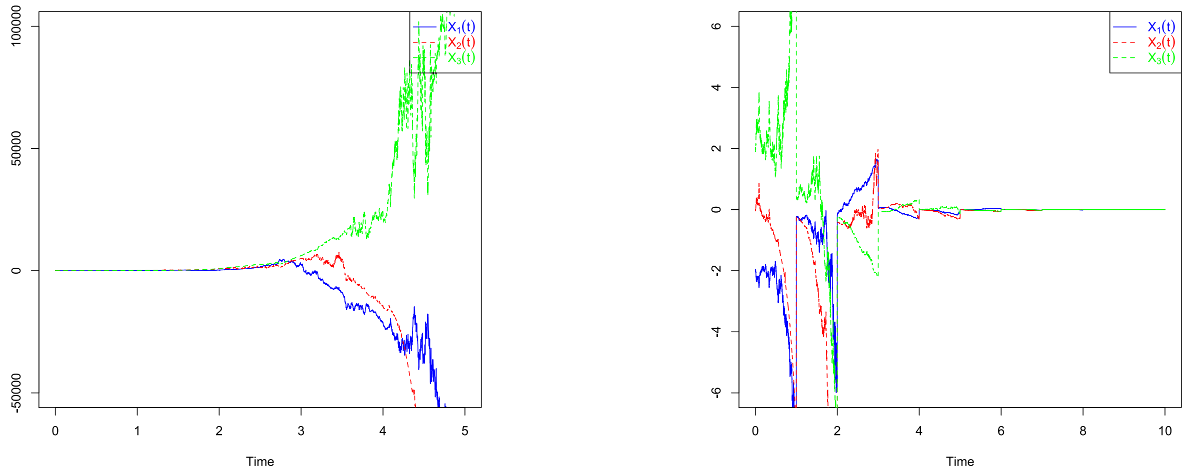
(a) *If $\Lambda(\eta_p) + \limsup_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\le t} p\ln|\delta_j|\Big] < 0$, then the trivial solution of Eq. (4.7) is pth moment exponentially stable.*

(b) *If $\Lambda(\eta_p) + \liminf_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\le t} p\ln|\delta_j|\Big] > 0$, then the trivial solution of Eq. (4.7) is pth moment exponentially unstable.*

(c) *If*

$$\sum_{i\in\mathcal{M}}\nu_i\bigg(\mu_i - \frac{\lambda_i^2}{2} + \limsup_{t\to\infty}\Big[\frac{1}{t}\sum_{j:t_j\le t}\ln|\delta_j|\Big]\bigg) < 0,$$

then the trivial solution of Eq. (4.7) is $\widetilde p$th moment exponentially stable for some $\widetilde p > 0$.



(a) (b)

Fig. 1. (a) Trajectories of Eq. (4.5) with $X(0) = (-2, 0, 2)^\top$ and $\alpha(0) = 1$; (b) Trajectories of Eq. (4.6) with $X(0) = (-2, 0, 2)^\top$, $\alpha(0) = 1$, $t_k = k$ and $\delta_k = 0.03$ for any $k \in \mathbb{N}$.

Proof. It can be seen that part (a) is a direct consequence of Theorem 3.3. Part (b) follows from Theorem 3.10. Part (c) is a consequence of Theorem 3.5. □

Example 4.6. We consider a neural network proposed by Hopfield [30]

$$C_i\dot{u}_i(t) = -\frac{1}{R_i}u_i(t) + \sum_{j=1}^{d} T_{ij}g_j\big(u_j(t)\big) \quad \text{for} \quad i = 1, 2, \ldots, d, \quad t \geq 0; \tag{4.8}$$

see also [2, Section 10.5]. Here $u_i(t)$ represents the voltage on the input of the ith neuron. Each neuron is characterized by an input capacitance C_i and a transfer function $g_i(u)$. The connection matrix element T_{ij} has a value either $1/R_{ij}$ or $-1/R_{ij}$ depending on whether the noninverting or inverting output of the jth neuron is connected to the input of the ith neuron through a resistance R_{ij}. The parallel resistance at the input of the ith neuron is $R_i = 1/\big(\sum_{j=1}^d |T_{ij}|\big)$. The function $g_i(u)$ is a nondecreasing Lipschitz continuous function with properties that

$$ug_i(u) \geq 0, \quad |g_i(u)| \leq 1 \wedge \big(\beta_i|u|\big) \quad \text{for} \quad u \in \mathbb{R},$$

where $\beta_i > 0$ is the slope of $g_i(u)$ at $u = 0$. Eq. (4.8) can be rewritten as

$$\dot{u}(t) = -Fu(t) + Ag(u(t)), \tag{4.9}$$

where

$$\begin{aligned} f_i &= \frac{1}{C_iR_i}, \quad a_{ij} = \frac{T_{ij}}{C_i}, \quad u(t) = \big(u_1(t), \ldots, u_d(t)\big)^\top, \\ F &= \operatorname{diag}(f_1, \ldots, f_d), \quad A = (a_{ij})_{d\times d}, \quad g(u) = \big(g_1(u), \ldots, g_d(u)\big)^\top, \end{aligned} \tag{4.10}$$

Neural networks have been successfully employed in various areas such as pattern recognition, associative memory and combinatorial optimization; see [10, Section 4.2] and references therein. In practice, neural networks are subject to various types of noise and abrupt jumps at certain instants. In [2, Section 10.5], the authors took the white noise and color noise (Markovian switching) into account. In [9], the authors recognized that many sudden and sharp changes occur instantaneously, in the form of impulses. Therefore, the authors proposed and studied neural networks with impulsive perturbations; see also [10, Section 4.2] and references therein for further discussions. Taking into consideration noise and impulsive effects, we focus on a stochastic neural network given by

$$\begin{aligned} dX(t) &= \big[-F(\alpha(t))X(t) + A(\alpha(t))g(X(t))\big]dt \\ &\quad + \sigma(X(t), \alpha(t))dw(t), \quad t \geq 0, \\ X(t_k) &= I_k\big(X(t_k^-), \alpha(t_k^-)\big), \quad k \in \mathbb{N}. \end{aligned} \tag{4.11}$$

Here $\alpha(\cdot)$, $I_k(\cdot)$, $\{t_k\}_{k\in\mathbb{N}}$ are defined as in Eq. (2.2) and for each regime $i \in \mathcal{M}$, $F(i)$ and $A(i)$ correspond to F and A in Eq. (4.9), respectively.

As pointed out in [2,10], it is critical to know whether the networks are stable or not under perturbations. By using the results developed in this paper, we can find sufficient conditions for moment exponential stability of the neutral networks. To proceed, suppose for each $i \in \mathcal{M}$, there exist real numbers λ_i, μ_i, ρ_i and a sequence of positive numbers $\{\delta_k\}_{k\in\mathbb{N}}$ such that

$$\begin{aligned} &x^\top\big[-F(i)x + A(i)g(x)\big] \leq \lambda_i|x|^2, \qquad |\sigma(x, i)|^2 \leq \mu_i|x|^2, \\ &\rho_i|x|^4 \leq \big|x^\top\sigma(x, i)\big|^2, \qquad \big|I_k(x, i)\big| \leq \delta_k|x|, \end{aligned}$$

for all $(x, i) \in \mathbb{R}^d \times \mathcal{M}$ and $k \in \mathbb{N}$. Suppose that

$$\sum_{i\in\mathcal{M}} \nu_i\bigg(\lambda_i + \frac{1}{2}\mu_i - \rho_i + \limsup_{t\to\infty}\bigg[\frac{1}{t}\sum_{j:t_j\leq t} \ln|\delta_j|\bigg]\bigg) < 0.$$

By Theorem 4.1, the trivial solution of Eq. (4.11) is $\widetilde{p}$th moment exponentially stable for some $\widetilde{p} > 0$. Moreover, by virtue of Theorem 3.13, the trivial solution of Eq. (4.11) is almost surely exponentially stable.

5. Concluding remarks

This paper has been devoted to the study of SDEs with impulsive perturbations and Markovian switching. We have established new explicit criteria for pth moment exponential stability. Stabilization effects of the Markov chain and impulsive perturbations have been revealed. Stabilization of switching diffusions by additional impulsive perturbations has been discussed. Although the paper is devoted to a specific class of switching SDEs, one can adopt the same approach to treat hybrid systems with impulsive perturbations.

CRediT authorship contribution statement

Ky Q. Tran: Conceptualization, Methodology, Formal analysis, Investigation, Writing. **Dang H. Nguyen:** Conceptualization, Methodology, Formal analysis, Investigation, Writing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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