INVOLUTIONS UNDER BRUHAT ORDER AND LABELED MOTZKIN PATHS

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Abstract. In this note, we introduce a statistic on Motzkin paths that describes the rank generating function of Bruhat order for involutions. Our proof relies on a bijection introduced by P. Biane from permutations to certain labeled Motzkin paths and a recently introduced interpretation of this rank generating function in terms of visible inversions. By restricting our identity to fixed-point-free (FPF) involutions, we recover an identity due to L. Billera, L. Levine and K. Mészáros with a previous bijective proof by M. Watson. Our work sheds new light on the Ethiopian dinner game.

1. Introduction

Let $\mathfrak{S}_n$ be the symmetric group on n elements, $\mathcal{I}_n$ be the subset of involutions and $\mathcal{I}_n^{FPF}$ (n even) be the subset of fixed-point-free (FPF) involutions. Bruhat order on the symmetric group $(\mathfrak{S}_n, \leq)$ is a graded poset whose rank function counts the number of inversions. The restriction of Bruhat order to $\mathcal{I}$ and $\mathcal{I}^{FPF}$, first considered by RW Richardson and T. A. Springer due to their relation with K–orbit closures [RS90], are also graded partial orders (see [DS01, Inc04]). Let $R_{\mathcal{I}_n}(q)$ and $R_{\mathcal{I}_{2n}^{FPF}}(q)$ be the rank generating functions of $(\mathcal{I}_n, \leq)$ and $(\mathcal{I}_{2n}^{FPF}, \leq)$, respectively. For $\mathcal{M}_n$ the set of Motzkin paths of length n and $\mu \in \mathcal{M}_n$, we introduce a generating function $H[\mu; q]$ in Equation (2.2) satisfying the identity:

Theorem 1.1. *For all $n \in \mathbb{N}$,*

$$\sum_{\mu \in \mathcal{M}_n} H[\mu; q] = R_{\mathcal{I}_n}(q),$$

Our proof relies on a bijection due to P. Biane [Bia93] that, as observed in [BBS11], maps involutions to Motzkin paths with labeled down steps. In [HMP19], Z. Hamaker, E. Marberg and B. Pawlowski introduce *visible inversions* as a combinatorial interpretation of rank in $(\mathcal{I}_n, \leq)$. We show $H[\mu; q]$ counts visible inversions for the involutions corresponding to μ. Similarly, Biane's bijection maps $\mathcal{I}_{2n}^{FPF}$ to $\mathcal{D}_{2n}$, the set of Dyck paths with length $2n$ and labeled down steps. As a consequence, we recover the following identity:

Corollary 1.2. *For all $n \in \mathbb{N}$,*

$$\sum_{\delta \in \mathcal{D}_{2n}} H[\delta; q] = q^n R_{\mathcal{I}_{2n}^{FPF}}(q) = q^n \prod_{k=1}^{n} [2k-1]_q.$$

An equivalent form of Corollary 1.2 (see Equation (3.1)) is [BLM15, Cor. 8], where a discussion of related results appears. M. Watson gives a bijective proof [Wat14], but his argument is more involved than ours since it proves a stronger statement. He introduces a partial order on full rook placements for certain diagrams and shows it is isomorphic to Bruhat order on $\mathcal{I}^{FPF}$ using his bijection. In fact, Watson's bijection is equivalent to Biane's when restricted to $\mathcal{I}^{FPF}$, and his approach can be extended to $\mathcal{I}_n$ as explained in Section 4.1. In Section 4.2 we observe that the bijection in [BLM15], which is not used directly in their proof of Corollary 1.2, is also equivalent to Biane's when restricted to $\mathcal{I}_{2n}^{FPF}$. Their proof arises from combinatorics related to the Ethiopian dinner game introduced in [LS12], and we explain how to interpret Corollary 1.2 in this context.

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2. Combinatorial Structures

2.1. **Permutations, Bruhat order, and visible inversions.** Let $\mathfrak{S}_n$ be the set of permutations from $[n] = \{1, 2, \ldots, n\}$ to itself. All elements of $\mathfrak{S}_n$ can be written as a product of disjoint cycles. For $\pi \in \mathfrak{S}_n$, let $\mathrm{Inv}(\pi) = \{(i, j) \in [n]^2 \mid i < j \text{ and } \pi(i) > \pi(j)\}$ be the set of *inversions* of π and $\ell(\pi) = |\mathrm{Inv}(\pi)|$. *Bruhat order* $(\mathfrak{S}_n, \leq)$ is a partial order defined as the transitive closure of the relations: $\pi \lessdot (i\ j)\pi$ if $\ell((i\ j)\pi) = \ell(\pi) + 1$ with $(i\ j) \in \mathfrak{S}_n$. The rank function of Bruhat order is ℓ.

A permutation π is an *involution* if $\pi^2 = \mathrm{id}_n$, and is *fixed-point-free* (FPF) if $\pi(i) \neq i$ for all $i \in [n]$. Let $\mathcal{I}_n$ be the set of involutions of size n and $\mathcal{I}_{2n}^{FPF}$ be the set of FPF involutions of size $2n$. Bruhat order induces partial orders on $\mathcal{I}_n$ and $\mathcal{I}_{2n}^{FPF}$ (see Figure 1). For $\sigma \in \mathcal{I}_n$, define $\mathrm{Cyc}(\sigma) = \{(i, j) \in [n] \times [n] : i < j = \sigma(i)\}$, $c(\sigma) = |\mathrm{Cyc}(\sigma)|$ and

$$\hat{\ell} = \frac{\ell + c}{2} \quad \text{and} \quad \hat{\ell}^{FPF} = \hat{\ell} - c = \frac{\ell - c}{2}, \quad \text{respectively.}$$

Proposition 2.1 ([Inc04, Thm. 5.2] and [DS01, Thm. 1.3])**.** *Both $(\mathcal{I}_n, \leq)$ and $(\mathcal{I}_{2n}^{FPF}, \leq)$ are graded posets with respective rank functions $\hat{\ell}$ and $\hat{\ell}^{FPF}$.*

Note that $\hat{\ell}^{FPF}$ is well-defined for $\sigma \in \mathcal{I}_n$, where it is studied in [DS01]. See Equation (4.2) for further discussion. When $\tau \in \mathcal{I}_{2n}^{FPF}$, we have $c(\tau) = n$ so $\hat{\ell}^{FPF}(\tau) = \hat{\ell}(\tau) - n = \frac{\ell(\tau) - n}{2}$.

For $\pi \in \mathfrak{S}_n$, say an inversion $(i, j) \in \mathrm{Inv}(\pi)$ is *visible* if $\pi(j) \leq \min(i, \pi(i))$. Let $\widehat{\mathrm{Inv}}(\pi)$ be the set of visible inversions for π, respectively. For example, with $\sigma = (1, 5)(2)(3, 6)(4) = 526413$, we have

$$\widehat{\mathrm{Inv}}(\sigma) = \{(1, 5), (2, 5), (3, 5), (3, 6), (4, 5), (4, 6)\} \quad \text{while} \quad \mathrm{Inv}(\sigma) = \widehat{\mathrm{Inv}}(\sigma) \cup \{(1, 2), (1, 4), (1, 6), (3, 4)\}.$$

Since $c(\sigma) = 2$, we see $\hat{\ell}(\sigma) = \frac{10+2}{2} = 6 = |\widehat{\mathrm{Inv}}(\sigma)|$. Note $\mathrm{Cyc}(\sigma) \subseteq \widehat{\mathrm{Inv}}(\sigma)$ for all $\sigma \in \mathcal{I}_n$.

Proposition 2.2 ([HMP19, Lem. 4.11])**.** *For $\sigma \in \mathcal{I}_n$ we have*

$$\hat{\ell}(\sigma) = |\widehat{\mathrm{Inv}}(\sigma)| \quad \textit{hence} \quad \hat{\ell}^{FPF}(\sigma) = |\widehat{\mathrm{Inv}}(\sigma) \setminus \mathrm{Cyc}(\sigma)|.$$

For $(P, \leq)$ a graded poset with rank function r, let $R_P(q) = \sum_{p \in P} q^{r(p)}$ be the *rank generating function* of P. By Proposition 2.1, we have

$$R_{\mathcal{I}_n}(q) = \sum_{\sigma \in \mathcal{I}_n} q^{\hat{\ell}(\sigma)} \quad \text{and} \quad R_{\mathcal{I}_{2n}^{FPF}}(q) = \sum_{\tau \in \mathcal{I}_{2n}^{FPF}} q^{\hat{\ell}^{FPF}(\sigma)}. \tag{2.1}$$

For $n \geq 1$, let $[n]_q = \frac{1-q^n}{1-q}$. Using induction, it is easy to show that $R_{\mathcal{I}_{2n}^{FPF}}(q) = \prod_{k=1}^{n} [2k-1]_q := [2n-1]_q!!$, but $R_{\mathcal{I}_n}(q)$ does not have a simple closed form. However, it can be efficiently computed using the recurrence

$$R_{\mathcal{I}_n}(q) = R_{\mathcal{I}_{n-1}}(q) + q[n-1]_q R_{\mathcal{I}_{n-2}}(q).$$

2.2. **Labeled Motzkin path, statistics and bijections.** Recall a *Motzkin path* of *length* n is a lattice path from (0,0) to (n,0) consisting of up steps $U = (1, 1)$, down steps $D = (-1, 1)$, and horizontal steps $L = (0, 1)$ that does not go below the x-axis. Let $\mathcal{M}_n$ be the set of Motzkin paths of length n. For $\mu \in \mathcal{M}_n$, the *height* of the ith step is the larger of the step's y-coordinates, denoted $h_i(\mu)$.

For our purposes, a *labeled Motzkin path* is a Motzkin path $\mu = \mu_1 \ldots \mu_n$ where each down step $\mu_i = D$ is labeled with an integer $\lambda_i \in [h_i(\mu)]$. Let $\mathcal{M}_n^L$ be the set of labeled Motzkin paths of length n. As observed in [BBS11], the bijection in [Bia93] restricts to a simple bijection $\phi : \mathcal{I}_n \to \mathcal{M}_n^L$. For $\sigma \in \mathcal{I}_n$, let $\mu = \mu_1 \ldots \mu_n$ with

$$\mu_i = \begin{cases} U & i < \sigma(i) \\ L & i = \sigma(i) \\ D & i > \sigma(i) \end{cases}.$$

When $\mu_i = D$, let λ_i be the number of integers $j \geq i$ such that $\pi(j) \leq \pi(i)$ and define $\phi(\sigma) = (\mu, \lambda)$. Note that this description deviates from Biane's, in that his Motzkin paths receive two labels at every horizontal and down step. For involutions, the double labels are redundant: down steps are labeled twice by the same value and horizontal steps are labeled twice by their heights.

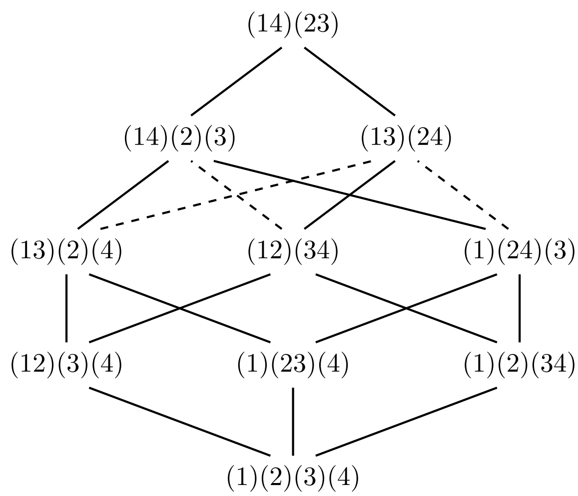


FIGURE 1. Bruhat order for $\mathcal{I}_4$. The dashed edges do not appear in $<_W$ (see Remark 3.2).

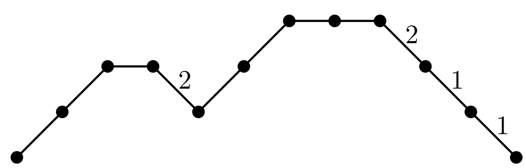


FIGURE 2. The labeled Motzkin path corresponding to $\sigma = (1\ 10)(2\ 4)(3)(5\ 9)(6\ 11)(7)(8)$.

For $(\mu, \lambda) \in \mathcal{M}_n^L$ with $\mu = \mu_1 \dots \mu_n$, define

$$H_i(\mu, \lambda) = \begin{cases} \lambda_i(\mu) - 1 & \mu_i = D \\ h_i(\mu) & \mu_i \neq D \end{cases} \quad \text{and} \quad H(\mu, \lambda) = \sum_{i=1}^{n} H_i(\mu, \lambda).$$

Next, define

$$H[\mu; q] = \sum_{\lambda:(\mu,\lambda)\in\mathcal{M}_n} q^{H(\mu,\lambda)}. \tag{2.2}$$

Alternatively, with

$$H_i[\mu; q] = \begin{cases} [h_i(\mu)]_q & \mu_i = D \\ q^{h_i(\mu)} & \text{else} \end{cases}, \quad \text{we have} \quad H[\mu; q] = \prod_{i=1}^{n} H_i[\mu; q]. \tag{2.3}$$

Note $H[\mu; 1]$ counts the number of labelings λ for μ. Define $\tilde{H}_i[\mu; q]$ and $\tilde{H}[\mu; q]$ analogously, omitting a factor of q on each upstep $\mu_i = U$. For example, with (μ, λ) as in Figure 2, we have $H(\mu, \lambda) = 18$ and

$$\{H_i[\mu; q]\}_{i=1}^{11} = \{q, q^2, q^2, 1 + q, q^2, q^3, q^3, q^3, 1 + q + q^2, 1 + q, 1\},$$

so $H[\mu; q] = q^{16}(1 + q)^2(1 + q + q^2)$ and $\tilde{H}[\mu; q] = q^{12}(1 + q)^2(1 + q + q^2)$.

A *Dyck path* is a Motzkin path without any horizontal steps. Necessarily, Dyck paths are always of even length. Let $\mathcal{D}_{2n}$ be the set of Dyck paths of length $2n$. By restricting ϕ to $\mathcal{I}_{2n}^{FPF}$, we obtain a bijection between $\mathcal{I}_{2n}^{FPF}$ and labeled Dyck paths of length $2n$, which are also known as *Hermite histories*. This restriction is well known when viewing $\mathcal{I}_{2n}^{FPF}$ as the set of perfect matchings of the complete graph K_{2n}. Since Dyck paths of length $2n$ always have n down steps, we have $H[\mu; q] = q^n \tilde{H}[\mu; q]$.

3. Proof of Main Results

Proof of Theorem 1.1. Let $\sigma \in \mathcal{I}_n$ and $(\mu, \lambda) = \phi(\sigma)$. We split $\widehat{\text{Inv}}(\sigma)$ into 2 disjoint classes S_1 and S_2.

Class 1: $S_1 = \{(i,j) \in \widehat{\mathrm{Inv}}(\sigma) \mid i \leq \sigma(i)\}$.
Fix $s \in [n]$ such that $s \leq \sigma(s)$. The number of integers t such that $(s,t) \in S_1$ is equal to the number of $(u,t) \in \mathrm{Cyc}(\sigma)$ such that $u \leq s < t$. As s corresponds to a horizontal or up step in μ, $h_s(\mu)$ counts the number of up steps in excess of the number of down step among the first s steps. Through the bijection ϕ, $h_s(\mu)$ counts the number of $(u, \sigma(u)) \in \mathrm{Cyc}(\sigma)$ such that $u \leq s < \sigma(u)$. Thus, $|S_1|$ is equal to the sum of the heights of all horizontal and up steps of μ.

Class 2: : $S_2 = \{(i,j) \in \widehat{\mathrm{Inv}}(\sigma) \mid i > \sigma(i)\}$.
As before, fix $s \in [n]$ such that $s > \sigma(s)$ and count the number of integers t such that $(s,t) \in S_2$. This requires counting the number of $(u,t) \in \mathrm{Cyc}(\sigma)$ such that $u \leq \sigma(s) < s < t$. Note that the weak inequality is changed to strict as $s < t$. As s corresponds to an down step in μ, the s-th step is labeled by some integer $1 \leq \lambda_s \leq h_s(\mu)$. Note that λ_s refers to the number of integers $t \geq s$ such that $\sigma(t) \leq \sigma(s)$. Thus, $\lambda_s - 1$ refers to the number of $(\sigma(t), t) \in \mathrm{Cyc}(\sigma)$ such that $\sigma(t) < \sigma(s) < s < t$. Thus, $|S_2|$ is equal to the sum of the labels (minus 1) of all down steps of μ.

As a consequence, we see $|\widehat{\mathrm{Inv}}(\sigma)| = H(\mu, \lambda)$. Therefore since $\hat{\ell}(\sigma) = |\widehat{\mathrm{Inv}}(\sigma)|$ we have

$$R_{\mathcal{I}_n}(q) = \sum_{\mu \in \mathcal{M}_n} H[\mu; q],$$

which completes our proof. □

Example 3.1. In Figure 2, σ has eighteen visible inversions. The two from Class 2 are $(4, 10)$ and $(9, 10)$, corresponding to the labels 2 in positions 4 and 9, respectively. The remaining sixteen visible inversions are from Class 1, and are encoded in the underlying Motzkin path.

One could also derive Theorem 1.1 from [Bia93, §3.2] and Proposition 2.1, but this would require a non-trivial modification of Biane's statistics. We prefer a self-contained proof, given its ease and brevity.

Proof of Corollary 1.2. The restriction of ϕ to $\mathcal{I}_{2n}^{FPF}$ maps FPF involutions to labeled Dyck paths of length $2n$. For $\tau \in \mathcal{I}_{2n}^{FPF}$, note that $\hat{\ell}(\tau) - \hat{\ell}^{FPF}(\tau) = \frac{1}{2}(|\mathrm{Cyc}(\tau)| + n) = n$. Thus, weighting $\mathcal{I}^{FPF}$ by $\hat{\ell}$ rather than $\hat{\ell}^{FPF}$ results in $q^n R_{\mathcal{I}_{2n}^{FPF}}(q)$ as desired. □

If we instead sum over the statistic $\tilde{H}[\mu; q]$ defined after Equation (2.3), we obtain

$$\sum_{\delta \in \mathcal{D}_{2n}} \tilde{H}[\delta; q] = R_{\mathcal{I}_{2n}^{FPF}}(q) = \prod_{k=1}^{n} [2k-1]_q. \tag{3.1}$$

Remark 3.2. An alternate approach to proving Theorem 1.1, used in [Wat14] to prove Corollary 1.2, is to describe the image of $(\mathcal{I}_n, \leq)$ under ϕ. One may show for an arbitrary cover relation $\tau \lessdot \sigma$ that $H(\phi(\sigma)) = H(\phi(\tau)) + 1$. An explicit description of cover relations appears in [Inc04, Tab. 1]. This strategy is easier to implement for the *weak order for involutions*, denoted $(\mathcal{I}_n, \leq_W)$, another partial order introduced in [RS90] with the same rank generating function $R_{\mathcal{I}_n}(q)$ but fewer cover relations. Using Figure 4.1, one can verify for $\tau \lessdot_W \sigma$ that $H(\phi(\sigma)) = H(\phi(\tau)) + 1$.

4. Connections to related work

4.1. **Watson's description.** We explain the relationship between Corollary 1.2 and [Wat14, Thm. 1]:

$$\sum_{\delta \in \mathcal{D}_{2n}} \prod_{i \in [n]} q^{d_i(\delta) - 2i} [d_i(\delta) - 2i + 1]_q = \prod_{k=1}^{n} [2k-1]_q., \tag{4.1}$$

where $d_i(\delta)$ is the position of the i-th down step in δ. If $d_i(\delta) = i'$, then the height of the i'-th step would the number of up steps $(d_i(\delta) - i)$ before i' in excess of the number of down step $(i-1)$ before i'. Thus, we have that the height of the i-th down step is $d_i(\delta) - 2i + 1$, so Equations (3.1) and (4.1) are equivalent.

Watson interprets Equation (4.1) in terms of fully packed rook placements on Young diagrams. These correspond to labeled Dyck paths as follows. For $(\delta = \delta_1 \ldots \delta_{2n}, \lambda)$ a labeled Dyck path, the corresponding Young diagram is cut out by the lattice path from $(0, n)$ to $(n, 0)$ whose i-th step is vertical if $\delta_i = D$ and horizontal if $\delta_i = U$. Each downward step is associated with a row on the diagram. Starting with the first downward step, place a rook in the λ-th leftmost valid (not in the same column as another rook) spot in

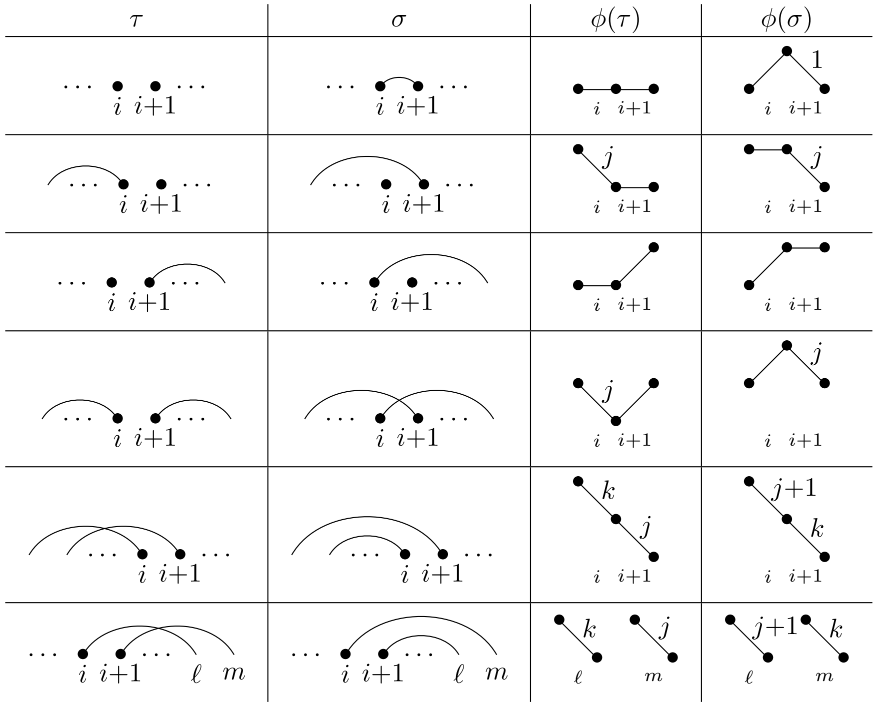


FIGURE 3. Cover relations $\tau \lessdot_W \sigma$ for $\tau, \sigma \in \mathcal{I}_n$, with involutions depicted as partial matchings.

the associated row. To show this is reversible, note that δ can be recovered from the shape of the diagram and the down step labeling can be iteratively recovered by the rooks.

4.2. $\mathcal{I}_{2n}^{FPF}$ **and the Ethiopian dinner game.** In [BLM15], the authors study the *Ethiopian dinner game*. Alice and Bob are sharing a meal with morsels $\{1, \ldots, 2n\}$. Bob prefers larger-valued morsels, while Alice prefers larger values π_k for some permutation $\pi = \pi_1 \ldots \pi_{2n}$. The players alternate choice of morsel, beginning with Alice. The optimal strategy for both players is best explained by describing the reverse order in which morsels are chosen – Bob chooses the smallest unselected morsel in $\pi_1 \ldots \pi_{2n}$, then Alice chooses the leftmost unselected morsel and so on, resulting in an allocation function $w : [2n] \to \{A, B\}$ [KC71, LS12].

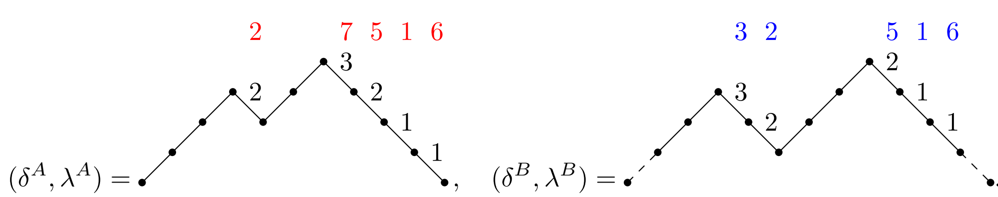
The main result [BLM15, Thm. 1] is a bijection from permutations in S_{2n} to pairs of labeled Dyck paths obtained by analyzing the Ethiopian dinner game. Given a permutation $\pi = \pi_1 \ldots \pi_{2n}$ with optimal allocation w, construct the Dyck paths $\delta^A(\pi) = \delta_1^A \ldots \delta_{2n}^A$ and $\delta^B(\pi) = \delta_1^B \ldots \delta_{2n+2}^B$ by setting

$$\delta_i^A = \begin{cases} D & w(\pi_i) = A \\ U & \text{else,} \end{cases} \quad \text{and} \quad \delta_j^B = \begin{cases} D & w(j+1) = B \text{ or } j = 2n+2 \\ U & \text{else.} \end{cases}$$

Note by construction that $\delta_1^B = U$. For $\delta_i^A = D$ and $\delta_j^B = D$ ($j \neq 2n+2$), define labels λ^A and λ^B by

$$\lambda^A(i) = 1 + \#\{k < i : \delta_k^A = D, \pi_k^{-1} > \pi_i^{-1}\} \quad \text{and} \quad \lambda^B(j) = 1 + \#\{\ell > j : \delta_\ell^B = D, \pi_{j-1} > \pi_{\ell-1}\}.$$

For example, with $\tau = 94328\ 10\ 7516$, we have $w = AABBAAABBB$ and



Here, the red integers indicate σ_i^{-1} for $\delta_i^A = D$ and the blue integers indicate σ_{j-1} for $\delta_j^B = D$.

Let $(\hat{\delta}^B, \lambda^B)$ be the lattice path obtained by ignoring the first and last step of δ^B.

Proposition 4.1. *For $\pi \in S_{2n}$, $(\delta^A(\pi), \lambda^A) = (\hat{\delta}^B(\pi), \lambda^B)$ if and only if $\pi \in \mathcal{I}_{2n}^{FPF}$.*

Proof. Let w be the allocation function for π, and write

$$w^{-1}(A) = \{a_1 < \cdots < a_n\}, \quad w^{-1}(B) = \{b_1 < \cdots < b_n\}.$$

By construction $\delta^A = \hat{\delta}^B$ if and only if both $\pi(w^{-1}(B) = w^{-1}(A))$, hence $\pi^{-1}(w^{-1}(B)) = w^{-1}(A)$. Moreover, $\lambda^A = \lambda^B$ if and only if $\pi(b_i) < \pi(b_j)$ for $i < j$. This is equivalent to saying $\pi(a_i) = b_i$ and $\pi(b_i) = a_i$, that is $\pi \in \mathcal{I}_{2n}^{FPF}$. □

We have the surprising but easy consequence that for $\mathcal{I}_{2n}^{FPF}$ Biane's bijection coincides with theirs.

Corollary 4.2. *For $\tau \in \mathcal{I}_{2n}^{FPF}$, $\phi(\tau) = (\delta^A(\tau), \lambda^A) = (\hat{\delta}^B(\tau), \lambda^B)$.*

Call a permutation π *fair* if Alice and Bob eat the same morsels regardless of who chooses first. For a fixed-point-free involution, Proposition 4.1 implies whenever Alice picks the morsel with value i that Bob will next pick the morsel at position i. This guarantees any $\tau \in \mathcal{I}_{2n}^{FPF}$ is fair.

Proposition 4.3. *A permutation $\pi \in \mathfrak{S}_{2n}$ is fair if and only if $\hat{\delta}^B(\pi)$ is a labeled Dyck path.*

Proof. Let a_i and b_i be the position of the i-th morsel eaten by Alice and Bob respectively. These values can be recovered from $\hat{\delta}^B(\pi)$. Based on the optimal strategy, Alice ate her morsels starting from the right, so a_i is the position of the i-th rightmost up step in $\hat{\delta}^B(\pi)$. To recover Bob's moves, start from the leftmost down step and move right. If a down step is labelled k, match this down step with the k-th leftmost up step that has not been matched yet. Then b_i is the position of the down step matched with the up step at a_i. Note that $\hat{\delta}^B(\pi)$ is a labeled Dyck path if and only if a_i is to the left of b_i for all $i \in [n]$.

Let a'_i and b'_i be the position of the i-th morsel eaten by Alice and Bob respectively when Bob started first. Similar to the Alice-first variant, the optimal strategy is found starting with the last moves. Suppose that $a'_n, b'_n, \ldots, a'_{i+1}, b'_{i+1}$ are equal to their counterparts of the original game. Then $a'_i = \min\{a_1, b_1, \ldots, a_i, b_i\} = \min\{a_i, b_i\}$. Therefore if $a'_i = a_i$ then $a_i < b_i$. Now, suppose that $a_n, b_n, \ldots, b_{i+1}, a_i$ are equal to their counterparts. Then b'_i is Alice's least favorite morsel among $\{b_1, a_1, \ldots, a_{i-1}, b_i\}$, but b_i is known to be Alice's least favorite morsel in the larger set $\{a_1, b_1, \ldots, b_i, a_i\}$, so $b'_i = b_i$. Thus by induction π is fair if and only if $a_i < b_i$ for all $i \in [n]$, which is equivalent to $\hat{\delta}^B(\pi)$ being a labelled Dyck path. □

Remark 4.4. As a consequence of the crossout procedure in [BLM15], Alice will always eat morsels from right to left and Bob will always eat morsels from highest to lowest. Therefore, for a fair permutation Alice and Bob will eat the same morsels in the same order regardless of who chooses first.

Corollary 4.5. *The number of fair permutations of length $2n$ is* $(2n-1)!!^2$.

Proof. By Proposition 4.3 and [BLM15, Thm. 1], counting fair permutations is equivalent to counting pairs of labelled Dyck paths of length n. Setting $q = 1$ in Corollary 1.2, this is $(2n-1)!!^2$. □

Call a permutation *k-fair* if Alice eats all but k of the same morsels when going second. Note the fair permutations are precisely the 0-fair permutation. It would be interesting to study enumerative properties of k-fair permutations, and structural properties of their corresponding Dyck paths.

4.3. **A final identity.** The following equation is [DS01, Thm. 1.2] and its translation to Motzkin paths:

$$\binom{n}{k}_q = \sum_{\sigma \in \mathcal{I}_n} (q-1)^{c(\sigma)} q^{\hat{\ell}^{FPF}(\sigma)} \binom{n-2c(\sigma)}{k-c(\sigma)} = \sum_{\mu \in \mathcal{M}_n} (q-1)^{s(\mu)} \tilde{H}[\mu; q] \binom{n-2s(\mu)}{k-s(\mu)} \tag{4.2}$$

where $s(\mu)$ is the number of down steps in μ. The proof of Equation (4.2) follows from generating function manipulations, and it would be interesting to give a direct combinatorial proof using Motzkin paths.

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