

Graceful Wind Chimes

Jocelyn R. Bell*, Lotenna Nwobbi†, Connor Parrow‡,
Alexandria Wheeler§ and Ephrata Zelleke¶

Abstract

A graceful labeling of a graph with n edges is an assignment to each vertex a unique label between 0 and n , inclusive, satisfying the additional requirement that if each edge is labeled by the absolute value of the difference of its incident vertices, then the set of edge labels is exactly $\{1, 2, \dots, n\}$. It has been an open question for more than 50 years whether or not every tree has a graceful labeling. In fact, it is unknown whether every tree with maximum degree 3 has a graceful labeling. In this paper we introduce a class of maximum degree 3 trees we call wind chimes. We prove that all wind chimes satisfying certain criteria have a graceful labeling.

1 Introduction

In 1967, A. Rosa introduced the concept of a graceful labeling of a graph [8]. A labeling of a graph with n edges is said to be graceful if each vertex is assigned a distinct number in $\{0, 1, \dots, n\}$ and when each edge is labeled by the absolute value of the difference of the labels of its incident vertices, each number in $\{1, 2, \dots, n\}$ appears on exactly one edge. See Figure 1.

A *graph* G is a pair $(V(G), E(G))$ consisting of vertices $V(G)$ and edges $E(G)$ between vertices of G . If u and v are vertices of a graph G and there is

*Hobart and William Smith Colleges bell@hws.edu

†Harvey Mudd College lnwobbi@g.hmc.edu

‡William Smith College connor.parrow@hws.edu

§Carthage College, awheeler@carthage.edu

¶Goucher College zephrrata@gmail.com

This material is based upon work supported by the National Science Foundation under Grant No. 1757616.

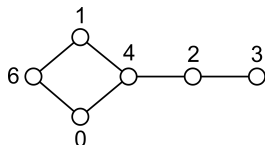


Figure 1: A gracefully labeled graph

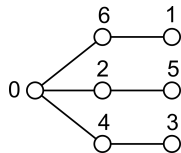


Figure 2: A gracefully labeled tree

an edge between u and v , this edge is denoted uv and we write $uv \in E(G)$. A *tree* is a connected graph with no cycles. The graph in Figure 1 contains a cycle while the graph in Figure 2 is a tree. The *degree* of a vertex v is the number of edges incident with v . For example, in Figure 2, the vertex labeled 0 has degree 3. For definitions omitted we refer the reader to West [10].

In this paper, a *labeling* of a graph G is injection from $V(G)$ to $\{0, 1, 2, \dots\}$. A *labeled graph* is a pair (G, θ) where G is a graph and θ is a labeling of G . If (G, θ) is a labeled graph, we sometimes refer to the vertex of G that is labeled k as the k -vertex of G .

Every labeling θ of a graph G induces an edge labeling as follows: for every edge $uv \in E(G)$, define $\theta(uv) = |\theta(u) - \theta(v)|$. A labeled graph (G, θ) with n edges is *gracefully labeled*, and θ is called a *graceful labeling* of G , provided the range of θ is contained in $\{0, 1, \dots, n\}$ and $\{\theta(uv) : uv \in E(G)\} = \{1, 2, \dots, n\}$. A graph for which there exists a graceful labeling is called *graceful*. The graphs appearing in Figures 1 and 2 are both graceful.

While there are many graphs that are not graceful, it has been an open question for more than 50 years whether or not every tree is graceful. That every tree can be gracefully labeled is known as the graceful tree conjecture. There are hundreds of papers written on the topic of graceful labeling [5]. In fact, as J. Gallian writes,

Despite the efforts of many, the graceful tree conjecture remains open even for trees with maximum degree 3 [5].

In this paper we take a step towards solving the graceful tree conjecture for trees with maximum degree 3.

Trees with maximum degree 2 are *paths*, and every path is graceful [8]. We denote the path with n vertices and $n - 1$ edges by P_n . We will write the path P_n as $P_n = (v_1, v_2, \dots, v_n)$, by which we mean that the vertices of P_n are $\{v_1, \dots, v_n\}$ and the edges of P_n are $\{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$. The vertices v_1 and v_n will be called *end-vertices*.

In this paper we will be concerned with a certain family of maximum degree 3 trees. Choose a natural number m , and to each vertex v_i of the path $P_m = (v_1, v_2, \dots, v_m)$, attach a path P_{l_i} by adding an edge between v_i and an end-vertex of P_{l_i} . We call the resulting tree W a *wind chime*, and for brevity denote W by $(l_1, l_2, \dots, l_m)$. The vertices $(v_1, v_2, \dots, v_m)$ of P_m form the *spine* of $(l_1, l_2, \dots, l_m)$. See Figure 3. Since, for example $(4, 3, 3, 2)$ and $(2, 0, 0, 3, 3, 0, 1)$ represent the same wind chime, to eliminate multiple names for the same tree, if

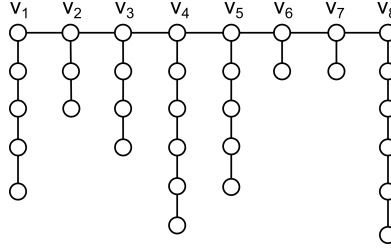


Figure 3: The wind chime $(4, 2, 3, 5, 4, 1, 1, 5)$

W is a wind chime one can require m to be minimal such that $W = (l_1, \dots, l_m)$ and $(l_1, \dots, l_m)$ is minimal with respect to the lexicographic ordering on m -tuples.

Every wind chime is a tree with maximum degree 3. While the graceful tree conjecture for trees with maximum degree 3 is seemingly difficult, we offer the following simplification, which may prove to be more tractable.

Conjecture. *Every wind chime is graceful.*

Although this paper falls short of settling this conjecture, we will show that wind chimes that satisfy certain congruence conditions can be gracefully labeled. Some wind chimes have previously been shown to be graceful. Huang, Kotzig and Rosa [6] have shown that any tree with at most 4 vertices of degree 1 is graceful, and therefore any wind chime with at most 4 spine vertices is graceful. Some other graceful wind chimes appear in [2]; in this paper we generalize the wind chime results due to Barrientos [2].

Our results will require a type of labeling stronger than graceful, which was also introduced by A. Rosa [8]. A labeling θ of a graph G with n edges is said to be an α -labeling provided θ is graceful and there exists a number λ so that for every edge uv of G , either $\theta(u) \leq \lambda < \theta(v)$ or $\theta(v) \leq \lambda < \theta(u)$. This λ is known as the *boundary value* of the α -labeling θ . The value λ is the largest of the “small” labels and the value $\lambda + 1$, which will appear often in this paper, is the smallest of the “large” labels. To avoid confusion, we sometimes denote the boundary value λ of the α -labeling θ by λ_θ . See Figure 1; this labeling is in fact an α -labeling with boundary value 2.

There are trees that cannot be α -labeled. In fact, there are trees with degree at most 3 that have no α -labeling: the spider with three legs of length two appearing in Figure 2 is one such tree. Graceful labelings of spider trees have been studied rather extensively; see, for example, [1, 3].

In fact, the tree in Figure 2 is a wind chime: it is the wind chime $(1, 2, 1)$. Therefore, there is at least one wind chime that cannot be α -labeled. However, the tree in Figure 2 is the only spider tree with three legs that does not have an α -labeling [6]. Hence any wind chime $(l_1, l_2, l_3) \neq (1, 2, 1)$ has an α -labeling.

2 Preliminaries

In this paper all graphs are simple and connected. A graph G is *bipartite* if there exists a partition of the vertices of G into two sets X and Y satisfying that every edge of G joins a vertex in X with a vertex in Y . X and Y are known as the *parts* of G . Every tree is bipartite.

If G is a graph with n edges and θ is a graceful labeling of G , the *complementary labeling* θ' of θ is defined by setting $\theta'(v) = n - \theta(v)$ for each vertex v of G . The labeling θ' is also graceful; moreover, if θ is an α -labeling with boundary value λ_θ then θ' is an α -labeling with boundary value $\lambda_{\theta'} = n - \lambda_\theta - 1$.

If G is a graph with n edges and θ is an α -labeling of G with boundary value λ_θ , then the *inverse labeling* θ^* of θ is defined by $\theta^*(v) = \lambda_\theta - \theta(v) \bmod n + 1$ for each $v \in V(G)$ [9]. The inverse labeling θ^* of an α -labeling θ is also an α -labeling with the same boundary value as θ .

We will make use of the following proposition. This result is not new; it is a modification of a result of Huang, Kotzig and Rosa [6] and also appears in [11]. It allows for two α -labeled graphs to be combined to produce another α -labeled graph, by adding an edge between the 0-vertex of one graph and the vertex labeled by the smallest of the large labels in the other. We have highlighted a specific detail as it will be needed to prove our main results.

Proposition 1. *Let (G_1, θ_1) be an α -labeled graph with boundary value λ_1 and let (G_2, θ_2) be an α -labeled graph with boundary value λ_2 . Let $v_1 \in V(G_1)$ satisfy $\theta_1(v_1) = \lambda_1 + 1$ and let $v_2 \in V(G_2)$ satisfy $\theta_2(v_2) = 0$. Let G be the graph formed by adding the edge v_1v_2 to $G_1 \cup G_2$. Then there exists an α -labeling ψ of G with boundary value λ_ψ satisfying $\psi(v^*) = \lambda_\psi + 1$, where $v^* \in V(G_2)$ satisfies $\theta_2(v^*) = \lambda_2 + 1$.*

Proof. Suppose G_2 has m edges. If ψ is defined as follows:

$$\psi(v) = \begin{cases} \theta_1(v) & \text{if } v \in V(G_1) \text{ and } \theta_1(v) \leq \lambda_1 \\ \theta_1(v) + m + 1 & \text{if } v \in V(G_1) \text{ and } \theta_1(v) > \lambda_1 \\ \theta_2(v) + \lambda_1 + 1 & \text{if } v \in V(G_2) \end{cases} \quad (1)$$

then ψ is an α -labeling of G [11]. One can check that the boundary value of ψ is $\lambda_\psi = \lambda_1 + \lambda_2 + 1$. It remains to show that $\psi(v^*) = \lambda_\psi + 1$. By the definition of ψ , since $v^* \in V(G_2)$ and $\theta_2(v^*) = \lambda_2 + 1$, $\psi(v^*) = \theta_2(v^*) + \lambda_1 + 1 = \lambda_2 + 1 + \lambda_1 + 1 = \lambda_\psi + 1$. $\square$

See Figure 4 for an example of Proposition 1 applied to two disjoint copies of the labeled graph in Figure 1, adding an edge between the 3-vertex (where 3 is the smallest of the large labels) of one copy and the 0-vertex of the other.

3 Labeled Paths

As wind chimes are built from paths, paths will play an important role in our results. Paths are trees and are therefore bipartite graphs. If $n = 2k$, then the

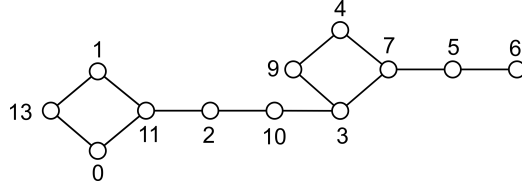


Figure 4: Proposition 1 applied to two copies of Figure 1

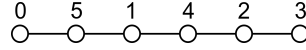


Figure 5: P_6 labeled as in Equation 2

parts of P_n both have size k , and any α -labeling has boundary value $k - 1$. If $n = 2k + 1$, one part of P_n contains k vertices while the other contains $k + 1$ vertices. In this case, the boundary value of an α -labeling of the path P_n is either $k - 1$ (if the 0-vertex is in the smaller part) or k (if the 0-vertex is in the larger part).

Any path P_n has an α -labeling θ with an end-vertex labeled 0 [8]; explicitly, θ is as follows:

$$\theta(v_i) = \begin{cases} \frac{i-1}{2} & \text{if } i \text{ is odd} \\ n - \frac{i}{2} & \text{if } i \text{ is even.} \end{cases} \quad (2)$$

Note that if n is even, then $\theta(v_n) = \lambda + 1$ where $\lambda = \frac{n-2}{2}$ is the boundary value of θ . See Figure 5. Taking the complementary labeling θ' of θ yields an α -labeling of P_n with an end-vertex labeled $n - 1$. Explicitly,

$$\theta'(v_i) = \begin{cases} n - \frac{i+1}{2} & \text{if } i \text{ is odd} \\ \frac{i-2}{2} & \text{if } i \text{ is even.} \end{cases} \quad (3)$$

Note that if n is odd, then $\theta'(v_n) = \lambda + 1$ where $\lambda = \frac{n-3}{2}$ is the boundary value of θ' . See Figure 6.

There is quite a bit of freedom in the placement of the 0-vertex in an α -labeling of a path. A. Rosa showed that for every n and for any vertex v of P_n there exists an α -labeling θ of P_n with $\theta(v) = 0$, except for the case that v is the central vertex of P_5 [9]. In fact, with few exceptions, for any vertex v of P_n and each number j with $1 \leq j \leq n - 1$, there exists an α -labeling of P_n in which v is labeled j [4]. We will require more, however; for our purposes it

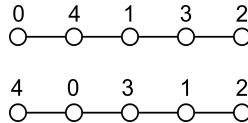


Figure 6: P_5 with the labelings in Equation 2 and Equation 3

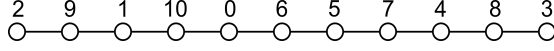


Figure 7: P_{11} with the labeling in the proof of Lemma 2

will be important that the vertex labeled with the smallest of the large labels is adjacent to the 0-vertex. The next three lemmas provide criteria for when this is possible. Although these lemmas easily follow from the proof of the main result in [9], the fact that these labelings also satisfy our additional condition requires proof.

Lemma 2. *If $n = 2k + 1$ and $n \equiv 3 \pmod{4}$, there exists an α -labeling ψ of $P_n = (v_1, \dots, v_n)$ with boundary value k satisfying $\psi(v_k) = 0$ and $\psi(v_{k+1}) = k + 1$.*

Proof. Since $n \equiv 3 \pmod{4}$, if $n = 2k + 1$ then k is odd. Let θ be the α -labeling of P_n as depicted in Figure 3 of [9]. No explicit formula is given in [9], but it can be written as follows:

$$\theta(v_i) = \begin{cases} k - \frac{k-i}{2} & \text{if } i \leq k \text{ and } i \text{ is odd} \\ k + \frac{k-i+1}{2} & \text{if } i \leq k \text{ and } i \text{ is even} \\ \frac{i-k}{2} - 1 & \text{if } i > k \text{ and } i \text{ is odd} \\ n - \frac{i-k+1}{2} & \text{if } i > k \text{ and } i \text{ is even.} \end{cases}$$

Then, since $\theta(v_{k+2}) = 0$, it follows that the boundary value of θ is k , since the part of P_n which contains v_{k+2} is the larger one. Now let ψ be the inverse labeling of θ ; since P_n has $n - 1$ edges, $\psi(v) = k - \theta(v) \pmod{n}$. Then $\psi(v_k) = k - k \pmod{n} = 0$ and

$$\psi(v_{k+1}) = k - \left(n - \frac{k+1-k+1}{2} \right) \pmod{n} = k + 1$$

and since the boundary value of ψ is k , ψ satisfies the conditions of the lemma. $\square$

See Figure 7 for the labeling of P_{11} as in the proof of Lemma 2.

Lemma 3. *If $n = 2k$, there exists an α -labeling ψ of $P_n = (v_1, \dots, v_n)$ with boundary value $k - 1$ satisfying $\psi(v_k) = 0$ and $\psi(v_{k+1}) = k$.*

Proof. Suppose $n = 2k$, and let θ be the α -labeling of P_n as in Figure 1 of [9]; no explicit formula is given, however this labeling can be written as follows: if k is odd,

$$\theta(v_i) = \begin{cases} k - 1 - \frac{k-i}{2} & \text{if } i \leq k \text{ and } i \text{ is odd} \\ k - 1 + \frac{k-i+1}{2} & \text{if } i \leq k \text{ and } i \text{ is even} \\ \frac{i-k}{2} - 1 & \text{if } i > k \text{ and } i \text{ is odd} \\ n - \frac{i-k+1}{2} & \text{if } i > k \text{ and } i \text{ is even} \end{cases}$$

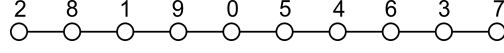


Figure 8: P_{10} with the labeling in the proof of Lemma 3

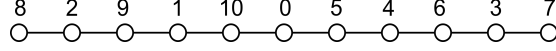


Figure 9: P_{11} with the labeling in the proof of Lemma 4

and if k is even,

$$\theta(v_i) = \begin{cases} k + \frac{k-i-1}{2} & \text{if } i \leq k \text{ and } i \text{ is odd} \\ k - 1 - \frac{k-i}{2} & \text{if } i \leq k \text{ and } i \text{ is even} \\ n - \frac{i-k+1}{2} & \text{if } i > k \text{ and } i \text{ is odd} \\ \frac{i-k}{2} - 1 & \text{if } i > k \text{ and } i \text{ is even.} \end{cases}$$

In either case, $\theta(v_k) = k - 1$ and $\theta(v_{k+1}) = n - 1$, and any α -labeling of P_n has boundary value $k - 1$. Let ψ be the inverse labeling of θ , so that $\psi(v) = k - 1 - \theta(v) \bmod n$. Then $\psi(v_k) = 0$ and $\psi(v_{k+1}) = k - 1 - (n - 1) \bmod n = k$, as required. $\square$

See Figure 8 for the labeling of P_{10} as in the proof of Lemma 3.

Lemma 4. *If $n = 2k + 1$ and $n \equiv 3 \bmod 4$ there exists an α -labeling ψ of $P_n = (v_1, \dots, v_n)$ with boundary value $k - 1$ satisfying $\psi(v_{k+1}) = 0$ and $\psi(v_{k+2}) = k$.*

Proof. Suppose $n = 2k + 1$, $n \equiv 3 \bmod 4$, $P_n = (u_1, \dots, u_n)$ and let θ be the α -labeling of P_n satisfying $\theta(u_k) = 0$ and $\theta(u_{k+1}) = k + 1$ with boundary value k found in the proof of Lemma 2. Since P_n has $n - 1$ edges, the complementary labeling θ' of θ is an α -labeling satisfying $\theta'(u_k) = (n - 1) - 0 = n - 1$ and $\theta'(u_{k+1}) = (n - 1) - (k + 1) = k - 1$. The boundary value of θ' is $(n - 1) - k - 1 = k - 1$. Then, the inverse labeling f of θ' satisfies

$$f(u_k) = \lambda_{\theta'} - \theta'(u_k) \bmod n = (k - 1) - (n - 1) \bmod n = k$$

$$f(u_{k+1}) = \lambda_{\theta'} - \theta'(u_{k+1}) \bmod n = (k - 1) - (k - 1) \bmod n = 0.$$

Then, reversing the order of the vertices of P_n by setting $v_i = u_{n+1-i}$ for $1 \leq i \leq n$ and setting $\psi(v_i) = f(u_{n+1-i})$ yields the result, since $\psi(v_{k+1}) = 0$ and $\psi(v_{k+2}) = k$. $\square$

See Figure 9 for the labeling of P_{11} as in the proof of Lemma 4.

4 Wind Chimes

A tree is said to be a *caterpillar* if, after all of its degree 1 vertices are removed, the remaining tree is a path. All caterpillars can be α -labeled [8]. Therefore,

any wind chime of the form $(l_1, \dots, l_m)$ where $l_i \leq 1$ for each i has an α -labeling. It follows from a construction of Koh, Rogers and Tan [7] that all wind chimes of the form $(n, n, n, \dots, n)$ are graceful. It is known that every wind chime of the form $(k, k+1, k+2, \dots, k+n)$ has an α -labeling [2] (see Figure 12). We prove more general results which include these three results as special cases.

Theorem 5. *The wind chime $(l_1, l_2, \dots, l_m)$ has an α -labeling provided for each i with $1 \leq i \leq \lfloor \frac{m-1}{2} \rfloor$, $|l_{2i} - l_{2i+1}| \leq 1$ and $l_{2i+1} + l_{2i} \not\equiv 3 \pmod{4}$.*

Proof. Suppose W is a wind chime with an odd number of spine vertices. We prove that W has an α -labeling by induction.

Any wind chime W with 1 spine vertex is a path $P_n = (u_1, u_2, \dots, u_n)$. Choose the spine vertex to be $v_1 = u_n$. If n is even, choose ϕ to be the labeling of P_n as in Equation 2, and if n is odd, choose ϕ to be the labeling as in Equation 3. Then ϕ is an α -labeling of W satisfying $\phi(v_1) = \lambda_\phi + 1$ where λ_ϕ is the boundary value of the labeling ϕ .

Next, assume that any wind chime with $2k-1$ spine vertices $v_1, \dots, v_{2k-1}$ satisfying the conditions of the theorem has an α -labeling ϕ satisfying $\phi(v_{2k-1}) = \lambda_\phi + 1$, where λ_ϕ is the boundary value of ϕ .

Now suppose $m = 2k + 1$ and let $(l_1, \dots, l_m)$ be any wind chime satisfying the hypotheses of the theorem with spine vertices $v_1, \dots, v_m$. By the induction hypothesis, $(l_1, \dots, l_{m-2})$ has an α -labeling ϕ satisfying that $\phi(v_{m-2}) = \lambda_\phi + 1$ where λ_ϕ is the boundary value of ϕ .

By the hypotheses of the theorem, (l_{m-1}, l_m) is the path $P_j = (w_1, \dots, w_j)$ where j is even (if $|l_{m-1} - l_m| = 0$), or where $j \equiv 3 \pmod{4}$ (if $|l_{m-1} - l_m| = 1$). Let v_{m-1} and v_m be the “spine” vertices of (l_{m-1}, l_m) ; that is, if $q = l_{m-1} + 1$ then $v_{m-1} = w_q$ and $v_m = w_{q+1}$. See Figure 10. If $l_{m-1} = l_m$ apply Lemma 3 to P_j ; if $l_{m-1} + 1 = l_m$ apply Lemma 2; or if $l_{m-1} = l_m + 1$ apply Lemma 4 to P_j . This yields an α -labeling θ of $P_j = (l_{m-1}, l_m)$ satisfying $\theta(v_{m-1}) = 0$ and $\theta(v_m) = \lambda_\theta + 1$ where λ_θ is the boundary value of the labeling θ .

Now apply Proposition 1 to $(G_1, \theta_1) = ((l_1, \dots, l_{m-2}), \phi)$ and $(G_2, \theta_2) = ((l_{m-1}, l_m), \theta)$. Since $\phi(v_{m-2}) = \lambda_\phi + 1$ and $\theta(v_{m-1}) = 0$, the resulting graph includes an edge between v_{m-2} and v_{m-1} , so that the resulting graph is the wind chime $(l_1, \dots, l_m)$. The resulting α -labeling ψ of $(l_1, \dots, l_m)$ is as in Equation 1. Since $\theta(v_m) = \lambda_\theta + 1$, by Proposition 1 we have $\psi(v_m) = \lambda_\psi + 1$, where λ_ψ is the boundary value of ψ . This completes the induction.

Now, for the case of an even number of spine vertices, suppose $m = 2k$ and $(l_1, l_2, \dots, l_m)$ satisfies the hypotheses of the theorem. By the preceding argument, $(l_1, l_2, \dots, l_{m-1})$ with spine vertices $v_1, \dots, v_{m-1}$ has an α -labeling ϕ satisfying $\phi(v_{m-1}) = \lambda_\phi + 1$. Let $n = l_m + 1$ and let φ be an α -labeling of $P_n = (u_1, \dots, u_n)$ so that $\varphi(u_1) = 0$. Now apply Proposition 1 with $(G_1, \theta_1) = ((l_1, l_2, \dots, l_{m-1}), \phi)$ and $(G_2, \theta_2) = (P_n, \varphi)$ to complete the proof. $\square$

The wind chime $(4, 2, 3, 5, 4, 1, 1, 5)$ in Figure 3 satisfies the criteria of Theorem 5; see Figure 11 for the labeling of this tree produced as in the proof of Theorem 5.

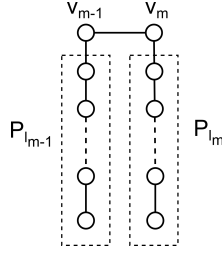


Figure 10: Schematic diagram for the proof of Theorem 5

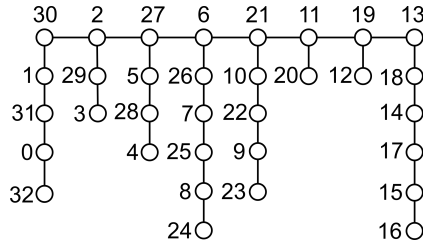


Figure 11: Labeling of $(4, 2, 3, 5, 4, 1, 1, 5)$ as in Theorem 5

Theorem 6. *If $m \geq 2$, the wind chime $(l_1, l_2, \dots, l_m)$ has an α -labeling provided for each i with $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$, $|l_{2i-1} - l_{2i}| \leq 1$ and $l_{2i-1} + l_{2i} \not\equiv 3 \pmod{4}$.*

Proof. The proof is very similar to the proof of Theorem 5. Suppose W is a wind chime with an even number of spine vertices. We prove that W has an α -labeling by induction.

Any wind chime of the form $W = (l_1, l_2)$ satisfying the hypotheses of the theorem satisfies that $l_1 + l_2 \not\equiv 3 \pmod{4}$, and therefore W is the path $P_n = (u_1, \dots, u_n)$ where $n = l_1 + l_2 + 2$, so that $n \not\equiv 1 \pmod{4}$. Let $v_1 = u_{l_1+1}$ and $v_2 = u_{l_1+2}$. Then, by either Lemma 2, Lemma 3, or Lemma 4, $W = P_n$ has an α -labeling ϕ satisfying that $\phi(v_1) = 0$ and $\phi(v_2) = \lambda_\phi + 1$, where λ_ϕ is the boundary value of the labeling ϕ .

Next, assume that any wind chime W satisfying the conditions of the theorem with $2k$ spine vertices $v_1, \dots, v_{2k}$ has an α -labeling ϕ satisfying that $\phi(v_{2k}) = \lambda_\phi + 1$, where λ_ϕ is the boundary value of the labeling ϕ .

Suppose $m = 2k + 2$ and let $(l_1, \dots, l_m)$ be any wind chime satisfying the hypotheses of the theorem with spine vertices $v_1, \dots, v_m$. By the induction hypothesis, $(l_1, \dots, l_{m-2})$ has an α -labeling ϕ satisfying that $\phi(v_{m-2}) = \lambda_\phi + 1$ where λ_ϕ is the boundary value of ϕ . As in the proof of Theorem 5, by either Lemma 2, Lemma 3, or Lemma 4, the wind chime (l_{m-1}, l_m) with “spine” vertices v_{m-1}, v_m has an α -labeling θ satisfying $\theta(v_{m-1}) = 0$ and $\theta(v_m) = \lambda_\theta + 1$, where λ_θ is the boundary value of the labeling θ . To complete the induction, apply Proposition 1 to $(G_1, \theta_1) = ((l_1, \dots, l_{m-2}), \phi)$ and $(G_2, \theta_2) = ((l_{m-1}, l_m), \theta)$, which adds an edge between v_{m-2} and v_{m-1} , to obtain an α -labeling ψ of $(l_1, \dots, l_m)$ satisfying $\psi(v_m) = \lambda_\psi + 1$, where λ_ψ is the boundary value of ψ .

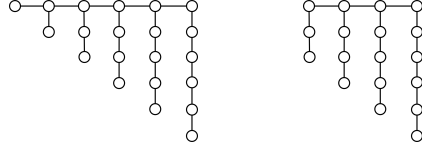


Figure 12: The triangular tree $(0, 1, 2, 3, 4, 5)$ and the truncated triangular tree $(2, 3, 4, 5)$

Next, suppose $m = 2k + 1$ and $(l_1, \dots, l_m)$ with spine vertices $v_1, \dots, v_m$ satisfies the hypotheses of the theorem. By the preceding argument, the wind chime $(l_1, \dots, l_{m-1})$ has an α -labeling ϕ satisfying $\phi(v_{m-1}) = \lambda_\phi + 1$, where λ_ϕ is the boundary value of ϕ . Let $n = l_m + 1$ and let φ be an α -labeling of $P_n = (u_1, \dots, u_n)$ so that $\varphi(u_1) = 0$. Proceed, as in the proof of Theorem 5, to apply Proposition 1 to $(G_1, \theta_1) = ((l_1, l_2, \dots, l_{m-1}), \phi)$ and $(G_2, \theta_2) = (P_n, \varphi)$ to finish the proof. $\square$

Next, we shall examine how our results imply each of the results on wind chimes due to Barrientos [2]. In [2], a wind chime of the form $(0, 1, 2, \dots, n)$ is called a triangular tree, and a wind chime of the form $(k, k + 1, k + 2, \dots, k + n)$ where $k \geq 1$ is called a truncated triangular tree. See Figure 12. Theorem 2.1 of [2] states that every triangular tree has an α -labeling, and Theorem 2.2 of [2] states that every truncated triangular tree has an α -labeling.

Consider any wind chime of the form $W = (k, k + 1, k + 2, \dots, k + n)$ for $k \geq 0$. Then $W = (l_1, l_2, \dots, l_{n+1})$ where $l_i = k + i - 1$. Then $|l_j - l_{j+1}| = 1$ for $1 \leq j \leq n$. If k is odd, then $k = 2q + 1$ and

$$l_{2i+1} + l_{2i} = 2k + 4i - 1 = 2(2q + 1) + 4i - 1 = 4(q + i) + 1 \equiv 1 \pmod{4}$$

so that W satisfies the hypotheses of Theorem 5. If k is even, then $k = 2q$ and

$$l_{2i} + l_{2i-1} = 2k + 4i - 3 = 4(q + i) - 3 \equiv 1 \pmod{4}$$

so that W satisfies the hypotheses of Theorem 6. Therefore Theorems 2.1 and 2.2 of [2] follow from our results.

In [2], pairs of triangular trees are combined to produce wind chimes in three different ways. In the first construction in [2], two copies of the same triangular tree $(0, 1, \dots, n)$ are connected by adding an edge between the last spine vertices of each tree, as in Figure 13. The resulting tree is a wind chime of the form

$$W_1 = (0, 1, 2, \dots, n - 1, n, n, n - 1, \dots, 2, 1, 0).$$

Theorem 3.1 of [2] states that any such wind chime has an α -labeling. If $W_1 = (l_1, \dots, l_m)$ then $|l_{2i} - l_{2i-1}| \leq 1$, and $l_{2i-1} + l_{2i} \equiv 1 \pmod{4}$ if $i \neq \frac{n+2}{2}$. If $i = \frac{n+2}{2}$, then $l_{2i-1} + l_{2i} = 2n$, so in either case $l_{2i-1} + l_{2i} \not\equiv 3 \pmod{4}$. Thus W_1 satisfies the hypotheses of Theorem 6, and therefore Theorem 3.1 of [2] follows from Theorem 6.

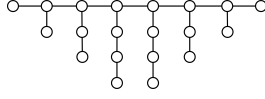


Figure 13: The wind chime $(0, 1, 2, 3, 3, 2, 1, 0)$

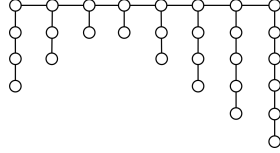


Figure 14: The wind chime $(3, 2, 1, 1, 2, 3, 4, 5)$

In the second construction in [2], two triangular trees $(0, 1, \dots, m)$ and $(0, 1, \dots, n)$ where m and n are both odd are connected by adding an edge between the last spine vertices of each tree. This results in a wind chime of the form

$$W_2 = (0, 1, \dots, m-1, m, n, n-1, \dots, 1, 0).$$

Theorem 3.2 of [2] states that any tree of this type has an α -labeling. If $W_2 = (l_1, \dots, l_{m+n+2})$ then since m is odd, $m = l_{m+1} \neq l_{2i-1}$ for any i , and so $|l_{2i-1} - l_{2i}| = 1$ for $1 \leq i \leq \frac{m+n+2}{2}$. Since n is odd, for $i > \frac{m+1}{2}$ we have $l_{2i-1} + l_{2i} \equiv 1 \pmod{4}$, and for $i \leq \frac{m+1}{2}$ we also have $l_{2i-1} + l_{2i} \equiv 1 \pmod{4}$. Therefore a wind chime of the form W_2 , where m and n are both odd, satisfies the hypotheses of Theorem 6, and thus Theorem 3.2 of [2] follows from Theorem 6.

In the third construction in [2], two triangular trees $(0, 1, \dots, m)$ and $(0, 1, \dots, n)$ are connected by identifying the first spine edge of each tree, as in Figure 14. The result is a wind chime of the form

$$W_3 = (n, n-1, \dots, 2, 1, 1, 2, \dots, m).$$

Theorem 3.3 of [2] states that any tree of this variety has an α -labeling. If $W_3 = (l_1, l_2, \dots, l_{n+m})$ then $|l_j - l_{j+1}| \leq 1$, and if n is even, $l_{2i} + l_{2i+1} \not\equiv 3 \pmod{4}$ so that W_3 satisfies the hypotheses of Theorem 5. If n is odd, then $l_{2i-1} + l_{2i} \not\equiv 3 \pmod{4}$ so that W_3 satisfies the hypotheses of Theorem 6. Therefore Theorem 3.3 of [2] follows from our results.

Note, however, that our results are strictly stronger than the results on wind chimes in [2], as no result in [2] produces an α -labeling of the tree in Figure 11. Moreover, our results allow for arbitrarily many truncated triangular trees to be combined, subject to a congruence condition, while the results in [2] allow for just two triangular trees to be combined.

It is tempting to try to prove that all wind chimes except $(1, 2, 1)$ are graceful using the technique in the proof of Theorem 5. However, this approach is doomed to fail, as the next theorem demonstrates.

Theorem 7. *If $n = 2k + 1$ and $n \equiv 1 \pmod{4}$ then no graceful labeling θ of P_n satisfies $\theta(v_k) = 0$ and $\theta(v_{k+1}) = k$.*

Proof. Suppose $n \equiv 1 \pmod{4}$ and $n = 2k + 1$ where $n > 1$ and for contradiction that θ is a graceful labeling of $P_n = (v_1, \dots, v_n)$ satisfying $\theta(v_k) = 0$ and $\theta(v_{k+1}) = k$. For ease of bookkeeping, for $1 \leq i \leq k$ let $u_i = v_{k+1-i}$ so that the vertices of P_n are written as follows:

$$P_n = (u_k, u_{k-1}, \dots, u_2, u_1, v_{k+1}, \dots, v_n).$$

For $1 \leq i < k$ let $e_i = u_{i+1}u_i$. We claim that for $i \leq k$,

$$\theta(u_i) = \begin{cases} \frac{i-1}{2} & \text{if } i \text{ is odd} \\ n - \frac{i}{2} & \text{if } i \text{ is even} \end{cases} \quad (4)$$

and if $i < k$ then $\theta(e_i) = n - i$. The proof is by induction on i .

By assumption $\theta(u_1) = 0$. The path (P_n, θ) must have an edge labeled $n - 1$ as P_n has $n - 1$ edges. By assumption $\theta(v_{k+1}) = k$, so the only possibility is that $\theta(u_2) = n - 1$, since the $(n - 1)$ -vertex must be adjacent to the 0-vertex in any graceful labeling of a graph with $n - 1$ edges. Thus $\theta(e_1) = n - 1$ and the base case holds.

Next suppose that $1 \leq j < k$ and that for $1 \leq i \leq j$, $\theta(u_i)$ is as defined above in Equation 4 and for $i < j$ we have $\theta(e_i) = n - i$. There are two cases, depending on the parity of j .

If j is odd, then $\theta(u_j) = \frac{j-1}{2}$ and the vertex labels of $u_1, \dots, u_j$ are

$$\left\{0, 1, \dots, \frac{j-1}{2}\right\} \cup \left\{\left(n - \frac{j-1}{2}\right), \dots, n-1\right\}.$$

Therefore if v is a vertex of P_n which is not an element of $\{u_1, u_2, \dots, u_j\}$ then $\frac{j-1}{2} < \theta(v) < n - \frac{j-1}{2}$. Therefore if uv is an edge of P_n and neither u nor v is an element of $\{u_1, u_2, \dots, u_j\}$, then

$$\theta(uv) \leq \left(n - \frac{j-1}{2} - 1\right) - \left(\frac{j-1}{2} + 1\right) = n - j - 1.$$

Now, the edge labels appearing on $e_1, \dots, e_{j-1}$ are $n - 1, n - 2, \dots, n - (j - 1)$ respectively. P_n must contain an edge with the label $n - j$; the only possibility is the edge $e_j = u_{j+1}u_j$, since by assumption $\theta(u_1v_{k+1}) = k$. Therefore it must be that $\theta(u_{j+1}) = n - \frac{j-1}{2} - 1 = n - \frac{j+1}{2}$, which makes $\theta(e_j) = n - j$.

If j is even, then $\theta(u_j) = n - \frac{j}{2}$ and the vertex labels of $u_1, \dots, u_j$ are

$$\left\{0, 1, \dots, \frac{j-2}{2}\right\} \cup \left\{\left(n - \frac{j}{2}\right), \dots, n-1\right\}.$$

If v is a vertex of P_n satisfying $v \notin \{u_1, u_2, \dots, u_j\}$ then $\frac{j-2}{2} < \theta(v) < n - \frac{j}{2}$. Therefore if uv is an edge of P_n and neither u nor v is an element of

$\{u_1, u_2, \dots, u_j\}$ then $\theta(uv) \leq n-j-1$. The edge labels appearing on $e_1, \dots, e_{j-1}$ are $n-1, n-2, \dots, n-(j-1)$, respectively. P_n must contain an edge labeled $n-j$. Therefore it must be that $\theta(e_j) = \theta(u_{j+1}u_j) = n-j$, which makes $\theta(u_{j+1}) = \frac{j}{2} = \frac{(j+1)-1}{2}$. This completes the induction.

Since $n = 2k+1$ and $n \equiv 1 \pmod{4}$ it follows that k is even. By the previous argument the vertices $u_k, \dots, u_1$ have the labels

$$\left\{0, 1, \dots, \frac{k-2}{2}\right\} \cup \left\{\left(n - \frac{k}{2}\right), \dots, n-1\right\}.$$

Thus if v is a vertex of P_n which is not an element of $\{u_1, \dots, u_k\}$ then

$$\frac{k-2}{2} + 1 \leq \theta(v) \leq n - \frac{k}{2} - 1.$$

Therefore if uv is an edge of P_n and neither u nor v is an element of $\{u_1, \dots, u_k\}$, then

$$\theta(uv) \leq \left(n - \frac{k}{2} - 1\right) - \left(\frac{k-2}{2} + 1\right) = n - k - 1 = k.$$

By the preceding argument the labels $n-1, n-2, \dots, k+2$ appear on the edges $e_1, \dots, e_{k-1}$. By assumption, $\theta(u_1v_{k+1}) = k$. Therefore there is no edge of P_n with the label $k+1$, contradicting that θ is graceful. $\square$

Acknowledgment

The authors would like to thank the anonymous referee, whose valuable comments greatly improved this paper.

References

- [1] P. Bahls, S. Lake and A. Wertheim, Gracefulness of families of spiders, *Involve* **3** (2010), 241-247.
- [2] C. Barrientos, Alpha graphs with different pendent paths, *Electronic Journal of Graph Theory and Applications* **8.2** (2020), 301-317.
- [3] J. Bell and D. F. Cummins, Graceful pairings, *Australasian Journal of Combinatorics* **75.3** (2019): 343-356.
- [4] R. Cattell, Graceful labellings of paths, *Discrete mathematics* **307.24** (2007), 3161-3176.
- [5] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* **DS6** (2018), 1-502.
- [6] C. Huang, A. Kotzig and A. Rosa, Further results on tree labellings, *Util. Math.* **21** (1982), 31-48.

- [7] K. M. Koh, T. Tan and D. G. Rogers, Two theorems on graceful trees, *Discrete Math.* **25** (1979), 141-148.
- [8] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, New York; Dunod, Paris (1967), 349–355.
- [9] A. Rosa, Labelling snakes, *Ars Combin.* **3** (1977), 67-74.
- [10] D. B. West, Introduction to graph theory. Vol. 2. Upper Saddle River: Prentice hall, 2001.
- [11] S-L. Wu and H-C. Lu, On the Constructions of New Families of Graceful Graphs, *Ars Combin.* **106** (2012), 235-246.