Isospectrality and matrices with concentric circular higher rank numerical ranges

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Abstract

We characterize under what conditions $n \times n$ Hermitian matrices A_1 and A_2 have the property that the spectrum of $\cos tA_1 + \sin tA_2$ is independent of t (thus, the trigonometric pencil $\cos tA_1 + \sin tA_2$ is isospectral). One of the characterizations requires the first $\lceil \frac{n}{2} \rceil$ higher rank numerical ranges of the matrix $A_1 + iA_2$ to be circular disks with center 0. Finding the unitary similarity between $\cos tA_1 + \sin tA_2$ and, say, A_1 involves finding a solution to Lax's equation.

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1 Introduction

Questions regarding rotational symmetry of the classical numerical range as well as the $C-$numerical range have been studied in [1, 4, 6, 7, 8]; there is a natural connection with isospectral properties. In this paper we study the one parameter pencil $\mathrm{Re}(e^{-it}B) = \cos tA_1 + \sin tA_2$, where $A_1 = \mathrm{Re}B = \frac{1}{2}(B + B^*)$ and $A_2 = \frac{1}{2i}(B - B^*)$. We say that the pencil is *isospectral* when the spectrum $\sigma(\mathrm{Re}(e^{it}B))$ of $\mathrm{Re}(e^{it}B)$ is independent of $t \in [0, 2\pi)$; recall that the spectrum of a square matrix is the set of its eigenvalues, counting algebraic multiplicity. As our main result (Theorem 1.1) shows there is a natural connection between isospectrality and the rotational symmetry of the higher rank numerical ranges of B.

Recall that the *rank-k numerical range* of a square matrix B is defined by

$$\Lambda_k(B) = \{\lambda \in \mathbb{C} : PBP = \lambda P \text{ for some rank } k \text{ orthogonal projection } P\}.$$

This notion, which generalizes the classical numerical range when $k = 1$ and is motivated by the study of quantum error correction, was introduced in [2]. In [3, 10] it was shown that $\Lambda_k(B)$ is convex. Subsequently, in [7] a different proof of convexity was given by showing the equivalence

$$z \in \Lambda_k(B) \iff \mathrm{Re}(e^{-it}z) \leq \lambda_k(\mathrm{Re}(e^{-it}B)) \text{ for all } t \in [0, 2\pi). \tag{1}$$

Here $\lambda_k(A)$ denotes the kth largest eigenvalue of a Hermitian matrix A.

In order to state our main result, we consider words w in two letters. For instance, PPQ, $PQPQPP$ are words in the letters P and Q. The length of a word w is denoted by $|w|$. When we write $\mathrm{na}(w, P) = l$ we mean that P appears l times in the word w (na=number of appearances). The trace of a square matrix A is denoted by Tr A.

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Theorem 1.1. *Let $B \in \mathbb{C}^{n\times n}$. The following are equivalent.*

(i) The pencil $\mathrm{Re}(e^{-it}B) = \cos t\ \mathrm{Re}B + \sin t\ \mathrm{Im}B$ *is isospectral.*

(ii) $\sum_{|w|=k,\mathrm{na}(w,B^*)=l} \mathrm{Tr}\ w(B,B^*) = 0$, $0 \le l < \frac{k}{2}$, $1 \le k \le n$.

(iii) For $1 \le k \le \lceil n/2 \rceil$ the rank-k numerical range of B is a circular disk with center 0, and $\mathrm{rank}\,\mathrm{Re}(e^{-it}B)$ *is independent of t.*

(iv) $\mathrm{Re}(e^{-it}B)$ *is unitarily similar to* $\mathrm{Re}(B)$ *for all $t \in [0, 2\pi)$.*

Any of the conditions (i)-(iv) imply that B is nilpotent.

The paper is organized as follows. In Section 2 we prove our main result. In Section 3 we discuss the connection with Lax pairs.

2 Isospectral paths

We will use the following lemma.

Lemma 2.1. *Let $M(t) \in \mathbb{C}^{n\times n}$ for t ranging in some domain. Then the spectrum $\sigma(M(t))$ is independent of t if and only if* $\mathrm{Tr}M(t)^k$*, $k = 1, \ldots, n$, are independent of t.*

Proof. The forward direction is trivial. For the other direction, use Newton's identities to see that the first n moments of the zeros of a degree n monic polynomial uniquely determine the coefficients of the polynomial, and thus the zeros of the polynomial. This implies that $\mathrm{Tr}M(t)^k$, $k = 1, \ldots, n$, uniquely determine the eigenvalues of the $n \times n$ matrix. Thus, if $\mathrm{Tr}M(t)^k$, $k = 1, \ldots, n$, are independent of t, then the spectrum of $M(t)$ is independent of t. $\square$

Proof of Theorem 1.1. Consider the trigonometric polynomials $f_k(t) = 2^k\mathrm{Tr}[\mathrm{Re}(e^{-it}B)]^k$, $k = 1, \ldots, n$. The coefficient of $e^{i(2l-k)t}$ in $f_k(t)$ is given by $\sum_{|w|=k,\mathrm{na}(w,B^*)=l} \mathrm{Tr}\ w(B,B^*)$. By Lemma 2.1 the spectrum of $\mathrm{Re}(e^{-it}B)$ is independent of t if and only for $k = 1, \ldots, n$ and $2l \neq k$ the coefficient of $e^{i(2l-k)t}$ in $f_k(t)$ is 0. Due to symmetry, when they are 0 for $2l < k$ they will be 0 for $2l > k$. This gives the equivalence of (i) and (ii).

In particular note that when $l = 0$, we find that $\mathrm{Tr}B^k = 0$, $k = 1, \ldots, n$, and thus B is nilpotent.

Next, let us prove the equivalence of (i) and (iii). Assuming (i) we have that $\mathrm{Re}B$ and $-\mathrm{Re}B$ have the same spectrum, so $\mathrm{Re}B$ has $\lceil n/2 \rceil$ nonnegative eigenvalues. As the spectrum of $\mathrm{Re}(e^{-it}B)$ is independent of t, we have that $\mathrm{Re}(e^{-it}B)$ has $\lceil n/2 \rceil$ nonnegative eigenvalues for all t, guaranteeing the rank-k numerical range is nonempty for $k \le \lceil n/2 \rceil$. Next, since $\lambda_k(\mathrm{Re}(e^{-it}B))$ is independent of t, it immediately follows from the characterization (1) that $\Lambda_k(B)$, $1 \le k \le \lceil n/2 \rceil$, is a circle with center 0. Also, (i) clearly implies that $\mathrm{rank}(e^{-it}B)$ is independent of t.

Conversely, let us assume (iii). If the rank k-numerical range of B is $\{z : |z| \le r\}$ for some $r > 0$ then $\lambda_k(\mathrm{Re}(e^{-it}B))$ is constant. This also yields that $\lambda_{n+1-k}(\mathrm{Re}(e^{-it}B)) = -\lambda_k(-\mathrm{Re}(e^{-it}B))$. When for $1 \le k \le \lceil n/2 \rceil$ we have that $\Lambda_k(B)$ has a positive radius, we obtain that (i) holds. Next, let us suppose $\Lambda_\ell(B)$ has radius zero, and ℓ is the least integer with this property. Then, as before, we may conclude that $\lambda_k(\mathrm{Re}(e^{-it}B))$ is a positive constant for $1 \le k < \ell$. We also have, for $\ell \le k \le \lceil n/2 \rceil$, that $\lambda_k(\mathrm{Re}(e^{-it}B)) = 0$ for some t. As we require $\mathrm{rank}\,\mathrm{Re}(e^{-it}B)$ to be independent of t, we find that for $\ell \le k \le \lceil n/2 \rceil$, $\lambda_k(\mathrm{Re}(e^{-it}B)) = 0$ for all t. Again using $\lambda_{n+1-k}(\mathrm{Re}(e^{-it}B)) = -\lambda_k(-\mathrm{Re}(e^{-it}B))$, we arrive at (i).

The equivalence of (i) and (iv) is obvious. $\square$

Remark. The condition that rank $\mathrm{Re}(e^{-it}B)$ is independent of t in Theorem 1.1(iii) is there to handle the case when $\Lambda_k(B)$ has a zero radius. Indeed, it can happen that $\Lambda_k(B) = \{0\}$ without $\lambda_k(\mathrm{Re}(e^{-it}B))$ being independent of t; one such example is a diagonal matrix with eigenvalues $1, 0, -1, i$. It is unclear whether this can happen for a matrix whose higher rank numerical ranges are disks centered at 0.

For sizes 2, 3, and 4, the conditions in Theorem 1.1 are equivalent to B being nilpotent and the numerical range of B being rotationally symmetric.

Corollary 2.2. *Let $B \in \mathbb{C}^{n\times n}$, $n \leq 4$. Then the spectrum of $\mathrm{Re}(e^{-it}B) = \cos t\ \mathrm{Re}B + \sin t\ \mathrm{Im}B$ is independent of t if and only if B is nilpotent and the numerical range is a disk centered at 0.*

Proof. When $n = 2$, condition (ii) in Theorem 1.1 comes down to $\mathrm{Tr}B = \mathrm{Tr}B^2 = 0$. When $n = 3$ we get the added conditions that $\mathrm{Tr}B^3 = \mathrm{Tr}B^2B^* = 0$. When $n = 4$, we also need to add the conditions $\mathrm{Tr}B^4 = \mathrm{Tr}B^3B^* = 0$. The condition that $\mathrm{Tr}B^k = 0$, $1 \leq k \leq n$, is equivalent to B being nilpotent. The corollary now easily follows by invoking Remarks 1-3 in [6]. □

When $n = 5$, condition Theorem 1.1(ii) says that B is nilpotent and satisfies

$$\mathrm{Tr}B^2B^* = \mathrm{Tr}B^3B^* = \mathrm{Tr}B^4B^* = \mathrm{Tr}B^3B^{2*} + \mathrm{Tr}B^2B^*BB^* = 0.$$

Now to show that Corollary 2.2 does not hold for $n \geq 5$, note that the following example from [6],

$$B = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

is nilpotent, has the unit disk as its numerical range, but $\mathrm{Tr}B^2B^* = 1 \neq 0$.

3 Connection with Lax pairs

A *Lax pair* is a pair $L(t), P(t)$ of Hilbert space operator valued functions satisfying Lax's equation:

$$\frac{dL}{dt} = [P, L],$$

where $[X, Y] = XY - YX$. The notion of Lax pairs goes back to [5]. If we start with $P(t)$, and one solves the initial value differential equation

$$\frac{d}{dt}U(t) = P(t)U(t), \qquad U(0) = I, \tag{2}$$

then $L(t) := U(t)L(0)U(t)^{-1}$ is a solution to Lax's equation. Indeed,

$$L'(t) = \frac{d}{dt}[U(t)L(0)U(t)^{-1}] =$$

$$P(t)U(t)L(0)U(t)^{-1} - U(t)L(0)U(t)^{-1}P(t)U(t)U(t)^{-1} = P(t)L(t) - L(t)P(t).$$

This now yields that $L(t)$ is isospectral. When $P(t)$ is skew-adjoint, then $U(t)$ is unitary.

In our case we have that $L(t) = \mathrm{Re}(e^{-it}B)$, and our $U(t)$ will be unitary. This corresponds to $P(t)$ being skew-adjoint. When we are interested in the case when $P(t) \equiv K$ is constant, we have that $U(t) = e^{tK}$. Thus, we are interested in finding K so that $e^{-tK}L(t)e^{tK} = L(0)$, where $L(t) = A_1 \cos t + A_2 \sin t$. If we now differentiate both sides, we find

$$-e^{-tK}KL(t)e^{tK} + e^{-tK}L'(t)e^{tK} + e^{-tK}L(t)Ke^{tK} = 0.$$

Multiplying on the left by e^{tK} and on the right by e^{-tK}, we obtain

$$-A_1 \sin t + A_2 \cos t = L'(t) = [K, L(t)] = [K, A_1 \cos t + A_2 \sin t].$$

This corresponds to $[K, A_1] = A_2$ and $[K, A_2] = -A_1$, which is equivalent to $[K, B] = -iB$. We address this case in the following result, which is partially due to [8].

Theorem 3.1. *Let $B \in \mathbb{C}^{n\times n}$. The following are equivalent.*

(i) $e^{it}B$ *is unitarily similar to* B *for all* $t \in [0, 2\pi)$.

(ii) $\mathrm{Tr}\ w(B, B^*) = 0$ *for all words* w *with* $\mathrm{na}(w, B) \neq \mathrm{na}(w, B^*)$.

(iii) There exists a skew-adjoint matrix K *satisfying* $[K, B] = -iB$.

(iv) There exists a unitary matrix U *such that* $UBU^* = B_1 \oplus \cdots \oplus B_r$ *is block diagonal and each submatrix* B_j *is a partitioned matrix (with square matrices on the block diagonal) whose only nonzero blocks are on the block superdiagonal.*

Recall that Specht's theorem [9] says that A is unitarily similar to B if and only if $\mathrm{Tr}\ w(A, A^*) = \mathrm{Tr}\ w(B, B^*)$ for all words w.

Proof. By Specht's theorem $e^{it}B$ is unitarily similar to B for all t if and only if $\mathrm{Tr}\ w(e^{it}B, e^{-it}B^*) = \mathrm{Tr}\ w(B, B^*)$ for all t and all words. When $\mathrm{na}(w, B) \neq \mathrm{na}(w, B^*)$ this can only happen when $\mathrm{Tr}\ w(B, B^*) = 0$. When $\mathrm{na}(w, B) = \mathrm{na}(w, B^*)$, we have that $\mathrm{Tr}\ w(e^{it}B, e^{-it}B^*)$ is automatically independent of t. This proves the equivalence of (i) and (ii).

The equivalence of (i) and (iv) is proven in [8, Theorem 2.1]. We will finish the proof by proving (iv) $\to$ (iii) $\to$ (i).

Assuming (iv), let K_j be a block diagonal matrix partitioned in the same manner as B_j and whose mth diagonal block equals imI. Then $[K_j, B_j] = -iB_j$. Let $K = U^*(K_1 \oplus \cdots \oplus K_r)U$. Then $[K, B] = -iB$, proving (iii).

When (iii) holds, let $U(t) = e^{-Kt}$. Denote $\mathrm{ad}_X Y = [X, Y]$. Then $e^X Y e^{-X} = \sum_{m=0}^{\infty} \frac{1}{m!}\mathrm{ad}_X^m Y$, and (iii) yields that

$$U(t)BU(t)^* = e^{-tK}Be^{tK} = \sum_{m=0}^{\infty} \frac{1}{m!}\mathrm{ad}_{-tK}^m B = \sum_{m=0}^{\infty} \frac{(it)^m}{m!} B = e^{it}B,$$

yielding (i). $\square$

It is clear that if B satisfies Theorem 3.1(i) it certainly satisfies Theorem 1.1(i). In general the converse will not be true, and it is not hard to deduce that the size of such an example will have to be at least 4. Indeed, if B is a strictly upper triangular 3×3 matrix with $\mathrm{Tr} B^2 B^* = 0$ at least one of the entries above the diagonal is zero, making B satisfy Theorem 3.1(iv). An example that satisfies the conditions of Theorem 1.1 but does not satisfy those of Theorem 3.1 is

$$B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

Indeed, it is easy to check that $\mathrm{Tr} B^2 B^* = \mathrm{Tr} B^3 B^* = 0$, but $\mathrm{Tr} B^3 B^* B B^* = -1 \neq 0$.

When B satisfies the conditions of Theorem 3.1, the K from Theorem 3.1(iii) will yield the unitary similarity $\mathrm{Re}(e^{it}B) = e^{-tK}(\mathrm{Re} B)e^{tK}$. It is easy to find $K = -K^*$ satisfying $[K, B] = -iB$ as it amounts to solving a system of linear equations (with the unknowns the entries in the lower triangular part of K).

When B satisfies the conditions of Theorem 1.1, but not those of Theorem 3.1, finding a unitary similarity $U(t)$ so that $\mathrm{Re}(e^{it}B) = U(t)(\mathrm{Re} B)U(t)^*$ becomes much more involved. To go about this one could first find a solution $P(t)$ to Lax's equation

$$-A_1 \sin t + A_2 \cos t = L'(t) = [P(t), L(t)] = [P(t), A_1 \cos t + A_2 \sin t],$$

which now will not be constant. Next, one would solve the initial value ordinary differential matrix equation (2).

To illustrate what a solution $P(t)$ may look like, we used Matlab to produce a solution when $A_1 = \mathrm{Re}\ B$ and $A_2 = \mathrm{Im}\ B$ with B as in (3). We found the following solution in a neighborhood of 0, where we abbreviate $c = \cos\ t, s = \sin\ t$:

$$\begin{pmatrix} -i & 0 & 0 & -\frac{(s+c\,i)^3}{2\,c} \\ 0 & -i & \frac{s}{2\,c} + \frac{1}{2}i & 0 \\ 0 & -\frac{s}{2\,c} + \frac{1}{2}i & 0 & 0 \\ \frac{(s-c\,i)^3}{2\,c} & 0 & 0 & 0 \end{pmatrix}.$$

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