Modal analysis in curvilinear coordinates

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Summary Modal analysis techniques have proven useful in understanding and modeling turbulent phenomena [1]. However, these techniques are more efficient in parallel flows where Fourier transforms can be taken along homogeneous directions. We suggest that quasi-1D methods can be applied to mildly non-canonical flows by using a curvilinear coordinate system. For a given base flow, we identify a curvilinear coordinate system that allows the Fourier-transformed equations of motion to be simplified into a quasi-1D system that can be efficiently analyzed.

When performing reduced-order modeling or stability analysis in parallel flows, significant order reduction is possible using Fourier-transforms along the homogeneous directions [4, 1]. But when one transitions to a non-parallel flow, these order reductions are lost, and one instead must do a global analysis [2], requiring more intensive computation. Particularly for weakly non-parallel flows, the exact solution of modes using 1D analysis techniques may be possible if one changes the coordinate system within which the equations are written. We aim to formulate local stability and resolvent analyses using curvilinear forms of the Navier-Stokes equations by iteratively solving for an appropriate coordinate system given a particular base flow. We seek a coordinate system within which a Fourier transform can be usefully performed.

We define the following transformation between Cartesian coordinates (x^i) and orthogonal curvilinear coordinates (Z^i) such that,

$$Z^i = Z^i(x^1, x^2, x^3), \tag{1}$$
$$x^i = x^i(Z^1, Z^2, Z^3). \tag{2}$$

In the curvilinear coordinate system, we write the Navier-Stokes equations as,

$$\begin{gathered}\frac{\partial U_k}{\partial t} + U^h \frac{\partial U_k}{\partial Z^h} + U_j \Gamma^j_{hk} U^h = -\frac{1}{\rho}\frac{\partial P}{\partial Z^k} + \nu \left[\nabla^2 U_k - 2 g^{ij} \Gamma^h_{jk} \frac{\partial U_h}{\partial Z^i} + U_h \frac{\partial}{\partial Z^k}\left(\frac{1}{\sqrt{g}} \frac{\partial}{\partial Z^i}(\sqrt{g} g^{ih}) \right) \right], \\ g^{kk}\left(\frac{\partial U_k}{\partial Z^k} - U_h \Gamma^h_{kk} \right) = 0 \end{gathered} \tag{3}$$

where U_i and U^i are the covariant and contravariant components of velocity respectively, ρ is fluid density, P is the scalar pressure, ν is fluid kinematic viscosity, g^{ij} is the metric tensor, and Γ^i_{jk} are the Christoffel symbols of the second kind [3]. The metric tensor and Christoffel symbols are additional variables introduced to keep track of the changing length and direction of the basis vectors at every point in a curvilinear coordinate system, whereas the basis remains constant everywhere in Cartesian coordinates.

At this point, a choice of the specific curvilinear coordinate system has not been made. The coordinate transformation should be chosen such that, when the governing equations are cast in a form appropriate to perform stability analysis or resolvent analysis, it will allow taking a Fourier transform along at least one of the non-parallel flow directions. To identify the suitable constraints on such a coordinate system, we decompose the flow as $U_i = \bar{U}_i + u_i$ and expand the curvilinear Navier-Stokes equations in these terms. $\bar{U}_i$ is the base flow for stability analysis or mean flow for resolvent analysis, and u_i is the fluctuation about the base flow. In equation 4, we show the left hand side of the resulting equation in the Z^1 direction for a two-dimensional flow, without including the terms containing only the base flow for the sake of compactness. Terms that contain only the base flow can be directly computed.

$$\begin{aligned} &\frac{\partial u_1}{\partial t} + \left[\bar{U}_1 \frac{\partial u_1}{\partial Z^1} + u_1 \frac{\partial \bar{U}_1}{\partial Z^1} + u_1 \frac{\partial u_1}{\partial Z^1} \right] \sqrt{\frac{g^{11}}{g_{11}}} + \left[\bar{U}_2 \frac{\partial u_1}{\partial Z^2} + u_2 \frac{\partial \bar{U}_1}{\partial Z^2} + u_2 \frac{\partial u_1}{\partial Z^2} \right] \sqrt{\frac{g^{22}}{g_{22}}} \\ &- \left[{u_1}^2 + 2\bar{U}_2 u_1 \right] \sqrt{\frac{g^{11}}{g_{11}}} \left(\frac{1}{h_1} \frac{\partial h_1}{\partial Z^1} \right) - \left[{u_2}^2 + 2\bar{U}_2 u_2 \right] \sqrt{\frac{g^{22}}{g_{22}}} \left(\frac{1}{h_2} \frac{\partial h_2}{\partial Z^1} \right) \\ &- \left[\bar{U}_1 u_2 + u_1 \bar{U}_2 + u_1 u_2 \right] \left(\sqrt{\frac{g^{11}}{g_{11}}} \left(-\frac{h_1}{{h_2}^2} \frac{\partial h_1}{\partial Z^2} \right) + \sqrt{\frac{g^{22}}{g_{22}}} \left(\frac{h_1}{h_2} \frac{\partial h_1}{\partial Z^2} \right) \right) \end{aligned} \tag{4}$$

To efficiently represent the fluctuating velocity field using a Fourier transform in the Z^1 direction, $\bar{U}_1$ and $\bar{U}_2$ must not be functions of Z^1. For example, consider the Fourier transform of a term in equation 4,

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$$\int_{-\infty}^{\infty} \bar{U}_1 \frac{\partial u_1}{\partial Z^1} \cdot e^{-ik_1 Z^1} dZ^1 = \begin{cases} \bar{U}_1(-ik_1)\hat{u}_1, & \text{if } \bar{U}_1 \text{ is constant in } Z^1, \\ \text{does not simplify}, & \text{if } \bar{U}_1 \text{ is a function of } Z^1, \end{cases} \tag{5}$$

where $\hat{u}_1 = \hat{u}_1(k_1, Z^2, Z^2)$ is the Fourier transform of u_1 in the Z^1 direction. When $\bar{U}_1$ is not a function of Z^1, the expression on the left hand side of equation 5 simplifies to an algebraic expression.

To identify the appropriate coordinate system $(Z^1, Z^2) = (T, B)$ in which $\bar{U}_T$ and $\bar{U}_N$ are constant along T, we use an iterative scheme. Given a simple, non-parallel base flow initially defined in (x, y) such that $\bar{U} = [\bar{U}_x \;\; \bar{U}_y]'$ (schematically illustrated in Fig.1), we approximate the appropriate coordinate system (T, N) by defining curves along which $\bar{U}_x$, $\bar{U}_y$ are constant. With the approximate (T, N), we compute the components of our velocity field in the new coordinate directions, $\bar{U}_T, \bar{U}_N$. We iterate between defining the coordinate system (T, N) based on the components of the velocity field $\bar{U}_T, \bar{U}_N$ and defining the components of the velocity field based upon the coordinate system (T, N) until the process converges to a consistent coordinate system.

We will test our algorithm by re-deriving Cartesian and cylindrical coordinate systems for input parallel and axisymmetric base flows. We will report on the coordinate system found, and / or on the constraints that prevent the identification of such coordinate systems. We will additionally look for and report any assumptions that could be used to approximately use such coordinate systems, enabling approximate use of local analysis for non-parallel flows.

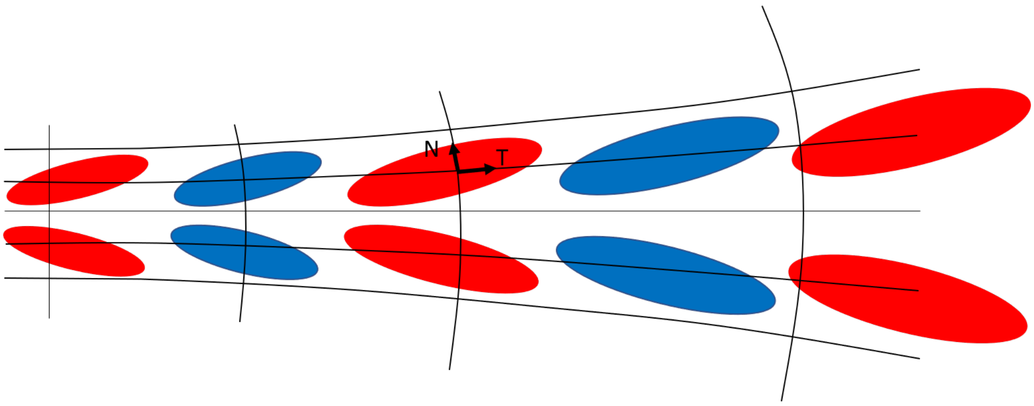

Figure 1: Illustration of modes following a curvilinear coordinate system.

References

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