

C^1 ACTIONS ON MANIFOLDS BY LATTICES IN LIE GROUPS

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ABSTRACT. In this paper we study Zimmer’s conjecture for C^1 actions of lattice subgroup of a higher-rank simple Lie group with finite center on compact manifolds. We show that when the rank of a uniform lattice is larger than the dimension of the manifold, then the action factors through a finite group. For lattices in $SL(n, \mathbb{R})$, the dimensional bound is sharp.

1. INTRODUCTION

Zimmer’s conjecture for actions of higher-rank lattice on compact manifolds says that if the group is large with respect to the dimension of the manifold, then any such action should factor through a finite group. This conjecture is motivated by a long history of research, including the local rigidity results of Selberg [40] and Weil [43] on linear representation theory, the global rigidity results of Mostow [36], the superrigidity theorem of Margulis [33], and the cocycle superrigidity theorem of Zimmer [46]. Since its introduction, Zimmer’s conjecture has attracted considerable interests.

For C^0 actions on the circle, the above conjecture is confirmed by Lifschitz, Witte Morris [29, 44] for many non-uniform lattices. For C^1 actions on the circle, Burger-Monod [8] and Ghys [19] showed similar results for many other cases, including all lattices in higher rank simple Lie groups. For C^1 area preserving actions on closed orientable surface with genus at least 2, Zimmer’s conjecture is proved by Polterovich [38] for non-uniform lattices. His result is then generalised by Franks-Handel in [18] to any C^1 action which preserves a Borel measure. For analytic actions, Ghys [19] studied the case where the manifold is a circle; Farb-Shalen [14] studied this conjecture under additional assumptions on the group and the manifold. For a very detailed survey on other earlier results on Zimmer’s program, we refer the readers to [16].

In recent breakthrough [2, 3], Brown-Fisher-Hurtado proved the C^2 version¹ of Zimmer’s conjecture for all co-compact lattices² in real split simple Lie group and $SL(n, \mathbb{Z})$ using some previous progress made by Brown-Rodriguez Hertz-Wang in [5, 6] and Lafforgue, de Laat and de la Salle in [23, 11, 12]. We refer the reader

Date: February 7, 2021.

Key words and phrases. Lattice actions, Zimmer’s conjecture, rigidity.

Brown was supported by NSF No.1752675.

Damjanović was supported by Swedish Research Council grant VR2015-04644.

Zhang was supported by the National Science Foundation under Grant No. DMS-1638352. Zhang also thanks the support and hospitality of IAS and KTH since part of this work is done during his postdoc there.

¹Their result can be improved with a bit more work to include $C^{1+\epsilon}$ -actions.

²These results are generalized recently in [4] to all non-uniform lattices.

to Fisher's paper [15] for an excellent survey of the history and recent progress on Zimmer's conjecture. The purpose of the present paper is to extend the results in [2, 3, 4] to C^1 actions, when the rank of the acting group is sufficiently large.

Compared to the previous results, there are 2 new ideas here. First is that while many results in Non-uniform Hyperbolic Theory fail or remain unknown in the C^1 setting, some of them continue to hold under the presence of suitable continuous splitting. In our case, we can apply a variant of Avila-Viana's invariance principle to an element in the kernel of all Lyapunov functionals to obtain the extra invariance needed to conclude the proof. For C^2 action, the idea to use action by an element in the kernel of all fiberwise exponents was originally due to Sebastian Hurtado and appears in the Bourbaki notes of Cantat [9]. The second one is that we use the information extracted by using strong property (T) to control the L^p norms of the derivatives for sufficiently large p . This allows us to show that C^1 action is uniformly bounded under certain Hölder norm. Then we use the resolution of the Hilbert-Smith conjecture for sufficiently Hölder actions to conclude the proof.

2. STATEMENT OF THE MAIN RESULTS

We first recall the statement of Zimmer's conjecture.

For a real semisimple Lie group G with Lie algebra \mathfrak{g} , let

- $v(G)$ denote the minimal codimension of proper parabolic subalgebras of \mathfrak{g} ;
- $d(G)$ denote the minimal codimension of proper subalgebras of the compact real form of $\mathfrak{g}_{\mathbb{C}}$;
- $n(G)$ denote the minimal dimension of nontrivial real representations of \mathfrak{g} .

It is proved in [41] that $v(G) < n(G)$.³

CONJECTURE 1. *Let G be a connected real semisimple Lie group with finite center and without almost-simple factors of real rank less than 2. Let $\Gamma < G$ be a lattice, M be a compact manifold, $\alpha : \Gamma \rightarrow \text{Diff}(M)$ be an action.*

- (1) *If $\dim(M) < v(G)$, then α preserves a Riemannian metric.*
- (2) *If $\dim(M) < \min\{v(G), d(G)\}$, then $\alpha(\Gamma)$ is finite.*
- (3) *If $\dim(M) < n(G)$ and α preserves a volume density, then α preserves a Riemannian metric.*
- (4) *If $\dim(M) < \min\{n(G), d(G)\}$ and α preserves a volume density, then $\alpha(\Gamma)$ is finite.*

The main result of this paper is the following generalisation of the results in [2, 4] to C^1 regularity.

THEOREM 1. *Let M be a compact manifold. Let G be an almost simple real Lie group with finite center and with real-rank at least 2, and let $\Gamma < G$ be a lattice. Let $\alpha : \Gamma \rightarrow \text{Diff}^1(M)$ be a group homomorphism. Assume that Γ is an uniform lattice or $\Gamma = \text{SL}(n, \mathbb{Z})$, and assume either that $\dim M < \min(\text{rank}_{\mathbb{R}}(G), d(G))$, or that $\dim M \leq \min(\text{rank}_{\mathbb{R}}(G), d(G) - 1)$ and $\alpha(\Gamma) \subset \text{Diff}^1(M, \text{vol})$. Then α has finite image.*

Compared to the main result in [2, 4] for almost-simple real Lie groups, in Theorem 1 we have posed a different requirement on the dimension of the manifold.

³We thank Jinpeng An for this remark.

Indeed, we can deduce from [2, Theorem 2.7] that for a group homomorphism $\alpha : \Gamma \rightarrow \text{Diff}^2(M)$, the conclusion of Theorem 1 is true if $\text{rank}_{\mathbb{R}} G$ is replaced by the *minimal resonant codimension* $r(G)$ (see [2, Definition 2.1]). We remark that under the conditions of Theorem 1, we always have that

$$r(G) \geq \text{rank}_{\mathbb{R}} G.$$

COROLLARY A. *Let M be a compact manifold. Let $\Gamma < G$ be a lattice. Let $\alpha : \Gamma \rightarrow \text{Diff}^1(M)$ (resp. $\text{Diff}^1(M, \text{vol})$) be a group homomorphism. Assume that Γ is an uniform lattice or $\Gamma = SL(n, \mathbb{Z})$, and assume that one of the following is true:*

- (1) $G = SL(n, \mathbb{R})$, $\dim M < n - 1$ (resp. $\leq n - 1$) and $n \geq 3$;
- (2) $G = Sp(2n, \mathbb{R})$, $\dim M < n$ (resp. $\leq n$) and $n \geq 2$;
- (3) $G = SO(n, n)$, $\dim M < n$ (resp. $\leq n$) and $n \geq 4$;
- (4) $G = SO(n, n + 1)$, $\dim M < n - 1$ (resp. $\leq n - 1$) and $n \geq 3$.

Then α has finite image.

When α is a C^2 action, the conclusion of Theorem 1 is already obtained in [2, 4]. Moreover, when $G = Sp(2n, \mathbb{R})$, $SO(n, n)$ or $SO(n, n + 1)$, the dimension bound in Corollary A is not optimal. However, when $G = SL(n, \mathbb{R})$, we have

$$r(G) = \text{rank}_{\mathbb{R}} G = n - 1.$$

By considering the actions of $SL(n, \mathbb{R})$ by projective transformations on $\mathbb{P}(\mathbb{R}^n)$, and by the affine transformations on \mathbb{T}^n , we see that Corollary A has optimal bounds for $G = SL(n, \mathbb{R})$. We note that for C^0 action by $SL(n, \mathbb{Z})$, ($n \geq 3$) on compact manifold with $\chi(M) \not\equiv 0 \pmod{3}$, the finite image property of α is proved by Ye in [45].

The proofs of the results in this paper follow closely the strategy in [2]. We recommend the reader to have this paper close at hand as we make many references to these works, although we also repeat some of the main arguments for reader's convenience. Below we first describe the general strategy of the proofs in [2, 3, 4], and then we point out the main new ideas and modifications we make here in order to obtain results in C^1 regularity.

3. REVIEW OF BFH AND OUTLINE OF THE PROOF

Step 1: Uniform subexponential growth.

We fix a finite set of symmetric generators for Γ , denoted by $S = \{\gamma_i\}$. For any $\gamma \in \Gamma$, we let $\ell(\gamma)$ denote the word-length distance from γ to the identity relative to S . In other words, $\ell(\gamma)$ is the smallest integer k such that γ may be represented by a product $\zeta_1 \cdots \zeta_k$ where $\zeta_j \in S$ for each $1 \leq j \leq k$.

We first recall the following notion.

DEFINITION 1. Let $\alpha : \Gamma \rightarrow \text{Diff}^1(M)$ be an action of Γ on a compact manifold M by C^1 diffeomorphisms. We fix a background C^∞ Riemannian metric on M . We say that α has *uniform subexponential growth of derivatives* if for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that for all $\gamma \in \Gamma$ we have

$$\|D\alpha(\gamma)\| \leq C_\varepsilon e^{\varepsilon \ell(\gamma)}.$$

It is clear that the above definition is independent of the choice of the metric on M or the generating set S .

The main result of Step 1 is the following.

PROPOSITION 1. *Let M be a compact manifold, and let G be a connected, almost-simple real Lie group with finite center and whose real-rank is at least 2. Let $\Gamma < G$ be a lattice. Let $\alpha : \Gamma \rightarrow \text{Diff}^1(M)$ be a group homomorphism. Assume Γ is an uniform lattice or $\Gamma = SL(n, \mathbb{Z})$, and assume either that $\dim M < \text{rank}_{\mathbb{R}}(G)$, or that $\dim M \leq \text{rank}_{\mathbb{R}}(G)$ and $\alpha(\Gamma) \subset \text{Diff}^1(M, \text{vol})$. Then α has uniform subexponential growth of derivatives.*

We prove Proposition 1 following the same scheme in [2]. As in [2], we define the suspension space M^α as the quotient of $G \times M$ by Γ -action $(g, x) \mapsto (g\gamma, \alpha(\gamma^{-1})x)$. We recall that M^α is a fiber bundle over G/Γ with fibers modeled on M . Moreover M^α is equipped with a left G -action, denoted by $\tilde{\alpha}$, by diffeomorphisms which preserves the foliation into fibers. We present the construction of M^α and its further properties in Section 4.1.

As the G -action preserves the foliation into fibers of M^α , we may consider the restriction of $D\tilde{\alpha}$ to the subbundle $E^F := \text{Ker}(D\pi)$ tangent to the fibers of M^α . Let A be the maximal split torus of G , and let μ be an A -ergodic A -invariant measure on M^α . We can associate to μ and the derivative A -cocycle $D\tilde{\alpha}|_{E^F}$ a set of fiberwise Lyapunov functionals $\lambda_i^F : \text{Lie}(A) \rightarrow \mathbb{R}$, $1 \leq i \leq k$ by the higher-rank Oseledec's theorem (see, e.g., [7, Theorem 2.4]). We refer the reader to [2, Proposition 3.3] for the definition and properties of Lyapunov functionals. The maximal fiberwise Lyapunov exponent for $a \in A$ with respect to an a -invariant probability measure μ is defined as

$$\lambda_+^F(a, \mu) = \inf_{n \rightarrow \infty} \frac{1}{n} \int \log \|D\tilde{\alpha}(a^n)|_{E^F(x)}\| d\mu(x).$$

By [2, Proposition 3.7], we have

PROPOSITION 2. *Suppose that Γ is an uniform lattice or $\Gamma = SL(n, \mathbb{Z})$, and α fails to have uniform subexponential growth of derivatives. There exists an $s \in A$ and an A -invariant Borel probability measure μ on M^α with $\lambda_+^F(s, \mu) > 0$ such that $\pi_*\mu$ is the Haar measure on G/Γ .*

This is the only place where we have used the hypothesis that Γ is an uniform lattice or $\Gamma = SL(n, \mathbb{Z})$. In a work in progress of Brown-Fisher-Hurtado [4], they have generalised Proposition 2 to any lattice in G . Admitting their results, all of the results in the present paper hold for arbitrary lattices.

To complete the proof of Proposition 1, it remains to show the following.

PROPOSITION 3. *Let μ be an A -invariant Borel probability measure on M^α such that $\pi_*\mu$ is the Haar measure on G/Γ . If either that $\text{rank}_{\mathbb{R}} G > \dim M$, or that $\text{rank}_{\mathbb{R}} G \geq \dim M$ and $\alpha(\Gamma) \subset \text{Diff}^1(M, \text{vol})$, then μ is G -invariant.*

Let $a \in G$ be a \mathbb{R} -semisimple element. The unstable, resp. stable, subgroup for a are respectively

$$\begin{aligned} H^u &:= \{g \mid \lim_{n \rightarrow -\infty} a^n g a^{-n} = e\}, \\ H^s &:= \{g \mid \lim_{n \rightarrow +\infty} a^n g a^{-n} = e\}. \end{aligned}$$

PROPOSITION 4. *Let $a \in A$ be an \mathbb{R} -semisimple element. Suppose μ is an a -invariant a -ergodic probability measure on M^α such that*

- (1) $\pi_*\mu$ is the Haar measure on G/Γ , and
- (2) all fiberwise Lyapunov exponents of $D\tilde{\alpha}(a)$ are non-positive.

Then μ is H^u -invariant.

The proof of Proposition 4 will be given in Section 4. We are ready to deduce Proposition 3 from Proposition 4.

Proof of Proposition 3. We can assume without loss of generality that μ is A -ergodic, otherwise we may replace μ by any one of its A -ergodic components. This is because any A -ergodic component of μ projects to some A -ergodic component of $\pi_*\mu$; while by hypothesis $\pi_*\mu$ is the Haar measure on G/Γ which is itself A -ergodic by Moore's ergodicity theorem (see for instance [35] or [47, Theorem 2.2.6]). This allows us to define fiberwise Lyapunov functionals. We denote by $\lambda_1^F, \dots, \lambda_k^F$ the total collection of distinct fiberwise Lyapunov functionals. We have that $k \leq \dim M$. Moreover, notice that when $\alpha(\Gamma) \subset \text{Diff}^1(M, \text{vol})$, the sum of all Lyapunov functionals (considered with multiplicities) is zero. Then under the condition of the proposition, we can pick an arbitrary element $a \in (\cap_{i=1}^k \exp(\text{Ker}(\lambda_i^F))) \setminus \{e\}$ such that

$$\lambda_+^F(a, \mu) = \lambda_+^F(a^{-1}, \mu) = 0.$$

Then all a -ergodic components of μ have vanishing fiberwise Lyapunov exponents. By Proposition 4, we deduce that μ is H^u -invariant. By symmetry, we also have that μ is H^s -invariant. As G is almost-simple, G is generated by H^u and H^s . Consequently, μ is G -invariant. \square

Proof of Proposition 1. Assume that α fails to have uniform subexponential growth of derivatives. Then by Proposition 2, there is a $s \in A$ and an A -invariant measure μ such that $\lambda_+^F(s, \mu) > 0$ and $\pi_*\mu$ is the Haar measure on G/Γ . By Proposition 3, we deduce that μ is G -invariant. Recall that $n(G) > \text{rank}_{\mathbb{R}} G$ where $n(G)$ denotes the minimal dimension of a non-trivial real representation of the Lie algebra of G . By Zimmer's cocycle superrigidity theorem (we use the version by Fisher-Margulis in [17, Theorem 1.4]. We refer the readers to [46, 47, 48] for some earlier results), the G -action preserves a measurable metric on E^F . This contradicts that $\lambda_+^F(s, \mu) > 0$. Thus α must have uniform subexponential growth of derivatives. \square

Step 2: Strong property (T) and averaging.

In this step, we follow [2] to construct a Γ -invariant continuous distance by using the strong property (T) of Γ proved by Lafforgue, de Laat and de la Salle in [23, 11, 12]. The main result of this step is the following proposition whose proof will be given in Section 5.

PROPOSITION 5. *If α has uniform subexponential growth of derivatives, then there exists a distance $\bar{d} : M \times M \rightarrow [0, \infty)$ that is invariant by the Γ -action α . Moreover, for any $\beta \in (0, 1)$, the set $\alpha(\Gamma)$ is precompact in $\text{Hol-Homeo}^\beta(M)$, the space of β -bi-Hölder homeomorphisms of M with respect to the background Riemannian distance.*

Proposition 5 replaces [2, Theorem 2.9]. In [2], the authors study a C^2 -action of Γ , and the induced Γ action on $W^{1,p}(S^2(T^*M))$, the Sobolev space of all the sections φ of the bundle of symmetric two forms $S^2(T^*M)$ such that both φ and $D\varphi$ are L^p with respect to the Lebesgue measure. Then the strong property (T) and the uniform subexponential growth of derivatives give us the Γ -invariant section in $W^{1,p}(S^2(T^*M))$ which is continuous if p is sufficiently large. The above method can be adapted to the case where the action is $C^{1+\epsilon}$ for any $\epsilon > 0$.

In our case, α is only C^1 , and consequently α does not induce a Γ action on $W^{1,p}(S^2(T^*M))$. We consider instead the induced Γ -action on $L^p(S^2(T^*M))$, and obtain a L^p α -invariant section of $S^2(T^*M)$. We use the exponential convergence inherited from the strong property (T) and Cauchy inequality to bound the Sobolev norms of the Γ -action.

To make use of Proposition 5, we also need the solution of Hilbert-Smith conjecture for sufficiently Hölder actions proved in [39, 31]. We recall the statement here.

LEMMA 1. *For any $\beta \in (\frac{\dim M}{\dim M+1}, 1)$ the following is true: let H be a compact topological group which admits a faithful action on M by β -Hölder homeomorphisms. Then H is a Lie group.*

COROLLARY B. *Let G, Γ, μ, α be as in Theorem 1. Assume either that $\dim M < \text{rank}_{\mathbb{R}}(G)$, or that $\dim M \leq \text{rank}_{\mathbb{R}}(G)$ and $\alpha(\Gamma) \subset \text{Diff}^1(M, \text{vol})$. Then α factors through a compact Lie group. That is, there exist: a compact Lie group H ; an injective group homomorphism $\iota : H \rightarrow \text{Homeo}(M)$; and a group homomorphism $\phi : \Gamma \rightarrow H$ such that $\alpha = \iota \circ \phi$.*

Proof. By Proposition 1, the action α has uniform subexponential growth of derivatives. We fix any $\beta \in (\frac{\dim M}{\dim M+1}, 1)$. By Proposition 5, the closure of $\alpha(\Gamma)$ in $\text{Hol-Homeo}^\beta(M)$, denoted by K_0 , is a compact topological subgroup of $\text{Homeo}(M)$. By Lemma 1, we see that K_0 is a compact Lie group. \square

Step 3: Margulis superrigidity with compact codomain.

After Step 1 and 2, we can apply precisely the same method as in [2] to show the finite image property. We refer the reader to [2, Section 7] for details.

Proof of Theorem 1. The proof is essentially contained in [2, Section 7]. We reproduce it below for the convenience of the readers.

Let H be the compact Lie group given by Corollary B, and let $\iota : H \rightarrow \text{Homeo}(M)$ and $\phi : \Gamma \rightarrow H$ be the associated group homomorphisms. Assume that $\alpha = \iota \circ \phi$ has infinite image. Then by Margulis' arithmeticity theorem and superrigidity theorem, each almost simple factor of H is a compact form of G . Since $\iota : H \rightarrow \text{Homeo}(M)$ is injective, there is some $x \in M$ such that $\iota(H)x$ contains a compacta homeomorphic to K/C where K is a compact form of G and C is a closed proper subgroup of K . This is impossible since by hypothesis $\dim(K/C) \geq d(G) > \dim M$. \square

4. PROOF OF PROPOSITION 4

4.1. Suspension space. In this subsection, we recall the suspension construction and the induced G -action in [6, Section 2].

Let α be a Γ -action on M by C^1 diffeomorphisms, i.e., $\alpha(gh) = \alpha(g)\alpha(h)$. We consider the right action by Γ on $G \times M$ defined as

$$(g, x) \cdot \gamma = (g\gamma, \alpha(\gamma^{-1})(x)), \quad \forall \gamma \in \Gamma$$

and the left G -action

$$a \cdot (g, x) = (ag, x), \quad \forall a \in G.$$

Define the quotient manifold $M^\alpha := (G \times M)/\Gamma$. Since the left G -action commutes with the right Γ -action, the left G -action descends to a left G -action on M^α , denoted

by $\tilde{\alpha}$. Since α is a C^1 action, M^α is naturally equipped with a C^1 manifold structure. The action $\tilde{\alpha}$ is given by C^1 diffeomorphisms of M^α . Moreover, denote by $\pi : M^\alpha \rightarrow G/\Gamma$ the projection induced by $G \times M \rightarrow G$, then M^α is a C^1 fiber bundle over G/Γ induced by π with fibers diffeomorphic to M .

With a slight abuse of notation, we use $d(\cdot, \cdot)$ to denote both the right-invariant metric on G , and the quotient metric on G/Γ . We denote by ν the normalised left Haar measure on G/Γ .

By the construction in [3, Section 2.2] (see also [6, Section 2.1] for the details), there exists a C^1 Riemannian metric $\langle \cdot, \cdot \rangle$ on $G \times M$ with the following properties:

- (1) $\langle \cdot, \cdot \rangle$ is invariant under the right Γ -action,
- (2) for each $(g, x) \in G \times M$, under the canonical identification of the G -orbit of (g, x) with G , the restriction of $\langle \cdot, \cdot \rangle$ to the G -orbit of (g, x) coincides with d_G ,
- (3) There exist a Siegel fundamental set $D \subset G$ for the right Γ -action (see [33, VIII.1] for the definition) containing the identity $e \in G$, and a constant $C_1 > 1$ such that for any $g_1, g_2 \in D$, the map $(g_1, x) \mapsto (g_2, x)$ distorts the restrictions of $\langle \cdot, \cdot \rangle$ to $\{g_1\} \times M$ and $\{g_2\} \times M$ by at most C_1 .

We use $\langle \cdot, \cdot \rangle_g$ to denote the restriction of $\langle \cdot, \cdot \rangle$ to $\{g\} \times M$, and view it as a metric on M . By item (1) above, we can equip M^α with the quotient metric of $\langle \cdot, \cdot \rangle$.

We fix $\{\gamma_i\}$, a finite symmetric generating set for Γ . Let ℓ denote the word-length distance on Γ relative to $\{\gamma_i\}$. Given a fundamental domain $F_D \subset D$ for the right Γ -action on G , i.e., $G = F_D\Gamma$ and $F_D\gamma \cap F_D = \emptyset$ for $\forall \gamma \in \Gamma \setminus \{e\}$, the return cocycle $\beta : G \times G/\Gamma \rightarrow \Gamma$ associated to F_D is defined as follows. For any $g \in G$, $x \in G/\Gamma$, we set $\beta(g, x)$ to be the unique element $\gamma \in \Gamma$ such that $g\tilde{x} \in F_D\gamma$, where \tilde{x} is the lift of x in F_D . The following are from [3] whose proofs rely on [30].

LEMMA 2. *If $F_D \subset D$ is a fundamental domain for the right Γ -action on G such that $e \in F_D$, then there is a constant $C > 0$ such that for any $g \in G$, any $x \in G/\Gamma$,*

$$\ell(\beta(g, x)) < Cd(g, e) + Cd(x, \Gamma) + C.$$

LEMMA 3. *There is a constant $C > 0$ such that the following is true. For any $g \in G$, any $x \in G/\Gamma$, any $p \in \pi^{-1}(x)$ we have*

$$\log \|D_p \tilde{\alpha}(g)\| < Cd(g, e) + Cd(x, \Gamma) + C.$$

4.2. Smooth cocycle. Let a be as in Proposition 4. In various statements about typical points in G/Γ in this rest of this section, we will always refer to the Haar measure ν .

Following [26] and [34], we may find a measurable partition ζ of G/Γ with the following properties:

- (1) ζ is subordinate to the partition of G/Γ into H^u orbits: for a.e. $x \in G/\Gamma$,
 - (a) the atom $\zeta(x)$ is contained in the orbit $H^u \cdot x$,
 - (b) the atom $\zeta(x)$ is precompact in the orbit $H^u \cdot x$,
 - (c) the atom $\zeta(x)$ contains a neighborhood of x in the orbit $H^u \cdot x$,
- (2) ζ is a -decreasing, i.e., $a(\zeta) \leq \zeta$.

We also require that ζ satisfies the following additional property:

- (3) There is a compact set $W \subset H^u$ such that for a.e. x

$$\zeta(x) \subset W \cdot x.$$

To build a partition ζ satisfying (1)–(3), we first let ζ_0 be a partition satisfying (1) and (2). Select a ζ_0 -measurable subset $S \subset G/\Gamma$ with positive ν -measure such that the diameter of $\zeta_0(x)$ is uniformly bounded in the $H^u \cdot x$ -orbit for all $x \in S$. It is well-known that a is ergodic with respect to the Haar measure ν . Thus for a.e. $x \in G/\Gamma$, the following number is well-defined:

$$n_x = \inf\{n \in \mathbb{N} \mid a^n \cdot x \in S\}.$$

We set

$$\zeta(x) = a^{-n_x} \zeta_0(a^{n_x} \cdot x).$$

Then ζ still satisfies (1) and (2). Since $\text{Ad}(a^{-1})$ is a contraction restricted to the Lie algebra of H^u , ζ also satisfies (3).

Since ζ is measurable, we may apply [1, Prop 4.5] to find a measurable selection: there is a measurable map $\psi : G/\Gamma \rightarrow G/\Gamma$ such that ψ is constant on every atom of ζ , and $\psi(x) \in \zeta(x)$ for ν -a.e. x . Recall our choice of a Siegel fundamental set $D \subset G$ and fix a fundamental domain $F_D \subset D$ such that $e \in F_D$. Let $\tilde{\psi} : G/\Gamma \rightarrow G$ be the map that assigns $x \in G/\Gamma$ the unique $g \in F_D$ with $\psi(x) = g\Gamma$. Note that $\tilde{\psi}$ is ζ -measurable.

Since H^u is horospherical for a , for a.e. $x \in G/\Gamma$ the map $H^u \rightarrow G/\Gamma, h \mapsto h \cdot x$ is injective. Indeed, for a μ -typical $x \in G/\Gamma$, there is a sequence $\{t_m\}_{m \geq 0}$ of positive numbers that tends to infinity such that $\{a^{-t_m} \cdot x\}_{m \geq 0}$ is precompact. Then $h \mapsto h \cdot x$ must be injective on H^u since each H^u -orbit is contracted by the backward iterates of a , and $G \rightarrow G/\Gamma$ is a local homeomorphism. For any such x , we let W_x be the inverse image of $\zeta(x)$ under the map $H^u \rightarrow G/\Gamma, h \mapsto h \cdot \psi(x)$; and let $\zeta_1(x) = W_x \tilde{\psi}(x)$. Notice that by definition $\pi(\zeta_1(x)) = \zeta(x)$, and $\zeta_1(x) \cap F_D \neq \emptyset$.

As F_D is a fundamental domain contained in D , we can choose a Borel trivialization associated to F_D , denoted by

$$\iota : M^\alpha \rightarrow F_D \times M$$

where for each $x \in G/\Gamma$, we identify $\iota|_{\pi^{-1}(x)}$ with a diffeomorphism $\iota_x : \pi^{-1}(x) \rightarrow M$. Moreover, by the construction of the metric $\langle \cdot, \cdot \rangle$ on $D \times M$, we may assume that $\|\iota_x\|_{C^1}$ is uniformly bounded over all $x \in G/\Gamma$.

Given a typical $x \in G/\Gamma$, let $u_x \in H^u$ be such that $x = u_x \cdot \psi(x)$. Set $g_x : \pi^{-1}(x) \rightarrow \pi^{-1}(\psi(x))$ to be

$$g_x(y) = \tilde{a}(u_x^{-1})(y).$$

Given $x \in G/\Gamma$, set $F_x : M \rightarrow M$ to be

$$(4.1) \quad F_x(y) = \iota_{\psi(a^{-1} \cdot x)}(g_{a^{-1} \cdot \psi(x)}(\tilde{a}(a^{-1})(\iota_{\psi(x)}^{-1}(y)))).$$

Let $F : G/\Gamma \times M \rightarrow G/\Gamma \times M$ be the measurable map

$$(4.2) \quad F(x, y) = (a^{-1} \cdot x, F_x(y)).$$

Using $\{g_x\}$, we define a measurable map $\Phi : M^\alpha \rightarrow G/\Gamma \times M$ as follows:

$$(4.3) \quad \Phi(y) = (\pi(y), \iota_{\psi(\pi(y))} g_{\pi(y)}(y)).$$

Let μ be the a -ergodic a -invariant measure in Proposition 4, let $\mu^* = \Phi_* \mu$.

CLAIM 1. Φ is a Borel isomorphism. Moreover, for μ -a.e. $x \in G/\Gamma$, Φ is a C^1 diffeomorphism from $\pi^{-1}(x)$ to M , and we have

$$F \cdot \Phi = \Phi \cdot \tilde{a}(a^{-1}).$$

Proof. We set $x = \pi(z)$. Then we have

$$\pi(a^{-1} \cdot z) = a^{-1} \cdot \pi(z) = a^{-1} \cdot x.$$

Then

$$F\Phi(z) = (a^{-1} \cdot x, \iota_{\psi(a^{-1} \cdot x)} g_{a^{-1} \cdot \psi(x)}(\tilde{\alpha}(a^{-1})(g_x(z))))$$

and

$$\Phi(\tilde{\alpha}(a^{-1})(z)) = (a^{-1} \cdot x, \iota_{\psi(a^{-1} \cdot x)} g_{a^{-1} \cdot x}(\tilde{\alpha}(a^{-1})(z))).$$

Then by definition, it suffices to show that

$$au_{a^{-1} \cdot x} = u_x au_{a^{-1} \cdot \psi(x)}.$$

By definition,

$$au_{a^{-1} \cdot x} \cdot \psi(a^{-1} \cdot x) = a \cdot a^{-1} \cdot x = x.$$

We also notice that $a^{-1} \cdot \psi(x) \in a^{-1} \cdot \zeta(x) \subset \zeta(a^{-1} \cdot x)$. Thus

$$\psi(a^{-1} \cdot \psi(x)) = \psi(a^{-1} \cdot x).$$

Then

$$u_x au_{a^{-1} \cdot \psi(x)} \cdot \psi(a^{-1} \cdot x) = u_x a \cdot a^{-1} \cdot \psi(x) = x.$$

This completes the proof. \square

Let $\{\mu_x^*\}$ be the disintegration of μ^* with respect to the partition of $G/\Gamma \times M$ into fibers. The following properties follow immediately from the above constructions and observations.

PROPOSITION 6. *We have*

- (1) *for a.e. $x \in G/\Gamma$ and every $x' \in \zeta(x)$, $F_x = F_{x'}$; in particular, $x \mapsto F_x$ is ζ -measurable.*
- (2) *The function $x \mapsto \log \|F_x^{-1}\|_{C^1}$ is in $L^1(G/\Gamma, \nu)$.*
- (3) *Φ is a measurable conjugacy between the dynamics of a^{-1} on M^a and of F on $G/\Gamma \times M$.*
- (4) *The fiberwise Lyapunov exponents for Da with respect to μ are all non-positive if, and only if, the fiberwise Lyapunov exponents of F with respect to μ^* are all non-negative.*
- (5) *μ is H^u -invariant if and only if the map $x \mapsto \mu_x^*$ is ζ -measurable.*

Proof. Item (1) follows immediately from the construction. Item (3) is given by Claim 1. Item (4) follows from item (3) and our hypothesis on a in Proposition 4: all fiberwise Lyapunov exponents of $D\tilde{\alpha}(a)$ are non-positive.

To show item (2), we first notice that by (4.1) for ν -a.e. $x \in G/\Gamma$, we have

$$\|F_x^{-1}\|_{C^1} \leq \|\tilde{\alpha}(u_{a^{-1} \cdot \psi(x)})|_{\pi^{-1}(\psi(a^{-1} \cdot x))}\|_{C^1} \|\tilde{\alpha}(a)|_{\pi^{-1}(a^{-1} \cdot \psi(x))}\|_{C^1}.$$

By Lemma 3, we have

$$\begin{aligned} \log \|\tilde{\alpha}(a)|_{\pi^{-1}(a^{-1} \cdot \psi(x))}\|_{C^1} &\leq Cd(a, e) + Cd(a^{-1} \cdot \psi(x), x_0) + C, \\ \log \|\tilde{\alpha}(u_{a^{-1} \cdot \psi(x)})|_{\pi^{-1}(\psi(a^{-1} \cdot x))}\|_{C^1} &\leq C \sup_{b \in W} d(b, e) + Cd(\psi(a^{-1} \cdot x), x_0) + C. \end{aligned}$$

Note that there are $u_1, u_2 \in W$ such that

$$a^{-1} \cdot \psi(x) = a^{-1} u_1^{-1} x, \quad \psi(a^{-1} \cdot x) = u_2^{-1} a^{-1} x.$$

Then we have

$$d(a^{-1} \cdot \psi(x), x_0), d(\psi(a^{-1} \cdot x), x_0) \leq Cd(x, x_0) + C'$$

for some C depending only on G, Γ , and some C' depending only on W and a . Thus item (2) follows from the fact that

$$(x \mapsto d(x, x_0)) \in L^1(G/\Gamma, \nu).$$

The “only if” part of Item (5) follows by definition. We assume that $x \mapsto \mu_x^*$ is ξ -measurable. Then for μ -a.e. x , for any $h \in H^u$ such that $h(\pi(x)) \in \xi(\pi(x))$, we have $\tilde{\alpha}(h)_* \mu_{\pi(x)} = \mu_{h(\pi(x))}$ where $\{\mu_z\}_{z \in G/\Gamma}$ is the disintegration of μ along the fibers. Moreover by Claim 1, we see that $x \mapsto \mu_x^*$ is $a^n(\xi)$ -measurable for any $n \geq 1$. We can use the above argument for $a^n(\xi)$ instead of ξ (for all $n \geq 1$) to show that μ is H^u -invariant. \square

4.3. Avila-Viana’s invariance principle. We will use a variant of [1, Theorem B] to conclude the proof of Proposition 4. Let us first briefly recall the setting in [1].

Let $(\hat{X}, \hat{\mathcal{B}}, \hat{\mu})$ be a probability space, and let $\hat{f} : \hat{X} \rightarrow \hat{X}$ be an invertible $\hat{\mu}$ -preserving measurable transformation. Let N be a compact Riemannian manifold. We set $\hat{\mathcal{E}} = \hat{X} \times N$, and denote by $\hat{P} : \hat{\mathcal{E}} \rightarrow \hat{X}$ the projection to the first coordinate. We say that $\hat{F} : \hat{\mathcal{E}} \rightarrow \hat{\mathcal{E}}$ is a *smooth cocycle* over \hat{f} if \hat{F} is of form $\hat{F}(\hat{x}, \hat{y}) = (\hat{f}(\hat{x}), \hat{F}_{\hat{x}}(\hat{y}))$, where $\hat{F}_{\hat{x}}$ is a diffeomorphism of N for each \hat{x} . We also assume the following:

$$(4.4) \quad \int |\log(\sup_{\hat{y}} \|D\hat{F}_{\hat{x}}(\hat{y})^{-1}\|)| d\hat{\mu}(\hat{x}) < \infty.$$

In this case, for any \hat{F} -invariant probability measure \hat{m} on $\hat{\mathcal{E}}$ that projects to $\hat{\mu}$ under \hat{P} , the minimal Lyapunov exponent is a well-defined quantity at \hat{m} -almost every (\hat{x}, \hat{y}) by the following formula:

$$\lambda_-(\hat{F}, \hat{x}, \hat{y}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|D\hat{F}_{\hat{x}}^n(\hat{y})^{-1}\|^{-1}.$$

Let $H : \hat{\mathcal{E}} \rightarrow \hat{\mathcal{E}}$ be a map of form $H(\hat{x}, \hat{y}) = (\hat{x}, H_{\hat{x}}(\hat{y}))$ where $H_{\hat{x}}$ is a C^1 diffeomorphism of N which depends measurably on \hat{x} , and satisfies that

$$(4.5) \quad \int [|\log(\sup_{\hat{y}} \|D\hat{H}_{\hat{x}}(\hat{y})^{-1}\|)| + |\log(\sup_{\hat{y}} \|D\hat{H}_{\hat{x}}(\hat{y})\|)] d\hat{\mu}(\hat{x}) < \infty.$$

Then we call the map defined as follow

$$\tilde{F}(\hat{x}, \hat{y}) = (\hat{f}(\hat{x}), \tilde{F}_{\hat{x}}(\hat{y})) = H \circ \hat{F} \circ H^{-1}(\hat{x}, \hat{y}),$$

a *deformation of a smooth cocycle* \hat{F} , and denote $\tilde{m} = H_* \hat{m}$. The decomposition of $\hat{\mathcal{E}}$ into fibers gives rise to a disintegration of \tilde{m} into a collection of conditional measures $\{\tilde{m}_{\hat{x}}\}_{\hat{x} \in \hat{X}}$ on the fibers. We emphasize that the assumption that $\{H_{\hat{x}}\}_{\hat{x} \in X}$ are diffeomorphisms simplifies our argument (in [1] the authors worked under the weaker assumption that $\{H_{\hat{x}}\}_{\hat{x} \in X}$ are uniformly Holder homeomorphisms). Actually in this paper we have only used the case where $H = \text{Id}$.

The following theorem, whose proof is deferred to Appendix A, is a variant of [1, Theorem B].

THEOREM 2. *Let \hat{m} be an \hat{F} -invariant measure on $\hat{\mathcal{E}}$ which projects down to $\hat{\mu}$. Let $\mathcal{B}_0 \subset \hat{\mathcal{B}}$ be a σ -algebra which generates $\hat{\mathcal{B}} \bmod 0$ under \hat{f} . Assume that both \hat{f} and $\hat{x} \mapsto \hat{F}_{\hat{x}}$ are \mathcal{B}_0 -measurable $\bmod 0$, and $\lambda_-(\hat{F}, \hat{x}, \hat{y}) \geq 0$ for \hat{m} -almost every (\hat{x}, \hat{y}) , then the disintegration $\hat{x} \mapsto \hat{m}_{\hat{x}}$ of the measure \hat{m} is \mathcal{B}_0 -measurable $\bmod 0$.*

4.4. Completing the proof. We can now finish the proof of Proposition 4.

Proof of Proposition 4. By Proposition 6, the hypothesis of Theorem 2 is satisfied with $(\hat{X}, N, \mathcal{B}_0, \hat{\mu}, \hat{m}, \hat{f}, \hat{F}, \hat{H})$ being $(G/\Gamma, M, \mathcal{B}_{\zeta}, \mu, \nu, a^{-1}, F, \text{Id})$. Here \mathcal{B}_{ζ} denotes the σ -algebra generated by the partition ζ . Then by Theorem 2, the map $x \mapsto \mu_x^*$ is ζ -measurable. Proposition 4 then follows from Proposition 6(5). \square

5. PROOF OF PROPOSITION 5

Recall that we fixed a finite set of symmetric generators $\{\gamma_i\}$ for Γ . The word distance ℓ on Γ is defined in Section 4.1.

Proof of Proposition 5. We let $\|\cdot\|_g$ denote the background Riemannian metric g on TM , and let Vol_g denote the volume form induced by $\|\cdot\|_g$. There is a C^∞ Riemannian metric on $S^2(T^*M)$ associated to $\|\cdot\|_g$. We denote by $L^p(M, \text{Vol}_g, S^2(T^*M))$ the space of L^p sections of the tensor bundle $S^2(T^*M)$ with respect to Vol_g .

Since α has uniform subexponential growth of derivatives, by the strong property (T) of the lattice Γ (proved in [23, 11, 12]), we can adapt the argument in [2] to show that there exist:⁴

- (1) constants $C_p''', \sigma_p > 0$ for every $1 \leq p < \infty$;
- (2) $\bar{g} \in L^p(M, \text{Vol}_g, S^2(T^*M))$ for all $1 \leq p < \infty$, which is non-degenerate, i.e., $\|v\|_{\bar{g}} > 0$ for Vol_g -a.e. $x \in M$, and every non-zero $v \in T_x M$;
- (3) a sequence of probability measures on Γ , denoted by $\{\omega_n\}_n$, satisfying $\text{supp}(\omega_n) \subset B_{\text{word}}(e, n) \subset \Gamma$ for every n , where $B_{\text{word}}(e, n)$ denotes the radius n open ball in Γ centered at e with respect to the word distance,

such that, setting $g_n = \int \alpha(\gamma)^* g d\omega_n(\gamma)$, then we have

$$(5.1) \quad \|g_n - \bar{g}\|_{L^p} < C_p''' e^{-n\sigma_p}, \quad \forall 1 \leq p < \infty.$$

As a consequence, denote by $\text{Vol}_{\bar{g}}$ the measurable volume form induced by $\|\cdot\|_{\bar{g}}$, then the measure $d\text{Vol}_{\bar{g}}$ is absolutely continuous with respect to $d\text{Vol}_g$, and the density function $\frac{d\text{Vol}_{\bar{g}}}{d\text{Vol}_g}$ has full support.

We define Lebesgue measurable functions $\bar{R}, \underline{R} : M \rightarrow \mathbb{R}_+$ as follows. Set

$$\bar{R}(x) = \sup_{v \in T_x M, \|v\|_g=1} \|v\|_{\bar{g}}, \quad \underline{R}(x) = \inf_{v \in T_x M, \|v\|_g=1} \|v\|_{\bar{g}}.$$

It is direct to see that for $d\text{Vol}_g$ -a.e. $x \in M$,

$$\frac{d\text{Vol}_{\bar{g}}}{d\text{Vol}_g}(x) < \bar{R}^{\dim M}(x), \quad \frac{d\text{Vol}_g}{d\text{Vol}_{\bar{g}}}(x) < \underline{R}^{-\dim M}(x).$$

We have the following lemma.

⁴We obtain (1)–(3) for $p \in (1, \infty)$ by strong property (T), then the case for $p = 1$ follows from Cauchy's inequality.

LEMMA 4. For every $1 \leq p < \infty$, there is $C_p > 0$ such that

$$\int \underline{R}^{-p} d\text{Vol}_g < C_p, \quad \int \overline{R}^p d\text{Vol}_g < C_p.$$

Proof. The second inequality follows immediately from the fact that $\bar{g} \in L^p(M, \text{Vol}_g, S^2(T^*M))$. It remains to prove the first inequality.

We define for every $n \geq 1$,

$$\underline{R}_n(x) = \inf_{v \in T_x M, \|v\|_g=1} \|v\|_{g_n}, \quad \forall x \in M,$$

and $\Omega_n = \{x \mid \underline{R}(x) \geq \frac{1}{2} \underline{R}_n(x)\}$.

For the convenience of the notation, we set $\Omega_0 = \emptyset$. It is clear that $\cup_n \Omega_n$ is a $d\text{Vol}_g$ -conull subset of M .

By the uniform subexponential growth of derivatives, for every $\varepsilon > 0$ there is $C''_\varepsilon > 0$ such that

$$(5.2) \quad \sup_{x \in M} (\underline{R}_n(x)^{-1}) < C''_\varepsilon e^{n\varepsilon}, \quad \forall n \geq 1.$$

By (5.1) and (5.2), for every $\varepsilon > 0$ we have

$$\begin{aligned} \text{Vol}_g(\Omega_n^c) &\leq \text{Vol}_g(\{x \mid |\underline{R}(x) - \underline{R}_n(x)| > \frac{1}{2} \underline{R}_n(x)\}) \\ &\leq 2 \sup_{x \in M} (\underline{R}_n(x)^{-1}) \int |\underline{R}(x) - \underline{R}_n(x)| d\text{Vol}_g(x) \\ &\leq 2C''_\varepsilon C'''_1 e^{n\varepsilon - n\sigma_1}. \end{aligned}$$

Then for each $1 \leq p < \infty$, we take $\varepsilon = \sigma_1 / (10p)$, and we obtain

$$\begin{aligned} \int \underline{R}(x)^{-p} d\text{Vol}_g(x) &\leq 2^p \sum_{n=0}^{\infty} \int_{\Omega_{n+1} \setminus \Omega_n} \underline{R}_{n+1}(x)^{-p} d\text{Vol}_g(x) \\ &\leq 2^p \sum_{n=0}^{\infty} \sup_x (\underline{R}_{n+1}(x)^{-p}) \text{Vol}_g(\Omega_n^c) \\ &\leq 2^{p+1} (C''_\varepsilon)^{p+1} C'''_1 \sum_{n=0}^{\infty} e^{(n+1)p\varepsilon - n(\sigma_1 - \varepsilon)} := C_p < \infty. \end{aligned}$$

□

LEMMA 5. For every $1 \leq p < \infty$, there exists $D_p > 0$ such that for every $\gamma \in \Gamma$,

$$\int_M \|D_x \alpha(\gamma)\|_g^p d\text{Vol}_g(x) \leq D_p.$$

Proof. Take an arbitrary $\gamma \in \Gamma$, and set $F = \alpha(\gamma)$. We recall that F preserves \bar{g} . That is, for $d\text{Vol}_{\bar{g}}$ -a.e. x , for every $v \in T_x M$, we have $\|v\|_{\bar{g}} = \|D_x F(v)\|_{\bar{g}}$. Hence the measure $d\text{Vol}_{\bar{g}}$ is F -invariant.

Notice that for $d\text{Vol}_g$ -a.e. $x \in M$,

$$\begin{aligned}
\|D_x F\|_g &= \sup_{v \in T_x M, \|v\|_g=1} \|D_x F(v)\|_g \\
&= \sup_{v \in T_x M, \|v\|_g=1} \|D_x F(v)\|_{\bar{g}} \frac{\|D_x F(v)\|_g}{\|D_x F(v)\|_{\bar{g}}} \\
&= \sup_{v \in T_x M, \|v\|_g=1} \|v\|_{\bar{g}} \frac{\|D_x F(v)\|_g}{\|D_x F(v)\|_{\bar{g}}} \\
&\leq \bar{R}(x) \underline{R}(F(x))^{-1}.
\end{aligned}$$

Then by Cauchy's inequality,

$$\int \|D_x F\|_g^p d\text{Vol}_g(x) \leq \left(\int \bar{R}(x)^{2p} d\text{Vol}_g(x) \right)^{1/2} \left(\int \underline{R}(F(x))^{-2p} d\text{Vol}_g(x) \right)^{1/2}.$$

Also

$$\begin{aligned}
\int \underline{R}(F(x))^{-2p} d\text{Vol}_g(x) &= \int \underline{R}(F(x))^{-2p} \frac{d\text{Vol}_g}{d\text{Vol}_{\bar{g}}}(x) d\text{Vol}_{\bar{g}}(x) \\
&\leq \left(\int \underline{R}(F(x))^{-4p} d\text{Vol}_{\bar{g}}(x) \right)^{1/2} \left(\int \left(\frac{d\text{Vol}_g}{d\text{Vol}_{\bar{g}}}(x) \right)^2 d\text{Vol}_{\bar{g}}(x) \right)^{1/2} \\
&\leq \left(\int \underline{R}(F(x))^{-4p} d\text{Vol}_{\bar{g}}(x) \right)^{1/2} \left(\int \frac{d\text{Vol}_g}{d\text{Vol}_{\bar{g}}}(x) d\text{Vol}_g(x) \right)^{1/2} \\
&\leq \left(\int \underline{R}(x)^{-4p} d\text{Vol}_{\bar{g}}(x) \right)^{1/2} \left(\int \underline{R}^{-\dim M}(x) d\text{Vol}_g(x) \right)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
\int \underline{R}(x)^{-4p} d\text{Vol}_{\bar{g}}(x) &= \int \underline{R}(x)^{-4p} \frac{d\text{Vol}_{\bar{g}}}{d\text{Vol}_g}(x) d\text{Vol}_g(x) \\
&\leq \left(\int \underline{R}(x)^{-8p} d\text{Vol}_g(x) \right)^{1/2} \left(\int \left(\frac{d\text{Vol}_{\bar{g}}}{d\text{Vol}_g}(x) \right)^2 d\text{Vol}_g(x) \right)^{1/2} \\
&\leq \left(\int \underline{R}(x)^{-8p} d\text{Vol}_g(x) \right)^{1/2} \left(\int \bar{R}^{2 \dim M}(x) d\text{Vol}_g(x) \right)^{1/2}.
\end{aligned}$$

By Lemma 4,

$$\int \|D_x F(v)\|_g^p d\text{Vol}_g(x) \leq C_{2p}^{1/2} C_{8p}^{1/8} C_{2 \dim M}^{1/8} C_{\dim M}^{1/4}.$$

Since γ is chosen arbitrarily, we can conclude the proof by taking D_p to be the right hand side of the last inequality. \square

We fix an embedding $\iota : M \rightarrow \mathbb{R}^N$ for some integer N . Let $\pi_i : \mathbb{R}^N \rightarrow \mathbb{R}$ be the projection to the i -th coordinate. We have seen that for every $1 \leq p < \infty$, there exists a constant $C'_p > 0$ such that for every $1 \leq i \leq N$, for every $\gamma \in \Gamma$,

$$\int |D_x(\pi_i \iota \alpha(\gamma))|^p d\text{Vol}_g(x) < C'_p.$$

Take $p > \dim M / (1 - \beta)$. Then by Sobolev's embedding theorem, we can see that the set $\{\alpha(\gamma) \mid \gamma \in \Gamma\}$ is pre-compact in $\text{Hol-Homeo}^\beta(M)$. We know that any pre-compact subset of $\text{Hol-Homeo}^\beta(M)$ is equicontinuous in $\text{Homeo}(M)$. Thus the closure of $\alpha(\Gamma)$ in $\text{Homeo}(M)$ is a compact topological group K_0 , and it is direct to verify by definition that $K_0 \subset \text{Hol-Homeo}^\beta(M)$. It is then direct to construct a Γ -invariant continuous distance on M by averaging. \square

APPENDIX A.

We now give the proof of Theorem 2 in this appendix. We recall the construction in [1, Section 3]. There is a Lebesgue space (X, \mathcal{B}, μ) obtained by identifying any two points of \hat{X} which are not distinguished by any element of \mathcal{B} ; and a projection $\pi : \hat{X} \rightarrow X$ such that $\mathcal{B} = \pi_* \mathcal{B}_0$ and $\mu = \pi_* \hat{\mu}$. Since \hat{f} is \mathcal{B}_0 -measurable mod 0, there exists a \mathcal{B} -measurable mod 0 transformation $f : X \rightarrow X$ such that $\pi \circ \hat{f} = f \circ \pi$. Let $\mathcal{E} = X \times N$ and $P : \mathcal{E} \rightarrow X$ the canonical projection. Since \tilde{F} is \mathcal{B}_0 -measurable mod 0, we may write $F_{\pi(\hat{x})} = \tilde{F}_{\hat{x}}$ for some \mathcal{B} -measurable mod 0 fiber bundle morphism $F : \mathcal{E} \rightarrow \mathcal{E}$ over f . The measure $m = (\pi \times id)_* \hat{m}$ is F -invariant and projects down to μ . Denote by $\{m_x\}_{x \in X}$ the measure disintegration of m corresponding to the partition of \mathcal{E} into the fibers. By the F -invariance of m we deduce that for μ -a.e. $x \in X$,

$$(A.1) \quad m_{f(x)} = \int (F_{x'})_* m_{x'} d\mu_x^{f^{-1}(\mathcal{B}_0)}(x').$$

Notice that, by our assumption $\tilde{F} = H \circ \hat{F} \circ H^{-1}$; and that $H_{\hat{x}}$ is a C^1 diffeomorphism for a typical \hat{x} , we deduce that

$$\lambda_-(\tilde{F}) = \lambda_-(\hat{F}).$$

This equality makes our argument slightly easier than that of [1].

We define $J : \mathcal{E} \rightarrow [0, \infty)$ by considering the Lebesgue decomposition of $(F_x^{-1})_* m_{f(x)}$ relative to m_x :

$$(F_x^{-1})_* m_{f(x)} = J(x, \cdot) m_x + \eta_x.$$

Define

$$h(F, m) = \int -\log J dm.$$

Following the proof of [1, Theorem B], we will show that $m(\{J = 0\}) = 0$ and in addition the following is true.

PROPOSITION 7. *We have*

$$0 \leq h(F, m) \leq -\dim N \int \min\{0, \lambda_-(\hat{F})\} d\hat{m}.$$

The statement of Proposition 7 is the same as [1, Proposition 3.1], except that we are now assuming (4.4) and (4.5) while in [1] the authors assume that $\log \|D\hat{F}_{\hat{x}}(\hat{y})^{-1}\|$, $\log \|D\hat{H}_{\hat{x}}(\hat{y})\|$ and $\log \|D\hat{H}_{\hat{x}}(\hat{y})^{-1}\|$ are all uniformly bounded, and the dependence of $D\hat{F}_{\hat{x}}(\hat{y})$, $D\hat{H}_{\hat{x}}(\hat{y})$ on \hat{x}, \hat{y} are uniformly continuous. Thus we will need to make some adjustments to the proof in [1] (see also [25]).

Under the hypothesis of Theorem 2, we can conclude by Proposition 7 that $h(F, m)$ vanishes. Once we know that $h(F, m)$ vanishes, we can apply [1, Proposition 3.2] to conclude the proof of Theorem 2. We recall the statement below.

PROPOSITION 8. *If $h(F, m) = 0$ then $\hat{x} \mapsto \hat{m}_{\hat{x}}$ is \mathcal{B}_0 -measurable mod 0.*

The proof of Proposition 8 is rather general and the conditions (4.4) and (4.5) suffice. Now it suffices to give the proof of Proposition 7. The proof here follows essentially the scheme in [25].

Proof of Proposition 7. By the same argument in [1, Section 3.2], we may assume without loss of generality that \hat{m} is ergodic for \hat{F} . In this case, $\min(0, \lambda_-(\hat{F}))$ is a constant \hat{m} -almost everywhere, and is denoted by $-\lambda \leq 0$.

For any integer $k > 0$, for any $(x, \xi) \in X \times N$ we define

$$F_x^k = \begin{cases} F_{f^{k-1}(x)} \cdots F_x & k \geq 0, \\ F_{f^{-k}(x)}^{-1} \cdots F_{f^{-1}(x)}^{-1} & k < 0, \end{cases}$$

and

$$L_k(x, \xi) = \|D_{\xi} F_{f^k(x)}^{-k}\|, \quad C_k(x) = \sup_{\xi \in N} L_k(x, \xi), \quad \tilde{C}_k(x, \xi) = C_k(x).$$

Notice that we have

$$(A.2) \quad 0 \leq \log \tilde{C}_k(x, \xi) \leq \sum_{i=0}^{k-1} \log \tilde{C}_1(F^i(x, \xi)).$$

Given $(x, \xi) \in X \times N$, we denote by $B(\xi, \delta)$ the ball in N centered at ξ of radius $\delta > 0$ and write

$$B((x, \xi), \delta) = \{x\} \times B(\xi, \delta).$$

For each integer $l \geq 0$, we write

$$J_l(x, \xi, \delta) = \frac{(F_{f^l(x)}^{-l})_* m_{f^l(x)}(B(\xi, \delta))}{m_x(B(\xi, \delta))}$$

and

$$J_l^*(x, \xi) = \max_{\delta > 0} J_l(x, \xi, \delta).$$

It is clear that $J_l \geq 0$ and $J_l^* \geq 1$.

We fix some $\epsilon > 0$. Then there is $\beta_1 = \beta_1(\epsilon) > 0$ so that for any set $A \subset X \times N$ with $m(A) < \beta_1$, we have

$$(A.3) \quad \int_A \log \tilde{C}_1 d\mu < \epsilon.$$

Fix some integer $l > 0$ such that the measurable set $\Lambda_1 \subset X \times N$ defined by

$$\Lambda_1 = \{(x, \xi) \mid L_l(x, \xi) \leq e^{(\lambda + \epsilon)l}\}$$

satisfies $m(\Lambda_1) > 1 - \beta_1/2$. Then there is a subset $\Lambda \subset \Lambda_1$ with

$$(A.4) \quad m(\Lambda) > 1 - \beta_1$$

such that the derivatives $D_{\xi} F_x$ are uniformly continuity in ξ over all $x \in \Lambda$, and for some $\delta_1 = \delta_1(\epsilon, l, \Lambda) > 0$, for any $x \in \Lambda$, for any $\delta \in (0, \delta_1(\epsilon))$ we have

$$(A.5) \quad F_x^{-l}(B(\xi, \delta)) \subset B(F_x^{-l}(\xi), e^{(\lambda + 2\epsilon)l}\delta).$$

We denote by E_l the collection of ergodic component of m for F^l . Since μ is F -ergodic, we deduce that E_l is finite and F induces a cyclic permutation of E_l .

Moreover, for m -almost every (x, ξ) we denote by $m_{(x, \xi)}$ the ergodic component at (x, ξ) .

By [25, Prop 5], we know that

$$\log J_l^* \in L^1(\mathcal{E}, m).$$

More precisely, we have the following.

LEMMA 6. *For m -a.e. (x, ξ) , we have*

$$J_l(x, \xi) = \prod_{i=0}^{l-1} J(F^i(x, \xi)).$$

Consequently, for any $m' \in E_l$, we have

$$(A.6) \quad lh(F, m) = - \int \log J_l(x, \xi) dm'(x, \xi).$$

Proof. This is clear that the first equality holds when $l = 1$. For any $l > 1$, we have

$$\begin{aligned} (F_{f^l(x)}^{-l})_* m_{f^l(x)} &= (F_x^{-1})_* (F_{f^l(x)}^{-l+1})_* m_{f^l(x)} \\ &= (F_x^{-1})_* (J_{l-1}(F(x, \xi)) m_{f(x)} + \eta_{f(x)}^{(l-1)}) \\ &= J_{l-1}(F(x, \xi)) \cdot J(x, \xi) m_x + J_{l-1}(F(x, \xi)) \eta_x + (F_x^{-1})_* \eta_{f(x)}^{(l-1)}. \end{aligned}$$

Here $\eta_{f(x)}^{(l-1)}$ is the singular component of $(F_{f^l(x)}^{-l+1})_* m_{f^l(x)}$ with respect to $m_{f(x)}$. By definition, η_x is singular with respect to m_x . Moreover, $(F_x^{-1})_* \eta_{f(x)}^{(l-1)}$ is also singular with respect to m_x for m -a.e. x . Otherwise, we would know that $\eta_{f(x)}^{(l-1)}$ is not singular with respect to $(F_x)_* m_x$; then by (A.1) we would know that $\eta_{f(x)}^{(l-1)}$ is not singular with respect to $m_{f(x)}$. A contradiction. Consequently, we see that

$$J_l = J_{l-1} \circ F \cdot J.$$

We then conclude the proof of the first equality by induction. The equality (A.6) in the lemma is an immediate consequence of the first equality, and the fact that $m = \frac{1}{l} \sum_{i=0}^{l-1} (F^i)_* m'$ for any $m' \in E_l$. \square

We choose some $\beta_2 = \beta_2(\epsilon, l) > 0$ such that for $m' \in E_l$, and for every $A \subset X \times N$ with $m'(A) < \beta_2$, we have

$$(A.7) \quad \int_A (\log J_l^* + \log J_l) dm' < \epsilon.$$

We define

$$Z = \{(x, \xi) \mid J_l(x, \xi) = 0\}, \quad G = Z^c = \{(x, \xi) \mid J_l(x, \xi) > 0\}.$$

We fix a large constant $D > 0$. Given a constant $\delta > 0$, we define

$$\begin{aligned} G(\delta) &= \{(x, \xi) \in G \mid \log J_l(x, \xi; \delta') \leq \log J_l(x, \xi) + \epsilon \quad \forall \delta' \in (0, \delta)\}, \\ Z(\delta) &= \{(x, \xi) \mid \log J_l(x, \xi; \delta') \leq -D \quad \forall \delta' \in (0, \delta)\}. \end{aligned}$$

We fix some $\delta_2 = \delta_2(\epsilon, l) > 0$ sufficiently small so that for every $m' \in E_l$,

$$(A.8) \quad m'(G \setminus G(\delta_2)) < \beta_2.$$

We take an arbitrary $\delta_0 \in (0, \min(\delta_1, \delta_2))$. Given $(x, \xi) \in X \times N$, define $\delta_l(x, \xi; 0) = \delta_0$ and for $k \geq 1$ recursively define

$$\delta_l(x, \xi; k) = \begin{cases} e^{(-\lambda - 2\epsilon)l} \delta_l(x, \xi; k-1) & F^{kl}(x, \xi) \in \Lambda, \\ [\tilde{C}_l(F^{kl}(x, \xi))]^{-1} \delta_l(x, \xi; k-1) & F^{kl}(x, \xi) \notin \Lambda. \end{cases}$$

Observe that we have

$$\delta_l(x, \xi; k+1) \leq \delta_l(x, \xi; k) \leq \delta_0 \quad \forall k \geq 0$$

and by (A.5) and the definition of \tilde{C} we deduce that

$$F^{-l}(B(F^{(k+1)l}(x, \xi), \delta_l(x, \xi; k+1))) \subset B(F^{kl}(x, \xi), \delta_l(x, \xi; k)) \quad k \geq 0.$$

LEMMA 7. For m -a.e. (x, ξ) , we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \delta_l(x, \xi; n) \geq (-\lambda - 3\epsilon)l.$$

Proof. By definition, we have

$$\log \delta_l(x, \xi; n) = \log \delta_0 + \sum_{k=0}^{n-1} (l(-\lambda - 2\epsilon)1_{F^{kl}(x, \xi) \in \Lambda} - \log \tilde{C}_l(F^{kl}(x, \xi))1_{F^{kl}(x, \xi) \notin \Lambda}).$$

Then by Pointwise Ergodic Theorem, we have

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \delta_l(x, \xi; n) \geq -(\lambda + 2\epsilon)lm_{(x, \xi)}(\Lambda) - \int_{\Lambda^c} \log \tilde{C}_l dm_{(x, \xi)}.$$

By (A.2), (A.3) and (A.4), we obtain

$$\begin{aligned} \int_{\Lambda^c} \log \tilde{C}_l dm_{(x, \xi)} &\leq \sum_{i=0}^{l-1} \int_{\Lambda^c} \log \tilde{C}_1 \circ F^i dm_{(x, \xi)} \\ &\leq \sum_{i=0}^{l-1} \int_{\Lambda^c} \log \tilde{C}_1 d(F^i)_* m_{(x, \xi)} \\ &= l \int_{\Lambda^c} \log \tilde{C}_1 dm \leq l\epsilon. \end{aligned}$$

The equality follows from $m = \frac{1}{l} \sum_{i=0}^{l-1} (F^i)_* m_{(x, \xi)}$ for m -a.e. (x, ξ) . Then

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \delta_l(x, \xi; n) \geq -(\lambda + 3\epsilon)l.$$

□

Recall that by [25, Proposition 5], for any Borel probability measure ν on N , we have

$$\limsup_{r \rightarrow 0} \frac{\log \nu(B(\xi, r))}{\log r} \leq \dim N, \quad \nu - a.e. \xi.$$

Thus we may pick a subset $\Omega_1 \subset X \times N$ such that

$$\limsup_{r \rightarrow 0} \frac{\log \inf_{(x, \xi) \in \Omega_1} m_x(B(\xi, r))}{\log r} \leq \dim M$$

and $m'(\Omega_1) > 0$ for every $m' \in E_l$. Then for m' -almost every (x, ξ) , there is an infinite sequence of n such that $F^{nl}(x, \xi) \in \Omega_1$. Then for all sufficiently large n in such sequence we have

$$(A.9) \quad \begin{aligned} \frac{1}{n} \log m_{f^{nl}(x)}(B(F^{nl}(x, \xi), \delta_l(x, \xi; n))) &\geq \frac{1}{n} \log \delta_l(x, \xi; n)(\dim N + \epsilon) \\ &\geq (-\lambda - 4\epsilon)(\dim N + \epsilon)l. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &m_{f^{nl}(x)}(B(F^{nl}(x, \xi), \delta_l(x, \xi; n))) \\ &= m_x(B(x, \delta_l(x, \xi; 0))) \\ &\quad \cdot \prod_{j=0}^{n-1} \frac{m_{f^{(j+1)l}(x)}(B(F^{(j+1)l}(x, \xi), \delta_l(x, \xi; j+1)))}{m_{f^{jl}(x)}(B(F^{jl}(x, \xi), \delta_l(x, \xi; j)))} \\ &\leq \prod_{j=0}^{n-1} \frac{m_{f^{(j+1)l}(x)}(B(F^{(j+1)l}(x, \xi), \delta_l(x, \xi; j+1)))}{m_{f^{jl}(x)}(B(F^{jl}(x, \xi), \delta_l(x, \xi; j)))} \\ &\leq \prod_{j=0}^{n-1} \frac{m_{f^{(j+1)l}(x)}(F^l(B(F^{jl}(x, \xi), \delta_l(x, \xi; j))))}{m_{f^{jl}(x)}(B(F^{jl}(x, \xi), \delta_l(x, \xi; j)))} \\ &= \prod_{j=0}^{n-1} J_l(F^{jl}(x, \xi), \delta_l(x, \xi; j)). \end{aligned}$$

Take an arbitrary $m' \in E_l$. Then for m' -almost every (x, ξ) we have

$$\begin{aligned} &\limsup \frac{1}{n} \log m_{f^{nl}(x)}(B(F^{nl}(x, \xi), \delta_l(x, \xi; n))) \\ &\leq \limsup \frac{1}{n} \sum_{j=0}^{n-1} \log J_l(F^{jl}(x, \xi), \delta_l(x, \xi; j)) \\ &\leq \int_{Z(\delta_0)} \log J_l dm' + \int \log J_l^* dm' \\ &\leq -Dm'(Z(\delta_0)) + l \int \log J^* dm \end{aligned}$$

The last inequality follows from Lemma 6 and the definition of $Z(\delta_0)$. Combine the above inequality with (A.9) we conclude that

$$m'(Z(\delta_0)) \leq \frac{l(\int \log J^* dm + (\lambda + 4\epsilon)(\dim N + \epsilon))}{D}.$$

Since the above holds for any δ_0 sufficiently small and for any $m' \in E_l$, we deduce that

$$m(Z) \leq \limsup_{\delta_0 \rightarrow 0} m(Z(\delta_0)) \leq \frac{l(\int \log J^* dm + (\lambda + 4\epsilon)(\dim N + \epsilon))}{D}.$$

By letting D tend to infinity, we conclude that $m(Z) = 0$, and consequently $m(G) = 1$.

Now notice that for m' -almost every (x, ξ) we have

$$\begin{aligned} & \limsup \frac{1}{n} \log m_{f^{nl}(x)}(B(F^{nl}(x, \xi), \delta_l(x, \xi; n))) \\ & \leq \limsup \frac{1}{n} \sum_{j=0}^{n-1} \log J_l(F^{jl}(x, \xi), \delta_l(x, \xi; j)) \\ & \leq \int_{G(\delta_0)} (\log J_l + l\epsilon) dm' + \int_{G(\delta_0)^c} \log J_l^* dm' \\ & \leq \int \log J_l dm' + 5l\epsilon. \end{aligned}$$

The last inequality follows from (A.8), $G(\delta_2) \subset G(\delta_0)$ and (A.7). Combine the above inequality with (A.9) and (A.6) in Lemma 6, we obtain

$$h(F, m) - 5\epsilon \leq (\lambda + 4\epsilon)(\dim N + \epsilon).$$

Since ϵ is arbitrary, we conclude the proof of Proposition 7. \square

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