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Barycentric gluing and geometry of stable metrics

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Abstract

We discuss various aspects of a local-to-global embedding technique and the metric geometry of stable metric spaces, in particular two of its important subclasses: locally finite spaces and proper spaces. We explain how the barycentric gluing technique, which has been mostly applied to bi-Lipschitz embedding problems pertaining to locally finite spaces, can be implemented successfully in a much broader context. For instance, we show that the embeddability of an arbitrary metric space into ℓ_p is determined by the embeddability of its balls. We also introduce the notion of upper stability. This new metric invariant lies formally between Krivine–Maurey (isometric) notion of stability and Kalton's property $\mathcal{Q}$. We show that several results of Raynaud and Kalton for stable metrics can be extended to the broader context of upper stable metrics and we point out the relevance of upper stability to a long standing embedding problem raised by Kalton. Applications to compression exponent theory are highlighted and we recall old, and state new, important open problems. This article was written in a style favoring clarity over conciseness in order to make the material appealing, accessible, and reusable to geometers from a variety of backgrounds, and not only to Banach space geometers.

Keywords Barycentric gluing · Locally finite · proper · stable metrics · Property $\mathcal{Q}$ · Bi-Lipschitz · Coarse · Uniform embeddings · Compression exponents

Mathematics Subject Classification 51F30 · 46B20 · 46B85 · 46B99

Contents

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1 Introduction

The idea of finding a faithful representation, or embedding, of an intractable space into a tractable host space is an extremely powerful and versatile tool with far reaching applications. The class of spaces of interest and the relevant notions of faithful embeddability are dictated by the problems under scrutiny. This line of thoughts motivated the study of rich and deep embedding theories. Some of the most famous applications of this geometric approach arise in theoretical computer science, geometric group theory, topology, and noncommutative geometry. We refer to [8,22,34,41,56,57,62–65,69,76,83,89] (where comprehensive lists of references can be found) for an extensive account of these rapidly growing research directions.

The main objective in this article is to display various trade-offs when embedding faithfully every member of some interesting classes of metric spaces into a Banach space that can be drawn for a class of Banach spaces with desirable geometric properties. When we think of a Banach space with the best possible geometric properties, Hilbert space is most likely the first space that comes to mind. In most situations (but not all!), Hilbert space exhibits the strongest, or optimal, geometric behavior and we will indeed think of it as a Banach space with highly sought geometric properties. The Lebesgue sequence spaces ℓ_p and function spaces $L_p := L_p[0,1]$ in the super-reflexive range $p \in (1,\infty)$, and more generally super-reflexive Banach spaces[1], are targets of choice due to fundamental applications in non-commutative geometry and topology that will be explained shortly. Despite not possessing the strong geometric features of super-reflexive spaces, ℓ_1 and L_1 nevertheless play a pivotal role in the design of approximation algorithms.

We will mainly consider embeddings of metric spaces into Banach spaces and give most definitions for this setting, but obviously these can be generalized to metric space targets. The strongest notion of faithfulness for metric spaces is isometric embeddability. A metric space $(\mathsf{X}, \mathsf{d}_\mathsf{X})$ admits an *isometric embedding* into a Banach space $(\mathscr{Y}, \|\cdot\|_\mathscr{Y})$ if there exists $f\colon \mathsf{X} \to \mathscr{Y}$ such that $\|f(x) - f(y)\|_\mathscr{Y} = \mathsf{d}_\mathsf{X}(x,y)$, for all $x, y \in \mathsf{X}$. It follows from a result of Godefroy and Kalton [31] that if a Banach space $\mathscr{Y}$ contains an isometric copy of every separable metric space than it must contains a linear isometric copy of every separable Banach space, and thus we cannot expect $\mathscr{Y}$ to have any non-trivial geometric features; in particular $\mathscr{Y}$ cannot be reflexive and even less so super-reflexive. Luckily, for the applications that are of interest to us we only need to consider certain restricted classes of metric spaces and weaker notion of faithfulness. A natural relaxation of isometric embeddability allows for

[1] A Banach space is super-reflexive if all its ultrapowers are reflexive.

some distortion of the distances. A *bi-Lipschitz embedding* from $(\mathsf{X}, \mathrm{d}_{\mathsf{X}})$ into $(\mathscr{Y}, \|\cdot\|_{\mathscr{Y}})$ is a map $f\colon \mathsf{X} \to \mathscr{Y}$ satisfying

$$s \cdot \mathrm{d}_{\mathsf{X}}(x, y) \leq \|f(x) - f(y)\|_{\mathscr{Y}} \leq D \cdot s \cdot \mathrm{d}_{\mathsf{X}}(x, y). \tag{1}$$

for all $x, y \in \mathsf{X}$ and some constants $s \in (0, \infty)$ and $D \in [1, \infty)$ that do not depend on the points x and y. Note that in our setting the target space being a Banach space, one can always take $s = 1$. A bi-Lipschitz embedding accounts for the geometry at *all scales* and is the faithfulness notion that is ubiquitous in theoretical computer science, in particular in the design of approximation algorithms. Unlike the situation in the isometric category, it is not true that if a Banach space must contain a bi-Lipschitz copy of every separable metric space then it must contain a linear isomorphic copy of every separable Banach space. Indeed, a landmark result of Aharoni [1] states that every separable metric space bi-Lipschitzly embeds into c_0, however it has long been known that every subspace of c_0 contains an isomorphic copy of c_0 [5, Chapter 12, Theorem 1]. Moreover, a Banach space that contains a bi-Lipschitz copy of every separable metric space cannot be super-reflexive, yet even reflexive or with the Radon–Nikodým property. These claims follow from classical differentiability arguments which are thoroughly discussed in [16]. Thus, we must once more sacrifice on either the largeness of the class of metric spaces considered or the faithfulness of the embedding. Theoretical computer science is mostly concerned with finite objects, in particular with finite metric spaces, and low-distortion bi-Lipschitz embeddability of finite metric spaces (and subclasses thereof) into finite-dimensional Banach spaces, in particular ℓ_p^k for $p \in \{1, 2\}$ and a small dimension k, has proven to be a crucial tool (see [2]). We will not expand more on this fascinating area here. We will instead focus our attention to infinite metric spaces and on a relaxation of bi-Lipschitz embeddability, called coarse embeddability, which accounts for the large scales only, and its small-scale counterpart, namely uniform embeddability. Let $(\mathsf{X}, \mathrm{d}_{\mathsf{X}})$ be a metric space, $(\mathscr{Y}, \|\cdot\|_{\mathscr{Y}})$ be a Banach space and $f\colon \mathsf{X} \to \mathscr{Y}$ be a map. We define for $t \geq 0$,

$$\rho_f(t) := \inf\{\|f(x) - f(y)\|_{\mathscr{Y}} : \mathrm{d}_{\mathsf{X}}(x, y) \geq t\},$$

and

$$\omega_f(t) := \sup\{\|f(x) - f(y)\|_{\mathscr{Y}} : \mathrm{d}_{\mathsf{X}}(x, y) \leq t\}.$$

We will refer to the map f as an embedding of X into $\mathscr{Y}$ and for every $x, y \in \mathsf{X}$, it is easy to verify that

$$\rho_f(\mathrm{d}_{\mathsf{X}}(x, y)) \leq \|f(x) - f(y)\|_{\mathscr{Y}} \leq \omega_f(\mathrm{d}_{\mathsf{X}}(x, y)).$$

The moduli ρ_f and ω_f will be called the *compression modulus* and the *expansion modulus* [2], respectively, of the embedding. A map $f\colon \mathsf{X} \to \mathscr{Y}$ is said to be a *uniform embedding* if

$$\lim_{t \to 0} \omega_f(t) = 0 \text{ and } \rho_f(t) > 0 \text{ for all } t > 0. \tag{2}$$

This type of embedding is designed to describe the microscopic structure of X since only the behavior of f with respect to pairs of points whose distance to each other is small is taken into account. It can be seen as a quantitative version of a topological embedding when the spaces carry a metric structure. Uniform embeddability is irrelevant for uniformly

[2] The expansion modulus is usually called modulus of uniform continuity when f is a uniformly continuous map.

discrete[3] spaces and we can always assume that the metric is bounded, by replacing d_X with the uniformly equivalent metric $\min\{d_X, 1\}$ for instance. Uniform embeddings have been extensively studied in nonlinear Banach space theory for about 50 years, and they recently made a quite unexpected appearance in sketching theory in theoretical computer science (see [3]).

On the other hand, we say that X admits a *coarse embedding*[4] into $\mathscr{Y}$ if there is a map $f\colon \mathsf{X} \to \mathscr{Y}$ such that

$$\lim_{t\to\infty} \rho_f(t) = \infty \text{ and } \omega_f(t) < \infty \text{ for all } t > 0. \tag{3}$$

It will be interesting to consider embeddings which are *simultaneously* coarse and uniform. A map $f\colon \mathsf{X} \to \mathscr{Y}$ satisfying (2) and (3) is usually called a *strong embedding*.

The notion of coarse embedding is not relevant for bounded metric spaces, since any map sending all the points of a bounded space onto a single point of the target space is trivially a coarse embedding. In particular, a coarse embedding is not necessarily injective and it is worth noting that we can always assume that X is uniformly discrete by passing to one of its skeletons. A subset S of a metric space (X, d_X) is called an (δ_s, δ_m)-*skeleton* if there exist $0 < \delta_s \le \delta_m < \infty$ such that S is δ_s-separated and $\sup_{x\in \mathsf{X}} d_X(x, S) \le \delta_m$. A classical and simple application of Zorn's lemma shows that every non-empty infinite metric space X admits a (δ, δ)-skeleton for every $\delta \in (0, \mathrm{diam}(\mathsf{X}))$. The following easy and well-known fact tells us that coarsely embedding a metric space or one of its skeleton is essentially the same, up to some usually inessential loss in the faithfulness of the embedding.

Lemma 1.1 *Let (X, d_X) be a metric space and S a (δ_s, δ_m)-skeleton of X, then any map $c\colon X \to S$ which maps a point in X to its closest point in S satisfies for all $x, y \in X$*

$$d_X(x, y) - 2\delta_m \le d_X(c(x), c(y)) \le d_X(x, y) + 2\delta_m. \tag{4}$$

The closest point map above (ties are arbitrarily resolved) is a very faithful coarse embedding[5], and it is an example of what is called a *quasi-isometric embedding* in geometric group theory or a *coarse bi-Lipschitz embedding* in nonlinear Banach space theory[6].

A map $f\colon \mathsf{X} \to \mathscr{Y}$ is called a *coarse bi-Lipschitz embedding* or a *quasi-isometric embedding* if it is bi-Lipschitz up to some additive constants, i.e. if there are constants $A \in [1, \infty)$ and $B \in (0, \infty)$ such that for all $x, y \in \mathsf{X}$,

$$\frac{1}{A} d_X(x, y) - B \le \|f(x) - f(y)\|_{\mathscr{Y}} \le A d_X(x, y) + B.$$

The geometric group theory terminology is well-established and predates the nonlinear Banach space theory terminology. However, coarse bi-Lipschitz embeddings are more tightly connected to bi-Lipschitz embeddings than with isometric embeddings and in the remainder of this article we will favor the terminology "quasi-isometric" when groups are involved and

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[3] X is uniformly discrete if there exists a constant $\delta_s \in (0, \infty)$ such that $d_X(x, y) \ge \delta_s$ for all $x, y \in \mathsf{X}$. If we want to emphasize on the separation parameter δ_s we will talk about a δ_s-separated metric space.

[4] For quite some time, in the geometric group theory and noncommutative geometry communities, a "coarse embedding" was simply called a "uniform embedding", a shorter version of Gromov's original terminology "uniform embedding at infinity" [35], but it seems now that the terminology "coarse embedding" has been widely adopted.

[5] An embedding satisfying (4) is called a *near-isometry* in [84, Definition 10, page 48]

[6] In nonlinear Banach space theory it was customary to say that a space quasi-isometrically embeds if for every $\epsilon > 0$ there exists a bi-Lipschitz embedding with distortion at most $1 + \epsilon$, but this property is now most commonly referred to as almost isometric embeddability.

the terminology "coarse bi-Lipschitz" when treating the case of general metric spaces. Again, the possibility of incorporating a positive constant B (which is not allowed for bi-Lipschitz embedding) allows non injective maps and the small-scale structure can be forgotten in the process. In particular, an easy application of Lemma 1.1 shows that the concepts of coarse bi-Lipschitz embeddability and bi-Lipschitz embeddability for large distances are equivalent, where a map $f\colon \mathsf{X} \to \mathscr{Y}$ is said to be a *bi-Lipschitz embedding at large distances* if there exist $\tau \in (0, \infty)$ and $D_\tau \in [1, \infty)$ such that $\mathrm{d}_\mathsf{X}(x, y) \geq \tau$ implies that

$$\frac{1}{D_\tau}\mathrm{d}_\mathsf{X}(x, y) \leq \|f(x) - f(y)\|_\mathscr{Y} \leq D_\tau \mathrm{d}_\mathsf{X}(x, y).$$

The origins of geometric group theory go back at least as early as Mostow's rigidity theorem and it received a tremendous impetus under Gromov's lead [35]. For instance, quasi-isometric rigidity of groups has become a prominent branch of geometric group theory [50]. The much weaker notion of coarse embeddability is crucial for applications in noncommutative geometry and topology. Gromov suggested in [36] that a space whose large scale geometry is compatible in a certain sense with the geometry of a super-reflexive Banach space is very likely to satisfy a conjecture of Novikov. Building upon a groundbreaking work of Guoliang Yu [88], Gromov's intuition was proved to be true by Kasparov and Yu [51]. They showed that if a metric space with bounded geometry[7] admits a coarse embedding into a super-reflexive Banach space, then it satisfies the coarse geometric Novikov conjecture. We refer to [27,86,87], for instance, for more information on the Novikov conjectures and the noncommutative geometry of groups. In the late 1990s, the attention was then drawn on the coarse geometry of Banach spaces, which had been little considered at that time, contrary to its uniform counterpart.

In light of Kasparov–Yu result it is natural to ask whether every metric space with bounded geometry admits a coarse embedding into some super-reflexive Banach space. A positive answer to this question would imply the astounding statement that the coarse Novikov conjecture holds for every bounded geometry metric space! This hope was dashed quickly. As observed by Gromov [33], an infinite disjoint union (which has bounded geometry) of a sequence of (finite and regular) expander graphs does not admit a coarse embedding into a Hilbert space. It is much more difficult, but nevertheless true, that there are infinite metric spaces with bounded geometry that do not coarsely embed into *any* super-reflexive Banach space. Two delicate constructions were given by Lafforgue [53] (via an algebraic approach), and Mendel and Naor [61] (via a graph theoretic approach). However, Brown and Guentner [19] showed that if we relax the super-reflexivity condition then the situation improves.

Theorem 1.2 *Every metric space with bounded geometry admits a coarse embedding into a reflexive Banach space* $\left(\sum_{n=1}^{\infty} \ell_{p_n}\right)_2$, *where* $\lim_{n\to\infty} p_n = +\infty$.

Unfortunately, spaces $\left(\sum_{n=1}^{\infty} \ell_{p_n}\right)_2$ as above are in many aspects far from super-reflexive spaces. Motivated by the discussion above, constructing embeddings into Banach spaces whose geometric behavior mimics to the greatest extent possible the behavior of super-reflexive spaces has become a fundamental endeavor. Since the appearance of Brown-Guentner embedding result, the theory of metric embeddings of general metric spaces into Banach spaces has sustained a steady growth. An underlying goal of this article if to showcase these developments by emphasizing on the tradeoff between the faithfulness of the embeddings and the strength of the geometric features of the host spaces.

[7] A metric space has bounded geometry if the number of points in any ball of finite radius is finite and admits a (finite) upper bound that does not depend on the center of the ball, but eventually on the radius.

In Sects. 2.1–2.2 we present a detailed treatment of the barycentric gluing technique introduced in [6]. In particular we expand its implementation to arbitrary metric spaces and arbitrary embeddings. Rather general new local-to-global embedding results are then obtained, some having already been used in applications ([20, Theorem 44] for instance). In Sects. 2.3–3, which are essentially expository, we review top-of-the-art embedding results for the classes of locally finite, proper, and stable metric spaces. Section 4 focuses on the geometry of stable metric spaces. It is shown that stability, which is an isometric notion, can be relaxed into a bi-Lipschitz invariant, called upper stability, that allows us to revisit and extend works of Raynaud and Kalton about stable spaces.

2 Barycentric gluing: a versatile local-to-global embedding technique

Brown and Guentner [19] showed that every bounded geometry metric space admits a coarse embedding into a reflexive Banach space $\left(\sum_{n=1}^{\infty} \ell_{p_n}\right)_2$ where $\lim_{n\to\infty} p_n = \infty$. Brown and Guentner embedding technique is a modification of the construction of coarse embeddings using Yu's property A, and the compression rate is of the order of $\sqrt{t}$. Using a different embedding technique, Ostrovskii significantly refined Brown and Guentner result when he showed [72] that every locally finite[8] metric space can be embedded into any Banach space that contains almost isometric copies of ℓ_∞^n for all $n \geq 1$ (with a slight abuse of terminology we will simply say "contains the ℓ_∞^n's" in the sequel). It is worth mentioning that a deep result of Maurey and Pisier [60] states that a Banach space $\mathscr{Y}$ contains the ℓ_∞^n's if and only if $\mathscr{Y}$ does not have finite Rademacher cotype. In particular the host space in Ostrovskii's result can be taken to be $\left(\sum_{n=1}^{\infty} \ell_{p_n}\right)_2$ with $\lim_{n\to\infty} p_n = \infty$ but also, and most importantly, $\left(\sum_{n=1}^{\infty} \ell_\infty^n\right)_2$. This Banach space is a reflexive Banach space that has the same asymptotic structure as Hilbert space. So from the asymptotic point of view the host Banach space is extremely close to the "most" super-reflexive Banach space. Moreover, Ostrovskii's embedding compression rate is of the order of[9] $t^{\log\frac{3}{2}}$. Whether it is possible to upgrade the previous coarse embeddings to bi-Lipschitz embeddings was unclear at that time and a new idea was needed. This idea, that will be referred to as *barycentric gluing*, appeared first in [6] and was inspired by Ribe's proof [82] of the uniform equivalence between $\left(\sum_{n=1}^{\infty} \ell_{q_n}\right)_2$ and $\left(\sum_{n=1}^{\infty} \ell_{q_n}\right)_2 \oplus \ell_1$ where $q_n \to 1$ when $n \to \infty$. The original motivation to introduce barycentric gluing was to improve one implication in Bourgain's metric characterization of super-reflexivity in terms of bi-Lipschitz embeddings of finite binary trees [17]. Soon after [6], it was realized that barycentric gluing was amenable to the study of the bi-Lipschitz geometry of locally finite metric spaces and in [10] it was shown that Ostrovskii's result holds for the significantly stronger notion of bi-Lipschitz embeddings[10]. Subsequent implementations of barycentric gluing, which culminated with Ostrovskii's beautiful finite determinacy theorem, relied heavily on the local finiteness of the space to be embedded [7,10,75] or on specific properties of the local embeddings [9]. Barycentric gluing was recently applied in [68] in connection with embeddings of the finitely generated 3-dimensional Heisenberg group. As it will be shown in this section, it turns out

[8] A metric space is locally finite if every ball is finite.

[9] An inspection of the proof reveals that the exponent could be any number strictly less than 1.

[10] Unfortunately, Ostrovskii's result was not cited in [10] since the authors were not aware of it at that time. While the result of Brown and Guentner is widely known, Ostrovkii's result seems to have been unintentionaly but unduly overlooked.

that barycentric gluing is a much more versatile local-to-global embedding technique than originally perceived.

2.1 Embeddability into ℓ_p-spaces is locally determined

The following definition is reminiscent of the notion of finite representability in Banach space theory.

Definition 2.1 Let $\lambda \in [1,\infty)$. We will say that a metric space $(\mathsf{X}, \mathrm{d}_\mathsf{X})$ is *locally λ-bi-Lipschitz representable in* a metric space $(\mathsf{Y}, \mathsf{d}_\mathsf{Y})$ if for every *metric ball B* in X there is a map $f_B\colon B \to \mathsf{Y}$ and a scaling factor $s_B > 0$ such that for all $x, y \in B$

$$s_B \cdot \mathrm{d}_\mathsf{X}(x,y) \le \mathrm{d}_\mathsf{Y}(f_B(x), f_B(y)) \le \lambda s_B \cdot \mathrm{d}_\mathsf{X}(x,y). \tag{5}$$

We simply say that $(\mathsf{X}, \mathrm{d}_\mathsf{X})$ is *locally bi-Lipschitz representable* if $(\mathsf{X}, \mathrm{d}_\mathsf{X})$ is locally λ-bi-Lipschitz representable for some $\lambda \ge 1$.

In the proof of the next theorem we present the mechanism of the barycentric gluing technique in its most elementary and general implementation.

Theorem 2.2 *Let $p \in [1,\infty)$. If $(X, \mathrm{d}_\mathsf{X})$ is locally bi-Lipschitz representable in a Banach space $(\mathscr{Y}, \|\cdot\|)$, then X bi-Lipschitzly embeds into $\ell_p(\mathscr{Y})$.*

The proof will actually show that if X is locally λ-bi-Lipschitz representable in $\mathscr{Y}$, then X bi-Lipschitzly embeds into $\ell_p(\mathscr{Y})$ with distortion at most $30 \cdot 5^{1-1/p}\lambda$.

Proof Let $\gamma \stackrel{\mathrm{def}}{=} 1/\lambda$ and fix $x_0 \in \mathsf{X}$. Assume, after rescaling and translating if necessary, that for every $k \in \mathbb{Z}$ there exists $f_k\colon B_\mathsf{X}(x_0, 2^k) \to \mathscr{Y}$ such that $f_k(x_0) = 0$ and for all $x, y \in B_\mathsf{X}(x_0, 2^k)$,

$$\gamma \mathrm{d}_\mathsf{X}(x,y) \le \|f_k(x) - f_k(y)\| \le \mathrm{d}_\mathsf{X}(x,y). \tag{6}$$

Note that for all $x \in B_\mathsf{X}(x_0, 2^k)$ one has

$$\gamma \mathrm{d}_\mathsf{X}(x_0,x) \overset{7-\mathrm{low}}{\le} \|f_k(x)\| \overset{7-\mathrm{up}}{\le} \mathrm{d}_\mathsf{X}(x_0,x).$$

The only reason to assume that the local embeddings are 1-Lipschitz is to optimize the distortion loss (otherwise the proof below, which uses (7)-up to estimate the Lipschitz but also the co-Lipschitz constant, will give a distortion of the order of $O(\lambda^2)$).

Define the barycentric map $f\colon \mathsf{X} \to \ell_p(\mathscr{Y})$ by

$$f(x) = (0, \dots, 0, \mu_x f_k(x), (1-\mu_x) f_{k+1}(x), 0, \dots) \tag{7}$$

if $\mathrm{d}_\mathsf{X}(x_0,x) \in [2^{k-1}, 2^k)$ where $\mu_x = \frac{2^k - \mathrm{d}_\mathsf{X}(x_0,x)}{2^{k-1}}$. The barycentric map will actually fall short of providing the desired embedding. Nevertheless, we proceed to estimate the distortion of the map f by distinguishing several cases and we will slightly modify the barycentric map later in order to obtain a genuine bi-Lipschitz embedding. For simplicity, we work out the proof in the case $p = 1$ and we will explain the minor modifications that are required when $p \in (1,\infty)$.

Let $x, y \in \mathsf{X}$ and assume without loss of generality that $\mathrm{d}_\mathsf{X}(x_0,x) \le \mathrm{d}_\mathsf{X}(x_0,y)$. Observe first that if $k \in \mathbb{Z}$ and $x \in X$ are such that $2^{k-1} \le \mathrm{d}_\mathsf{X}(x_0,x) < 2^k$, then

$$2^k - \mathrm{d}_\mathsf{X}(x_0,x) \le \mu_x \mathrm{d}_\mathsf{X}(x_0,x) \le 2(2^k - \mathrm{d}_\mathsf{X}(x_0,x)) \tag{8}$$

and

$$\mathrm{d_X}(x_0, x) - 2^{k-1} \leq (1-\mu_x)\mathrm{d_X}(x_0, x) \leq 2(\mathrm{d_X}(x_0, x) - 2^{k-1}) \tag{9}$$

Case 1 Assume that $2^{k-1} \leq \mathrm{d_X}(x_0, x) < 2^k < 2^{r-1} \leq \mathrm{d_X}(x_0, y) < 2^r$ with $r \geq k+2$.
Then,

$$\|f(x) - f(y)\| = \mu_x \|f_k(x)\| + (1-\mu_x)\|f_{k+1}(x)\| + \mu_y \|f_r(y)\| + (1-\mu_y)\|f_{r+1}(y)\|, \tag{10}$$

and it follows from (7)-up that

$$\|f(x) - f(y)\| \leq \mathrm{d_X}(x_0, x) + \mathrm{d_X}(x_0, y).$$

But in this case, one has $\mathrm{d_X}(x_0, x) \leq \frac{\mathrm{d_X}(x_0, y)}{2}$, therefore

$$\frac{\mathrm{d_X}(x_0, y)}{2} \leq \mathrm{d_X}(x_0, y) - \mathrm{d_X}(x_0, x) \leq \mathrm{d_X}(x, y) \leq \mathrm{d_X}(x_0, x) + \mathrm{d_X}(x_0, y) \leq \frac{3\mathrm{d_X}(x_0, y)}{2}, \tag{11}$$

and hence

$$\|f(x) - f(y)\| \leq 3\mathrm{d_X}(x, y). \tag{12}$$

The lower bound follows easily from (10) and (7)-low. Indeed,

$$\begin{aligned}\|f(x) - f(y)\| &\geq \mu_x \gamma d(x_0, x) + (1-\mu_x)\gamma d(x_0, x) + \mu_y \gamma d(x_0, y) + (1-\mu_y)\gamma d(x_0, y)\\ &\geq \gamma d(x, y).\end{aligned}$$

Case 2 Assume that $2^{k-1} \leq \mathrm{d_X}(x_0, x) < 2^k \leq \mathrm{d_X}(x_0, y) < 2^{k+1}$.
In this case,

$$\|f(x) - f(y)\| = \mu_x \|f_k(x)\| + \|(1-\mu_x) f_{k+1}(x) - \mu_y f_{k+1}(y)\| + (1-\mu_y)\|f_{k+2}(y)\|, \tag{13}$$

and it follows from (7)-up and the triangle inequality that

$$\|f(x) - f(y)\| \leq \|f_{k+1}(x) - f_{k+1}(y)\| + 2\mu_x \mathrm{d_X}(x_0, x) + 2(1-\mu_y)\mathrm{d_X}(x_0, y).$$

Invoking (6), (8) and (9) we obtain

$$\|f(x) - f(y)\| \leq \mathrm{d_X}(x, y) + 4(2^k - \mathrm{d_X}(x_0, x)) + 4(\mathrm{d_X}(x_0, y) - 2^k).$$

It follows that

$$\|f(x) - f(y)\| \leq \mathrm{d_X}(x, y) + 4(\mathrm{d_X}(x_0, y) - \mathrm{d_X}(x_0, x)) \leq 5\mathrm{d_X}(x, y). \tag{14}$$

For the lower bound, (13) gives

$$\begin{aligned}\|f(x) - f(y)\| &\geq \|f_{k+1}(x) - f_{k+1}(y)\| - \|\mu_x f_{k+1}(x) + (1-\mu_y) f_{k+1}(y)\|\\ &\geq \|f_{k+1}(x) - f_{k+1}(y)\| - \mu_x \mathrm{d_X}(x_0, x) - (1-\mu_y)\mathrm{d_X}(x_0, y) && (15)\\ &\geq \gamma \mathrm{d_X}(x, y) - 2(2^k - \mathrm{d_X}(x_0, x)) - 2(\mathrm{d_X}(x_0, y) - 2^k) && (16)\\ &\geq \gamma \mathrm{d_X}(x, y) - 2(\mathrm{d_X}(x_0, y) - \mathrm{d_X}(x_0, x)), && (17)\end{aligned}$$

where we used (7) in (15), and (8) and (9) for (16).

Now observe that based on (13), (8), and 9 we have

$$\|f(x)-f(y)\| \geq \mu_x \gamma \mathrm{d_X}(x_0,x) \geq \gamma(2^k - \mathrm{d_X}(x_0,x)) \tag{18}$$

and

$$\|f(x)-f(y)\| \geq (1-\mu_y)\gamma \mathrm{d_X}(x_0,y) \geq \gamma(\mathrm{d_X}(x_0,y) - 2^k), \tag{19}$$

and summing (18) and (19) gives

$$2\|f(x)-f(y)\| \geq \gamma(\mathrm{d_X}(x_0,y) - \mathrm{d_X}(x_0,x)). \tag{20}$$

Combining (20) and (17) we get

$$\|f(x)-f(y)\| \geq \frac{\gamma^2}{4+\gamma}\mathrm{d_X}(x,y) \geq \frac{\gamma^2}{5}\mathrm{d_X}(x,y).$$

Note that we will be able to improve this (temporary) co-Lipschitz constant once we modify the barycentric map.

Case 3 Assume that $2^{k-1} \leq \mathrm{d_X}(x_0,x) \leq \mathrm{d_X}(x_0,y) < 2^k$.

In this configuration,

$$\|f(x)-f(y)\| = \|\mu_x f_k(x) - \mu_y f_k(y)\| + \|(1-\mu_x)f_{k+1}(x) - (1-\mu_y)f_{k+1}(y)\| \tag{21}$$

and the triangle inequality provides

$$\begin{aligned}\|f(x)-f(y)\| \leq \mu_x\|f_k(x)-f_k(y)\| + |\mu_x-\mu_y|\cdot\|f_k(y)\| + |\mu_x-\mu_y|\cdot\|f_{k+1}(y)\| \\ + (1-\mu_x)\|f_{k+1}(x)-f_{k+1}(y)\|.\end{aligned}$$

Noticing that

$$|\mu_x-\mu_y| = \mu_x-\mu_y = \frac{2^k-\mathrm{d_X}(x_0,x)}{2^{k-1}} - \frac{2^k-\mathrm{d_X}(x_0,y)}{2^{k-1}} = \frac{\mathrm{d_X}(x_0,y)-\mathrm{d_X}(x_0,x)}{2^{k-1}}, \tag{22}$$

it follows from (22) combined with (6) and (7)-up that

$$\begin{aligned}\|f(x)-f(y)\| &\leq \mathrm{d_X}(x,y) + 2\mathrm{d_X}(x_0,y)\frac{\mathrm{d_X}(x_0,y)-\mathrm{d_X}(x_0,x)}{2^{k-1}} \\ &\leq \mathrm{d_X}(x,y) + 4(\mathrm{d_X}(x_0,y)-\mathrm{d_X}(x_0,x)) \\ &\leq 5\mathrm{d_X}(x,y).\end{aligned} \tag{23}$$

On the other hand,

$$\begin{aligned}\|f(x)-f(y)\| &\geq \mu_x\|f_k(x)-f_k(y)\| + (1-\mu_x)\|f_{k+1}(x)-f_{k+1}(y)\| \\ &\quad - |\mu_x-\mu_y|\|f_k(x)\| - |\mu_x-\mu_y|\cdot\|f_{k+1}(y)\| \\ &\geq \gamma\mathrm{d_X}(x,y) - 2(\mu_x-\mu_y)\mathrm{d_X}(x_0,y) \\ &\geq \gamma\mathrm{d_X}(x,y) - 4(\mathrm{d_X}(x_0,y)-\mathrm{d_X}(x_0,x)).\end{aligned} \tag{24}$$

If $\mathrm{d_X}(x_0,y) - \mathrm{d_X}(x_0,x) \leq \frac{\gamma}{5}\mathrm{d_X}(x,y)$ it follows from (24) that

$$\|f(x)-f(y)\| \geq \frac{\gamma}{5}\mathrm{d_X}(x,y). \tag{25}$$

If $\mathrm{d_X}(x_0, y) - \mathrm{d_X}(x_0, x) \geq \frac{\gamma}{5}\mathrm{d_X}(x, y)$ we cannot conclude. To be able to take care of this inconclusive case we slightly modify the barycentric map f as follows.

Let $\hat{f}: X \to \ell_p(\mathscr{Y})\oplus_p\mathbb{R}$ defined by $\hat{f}(x) = (f(x), \mathrm{d_X}(x_0, x))$. Observe that $\ell_p(\mathscr{Y})\oplus_p\mathbb{R}$ is isometric to a subset of $\ell_p(\mathscr{Y})$. Since (we are still considering the case $p = 1$)

$$\|\hat{f}(x) - \hat{f}(y)\| = \|f(x) - f(y)\| + |\mathrm{d_X}(x_0, x) - \mathrm{d_X}(x_0, y)| \leq \|f(x) - f(y)\| + \mathrm{d_X}(x, y),$$

it follows from (12), (14), and (23) that for every $x, y \in X$

$$\|\hat{f}(x) - \hat{f}(y)\| \leq 6\mathrm{d_X}(x, y).$$

Note also that since $\|\hat{f}(x) - \hat{f}(y)\| \geq \|f(x) - f(y)\|$ the previous co-Lipschitz constant estimates for f translate into estimates for $\hat{f}$. Moreover, in the troublesome Case 3 when $\mathrm{d_X}(x_0, y) - \mathrm{d_X}(x_0, x) \geq \frac{\gamma}{5}\mathrm{d_X}(x, y)$ we can now infer that

$$\|\hat{f}(x) - \hat{f}(y)\| \geq |\mathrm{d_X}(x_0, x) - \mathrm{d_X}(x_0, y)| \geq \frac{\gamma}{5}\mathrm{d_X}(x, y).$$

Looking back at the (suboptimal) computation of the co-Lipschitz constant of the barycentric map in Case 2, the following improvement can now be achieved for the modified function $\hat{f}$. Recall that inequality (17) implies that

$$\|\tilde{f}(x) - \tilde{f}(y)\| \geq \|f(x) - f(y)\| \geq \gamma\mathrm{d_X}(x, y) - 2(\mathrm{d_X}(x_0, y) - \mathrm{d_X}(x_0, x)),$$

If $\mathrm{d_X}(x_0, y) - \mathrm{d_X}(x_0, x) \leq \frac{\gamma}{3}\mathrm{d_X}(x, y)$ then

$$\|\tilde{f}(x) - \tilde{f}(y)\| \geq \frac{\gamma}{3}\mathrm{d_X}(x, y),$$

otherwise

$$\|\hat{f}(x) - \hat{f}(y)\| \geq |\mathrm{d_X}(x_0, x) - \mathrm{d_X}(x_0, y)| \geq \frac{\gamma}{3}\mathrm{d_X}(x, y).$$

Ultimately we proved that there exists a map $\hat{f}: \mathsf{X} \to \ell_p(\mathscr{Y})$ such that for all $x, y \in \mathsf{X}$ (and when $p = 1$)

$$\frac{\gamma}{5}\mathrm{d_X}(x, y) \leq \|\hat{f}(x) - \hat{f}(y)\| \leq 6\mathrm{d_X}(x, y).$$

To finish the proof in the general case, observe that for fixed x, y in X, $\hat{f}(x)$ and $\hat{f}(y)$ together involve at most 5 coordinates. Since for all $z \in \mathbb{R}^N$, $\|z\|_p \leq \|z\|_1$, the Lipschitz constant of $\hat{f}$ is still 6 in the case of an arbitrary $p \in [1, \infty)$. As for the co-Lipschitz constant, Hölder's inequality implies that for all $z \in \mathbb{R}^N$, $\|z\|_1 \leq \|z\|_p N^{1-1/p}$, and hence the lower bound estimate for a general $p \in [1, \infty)$ becomes $\frac{\gamma \mathrm{d_X}(x,y)}{5^{2-1/p}} \leq \|\hat{f}(x) - \hat{f}(y)\|$. □

It follows from the classical fact that $\ell_p(\ell_p)$ is linearly isometric to ℓ_p, that the bi-Lipschitz embeddability of a metric space into ℓ_p is locally determined.

Corollary 2.3 *Let $p \in [1, \infty)$ and $(\mathsf{X}, \mathrm{d_X})$ be a metric space. If X is locally bi-Lipschitz representable in ℓ_p, then X bi-Lipschitzly embeds into ℓ_p.*

In Corollary 2.3, we could have replaced ℓ_p by any Banach space $\mathscr{Y}$ such that $\mathscr{Y}$ is isomorphic to $\ell_p(\mathscr{Y})$ for some p. Theorem 2.2, and the variety of corollaries that can be deduced from it, are usually sufficient if we are mainly interested in embeddings in the host spaces ℓ_p or $L_p[0, 1]$.

In general the host space of the global embedding in Theorem 2.2 will not be isomorphic to the host space of the local embeddings. Nevertheless, in various situations this problem can be overcome. In [6,7,9,10,74], barycentric gluing is combined with structural manipulations that can be performed internally to the host space, thus avoiding the use of external ℓ_p-sums. We now prove a general result where barycentric gluing is implemented internally. Since we will use classical finite-dimensional Schauder decomposition technique, it is crucial that the images of the local embeddings live in finite-dimensional subspaces of the host space. Note that this will always happen when the domain space is locally finite, and it partially explains why barycentric gluing was mainly implemented in the context of locally finite spaces.

Theorem 2.4 *Let $(X, \mathrm{d_X})$ be a metric space and $(\mathscr{Y}, \|\cdot\|)$ a Banach space. Assume that for some $\lambda \geq 1$,*

(i) *X is locally λ-bi-Lipschitz representable in every finite-codimensional subspace of $\mathscr{Y}$,*
(ii) *for every ball in X, its image under any of the local embeddings from (i) spans a finite-dimensional subspace.*

Then, X bi-Lipschitzly embeds into $\mathscr{Y}$.

Proof Let $\gamma \stackrel{\text{def}}{=} 1/\lambda$ and fix $x_0 \in X$. We will build a finite-dimensional Schauder decomposition inside $\mathscr{Y}$ that will be a substitute for the external ℓ_p-sum in Theorem 2.2. Let $\eta > 0$ and $B_k \stackrel{\text{def}}{=} B_X(x_0, 2^k)$. Pick a sequence $(\eta_j)_{j=0}^{\infty}$ with $\eta_j > 0$ and $\Pi_{j=0}^{\infty}(1+\eta_j) \leq 1+\eta$. Choose first a unit vector $v \in \mathscr{Y}$ and let $\mathbb{R}v$ be the dimension 1 subspace of $\mathscr{Y}$ generated by v. Then, using the standard Mazur technique, we can find a finite-codimensional subspace $\mathscr{H}_0$ of $\mathscr{Y}$ so that

$$\forall y \in \mathbb{R}v,\ \forall z \in \mathscr{H}_0,\ \|y\|_{\mathscr{Y}} \leq (1+\eta_0)\|y+z\|_{\mathscr{Y}}.$$

By our assumption, and without loss of generality, there exists $f_0 \colon B_0 \to \mathscr{H}_0$ such that $f_0(x_0) = 0$ and for all $x, y \in B_0$,

$$\gamma \mathrm{d_X}(x, y) \leq \|f_0(x) - f_0(y)\|_{\mathscr{Y}} \leq \mathrm{d_X}(x, y).$$

Since by assumption the linear span of the set $f_0(B_0)$, denoted by $\mathscr{F}_0$, is a finite-dimensional subspace of $\mathscr{H}_0$, by Mazur's technique we can find one more time a finite-codimensional subspace $\mathscr{H}_1$ of $\mathscr{Y}$ so that

$$\forall y \in \mathbb{R}v \oplus \mathscr{F}_0,\ \forall z \in \mathscr{H}_1,\ \|y\|_{\mathscr{Y}} \leq (1+\eta_1)\|y+z\|_{\mathscr{Y}}.$$

Continuing this process, we can thus construct a sequence $(\mathscr{F}_k)_{k=-1}^{\infty}$ of finite-dimensional subspaces of $\mathscr{Y}$ (where $\mathscr{F}_{-1} \stackrel{\text{def}}{=} \mathbb{R}v$) and maps $f_k \colon B_k \to \mathscr{F}_k$ such that $f_k(x_0) = 0$ and for all $x, y \in B_k$,

$$\gamma \mathrm{d_X}(x, y) \leq \|f_k(x) - f_k(y)\|_{\mathscr{Y}} \leq \mathrm{d_X}(x, y).$$

By construction, $(\mathscr{F}_j)_{j=-1}^{\infty}$ is a finite-dimensional Schauder decomposition of its closed linear span, that we denote by $\mathscr{Z}$, and there are projections P_j from $\mathscr{Z}$ onto $\mathscr{F}_{-1} \oplus \cdots \oplus \mathscr{F}_j$ with kernel $\overline{\operatorname{span}}\,(\bigcup_{i=j+1}^{\infty} \mathscr{F}_i)$ with $\|P_j\| \leq 1+\eta$. Then we define, $f \colon \mathsf{X} \to \oplus_{i=-1}^{\infty}\mathscr{F}_i \subset \mathscr{Y}$ by

$$f(x) = \mathrm{d_X}(x_0, x)v \oplus 0 \oplus \cdots \oplus 0 \oplus \mu_x f_k(x) \oplus (1-\mu_x) f_{k+1}(x) \oplus 0 \oplus \cdots$$

if $d_X(x_0, x) \in [2^{k-1}, 2^k)$ where $\mu_x = \frac{2^k - d_X(x_0,x)}{2^{k-1}}$. The proof can be completed by performing computations almost identical to the ones in the proof of Theorem 2.4 and the details are left to the studious reader. The only difference is that the projections are not of norm 1 but only bounded above by $1 + \eta$. □

Even though the assumptions of Theorem 2.4 might seem quite restrictive, it can be applied successfully in a variety of situations. Observe first that for locally finite metric spaces local bi-Lipschitz representability coincides with the following notion already introduced in [55, page 1612] (under the terminology *finite representability*).

Definition 2.5 Let $\lambda \in [1, \infty)$. We will say that a metric space $(\mathsf{X}, d_{\mathsf{X}})$ is *finitely λ-bi-Lipschitz representable in* a metric space $(\mathsf{Y}, d_{\mathsf{Y}})$ if for every *finite subset* F in X there is a map $f_F \colon F \to \mathsf{Y}$ and a scaling factor $s > 0$ such that for all $x, y \in F$

$$s \cdot d_X(x, y) \leq d_Y(f_F(x), f_F(y)) \leq \lambda s \cdot d_X(x, y).$$

Therefore, if X is locally finite and satisfies assumption (i) in Theorem 2.4 then it automatically satisfies assumption (ii). Bourgain [17] showed that there exists $\lambda \geq 1$ such that the infinite binary tree is finitely λ-bi-Lipschitz representable in every non-superreflexive Banach space X. Barycentric gluing was originally introduced to show that this implies the bi-Lipschitz embeddability of the infinite binary tree.

Corollary 2.6 *The infinite binary tree bi-Lipschitzly embeds into every non-superreflexive Banach space.*

Since non-superreflexivity is a finite-codimensional hereditary property, meaning that every co-dimensional subspace of a non-superreflexive Banach space is non-superreflexive, Corollary 2.6, which originally appeared in [6], is an immediate corollary of Theorem 2.4 and the fact that the infinite binary tree is locally finite.

An immediate consequence of Fréchet's embedding [29] is that every metric space is finitely bi-Lipschitz representable in every Banach which contains the ℓ_∞^n's. It is an easy consequence of Maurey-Pisier theorem that the property of containing the ℓ_∞^n's is a finite co-dimensional hereditary property (see [11] for a proof) and we can deduce from Theorem 2.4 the main result of [10].

Corollary 2.7 *Every locally finite metric space bi-Lipschitzly embeds into every Banach space which contains the ℓ_∞^n's. In particular, every localy finite metric space admits a bi-Lipschitzly embedding into the reflexive and asymptotically-ℓ_2 Banach space $\left(\sum_{n=1}^{\infty} \ell_\infty^n\right)_2$.*

Since finite-dimensional subspaces of Hilbert space are simply Euclidean spaces and since every infinite-dimensional Banach space contains isomorphic copies of the ℓ_2^n's (with uniformly bounded Banach-Mazur distortion, in fact almost isometrically) by Dvoretzky's theorem [23], it follows that Hilbert space is finitely bi-Lipschitz representable in every infinite-dimensional Banach space. An immediate consequence of this observation and Theorem 2.4, gives the theorem below which first appeared in [74].

Corollary 2.8 *Let $\mathscr{Y}$ be an infinite-dimensional Banach space. Then any locally finite subset of Hilbert space admits a bi-Lipschitz embedding into $\mathscr{Y}$.*

More consequences of Theorem 2.2 to geometric group theory will be discussed in Sect. 5.2.

2.2 Barycentric gluing in the context of general embeddings

In the previous section we focused on bi-Lipschitz embeddings. Remarkably, barycentric gluing can be applied to more general notions of embeddings. At this point it is necessary to introduce some terminology. This terminology detour will come in very handy later in order to formulate the results of this section in a condensed fashion. The following definition will be needed in order to quantify the faithfulness of an embedding.

Definition 2.9 [(ρ,ω)-embeddings] Given non-decreasing functions $\rho,\omega\colon[0,\infty)\to[0,\infty)$ and a map $f\colon(\mathsf{X},\mathrm{d}_\mathsf{X})\to(\mathsf{Y},\mathsf{d}_\mathsf{Y})$ such that for all $x,y\in\mathsf{X}$,

$$\rho\big(\mathrm{d}_\mathsf{X}(x,y)\big)\le\mathrm{d}_\mathsf{Y}\big(f(x),f(y)\big)\le\omega\big(\mathrm{d}_\mathsf{X}(x,y)\big),\tag{26}$$

we will say that f is a $[\rho,\omega]$-*embedding from* X *into* Y.

We will mostly focus on coarse and uniform embeddings, and compression and expansion rate. In order to declutter statements from irrelevant (multiplicative) constants we will use a convenient equivalence relation on real functions that capture their large scale and small scale behaviors. For two functions $g,h\colon[0,\infty)\to\mathbb{R}$ we write $g\ll h$ if there exist $c_1,c_2>0$ and $c_3,c_4\in\mathbb{R}$ such that $g(t)\le c_1h(c_2t+c_3)+c_4$ for every $t\in[0,\infty)$. If $g\ll h$ and $h\ll g$ then we write $g\asymp h$. The relation $\asymp$ is easily seen to be an equivalence relation and a function shall be identified with its equivalence class in the sequel. For instance, if we say that X admits a $[t/\log(t),t]$-embedding into Y this will mean that there exist constants $c_1,c_2,c_3>0$ and $c_4,c_5,c_6,c_7\in\mathbb{R}$ and a map $f\colon(\mathsf{X},\mathrm{d}_\mathsf{X})\to(\mathsf{Y},\mathsf{d}_\mathsf{Y})$ such that for all $x,y\in\mathsf{X}$,

$$\frac{c_1\mathrm{d}_\mathsf{X}(x,y)+c_4}{\log(c_2\mathrm{d}_\mathsf{X}(x,y)+c_5)}+c_6\le\mathrm{d}_\mathsf{Y}(f(x),f(y))\le c_3\mathrm{d}_\mathsf{X}(x,y)+c_7.\tag{27}$$

Remark 2.10 Note that Lemma 1.1 and the convention to identify a function with its equivalence class implies that if a skeleton of $(\mathsf{X},\mathrm{d}_\mathsf{X})$ admits a $[\rho,\omega]$-embedding into Y then X also admits a $[\rho,\omega]$-embedding into Y.

The notion of bi-Lipschitz representability can naturally be extended in order to accommodate (ρ,ω)-embeddings.

Definition 2.11 [(ρ,ω)-representability] Let $\rho,\omega\colon[0,\infty)\to[0,\infty)$ be non-decreasing maps. We will say that a metric space $(\mathsf{X},\mathrm{d}_\mathsf{X})$ is *locally* (ρ,ω)-*representable in* a metric space $(\mathsf{Y},\mathsf{d}_\mathsf{Y})$ if there exists for every metric ball B in X a map $f_B\colon B\to\mathsf{Y}$ such that for all $x,y\in B$

$$\rho\big(\mathrm{d}_\mathsf{X}(x,y)\big)\le\mathrm{d}_\mathsf{Y}\big(f_B(x),f_B(y)\big)\le\omega\big(\mathrm{d}_\mathsf{X}(x,y)\big).\tag{28}$$

The proof of Theorem 2.2 can be slightly modified to obtain the following more general theorem.

Theorem 2.12 *Let* $p\in[1,\infty)$. *If* $(\mathsf{X},\mathrm{d}_\mathsf{X})$ *is locally* $[\rho,t]$-*representable in a Banach space* $(\mathscr{Y},\|\cdot\|)$, *then* X *admits a* $[\rho,t]$-*embedding into* $\ell_p(\mathscr{Y})$.

Proof We only emphasize the elements of the proof of Theorem 2.2 that need to be adjusted. Fix $x_0\in\mathsf{X}$ and without loss of generality assume that for every $k\in\mathbb{Z}$ there exists $f_k\colon B_\mathsf{X}(x_0,2^k)\to\mathscr{Y}$ such that $f_k(x_0)=0$ and for all $x,y\in B_\mathsf{X}(x_0,2^k)$,

$$\rho(\mathrm{d}_\mathsf{X}(x,y))-\alpha\le\|f_k(x)-f_k(y)\|\le\lambda\mathrm{d}_\mathsf{X}(x,y)+\beta,$$

for some $\alpha, \beta \geq 0$ and $\lambda > 0$. Note that this time one has for all $x \in B_{\mathsf{X}}(x_0, 2^k)$

$$\rho(\mathrm{d}_{\mathsf{X}}(x_0, x)) - \alpha \overset{(29)-\mathrm{low}}{\leq} \|f_k(x)\| \overset{(29)-\mathrm{up}}{\leq} \lambda \mathrm{d}_{\mathsf{X}}(x_0, x) + \beta.$$

The embedding $\hat{f}: X \to \ell_p(\mathscr{Y}) \oplus_p \mathbb{R}$ is still defined by $\hat{f}(x) = (f(x), \mathrm{d}_{\mathsf{X}}(x_0, x))$ where $f: \mathsf{X} \to \ell_p(\mathscr{Y})$ is again

$$f(x) = (0, \ldots, 0, \mu_x f_k(x), (1 - \mu_x) f_{k+1}(x), 0, \ldots)$$

with $\mathrm{d}_{\mathsf{X}}(x_0, x) \in [2^{k-1}, 2^k)$ and $\mu_x = \frac{2^k - \mathrm{d}_{\mathsf{X}}(x_0,x)}{2^{k-1}}$. In Theorem 2.2 the estimations of the Lipschitz constant of $\hat{f}$ only use (7)-up. Substituting (7)-up with (29)-up we can verify that in

Case 1: $\|\hat{f}(x) - \hat{f}(y)\| \leq (3\lambda + 1)\mathrm{d}_{\mathsf{X}}(x, y) + 2\beta$,
Case 2: $\|\hat{f}(x) - \hat{f}(y)\| \leq (5\lambda + 1)\mathrm{d}_{\mathsf{X}}(x, y) + 5\beta$,
Case 3: $\|\hat{f}(x) - \hat{f}(y)\| \leq (5\lambda + 1)\mathrm{d}_{\mathsf{X}}(x, y) + 3\beta$.

Regarding the lower bound computations (again in the case $p = 1$), in Case 1 we will now have

$$\begin{aligned} \|\hat{f}(x) - \hat{f}(y)\| &\geq \|f(x) - f(y)\| \overset{(10)}{\geq} \mu_y \|f_r(y)\| + (1 - \mu_y)\|f_{r+1}(y)\| \\ &\overset{(29)-\mathrm{low}}{\geq} \rho(\mathrm{d}_{\mathsf{X}}(x_0, y)) - \alpha \overset{(11)}{\geq} \rho\Big(\frac{2\mathrm{d}_{\mathsf{X}}(x, y)}{3}\Big) - \alpha. \end{aligned}$$

In Case 2 and Case 3, substituting (7)-low and (7)-up with (29)-low and (29)-up in the corresponding computations will give

$$\|\hat{f}(x) - \hat{f}(y)\| \geq \|f(x) - f(y)\| \geq \rho(\mathrm{d}_{\mathsf{X}}(x, y)) - \alpha - 2\lambda(\mathrm{d}_{\mathsf{X}}(x_0, y) - \mathrm{d}_{\mathsf{X}}(x_0, x)) - 2\beta,$$

and

$$\|\hat{f}(x) - \hat{f}(y)\| \geq \|f(x) - f(y)\| \geq \rho(\mathrm{d}_{\mathsf{X}}(x, y)) - \alpha - 4\lambda(\mathrm{d}_{\mathsf{X}}(x_0, y) - \mathrm{d}_{\mathsf{X}}(x_0, x)) - 4\beta,$$

respectively.

In any case, if $\mathrm{d}_{\mathsf{X}}(x_0, y) - \mathrm{d}_{\mathsf{X}}(x_0, x) \leq \frac{\rho(\mathrm{d}_{\mathsf{X}}(x,y))}{8\lambda}$ it will follow that

$$\|\hat{f}(x) - \hat{f}(y)\| \geq \frac{\rho(\mathrm{d}_{\mathsf{X}}(x, y))}{2} - \alpha - 4\beta,$$

and otherwise

$$\|\hat{f}(x) - \hat{f}(y)\| \geq |\mathrm{d}_{\mathsf{X}}(x_0, x) - \mathrm{d}_{\mathsf{X}}(x_0, y)| \geq \frac{\rho(\mathrm{d}_{\mathsf{X}}(x, y))}{8\lambda}.$$

Therefore when $p = 1$ the map $\hat{f}: X \to \ell_p(\mathscr{Y})$ satisfies for all $x, y \in X$

$$\frac{\rho(2\mathrm{d}_{\mathsf{X}}(x, y)/3)}{8\lambda} - \alpha - 4\beta \leq \|\hat{f}(x) - \hat{f}(y)\| \leq (5\lambda + 1)\mathrm{d}_{\mathsf{X}}(x, y) + 5\beta.$$

The observations regarding the validity of the proof for an arbitrary $p \in [1, \infty]$ are still valid and the conclusion follows.

□

Notice that in the proof of Theorem 2.12 the only property of ρ that we use is its monotonicity and thus Theorem 2.12 can be applied in many situations. As a typical example, we wish to spell out a particularly interesting application of Theorem 2.12 to coarse geometry.

Recall first that a family of metric spaces $(\mathsf{X}_i)_{i\in I}$, such that $\sup_{i\in I}\operatorname{diam}(\mathsf{X}_i)=\infty$, is said to be *equi-coarsely embeddable* into $(\mathsf{Y},\mathsf{d}_\mathsf{Y})$ if there exist $\rho,\omega\colon[0,\infty)\to[0,\infty)$ non-decreasing with $\lim_{t\to\infty}\rho(t)=\infty$ and for every $i\in I$ a map $f_i:\mathsf{X}_i\to\mathsf{Y}$ such that for all $x,y\in\mathsf{X}_i$

$$\rho\big(\mathrm{d}_{\mathsf{X}_i}(x,y)\big)\le \mathrm{d}_{\mathrm{Y}}\big(f_i(x),f_i(y)\big)\le\omega\big(\mathrm{d}_{\mathrm{X}\,i}(x,y)\big).$$

Note that by Remark 2.10, the proof of Theorem 2.12 will work just fine if merely assume that for some $x_0\in\mathsf{X}$ the sequence of dyadic balls $\{B_\mathsf{X}(x_0,2^n)\}_{n\ge1}$ admits an equi-coarse embedding into a Banach space $\mathscr{Y}$ with compression function ρ and expansion function $\omega(t)\asymp t$.

Corollary 2.13 *Let $(X,\mathrm{d}_\mathrm{X})$ be a metric space. Assume that for some $x_0\in X$ the sequence of dyadic balls $\{B_\mathsf{X}(x_0,2^n)\}_{n\ge1}$ admits an equi-coarse embedding into a Banach space $\mathscr{Y}$ with compression function ρ and expansion function $\omega(t)\asymp t$, then for any $p\in[1,\infty]$ X admits a coarse embedding into $\ell_p(\mathscr{Y})$ with compression function $\tilde\rho\asymp\rho$ and expansion function $\tilde\omega(t)\asymp t$.*

Remark 2.14 For metrically convex spaces one can always assume that $\omega(t)\asymp t$, and thus for coarse embeddings of metrically convex spaces we only need to focus on the compression function ρ.

The proof of Theorem 2.12 can also be adjusted to obtain.

Theorem 2.15 *Let $(X,\mathrm{d}_\mathrm{X})$ be a metric space and $(\mathscr{Y},\|\cdot\|)$ a Banach space. Assume that*

(i) *X is locally $[\rho,t]$-representable in every finite-codimensional subspace of $\mathscr{Y}$,*
(ii) *for every ball in X, its image under any of the local embeddings from (i) spans a finite-dimensional subspace.*

Then, X admits a $[\rho,t]$-embedding into $\mathscr{Y}$.

2.3 Ostrovskii's finite determinacy theorem

In Sect. 2.1 we saw that an arbitrary metric space admits a bi-Lipschitz embedding into ℓ_p whenever it is locally bi-Lipschitz representable in ℓ_p. For arbitrary Banach space targets, the same conclusion holds under the stronger representability requirements of Theorem 2.12. If we restrict ourselves to the much smaller class of locally finite metric spaces, the representability requirements of Theorem 2.12 reduce to finite bi-Lipschitz representability into *every* finite-codimensional subspace. In 2012, Ostrovskii showed that this stronger representability requirement is superfluous.

Theorem 2.16 *If a locally finite metric space $(X,\mathrm{d}_\mathrm{X})$ is finitely bi-Lipschitz representable in a Banach space $(\mathscr{Y},\|\cdot\|)$ then X admits a bi-Lipschitz embedding into $\mathscr{Y}$.*

Theorem 2.16 will be referred to as *Ostrovskii's finite determinacy theorem*, and we will say that the bi-Lipschitz embeddability of locally finite metric spaces into Banach spaces is *finitely determined*.

Recall that a Banach space $\mathscr{X}$ is said to be *finitely λ-representable* in another Banach space $\mathscr{Y}$ if for every finite-dimensional subspace $\mathscr{F}$ of $\mathscr{X}$ there exists a finite-dimensional subspace $\mathscr{G}$ of $\mathscr{Y}$ and a bounded linear map $T\colon\mathscr{F}\to\mathscr{G}$ such that $\dim(\mathscr{F})=\dim(\mathscr{G})$ and $\|T\|\cdot\|T^{-1}\|\le\lambda$. We then say that $\mathscr{X}$ is *(crudely) finitely representable* in $\mathscr{Y}$ if it is finitely λ-representable for some $\lambda\in[1,\infty)$. The heart of the proof of Ostrovskii's finite determinacy theorem is the following result (which is implicit in [75]).

Theorem 2.17 *If a Banach space $\mathscr{X}$ is finitely representable into a Banach space $\mathscr{Y}$ then every locally finite subset of $\mathscr{X}$ admits a bi-Lipschitz embedding into $\mathscr{Y}$.*

The assumption of Theorem 2.17 implies that every locally finite subset M of $\mathscr{X}$ is finitely λ-bi-Lipschitz representable in $\mathscr{Y}$, for some $\lambda \geq 1$, and thus an external application of barycentric gluing (Theorem 2.2 obviously applies here) gives that M bi-Lipschitzly embeds into $\ell_p(\mathscr{Y})$. The key new idea of Ostrovskii was to show that, for locally finite metric spaces, it is possible to implement the barycentric gluing internally via a delicate compactness argument. The proof of Theorem 2.17 has many technical intricacies and we will only sketch the main lines of Ostrovskii's argument.

Sketch of proof of Theorem 2.17 Let M be a locally finite subset of $\mathscr{X}$. By translating and rescaling if necessary we can assume without loss of generality that $0 \in \mathsf{M}$ and that $\|x\|_{\mathscr{X}} \geq 1$ for all $x \in \mathsf{M} \setminus \{0\}$. It follows easily from the assumption of Theorem 2.17 that for some $\gamma > 0$, there exists for every $k \geq 1$ a map $f_k \colon B_k \stackrel{\text{def}}{=} \{x \in \mathsf{M};\, \|x\|_{\mathscr{X}} \leq 2^k\} \to \mathscr{Y}$ such that $f_k(0) = 0$ and for all $x, y \in B_k$

$$\gamma\|x - y\|_{\mathscr{X}} \leq \|f_k(x) - f_k(y)\|_{\mathscr{Y}} \leq \|x - y\|_{\mathscr{X}}. \tag{29}$$

We can thus define the "natural" barycentric map $f(x) = \mu_x f_k(x) + (1 - \mu_x) f_{k+1}(x)$, if $2^{k-1} \leq \|x\|_{\mathscr{X}} \leq 2^k$ where $\mu_x = \frac{2^k - \|x\|_{\mathscr{X}}}{2^{k-1}}$. Essentially the same computations as in the proof of Theorem 2.2 will show that f is a Lipschitz map. Computing the co-Lipschitz constant of f is much more troublesome here due to the fact that we cannot simply "split" the contributions of the components of the barycentric map using projections (as we did in Theorem 2.2 or Theorem 2.4). This is the main obstacle which has to be overcome. Another obstacle is related to the modification of the barycentric map in order to compute the co-Lipschitz constant in the case where x and y lie in the same dyadic annulus. We also need to do this here and we could modify f similarly and define $\tilde{f}(x) := (f(x), \|x\|_{\mathscr{X}}) \in \mathscr{Y} \oplus \mathbb{R}$. This would be fine if $\mathscr{Y} \oplus \mathbb{R}$ were isomorphic to $\mathscr{Y}$. Unfortunately, the existence of Banach spaces that are not isomorphic to any of their hyperplanes is an annoyance that can be dealt with the following lemma (see [75] for a proof).

Lemma 2.18 *Let f be the barycentric map defined above. There exist a numerical constnat $c \in (0, \infty)$ and a map $\tilde{f} \colon M \to \mathscr{Y}$ such that*

1. *$\tilde{f}$ is Lipschitz,*
2. *for every $x, y \in M$, we have*

$$\|\tilde{f}(x) - \tilde{f}(y)\|_{\mathscr{Y}} \geq \max\Big\{\|f(x) - f(y)\|_{\mathscr{Y}} - \big|\|x\|_{\mathscr{X}} - \|y\|_{\mathscr{X}}\big|, c\big|\|x\|_{\mathscr{X}} - \|y\|_{\mathscr{X}}\big|\Big\} \tag{30}$$

The map $\tilde{f}$ is the sum in $\mathscr{Y}$ of f and $\tau(\|\cdot\|_{\mathscr{X}})$ where the auxiliary map $\tau \colon [0, \infty) \to \mathscr{Y}$ is defined as the following piecewise-linear function:

$$\tau(s) \stackrel{\text{def}}{=} \begin{cases} s \cdot y_1 & \text{if } 0 \leq s \leq 2, \\ 2y_1 + \sum_{i=1}^{k-1} 2^i y_{i+1} + (s - 2^k) y_{k+1} & \text{if } s \in [2^k, 2^{k+1}] \text{ for some } k \geq 1, \end{cases}$$

and where $\{y_k\}_{k\in\mathbb{N}}$ is a sequence of unit vectors in $\mathscr{Y}$ constructed inductively with the help of Mazur's lemma and such that $\operatorname{dist}(y_k, \operatorname{span}(\{f(x) \colon \|x\|_{\mathscr{X}} \leq 2^{k+1}\} \cup \{y_i \colon 1 \leq i \leq k-1\})) = 1$.

Distinguishing again a couple of cases, one can estimate the co-Lipschitz constant of the modification $\tilde{f}$ of the barycentric map and show that $\tilde{f}$ is $\frac{c\gamma}{10}$-co-Lipschitz in Case 1 (points separated by at least one dyadic annulus) and Case 2 (points in two consecutive annuli).

Case 3 $2^{k-1} \leq \|x\|_{\mathscr{X}} < \|y\|_{\mathscr{X}} < 2^k$.

This configuration where the pair of points belongs to the same dyadic annulus is, as always, the most delicate. In this case, we do not have enough information on the f_k's to estimate the co-Lipschitz constant of $\tilde{f}$. Ostrovskii's solution to this substantial hurdle was to use auxiliary accumulation points of the local embeddings in order to perform the lower bound computations. It is a good time for the reader to convince himself that the arguments above are obviously still valid if in the definition of the barycentric map f the sequence $\{f_k\}_{k=1}^{\infty}$ was replaced by an arbitrary subsequence $\{f_{n_k}\}_{k=1}^{\infty}$; f_r satisfies (29) for all $r \geq k$ afterall. After a series of careful extractions, Ostrovskii showed that we could have assumed without loss of generality that the sequence of local embeddings satisfies the technical conditions in the following crucial lemma.

Lemma 2.19 *There is an isometric embedding of $\mathscr{Y}$ into a Banach space $\mathscr{Z}^*$ and a collection of vectors $\{f_\omega(x)\}_{x\in\mathscr{X}} \in \mathscr{Z}^*$ with the following properties:*

1. *for every $x, y \in M$ there exists $h_{x,y} \in S_{\mathscr{Z}}$ such that*

$$[f_\omega(x) - f_\omega(y)](h_{x,y}) \geq \frac{7}{8}\|f_\omega(x) - f_\omega(y)\|_{\mathscr{Z}^*}. \tag{31}$$

and for all $k \geq 1$ and $x \neq y \in B_k$ such that $f_\omega(x) \neq f_\omega(y)$, and any $r \in \{k, k+1\}$

$$\Big|[f_r(x) - f_r(y) - (f_\omega(x) - f_\omega(y))](h_{x,y})\Big| \leq \frac{1}{8}\|f_\omega(x) - f_\omega(y)\|_{\mathscr{Z}^*}. \tag{32}$$

2. *for all $k \geq 1$ and $x \neq y \in B_k$ there exists $g_k^{x,y} \in S_{\mathscr{Z}}$ such that*

$$[f_k(x) - f_k(y) - (f_\omega(x) - f_\omega(y))]\left(g_k^{x,y}\right) \geq \frac{7}{8}\|f_k(x) - f_k(y) - (f_\omega(x) - f_\omega(y))\|_{\mathscr{Z}^*}. \tag{33}$$

and

$$\Big|[f_{k+1}(x) - f_{k+1}(y) - (f_\omega(x) - f_\omega(y))]\left(g_k^{x,y}\right)\Big| \leq \frac{\gamma}{128}\|x - y\|_{\mathscr{Y}}. \tag{34}$$

Lemma 2.19 is decisive in Ostrovskii's implementation of barycentric gluing in the general setting, as it provides a powerful substitute for the ad'hoc Schauder decompositions that were available in the previous implementations of barycentric gluing. The fact that $\mathscr{Y}$ can be isometrically embedded into a (separable) dual space, allowed Ostrovskii to use weak$*$-accumulation points and successive extractions to obtain the lower bounds estimates (31)–(34) above, which can be thought of as substitutes for the projections that are used to compute lower bound estimates when Schauder decompositions are available. Assuming Lemma 2.19 for the moment, we complete the estimation of the co-Lipschitz constant. First observe that if $\|y\|_{\mathscr{X}} - \|x\|_{\mathscr{X}} \geq \frac{\gamma}{64}\|x - y\|_{\mathscr{X}}$ then

$$\|\tilde{f}(x) - \tilde{f}(y)\|_{\mathscr{Y}} \geq \frac{c\gamma}{64}\|x - y\|_{\mathscr{X}}.$$

We claim that it is sufficient to show that if $\|y\|_{\mathscr{X}} - \|x\|_{\mathscr{X}} \leq \frac{\gamma}{64}\|x - y\|_{\mathscr{X}}$ then

$$\|f(x) - f(y)\|_{\mathscr{Y}} \geq \frac{3\gamma}{128}\|x - y\|_{\mathscr{X}}. \tag{35}$$

Indeed, assuming that our claim is true then it follows from (30) that

$$\begin{aligned}\|\tilde{f}(x)-\tilde{f}(y)\|_{\mathscr{Y}} &\geq \frac{3\gamma}{128}\|x-y\|_{\mathscr{X}}-(\|y\|_{\mathscr{X}}-\|x\|_{\mathscr{X}})\\ &\geq \frac{\gamma}{64}\|x-y\|_{\mathscr{X}}.\end{aligned}$$

So from now on we assume that $\|y\|_{\mathscr{X}}-\|x\|_{\mathscr{X}}\leq \frac{\gamma}{64}\|x-y\|_{\mathscr{X}}$. We need to distinguish a couple more subcases.

Subcase 3.a. If $\|f_\omega(x)-f_\omega(y)\|_{\mathscr{X}^*}\geq \frac{\gamma}{8}\|x-y\|_{\mathscr{X}}$, then we rewrite

$$f(x)-f(y)\stackrel{\text{def}}{=}\mu_x f_k(x)+(1-\mu_x)f_{k+1}(x)-(\mu_y f_k(y)+(1-\mu_y)f_{k+1}(y))$$

as

$$\begin{aligned}f(x)-f(y)=f_\omega(x)-f_\omega(y)&+\mu_x\big(f_k(x)-f_k(y)-(f_\omega(x)-f_\omega(y))\big)\\ &+(1-\mu_x)\big(f_{k+1}(x)-f_{k+1}(y)-(f_\omega(x)-f_\omega(y))\big)\\ &+(\mu_x-\mu_y)f_k(y)+(\mu_y-\mu_x)f_{k+1}(y).\end{aligned}$$

Recalling that we identify $f(x)-f(y)\in\mathscr{Y}$ with its isometric image in $\mathscr{X}^*$ we have $\|f(x)-f(y)\|_{\mathscr{Y}}=\|f(x)-f(y)\|_{\mathscr{X}^*}$ and thus since $h_{x,y}\in S_{\mathscr{X}}$

$$\begin{aligned}\|f(x)-f(y)\|_{\mathscr{Y}} &\geq [f(x)-f(y)](h_{x,y})\\ &\geq [f_\omega(x)-f_\omega(y)](h_{x,y})+\mu_x[f_k(x)-f_k(y)-(f_\omega(x)-f_\omega(y))](h_{x,y})\\ &\quad+(1-\mu_x)[f_{k+1}(x)-f_{k+1}(y)-(f_\omega(x)-f_\omega(y))](h_{x,y})\\ &\quad+(\mu_x-\mu_y)f_k(y)(h_{x,y})+(\mu_y-\mu_x)f_{k+1}(y)(h_{x,y})\\ \stackrel{(31),(32),(29)}{\geq} &\frac{7}{8}\|f_\omega(x)-f_\omega(y)\|_{\mathscr{X}^*}-\mu_x\frac{1}{8}\|f_\omega(x)-f_\omega(y)\|_{\mathscr{X}^*}\\ &\quad-(1-\mu_x)\frac{1}{8}\|f_\omega(x)-f_\omega(y)\|_{\mathscr{X}^*}-2|\mu_x-\mu_y|\|y\|_{\mathscr{X}}\\ &\geq \frac{3}{4}\|f_\omega(x)-f_\omega(y)\|_{\mathscr{X}^*}-4(\|y\|_{\mathscr{X}}-\|x\|_{\mathscr{X}})\\ &\geq \frac{3\gamma}{32}\|x-y\|_{\mathscr{X}}-\frac{\gamma}{16}\|x-y\|_{\mathscr{X}}\\ &\geq \frac{\gamma}{32}\|x-y\|_{\mathscr{X}}.\end{aligned}$$

Subcase 3.b. Assume that $\|f_\omega(x)-f_\omega(y)\|_{\mathscr{X}^*}<\frac{\gamma}{8}\|x-y\|_{\mathscr{X}}$.

Subcase 3.b.i. If $\mu_x \| f_k(x) - f_k(y) - (f_\omega(x) - f_\omega(y)) \|_{\mathscr{Z}^*} \geq \frac{\gamma}{4} \|x - y\|_{\mathscr{X}}$, then

$$\begin{aligned}
\|f(x) - f(y)\|_{\mathscr{Y}} &\geq [f(x) - f(y)](g_k^{x,y}) \\
&\geq \mu_x [f_k(x) - f_k(y) - (f_\omega(x) - f_\omega(y))](g_k^{x,y}) + [f_\omega(x) - f_\omega(y)](g_k^{x,y}) \\
&\quad + (1 - \mu_x)[f_{k+1}(x) - f_{k+1}(y) - (f_\omega(x) - f_\omega(y))](g_k^{x,y}) \\
&\quad + (\mu_x - \mu_y) f_k(y)(g_k^{x,y}) + (\mu_y - \mu_x) f_{k+1}(y)(g_k^{x,y}) \\
(33), (34), (29) &\geq \mu_x \frac{7}{8} \|f_k(x) - f_k(y) - (f_\omega(x) - f_\omega(y))\|_{\mathscr{Z}^*} - \|f_\omega(x) - f_\omega(y)\|_{\mathscr{Z}^*} \\
&\quad - (1 - \mu_x) \frac{\gamma}{64} \|x - y\|_{\mathscr{X}} - 2|\mu_x - \mu_y| \cdot \|y\|_{\mathscr{X}} \\
&\geq \frac{7\gamma}{32} \|x - y\|_{\mathscr{X}} - \frac{\gamma}{8} \|x - y\|_{\mathscr{X}} - \frac{\gamma}{128} \|x - y\|_{\mathscr{X}} - \frac{\gamma}{16} \|x - y\|_{\mathscr{X}} \\
&\geq \frac{3\gamma}{128} \|x - y\|_{\mathscr{X}}.
\end{aligned}$$

Subcase 3.b.ii. Suppose that $\mu_x \| f_k(x) - f_k(y) - (f_\omega(x) - f_\omega(y)) \|_{\mathscr{Z}^*} < \frac{\gamma}{4} \|x - y\|_{\mathscr{X}}$.
Remark that for $n = k$ or $k + 1$, one has

$$\begin{aligned}
\|f_n(x) - f_n(y) - (f_\omega(x) - f_\omega(y))\|_{\mathscr{Z}^*} &\geq \|f_n(x) - f_n(y)\|_{\mathscr{Y}} - \|f_\omega(x) - f_\omega(y)\|_{\mathscr{Z}^*} \\
&\geq \gamma \|x - y\|_{\mathscr{X}} - \frac{\gamma}{8} \|x - y\|_{\mathscr{X}} \\
&\geq \frac{7\gamma}{8} \|x - y\|_{\mathscr{X}},
\end{aligned}$$

and hence $\mu_x < \frac{2}{7}$ and $1 - \mu_x \geq \frac{5}{7}$. Consequently,

$$\begin{aligned}
\|f(x) - f(y)\|_{\mathscr{Y}} &\geq (1 - \mu_x) \|f_{k+1}(x) - f_{k+1}(y) - (f_\omega(x) - f_\omega(y))\|_{\mathscr{Z}^*} \\
&\quad - \|f_\omega(x) - f_\omega(y)\|_{\mathscr{Z}^*} - \mu_x \|f_k(x) - f_k(y) - (f_\omega(x) - f_\omega(y))\|_{\mathscr{Z}^*} \\
&\quad - |\mu_x - \mu_y| \|f_k(y)\|_{\mathscr{Y}} - |\mu_y - \mu_x| \|f_{k+1}(y)\|_{\mathscr{Y}} \\
&\geq \frac{5}{7} \cdot \frac{7\gamma}{8} \|x - y\|_{\mathscr{X}} - \frac{\gamma}{8} \|x - y\|_{\mathscr{X}} - \frac{\gamma}{4} \|x - y\|_{\mathscr{X}} - \frac{\gamma}{16} \|x - y\|_{\mathscr{X}} \\
&\geq \frac{3\gamma}{16} \|x - y\|_{\mathscr{X}}.
\end{aligned}$$

□

It is well known that any ultrapower of a Banach space $\mathscr{X}$ is finitely representable in the space $\mathscr{X}$ itself.

Corollary 2.20 *Let $\mathscr{X}$ be a Banach space, and $\mathcal{U}$ be any non-principal ultrafilter, then every locally finite subset of $\mathscr{X}^{\mathcal{U}}$ admits a bi-Lipschitz embedding into $\mathscr{X}$.*

Theorem 2.16 now follows easily from Corollary 2.20.

Proof of Theorem 2.16 Let X be locally finite, and $\mathscr{Y}$ be a Banach space. Fix $x_0 \in \mathsf{X}$ and assume that X is finitely λ-bi-Lipschitz representable in Y. Then for every $k \in \mathbb{N}$ there exists $f_k \colon B(x_0, 2^k) \to \mathscr{Y}$ such that $f_k(x_0) = 0$ and for every $x, y \in B(x_0, 2^k)$,

$$d_{\mathsf{X}}(x, y) \leq \|f_k(x) - f_k(y)\|_{\mathscr{Y}} \leq \lambda d_{\mathsf{X}}(x, y).$$

For all $x \in \mathsf{X}$, let $k(x)$ the smallest integer such that $x \in B(x_0, 2^{k(x)})$. The sequence $(f_k(x))_{k \geq k(x)}$ is bounded and we define

$$\begin{aligned}
f \colon \mathsf{X} &\to \mathscr{Y}^{\mathcal{U}} \\
x &\mapsto (0, \ldots, 0, f_{k(x)}(x), f_{k(x)+1}(x), \ldots), \text{ if } d_{\mathsf{X}}(x, x_0) \leq 2^{k(x)}.
\end{aligned}$$

By definition of the norm of the ultrapower it is clear that for every $x, y \in \mathsf{X}$,

$$\mathrm{d}_{\mathsf{X}}(x, y) \leq \|f(x) - f(y)\|_{\mathscr{Y}^{\mathcal{U}}} \leq \lambda \mathrm{d}_{\mathsf{X}}(x, y).$$

Since, local finiteness is preserved under bi-Lipschitz embeddings, $f(\mathsf{X})$ is a locally finite subset of $Y^{\mathcal{U}}$. The composition of a bi-Lipschitz embedding with another bi-Lipschitz embedding being a bi-Lipschitz embedding, the conclusion follows from Corollary 2.20. □

While the statement of Theorem 2.16 is conceptually very appealing, we feel that Theorem 2.17 and Corollary 2.20 should be more useful in practice.

It is clear from the proof above that the embeddability of locally finite metric spaces is finitely determined for other notions of embeddability, as long as:

1. the composition with a bi-Lipschitz embedding does not change the nature of the embedding,
2. the induced notion of representability requires a uniform control on the expansion and compression moduli,
3. the embedding preserves local finiteness.

Corollary 2.21 *The coarse embeddability of a locally finite metric space into an arbitrary Banach space is finitely determined.*

3 Simultaneously coarse and uniform embeddings of proper and stable metrics

In the previous section we saw that every locally finite metric space (and thus every metric space with bounded geometry) admits a bi-Lipschitz embedding into every Banach space that contains the ℓ_∞^n's. It is natural to investigate if similar general embedding results hold for larger classes of metric spaces. In this section, which is of expository nature, we discuss satisfactory answers that have been given for proper metrics in [7,11] and for stable metrics in [46].

3.1 Almost bi-Lipschitz embeddability of proper metric spaces

A metric space is *proper* if all its closed balls are compact. It is plain that locally finite metrics are proper metrics and that the class of proper metrics strictly contains the class of locally finite metrics. Observing that any skeleton of a proper metric space is locally finite and recalling that every skeleton is coarse-Lipschitz equivalent to the original space, it follows that every embedding result in the previous section pertaining to locally finite metrics extends to an embedding result for proper metrics. We just single out one such example and leave other similar easy derivations to the reader.

Corollary 3.1 *Every proper metric space coarse bi-Lipschitzly embeds into every Banach space with trivial cotype. In particular, every proper metric space admits a coarse bi-Lipschitz embedding into the reflexive and asymptotically-ℓ_2 Banach space $\left(\sum_{n=1}^{\infty} \ell_\infty^n\right)_2$.*

An interesting problem for proper metric spaces is to preserve the small-scale structure (via uniform embeddings) or, more ambitiously, to preserve simultaneously the small-scale and the large-scale structure (via embeddings that are simultaneously uniform and coarse). Consider first the case of compact metric spaces. A compact metric space being bounded it does not make much sense to discuss its large-scale geometry. However, as far as its small-scale geometry is concerned, it was known since [16, Proposition 7.18] that for every compact metric space $(\mathsf{X}, d_{\mathsf{X}})$ there exist a constant $C > 0$ and a map $f : \mathsf{X} \to \left(\sum_{n=1}^{\infty} \ell_\infty^n\right)_2$ such that for all $x, y \in \mathsf{X}$,

$$d_{\mathsf{X}}(x, y) \leq \|f(x) - f(y)\| \leq C\sqrt{|\log d_{\mathsf{X}}(x, y)|}\, d_{\mathsf{X}}(x, y). \tag{36}$$

The original net-argument from [16, Proposition 7.18] induces the loss of a logarithmic factor, but with a bit more care it can be improved. The proof that we give below uses a slightly different embedding.

Proposition 3.2 *Let (X, d_{X}) be a compact metric space. For every continuous, decreasing function $\mu : (0, \operatorname{diam}(X)] \to [0, +\infty)$ such that $\mu(\operatorname{diam}(X)) = 0$ and $\lim_{t\downarrow 0} \mu(t) = +\infty$, there exists an embedding $f : X \to \left(\sum_{n=1}^{\infty} \ell_\infty^n\right)_2$ such that for all $x, y \in X$,*

$$\frac{1}{2}\frac{d_{\mathsf{X}}(x, y)}{\mu(d_{\mathsf{X}}(x, y))} \leq \|f(x) - f(y)\| \leq \frac{\pi}{\sqrt{6}} d_{\mathsf{X}}(x, y).$$

Proof Assume that the diameter of X is D. And let $\mu : (0, D] \to [0, +\infty)$ be a continuous, decreasing function such that $\mu(D) = 0$ and $\lim_{t\to 0} \mu(t) = +\infty$. Then the map μ has an inverse denoted $\mu^{-1} : [0, +\infty) \to (0, D]$ that is decreasing, with $\lim_{t\to+\infty} \mu^{-1}(t) = 0$. Fix x_0 in X, and denote $\sigma := \mu^{-1} : \mathbb{Z}^- \to (0, D]$. For any $k \in \mathbb{Z}^+$, let R_k be a maximal $\frac{\sigma(k)}{16}$-net of X. Observe that R_k is finite by compactness of X. Define the following 1-Lipschitz maps:

$$\begin{aligned} \varphi_k : \mathsf{X} &\to \ell_\infty(R_k) \\ x &\mapsto \left(d_{\mathsf{X}}(x, y) - d_{\mathsf{X}}(y, x_0)\right)_{y\in R_k}. \end{aligned}$$

The embedding is given by

$$\begin{aligned} f : \mathsf{X} &\to \left(\sum_{k=1}^{\infty} \ell_\infty(R_k)\right)_2 \\ x &\mapsto \sum_{k=1}^{\infty} \frac{\varphi_k(x)}{k}. \end{aligned}$$

It is clear that f is C-Lipschitz with $C = \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{\frac{1}{2}} = \frac{\pi}{\sqrt{6}}$.

Let $x \neq y \in \mathsf{X}$, then there exists $l \in \mathbb{Z}^+$ such that $\sigma(l+1) \leq d_{\mathsf{X}}(x, y) < \sigma(l)$, or equivalently $l + 1 \leq \mu(d_{\mathsf{X}}(x, y)) < l$. Since R_{l+1} is a $\frac{\sigma(l+1)}{4}$-net in X we can find $r_y \in R_{l+1}$ such that $d_{\mathsf{X}}(r_y, y) < \frac{\sigma(l+1)}{4} \leq \frac{d_{\mathsf{X}}(x,y)}{4}$. Therefore

$$\begin{aligned} \|f(x) - f(y)\| &\geq \frac{\|\varphi_{l+1}(x) - \varphi_{l+1}(y)\|_\infty}{l+1} \geq \frac{\sup_{r\in R_{l+1}} \left|d_{\mathsf{X}}(x, r) - d_{\mathsf{X}}(y, r)\right|}{l+1} \\ &\geq \frac{|d_{\mathsf{X}}(x, r_y) - d_{\mathsf{X}}(y, r_y)|}{l+1} \geq \frac{d_{\mathsf{X}}(x, y) - 2d_{\mathsf{X}}(y, r_y)}{l+1} \geq \frac{1}{2}\frac{d_{\mathsf{X}}(x, y)}{l+1}, \end{aligned}$$

and

$$\|f(x) - f(y)\| \geq \frac{1}{2}\frac{d_{\mathsf{X}}(x, y)}{\mu(d_{\mathsf{X}}(x, y))}.$$

□

Proposition 3.2 says that we can construct uniform embeddings which are arbitrarily close to bi-Lipschitz embeddings. Proposition 3.2 is optimal since there exists a compact metric space which does not admit a bi-Lipschitz embedding into any Banach space with the Radon-Nikodým property, and in particular into any reflexive Banach space (see [11] for more details).

In [7], the original net-argument from [16, Proposition 7.18] was combined with barycentric gluing to show that every proper metric space admits a $[t/\log^2 t, t]$-embedding into any Banach space with trivial cotype. This embedding which is simultaneously uniform and coarse was quite satisfactory but was suboptimal in two ways. On the one hand, it is not a coarse bi-Lipschitz embedding, but we know that every proper space admits a coarse bi-Lipschitz embedding into any Banach space which contains the ℓ_∞^n's. On the other hand, the compact balls only embed with suboptimal faithfulness in light of either (36) or Proposition 3.2. This suboptimality was due to the fact that the parameters of the nets and the radii of the dyadic balls used in the barycentric gluing were correlated in [7]. By "decorrelating" the construction, a much tighter embedding result was proved in [11] and the following notion naturally arose.

Definition 3.3 We say that (X, d_X) is *almost bi-Lipschitzly embeddable* into $(\mathsf{Y}, \mathsf{d_Y})$ if there exist a scaling factor $r \in (0, \infty)$ and a constant $D \in [1, \infty)$, such that for any continuous function $\varphi\colon [0, +\infty) \to [0, 1)$ satisfying $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$, there exists a map $f_\varphi\colon \mathsf{X} \to \mathsf{Y}$ such that for all $x, y \in \mathsf{X}$,

$$\varphi(d_X(x, y)) \cdot r \cdot d_X(x, y) \leq d_Y(f_\varphi(x), f_\varphi(y)) \leq D \cdot r \cdot d_X(x, y).$$

Proposition 3.2 essentially says that a compact metric space almost bi-Lipschiztly embeds into $\left(\sum_{n=1}^\infty \ell_\infty^n\right)_2$. It is clear from Definition 3.3 that if X admits a bi-Lipschitz embedding into Y, then X almost bi-Lipschitzly embeds into Y. Note also that if X almost bi-Lipschitzly embeds into Y, then X admits an embedding into Y that is simultaneously coarse and uniform. Moreover, if X almost bi-Lipschitzly embeds into Y, it is easy to see, by taking φ appropriately, that X admits a coarse bi-Lipschitz embedding into Y. The following theorem is the main result from [11].

Theorem 3.4 *Let $p \in [1, \infty]$. If ℓ_p is finitely representable in a Banach space $\mathscr{Y}$ then any proper subset of $L_p[0, 1]$ is almost bi-Lipschitzly embeddable into $\mathscr{Y}$. In particular, any proper subset of $L_p[0, 1]$ is almost bi-Lipschitzly embeddable into ℓ_p.*

The following corollaries can be easily derived from Theorem 3.4.

Corollary 3.5 *Let $\mathscr{Y}$ be a Banach space which contains the ℓ_∞^n's, and X be a proper metric space, then X almost bi-Lipschitzly embeds into $\mathscr{Y}$. In particular, every proper metric space almost bi-Lipschitzly embeds into the reflexive and asymptotically-ℓ_2 Banach space $\left(\sum_{n=1}^\infty \ell_\infty^n\right)_2$.*

Corollary 3.6 *Let $\mathscr{Y}$ be an infinite-dimensional Banach space. Then any proper subset of Hilbert space almost bi-Lipschitzly embeds into $\mathscr{Y}$.*

Theorem 3.4, Corollaries 3.5, 3.6 are tight in the sense that "almost bi-Lipschitz embeddability" cannot be upgraded to "bi-Lipschitz embeddability" (see [11] for more details).

3.2 Nearly isometric embeddability of stable metric spaces

Comparing Corollaries 2.7 and 3.5 we see that it is always possible to embed locally finite metric spaces into any Banach space that contains the ℓ_∞^n's, and also proper metrics albeit with a lesser degree of faithfulness. It is natural to wonder whether this trend continues if we enlarge once more the class of metrics considered. In order to relax the properness condition we can consider stable metrics. Before discussing embedding results for stable metrics, we make a little detour to introduce the fascinating notion of metric stability.

3.2.1 Stable metrics

Stable norms[11] were introduced, originally for separable Banach spaces, by Krivine and Maurey [52]. Whether every infinite-dimensional Banach space contains an isomorphic copy of c_0 or ℓ_p, for some $p \in [1, \infty)$, was a long-standing open problem in Banach space theory. In early 1970s, Tsirelson [85] built an example of a space that does not have this property. However, Krivine and Maurey showed that for stable Banach spaces such Tsirelson-type construction cannot happen since every stable Banach space contains an isomorphic copy (actually almost isometric) of ℓ_p for some $p \in [1, \infty)$. Krivine-Maurey result is one of many evidences that stable Banach spaces have a much more regular structure than arbitrary Banach spaces. We point the reader to [38] for an extensive account on the theory of stable Banach spaces. It seems that the natural extension of the notion of stability to arbitrary metric spaces was first studied by Garling [30].

Definition 3.7 A metric space $(\mathsf{X}, \mathrm{d}_\mathsf{X})$ is said to be *stable* if for any two bounded sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$, and any two non-principal ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$, the following equality holds:

$$\lim_{m\to\mathcal{U}} \lim_{n\to\mathcal{V}} \mathrm{d}_\mathsf{X}(x_m, y_n) = \lim_{n\to\mathcal{V}} \lim_{m\to\mathcal{U}} \mathrm{d}_\mathsf{X}(x_m, y_n). \tag{37}$$

A Banach space is stable if its canonical metric induced by its norm is stable. Stability is an isometric property and is inherited by subsets. Despite condition (37) seems quite restrictive the class of stable metrics is rather rich and contains lots of interesting metrics.

Example 3.8 Proper metric spaces, and in particular finite, compact, bounded geometry, or locally finite metric spaces, are stable.

For all the metric spaces in Example 3.8 the closed balls are either finite or compact, and if $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty$ are bounded sequences, then given any non-principal ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$, there exist $x, y \in \mathsf{X}$ such that $\lim_{n\to\mathcal{U}} x_n = x$ and $\lim_{n\to\mathcal{V}} y_n = y$. By continuity properties of the distance function we thus have,

$$\lim_{m\to\mathcal{U}} \lim_{n\to\mathcal{V}} \mathrm{d}_\mathsf{X}(x_m, y_n) = \mathrm{d}_\mathsf{X}(x, y) = \lim_{n\to\mathcal{V}} \lim_{m\to\mathcal{U}} \mathrm{d}_\mathsf{X}(x_m, y_n).$$

In particular, finite-dimensional Banach spaces, finitely generated groups equipped with their canonical word metric, compactly generated groups equipped with their canonical proper metric are stable metric spaces. It is a classical fact that if a Banach space is proper then it must be finite-dimensional. We now describe infinite-dimensional Banach spaces that are stable since it will provide us with examples of stable metric spaces which do not belong to the list of (trivially) stable spaces from Example 3.8.

[11] A norm is stable if for any two bounded sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$, $\lim_m \lim_n \|x_m + y_n\| = \lim_n \lim_m \|x_m + y_n\|$, whenever the limits exist.

Example 3.9 Hilbert space is stable.

It is fairly easy to show that Hilbert space is stable using the classical representation of Hilbert norm in terms of the scalar product. Assume that $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ are two bounded sequences in Hilbert space. For any two ultrafilters $\mathcal{U}$ and $\mathcal{V}$, since Hilbert space is reflexive, the two sequences are weakly convergent along $\mathcal{U}$ and $\mathcal{V}$ to, say, x and y, respectively. Denote $a := \lim_{n\to\mathcal{U}} \|x_n\|_2^2$ and $b := \lim_{n\to\mathcal{V}} \|y_n\|_2^2$. Then,

$$\begin{aligned}\lim_{m\to\mathcal{U}}\lim_{n\to\mathcal{V}} \|x_m - y_n\|_2^2 &= \lim_{m\to\mathcal{U}}\lim_{n\to\mathcal{V}} \left(\|x_m\|_2^2 + \|y_n\|_2^2 - 2\langle x_m, y_n\rangle\right)\\ &= \lim_{m\to\mathcal{U}} \|x_m\|_2^2 + \lim_{n\to\mathcal{V}} \|y_n\|_2^2 - 2\langle x, y\rangle\\ &= a + b - 2\langle x, y\rangle,\end{aligned}$$

where in the second equality we used the definition of weak convergence. Proceeding in a similar manner we can show that

$$\lim_{n\to\mathcal{V}}\lim_{m\to\mathcal{U}} \|x_m - y_n\|_H^2 = a + b - 2\langle x, y\rangle\,,$$

and this completes the proof.

Example 3.10 The Banach space ℓ_p, for $p \in [1, \infty)$, is stable.

There are several different approaches to prove the statement in Example 3.10. It is an elementary, but somewhat tedious, exercise to show that for $p \in [1, \infty)$, the ℓ_p-sum of stable Banach spaces is stable, and one can then argue that ℓ_p is by definition the ℓ_p-sum of countably many copies of $(\mathbb{R}, |\cdot|)$, which is a proper, and hence stable, space. This argument builds on the following classical lemma.

Lemma 3.11 *Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence in ℓ_p, and $\mathcal{U}$ a non-principal ultrafilter on $\mathbb{N}$. Suppose that $\lim_{n\to\mathcal{U}} e_i^*(x_n) = 0$ for all $i \in \mathbb{N}$, i.e. $\{x_n\}_{n=1}^{\infty}$ converges coordinatewise to 0 with respect to $\mathcal{U}$. Then, for every $z \in \ell_p$,*

$$\lim_{n\to\mathcal{U}} \|z + x_n\|_p^p = \|z\|_p^p + \lim_{n\to\mathcal{U}} \|x_n\|_p^p. \tag{38}$$

The conclusion of the lemma clearly holds if z and the x_n's have disjoint supports. Since $\{x_n\}_{n=1}^{\infty}$ converges coordinate-wise to 0 by reaching far out in the sequence one can select x_n such that the essential contributions to the norm of x_n and of z are supported on essentially disjoint supports, and a classical approximation and truncation argument gives the conclusion.

For every bounded sequence $\{y_n\}_{n=1}^{\infty}$ and every non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, there exist $y \in \ell_p$ and $\mu \geq 0$ so that if $z \in \ell_p$, then $\lim_{n\to\mathcal{U}} \|z - y_n\|_p^p = \|z - y\|^p + \mu^p$. Indeed, there exists $y \in \ell_p$ such that for all $i \in \mathbb{N}$ $\lim_{n\to\mathcal{U}} e_i^*(y_n) = e_i^*(y)$, and the conclusion follows from Lemma 3.11 with $\{x_n\}_{n=1}^{\infty} = \{y - y_n\}_{n=1}^{\infty}$, and $\mu = \lim_{n,\mathcal{U}} \|y - y_n\|$. We are now in position to prove that ℓ_p is stable. For a pair of bounded sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and a pair of non-principal ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$, there exist $x, y \in \ell_p$, $\mu, \nu \geq 0$ such that for all $z \in \ell_p$ the following equalities hold:

$$\lim_{n\to\mathcal{U}} \|z - x_n\|_p^p = \|z - x\|_p^p + \mu^p$$

and

$$\lim_{n\to\mathcal{V}} \|z - y_n\|_p^p = \|z - y\|_p^p + \nu^p.$$

It follows that

$$\lim_{m\to\mathcal{V}}\lim_{n\to\mathcal{U}}\|x_m-y_n\|_p^p=\|x-y\|_p^p+\mu^p+\nu^p=\lim_{n\to\mathcal{U}}\lim_{m\to\mathcal{V}}\|x_m-y_n\|_p^p.$$

Example 3.12 The Banach space c_0, and more generally any Banach space containing an isomorphic copy of c_0, is not stable.

The fact that c_0 is not stable can easily be checked directly by considering the sequences $x_n=-e_n$ and $y_n=\sum_{i=1}^n e_i$, $n\geq 1$, where $(e_n)_{n=1}^\infty$ denotes the canonical basis of c_0. Indeed, $\lim_{m\to\mathcal{V}}\lim_{n\to\mathcal{U}}\|x_n-y_m\|_\infty=2$ while $\lim_{n\to\mathcal{U}}\lim_{m\to\mathcal{V}}\|x_n-y_m\|_\infty=1$. In order to show the second part of Example 3.12 we need to invoke James' c_0-distortion theorem [44] which says that if a Banach space $\mathscr{Y}$ contains an isomorphic copy of c_0 then $\mathscr{Y}$ will contain for every $\epsilon>0$ a subspace that is $(1+\epsilon)$-isomorphic to c_0. More examples of spaces that are not stable will be given in Sect. 4.

The stability of the function space $L_p[0,1]$ is much more difficult to obtain than the stability of the sequence space ℓ_p since the validity of equality (38) for the sequence space does not hold in the function space case, and thus another argument is needed.

Example 3.13 The Banach space $L_p[0,1]$, for $p\in[1,\infty)$, is stable.

For $s\in(0,1)$, the metric space $(\mathsf{X},\mathsf{d}_\mathsf{X}^s)$ is commonly called the *s-snowflaking* of X. It is clear that a snowflaking of a metric pace is stable if and only if the original metric on the space is stable. For $p\in[1,2)$, $L_p[0,1]$ will be stable since it is well known that the $\frac{p}{2}$-snowflaking of $L_p[0,1]$ embeds isometrically into the Hilbert space $L_2[0,1]$. This argument fails for $p>2$, and the stability of $L_p[0,1]$ for $p>2$ is more difficult to prove and requires a fine understanding of the relationship between stability and reflexivity. Krivine and Maurey gave a representation of the norm of a stable Banach space which provides a direct relationship between stability and reflexivity.[12]

Theorem 3.14 *Let $\mathscr{X}$ be a Banach space and fix $p\in[1,\infty)$. Then, $\mathscr{X}$ is stable if and only if there exist a reflexive Banach space $\mathscr{Y}$, a dense subset B of the unit ball of $\mathscr{X}$, and maps $g\colon B\to\mathscr{Y}$, $h\colon B\to\mathscr{Y}^*$ so that for all $x,y\in B$ we have $\|x-y\|^p=\langle g(x),h(y)\rangle$ where $\langle\cdot,\cdot\rangle$ is the duality product between $\mathscr{Y}$ and $\mathscr{Y}^*$.*

The theorem below, which includes Example 3.13, follows from a similar but more delicate representation of stable norms that was also proved in [52]. For a Banach space $\mathscr{X}$ we denote by $L_p(\Omega,\mathcal{B},\mu;\mathscr{X})$, or simply $L_p(\Omega;\mathscr{X})$, the Banach space of Bochner equivalence classes of Bochner p-integrable and $\mathscr{X}$-valued functions defined on the measured space $(\Omega,\mathcal{B},\mu)$.

Theorem 3.15 *Let $p\in[1,\infty)$. If $\mathscr{X}$ is a stable Banach space then $L_p([0,1];\mathscr{X})$ is also stable.*

3.2.2 Kalton's embedding of stable metrics

It follows from the examples in the previous section that the class of stable metrics strictly extends the class of proper metrics. For stable metrics there is no analogue of Corollary 2.7 or Corollary 3.5 even if we are willing to significantly weaken the degree of faithfulness of the embeddings. Indeed, it was proved in [13] that Tsirelson space $\mathscr{T}^*$ (which is reflexive and

[12] The connection between stable functions and weak-compactness goes back at least to the work of Grothendieck [37].

contains the ℓ_∞^n's) does not contain a coarse copy nor a uniform copy of any of the (stable) Banach spaces ℓ_p for $p \in [1, \infty)$. The next course of action would be either to consider even weaker notions of embeddability or to settle for embeddings into Banach spaces with weaker geometric features.

A remarkable result of Kalton [46] states that a stable metric can always be embedded into a reflexive Banach space, which depends on the stable metric, in a way that distorts the distances by only a slight amount.

Theorem 3.16 *Let X be a stable metric space, and $s \in (0, 1)$, then there exist a reflexive Banach space $\mathscr{K}_s(X)$ and a map $f: X \to \mathscr{K}_s(X)$ such that for all $x, y \in X$*

$$\min\left\{d_X(x, y), d_X^s(x, y)\right\} \le \|f(x) - f(y)\|_{\mathscr{K}_s(X)} \le \max\left\{d_X(x, y), d_X^s(x, y)\right\}.$$

Unlike in Corollary 2.7 or Corollary 3.5, where the host space can be taken arbitrarily from a given class of Banach spaces, the reflexive space $\mathscr{K}_s(\mathsf{X})$ and the embedding f highly depend on s and X. Given $s \in (0, 1)$ the crux of Kalton's proof is to leverage the stability condition in order to construct a weakly compact operator $S: \ell_1(W) \to \mathrm{Lip}_0(\mathsf{X}, d_{\mathsf{X}}^s)$ for some well-chosen set W that is weakly relatively compact in $\mathrm{Lip}_0(\mathsf{X}, d_{\mathsf{X}}^s)$ the Lipschitz space over the s-snowflaking of X. Then, by Davis–Figiel–Johnson–Pełczyński factorization theorem [21], S factors through a reflexive Banach space $\mathscr{X}$ and $\mathscr{K}_s(\mathsf{X})$ is defined as the dual of $\mathscr{X}$. A careful inspection of Kalton's proof reveals that an embedding with tighter guarantees can be achieved. The adjustments were carried out in [11] were the following definition, which captures the quantitative improvements, was introduced.

Definition 3.17 Let $\mathcal{C}$ be a class of metric spaces. We say that $(\mathsf{X}, d_{\mathsf{X}})$ *nearly isometrically embeds* into $\mathcal{C}$ if for any pair of continuous functions $\rho, \omega : [0, +\infty) \to [0, +\infty)$ satisfying

(i) $\omega(0) = 0$, $t \le \omega(t)$ for $t \in [0, 1]$, and $\lim_{t \downarrow 0} \frac{\omega(t)}{t} = +\infty$,
(ii) $\omega(t) = t$ for $t \ge 1$,
(iii) $\rho(t) = t$ for $t \in [0, 1]$,
(iv) $\rho(t) \le t$ for $t \ge 1$ and $\lim_{t \to +\infty} \frac{\rho(t)}{t} = 0$,

there exist a space $(\mathsf{Y}, d_{\mathsf{Y}}) \in \mathcal{C}$ and a map $f: \mathsf{X} \to \mathsf{Y}$ such that for all $x, y \in \mathsf{X}$

$$\rho\big(d_{\mathsf{X}}(x, y)\big) \le d_{\mathsf{Y}}\big(f(x), f(y)\big) \le \omega\big(d_{\mathsf{X}}(x, y)\big).$$

If $\mathcal{C}$ is reduced to a single element $(\mathsf{Y}, d_{\mathsf{Y}})$ we say that $(\mathsf{X}, d_{\mathsf{X}})$ nearly isometrically embeds into $(\mathsf{Y}, d_{\mathsf{Y}})$.

Theorem 3.18 *Let (X, d_X) be a stable metric space, then X nearly isometrically embeds into the class of reflexive Banach spaces.*

We refer to [11] for the proof of Theorem 3.18. The terminology is motivated by the fact that if X nearly isometrically embeds into $\mathcal{C}$, then for every $0 < \delta \le \Delta < \infty$ there exist $\mathsf{Y} \in \mathcal{C}$ and $f: \mathsf{X} \to \mathsf{Y}$ such that for all $x, y \in \mathsf{X}$ satisfying $d_{\mathsf{X}}(x, y) \in [\delta, \Delta]$,

$$d_{\mathsf{Y}}\big(f(x), f(y)\big) = d_{\mathsf{X}}(x, y).$$

This property is not achieved with Kalton's original embedding.

4 Upper stability

Krivine-Maurey stability theorem can be used to provide examples of Banach spaces, other than c_0, without any stable renormings.

Theorem 4.1 *Every stable Banach space contains an isomorphic copy*[13] *of ℓ_p for some $p \in [1, \infty)$.*

Therefore, Tsirelson-like spaces, which do no contain any isomorphic copy of ℓ_p for any $p \in [1, \infty)$, do not admit any equivalent stable norm. Note also that if a Banach space contains an isomorphic copy of ℓ_p for some $p \in [1, \infty)$, then it will contain an unconditional basic sequence. Hence, Banach spaces with no unconditional basic sequences, in particular hereditarily indecomposable Banach spaces (see [59] for a discussion of how to construct such spaces), also do not admit any equivalent norm that is stable.

In [80], Raynaud showed that the conclusion of Krivine-Maurey theorem is valid for any Banach space whose unit ball admits a uniformly equivalent stable metric. It was well known that spreading basic sequences in stable Banach spaces are unconditional, and Raynaud showed that this property still holds if we merely assume that the unit ball of the Banach space embeds uniformly into a stable space.

Theorem 4.2 *Let $\mathscr{X}$ be a Banach space. If the unit ball of $\mathscr{X}$ uniformly embeds into a stable metric space, then every spreading basic sequence in $\mathscr{X}$ is unconditional.*

The following theorem from [80] follows then from the observation that the summing basis of c_0, i.e. the sequence $\{s_n\}_{n=1}^{\infty}$ where $s_n = \sum_{i=1}^{n} e_i$, is spreading but not unconditional, and extends a previous result of Enflo [25] which states that the unit ball of c_0 does not embed uniformly into Hilbert space.

Theorem 4.3 *The unit ball of* c_0 *does not uniformly embed into a stable metric space.*

The notion of stability made a spectacular come back in nonlinear embedding theory due to the work of Kalton [46] where Property $\mathcal{Q}$ was introduced (see also [12,26,77] for recent papers discussing and/or using stable metrics). In the sequel we will always write an element $\bar{n} = (n_1, \dots, n_k)$ of $[\mathbb{N}]^k$, the set of k-subsets of $\mathbb{N}$, in increasing order, i.e. $n_1 < n_2 < \cdots < n_k$. In [46], Kalton defined a graph structure on (the vertex set) $[\mathbb{N}]^k$ as follows: $\bar{m}$ and $\bar{n}$ are adjacent if and only if they are different and they interlace, meaning that either $m_1 \le n_1 \le m_2 \le n_2 \le \cdots \le m_k \le n_k$ or $n_1 \le m_1 \le n_2 \le m_2 \le \cdots \le n_k \le m_k$. This (non-locally finite) infinite graph is then equipped with its canonical graph metric, simply denoted $\mathrm{d_I}$. Kalton's property $\mathcal{Q}$ is the following concentration inequality on Kalton's interlacing graphs $\{[\mathbb{N}]^k, \mathrm{d_I}\}_{k \ge 1}$.

Definition 4.4 A metric space $(\mathsf{X}, \mathrm{d_X})$ has *property $\mathcal{Q}$* if there exists a constant $C > 0$ so that for every Lipschitz map $f : ([\mathbb{N}]^k, \mathrm{d_I}) \to \mathsf{X}$ there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$, such that for all $\bar{m}, \bar{n} \in [\mathbb{M}]^k$,

$$\mathrm{d_X}\big(f(\bar{m}), f(\bar{n})\big) \le C \cdot \mathrm{Lip}(f). \tag{39}$$

Note that for any infinite subset $\mathbb{M}$ of $\mathbb{N}$, the graph $([\mathbb{M}]^k, \mathrm{d_I})$ has diameter k, and it is immediate from the definitions that a metric space with property $\mathcal{Q}$ cannot equi-coarsely contain the sequence $\{([\mathbb{N}]^k, \mathrm{d_I})\}_{k \ge 1}$. Kalton's paper [46] was extremely influential. In particular, it initiated the use of concentration inequalities in the form of (39) in nonlinear embedding problems. In [13,47,49], several long-standing open problems were resolved using this approach.

[13] In fact it contains an almost linearly isometric copy.

Kalton's interlacing graph metric encapsulates the particular behavior of the summing basis of c_0 as it is a combinatorial realization of a metric that is naturally induced by the bi-monotone version of the summing norm given by

$$\left\|\sum_i a_i s_i\right\|_{\text{sum}} := \sup\left\{\left|\sum_{i=k}^{m} a_i\right| : k, m \in \mathbb{N},\ k \leq m\right\}. \tag{40}$$

It is a tedious task (cf. [54] or [12]) to show that if $\bar{m}, \bar{n} \in [\mathbb{N}]^k$ then

$$\mathrm{d}_{\mathrm{I}}(\bar{m}, \bar{n}) = \left\|\sum_{i=1}^{k} s_{m_i} - \sum_{i=1}^{k} s_{n_i}\right\|_{sum}. \tag{41}$$

It was shown in [46] that Kalton's property $\mathcal{Q}$ belongs to the scanty list of obstructions to coarse and uniform embeddability.

Lemma 4.5 *Let $\mathscr{X}$ be a Banach space. Assume that, either*

(i) $\mathscr{X}$ coarsely embeds into a metric space with property $\mathcal{Q}$ or
(ii) the unit ball of $\mathscr{X}$ uniformly embeds into a metric space with property $\mathcal{Q}$.

Then $\mathscr{X}$ has property $\mathcal{Q}$.

Since Kalton showed that stable metric spaces have property $\mathcal{Q}$, Raynaud's uniform non-embeddability result (and its coarse version) follows from the discussion above, which implies that c_0 does not have property $\mathcal{Q}$, and Lemma 4.5. The following crucial observation is inspired by [52, Théorème page 273] and [80, proof of Proposition 5.1] (cf. [16, Lemma 9.19] and [76, Lemma 6.17].

Lemma 4.6 *The following assertions are equivalent:*

(i) (X, d_X) is a stable metric space.
(ii) For every $k \geq 2$, every $1 \leq \ell < k$, every permutation $\pi : [k] := \{1, 2, \ldots, k\} \to [k]$ which preserves the order on $\{1, \ldots, \ell\}$ and $\{\ell+1, \ldots, k\}$, all non-principal ultrafilters $\mathcal{U}_1, \ldots, \mathcal{U}_k$ on $\mathbb{N}$, and all bounded maps $f : \mathbb{N}^\ell \to X$ and $g : \mathbb{N}^{k-\ell} \to X$, the following identity holds:

$$\begin{aligned}&\lim_{n_1\to\mathcal{U}_1} \ldots \lim_{n_k\to\mathcal{U}_k} \mathrm{d}_X\big(f(n_1, \ldots, n_\ell), g(n_{\ell+1}, \ldots, n_k)\big)\\ &= \lim_{n_{\pi^{-1}(1)}\to\mathcal{U}_{\pi^{-1}(1)}} \ldots \lim_{n_{\pi^{-1}(k)}\to\mathcal{U}_{\pi^{-1}(k)}} \mathrm{d}_X\big(f(n_1, \ldots, n_\ell), g(n_{\ell+1}, \ldots, n_k)\big).\end{aligned} \tag{42}$$

(iii) For every $k \geq 2$, every $1 \leq \ell < k$, every permutation $\pi : [k] \to [k]$ which preserves the order on $\{1, \ldots, \ell\}$ and $\{\ell+1, \ldots, k\}$, every non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and all bounded maps $f : \mathbb{N}^\ell \to X$ and $g : \mathbb{N}^{k-\ell} \to X$, the following identity holds:

$$\begin{aligned}&\lim_{n_1\to\mathcal{U}} \ldots \lim_{n_k\to\mathcal{U}} \mathrm{d}_X\big(f(n_1, \ldots, n_\ell), g(n_{\ell+1}, \ldots, n_k)\big)\\ &= \lim_{n_1\to\mathcal{U}} \ldots \lim_{n_k\to\mathcal{U}} \mathrm{d}_X\big(f(n_{\pi(1)}, \ldots, n_{\pi(\ell)}), g(n_{\pi(\ell+1)}, \ldots, n_{\pi(k)})\big).\end{aligned} \tag{43}$$

(iv) For every $k \geq 1$, every $1 \leq \ell < k$, every permutation $\pi : [k] \to [k]$ which preserves the order on $\{1, \ldots, \ell\}$ and $\{\ell+1, \ldots, k\}$, and all bounded maps $f : [\mathbb{N}]^\ell \to X$ and $g : [\mathbb{N}]^{k-\ell} \to X$ we have for every infinite subset $\mathbb{M}$ of $\mathbb{N}$,

$$\begin{aligned}&\inf_{\bar{n}\in[\mathbb{M}]^k} \mathrm{d}_X(f(n_1, \ldots, n_\ell), g(n_{\ell+1}, \ldots, n_k))\\ &\leq \sup_{\bar{n}\in[\mathbb{M}]^k} \mathrm{d}_X(f(n_{\pi(1)}, \ldots, n_{\pi(\ell)}), g(n_{\pi(\ell+1)}, \ldots, n_{\pi(k)})).\end{aligned} \tag{44}$$

Proof The proof of $(i) \implies (ii)$ is by induction on $k \geq 2$. If $k = 2$ the only non-trivial permutation to consider is $\pi := \binom{1\,2}{2\,1} = \pi^{-1}$. The identity

$$\lim_{n_1\to\mathcal{U}_1}\lim_{n_2\to\mathcal{U}_2} \mathrm{d_X}\big(f(n_1), g(n_2)\big) = \lim_{n_2\to\mathcal{U}_2}\lim_{n_1\to\mathcal{U}_1} \mathrm{d_X}\big(f(n_1), g(n_2)\big)$$

follows from the definition of stability.

Assume now that (42) holds for some $k \geq 2$. Let $1 \leq \ell < k+1$ and $\pi : [k+1] \to [k+1]$ be a permutation which preserves the order on $\{1, \ldots, \ell\}$ and $\{\ell+1, \ldots, k+1\}$ We think of the inverse permutation π^{-1} as a (reverse) ordering according to which the limits should be taken: first along the variable originally associated to $\mathcal{U}_{\pi^{-1}(k-1)}$, then along the variable originally associated to $\mathcal{U}_{\pi^{-1}(k)}$, and so on... The only requirement on the permutation π means that, in the ordering given by π^{-1}, we can choose to alternate between the first ℓ original limits and the last original $k+1-\ell$ limits as long as we do not swap the order of the the first ℓ original limits nor swap the order of the the last original $k+1-\ell$ limits. More formally, $[1 \leq \pi^{-1}(i) < \pi^{-1}(j) \leq \ell \implies i < j] \wedge [\ell+1 \leq \pi^{-1}(i) < \pi^{-1}(j) \leq k+1 \implies i < j]$. In the case that there is a pair of two consecutive limits amongst the first ℓ original limits which are still consecutive in reordering given by π^{-1}, i.e. if $2 \leq \pi^{-1}(i) + 1 = \pi^{-1}(i+1) \leq \ell$ for some $1 \leq i \leq k$, we consider $\mathcal{U}_{\pi^{-1}(i)} \otimes \mathcal{U}_{\pi^{-1}(i+1)}$ the ultrafilter tensor product (or Fubini product) defined by

$$\mathcal{U}_{\pi^{-1}(i)} \otimes \mathcal{U}_{\pi^{-1}(i+1)} := \Big\{ A \subseteq \mathbb{N}^2 \colon \big\{ m_i \in \mathbb{N} \colon \{m_{i+1} \in \mathbb{N} \colon (m_i, m_{i+1}) \in A\} \in \mathcal{U}_{\pi^{-1}(i+1)} \big\} \\ \times \in \mathcal{U}_{\pi^{-1}(i)} \Big\}.$$

As $\mathcal{U}_{\pi^{-1}(i)}$ and $\mathcal{U}_{\pi^{-1}(i+1)}$ are non-principal ultrafilters, $\mathcal{U}_{\pi^{-1}(i)} \otimes \mathcal{U}_{\pi^{-1}(i+1)}$ is a non-principal ultrafilter on $\mathbb{N}^2$, and moreover for any bounded function h on $\mathbb{N}^2$ it holds

$$\lim_{m_i\to\mathcal{U}_{\pi^{-1}(i)}}\ \lim_{m_{i+1}\to\mathcal{U}_{\pi^{-1}(i+1)}} h(m_i, m_{i+1}) = \lim_{(m_i, m_{i+1})\to\mathcal{U}_{\pi^{-1}(i)}\otimes\mathcal{U}_{\pi^{-1}(i+1)}} h(m_i, m_{i+1}).$$

Now if one picks a bijection $\varphi\colon \mathbb{N}^2 \to \mathbb{N}$ and write $\varphi^{-1} = (\varphi_1^{-1}, \varphi_2^{-1})$, then the pushforward-ultrafilter $\varphi_*(\mathcal{U}_{\pi^{-1}(i)} \otimes \mathcal{U}_{\pi^{-1}(i+1)})$ is a non-principal ultrafilter on $\mathbb{N}$ satisfying

$$\lim_{m_i\to\mathcal{U}_{\pi^{-1}(i)}}\ \lim_{m_{i+1}\to\mathcal{U}_{\pi^{-1}(i+1)}} h(m_i, m_{i+1}) = \lim_{n\to\varphi_*\left(\mathcal{U}_{\pi^{-1}(i)}\otimes\mathcal{U}_{\pi^{-1}(i+1)}\right)} h\left(\varphi_1^{-1}(n), \varphi_2^{-1}(n)\right).$$

Based on this discussion if $\tilde{f}\colon \mathbb{N}^{\ell-1} \to \mathsf{X}$ is given by

$$\tilde{f}(m_1, \ldots, m_i, \ldots, m_{\ell-1}) := f(m_1, \ldots, m_{i-1}, \varphi_1^{-1}(m_i), \varphi_2^{-1}(m_i), m_{i+1}, \ldots, m_{\ell-1}),$$

and

$$\tilde{\mathcal{U}_r} := \begin{cases} \mathcal{U}_r & \text{if } 1 \leq r \leq \pi^{-1}(i) - 1, \\ \varphi_*(\mathcal{U}_{\pi^{-1}(i)} \otimes \mathcal{U}_{\pi^{-1}(i+1)}) & \text{if } r = \pi^{-1}(i), \\ \mathcal{U}_{r+1} & \text{if } \pi^{-1}(i) + 1 \leq r \leq k, \end{cases}$$

then we have

$$\lim_{n_1\to\mathcal{U}_1} \ldots \lim_{n_{k+1}\to\mathcal{U}_{k+1}} \mathrm{d_X}\big(f(n_1, \ldots, n_\ell), g(n_{\ell+1}, \ldots, n_{k+1})\big) \\ = \lim_{m_1\to\tilde{\mathcal{U}}_1} \ldots \lim_{m_k\to\tilde{\mathcal{U}}_k} \mathrm{d_X}\big(\tilde{f}(m_1, \ldots, m_{\ell-1}), g(m_\ell, \ldots, m_k)\big). \tag{45}$$

Also, if

$$\tilde{\pi}(r) := \begin{cases} \pi(r) & \text{if } 1 \le r \le \pi^{-1}(i) \le \ell - 1, \\ \pi(r+1) & \text{if } \pi^{-1}(i)+1 \le r \le k \text{ and } \pi(r+1) < i, \\ \pi(r+1) - 1 & \text{if } \pi^{-1}(i)+1 \le r \le k \text{ and } \pi(r+1) > i, \end{cases}$$

then $\tilde{\pi}$ is a permutation of $[k]$ that preserves the order on $\{1, \dots, \ell-1\}$ and $\{\ell, \dots, k\}$ and

$$\tilde{\pi}^{-1}(s) = \begin{cases} \pi^{-1}(s) & \text{if } 1 \le s \le i \text{ and } \pi^{-1}(s) \le \pi^{-1}(i), \\ \pi^{-1}(s) - 1 & \text{if } 1 \le s < i \text{ and } \pi^{-1}(s) > \pi^{-1}(i), \\ \pi^{-1}(s+1) - 1 & \text{if } i+1 \le s \le k \text{ and } \pi^{-1}(s+1) > \pi^{-1}(i). \end{cases}$$

It is not difficult, thought a bit tedious, to verify that $\tilde{\pi}^{-1} \circ \tilde{\pi} = \tilde{\pi} \circ \tilde{\pi}^{-1} = id$. To help understand these permutations, below is an illustrative example with $\ell = 4$ and $k+1 = 7$. If

$$\pi = \left(\begin{smallmatrix} 1&2&3&4&5&6&7 \\ 1&3&4&6&2&5&7 \end{smallmatrix}\right) \text{ and hence } \pi^{-1} = \left(\begin{smallmatrix} 1&2&3&4&5&6&7 \\ 1&5&2&3&6&4&7 \end{smallmatrix}\right),$$

then

$$\tilde{\pi} = \left(\begin{smallmatrix} 1&2&3&4&5&6 \\ 1&3&5&2&4&6 \end{smallmatrix}\right) \text{ and } \tilde{\pi}^{-1} = \left(\begin{smallmatrix} 1&2&3&4&5&6 \\ 1&4&2&5&3&6 \end{smallmatrix}\right).$$

In this situation $i = 3$, $\pi^{-1}(2) = 2$ and $\pi^{-1}(4) = 3 = \pi^{-1}(2) + 1$. The permutations $\tilde{\pi}^{-1}$ and $\tilde{\pi}$ are obtained by identifying the consecutive limits and shifting accordingly the other entries. Moreover,

$$\begin{aligned} &\lim_{n_{\pi^{-1}(1)} \to \mathcal{U}_{\pi^{-1}(1)}} \dots \lim_{n_{\pi^{-1}(k+1)} \to \mathcal{U}_{\pi^{-1}(k+1)}} \mathrm{d_X}\big(f(n_1, \dots, n_\ell), g(n_{\ell+1}, \dots, n_{k+1})\big) \\ &= \lim_{m_{\tilde{\pi}^{-1}(1)} \to \tilde{\mathcal{U}}_{\tilde{\pi}^{-1}(1)}} \dots \lim_{m_{\tilde{\pi}^{-1}(k)} \to \tilde{\mathcal{U}}_{\tilde{\pi}^{-1}(k)}} \mathrm{d_X}\big(\tilde{f}(m_1, \dots, m_{\ell-1}), g(m_\ell, \dots, m_k)\big). \end{aligned} \tag{46}$$

We can now use the induction hypothesis to conclude that all the quantities in (45) and (46) are equal.

The case when there is a pair of consecutive limits amongst the last $k+1-\ell$ original limits which are still consecutive in the reordering given by π^{-1}, i.e. if $\ell+2 \le \pi^{-1}(i)+1 = \pi^{-1}(i+1) \le k+1$ for some $1 \le i \le k$, can be treated similarly.

Now, if there is no pair of consecutive limits amongst the first ℓ, nor the last $k+1-\ell$, original limits which are still consecutive in the reordering given by π^{-1}, i.e. for no $i \in \{1, \dots, \ell - 1\} \cup \{\ell+1, \dots, k\}$ do we have $\pi^{-1}(i+1) = \pi^{-1}(i) + 1$, then π^{-1} is necessarily of one of the following two types: (a) $\pi^{-1} = \left(\begin{smallmatrix} \cdots & \cdots & k-1 & k & k+1 \\ \cdots & \cdots & \ell-1 & k+1 & \ell \end{smallmatrix}\right)$ or (b) $\pi^{-1} = \left(\begin{smallmatrix} \cdots & \cdots & k-1 & k & k+1 \\ \cdots & \cdots & k & \ell & k+1 \end{smallmatrix}\right)$. In case (a), by stability we have that for all $(m_1, \dots, m_{k-1}) \in \mathbb{N}^{k-1}$

$$\begin{aligned} &\lim_{n_{k+1} \to \mathcal{U}_{k+1}} \lim_{n_\ell \to \mathcal{U}_\ell} \mathrm{d_X}\big(f(m_1, \dots, m_{\ell-1}, n_\ell), g(m_\ell, \dots, m_{k-1}, n_{k+1})\big) \\ &= \lim_{n_\ell \to \mathcal{U}_\ell} \lim_{n_{k+1} \to \mathcal{U}_{k+1}} \mathrm{d_X}\big(f(m_1, \dots, m_{\ell-1}, n_\ell), g(m_\ell, \dots, m_{k-1}, n_{k+1})\big), \end{aligned}$$

and hence

$$\begin{aligned} &\lim_{n_{\pi^{-1}(1)} \to \mathcal{U}_{\pi^{-1}(1)}} \dots \lim_{n_{\pi^{-1}(k+1)} \to \mathcal{U}_{\pi^{-1}(k+1)}} \mathrm{d_X}\big(f(n_1, \dots, n_\ell), g(n_{\ell+1}, \dots, n_{k+1})\big) \\ &= \lim_{n_{\pi^{-1}(1)} \to \mathcal{U}_{\pi^{-1}(1)}} \dots \lim_{n_{\ell-1} \to \mathcal{U}_{\ell-1}} \lim_{n_{k+1} \to \mathcal{U}_{k+1}} \lim_{n_\ell \to \mathcal{U}_\ell} \mathrm{d_X}\big(f(n_1, \dots, n_\ell), g(n_{\ell+1}, \dots, n_{k+1})\big) \\ &= \lim_{n_{\pi^{-1}(1)} \to \mathcal{U}_{\pi^{-1}(1)}} \dots \lim_{n_{\ell-1} \to \mathcal{U}_{\ell-1}} \lim_{n_\ell \to \mathcal{U}_\ell} \lim_{n_{k+1} \to \mathcal{U}_{k+1}} \mathrm{d_X}\big(f(n_1, \dots, n_\ell), g(n_{\ell+1}, \dots, n_{k+1})\big). \end{aligned}$$

We have thus reduced our problem to one of the previous situations already handled above. A similar reduction can be done in case (b).

The implication $(ii) \implies (iii)$ is just formal. To prove $(iii) \implies (iv)$ let $\mathbb{M}$ be an infinite subset of $\mathbb{N}$ and $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ containing $\mathbb{M}$. Since for every $\bar{n} \in [\mathbb{M}]^k$,

$$\begin{aligned} &\mathrm{d}_{\mathsf{X}}(f(n_{\pi(1)}, \dots, n_{\pi(\ell)}), g(n_{\pi(\ell+1)}, \dots, n_{\pi(k)})) \\ &\quad \le \sup_{\bar{n}\in[\mathbb{M}]^k} \mathrm{d}_{\mathsf{X}}(f(n_{\pi(1)}, \dots, n_{\pi(\ell)}), g(n_{\pi(\ell+1)}, \dots, n_{\pi(k)})) \end{aligned}$$

and since the ultrafilter $\mathcal{U}$ contains $\mathbb{M}$ we have

$$\begin{aligned} &\lim_{n_1\to\mathcal{U}} \dots \lim_{n_k\to\mathcal{U}} \mathrm{d}_{\mathsf{X}}(f(n_{\pi(1)}, \dots, n_{\pi(\ell)}), g(n_{\pi(\ell+1)}, \dots, n_{\pi(k)})) \\ &\quad \le \sup_{\bar{n}\in[\mathbb{M}]^k} \mathrm{d}_{\mathsf{X}}(f(n_{\pi(1)}, \dots, n_{\pi(\ell)}), g(n_{\pi(\ell+1)}, \dots, n_{\pi(k)})). \end{aligned}$$

By stability,

$$\begin{aligned} &\lim_{n_1\to\mathcal{U}} \dots \lim_{n_k\to\mathcal{U}} \mathrm{d}_{\mathsf{X}}(f(n_1, \dots, n_\ell), g(n_{\ell+1}, \dots, n_k)) \\ &\quad \le \sup_{\bar{n}\in[\mathbb{M}]^k} \mathrm{d}_{\mathsf{X}}(f(n_{\pi(1)}, \dots, n_{\pi(\ell)}), g(n_{\pi(\ell+1)}, \dots, n_{\pi(k)})), \end{aligned}$$

and thus for every $\varepsilon > 0$ there exists $\bar{n} \in [\mathbb{M}]^k$ (we again use that the ultrafilter contains $\mathbb{M}$) such that

$$\begin{aligned} \mathrm{d}_{\mathsf{X}}(f(n_1, \dots, n_\ell), g(n_{\ell+1}, \dots, n_k)) \le \sup_{\bar{n}\in[\mathbb{M}]^k} &\mathrm{d}_{\mathsf{X}}(f(n_{\pi(1)}, \dots, n_{\pi(\ell)}), \\ &\times g(n_{\pi(\ell+1)}, \dots, n_{\pi(k)})) + \varepsilon \end{aligned}$$

and the conclusion follows.

The implication $(iv) \implies (i)$ essentially follows from [52, Théorème page 276]. There, the argument is given for separable Banach spaces and for the convenience of the reader we just repeat the argument in the metric case. We will show that if $(\mathsf{X}, \mathrm{d}_{\mathsf{X}})$ is not stable then (44) is violated for $k = 2$ and π the interlacing permutation, which is the only non-trivial permutation in this case, i.e.

$$\inf_{n_1<n_2} \mathrm{d}_{\mathsf{X}}\big(f(n_1), g(n_2)\big) \le \sup_{n_1<n_2} \mathrm{d}_{\mathsf{X}}\big(f(n_2), g(n_1)\big). \tag{47}$$

Assume that there are ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ on $\mathbb{N}$, and two bounded functions $f, g\colon \mathbb{N} \to \mathsf{X}$ such that

$$\lim_{n_1\to\mathcal{U}_1} \lim_{n_2\to\mathcal{U}_2} \mathrm{d}_{\mathsf{X}}\big(f(n_1), g(n_2)\big) \ne \lim_{n_2\to\mathcal{U}_2} \lim_{n_1\to\mathcal{U}_1} \mathrm{d}_{\mathsf{X}}\big(f(n_1), g(n_2)\big).$$

We only treat the case

$$\lim_{n_1\to\mathcal{U}_1} \lim_{n_2\to\mathcal{U}_2} \mathrm{d}_{\mathsf{X}}\big(f(n_1), g(n_2)\big) < \lim_{n_2\to\mathcal{U}_2} \lim_{n_1\to\mathcal{U}_1} \mathrm{d}_{\mathsf{X}}\big(f(n_1), g(n_2)\big),$$

as the other case can be handle similarly. Let $\alpha, \beta > 0$ such that

$$\lim_{n_1\to\mathcal{U}_1} \lim_{n_2\to\mathcal{U}_2} \mathrm{d}_{\mathsf{X}}\big(g(n_2), f(n_1)\big) < \alpha < \beta < \lim_{n_2\to\mathcal{U}_2} \lim_{n_1\to\mathcal{U}_1} \mathrm{d}_{\mathsf{X}}\big(g(n_2), f(n_1)\big).$$

There is $B \in \mathcal{U}_2$ and $A \in \mathcal{U}_1$ such that $\lim_{n_1\to\mathcal{U}_1} \mathrm{d}_{\mathsf{X}}\big(g(n_2), f(n_1)\big) > \beta$ whenever $n_2 \in B$, and $\lim_{n_2\to\mathcal{U}_2} \mathrm{d}_{\mathsf{X}}\big(g(n_2), f(n_1)\big) < \alpha$ whenever $n_1 \in A$. We now construct recursively two sequences of integers $\{a_j\}_{j=1}^{\infty}$ and $\{b_j\}_{j=1}^{\infty}$ together with decreasing sequences of subsets

$\{A_j\}_{j=1}^{\infty}$ in $\mathcal{U}_1$ and $\{B_j\}_{j=1}^{\infty}$ in $\mathcal{U}_2$ as follows. Let $B_1 := B$ and pick $b_1 \in B_1$. Then let $A_1 := A \cap \{n_1 \in \mathbb{N}: \mathrm{d}_{\mathsf{X}}\big(g(b_1), f(n_1)\big) > \beta\}$ and observe that $A_1 \in \mathcal{U}_1$. Pick $a_1 \in A_1 \subset A$ and let $B_2 := B_1 \cap \{n_2 \in \mathbb{N}: \mathrm{d}_{\mathsf{X}}\big(g(n_2), f(a_1)\big) < \alpha\}$ which is clearly in $\mathcal{U}_2$. Assume now that a_k, A_k, and B_{k+1} have been defined. First, pick $b_{k+1} \in B_{k+1}$, then let $A_{k+1} := A_k \cap \{n_1 \in \mathbb{N}: \mathrm{d}_{\mathsf{X}}\big(g(b_{k+1}), f(n_1)\big) > \beta\}$, and pick $a_{k+1} \in A_{k+1}$. Finally, we let $B_{k+2} := B_{k+1} \cap \{n_2 \in \mathbb{N}: \mathrm{d}_{\mathsf{X}}\big(g(n_2), f(a_{k+1})\big) < \alpha\}$. By construction, it is plain that, for all $k \in \mathbb{N}$, we have $a_k \in A_k$ and $b_k \in B_k$. Moreover, if $k \le \ell$, then $a_\ell \in A_\ell \subseteq A_k$, which implies that $\mathrm{d}_{\mathsf{X}}\big(g(b_k), f(a_\ell)\big) > \beta$, and in turn $\inf_{k<\ell} \mathrm{d}_{\mathsf{X}}\big(g(b_k), f(a_\ell)\big) > \beta$. On the other hand, when $k > \ell$ we have $b_k \in B_k \subseteq B_{\ell+1}$, and hence $\mathrm{d}_{\mathsf{X}}\big(g(b_k), f(a_\ell)\big) < \alpha$. Therefore, $\sup_{k>\ell} \mathrm{d}_{\mathsf{X}}\big(g(b_k), f(a_\ell)\big) < \alpha$, and

$$\sup_{\ell<k} \mathrm{d}_{\mathsf{X}}\big(g(b_k), f(a_\ell)\big) < \alpha < \beta < \inf_{k<\ell} \mathrm{d}_{\mathsf{X}}\big(g(b_k), f(a_\ell)\big),$$

which is easily seen to contradict (47).

□

The notion of upper stability that we introduce in the definition below is a natural bi-Lipschitz variant of the isometric notion of stability and can be used to quantify the lack of stability.

Definition 4.7 Let $(\mathsf{X}, \mathrm{d}_{\mathsf{X}})$ be a metric space. We say that X is K_u-*upper stable* if for every $k \ge 1$, every $1 \le \ell < k$, every permutation $\pi\colon [k] \to [k]$ which preserves the order on $\{1, \dots, \ell\}$ and $\{\ell+1, \dots, k\}$, and every bounded maps $f\colon [\mathbb{N}]^{\ell} \to \mathsf{X}$ and $g\colon [\mathbb{N}]^{k-\ell} \to \mathsf{X}$ we have for every infinite subset $\mathbb{M}$ of $\mathbb{N}$

$$\begin{aligned}\inf_{\bar{n}\in[\mathbb{M}]^k} &\mathrm{d}_{\mathsf{X}}\big(f(n_1, \dots, n_\ell), g(n_{\ell+1}, \dots, n_k)\big)\\ &\le K_u \sup_{\bar{n}\in[\mathbb{M}]^k} \mathrm{d}_{\mathsf{X}}\big(f(n_{\pi(1)}, \dots, n_{\pi(\ell)}), g(n_{\pi(\ell+1)}, \dots, n_{\pi(k)})\big).\end{aligned} \tag{48}$$

And we say that X is *upper stable* if it is K_u-upper stable for some $K_u > 0$, and we will denote by $U_k(\mathsf{X})$ the smallest constant such that (48) holds.

Remark 4.8 It follows from Lemma 4.6 that a stable metric space is 1-upper stable.

If $\{s_n\}_{n\ge1}$ is the summing basis of c_0, it is easy to see for all $\bar{n} \in [\mathbb{N}]^{2k}$,

$$\Big\|\sum_{i=1}^{k} s_{n_i} - \sum_{i=k+1}^{2k} s_{n_i}\Big\|_\infty = k \quad \text{and} \quad \Big\|\sum_{i=1}^{k} s_{n_{2i-1}} - \sum_{i=1}^{k} s_{n_{2i}}\Big\|_\infty = 1.$$

This observation provides the main source of examples of non-upper stable spaces.

Example 4.9 The Banach space c_0 is not upper-stable. In fact, $U_k(\mathrm{c}_0) \ge k$.

In fact, it is immediate that $U_k(\mathsf{X}) \le \mathsf{c}_{\mathsf{Y}}(\mathsf{X}) U_k(\mathsf{Y})$ and because of (41), any metric space that contains bi-Lipschitz copies of the interlacing graphs will not be upper stable. The bi-Lipschitz nature of upper stability makes it a more convenient and versatile notion to work with, since results for stable metrics that are of isomorphic nature can be extended to the upper stable metrics. For instance, since upper stability is clearly a bi-Lipschitz invariant, it provides a quick way to show that c_0 does not admit any bi-Lipschitz embedding into a stable metric since upper stability generalizes stability.

Akin Property $\mathcal{Q}$, upper stability is also a coarse and uniform invariant for Banach spaces.

Lemma 4.10 *If a Banach space $\mathscr{X}$ coarsely, or uniformly, embeds into an upper stable metric space, then $\mathscr{X}$ is upper stable.*

Proof We prove first the coarse statement. Since $(\mathscr{X}, \|\cdot\|)$ coarsely embeds into an upper stable metric space $(\mathsf{Y}, \mathsf{d}_\mathsf{Y})$, there exist $\varphi\colon \mathscr{X} \to \mathsf{Y}$, and non-decreasing functions $\rho, \omega\colon [0,\infty) \to [0,\infty)$ such that

$$\rho(\|x-y\|) \leq \mathrm{d}_\mathrm{Y}(\varphi(x),\varphi(y)) \leq \omega(\|x-y\|),$$

with $\lim_{t\to\infty}\rho(t) = \infty$ and $\omega(t) < \infty$ for all $t > 0$. Let $1 \leq \ell < k$, $f\colon [\mathbb{N}]^\ell \to \mathscr{X}$ and $g\colon [\mathbb{N}]^{k-\ell} \to \mathscr{X}$ bounded maps, and $\pi\colon [k] \to [k]$ a permutation that preserves the order on $\{1,\dots,\ell\}$ and $\{\ell+1,\dots,k\}$, be given. Let $\mathbb{M}$ an infinite subset of $\mathbb{N}$ and let

$$\alpha \stackrel{\text{def}}{=} \sup_{\bar{n}\in[\mathbb{M}]^k} \|f(n_{\pi(1)},\dots,n_{\pi(\ell)}) - g(n_{\pi(\ell+1)},\dots,n_{\pi(k)})\|.$$

Then for every $(n_1,\dots,n_k) \in [\mathbb{M}]^k$

$$\mathrm{d}_\mathrm{Y}\left(\varphi\left(\frac{1}{\alpha}f(n_{\pi(1)},\dots,n_{\pi(\ell)})\right), \varphi\left(\frac{1}{\alpha}g(n_{\pi(\ell+1)},\dots,n_{\pi(k)})\right)\right) \leq \omega(1),$$

and since $\omega(1)$ is independent of $n_1 < \cdots < n_k$ we have

$$\sup_{\bar{n}\in[\mathbb{M}]^k} \mathrm{d}_\mathrm{Y}\left(\frac{1}{\alpha}\varphi\left(f(n_{\pi(1)},\dots,n_{\pi(\ell)})\right), \varphi\left(\frac{1}{\alpha}g(n_{\pi(\ell+1)},\dots,n_{\pi(k)})\right)\right) \leq \omega(1)$$

Let $\beta \stackrel{\text{def}}{=} \frac{1}{\alpha}\inf_{\bar{n}\in[\mathbb{M}]^k}\|f(n_1,\dots,n_\ell) - g(n_{\ell+1},\dots,n_k)\|$ and observe that for all $(n_1,\dots,n_k) \in [\mathbb{M}]^k$

$$\begin{aligned}\mathrm{d}_\mathrm{Y}&\left(\varphi\left(\frac{1}{\alpha}f(n_1,\dots,n_\ell)\right), \varphi\left(\frac{1}{\alpha}g(n_{\ell+1},\dots,n_k)\right)\right)\\ &\geq \inf\left\{\mathrm{d}_\mathrm{Y}\big(\varphi(x),\varphi(y)\big)\colon \|x-y\| \geq \beta\right\}\end{aligned}$$

and since the right hand side is independent of $n_1 < \cdots < n_k$,

$$\begin{aligned}\inf_{\bar{n}\in[\mathbb{M}]^k}\mathrm{d}_\mathrm{Y}&\left(\varphi\left(\frac{1}{\alpha}f(n_1,\dots,n_\ell)\right), \varphi\left(\frac{1}{\alpha}g(n_{\ell+1},\dots,n_k)\right)\right)\\ &\geq \inf\left\{\mathrm{d}_\mathrm{Y}\big(\varphi(x),\varphi(y)\big)\colon \|x-y\| \geq \beta\right\}\end{aligned}$$

The upper stability assumption gives

$$\inf\left\{\mathrm{d}_\mathrm{Y}\big(\varphi(x),\varphi(y)\big)\colon \|x-y\| \geq \beta\right\} \leq K_u\omega(1) < \infty. \tag{49}$$

Let $C > 0$ such that $\rho(C) \geq 2K_u\omega(1)$ (such a C exists since by assumption $\lim_{t\to\infty}\rho(t) = \infty$). If $\beta > C$ then whenever $\|x-y\| \geq \beta$ we have $\mathrm{d}_\mathrm{Y}\big(\varphi(x),\varphi(y)\big) \geq \rho\big(\|x-y\|\big) \geq \rho(C) \geq 2K_u\omega(1)$, but this contradicts (49). Therefore, $\beta \leq C$ necessarily, and this completes the proof since it precisely means that

$$\begin{aligned}\inf_{\bar{n}\in[\mathbb{M}]^k}&\|f(n_1,\dots,n_\ell) - g(n_{\ell+1},\dots,n_k)\|\\ &\leq C\sup_{\bar{n}\in[\mathbb{M}]^k}\|f(n_{\pi(1)},\dots,n_{\pi(\ell)}) - g(n_{\pi(\ell+1)},\dots,n_{\pi(k)})\|.\end{aligned}$$

Now, for the uniform case we assume that $\lim_{t\to 0}\omega(t)=0$ and $\rho(t)>0$ for all $t>0$. Let $\beta \stackrel{\text{def}}{=} \inf_{\bar{n}\in[\mathbb{M}]^k}\|f(n_1,\dots,n_\ell)-g(n_{\ell+1},\dots,n_k)\|$. Then for every $(n_1,\dots,n_k)\in[\mathbb{M}]^k$

$$\mathrm{d}_{\mathrm{Y}}\left(\varphi\left(\frac{1}{\beta}f(n_1,\dots,n_\ell)\right),\varphi\left(\frac{1}{\beta}g(n_{\ell+1},\dots,n_k)\right)\right)\geq\rho(1)>0$$

and since $\rho(1)$ is independent of $n_1<\cdots<n_k$ we have

$$\inf_{n\in[\mathbb{M}]^k}\mathrm{d}_{\mathrm{Y}}\left(\varphi\left(\frac{1}{\beta}f(n_1,\dots,n_\ell)\right),\varphi\left(\frac{1}{\beta}g(n_{\ell+1},\dots,n_k)\right)\right)\geq\rho(1)$$

Let

$$\alpha\stackrel{\text{def}}{=}\frac{1}{\beta}\sup_{\bar{n}\in[\mathbb{M}]^k}\|f(n_{\pi(1)},\dots,n_{\pi(\ell)})-g(n_{\pi(\ell+1)},\dots,n_{\pi(k)})\|,$$

and observe that for all $(n_1,\dots,n_k)\in[\mathbb{M}]^k$

$$\begin{aligned}&\mathrm{d}_{\mathrm{Y}}\left(\varphi\left(\frac{1}{\beta}f(n_{\pi(1)},\dots,n_{\pi(\ell)})\right),\varphi\left(\frac{1}{\beta}g(n_{\pi(\ell+1)},\dots,n_{\pi(k)})\right)\right)\\&\quad\leq\sup\{\mathrm{d}_{\mathrm{Y}}\left(\varphi(x),\varphi(y)\right):\|x-y\|\leq\alpha\}\end{aligned}$$

and since the right-hand side is independent of $n_1<\cdots<n_k$,

$$\begin{aligned}&\sup_{n\in[\mathbb{M}]^k}\mathrm{d}_{\mathrm{Y}}\left(\varphi\left(\frac{1}{\beta}f(n_{\pi(1)},\dots,n_{\pi(\ell)})\right),\varphi\left(\frac{1}{\beta}g(n_{\pi(\ell+1)},\dots,n_{\pi(k)})\right)\right)\\&\quad\leq\sup\{\mathrm{d}_{\mathrm{Y}}\left(\varphi(x),\varphi(y)\right):\|x-y\|\leq\alpha\}\end{aligned}$$

The upper stability assumption gives

$$\rho(1)\leq K_u\sup\{\mathrm{d}_{\mathrm{Y}}\big(\varphi(x),\varphi(y)\big):\|x-y\|\leq\alpha\}\tag{50}$$

Let $C>0$ such that $\omega(C)\leq\frac{\rho(1)}{2K_u}$ (such a C exists since by assumption $\lim_{t\to 0}\omega(t)=0$). If $\alpha<C$ then whenever $\|x-y\|\leq\alpha$ we have $\mathrm{d}_{\mathrm{Y}}\big(\varphi(x),\varphi(y)\big)\leq\omega\big(\|x-y\|\big)\leq\omega(C)\leq\frac{\rho(1)}{2K_u}$, but this contradicts (50). Therefore, $\alpha\geq C$ necessarily, and this completes the proof since it precisely means that

$$\begin{aligned}&\inf_{\bar{n}\in[\mathbb{M}]^k}\|f(n_1,\dots,n_\ell)-g(n_{\ell+1},\dots,n_k)\|\\&\quad\leq\frac{1}{C}\sup_{\bar{n}\in[\mathbb{M}]^k}\|f(n_{\pi(1)},\dots,n_{\pi(\ell)})-g(n_{\pi(\ell+1)},\dots,n_{\pi(k)})\|.\end{aligned}$$

□

It is not clear whether upper stability is preserved under uniform embeddings of the unit ball. The following lemma, which follows from a careful inspection of the proof in the uniform case of Lemma 4.10, will be needed to handle uniform embeddings of the unit ball.

Lemma 4.11 *Let* (X,d_X) *be a metric space that uniformly embeds, with compression function* ρ *and expansion function* ω*, into a* K_u*-upper stable metric space. If* $1\leq\ell<k$*,* $f:[\mathbb{N}]^\ell\to X$ *and* $g:[\mathbb{N}]^{k-\ell}\to X$ *are bounded maps, and* $\pi:[k]\to[k]$ *is a permutation that preserves the order on* $\{1,\dots,\ell\}$ *and* $\{\ell+1,\dots,k\}$ *then for every infinite subset* $\mathbb{M}$ *of* $\mathbb{N}$

$$\sup_{\bar{n}\in[\mathbb{M}]^k}\mathrm{d}_X\big(f(n_{\pi(1)},\dots,n_{\pi(\ell)}),g(n_{\pi(\ell+1)},\dots,n_{\pi(k)})\big)\geq C,$$

where C is any number satisfying $\omega(C) \le \frac{1}{2K_u}\rho(\alpha)$ with

$$\alpha \overset{\text{def}}{=} \inf_{\bar{n}\in[\mathbb{M}]^k} d_X\big(f(n_1,\dots,n_\ell), g(n_{\ell+1},\dots,n_k)\big).$$

We will now extend some results of Raynaud and Kalton to upper stable spaces. We start with Raynaud's result about unconditionality of spreading basic sequences in stable spaces. First recall some Banach space theoretic concepts mentioned above. A basic sequence $\{x_n\}_{n=1}^\infty$ in a Banach space $(\mathscr{X}, \|\cdot\|)$ is *spreading* if it is equivalent to all of its subsequences, in the sense that there exist $A_s, B_s \ge 1$ such that for all $k \ge 1$, $a_1, a_2, \dots, a_k \in \mathbb{R}$, and $1 \le m_1 < m_2 < \cdots < m_k \in \mathbb{N}$ we have

$$\frac{1}{B_s}\left\|\sum_{i=1}^k a_i x_i\right\| \le \left\|\sum_{i=1}^k a_i x_{n_i}\right\| \le A_s\left\|\sum_{i=1}^k a_i x_i\right\|. \tag{51}$$

A basic sequence $\{x_n\}_{n=1}^\infty$ is said to be *unconditional* if there exists $C_u \ge 1$ such that for all $k \ge 1$, $a_1, a_2, \dots, a_k \in \mathbb{R}$, and $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k \in \{-1, +1\}$ we have

$$\frac{1}{C_u}\left\|\sum_{n=1}^k a_n x_n\right\| \le \left\|\sum_{n=1}^k \varepsilon_n a_n x_n\right\| \le C_u\left\|\sum_{n=1}^k a_n x_n\right\|.$$

It is plain that the canonical basis $\{e_n\}_{n\ge1}$ of c_0 or ℓ_p, for $1 \le p < \infty$, are both spreading and unconditional[14] (both with constant 1).

Raynaud showed [80, Proposition 5.1] that if the unit ball of a Banach space $\mathscr{X}$ uniformly embeds into a stable metric space, then every spreading basic sequence in $\mathscr{X}$ is unconditional.

Theorem 4.12 *Let $\mathscr{X}$ be a Banach space. Assume that, either*

(i) $\mathscr{X}$ coarsely embeds into an upper stable metric space or
(ii) the unit ball of $\mathscr{X}$ uniformly embeds into an upper stable metric space.

Then every spreading basic sequence in $\mathscr{X}$ is unconditional.

Before we give the proof of Theorem 4.12, observe that the following corollary (cf. Theorem 4.3) follows from the fact that the summing basis of c_0 is spreading but not unconditional.

Corollary 4.13 c_0 *(resp. the unit ball of* c_0*) does not coarsely (resp. uniformly) embed into an upper stable metric space.*

Proof of Theorem 4.12 We begin by proving (i). Assume that the spreading basic sequence $\{x_n\}_{n=1}^\infty$ is C_b-basic, i.e.

$$\left\|\sum_{i=1}^j a_i x_i\right\| \le C_b\left\|\sum_{i=1}^k a_i x_i\right\|$$

for all $1 \le j \le k$, and $a_1, \dots, a_k \in \mathbb{R}$. Let A_s and B_s be the spreading constants as in (51) and K_u be the upper stability constant of the target metric space.

The following claim, which seems to be folklore, is implicitly used in Raynaud's proof. We could not locate a proof in the literature so we will include one that was graciously explained to us by Steve Dilworth.

[14] Sequences that are both spreading and unconditional are called subsymmetric.

Claim 4.14 *Let $\{x_n\}_{n=1}^{\infty}$ be a spreading sequence. If there exists a constant $\gamma > 0$ such that for all $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in \{-1, 1\}^k$,*

$$\left\| \sum_{i=1}^{k} x_i \right\| \leq \gamma \left\| \sum_{i=1}^{k} \varepsilon_i x_i \right\|, \tag{52}$$

then $\{x_n\}_{n=1}^{\infty}$ is unconditional.

Proof Assume that $\{x_n\}_{n=1}^{\infty}$ is spreading. If $\{x_n\}_{n=1}^{\infty}$ is weakly null, then it is well known that $\{x_n\}_{n=1}^{\infty}$ is unconditional (cf. [15, Proposition 2, page 17]). If $\{x_n\}_{n=1}^{\infty}$ is not weakly null, then there exist $\delta > 0$, a functional $x^* \in \mathscr{X}^*$ of norm 1, and integers $1 \leq n_1 < n_2 < \cdots < n_i < \cdots$ such that $x^*(x_{n_i}) \geq \delta$ for all $i \in \mathbb{N}$. Hence, for all $j \in \mathbb{N}$, $\| \sum_{i=1}^{j} x_{n_i} \| \geq j\delta$ and for all $\varepsilon = (\varepsilon_1, \dots, \varepsilon_j) \in \{-1, 1\}^j$,

$$\left\| \sum_{i=1}^{j} \varepsilon_i x_i \right\| \overset{(52)}{\geq} \gamma^{-1} \left\| \sum_{i=1}^{j} x_i \right\| \geq \frac{1}{\gamma A_s} \left\| \sum_{i=1}^{j} x_{n_i} \right\| \geq \frac{\delta}{\gamma A_s} j. \tag{53}$$

Therefore, there is a constant $c := \frac{\delta}{\gamma A_s} > 0$ such that $\frac{1}{2^j} \sum_{\varepsilon \in \{-1,1\}^j} \| \sum_{i=1}^{j} \varepsilon_i x_i \| \geq cj$, and it follows from Elton's theorem [24] that there are constants $\lambda, \mu > 0$, depending only on the constant c, such that for each $j \geq 1$ there exists $S_j \subseteq \{1, 2, \dots, j\}$ with $|S_j| \geq \lambda j$ and $\| \sum_{i \in S_j} a_i x_i \| \geq \mu \sum_{i \in S_j} |a_i|$ for all finite sequences of scalars $\{a_i\}_{i \in S_j}$. Consequently, the sequence $\{x_i\}_{i \in S_j}$ is μ-equivalent to the canonical basis of $\ell_1^{|S_j|}$. Since $\lim_{j \to \infty} |S_j| = \infty$ and $\{x_n\}_{n=1}^{\infty}$ is spreading, it follows that $\{x_n\}_{n=1}^{\infty}$ is equivalent to the canonical basis of ℓ_1, and hence $\{x_n\}_{n=1}^{\infty}$ is unconditional. $\square$

For any $\eta \in \{-1, 1\}^k$ let $N(\eta) \overset{\text{def}}{=} \| \sum_{i=1}^{k} \eta_i x_i \|$. By Claim 4.14 it suffices to show that there exists a constant γ such that for any $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k) \in \{-1, 1\}^k$,

$$N(\mathbf{1}) = \left\| \sum_{i=1}^{k} x_i \right\| \leq \gamma \left\| \sum_{i=1}^{k} \varepsilon_i x_i \right\| = \gamma N(\varepsilon). \tag{54}$$

Let $\ell \overset{\text{def}}{=} |\{i : \varepsilon_i = +1\}|$ and π the permutation on $[k]$ that maps $\{1, \dots, \ell\}$ onto $\{i : \varepsilon_i = +1\}$ and $\{\ell + 1, \dots, k\}$ onto $\{i : \varepsilon_i = -1\}$ while preserving the order on the respective sets. If we define for $(m_1, \dots, m_\ell) \in [\mathbb{N}]^\ell$

$$f(m_1, \dots, m_\ell) = \sum_{i=1}^{\ell} x_{m_i}$$

and for $(m_1, \dots, m_{k-\ell}) \in [\mathbb{N}]^{k-\ell}$

$$g(m_1, \dots, m_{k-\ell}) = \sum_{i=1}^{k-\ell} x_{m_i},$$

it follows from the proof of Lemma 4.10 that

$$\begin{aligned} &\inf_{\bar{n} \in [\mathbb{N}]^k} \| f(n_1, \dots, n_\ell) - g(n_{\ell+1}, \dots, n_k) \| \\ &\quad \leq C \sup_{\bar{n} \in [\mathbb{N}]^k} \| f(n_{\pi(1)}, \dots, n_{\pi(\ell)}) - g(n_{\pi(\ell+1)}, \dots, n_{\pi(k)}) \|, \end{aligned}$$

where C is any constant such that $\rho(C) \geq 2K_u\omega(1)$. Observe now that for all $(n_1, \ldots, n_k) \in [\mathbb{N}]^k$

$$A_s N(\varepsilon) \geq \left\| \sum_{i=1}^{k} \varepsilon_i x_{n_i} \right\| = \left\| \sum_{i=1}^{\ell} x_{n_{\pi(i)}} - \sum_{i=\ell+1}^{k} x_{n_{\pi(i)}} \right\|$$
$$= \| f(n_{\pi(1)}, \ldots, n_{\pi(\ell)}) - g(n_{\pi(\ell+1)}, \ldots, n_{\pi(k)}) \|.$$

and hence

$$\sup_{\bar{n} \in [\mathbb{N}]^k} \| f(n_{\pi(1)}, \ldots, n_{\pi(\ell)}) - g(n_{\pi(\ell+1)}, \ldots, n_{\pi(k)}) \| \leq A_s N(\varepsilon).$$

On the other hand, for all $n_1 < \cdots < n_k$,

$$N(\mathbf{1}) = \left\| \sum_{i=1}^{k} x_i \right\| \leq B_s \left\| \sum_{i=1}^{k} x_{n_i} \right\|, \tag{55}$$

and

$$\left\| \sum_{i=1}^{k} x_{n_i} \right\| \leq \left\| - \sum_{i=1}^{\ell} x_{n_i} + \sum_{i=\ell+1}^{k} x_{n_i} \right\| + 2 \left\| \sum_{i=1}^{\ell} x_{n_i} \right\|$$
$$\leq \left\| \sum_{i=1}^{\ell} x_{n_i} - \sum_{i=\ell+1}^{k} x_{n_i} \right\| + 2C_b \left\| \sum_{i=1}^{\ell} x_{n_i} - \sum_{i=\ell+1}^{k} x_{n_i} \right\|$$
$$= (2C_b + 1) \left\| \sum_{i=1}^{\ell} x_{n_i} - \sum_{i=\ell+1}^{k} x_{n_i} \right\|.$$

Combining the last inequality with (55) we have

$$N(\mathbf{1}) \leq B_s(2C_b + 1) \left\| \sum_{i=1}^{\ell} x_{n_i} - \sum_{i=\ell+1}^{k} x_{n_i} \right\|. \tag{56}$$

and hence,

$$\inf_{\bar{n} \in [\mathbb{N}]^k} \| f(n_1, \ldots, n_\ell) - g(n_{\ell+1}, \ldots, n_k) \| \geq \frac{N(\mathbf{1})}{B_s(2C_b + 1)}.$$

Consequently $N(\mathbf{1}) \leq C A_s B_s (2C_b + 1) N(\varepsilon)$ which is (54) with $\gamma = C A_s B_s (2C_b + 1)$.

The proof of (ii) goes as follows. If $N(\mathbf{1}) \leq N(\varepsilon)$ there is nothing to prove and in the sequel we assume that $N(\varepsilon) \leq N(\mathbf{1})$. We keep the same notation as in the proof of (i). First, observe that for every $1 \leq j \leq k$ and $m_1 < \cdots < m_j$,

$$\left\| \sum_{i=1}^{j} x_{m_i} \right\| \leq A_s \left\| \sum_{i=1}^{j} x_i \right\| \leq A_s C_b \left\| \sum_{i=1}^{k} x_i \right\| = A_s C_b N(\mathbf{1}).$$

Therefore, if for $1 \leq \ell < k$ we define for $(m_1, \ldots, m_\ell) \in [\mathbb{N}]^\ell$

$$f(m_1, \ldots, m_\ell) = \frac{1}{A_s C_b N(\mathbf{1})} \sum_{i=1}^{\ell} x_{m_i}$$

and for $(m_1, \dots, m_{k-\ell}) \in [\mathbb{N}]^{k-\ell}$

$$g(m_1, \dots, m_{k-\ell}) = \frac{1}{A_s C_b N(\mathbf{1})} \sum_{i=1}^{k-\ell} x_{m_i},$$

then f and g take values in the unit ball of $\mathscr{X}$. For any $n_1 < \cdots < n_k$

$$\begin{aligned} N(\mathbf{1}) &\overset{(56)}{\leq} B_s(2C_b+1) \left\| \sum_{i=1}^{\ell} x_{n_i} - \sum_{i=\ell+1}^{k} x_{n_i} \right\| \\ &= A_s B_s (2C_b+1) C_b N(\mathbf{1}) \| f(n_1, \dots, n_\ell) - g(n_{\ell+1}, \dots, n_k) \|, \end{aligned}$$

and hence

$$\inf_{\bar{n} \in [\mathbb{N}]^k} \| f(n_1, \dots, n_\ell) - g(n_{\ell+1}, \dots, n_k) \| \geq \frac{1}{A_s B_s (2C_b+1) C_b}.$$

If C is any positive number satisfying $0 < \omega(C) \leq \frac{1}{2K_u} \rho\Big(\frac{1}{A_s B_s (2C_b+1) C_b}\Big)$ it follows from Lemma 4.11 that

$$\sup_{\bar{n} \in [\mathbb{N}]^k} \| f(n_{\pi(1)}, \dots, n_{\pi(\ell)}) - g(n_{\pi(\ell+1)}, \dots, n_{\pi(k)}) \| \geq C,$$

and thus there exists $\bar{n} \in [\mathbb{N}]^k$ such that

$$\begin{aligned} A_s N(\varepsilon) = A_s \left\| \sum_{i=1}^{k} \varepsilon_i x_i \right\| &\geq \left\| \sum_{i=1}^{k} \varepsilon_i x_{n_i} \right\| = \left\| \sum_{i=1}^{\ell} x_{n_{\pi(i)}} - \sum_{i=\ell+1}^{k} x_{n_{\pi(i)}} \right\| \\ &\geq A_s C_b N(\mathbf{1}) \| f(n_{\pi(1)}, \dots, n_{\pi(\ell)}) - g(n_{\pi(\ell+1)}, \dots, n_{\pi(k)}) \| \\ &\geq A_s C_b N(\mathbf{1}) \frac{C}{2}. \end{aligned}$$

Taking $\gamma = \max\{1, \frac{2}{C \cdot C_b}\}$ in (54) concludes the proof. □

The next theorem shows that upper stability is a strengthening of Kalton's property $\mathcal{Q}$.

Theorem 4.15 *Every upper stable metric space has property $\mathcal{Q}$.*

Proof Let $(\mathsf{X}, \mathsf{d_X})$ be K_u-upper stable and fix $k \geq 1$ and a Lipschitz map $f\colon ([\mathbb{N}]^k, \mathsf{d_I}) \to \mathsf{X}$. Define

$$\mathcal{A} \overset{\text{def}}{=} \Big\{ \bar{n} \in [\mathbb{N}]^{2k} \colon \mathsf{d_X}\big(f(n_1, \dots, n_k), f(n_{k+1}, \dots, n_{2k})\big) \leq 2K_u \cdot \mathrm{Lip}(f) \Big\}.$$

By Ramsey theorem there exists an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that either $[\mathbb{M}]^{2k} \subset \mathcal{A}$ or $[\mathbb{M}]^{2k} \cap \mathcal{A} = \emptyset$. If the first possibility happens then for all $\bar{n} \in [\mathbb{M}]^{2k}$ we have

$$\mathsf{d_X}\big(f(n_1, \dots, n_k), f(n_{k+1}, \dots, n_{2k})\big) \leq 2K_u \cdot \mathrm{Lip}(f).$$

If $\bar{n}, \bar{m} \in [\mathbb{M}]^k$ we can choose $\bar{x} \in [\mathbb{M}]^k$ such that $x_1 > \max\{n_k, m_k\}$ and thus

$$\mathsf{d_X}\big(f(\bar{n}), f(\bar{m})\big) \leq \mathsf{d_X}\big(f(\bar{n}), f(\bar{x})\big) + \mathsf{d_X}\big(f(\bar{x}), f(\bar{m})\big) \leq 4K_u \cdot \mathrm{Lip}(f).$$

It remains to show that the second possibility cannot happen. Consider the interlacing permutation $\pi\colon [2k] \to [2k]$ defined by

$$\pi(i) = \begin{cases} 2i-1 & if \quad 1 \leq i \leq k \\ 2(i-k) & if \quad k+1 \leq i \leq 2k. \end{cases}$$

It is immediate to verify that π preserves the order on $\{1,\dots,k\}$ and on $\{k+1,\dots,2k\}$ and if $\bar{n}\in[\mathbb{M}]^{2k}$ then

$$\begin{aligned}&\mathrm{d}_{\mathrm{X}}(f(n_{\pi(1)},\dots,n_{\pi(k)}),f(n_{\pi(k+1)},\dots,n_{\pi(2k)}))\\&\quad=\mathrm{d}_{\mathrm{X}}(f(n_1,n_3,\dots,n_{2k-1}),f(n_2,n_4,\dots,n_{2k}))\le \mathrm{Lip}(f).\end{aligned}$$

Therefore,

$$\sup_{\bar{n}\in[\mathbb{M}]^{2k}}\mathrm{d}_{\mathrm{X}}(f(n_{\pi(1)},\dots,n_{\pi(k)}),f(n_{\pi(k+1)},\dots,n_{\pi(2k)}))\le \mathrm{Lip}(f),$$

and by K_u-upper stability it holds

$$\inf_{\bar{n}\in[\mathbb{M}]^{2k}}\mathrm{d}_{\mathrm{X}}(f(n_1,\dots,n_k),f(n_{k+1},\dots,n_{2k}))\le K_u\mathrm{Lip}(f).$$

In particular there exists $\bar{n}\in[\mathbb{M}]^{2k}$ such that

$$\mathrm{d}_{\mathrm{X}}(f(n_1,\dots,n_k),f(n_{k+1},\dots,n_{2k}))\le 2K_u\mathrm{Lip}(f),$$

but this implies that $[\mathbb{M}]^{2k}\cap\mathcal{A}\neq\emptyset$; a contradiction. $\square$

Recall that a Banach space $\mathscr{X}$ has trivial Rademacher type if and only if $\mathscr{X}$ contains the ℓ_1^n's (see [60]). It was shown by Guerre-Delabrière and Lapresté [39] that every non-reflexive stable Banach space contains an isomorphic copy of ℓ_1 and hence has trivial Rademacher type. Raynaud showed that every non-reflexive Banach space whose unit ball uniformly embeds into a stable metric has a spreading model isomorphic to ℓ_1. Without delving too deep into the theory of spreading models, a spreading model of $\mathscr{X}$ is a Banach space that can be associated to $\mathscr{X}$ and that is finitely representable in $\mathscr{X}$. Therefore the unit ball of a non-reflexive Banach space with non-trivial Rademacher type does not even embed uniformly into a stable space. Classical examples of non-reflexive Banach spaces with non-trivial Rademacher type are James non-reflexive space of type 2 [45] and Pisier-Xu interpolation spaces [79]. Raynaud's result was extended by Kalton. Indeed, it is a result from [46] that if $\mathscr{X}$ has property $\mathcal{Q}$ and is non-reflexive, then $\mathscr{X}$ has a spreading model isomorphic to ℓ_1. Kalton's proof relies on a result of Beauzamy [14] which says that a Banach space has the alternating Banach-Saks property if and only if none of its spreading models are isomorphic to ℓ_1. Recall that a Banach space $\mathscr{X}$ has the alternating Banach-Saks property if every bounded sequence $\{x_n\}_{n=1}^\infty\in\mathscr{X}$ has a subsequence $\{y_n\}_{n=1}^\infty$ so that the alternating Cesaro means $\frac{1}{n}\sum_{k=1}^n(-1)^k y_k$ converge in norm to 0. Note that the class of Banach spaces with the alternating Banach-Saks property strictly extends the class of Banach spaces with non-trivial type since c_0 has the alternating Banach-Saks property and trivial Rademacher type.

The conclusion of Raynaud and Kalton results obviously holds for non-reflexive upper stable spaces since upper stability implies property $\mathcal{Q}$. Therefore it follows from Kalton's result that a non-reflexive and upper stable Banach space has a spreading model isomorphic to ℓ_1, and thus contains the ℓ_1^n's. If we are merely interested in the containment of the ℓ_1^n's the elementary argument given below is all we need.

Theorem 4.16 *Let $\mathscr{X}$ be a Banach space which is non-reflexive. Assume that, either*

(i) $\mathscr{X}$ coarsely embeds into an upper stable metric space or
(ii) the unit ball of $\mathscr{X}$ uniformly embeds into an upper stable metric space.

Then $\mathscr{X}$ contains the ℓ_1^n's.

Proof Assertion (i) is a simple consequence of Lemma 4.10. Since $\mathscr{X}$ is not reflexive there exists a normalized James' sequence $\{x_n\}_{n=1}^{\infty}$ such that for all $1 \le \ell < k$, $b_1, \dots, b_k \ge 0$, and $\bar{n} \in [\mathbb{N}]^k$,

$$\|\sum_{i=1}^{\ell} b_i x_{n_i} - \sum_{i=\ell+1}^{k} b_i x_{n_i}\| \ge \frac{1}{2}\sum_{i=1}^{k} b_i \tag{57}$$

Let $a_1, \dots, a_k \in \mathbb{R}$ and choose $\varepsilon_1, \dots, \varepsilon_k \in \{\pm 1\}$ such that $\varepsilon_i |a_i| = a_i$ for all $1 \le i \le k$. Let $\ell \overset{\text{def}}{=} |\{i : \varepsilon_i = +1\}|$ and π the permutation on $[k]$ that maps $\{1, \dots, \ell\}$ onto $\{i : \varepsilon_i = +1\}$ and $\{\ell + 1, \dots, k\}$ onto $\{i : \varepsilon_i = -1\}$ while preserving the order on the respective sets. Let $\alpha \overset{\text{def}}{=} \sum_{i=1}^{k} |a_i|$ and define for $\bar{m} \in [N]^{\ell}$, $f(m_1, \dots, m_\ell) = \frac{1}{\alpha}\sum_{i=1}^{\ell} |a_{\pi(i)}| x_{m_i}$ and for $\bar{m} \in [\mathbb{N}]^{k-\ell}$, $g(m_1, \dots, m_{k-\ell}) = \frac{1}{\alpha}\sum_{i=1}^{k-\ell} |a_{\pi(\ell+i)}| x_{m_i}$. Observe that for all $n_1 < \cdots < n_k$,

$$\begin{aligned} \|f(n_1, \dots, n_\ell) - g(n_{\ell+1}, \dots, n_k)\| &= \frac{1}{\alpha}\left\|\sum_{i=1}^{\ell} |a_{\pi(i)}| x_{n_i} - \sum_{i=\ell+1}^{k} |a_{\pi(i)}| x_{n_i}\right\| \\ &\overset{(57)}{\ge} \frac{1}{2\alpha}\sum_{i=1}^{k} |a_{\pi(i)}| = \frac{1}{2\alpha}\sum_{i=1}^{k} |a_i| = \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} \|f(n_{\pi(1)}, \dots, n_{\pi(\ell)}) - g(n_{\pi(\ell+1)}, \dots, n_{\pi(k)})\| &= \frac{1}{\alpha}\left\|\sum_{i=1}^{\ell} |a_{\pi(i)}| x_{n_{\pi(i)}} - \sum_{i=\ell+1}^{k} |a_{\pi(i)}| x_{n_{\pi(i)}}\right\| \\ &= \frac{1}{\alpha}\left\|\sum_{i\colon a_i \ge 0} |a_i| x_{n_i} - \sum_{i\colon a_i \le 0} |a_i| x_{n_i}\right\| \\ &= \frac{1}{\alpha}\left\|\sum_{i=1}^{k} a_i x_{n_i}\right\|, \end{aligned}$$

It follows from Lemma 4.10 that there exists $C > 0$ such that

$$\begin{aligned} \inf_{\bar{n} \in [\mathbb{N}]^k} \|f(n_1, \dots, n_\ell) - g(n_{\ell+1}, \dots, n_k)\| \le C \sup_{\bar{n} \in [\mathbb{N}]^k} \|f(n_{\pi(1)}, \\ \dots, n_{\pi(\ell)}) - g(n_{\pi(\ell+1)}, \dots, n_{\pi(k)})\|, \end{aligned}$$

and hence

$$\frac{1}{C}\sum_{i=1}^{k} |a_i| \le \sup_{\bar{n} \in [\mathbb{N}]^k} \left\|\sum_{i=1}^{k} a_i x_{n_i}\right\|.$$

This implies that there exists $\bar{n} \in [\mathbb{N}]^k$ such that

$$\frac{1}{2C}\sum_{i=1}^{k} |a_i| \le \left\|\sum_{i=1}^{k} a_i x_{n_i}\right\| \le \sum_{i=1}^{k} |a_i|.$$

For assertion (ii) we can apply Lemma 4.11 since f and g take values in the unit ball of $\mathscr{X}$. □

In general, reflexivity is not preserved under coarse embeddability (resp. uniform embeddability of the unit ball) as it is well known that ℓ_1 coarsely embeds (resp. its unit ball uniformly embeds) into Hilbert space. The space ℓ_1 obviously contains the ℓ_1^n's and (part of) the following theorem, which is originally due to Kalton [46] under assumption (iii) or (iv), states that this is the only obstruction.

Theorem 4.17 *Let $\mathscr{X}$ be a Banach space that does not contain the ℓ_1^n's. Assume that at least one of the following conditions hold:*

(i) *$\mathscr{X}$ coarsely embeds into an upper stable metric space,*
(ii) *the unit ball of $\mathscr{X}$ uniformly embeds into an upper stable metric space,*
(iii) *$\mathscr{X}$ coarsely embeds into a reflexive Banach space,*
(iv) *the unit ball of $\mathscr{X}$ uniformly embeds into a reflexive Banach space.*

Then $\mathscr{X}$ is reflexive.

The content of Theorem 4.17 pertaining to upper stable metrics is logically equivalent to Theorem 4.16 and is an extension of [80, Proposition 5.2 (c)] by Remark 4.8. The statement regarding embeddings into reflexive spaces, which is surprisingly not explicitly stated in [46], follows from the fact that either (iii) or (iv) implies that $\mathscr{X}$ has property $\mathcal{Q}$ [46, Corollary 4.3] and that a non-reflexive space with property $\mathcal{Q}$ must contain the ℓ_1^n's [46, Corollary 4.6].

5 Final remarks and open problems

5.1 Optimal factor in Ostrovskii's finite determinacy theorem

Ostrovskii's finite determinacy theorem says that there is a universal constant $\alpha \in (0, \infty)$ such that for every Banach space $\mathscr{Y}$ and every locally finite metric space X, X admits a bi-Lipschitz embedding into $\mathscr{Y}$ with distortion at most $\alpha \cdot \beta$ whenever X is finitely β-bi-Lipschitz representable in $\mathscr{Y}$. A quick inspection of Ostrovskii's original proof gives that we can take α in the thousands (3000 will do). An example which shows that necessarily $\alpha > 1$ can be found in [48]. The parameter α was studied in [70,71]. Most notably it was shown that if the host space is an ℓ_p -sum of nested finite-dimensional spaces, then α can be taken to be arbitrarily close to 1, and examples are produced showing that we cannot take $\alpha = 1$. A very interesting new local-to-global embedding technique—logarithmic spiral gluing—was introduced to achieve this degree of precision in this special case.

5.2 $L_p[0, 1]$-compression exponent versus ℓ_p-compression exponent

The *compression exponent* of $(\mathsf{X}, \mathrm{d_X})$ in $(\mathsf{Y}, \mathrm{d_Y})$ (Y-compression of X in short) introduced by Guentner and Kaminker [32], is the parameter denoted by $\alpha_{\mathsf{Y}}(\mathsf{X})$ and defined as the supremum of all numbers $0 \le \alpha \le 1$ for which there exist $f\colon \mathsf{X} \to \mathsf{Y}$, $\tau \in (0, \infty)$, and $A \in [1, \infty)$ such that $\mathrm{d_X}(x, y) \ge \tau$ implies

$$\frac{1}{A}\mathrm{d_X}(x, y)^{\alpha} \le \mathrm{d_Y}(f(x), f(y)) \le A\mathrm{d_X}(x, y).$$

The compression exponent of an (unbounded) metric space is clearly invariant under coarse bi-Lipschitz embeddings. We will write L_p for $L_p[0, 1]$. For any metric space X, since ℓ_p embeds isometrically into L_p, and since L_p contains an isometric copy of L_2, the inequalities

$\alpha_{\ell_p}(\mathsf{X}) \leq \alpha_{L_p}(\mathsf{X})$ and $\alpha_2(\mathsf{X}) \leq \alpha_{L_p}(\mathsf{X})$ always hold for any metric space X. It is well known that finite subsets of L_p embed isometrically into ℓ_p and thus it follows that the L_p-compression of an unbounded sequence of finite metric spaces (defined in a natural way) and its ℓ_p-compression coincide. A similar reasoning does not work to prove an analogue statement if one is dealing with an (unbounded) infinite metric space since, for instance, L_4 does not coarse bi-Lipschitzly embeds into ℓ_4 [49]. Corollary 5.2, which obviously applies to finitely generated groups, answers negatively a question raised by Naor and Peres. According to Naor and Peres this subtlety between the L_p-compression and the ℓ_p-compression for infinite groups was first pointed out by Marc Bourdon, and Naor and Peres asked (Question 10.7 in [67]) whether there is a finitely generated group Γ such that $\alpha_{\ell_p}(\Gamma) \neq \alpha_{L_p}(\Gamma)$. Theorem 5.1 below is an easy consequence of Theorem 2.4.

Theorem 5.1 *Let $p \in [1,\infty]$. If ℓ_p is finitely representable in a Banach space $\mathcal{Y}$ then any locally finite subset of L_p admits a bi-Lipschitz embedding into $\mathcal{Y}$. In particular, any locally finite subset of L_p admits a bi-Lipschitz embedding into ℓ_p.*

In particular, Theorem 5.1 and the coarse Lipschitz invariance of compression exponents provides a complete answer for Naor and Peres question.

Corollary 5.2 *Let X be a proper metric space. Then, for every $p \in [1,+\infty]$*

$$\alpha_{\ell_p}(X) = \alpha_{L_p}(X).$$

In particular, if Γ is a finitely generated group with its canonical word metric, or a compactly generated group with its canonical proper metric, then

$$\alpha_{\ell_p}(\Gamma) = \alpha_{L_p}(\Gamma).$$

Remark 5.3 Corollary 5.2 also applies to locally compact second countable groups by the results of [40].

The cases $p = \infty$ and $p = 2$ of Theorem 5.1 are just reformulations of Corollaries 2.7 and 2.8, respectively. When $1 < p < \infty$ simply observe that every finite subset of L_p embeds isometrically into some ℓ_p^n, and thus L_p is finitely bi-Lipschitz representable in every Banach space which contains the ℓ_p^n's.

A coarse version of Theorem 5.1 seems to have appeared first in the unpublished manuscript [73]. After the appearance of [10], the coarse statement of [73] was upgraded to the bi-Lipschitz category, and only the explicit statement in Corollary 2.8 made it to the published paper [74, Theorem 6] while the explicit statement of Theorem 5.1 was surprisingly left out (but can be derived from [74, Proposition 1]). Theorem 5.1 appeared explicitly in [7] where the statement and the proof uses the convenient framework of $\mathcal{L}_p$-spaces.

Naor and Peres [66] proved that $\alpha_2^{\#}(\Gamma) \leq \alpha_{L_p}^{\#}(\Gamma)$, for any $p \in [1,\infty)$ and every finitely generated amenable group Γ, where $\alpha_{\mathsf{Y}}^{\#}$ denotes the Y-equivariant compression (see [66] for the definition). The following result follows from Theorem 2.17.

Corollary 5.4 *Let X be a proper metric space. If $\mathcal{X}$ and $\mathcal{Y}$ are two Banach spaces such that $\mathcal{X}$ is finitely representable in $\mathcal{Y}$, then $\alpha_{\mathcal{X}}(X) \leq \alpha_{\mathcal{Y}}(X)$. In particular, $\alpha_2(X) \leq \alpha_{\mathcal{Y}}(X)$ for every infinite-dimensional Banach space $\mathcal{Y}$.*

Note that the second part of the statement can be obtained with Corollary 2.8 only. We do not know if an equivariant analogue of Corollary 5.4 is true.

Proposition 5.5 *Let Γ be a finitely generated group. Do we have $\alpha_2^{\#}(\Gamma) \le \alpha_{\mathscr{Y}}^{\#}(\Gamma)$ for every infinite-dimensional Banach space $\mathscr{Y}$?*

The equivariant analogue of Corollary 5.2 also seems open.

Proposition 5.6 *Let Γ be a finitely generated group. Do we have for every $p \in [1, +\infty)$*

$$\alpha_{\ell_p}^{\#}(\Gamma) = \alpha_{L_p}^{\#}(\Gamma)?$$

5.3 Equivariant versions of the embedding results

Brown and Guentner showed that every countable discrete group admits a proper affine isometric action on a reflexive space $\left(\sum_{n=1}^{\infty} \ell_{p_n}\right)_2$ for some $p_n \to \infty$. This result was extended to locally compact second countable groups in [40]. A coarse equivariant version of Kalton's embedding result for stable metrics was proved by Rosendal [84, Theorem 22]. It would be interesting to know if there are equivariant versions of Corollary 2.7 for countable discrete groups, and of Corollaries 3.1 and 3.5 for compactly generated groups.

5.4 Embeddability of reflexive spaces into stable spaces

Note that $\mathscr{T}$ (Figiel–Johnson–Tsirelson space [28] that contains the ℓ_1^n's, and which is the dual of Tsirelson space $\mathscr{T}^*$ [85]) does not have the alternating Banach-Saks property, in a strong sense, since all its spreading models are isomorphic to ℓ_1. However there are reflexive Banach spaces with the alternating Banach-Saks property which fail to be stable. Problem 5.7 which was raised by Kalton [46, Problems 6.1, 6.2] asks for a converse to Theorem 3.16.

Proposition 5.7 *Let $\mathscr{X}$ be a separable reflexive Banach space.*

(i) *Does $\mathscr{X}$ coarsely embed into a stable metric space?*
(ii) *Does $\mathscr{X}$ uniformly embed into a stable metric space?*
(iii) *Does the unit ball of $\mathscr{X}$ uniformly embed into a stable metric space?*

The difficulty of Problem 5.7 can be partially explained by the deep work from [46] about the relationship between reflexivity and property $\mathcal{Q}$. Firstly, one of the main results from [46] is that every reflexive Banach space has property $\mathcal{Q}$ and this was used to resolve negatively the long-standing problem whether c_0 (resp. the unit ball of c_0) coarsely (resp. uniformly) embeds into a reflexive Banach space. Secondly, if $\mathscr{X}$ is a Banach space with the alternating Banach-Saks property, then under either assumption of Problem 5.7, $\mathscr{X}$ must necessarily be reflexive. We suspect the answer to assertion (i) and (ii) in Problem 5.7 are negative since we conjecture that there are reflexive spaces that are not upper stable. In particular, we conjecture that upper stability is strictly stronger than property $\mathcal{Q}$.

Note that it follows from [80] that the unit ball of a Tsirelson-like space does not uniformly embed into a super-stable space. It was shown in [18] that $\mathscr{T}^*$ (Tsirelson's original space [85] that contains the ℓ_∞^n's, and which is the dual of Figiel–Johnson–Tsirelson space $\mathscr{T}$ [28]) does not coarsely embeds into a super-stable space. However it is not known if this is true for all Tsirelson-like spaces since the following problem is still open [46, Problem 6.6].

Proposition 5.8 *If $\mathscr{X}$ coarsely embeds into a super-stable space, does $\mathscr{X}$ contain an isomorphic copy of ℓ_p for some $p \in [1, \infty)$?*

5.5 Reflexive asymptotic-ℓ_2 spaces, Baum-Connes and Novikov conjecture

To the best of our knowledge there is no example yet of bounded geometry spaces failing the coarse Baum-Connes conjecture or the coarse Novikov conjecture. In [51,88] the conjectures are proved for bounded geometry spaces admitting a weak embedding (coarse embedding) into very regular Banach spaces (Hilbert space or a uniformly convex Banach space with a uniformly convex dual). It would be interesting to investigate whether the conjectures can be proven under a much stronger embeddability requirement (bi-Lipschitz embedding) into Banach spaces with weaker geometric features. For instance, the space $\left(\sum_{n=1}^{\infty} \ell_\infty^n\right)_2$ is reflexive, asymptotic-ℓ_2, asymptotically uniformly convex with an asymptotically uniformly convex dual.

Proposition 5.9 *Let X be a discrete metric space with bounded geometry. Assume that X admits a bi-Lipschitz embedding into a reflexive asymptotic-ℓ_2 Banach space. Does X satisfy the coarse Baum-Connes or the coarse geometric Novikov conjecture?*

An important warning is in order here. By Corollary 2.7 a positive answer to Problem 5.9 will imply a positive answer to the respective conjectures for every metric space with bounded geometry! Nevertheless, it seems important to have a deeper understanding of the tradeoff between the faithfulness of the embedding and the geometric features of the host space in order to be able to confirm the conjectures.

5.6 More on stability

Kalton [46] proved that James space $\mathscr{J}$ and its dual $\mathscr{J}^*$ [42,43] do not have the $\mathcal{Q}$-property, and this gives two more examples of non-reflexive spaces that do not embed coarsely into a stable space.

The work of Rosendal [84] (and the references therein) contains lots of information about the existence of compatible stable metrics on groups.

To find more examples of Banach spaces which are not stable we need to resort to non-commutative L_p-spaces. We refer to [78] for a thorough discussion of non-commutative L_p-spaces. The commutative L_p-spaces belong to the class of non-commutative L_p-spaces. For instance, $L_p[0,1]$ can been represented as the non-commutative L_p-space associated to $L_\infty[0,1]$ considered as a von Neuman algebra on the Hilbert space $L_2[0,1]$. The simplest truly non-commutative L_p-spaces are the Schatten classes (denoted $\mathscr{S}_p(H)$). They are defined as the non-commutative L_p-spaces associated to $B(H)$, the algebra of all bounded operators on a Hilbert space H, equipped with be the usual trace on $B(H)$. The commutative theory can be satisfactorily extended to a large extent to the non-commutative setting, however there are some significant differences. The stability property of the non-commutative spaces is one of them.

Theorem 5.10 *Let $p \in [1,\infty)$, $p \neq 2$. Then a non-commutative L_p-space associated to a von Neumann algebra $\mathfrak{M}$ is stable if and only if $\mathfrak{M}$ is of type I.*

The "if" part of the theorem above was independently proved in [4,81] where it is shown that the non-commutative L_p-space associated to a von Neumann algebra $\mathfrak{M}$ of type I can be written as an ℓ_p-sum of commutative vector-valued L_p-spaces whose values fall into stable Banach spaces. The "only if" part comes from [58] and is proved in two steps as follows. Marcolino first showed that if $\mathfrak{M}$ is a von Neuman algebra not of type I, then for $1 \leq p \leq \infty$, the non-commutative L_p-space associated to the hyper finite II_1 factor is isometric to a (1-complemented) subspace of the non-commutative L_p-space associated to $\mathfrak{M}$. The conclusion follows from the fact that the non-commutative L_p-space associated to the hyper finite II_1 factor is not stable. Those spaces have a completely different linear structure compared to Tsirelson-like spaces since they contain copies of ℓ_p. The following problem seems open.

Proposition 5.11 *Does the non-commutative L_p-space associated to the hyper finite II_1 factor admit an equivalent stable norm?*

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