FIRST STABILITY EIGENVALUE OF SINGULAR HYPERSURFACES WITH CONSTANT MEAN CURVATURE IN SPHERES

NGUYEN THAC DUNG, JUNCHEOL PYO, AND HUNG TRAN

Abstract. In this paper, we study the first eigenvalue of the Jacobi operator on an integral n-varifold with constant mean curvature in the unit sphere $\mathbb{S}^{n+1}$. We found the optimal upper bound and prove a rigidity result characterizing the case when it is attained. This gives a new characterization for certain singular Clifford tori.

1. Introduction

Let M be an oriented smooth two-sided hypersurface immersed in the $(n+1)$-dimensional sphere. Let $\psi : M \to \mathbb{S}^{n+1}$ be the immersion and ν be a choice of a unit normal vector field along M. We consider a compact variation of the hypersurface, for any $t \in (-\varepsilon, \varepsilon)$,

$$\psi_t : M \to \mathbb{S}^{n+1}, \ \mathrm{supp}\psi_t \Subset M,$$

is an immersion with $\psi_0 = \psi$. The area of ψ_t is defined to be

$$A(t) = \int_M dA_t$$

where dA_t is the area element obtained via a pullback by the immersion ψ_t. The first variation formula of the area functional $A(t)$ is given by

$$\left.\frac{dA(t)}{dt}\right|_{t=0} = -\int_M nHf dA,$$

where $f = \left\langle \frac{\partial \psi_t}{\partial t}\Big|_{t=0}, \nu \right\rangle$ and H denotes the mean curvature. Hence, a compact hypersurface is minimal, namely, $H \equiv 0$, if and only if $\left.\frac{dA(t)}{dt}\right|_{t=0} = 0$. The second variation formula is given by

$$\left.\frac{d^2A(t)}{dt^2}\right|_{t=0} = -\int_M fJf dA,$$

where $J := \Delta + (|A|^2 + n) = \Delta + (|\Phi|^2 + n(1 + H^2))$. J is conventionally referred as the stability or Jacobi operator. Here A, Φ, H are the second fundamental form, the traceless second fundamental form and the mean curvature of ψ, respectively, and Δ is the Laplace-Beltrami operator. Let λ_1 be the the first eigenvalue of J,

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$Ju = -\lambda_1 u$ for some eigenfunction u. The variational characterization of λ_1 implies (see [Cha84])

$$\lambda_1 = \min\left\{\frac{\int_M fJf}{\int_M f^2}; f \in \mathcal{C}_0^\infty(M)\right\}.$$

The easiest minimal hypersurfaces to describe are the equators, i.e. the totally geodesic $(\mathbb{S}^n)'s$, and the Clifford tori defined by

$$\mathbb{S}^k\left(\sqrt{\frac{k}{n}}\right) \times \mathbb{S}^\ell\left(\sqrt{\frac{\ell}{n}}\right) \subset \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$$

with $k + \ell = n$. In his celebrated work [Sim68], Simons studied the first stability eigenvalue of a minimal closed hypersurface M^n immersed in $\mathbb{S}^{n+1}$. He proved that if M is not a totally geodesic sphere then $\lambda_1 \leq -2n$. This result enabled him to deduce that the only stable cones in $\mathbb{R}^n, n \leq 7$ are the ones that come from equators, i.e. hyperplanes. Later on, Wu [Wu93] characterized the equality $\lambda_1 = -2n$ case by showing that it holds only for the minimal Clifford torus. Shortly thereafter, Perdomo [Per02] provides a new proof of this spectral characterization. Afterwards, Alías, Barros, and Brasil Jr. [ABB04] extended these results to the case of constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$. They also characterized Clifford tori via the maximal value of λ_1. Recently, Chen and Cheng [CC17] obtained an optimal upper bound for λ_1 on non-totally umbilical compact hypersurfaces with constant mean curvature. The upper bound only depends on the mean curvature H and the dimension n. Furthermore, Cunha, Lima, and Santos [CLS17a, CLS17b] extended Chen and Cheng's result for the context of closed submanifold M^n immersed with nonzero parallel mean curvature vector field in the Euclidean unit sphere $\mathbb{S}^{n+p}$, the Euclidean space $\mathbb{R}^{n+p}$ or in the hyperbolic space $\mathbb{H}^{n+p}$.

On the other hand, there are important reasons to consider non-smooth hypersurfaces in geometric variational problems. First, in higher dimension, an area minimizing hypersurface spanning a given codimension-2 closed submanifold in Euclidean space may have a singular set. Second, in reality, it normally arises the need to model non-smooth physical objects.

Probably motivated by these considerations, F. Almgren [Alm66] introduced varifolds to prove, for every intermediate dimension, the existence of a generalized minimal surface (i.e., a surface with vanishing first variation of area) in a given compact smooth Riemannian manifold. Then, in 1972, an important partial regularity result for such varifolds was established by W. Allard [All72]. These pioneering works still have a strong influence in geometric analysis as well as related fields.

Now, we review some recent results. In [Wick14], the author gives a necessary and sufficient geometric structural condition for a stable co-dimension one integral varifold on a smooth Riemannian manifold to correspond to an embedded smooth hypersurface away from a small set of generally unavoidable singularities. He also derived regularity and compactness theorems and used them to show an optimal strong maximum principle for stationary codimension 1 integral varifolds. In [CM12], Colding and Minicozzi study the smoothness of a mean curvature flow that starts at a generic smooth closed embedded surface in $\mathbb{R}^3$ until it arrives at a singularity in a neighborhood of which the flow looks like concentric spheres or cylinders. In particular, they investigate the regularity of F-stable self-shrinkers and show that if the regular part of such an n-dimensional F-stationary integral varifold is orientable and F-stable and the singular set has finite $(n-2)$-dimensional

Hausdorff measure, then it is smooth. Recently, inspired by the study of Simons on the first eigenvalue of the Jacobi operator and the work of Colding and Minicozzi, Zhu extends the Simons' estimate to singular minimal hypersurface in $\mathbb{S}^{n+1}$, which is non-totally geodesic.

Motivated by aforementioned works, in this paper, our contribution is to extend Chen and Cheng's results to the setting of singular hypersurface with constant mean curvature in spheres. We consider n-varifolds satisfying the condition **(IV)** (see Section 2 for the precise definition) and obtain the following result.

Theorem 1.1. *Let V be an $n_{\geq 5}$-varifold in $\mathbb{S}^{n+1}$ satisfying the condition **(IV)** with orientable regular part $\mathrm{reg}V = M$. Let $\mathrm{sing}V$ be the singular part of M and assume that M has constant mean curvature and $\mathcal{H}^{n-4}(\mathrm{sing}V) = 0$ and M is non-totally umbilical.*

(1) *If $(nH)^2 < \frac{16(n-1)}{n^2(n-2)^2-16}$ then*

$$\lambda_1 \leq -n(1+H^2) - \frac{n\left(\sqrt{4(n-1)+n^2H^2}\ -(n-2)|H|\right)^2}{4(n-1)}$$

and the equality holds if and only if V is a Clifford torus $\mathbb{S}^1(r)\times\mathbb{S}^{n-1}(\sqrt{1-r^2})$ with

$$\frac{n}{(n-2)^2} > r^2 > \frac{1}{n}$$

or M is a Clifford torus $\mathbb{S}^{n-k}\left(\sqrt{\frac{n-k}{n}}\right)\times\mathbb{S}^k\left(\sqrt{\frac{k}{n}}\right)$ with $H=0$ for $k = 1,2,\ldots,n-1$.

(2) *If $(nH)^2 \geq \frac{16(n-1)}{n^2(n-2)^2-16}$ then*

$$\lambda_1 \leq -2(n-1)(1+H^2) + \frac{(n-2)^4}{8(n-1)}H^2$$

and the equality holds if and only if V is the Clifford torus $\mathbb{S}^1\left(\frac{\sqrt{n}}{n-2}\right)\times\mathbb{S}^{n-1}\left(\frac{\sqrt{(n-1)(n-4)}}{n-2}\right)$

Remark 1.2. As M is non-complete and non-compact, *a priori*, λ_1 might be $-\infty$.

We also note that, the condition (IV) is, in fact, a stationary condition with volume constraints. It is pointed out in [BW18] that if the varifold satisfies (IV) then its mean curvature is constant.

We would like to mention that to prove this theorem, we employ an approach in [CC17] choosing $f_\epsilon = (\varepsilon+|\Phi|^2)^\alpha$ to be the test function for variational formula of λ_1. However, since we work on a singular setting, it is non-trivial that the integration of f_ϵ converges. In fact, we have to choose the power α carefully leading to a divergence from the argument in [CC17]. To be precise, here we choose $\frac{1}{2}-\frac{1}{\sqrt{2n}} < \alpha \leq \frac{1}{2}$. In comparison, in [CC17], the range of α is $\left(\frac{1}{2}-\frac{1}{\sqrt{2n}}, \frac{1}{2}+\frac{1}{\sqrt{2n}}\right)$. Our main tools of dealing with the integration of f_ϵ come form [Zhu18] and [CM12] with significant modifications. They come from the fact that here we are dealing with the traceless second fundamental form and its associated Simons-type formula on a CMC hypersurface. In comparison, [Zhu18] is mostly concerned with the second fundamental form on a minimal hypersurface.

Here is the organization of our paper. In Section 2, we recall some basic background on hypersurfaces with constant mean curvature in both regular and singular setting. In Section 3, we obtain several estimates regarding the integration of $|\Phi|$ and f_ϵ. They are the key formulas in our proof of Theorem 1.1 in the last section.

2. Preliminaries

In this section, we recall basic notions about a varifold and discuss how we do integration on it.

2.1. **Brief Introduction to Varifolds.** In the following presentations, we use the approach in [BW18] (see also [Wick17])

To give a further discussion, let us recall some basic notations on varifolds (see [Sim83]). Let N be a countably n-rectifiable, $\mathcal{H}^n$-measurable subset of $\mathbb{R}^{n+p}$, where $\mathcal{H}^n$ is the n-dimensional Hausdorff measure in $\mathbb{R}^{n+p}$. Let θ be a positive locally $\mathcal{H}^n$-integrable function on N. Corresponding to such a pair (N,θ), we define the rectifiable n-varifold V to be simply the equivalence class of all pairs $(\widetilde{N},\widetilde{\theta})$ where $\widetilde{N}$ is countably n-rectifiable with $\mathcal{H}^n((N\setminus\widetilde{N})\cup(\widetilde{N}\setminus N))=0$ and $\theta=\widetilde{\theta}, \mathcal{H}^n$-a.e. on $N\cap\widetilde{N}$. θ is said to be the multiplicity function of V. Moreover, V is called an integer multiplicity rectifiable n-varifold (more briefly, an integer n-varifold) if the multiplicity function θ is integer-valued $\mathcal{H}^n$-a.e.

Associated to a rectifiable n-varifold V with the representative (N,θ) (as described above) there is a Radon measure (called the weight measure of V) defined by $\mu=\mathcal{H}^n\lfloor\theta$. Consequently, for $\mathcal{H}^n$ measurable A,

$$\mu_V(A)=\int\limits_{A\cap N}\theta d\mathcal{H}^n.$$

As in [Sim83], given a rectifiable n-varifold V, for any $q\in V$, we define the tangent space T_qV to be the approximate tangent space of μ_V whenever this exists (the reader are referred to Theorem 11.8 in [Sim83] for a discussion about the approximate tangent space). Note that $T_qV=T_qM, \mathcal{H}^n$-a.e.. We also define $\text{spt}_V:=\text{spt}\mu_V$. Thus, we can define the divergence almost everywhere by

$$\text{div}_V X(q)=\text{div}_{T_pV}X(q)=\sum_{i=1}^n\left\langle E_i,\overline{\nabla}_{E_i}X\right\rangle(q)$$

where $\{E_i\}$ is an orthonormal basic for T_qV and $\overline{\nabla}$ is the ambient connection. We have the following definition (see Definition 16.5 in [Sim83]).

Definition 2.1. Suppose H is a locally μ_V-integrable function on $M\cap U$ with values in $\mathbb{R}^{n+p}$ we say that V has generalized mean curvature H in U (U is an open set in $\mathbb{R}^{n+p}$) if

$$\int_U\text{div}_M X=-\int_U X\cdot Hd\mu_V$$

whenever X is a $\mathcal{C}^1$ vector field on U with compact support in U.

We note that when V is stationary in U then

$$\int_U\text{div}_M X=-\int_U X\cdot\overline{H}d\mu_V$$

where $\overline{H}(q)$ is the mean curvature vector on the approximate tangent space at q. This explains why H is said to be generalized mean curvature.

Definition 2.2 (Regular set reg$V$ and singular set sing$V$)**.** A point $x \in \mathcal{U}$ is a regular point of V if $x \in \mathrm{spt}\|V\|$ and if there exists $\sigma > 0$ such that $\mathrm{spt}\|V\| \cap B^{n+1}_\sigma(x)$ is an embedded smooth hypersurface of $B^{n+1}_\sigma(x)$. The regular set of V, denote by reg$V$, is the set of all regular points of V. The (interior) singular set of V, denote by sing$V$, is $(\mathrm{spt}\|V\| \setminus \mathrm{reg}V)$. By definition, reg$V$ is relatively open in $\mathrm{spt}\|V\|$ and sing$V$ is relative closed in $\mathrm{spt}\|V\|$. For convenience, we say that a varifold V is orientable if and only if reg$V$ is orientable.

Definition 2.3 (C^1-regular set $\mathrm{reg}_1 V$)**.** We define $\mathrm{reg}_1 V$ to be the set of all points $x \in \mathrm{spt}\|V\|$ with the property that there is $\sigma > 0$ such that $\mathrm{spt}\|V\| \cap B^{n+1}_\sigma(x)$ is an embedded hypersurface of $B^{n+1}_\sigma(x)$ of class C^1.

On a varifold V, we use a stationarity assumption as follows

: **(IV)**: Whenever $\mathcal{O} \subset (\mathcal{U} \setminus (\mathrm{spt}\|V\| \setminus \mathrm{reg}_1 V))$ is such that $\mathrm{reg}_1 V \cap \mathcal{O}$ is orientable, there exists an orientation $\hat{\nu}$ on $\mathrm{reg}_1 V \cap \mathcal{O}$ such that
$$\frac{d}{dt}\bigg|_{t=0} \|(\Psi_t)_\sharp V\| = 0$$
for any $X \in C^1_c(\mathcal{O})$ with $\int_{\mathrm{reg}_1 V \cap \mathcal{O}} X \cdot \hat{\nu} d\|V\| = 0$ and for any deformation Ψ_t with $\frac{d}{dt}\big|_{t=0} \Psi_t = X$.

Remark 2.4. As discussed in [BW18], the assumption **(IV)** implies that there exists a constant H such that $\vec{H} = H\hat{\nu}$, moreover $\mathrm{reg}_1 V = \mathrm{reg}V$, where $\vec{H}$ is the mean curvature of reg$V$.

Example 2.5. The following example is given in [BW18]. Consider the 1-dimension integral varifold V (higher dimension examples follow by a trivial product with a linear subspace) whose support is given by the set $D \subset \mathbb{R}^2$ defined by
$$D = \{y \geq -1, x^2 + (y+1)^2 = 1\} \cup \{y \leq 1, x^2 + (y-1)^2 = 1\},$$
where $(x, y) \in \mathbb{R}^2$, with multiplicity 2 on the portions $\{(x, y) \in \mathbb{R}^2 : -1 \leq y \leq 0, x \leq 0, x^2 + (y+1)^2 = 1\}$ and $\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1, x \geq 0, x^2 + (y-1)^2 = 1\}$, and multiplicity 1 on the rest. Then V is a CMC curve. For further examples, we refer the readers to Subsection 2.2 in [BW18].

2.2. **Integration on a Singular Hypersurface.** Throughout this paper, we will use a cut-off function introduced in [Zhu18] to integrate around the singular set. Consider an integral n-varifold V in $\mathbb{S}^{n+1}$. The singular set is closed and hence compact. So by definition of Hausdorff measure, if $\mathcal{H}^{n-q}(\mathrm{sing}V) = 0,\ q > 0$, then for any $\epsilon > 0$ we may cover the singular set by finitely many geodesic balls of $\mathbb{S}^{n+1}, \mathrm{sing}V \subset \bigcup_{i=1}^{m} B_{r_i}(p_i)$, where $\sum_{i=1}^m r_i^{n-q} < \epsilon$ and we may assume without loss of generality that $r_i << 1$ for each i.

Given such a covering we take smooth cutoff functions $0 \leq \varphi_{i,\epsilon} \leq 1$ on $\mathbb{S}^{n+1}$ with $\varphi_{i,\epsilon} = 1$ outside $B_{2r_i}(p_i), \varphi_{i,\epsilon} = 0$ inside $B_{r_i}(p_i)$ and $|\overline{\nabla}\varphi_{i,\epsilon}| \leq \frac{2}{r_i}$ in between. We then set $\varphi_\epsilon = \inf_i \varphi_{i,\epsilon}$ which is Lipschitz with compact support away from sing$V$, and $|\overline{\nabla}\varphi_\epsilon| \leq \sup_i |\overline{\nabla}\varphi_{i,\epsilon}|$. We now collect some properties of φ_ϵ. To state the result, let us recall a version of Theorem 17.6 in [Sim83].

Theorem 2.6 ([Sim83])**.** *Let U be an open subset in $\mathbb{R}^{n+k}$, V be an n-varifold with generalized mean curvature H in U. Given a fixed point $p \in U, 0 < \alpha \leq 1, \Lambda \geq 0$, if*

$$\frac{1}{\alpha}\int_{B_s(p)} |H| \leq \Lambda\left(\frac{s}{R}\right)^{\alpha-1}\mu_V(B_s(p)),$$

for all $s \in (0, R)$, where $\overline{B}_R(p) \subset U$, then

$$e^{\Lambda R^{1-\alpha}r^{\alpha}}r^{-n}\mu_V(B_r(p)) \leq e^{\Lambda R^{1-\alpha}s^{\alpha}}s^{-n}\mu_V(B_s(p)),$$

whenever $0 < r < s \leq R$.

Now, for $\alpha = 1, s = R \leq 1$, and V is an n-varifold satisfying **(IV)**, $M = \mathrm{reg}V$, we infer that H is constant. Hence, we can put $\Lambda = |H|$ in this theorem to obtain the following corollary.

Corollary 2.7. *Let U be an open subset in $\mathbb{R}^{n+k}$, V be an n-varifold satisfying **(IV)**. Given a fixed point $p \in U, B_r(p) \subset U$, then there is a constant C_V such that*

$$\mu_V(B_r(p)) \leq C_V r^n.$$

This corollary says that an n-varifold V satisfying **(IV)** in $\mathbb{S}^{n+1}$ has locally Euclidean volume growth. Using this property, the following useful results are proved in [Zhu18].

Proposition 2.8 ([Zhu18])**.** *Suppose that V satisfies **(IV)** and $\mathcal{H}^{n-q}(\mathrm{sing}V) = 0$ for some $q > 0$. Then on $M = \mathrm{reg}V$, we have*

(1)
$$\int_M |\nabla\varphi_\epsilon|^q \leq 2^{n+q}C_V\epsilon.$$

(2) *If $q \geq 1$ and $f \in L^p(M)$, where $p = \frac{q}{q-1}$, then*
$$\lim_{\epsilon\to 0}\int_M |f||\nabla\varphi_\epsilon| = 0.$$

(3) *If $q \geq 2$ and $f \in L^p(M)$, where $p = \frac{2q}{q-2}$, then*
$$\lim_{\epsilon\to 0}\int_M f^2|\nabla\varphi_\epsilon|^2 = 0.$$

Moreover, there is a version of integration by parts.

Lemma 2.9 ([Zhu18])**.** *Suppose $\mathcal{H}^{n-q}(\mathrm{sing}V) = 0$ for some $q \geq 0$. Assume that u, v are C^2 functions on $M = \mathrm{reg}V$ such that $|\nabla u|, |\nabla v|, |u\Delta v|$ is L^1 and $|u\nabla v|$ is $L^p, p = \frac{q}{q-1}$ then*

$$\int_M u\Delta v = -\int_M \langle\nabla u, \nabla v\rangle.$$

2.3. **Jacobi Operator and Its First Eigenvalue.** Throughout this paper, we assume that V is an n-varifold in $\mathbb{S}^{n+1}$ satisfying **(IV)** and let $M = \mathrm{reg}V$. By the above discussion, this assumption implies that M has constant mean curvature. We choose a local orthonormal frame $\{e_1, \ldots, e_{n+1}\}$ on $\mathbb{S}^{n+1}$ and its dual coframe $\{\omega_1, \ldots, \omega_{n+1}\}$ such that $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M. By Cartan's lemma, we have

$$\omega_{i(n+1)} = \sum_{j=1}^{n} h_{ij}\omega_j, \quad h_{ij} = h_{ji}.$$

The mean curvature H, the second fundamental form A, and the traceless second fundamental form Φ are defined, respectively, by

$$H = \frac{1}{n}\sum_{i=1}^{n} h_{ii}$$

$$A = \sum_{i,j=1}^{n} h_{ij}\omega_i \otimes \omega_j e_{n+1}$$

$$\Phi = \sum_{i,j=1}^{n} \Phi_{ij}\omega_i \otimes \omega_j e_{n+1},$$

where $\Phi_{ij} = h_{ij} - H\delta_{ij}$. Let $|A|^2$ be the square norm of the second fundamental form, then $|\Phi|^2 = |A|^2 - nH^2$. On M, we consider the Jacobi operator

$$Jf = \Delta f + (|A|^2 + n)f = \Delta f + |\Phi|^2 + n(1 + H^2)f.$$

Similarly to [CC17, Zhu18], we define the first stability eigenvalue to be

$$\begin{aligned}\lambda_1(M) = \inf_{\Omega} \lambda_1(\Omega) &= \inf_f \frac{\int_M |\nabla f|^2 - |A|^2 f^2 - nf^2}{\int_M f^2} \\ &= \inf_f \frac{\int_M |\nabla f|^2 - |\Phi|^2 f^2 - n(1+H^2)f^2}{\int_M f^2} \qquad (2.1)\end{aligned}$$

Here the infimum may be taken over Lipschitz functions f with compact support in M.

Recall that, as in [CC17], since H is constant, we have the following Simons type inequality on M.

$$\frac{1}{2}\Delta|\Phi|^2 = \frac{1}{2}\Delta|A|^2 = \sum_{i,j,k=1}^{n} h_{ijk}^2 + n|A|^2 - n^2H^2 + nHf_3 - |A|^4,$$

where $f_3 = \sum_{i=1}^{n} k_i^3$ and these $k_i, i = 1, \dots, n$ are the principal curvatures and $h_{ijk} := \nabla_{e_k} h_{ij}$. Moreover, f_3 can be written as

$$f_3 = \sum_{i=1}^{n}(k_i - H)^3 + 3H|\Phi|^2 + nH^3 = B_3 + 3H|\Phi|^2 + nH^3,$$

and

$$|B_3| \le \frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3.$$

Since H is constant, we have $\Phi_{ijk} = h_{ijk}$. By choosing the orientation, we may assume $H \ge 0$. The above discussion implies the following lemma.

Lemma 2.10.

$$|\nabla\Phi|^2 - |\Phi|^2 P_H(|\Phi|) \le \frac{1}{2}\Delta|\Phi|^2 \le |\nabla\Phi|^2 - |\Phi|^2 Q_H(|\Phi|),$$

where

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - n(1+H^2),$$

$$Q_H(x) = x^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - n(1+H^2).$$

We recall the following Kato type inequality.

Lemma 2.11 ([CLS17a]-Lemma2)**.**

$$|\nabla\Phi|^2 \geq \frac{n+2}{n}|\nabla|\Phi||^2.$$

3. Integral Estimates for $|\Phi|$

Throughout this section M^n will denote the regular part of an orientable n-varifold V in $\mathbb{S}^{n+1}$ satisfying **(IV)**. Moreover, by Lemma 2.7, we know that M has constant mean curvature H and is of locally Euclidean volume growth $\mu_V(B_r(p)) \leq C_V r^n$, for $r << 1$.

For any $a > 0$, we have

$$\frac{n(n-2)}{\sqrt{n(n-1)}}Hx \leq ax^2 + \frac{(n(n-2))^2}{4n(n-1)a}H^2.$$

This together with Lemma 2.10 and Lemma 2.11 implies

$$|\Phi|\Delta|\Phi| \geq \frac{2}{n}|\nabla|\Phi||^2 - (1+a)|\Phi|^4 - \left(\frac{(n-2)^2}{4n(n-1)a}(nH)^2 - n(1+H^2)\right)|\Phi|^2. \tag{3.1}$$

Define

$$c_1 = \frac{(n-2)^2}{4n(n-1)a}(nH)^2 - n(1+H^2).$$

Then the equation (3.1) can be rewritten as follows

$$|\Phi|\Delta|\Phi| \geq \frac{2}{n}|\nabla|\Phi||^2 - (1+a)|\Phi|^4 - c_1|\Phi|^2.$$

Next, we recall the variational characterization of $\lambda_1(M)$

$$\lambda_1 := \lambda_1(M) = \inf_f \frac{\int_M |\nabla f|^2 - |\Phi|^2 f^2 - n(1+H^2)f^2}{\int_M f^2}.$$

Hence, if $\lambda_1(M) > -\infty$ we have

$$\int_M |\Phi|^2 f^2 \leq \int_M |\nabla f|^2 - (\lambda_1 + n(1+H^2))\int_M f^2, \tag{3.2}$$

for any Lipschitz function f with compact support in M. First, we show that $|\Phi| \in L^2(M)$ as in the below lemma.

Lemma 3.1. *Suppose that $\mathcal{H}^{n-2}(\text{sing}V) = 0$ and let $M = \text{reg}V$. If $\lambda_1 > -\infty$ then $|\Phi| \in L^2(M)$.*

Proof. The proof follows by using the inequality (3.2). Let φ_ϵ as in Section 2 then plugging this function into (3.2), we conclude that the last term of the right hand side is bounded by $\mu_V(M)$ since $\varphi_\epsilon^2 \leq 1$. The gradient term is controlled by Lemma 2.8. Letting $\epsilon \to 0$, the conclusion is followed by Fatou's lemma. □

Lemma 3.2. *Let $M = \text{reg}V$, assume that $\mathcal{H}^{n-2}(\text{sing}V) = 0$ and $\lambda_1 > -\infty$. Then there exists $C = C(n, V, \lambda_1)$ such that for any $0 < r << 1, p \in \mathbb{S}^{n+1}$,we have*

$$\int_{M\cap B_r(p)} |\Phi|^2 \leq Cr^{n-2}.$$

Proof. The proof follows a similar argument in [Zhu18] by replacing $\|A\|$ by $|\Phi|$.

We choose a cutoff function $0 \leq \eta \leq 1$ such that $\eta = 1$ inside $B_r(p), \eta = 0$ outside $B_{2r}(p)$ and $\overline{\nabla}\eta \leq \frac{2}{r}$ in between. Then Note that Lemma 3.1 implies $|\Phi| \in L^2(M)$, therefore, we can use dominated convergence to approximate

$$\int_M |\Phi|^2\eta^2 = \lim_{\epsilon\to 0}\int_M |\Phi|^2\eta^2\varphi_\epsilon^2.$$

Using the stability inequality (3.2), for each fixed $\epsilon > 0$, we obtain

$$\int_M |\Phi|^2\eta^2\varphi_\epsilon^2 \leq 2\int_M \eta^2|\nabla\varphi_\epsilon|^2 + 2\int_M \varphi_\epsilon^2|\nabla\eta|^2 + C_{\lambda_1,H}\int_M \eta^2\varphi_\epsilon^2,$$

where $C_{\lambda_1,H} := |\lambda_1+n(1+H^2)|$. Observe that $\eta^2 \leq 1$, the assumption $\mathcal{H}^{n-2}(\mathrm{sing}V) = 0$ together with Proposition 2.8 implies the first term of the right hand side tends to zero as $\epsilon \to 0$. Since $\varphi_\epsilon^2 \leq 1$, we have

$$\int_M \eta\varphi_\epsilon^2 \leq \int_M \eta^2 \leq \int_{M\cap B_{2r}(p)} 1 \leq C_V 2^n r^n, \text{ and}$$

$$\int_M \varphi_\epsilon^2|\nabla\eta|^2 \leq \int_M |\nabla\eta|^2 \leq \int_{M\cap B_{2r}(p)} \frac{4}{r^2} \leq C_V 2^{n+2} r^{n-2},$$

thus, the proof is complete. □

Lemma 3.3. *Suppose that $\mathcal{H}^{n-4}(\mathrm{sing}V) = 0, n \geq 5$ and let $M = \mathrm{reg}V$. If $\lambda_1 > -\infty$, then $|\Phi| \in L^4(M)$ and $|\nabla|\Phi||, |\nabla\Phi| \in L^2(M)$.*

Moreover, for any $\beta \leq 1$ and $\eta > 0$, we have

$$(\eta + |\Phi|)^{2\beta-2}|\nabla|\Phi||^2 \in L^1(M).$$

Proof. The proof follows by using a strategy as in [SSY75] (see also [CM12, Zhu18]). As before, let $C := C_{\lambda_1,H} = |\lambda_1 + n(1 + H^2)|$. Suppose that f is a smooth function with compact support in M. We apply the stability inequality (3.2) with the test function $|\Phi|f$ and use the fundamental inequality $2xy \leq ax^2 + \frac{1}{a}y^2$ to obtain

$$\int_M |\Phi|^4 f^2 \leq (1+a)\int_M |\nabla|\Phi||^2 f^2 + \int_M |\Phi|^2\left(\left(1+\frac{1}{a}\right)|\nabla f|^2 + Cf^2\right), \tag{3.3}$$

where $a > 0$ is an arbitrary positive number to be chosen later.

On the other hand, the Simons type inequality in the equality (3.1) says that

$$|\Phi|\Delta|\Phi| \geq \frac{2}{n}|\nabla|\Phi||^2 - (1+a)|\Phi|^4 - c_1|\Phi|^2.$$

Multiplying both sides of this Simons type inequality by f^2, then using integration by parts and using the fundamental inequality $2xy \leq ax^2 + \frac{1}{a}y^2$, we obtain

$$\left(1+\frac{2}{n}-a\right)\int_M |\nabla|\Phi||^2 f^2 \leq (1+a)\int_M |\Phi|^4 f^2 + \frac{1}{a}\int_M |\Phi|^2|\nabla f|^2 + c_1\int_M |\Phi|^2 f^2.$$

Equivalently,

$$\int_M |\nabla|\Phi||^2 f^2 \leq \frac{(1+a)^2}{1+\frac{2}{n}-a}\int_M |\Phi|^4 f^2 + \frac{1}{a\left(1+\frac{2}{n}-a\right)}\int_M |\Phi|^2|\nabla f|^2 + \frac{c_1}{1+\frac{2}{n}-a}\int_M |\Phi|^2 f^2. \tag{3.4}$$

Combining (3.3) and (3.4), we yield

$$\int_M |\Phi|^4 f^2 \leq \frac{1+a}{1+\frac{2}{n}-a}\int_M |\Phi|^4 f^2 + C_{a,c_1}\int_M |\Phi|^2(|\nabla f|^2 + (C+1)f^2). \tag{3.5}$$

Choosing $a < \frac{1}{3n}$ then the first coefficient on the right hand side is less than 1. This implies that there is a constant $C_1 = C(c_1, n, H)$ such that

$$\int_M |\Phi|^4 f^2 \le C_1 \int_M |\Phi|^2 (f^2 + |\nabla f|^2). \tag{3.6}$$

Now, we apply this equality with $f = \varphi_\epsilon$ and $q = 4$. When ϵ approaches 0, the first term on the right hand side converges to $\int_M |\Phi|^2$, which is finite. We estimate the second term as follows:

$$\int_M |\Phi|^2 |\nabla \varphi_\epsilon|^2 \le \sum_i^m \frac{4}{r_i^2} \int_{M \cap B_{2r_i}(p_i) \setminus B_{r_i}(p_i)} |\Phi|^2.$$

Using Lemma 3.2, we have

$$\int_{M \cap B_{2r_i}(p_i) \setminus B_{r_i}(p_i)} |\Phi|^2 \le C' r_i^{n-2},$$

where C' depend on H and the volume bounds for M. Hence,

$$\int_M |\Phi|^2 |\nabla \varphi_\epsilon|^2 \le 4C' \sum_i r_i^{n-4} < 4C'\epsilon.$$

Here we note that the $r_i << 1$ were chosen so that $\sum_i r_i^{n-4} < \epsilon$. Letting $\epsilon \to 0$, we conclude that this term tends to zero. Hence, Lemma 3.1 together with this observation imply $|\Phi| \in L^4(M)$. As a consequence, the inequality (3.4) infers $|\nabla |\Phi|| \in L^2(M)$.

Finally, observe that Lemma 2.10 yields

$$|\nabla \Phi|^2 \le \frac{1}{2} \Delta |\Phi|^2 + |\Phi|^2 Q_H(|\Phi|),$$

where

$$Q_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n(1 + H^2).$$

Multiplying both sides of the above inequality by φ_ϵ^2 and integrating by parts, we obtain

$$\begin{aligned} \int_M \varphi_\epsilon^2 |\nabla \Phi|^2 \le & -2 \int_M \varphi_\epsilon |\Phi| \langle \nabla f, \nabla |\Phi| \rangle + \int_M \varphi_\epsilon |\Phi|^2 Q_H(|\Phi|) \\ \le & \int_M (\varphi_\epsilon^2 |\nabla |\Phi||^2 + |\Phi|^2 |\nabla \varphi_\epsilon|^2) + \int_M \varphi_\epsilon |\Phi|^2 Q_H(|\Phi|). \end{aligned}$$

As the above discussion, since $|\Phi|^2, |\Phi|^4 \in L^1(M)$, it is easy to see that $|\Phi|^2 Q_H(|\Phi|) \in L^1(M)$. Moreover, since $|\nabla |\Phi|| \in L^2(M)$, after letting $\epsilon \to 0$, we can conclude that $|\nabla \Phi| \in L^2(M)$. Now, since $\beta \le 1$ and $\eta > 0$ is fixed, we obtain

$$(\eta + |\Phi|)^{2\beta - 2} |\nabla |\Phi||^2 \le \eta^{2\beta - 2} |\nabla |\Phi||^2 \in L^1(M),$$

where we used $\beta - 1 \le 0$. The proof is complete. □

We have the following corollary.

Corollary 3.4. *Suppose that $\mathcal{H}^{n-4}(\text{sing}V) = 0$ and V satisfies **(IV)**. Let $M = \text{reg}V$, assume that $\lambda_1 > -\infty$. If $\varepsilon > 0$ and $f_\varepsilon := (\varepsilon + |\Phi|^2)^\alpha$ for some $0 < \alpha \le \frac{1}{2}$, then we have $f_\varepsilon \in L^4(M), |f_\varepsilon|, |\nabla f_\varepsilon| \in L^2(M)$.*

Proof. Since $0 < \alpha \leq \frac{1}{2}$, we have

$$(f_\varepsilon)^4 \leq \max\{1, (\varepsilon + |\Phi|^2)^2\} \leq \max\{1, 2\varepsilon^4 + 2|\Phi|^4\}.$$

Hence $f_\varepsilon \in L^4(M)$ since $|\Phi| \in L^4(M)$ as in Lemma 3.3. Next, we estimate $|\nabla f_\varepsilon|$ as follows.

$$\begin{aligned}|\nabla f_\epsilon|^2 &= 4\alpha^2(\varepsilon + |\Phi|^2)^{2\alpha-2}|\Phi|^2|\nabla|\Phi||^2 \\ &\leq 4\alpha^2(\varepsilon + |\Phi|^2)^{2\alpha-1}|\nabla|\Phi||^2 \leq 4\alpha^2\varepsilon^{2\alpha-1}|\nabla|\Phi||^2,\end{aligned}$$

where we used $2\alpha - 1 \leq 0$ and $\varepsilon + |\Phi|^2 \geq \varepsilon$ in the last inequality. Therefore, by Lemma 3.3 we obtain $|\nabla f_\varepsilon| \in L^2(M)$. □

Now we will show that we can apply the above integration by parts for $u = v = f_\varepsilon$, where $f_\epsilon = (\epsilon + |\Phi|^2)^\alpha, 0 < \alpha \leq \frac{1}{2}$ as in the previous corollary. Indeed, it is easy to show that

$$\begin{aligned}f_\varepsilon \Delta f_\varepsilon &= \alpha(\alpha-1)(\varepsilon + |\Phi|^2)^{2\alpha-2}|\nabla|\Phi|^2|^2 + \alpha(\varepsilon + |\Phi|^2)^{2\alpha-1}\Delta|\Phi|^2 \\ &\leq 4\alpha^{-1}(1-\alpha)|\nabla f_\varepsilon|^2 + \alpha\varepsilon^{2\alpha-1}|\Delta|\Phi|^2|,\end{aligned}$$

where we used $2\alpha - 1 \leq 0$. By Lemma 3.1 and Lemma 3.3, we have that $|\Phi| \in L^2(M) \cap L^4(M)$. Moreover, Lemma 2.10 asserts that

$$\frac{1}{2}\Delta|\Phi|^2 \leq |\nabla\Phi|^2 - |\Phi|^2\left(|\Phi|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi| - n(1+H^2)\right).$$

Observe that as in Lemma 3.3, we have $|\nabla\Phi| \in L^2(M), |\Phi| \in L^2(M) \cap L^4(M)$, hence $|\Delta|\Phi|^2| \in L^1(M)$. This together with Corollary 3.4 implies $f_\varepsilon \Delta f_\varepsilon \in L^1(M)$. Now, by Young's inequality, we have

$$(f_\epsilon|\nabla f_\epsilon|)^{4/3} \leq \frac{1}{3}f_\epsilon^4 + \frac{2}{3}|\nabla f_\epsilon|^2,$$

since f_ε is L^4 and $|\nabla f_\varepsilon|$ is L^2, we conclude that $|f_\varepsilon||\nabla f_\varepsilon|$ is L^p for $p = \frac{4}{3}$ (hence $q = \frac{p}{p-1} = 4$). Using Lemma 2.9, we have verified the following result.

Lemma 3.5. *Let V is an n-varifold satisfying **(IV)**. Suppose $\mathcal{H}^{n-4}(\mathrm{sing}V) = 0$, then for $f_\varepsilon = (\varepsilon + |\Phi|^2)^\alpha$, where $0 < \alpha \leq \frac{1}{2}$ and any given $\varepsilon > 0$,we have*

$$\int_M f_\varepsilon \Delta f_\varepsilon = -\int_M |\nabla f_\varepsilon|^2.$$

4. First stability eigenvalue

First, we observe that the first stability eigenvalue can be obtained as follows.

Lemma 4.1. *Let V be an $n_{\geq 5}$-varifold in $\mathbb{S}^{n+1}$ with orientable regular part $M = \mathrm{reg}V$. Assume that M has constant mean curvature and $\mathcal{H}^{n-4}(\mathrm{sing}V) = 0$. Then, we can get the same λ_1 by taking the infimum over Lipschitz functions f on M such that $f \in W^{1,2} \cap L^4$.*

Proof. This proof follows a similar argument in [Zhu18] but we use Φ, the traceless second fundamental form, instead of A.

Observe that if $\lambda_1(M) = -\infty$ then the definition (2.1) of λ implies that for any large real number $K > 0$, there exists a Lipschitz function with compact support on M such that

$$\frac{\int_M |\nabla f|^2 - |\Phi|^2 f^2 - n(1+H^2)f^2}{\int_M f^2} < -K.$$

Since f is Lipschitz, it must be almost everywhere differentiable. This together with the fact that f has compact support implies $f \in W^{1,2} \cap L^4$. Hence, the proof is complete.

Now, we assume $\lambda_1 > -\infty$. Since $f \in L^4(M)$, Lemma 3.1 together with this assumption implies $|\Phi|f$ is also L^2. We use the functions $g_\epsilon = f\varphi_\epsilon$ which are compactly supported away from the singular set, in the definition of λ_1. Since f and $|\Phi|f$ are in L^2, dominated convergence gives that

$$\int_M g_\epsilon^2 \to \int_M f^2, \text{ and } \int_M |\Phi|^2 g_\epsilon^2 \to \int_M |\Phi|^2 f^2$$

as $\epsilon \to 0$. For the gradient term, we have

$$\int_M |\nabla g_\epsilon|^2 = \int_M (\varphi_\epsilon^2 |\nabla f|^2 + 2\langle \nabla f, \nabla \varphi_\epsilon \rangle + f^2 |\nabla \varphi_\epsilon|^2).$$

Note that $f \in L^4(M)$, the properties of φ_ϵ in Lemma 2.8 imply that the second and third terms on the right hand side tend to zero when ϵ approaches 0. On the other hand, the first term on the right hand side tends to $\int_M |\nabla f|^2$ by the dominated convergence theorem. In summary, we have verified that $\int_M |\nabla g_\epsilon|^2 \to \int_M |\nabla f|^2$. Therefore, the proof of Lemma 4.1 is complete. □

We are now ready to provide the proof of the main theorem.

Proof of Theorem 1.1. For any $\varepsilon > 0$ and arbitrary number $\frac{1}{2} - \frac{1}{\sqrt{2n}} < \alpha \le \frac{1}{2}$, let $f_\varepsilon = (\varepsilon + |\Phi|^2)^\alpha$. By a direct computation, we obtain

$$\Delta f_\varepsilon = \alpha(\alpha-1)(\varepsilon + |\Phi|^2)^{\alpha-2} |\nabla |\Phi|^2|^2 + \alpha(\varepsilon + |\Phi|^2)^{\alpha-1} \Delta |\Phi|^2. \tag{4.1}$$

By Corollary 3.4 and Lemma 4.1, we can use f_ε as a test function for λ_1. Hence, by a calculation as in [CLS17a] and Lemma 3.5, we arrive at

$$\begin{aligned}\lambda_1 \int_M f_\varepsilon^2 \le & \alpha \int_M (\varepsilon + |\Phi|^2)^{2\alpha-2} \big((1+2\alpha\beta-\beta-\alpha)|\nabla|\Phi|^2|^2 - 2(1-\beta)(\varepsilon+|\Phi|^2)|\nabla\Phi|^2\big) \\ & + \int \left(\frac{2\alpha(1-\beta)}{\varepsilon+|\Phi|^2}|\Phi|^2 P_H(|\Phi|)\right) f_\varepsilon^2 - \int_M \big(|\Phi|^2 + n(1+H^2)\big) f_\varepsilon^2.\end{aligned}$$

Note that $\alpha > \frac{1}{2} - \frac{1}{\sqrt{2n}} \ge \frac{n-2}{4n}$, for all $n \in \mathbb{N}$, we now take β such that

$$1 - \beta = \frac{n+2}{n} . \frac{2n\alpha}{4n\alpha - (n-2)} > \frac{2n\alpha}{4n\alpha - (n-2)} \tag{4.2}$$

so we have

$$\begin{aligned} 1 + 2\alpha\beta - \beta - \alpha &= (1-2\alpha)(1-\beta) + \alpha \\ &\ge (1-2\alpha)\frac{2n\alpha}{4n\alpha-(n-2)} + \alpha = \frac{(n+2)\alpha}{4n\alpha - (n-2)} > 0. \end{aligned}$$

Using Kato inequality as in Lemma 2.11, we obtain

$$\begin{aligned} &(1+2\alpha\beta-\beta-\alpha)|\nabla|\Phi||^2 - 2(1-\beta)(\varepsilon+|\Phi|^2)|\nabla\Phi|^2 \\ &\le \frac{4n(1+2\alpha\beta-\beta-\alpha)}{n+2}|\Phi|^2|\nabla\Phi|^2 - 2(1-\beta)(\varepsilon+|\Phi|^2)|\nabla\Phi|^2 \\ &\le \frac{2}{n+2}|\Phi|^2|\nabla\Phi|^2\big(2n(1+2\alpha\beta-\beta-\alpha) - (n+2)(1-\beta)\big) \\ &= \frac{2}{n+2}|\Phi|^2|\nabla\Phi|^2\big((n-2)(1-\beta) - 4n\alpha(1-\beta) + 2n\alpha\big) \le 0 \end{aligned}$$

where in the last inequality we used $1-\beta > \frac{2n\alpha}{4n\alpha-(n-2)}$. Therefore, as in [CLS17a], it turns out that

$$\lambda_1\int_M f_\varepsilon^2 \le \int_M \frac{|\Phi|^2}{\varepsilon+|\Phi|^2}\Big((2\alpha(1-\beta)-1)|\Phi|^2+\frac{2\alpha(1-\beta)(n-2)}{\sqrt{n(n-1)}}(nH)|\Phi|-\varepsilon\Big)f_\varepsilon^2$$
$$-2n\alpha(1-\beta)(1+H^2)\int_M \frac{|\Phi|^2}{\varepsilon+|\Phi|^2}f_\varepsilon^2-n(1+H^2)\int_M f_\varepsilon^2 \tag{4.3}$$

Assume that $2\alpha(1-\beta)<1$, we have

$$\frac{2\alpha(1-\beta)(n-2)}{\sqrt{n(n-1)}}(nH)|\Phi| \le (1-2\alpha(1-\beta))|\Phi|^2+\frac{\alpha^2(1-\beta)^2(n-2)^2}{n(n-1)(1-2\alpha(1-\beta))}(nH)^2.$$

This implies

$$\lambda_1\int_M f_\varepsilon^2 \le \int_M \frac{|\Phi|^2}{\varepsilon+|\Phi|^2}\left(\frac{\alpha^2(1-\beta)^2(n-2)^2}{n(n-1)(1-2\alpha(1-\beta))}(nH)^2-\varepsilon\right)f_\varepsilon^2$$
$$-2n\alpha(1-\beta)(1+H^2)\int_M \frac{|\Phi|^2}{\varepsilon+|\Phi|^2}f_\varepsilon^2-n(1+H^2)\int_M f_\varepsilon^2. \tag{4.4}$$

Since M is non-totally umbilical, we have

$$\lim_{\varepsilon\to 0}\int_M f_\varepsilon^2=\int_M |\Phi|^{4\alpha}>0.$$

Letting $\varepsilon\to 0$, this observation together with (4.4) infers

$$\lambda_1 \le -n(1+2\alpha(1-\beta))(1+H^2)+\frac{\alpha^2(1-\beta)^2(n-2)^2}{n(n-1)(1-2\alpha(1-\beta))}(nH)^2. \tag{4.5}$$

(1) Now, if $n^2H^2 < \frac{64(n-1)}{n^2(n-2)^2-16}$ then a direct computation shows that

$$\frac{1}{2}\ge\frac{1}{2}\left(1-\frac{(n-2)H}{\sqrt{n^2H^2+4n(n-1)}}\right)>\frac{1}{2}-\frac{2}{n^2}.$$

Defining

$$w(\alpha)=\alpha(1-\beta)=\frac{2(n+2)\alpha^2}{4n\alpha-(n-2)},$$

so $w(\alpha)$ is an increasing function of α, for $\alpha>\frac{1}{2}-\frac{1}{n}$. Observe that

$$w\left(\frac{1}{2}-\frac{1}{n}\right)=\frac{1}{2}-\frac{2}{n^2},\quad w\left(\frac{1}{2}\right)=\frac{1}{2}.$$

This implies that there exists an $\alpha\in\left(\frac{1}{2}-\frac{1}{n},\frac{1}{2}\right]$ such that

$$w(\alpha)=\alpha(1-\beta)=\frac{1}{2}\left(1-\sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}}\right).$$

For these α and β, we have

$$2\alpha(1-\beta)=1-\sqrt{\frac{(n-2)^2H^2}{4(n-1)+n^2H^2}}$$

Plugging it into (4.5) and doing the same computation as in [CC17], we arrive at

$$\lambda_1 \leq -n(1+H^2) - \frac{n\left(\sqrt{n^2H^2+4(n-1)} - (n-2)|H|\right)^2}{4(n-1)}.$$

Now, suppose that the equality holds true, we can repeat the arguments as in [CC17] to show that M (hence V) is a Clifford torus.

(2) If $n^2H^2 \geq \frac{64(n-1)}{n^2(n-2)^2-16}$, we first modify the equation (4.2) to choose β such that

$$1-\beta = \frac{2n\alpha}{4n\alpha-(n-2)}$$

then we process s in [CC17] or [CLS17a] to obtain (4.5) again, namely, we have

$$\lambda_1 \leq -n(1+2\alpha(1-\beta))(1+H^2) + \frac{\alpha^2(1-\beta)^2(n-2)^2}{n(n-1)(1-2\alpha(1-\beta))}(nH)^2.$$

Now, we choose

$$\alpha(1-\beta) = \frac{n-2}{2n}, \text{ saying } \alpha = \frac{1}{2} - \frac{1}{n}, \quad \text{and} \quad \beta = 0.$$

Therefore, we can conclude that

$$\lambda_1 \leq -2(n-1)(1+H^2) + \frac{(n-2)^4}{8(n-1)}H^2.$$

Now the proof follows by applying the argument as in [CC17, CLS17a]. We omit the detail here. □

Theorem 4.2. *Let V be an $n_{\geq 5}$-varifold in $\mathbb{S}^n$ with orientable regular part $M = \text{reg}V$. Assume that M has constant mean curvature and $\mathcal{H}^{n-4}(\text{sing}V) = 0$ and M is non-totally umbilical. Then, we have*

$$\lambda_1 \leq -2n(1+H^2) + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|\frac{\int_M |\Phi|^3}{\int_M |\Phi|^2}.$$

If the equality holds then V is a Clifford torus.

Proof. We let $\alpha = \frac{1}{2}$, and $\beta = 0$ in (4.3) then let ε tend to 0, we obtain the conclusion of Theorem 4.2. Now, assume that the equality holds true, we can apply the argument in the proof of Theorem 2.2 in [ABB04] to show that M (hence V) is a Clifford torus. □

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