

SMOOTHNESS OF COLLAPSED REGIONS IN A CAPILLARITY MODEL FOR SOAP FILMS

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ABSTRACT. We study generalized minimizers in the soap film capillarity model introduced in [MSS19, KMS19]. Collapsed regions of generalized minimizers are shown to be smooth outside of dimensionally small singular sets, which are thus empty in physical dimensions. Since generalized minimizers converge to Plateau’s type surfaces in the vanishing volume limit, the fact that collapsed regions cannot exhibit Y -points and T -points (which are possibly present in the limit Plateau’s surfaces) gives the first strong indication that singularities of the limit Plateau’s surfaces should always be “wetted” by the bulky regions of the approximating generalized minimizers.

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1. INTRODUCTION

1.1. Overview. We continue the investigation of a model for soap films based on capillarity theory which was recently introduced by A. Scardicchio and the authors in [MSS19, KMS19]. Soap films are usually modeled as minimal surfaces with a prescribed boundary: this idealization of soap films gives a model without length scales, which cannot capture those behaviors of soap films determined by their three-dimensional features, e.g. by their thickness. Regarding enclosed volume, rather than thickness, as a more basic geometric property of soap films, in [MSS19, KMS19] we have started the study of soap films through capillarity theory, by proposing a **soap film capillarity model** (see (1.4) below). In this model, one looks for surface tension energy minimizers enclosing a fixed small volume, and satisfying a spanning condition with respect to a given wire frame. In [KMS19] we have proved the existence of minimizers, and have shown their convergence to minimal surfaces satisfying the same spanning condition as the volume constraint converges to zero (**minimal surfaces limit**). Although minimizers in the soap film capillarity model are described by regions of positive volume, these regions may fail to have uniformly positive thickness: indeed, in order to satisfy the spanning condition, minimizers may locally collapse onto surfaces. Understanding these collapsed surfaces, as well as their behavior in the minimal surfaces limit, is an important step in the study of the soap film capillarity model. In this paper we obtain a decisive progress on this problem, by showing the smoothness of collapsed surfaces, up to possible singular sets of codimension at least 7. In particular,

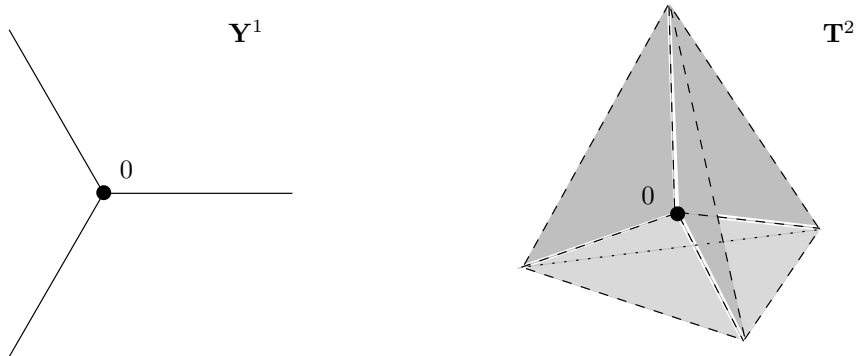


FIGURE 1.1. When $n = 1$, Almgren minimal sets are locally *isometric* either to lines or to $\mathbf{Y}^1 \subset \mathbb{R}^2$, the cone with vertex at the origin spanned by $(1, 0)$, $e^{i2\pi/3}$ and $e^{i4\pi/3}$. When $n = 2$, Almgren minimal sets are locally diffeomorphic either to planes (and locally at these points they are smooth minimal surfaces), or to $\mathbf{Y}^1 \times \mathbb{R}$, or to $\mathbf{T}^2$, the cone with vertex at the origin spanned by the edges of a reference regular tetrahedron (for the purposes of this paper, there is no need to specify this reference choice).

we show that, in physical dimensions, collapsed regions are smooth, thus providing strong evidence that, in the minimal surfaces limit, any singularities of solutions of the Plateau's problem should be “wetted” by the bulky parts of capillarity minimizers.

1.2. The soap film capillarity model. We start by recalling the formulation of our model for soap films hanging from a wire frame, together with the main results obtained in [KMS19, KMS20]. The wire frame is a compact set $W \subset \mathbb{R}^{n+1}$, $n \geq 1$, and the region of space accessible to soap films is the open set

$$\Omega = \mathbb{R}^{n+1} \setminus W.$$

A **spanning class** in Ω is a non-empty family $\mathcal{C}$ of smooth embeddings $\gamma: \mathbb{S}^1 \rightarrow \Omega$ which is homotopically closed¹ in Ω ; correspondingly, a relatively closed subset S of Ω is **$\mathcal{C}$ -spanning** W if $S \cap \gamma \neq \emptyset$ for every $\gamma \in \mathcal{C}$. Given choices of W and $\mathcal{C}$, we obtain a formulation of **Plateau's problem** (area minimization with a spanning condition) following Harrison and Pugh [HP16, HP17] (see also [DLGM17]), by setting

$$\ell = \ell(W, \mathcal{C}) = \inf \{ \mathcal{H}^n(S) : S \in \mathcal{S} \}, \quad (1.1)$$

where $\mathcal{H}^n$ denotes the n -dimensional Hausdorff measure on $\mathbb{R}^{n+1}$, and where

$$\mathcal{S} = \{ S \subset \Omega : S \text{ is relatively closed and } \mathcal{C}\text{-spanning } W \}. \quad (1.2)$$

Minimizers S of ℓ exist as soon as $\ell < \infty$. They are, in the jargon of Geometric Measure Theory, **Almgren minimal sets in Ω** , in the sense that they minimize area with respect to local Lipschitz deformations

$$\mathcal{H}^n(S \cap B_r(x)) \leq \mathcal{H}^n(f(S) \cap B_r(x)), \quad (1.3)$$

whenever f is a Lipschitz map with $\{f \neq \text{id}\} \subset\subset B_r(x) \subset\subset \Omega$ and $f(B_r(x)) \subset B_r(x)$ (here $B_r(x)$ is the open ball of center x and radius r in $\mathbb{R}^{n+1}$). This minimality property is crucial in establishing that, in the physical dimensions $n = 1, 2$, minimizers of ℓ satisfy the celebrated **Plateau's laws**, and are thus realistic models for actual soap films; see [Alm76, Tay76], section 7.2, and Figure 1.1.

¹If γ_0, γ_1 are smooth embeddings $\mathbb{S}^1 \rightarrow \Omega$ with $\gamma_0 \in \mathcal{C}$ and $f: [0, 1] \times \mathbb{S}^1 \rightarrow \Omega$ is a continuous mapping such that $f(0, \cdot) = \gamma_0$ and $f(1, \cdot) = \gamma_1$ then also $\gamma_1 \in \mathcal{C}$.

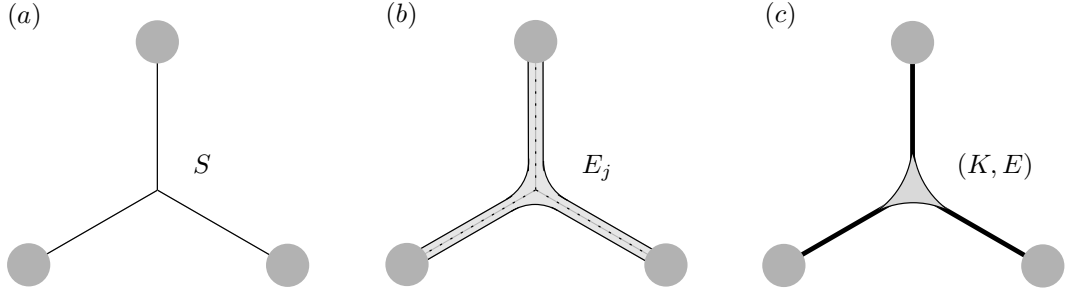


FIGURE 1.2. The soap film capillarity problem in the case when W consists of three small disks centered at the vertexes of an equilateral triangle, and $\mathcal{C}$ is generated by three loops, one around each disk in W : (a) the unique minimizer S of ℓ consists of three segments meeting at 120-degrees at a Y -point; (b) a minimizing sequence $\{E_j\}_j$ for $\psi(\varepsilon)$ will partly collapse along the segments forming S ; (c) the resulting generalized minimizer (K, E) , where $K \setminus \partial E$ consists of three segments (whose area is weighted by $\mathcal{F}$ with multiplicity 2, and which are depicted by bold lines), and where E is a negatively curved curvilinear triangle enclosing a volume ε , and “wetting” the Y -point of S .

In capillarity theory (neglecting gravity and working for simplicity with a null adhesion coefficient) regions E occupied by a liquid at equilibrium inside a container Ω can be described by minimizing the area $\mathcal{H}^n(\Omega \cap \partial E)$ of the boundary of E lying inside the container while keeping the volume $|E|$ of the region fixed. When the fixed amount of volume $\varepsilon = |E|$ is small, minimizers in the capillarity problem take the form of small almost-spherical droplets sitting near the points of highest mean curvature of $\partial\Omega$, see [BR05, Fal10, MM16]. To observe minimizers with a “soap film geometry”, we impose the $\mathcal{C}$ -spanning condition on $\Omega \cap \partial E$, and come to formulate the **soap film capillarity problem** $\psi(\varepsilon) = \psi(\varepsilon, W, \mathcal{C})$, by setting

$$\psi(\varepsilon) = \inf \{ \mathcal{H}^n(\Omega \cap \partial E) : E \in \mathcal{E}, |E| = \varepsilon, \text{ and } \Omega \cap \partial E \text{ is } \mathcal{C}\text{-spanning } W \}, \quad (1.4)$$

where

$$\mathcal{E} = \{ E \subset \Omega : E \text{ is an open set and } \partial E \text{ is } \mathcal{H}^n\text{-rectifiable} \}. \quad (1.5)$$

Of course, a minimizing sequence $\{E_j\}_j$ for $\psi(\varepsilon)$ may find energetically convenient to locally “collapse” onto lower dimensional regions, see Figure 1.2. Hence, we do not expect to find minimizers of $\psi(\varepsilon)$ in $\mathcal{E}$, but rather to describe limits of minimizing sequences in the class

$$\mathcal{K} = \left\{ (K, E) : E \subset \Omega \text{ is open with } \Omega \cap \text{cl}(\partial^* E) = \Omega \cap \partial E \subset K, \right. \\ \left. K \in \mathcal{S} \text{ and } K \text{ is } \mathcal{H}^n\text{-rectifiable} \right\}, \quad (1.6)$$

(where $\partial^* E$ is the reduced boundary of E , and cl stands for topological closure in $\mathbb{R}^{n+1}$), and to compute the limit of their energies with the relaxed energy functional $\mathcal{F}$ defined on $\mathcal{K}$ as

$$\mathcal{F}(K, E) = \mathcal{H}^n(\Omega \cap \partial^* E) + 2 \mathcal{H}^n(K \setminus \partial^* E) \quad \text{for } (K, E) \in \mathcal{K}. \quad (1.7)$$

Notice the factor 2 appearing as a weight for the area of $K \setminus \partial^* E$, due to the fact that $K \setminus \partial^* E$ originates as the limit of collapsing boundaries of $\Omega \cap \partial E_j$. We can now recall the two main results proved in [KMS19, KMS20], which state the existence of (generalized) minimizers of $\psi(\varepsilon)$ and prove the convergence of $\psi(\varepsilon)$ to the Plateau’s problem ℓ when $\varepsilon \rightarrow 0^+$.

Theorem 1.1 (Existence of generalized minimizers [KMS19, Theorem 1.4] and [KMS20, Theorem 5.1]). *Assume that $\ell = \ell(W, \mathcal{C}) < \infty$, Ω has smooth boundary, and that*

$$\exists \tau_0 > 0 \text{ such that } \mathbb{R}^{n+1} \setminus I_\tau(W) \text{ is connected for all } \tau < \tau_0, \quad (1.8)$$

where $I_\tau(W)$ is the closed τ -neighborhood of W .

If $\varepsilon > 0$ and $\{E_j\}_j$ is a minimizing sequence for $\psi(\varepsilon)$, then there exists $(K, E) \in \mathcal{K}$ with $|E| = \varepsilon$ such that, up to possibly extracting subsequences, and up to possibly modifying each E_j outside a large ball containing W (with both operations resulting in defining a new minimizing sequence for $\psi(\varepsilon)$, still denoted by $\{E_j\}_j$), we have that,

$$\begin{aligned} E_j &\rightarrow E \text{ in } L^1(\Omega), \\ \mathcal{H}^n \llcorner (\Omega \cap \partial E_j) &\xrightarrow{*} \theta \mathcal{H}^n \llcorner K \quad \text{as Radon measures in } \Omega \end{aligned} \quad (1.9)$$

as $j \rightarrow \infty$, where $\theta : K \rightarrow \mathbb{R}$ is upper semicontinuous and satisfies

$$\theta = 2 \text{ } \mathcal{H}^n\text{-a.e. on } K \setminus \partial^* E, \quad \theta = 1 \text{ on } \Omega \cap \partial^* E. \quad (1.10)$$

Moreover, $\psi(\varepsilon) = \mathcal{F}(K, E)$ and, for a suitable constant C , $\psi(\varepsilon) \leq 2\ell + C\varepsilon^{n/(n+1)}$.

Remark 1.2. Based on Theorem 1.1, we say that $(K, E) \in \mathcal{K}$ is a **generalized minimizer** of $\psi(\varepsilon)$ if $|E| = \varepsilon$, $\mathcal{F}(K, E) = \psi(\varepsilon)$ and there exists a minimizing sequence $\{E_j\}_j$ of $\psi(\varepsilon)$ such that (1.9) and (1.10) hold.

Theorem 1.3 (Minimal surfaces limit, [KMS19, Theorem 1.9] and [KMS20, Theorem 5.1]). *Assume that $\ell = \ell(W, \mathcal{C}) < \infty$, Ω has smooth boundary, and that (1.8) holds. Then ψ is lower semicontinuous on $(0, \infty)$ and $\psi(\varepsilon) \rightarrow 2\ell$ as $\varepsilon \rightarrow 0^+$. Moreover, if $\{(K_h, E_h)\}_h$ are generalized minimizers of $\psi(\varepsilon_h)$ corresponding to $\varepsilon_h \rightarrow 0^+$ as $h \rightarrow \infty$, then there exists a minimizer S of ℓ such that, up to extracting a subsequence in h , and as $h \rightarrow \infty$,*

$$2 \mathcal{H}^n \llcorner (K_h \setminus \partial^* E_h) + \mathcal{H}^n \llcorner (\Omega \cap \partial^* E_h) \xrightarrow{*} 2 \mathcal{H}^n \llcorner S, \quad \text{as Radon measures in } \Omega.$$

Theorem 1.1 and Theorem 1.3 open of course several questions on the properties of generalized minimizers at fixed ε , and on their behavior in the minimal surfaces limit $\varepsilon \rightarrow 0^+$. The two themes are very much intertwined, and in this paper we focus on the former, having in mind future developments on the latter. Before presenting our new results, we recall from [KMS19] one of the most basic properties of generalized minimizers of $\psi(\varepsilon)$, namely, that they actually minimize the relaxed energy $\mathcal{F}$ among their (volume-preserving) diffeomorphic deformations. In particular, they satisfy a certain Euler-Lagrange equation which, by Allard's regularity theorem [All72], implies a basic degree of regularity of K .

Theorem 1.4 ([KMS19, Theorem 1.6] and [KMS20, Theorem 5.1]). *Assume that $\ell = \ell(W, \mathcal{C}) < \infty$, Ω has smooth boundary, and that (1.8) holds. If (K, E) is a generalized minimizer of $\psi(\varepsilon)$ and $f : \Omega \rightarrow \Omega$ is a diffeomorphism with $|f(E)| = |E|$, then*

$$\mathcal{F}(K, E) \leq \mathcal{F}(f(K), f(E)). \quad (1.11)$$

In particular:

(i) *there exists $\lambda \in \mathbb{R}$ such that, for every $X \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ with $X \cdot \nu_\Omega = 0$ on $\partial\Omega$,*

$$\lambda \int_{\partial^* E} X \cdot \nu_E d\mathcal{H}^n = \int_{\partial^* E} \operatorname{div}^K X d\mathcal{H}^n + 2 \int_{K \setminus \partial^* E} \operatorname{div}^K X d\mathcal{H}^n \quad (1.12)$$

where div^K denotes the tangential divergence operator along K ;

(ii) *there exists $\Sigma \subset K$, closed and with empty interior in K , such that $K \setminus \Sigma$ is a smooth hypersurface, $K \setminus (\Sigma \cup \partial E)$ is a smooth embedded minimal hypersurface, $\mathcal{H}^n(\Sigma \setminus \partial E) = 0$, $\Omega \cap (\partial E \setminus \partial^* E) \subset \Sigma$ has empty interior in K , and $\Omega \cap \partial^* E$ is a smooth embedded hypersurface with constant scalar mean curvature λ (defined with respect to the outer unit normal ν_E of E).*

1.3. The exterior collapsed region of a generalized minimizer. In [KMS20] we have started the study of the **exterior collapsed region**

$$K \setminus \text{cl}(E)$$

of a generalized minimizer (K, E) of $\psi(\varepsilon)$. Indeed, the main result of [KMS20] is that if $K \setminus \text{cl}(E) \neq \emptyset$, then the Lagrange multiplier λ appearing in (1.12) is non-positive, a fact that, in turn, implies the validity of the convex hull inclusion $K \subset \text{conv}(W)$; see [KMS20, Theorem 2.8, Theorem 2.9]. In this paper, we continue the study of $K \setminus \text{cl}(E)$ by looking at its regularity. The basic fact that the multiplicity-one n -varifold defined by K is stationary in $\Omega \setminus \text{cl}(E)$, see (1.12), implies the existence of a relatively closed subset Σ of $K \setminus \text{cl}(E)$ such that

$$K \setminus (\text{cl}(E) \cup \Sigma) \text{ is a smooth minimal hypersurface} \quad (1.13)$$

and $\mathcal{H}^n(\Sigma) = 0$. The main result of this paper greatly improves this picture, by showing that Σ is much smaller than $\mathcal{H}^n$ -negligible.

Theorem 1.5 (Sharp regularity for the exterior collapsed region). *Assume that $\ell = \ell(W, \mathcal{C}) < \infty$, Ω has smooth boundary, and that (1.8) holds. If (K, E) is a generalized minimizer of $\psi(\varepsilon)$, then there exists a closed subset Σ of $K \setminus \text{cl}(E)$ such that $K \setminus (\Sigma \cup \text{cl}(E))$ is a smooth minimal hypersurface,*

$$\Sigma = \emptyset \quad \text{if } 1 \leq n \leq 6,$$

Σ is locally finite in $\Omega \setminus \text{cl}(E)$ if $n = 7$, and Σ is countably $(n - 7)$ -rectifiable (and thus has Hausdorff dimension $\leq n - 7$) if $n \geq 8$. In particular, in the physically relevant cases $n = 1$ and $n = 2$, the exterior collapsed region $K \setminus \text{cl}(E)$ is a smooth stable minimal hypersurface in $\Omega \setminus \text{cl}(E)$.

Remark 1.6 (Uniform local finiteness of the singular set). In fact, when $n \geq 7$ and Σ is possibly non-empty, we will show that Σ has locally finite $(n - 7)$ -dimensional Minkowski content (and thus locally finite $\mathcal{H}^{n-7}$ -measure) in $\Omega \setminus \text{cl}(E)$; see section 8.

Remark 1.7 (Consequences for the minimal surfaces limit). A striking consequence of Theorem 1.5 is that the exterior collapsed region $K \setminus \text{cl}(E)$ is dramatically *more regular* than the generic minimizer of Plateau's problem ℓ . For instance, in the physical dimension $n = 2$, one can apply the work of Taylor [Tay76], as detailed for example in section 7.2, to conclude that a minimizer S of ℓ (which is known to be an Almgren minimal set in Ω as defined in (1.3)) is locally diffeomorphic either to a plane (in which case, S is locally a smooth minimal surface), or to the cone $\mathbf{Y}^1 \times \mathbb{R}$ (Y -points), or to the cone $\mathbf{T}^2$ (T -points); and, indeed these singularities are easily observable in soap films. At the same time, by Theorem 1.5, when $n = 2$ the singular set of the exterior collapsed region is empty. Similarly, in arbitrary dimensions, the singular set of an n -dimensional minimizer S of ℓ could have codimension one in S , while, by Theorem 1.5, the singular set of $K \setminus \text{cl}(E)$ has *at least* codimension *seven* in $K \setminus \text{cl}(E)$. This huge regularity mismatch between the exterior collapsed region and the typical minimizer in Plateau's problem has a second point of interest, as it provides strong evidence towards the conjecture that, in the minimal surfaces limit " $(K_h, E_h) \rightarrow S$ " described in Theorem 1.3, low codimension singularities of minimizers S of ℓ may be contained (or even coincide, as it seems to be the case when $n = 1$) with the set of accumulation points of the bulky regions E_h . This implication is of course not immediate, and will require further investigation.

1.4. Outline of the proof of Theorem 1.5. The proof of Theorem 1.5 is based on a mix of regularity theorems from Geometric Measure Theory, combined with two steps which critically hinge upon the specific structure of the variational problem $\psi(\varepsilon)$. A breakdown of the argument is as follows:

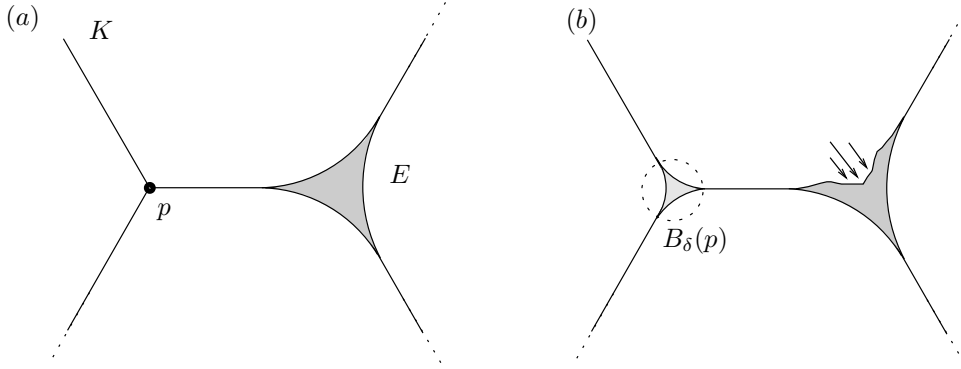


FIGURE 1.3. Wetting competitors: (a) a local picture of a generalized minimizer (K, E) when $n = 1$, with a point p of type Y ; (b) the wetting competitor is obtained by first modifying K at a scale δ near p , so to save an $O(\delta)$ of length at the expense of an increase of $O(\delta^2)$ in area; the added area can be restored by pushing inwards E at some point in $\partial^* E$, with a linear tradeoff between subtracted area and added length: in other words, to subtract an area of $O(\delta^2)$, we are increasing length by an $O(\delta^2)$ (whose size is proportional to the absolute value of the Lagrange multiplier λ of (K, E)). If δ is small enough in terms of λ , the $O(\delta)$ savings in length will eventually beat the $O(\delta^2)$ length increase used to restore the total area. In higher dimensions (where length and area become $\mathcal{H}^n$ -measure and volume/Lebesgue measure respectively), wetting competitors are obtained by repeating this construction in the cylindrical geometry defined by the spine $\{0\} \times \mathbb{R}^{n-1}$ of $\mathbf{Y}^1 \times \mathbb{R}^{n-1}$ near the Y -point p .

Step one ($K \setminus \text{cl}(E)$ is Almgren minimal in $\Omega \setminus \text{cl}(E)$): In (1.11) we have proved that (K, E) minimizes $\mathcal{F}$ against diffeomorphic images which preserve the volume of E , an information which implies $\mathcal{H}^n(\Sigma) = 0$ for the set Σ in (1.13). In this first step we greatly improve this information in the region away from E , by allowing for arbitrary Lipschitz deformations. Precisely, we show that $K \setminus \text{cl}(E)$ is an Almgren minimal set in $\Omega \setminus \text{cl}(E)$, i.e.

$$\mathcal{H}^n(K \cap B_r(x)) \leq \mathcal{H}^n(f(K) \cap B_r(x)) \quad (1.14)$$

for every Lipschitz map $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ such that $\{f \neq \text{id}\} \subset B_r(x)$ and $f(B_r(x)) \subset B_r(x)$, with $B_r(x) \subset\subset \Omega \setminus \text{cl}(E)$. Proving (1.14) is delicate, as discussed below.

Step two ($K \setminus \text{cl}(E)$ has no Y -points in $\Omega \setminus \text{cl}(E)$): We construct “wetting” competitors, that cannot be realized as Lipschitz images of K , to rule out the existence of Y -points in Σ , that is points where K is locally diffeomorphic to the cone $\mathbf{Y}^1 \times \mathbb{R}^{n-1}$; see Figure 1.3.

Step three: We combine the Almgren minimality of $K \setminus \text{cl}(E)$ in $\Omega \setminus \text{cl}(E)$ and the absence of Y -points in $K \setminus \text{cl}(E)$ with some regularity theorems by Taylor [Tay76] and Simon [Sim93], to conclude that the singular set Σ of $K \setminus \text{cl}(E)$ is $\mathcal{H}^{n-1}$ -negligible. At the same time, (1.11) implies that the multiplicity-one varifold associated to $K \setminus \text{cl}(E)$ is not only stationary, but also stable in $\Omega \setminus \text{cl}(E)$. Therefore, we can exploit Wickramasekera’s far reaching extension [Wic14] of a classical theorem of Schoen and Simon [SS81] to conclude that Σ is empty if $1 \leq n \leq 6$, is locally finite if $n = 7$, and is $\mathcal{H}^{n-7+\eta}$ -negligible for every $\eta > 0$ if $n \geq 8$. This last information, combined with the Naber-Valtorta theorem [NV15, Theorem 1.5] implies that when $n \geq 8$, Σ is countably $(n - 7)$ -rectifiable, thus completing the proof of the theorem. We notice here that when $n = 1, 2$, one can implement this strategy by relying solely on Taylor’s theorem [Tay76], thus avoiding the use of the Schoen-Simon-Wickramasekera theory, see Remark 7.8. Also, one can somehow rely on [SS81] only (rather than on the full strength of [Wic14]), see Remark 7.9. Finally, we

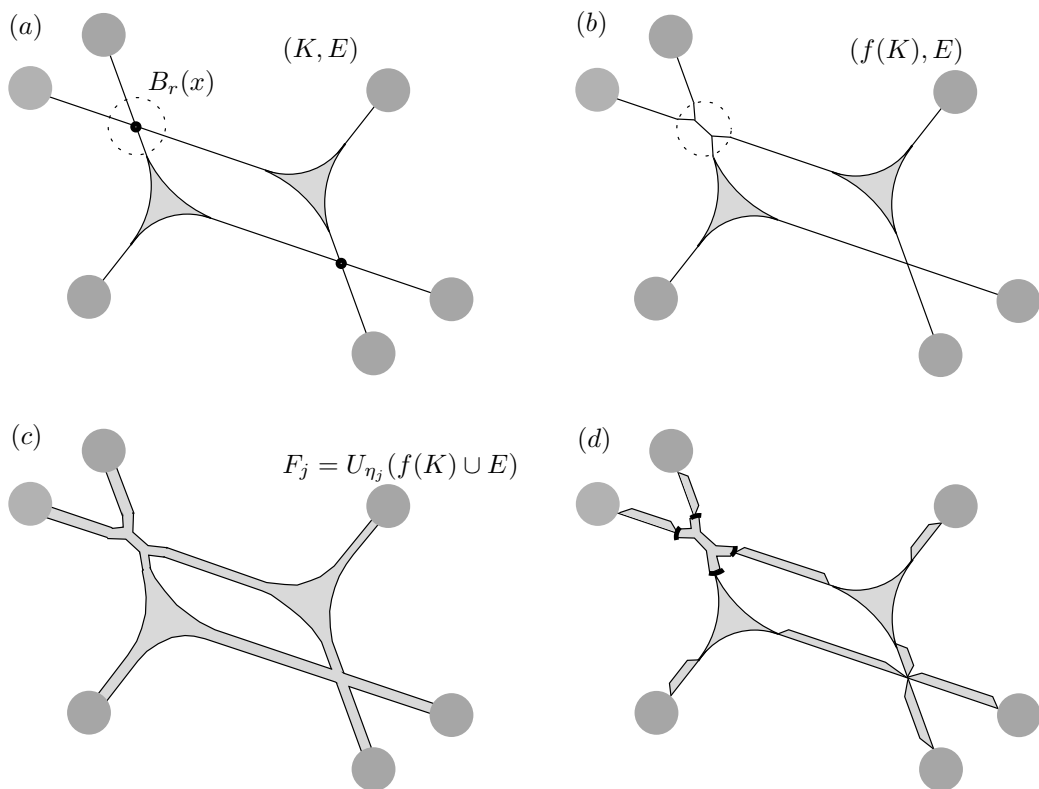


FIGURE 1.4. Proving that the exterior collapsed region is an Almgren minimal set.

notice that when $n \geq 8$, by further refining the above arguments, we can also show that Σ is locally $\mathcal{H}^{n-7}$ -finite: this is discussed in section 8.

We close this introduction by further discussing the construction of the competitors needed in carrying over step one of the above scheme. Indeed, this is a delicate point of the argument where we have made some non-obvious technical choices.

Discussion of step one: We illustrate the various aspects of the proof of (1.14) by means of Figure 1.4. In panel (a) we have a schematic representation of a generalized minimizer (K, E) whose exterior collapsed region $K \setminus \text{cl}(E)$ consists of various segments, intersecting along a singular set Σ which is depicted by two black disks. We center $B_r(x)$ at one of the points in Σ , pick r so that $B_r(x)$ is disjoint from $\text{cl}(E) \cup W$, and in panel (b) we depict the effect on $K \cap B_r(x)$ of a typical area-decreasing Lipschitz deformation supported in $B_r(x)$ (notice that such a map is not injective, so (1.11) is of no help here). As it turns out, one has $(f(K), E) \in \mathcal{K}$: the only non-trivial point is showing that $f(K)$ is $\mathcal{C}$ -spanning W , but this follows quite directly by arguing as in [DLGM17, Proof of Theorem 4, Step 3]. Now, in order to deduce (1.14) from $\psi(\varepsilon) = \mathcal{F}(K, E)$ we need to find a sequence $\{F_j\}_j$ in the competition class of $\psi(\varepsilon)$ such that $\mathcal{H}^n(\Omega \cap \partial F_j) \rightarrow \mathcal{F}(f(K), E)$ as $j \rightarrow \infty$. The obvious choice, at least in the situation depicted in Figure 1.4, would be taking ²

$$F_j = U_{\eta_j}(f(K) \cup E),$$

for some $\eta_j \rightarrow 0^+$, where $U_\eta(S)$ denotes the open η -tubular neighborhood of the set S , see panel (c); for such a set F_j , we want to show that (i) $\mathcal{H}^n(B_r(x) \cap \partial F_j) \rightarrow 2 \mathcal{H}^n(B_r(x) \cap f(K))$

²In the general situation, with $K \cap E$ possibly not empty, one should modify the formula for F_j by removing $I_{\eta_j}(K \cap E)$. This fact is taken into account in the actual proof when we consider the set A_1 in Lemma 5.2 below.

as $j \rightarrow \infty$; and (ii) that $\Omega \cap \partial F_j$ is $\mathcal{C}$ -spanning W . Concerning problem (i), taking into account that

$$\mathcal{H}^n(\Omega \cap \partial F_\eta) \approx \frac{|U_\eta(f(K) \cup E)|}{\eta} \quad \text{as } \eta \rightarrow 0^+,$$

one wants first to show that $f(K)$ is Minkowski regular, in the sense that

$$\lim_{\eta \rightarrow 0^+} \frac{|U_\eta(f(K))|}{2\eta} = \mathcal{H}^n(f(K)),$$

and then to discuss the relation between $|U_\eta(f(K))|$ and $|U_\eta(f(K) \cup E)|$, which needs to keep track of those “volume cancellations” due to the parts of $U_\eta(f(K))$ which are contained in E . Discussing such cancellations is indeed possible through a careful adaptation of some recent works by Ambrosio, Colesanti and Villa [ACV08, Vil09]. Addressing the Minkowski regularity of $f(K)$ requires instead the merging of two basic criteria for Minkowski regularity: “Lipschitz images of compact subsets in $\mathbb{R}^n$ are Minkowski regular” (Kneser’s Theorem [Kne55], see also [Fed69, 3.2.28-29] and [AFP00, Theorem 2.106]) and “compact $\mathcal{H}^n$ -rectifiable sets with uniform density estimates are Minkowski regular” (due to Ambrosio, Fusco and Pallara [AFP00, Theorem 2.104]). In section 3, see in particular Theorem 3.4 below, we indeed merge these criteria by showing that “Lipschitz images of compact $\mathcal{H}^n$ -rectifiable sets with uniform density estimates are Minkowski regular”. (To apply this theorem to $f(K)$ we need of course to obtain uniform density estimates for K , which are discussed in section 4, Theorem 4.1). We can thus come to a satisfactory solution of problem (i). However, we have not been able to solve problem (ii): in other words, it remains highly non-obvious if a set like $\Omega \cap \partial F_j$ is always $\mathcal{C}$ -spanning W , given the possibly subtle interactions between the geometries of E and $f(K)$ and the operation of taking open neighborhoods. To overcome the spanning problem, we explore the possibility of defining F_j as a *one-sided neighborhood* of $f(K)$ (which automatically contains the $\mathcal{C}$ -spanning set $f(K)$ in its boundary), rather than as an open neighborhood of $f(K)$ (which contains the $\mathcal{C}$ -spanning set $f(K)$ in its interior). As shown in [KMS20, Lemma 3.2], we can define $\mathcal{C}$ -spanning one-sided neighborhoods of a pair $(K, E) \in \mathcal{K}$ whenever K is smoothly orientable outside of a meager closed subset of K . Thanks to Theorem 1.4-(ii) a generalized minimizer (K, E) has enough regularity to define the required one-sided neighborhoods of K , but this regularity may be lost after applying the Lipschitz map f to K : it thus seems that neither approach is going to work. The solution comes *by mixing the two methods*, as depicted in panel (d): inside $B_r(x)$, we define F_j by taking an η_j -neighborhood of $f(K)$ – which is fine, in terms of proving the $\mathcal{C}$ -spanning condition, given the simple geometry of the ball and the care we will put in making sure that $\Omega \cap \partial F_j$ contains the spherical subsets $\partial B_r(x) \cap U_{\eta_j}(f(K))$; inside $\Omega \setminus \text{cl}(B_r(x))$ we will define F_j by the one-sided neighborhood construction – notice that we have enough regularity in this region because $f(K)$ and K coincide on $\Omega \setminus \text{cl}(B_r(x))$. We will actually need a variant of the one-sided neighborhood lemma [KMS20, Lemma 3.2], to guarantee that $\partial B_r(x) \cap U_{\eta_j}(f(K))$ is contained in $\Omega \cap \partial F_j$, see Lemma 5.2.

Organization of the paper: Section 2 contains a summary of the notation used in the paper. In section 3 we obtain the criterion for Minkowski regularity merging Kneser’s theorem with [AFP00, Theorem 2.104], see Theorem 3.4. In section 4 we discuss the lower density bounds up to the boundary wire frame needed to apply Theorem 3.4 to K , while in section 5 we put together all these results to show the Almgren minimality of $K \setminus \text{cl}(E)$ in $\Omega \setminus \text{cl}(E)$. In section 6 we construct the wetting competitors needed to exclude the presence of Y -points of $K \setminus \text{cl}(E)$ in $\Omega \setminus \text{cl}(E)$, and finally, in section 7, we illustrate the application of various regularity theorems [Tay76, Sim93, SS81, Wic14, NV15] needed to deduce Theorem 1.5 from our variational analysis. Finally, in section 8, we exploit

more specifically the Naber-Valtorta results on the quantitative stratification of stationary varifolds and prove the local $\mathcal{H}^{n-7}$ -estimates for Σ mentioned in Remark 1.6.

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2. NOTATION AND TERMINOLOGY

We summarize some basic definitions, mostly following [Sim83, Mag12].

Radon measures and rectifiability: We work in the Euclidean space $\mathbb{R}^{n+1}$ with $n \geq 1$. For $A \subset \mathbb{R}^{n+1}$, $\text{cl}(A)$ denotes the topological closure of A in $\mathbb{R}^{n+1}$, while $U_\eta(A)$ and $I_\eta(A)$ are the open and closed η -tubular neighborhoods of A , respectively. The open ball centered at $x \in \mathbb{R}^{n+1}$ with radius $r > 0$ is denoted $B_r(x)$; given $1 \leq k \leq n$ and a k -dimensional linear subspace $L \subset \mathbb{R}^{n+1}$, $B_r^L(x)$ denotes instead the open disc $B_r(x) \cap (x + L)$, and $B_r^k(x)$ is the corresponding shorthand notation when the subspace L is clear from the context. We use the shorthand notation $B_r = B_r(0)$ and $B_r^k = B_r^k(0)$. If $A \subset \mathbb{R}^{n+1}$ is (Borel) measurable, then $|A| = \mathcal{L}^{n+1}(A)$ and $\mathcal{H}^s(A)$ denote its Lebesgue and s -dimensional Hausdorff measures, respectively, and we set $\omega_k = \mathcal{H}^k(B_1^k)$. If μ is a Radon measure in $\mathbb{R}^{n+1}$, $A \subset \mathbb{R}^{n+1}$ is Borel, and $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^d$ is continuous and proper, then $\mu \llcorner A$ and $f_\# \mu$ denote the restriction of μ to A and the push-forward of μ through f , respectively defined by $(\mu \llcorner A)(E) = \mu(A \cap E)$ for every Borel $E \subset \mathbb{R}^{n+1}$ and $(f_\# \mu)(F) = \mu(f^{-1}(F))$ for every Borel $F \subset \mathbb{R}^d$. The **Hausdorff dimension** of A is denoted $\dim_{\mathcal{H}}(A)$: it is the infimum of all real numbers $t \geq 0$ such that $\mathcal{H}^s(A) = 0$ for all $s > t$. Given an integer $1 \leq k \leq n+1$, a Borel measurable set $M \subset \mathbb{R}^{n+1}$ is **countably k -rectifiable** if it can be covered by countably many Lipschitz images of $\mathbb{R}^k$ up to a set of zero $\mathcal{H}^k$ measure; M is **(locally) $\mathcal{H}^k$ -rectifiable** if it is countably k -rectifiable and, in addition, its $\mathcal{H}^k$ measure is (locally) finite. If M is locally $\mathcal{H}^k$ -rectifiable, then for $\mathcal{H}^k$ -a.e. $x \in M$ there exists a unique k -dimensional linear subspace of $\mathbb{R}^{n+1}$, denoted $T_x M$, with the property that $\mathcal{H}^k \llcorner ((M-x)/r) \xrightarrow{*} \mathcal{H}^k \llcorner T_x M$ in the sense of Radon measures in $\mathbb{R}^{n+1}$ as $r \rightarrow 0^+$: $T_x M$ is called the approximate tangent space to M at x . If $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ is locally Lipschitz and M is locally $\mathcal{H}^k$ -rectifiable then the tangential gradient $\nabla^M f$ and the tangential jacobian $J^M f$ are well defined at $\mathcal{H}^k$ -a.e. point in M .

Sets of finite perimeter: A Borel set $E \subset \mathbb{R}^{n+1}$ is: of **locally finite perimeter** if there exists an $\mathbb{R}^{n+1}$ -valued Radon measure μ_E such that $\langle \mu_E, X \rangle = \int_E \text{div}(X) dx$ for all vector fields $X \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$; of **finite perimeter** if, in addition, $P(E) = |\mu_E|(\mathbb{R}^{n+1})$ is finite. For any Borel set $F \subset \mathbb{R}^{n+1}$, the relative perimeter of E in F is then defined by $P(E; F) = |\mu_E|(F)$. The **reduced boundary** of a set E of locally finite perimeter is the set $\partial^* E$ of all points $x \in \mathbb{R}^{n+1}$ such that the vectors $|\mu_E|(B_r(x))^{-1} \mu_E(B_r(x))$ converge, as $r \rightarrow 0^+$, to a vector $\nu_E(x) \in \mathbb{S}^n$: $\nu_E(x)$ is called the **outer unit normal** to $\partial^* E$ at x . By De Giorgi's structure theorem, $\partial^* E$ is locally $\mathcal{H}^n$ -rectifiable, with $\mu_E = \nu_E \mathcal{H}^n \llcorner \partial^* E$ and $|\mu_E| = \mathcal{H}^n \llcorner \partial^* E$.

Integral varifolds: An **integral n -varifold** V on an open set $U \subset \mathbb{R}^{n+1}$ is a continuous linear functional on $C_c^0(U \times G_n^{n+1})$ (where G_n^{n+1} is the set of unoriented n -dimensional planes in $\mathbb{R}^{n+1}$) corresponding to a locally $\mathcal{H}^n$ -rectifiable set M in U , and a non-negative, integer valued function $\theta \in L_{\text{loc}}^1(\mathcal{H}^n \llcorner M)$, so that

$$V(\varphi) = \mathbf{var}(M, \theta)(\varphi) = \int_M \varphi(x, T_x M) \theta(x) d\mathcal{H}^n(x) \quad \text{for all } \varphi \in C_c^0(U \times G_n^{n+1}).$$

The function θ , which is uniquely defined only $\mathcal{H}^n$ -a.e. on M is called the **multiplicity** of V , while the Radon measure $\|V\| = \theta \mathcal{H}^n \llcorner M$ is the **weight** of V and $\text{spt } V = \text{spt } \|V\|$ is the **support** of V . If $\Phi: U \rightarrow U'$ is a diffeomorphism, the **push-forward of $V = \mathbf{var}(M, \theta)$ through Φ** is the integral n -varifold $\Phi_\# V = \mathbf{var}(\Phi(M), \theta \circ \Phi^{-1})$ on U' . If

$X \in C_c^1(U; \mathbb{R}^{n+1})$, then $\operatorname{div}^T X = \varphi(x, T)$ defines a function $\varphi \in C_c^0(U \times G_n^{n+1})$: correspondingly, one says that $\vec{H} \in L_{\operatorname{loc}}^1(U; \mathbb{R}^{n+1})$ is the **generalized mean curvature vector** of V if

$$\int_M \theta \operatorname{div}^M X \, d\mathcal{H}^n = \int_M X \cdot \vec{H} \theta \, d\mathcal{H}^n \quad \forall X \in C_c^1(U; \mathbb{R}^{n+1}). \quad (2.1)$$

When $\vec{H} = 0$ we say that V is **stationary** in U : for example, if M is a minimal hypersurface in U , then $V = \mathbf{var}(M, 1)$ is stationary in U . Area monotonicity carries over from minimal surfaces to stationary varifolds, in the sense that the density ratios

$$\frac{\|V\|(B_r(x))}{\omega_n r^n} \quad \text{are increasing in } r \in (0, \operatorname{dist}(x, \partial U)),$$

with limit value as $r \rightarrow 0^+$ denoted by $\Theta_V(x)$ and called the **density** of V at x .

3. MINKOWSKI CONTENT OF RECTIFIABLE SETS

The goal of this section is merging two well-known criteria for Minkowski regularity, Kneser's Theorem [Kne55] and [AFP00, Theorem 2.104], into Theorem 3.4 below. As explained in the introduction, this result will then play a crucial role in proving the Almgren minimality of exterior collapsed regions. It is convenient to introduce the following notation: given a compact set $Z \subset \mathbb{R}^d$ and an integer $k \in \{0, \dots, d\}$, we define the **upper and lower k -dimensional Minkowski contents** of Z as

$$\begin{aligned} \mathcal{UM}^k(Z) &= \limsup_{\eta \rightarrow 0^+} \frac{|U_\eta(Z)|}{\omega_{d-k} \eta^{d-k}}, \\ \mathcal{LM}^k(Z) &= \liminf_{\eta \rightarrow 0^+} \frac{|U_\eta(Z)|}{\omega_{d-k} \eta^{d-k}}. \end{aligned}$$

When $\mathcal{UM}^k(Z) = \mathcal{LM}^k(Z)$ we denote by $\mathcal{M}^k(Z)$ their common value, and call it the **k -dimensional Minkowski content** of Z . If the k -dimensional Minkowski content of Z exists, we say further that Z is **Minkowski k -regular** provided

$$\mathcal{M}^k(Z) = \mathcal{H}^k(Z). \quad (3.1)$$

It is easily seen that any k -dimensional C^2 -surface with boundary in $\mathbb{R}^d$ is Minkowski k -regular, but, as said, more general criteria are available.

Theorem 3.1 (Kneser's Theorem). *If $Z \subset \mathbb{R}^k$ is compact and $f : \mathbb{R}^k \rightarrow \mathbb{R}^d$ is a Lipschitz map, then $f(Z)$ is Minkowski k -regular.*

Theorem 3.2 ([AFP00, Theorem 2.106]). *If Z is a compact, countably k -rectifiable set in $\mathbb{R}^d$, and if there exists a Radon measure ν on $\mathbb{R}^d$ with $\nu \ll \mathcal{H}^k$ and*

$$\nu(B_r(x)) \geq c r^k, \quad \forall x \in Z, \forall r < r_0,$$

for positive constants c and r_0 , then Z is Minkowski k -regular.

Remark 3.3. For the reader's convenience we observe that Theorem 3.2 has the same statement as [AFP00, Theorem 2.104], although it should be noted that the existence of ν implies that $\mathcal{H}^k(Z) < \infty$, and thus that Z is $\mathcal{H}^k$ -rectifiable. For this reason we shall directly work with $\mathcal{H}^k$ -rectifiable sets.

We now prove a result that mixes elements of both Theorem 3.1 and Theorem 3.2, but that apparently does not follow immediately from them.

Theorem 3.4. *If Z is a compact and $\mathcal{H}^k$ -rectifiable set in $\mathbb{R}^d$ such that*

$$\mathcal{H}^k(Z \cap B_r(x)) \geq c r^k \quad \forall x \in Z, \forall r < r_0, \quad (3.2)$$

and if $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Lipschitz map, then $f(Z)$ is Minkowski k -regular.

We present a proof of Theorem 3.4 which follows the argument used in [AFP00] to prove Theorem 3.1. We premise two propositions to the main argument.

Proposition 3.5. *If Z is a compact and $\mathcal{H}^k$ -rectifiable set in $\mathbb{R}^d$ such that*

$$\mathcal{H}^k(Z \cap B_r(x)) \geq c r^k \quad \forall x \in Z, \forall r < r_0,$$

and if $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Lipschitz map with $J^Z f > 0$ a.e. on Z , then $f(Z')$ is Minkowski k -regular for any compact subset $Z' \subset Z$.

Proof. Let

$$\nu = f_{\#}(\mathcal{H}^k \llcorner Z).$$

If $y = f(x) \in f(Z)$ and $r \leq \text{Lip}(f) r_0$, then

$$\nu(B_r(y)) = \mathcal{H}^k(Z \cap f^{-1}(B_r(y))) \geq \mathcal{H}^k(Z \cap B_{r/\text{Lip}(f)}(x)) \geq \frac{c}{(\text{Lip}(f))^k} r^k.$$

Moreover $\nu \ll \mathcal{H}^k$, since if $E \subset \mathbb{R}^d$ with $\mathcal{H}^k(E) = 0$, then by $J^Z f > 0$ on Z we get

$$\nu(E) = \mathcal{H}^k(Z \cap f^{-1}(E)) = \int_{Z \cap f^{-1}(E)} \frac{J^Z f}{J^Z f} d\mathcal{H}^k = \int_{E \cap f(Z)} d\mathcal{H}^k(y) \int_{f^{-1}(y) \cap Z} \frac{d\mathcal{H}^0}{J^Z f} = 0.$$

We can thus apply Theorem 3.2 to $f(Z')$ for every $Z' \subset Z$ compact. $\square$

Proposition 3.6. *If Z is a compact and $\mathcal{H}^k$ -rectifiable set in $\mathbb{R}^d$ such that*

$$\mathcal{H}^k(Z \cap B_r(x)) \geq c r^k \quad \forall x \in Z, \forall r < r_0,$$

and if $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Lipschitz map, then

$$\mathcal{M}^k(f(Z')) = 0$$

whenever $Z' \subset Z$ is compact with $J^Z f = 0$ $\mathcal{H}^k$ -a.e. on Z' .

Proof. Let us define $f_\varepsilon : \mathbb{R}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ by setting

$$f_\varepsilon(x) = (f(x), \varepsilon x).$$

If $x \in Z$ is such that f is tangentially differentiable at x along Z , then f_ε is tangentially differentiable at x along Z , and thus

$$J^Z f_\varepsilon(x) > 0,$$

with $J^Z f_\varepsilon(x) \rightarrow J^Z f(x)$ as $\varepsilon \rightarrow 0^+$ and $J^Z f_\varepsilon \leq (\text{Lip}(f))^k + 1$ for $\varepsilon < \varepsilon_0$. By Proposition 3.5 and the area formula, since f_ε is injective we get

$$\mathcal{M}^k(f_\varepsilon(Z')) = \mathcal{H}^k(f_\varepsilon(Z')) = \int_{Z'} J^Z f_\varepsilon \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+,$$

where in computing the limit we have used $J^Z f = 0$ $\mathcal{H}^k$ -a.e. on Z' . Thus,

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{M}^k(f_\varepsilon(Z')) = 0.$$

At the same time if $\eta > 0$, then

$$\left\{ (y, z) \in \mathbb{R}^d \times \mathbb{R}^d : y \in B_\eta(f(x)), z \in B_\eta(\varepsilon x), x \in Z' \right\} \subset U_{2\eta}(f_\varepsilon(Z')),$$

so that Fubini's theorem gives

$$|U_\eta(f(Z'))| \omega_d \eta^d \leq |U_{2\eta}(f_\varepsilon(Z'))|.$$

Dividing by η^{2d-k} we get

$$\mathcal{U}\mathcal{M}^k(f(Z')) \leq C(d, k) \limsup_{\eta \rightarrow 0^+} \frac{|U_{2\eta}(f_\varepsilon(Z'))|}{\eta^{2d-k}} \leq C(d, k) \mathcal{M}^k(f_\varepsilon(Z')),$$

and letting $\varepsilon \rightarrow 0^+$ we conclude the proof. $\square$

Proof of Theorem 3.4. For brevity, set $S = f(Z)$. The rectifiability of S gives $\mathcal{LM}^k(S) \geq \mathcal{H}^k(S)$, see e.g. [AFP00, Proposition 2.101], so that we only need to prove $\mathcal{UM}^k(S) \leq \mathcal{H}^k(S)$. We set $F = \{J^Z f > 0\}$ (that is f is tangentially differentiable along Z with positive tangential Jacobian on F) and pick $Z_0 \subset \{J^Z f = 0\} \subset Z \setminus F$ compact with the property that

$$\mathcal{H}^k(Z \setminus (F \cup Z_0)) < \sigma, \quad (3.3)$$

for some $\sigma > 0$. In this way, by Proposition 3.6, we have

$$\mathcal{M}^k(S_0) = 0 \quad \text{where } S_0 = f(Z_0). \quad (3.4)$$

Since S is compact and $\mathcal{H}^k$ -rectifiable we can find a countable disjoint family $\{S_i\}_i$ of compact subsets of S , which covers S modulo $\mathcal{H}^k$, and such that $S_i = f_i(Z_i)$ for injective Lipschitz maps f_i with uniformly positive Jacobian on $\mathbb{R}^k$. By Theorem 3.1,

$$\mathcal{M}^k(S_i) = \mathcal{H}^k(S_i) \quad \text{for every } i. \quad (3.5)$$

We pick N so that

$$\mathcal{H}^k\left(S \setminus \bigcup_{i=1}^N S_i\right) < \delta, \quad (3.6)$$

for δ to be chosen depending on σ , and set

$$S^* = S \setminus \left(S_0 \cup \bigcup_{i=1}^N S_i\right). \quad (3.7)$$

Next, we further distinguish points in S^* depending on their distance from $\bigcup_{i=0}^N S_i$. More precisely, for any arbitrary $\lambda \in (0, 1)$ we define the compact set

$$S^{**} = S \setminus U_{\lambda\eta}\left(\bigcup_{i=0}^N S_i\right), \quad (3.8)$$

and then we apply the Besicovitch covering theorem to cover

$$S^{**} \subset \bigcup_{j \in J} B_{\lambda\eta}(y_j), \quad (3.9)$$

where J is a finite set of indexes, each $y_j \in S^{**}$, and each point of S^{**} belongs to at most $\xi(d)$ distinct balls in the covering. Notice that

$$\begin{aligned} \bigcup_{j \in J} B_{\lambda\eta}(y_j) &\subset U_{\lambda\eta}(S) \setminus \bigcup_{i=0}^N S_i, \\ S \cap \bigcup_{j \in J} B_{\lambda\eta}(y_j) &\subset S^*. \end{aligned}$$

Furthermore, $B_{\lambda\eta/\text{Lip } f}(x_j) \subset f^{-1}(B_{\lambda\eta}(y_j))$ for some $x_j \in Z$ such that $y_j = f(x_j)$. The lower density bound (3.2) then yields

$$\begin{aligned} \#(J) \frac{c(\lambda\eta)^k}{(\text{Lip } f)^k} &\leq \sum_{j \in J} \mathcal{H}^k\left(Z \cap B_{\lambda\eta/\text{Lip } f}(x_j)\right) \\ &\leq \sum_{j \in J} \mathcal{H}^k\left(Z \cap f^{-1}(B_{\lambda\eta}(y_j))\right) \\ &\leq \xi(d) \mathcal{H}^k\left(Z \cap f^{-1}\left(\bigcup_{j \in J} B_{\lambda\eta}(y_j)\right)\right) \\ &\leq \xi(d) \mathcal{H}^k(Z \cap f^{-1}(S^*)). \end{aligned}$$

However,

$$\begin{aligned}\mathcal{H}^k(Z \cap f^{-1}(S^*)) &= \mathcal{H}^k(Z \cap f^{-1}(S^*) \cap F) + \mathcal{H}^k(Z \cap f^{-1}(S^*) \setminus F) \\ &\leq \nu(S^*) + \mathcal{H}^k(Z \setminus (F \cup Z_0)) \leq \nu(S^*) + \sigma,\end{aligned}$$

provided we set

$$\nu = f_{\#}[\mathcal{H}^k \llcorner (Z \cap F)].$$

Since $J^Z f > 0$ on F , we have, for any Borel set $A \subset \mathbb{R}^d$

$$\nu(A) = \int_{Z \cap F \cap f^{-1}(A)} \frac{J^Z f}{J^Z f} = \int_A d\mathcal{H}^k(y) \int_{f^{-1}(y) \cap Z \cap F} \frac{d\mathcal{H}^0}{J^Z f}$$

so that $\nu \ll \mathcal{H}^k$. Therefore a suitable choice of $\delta = \delta(\sigma)$ in (3.6) gives

$$\nu(S^*) < \sigma,$$

and we have thus proved that

$$\#(J) \leq C(d, c, \text{Lip}(f)) \sigma \lambda^{-k} \eta^{-k}. \quad (3.10)$$

We can now conclude the argument. From the definition of S^* it follows that

$$S = \bigcup_{i=0}^N S_i \cup S^*, \quad (3.11)$$

and thus that

$$U_{\eta}(S) \subset \bigcup_{i=0}^N U_{\eta}(S_i) \cup U_{\eta}(S^*) \quad \text{for all } \eta > 0. \quad (3.12)$$

On the other hand, by the definition of S^{**}

$$U_{\eta}(S^*) \subset U_{\eta}(S^{**}) \cup \bigcup_{i=0}^N U_{(1+\lambda)\eta}(S_i), \quad (3.13)$$

which, together with (3.9), gives

$$U_{\eta}(S) \subset \bigcup_{i=0}^N U_{(1+\lambda)\eta}(S_i) \cup \bigcup_{j \in J} B_{(1+\lambda)\eta}(y_j). \quad (3.14)$$

By means of (3.10) we achieve

$$|U_{\eta}(S)| \leq \sum_{i=0}^N |U_{(1+\lambda)\eta}(S_i)| + C(d, c, \text{Lip}(f)) \sigma \lambda^{-k} (1 + \lambda)^d \eta^{d-k}, \quad (3.15)$$

so that, dividing by $\omega_{d-k} \eta^{d-k}$, taking the limit as $\eta \rightarrow 0^+$, and using (3.4) and (3.5) we obtain

$$\begin{aligned}\mathcal{UM}^k(S) &\leq (1 + \lambda)^{d-k} \sum_{i=1}^N \mathcal{H}^k(S_i) + C(d, k, c, \text{Lip}(f)) \sigma \lambda^{-k} (1 + \lambda)^d \\ &\leq (1 + \lambda)^{d-k} \mathcal{H}^k(S) + C(d, k, c, \text{Lip}(f)) \sigma \lambda^{-k} (1 + \lambda)^d.\end{aligned} \quad (3.16)$$

The conclusion follows by letting first $\sigma \rightarrow 0^+$ and then $\lambda \rightarrow 0^+$. $\square$

We close this section by proving a useful localization statement.

Proposition 3.7 (Localization of Minkowski content). *If Z is a compact and $\mathcal{H}^k$ -rectifiable set in $\mathbb{R}^d$ such that*

$$\mathcal{M}^k(Z) = \mathcal{H}^k(Z),$$

then

$$\lim_{\eta \rightarrow 0^+} \frac{|U_\eta(Z) \cap E|}{\omega_{d-k} \eta^{d-k}} = \mathcal{H}^k(Z \cap E)$$

whenever E is a Borel set with $\mathcal{H}^k(K \cap \partial E) = 0$.

Proof. If we set

$$\mu_\eta = \frac{\mathcal{L}^d \llcorner U_\eta(Z)}{\omega_{d-k} \eta^{d-k}}, \quad \mu = \mathcal{H}^k \llcorner Z,$$

then we just need to prove that, as $\eta \rightarrow 0^+$, $\mu_\eta \xrightarrow{*} \mu$ in $\mathbb{R}^d$. To this end, we first consider an open set A , set $A_\eta = \{x \in A : \text{dist}(x, \partial A) \geq \eta\}$, and notice that, for $\eta < \eta_0$,

$$\begin{aligned} \mu_\eta(A) &= \frac{|U_\eta(Z) \cap A|}{\omega_{d-k} \eta^{d-k}} \geq \frac{|U_\eta(Z \cap A_\eta)|}{\omega_{d-k} \eta^{d-k}} \\ &\geq \frac{|U_\eta(Z \cap A_{\eta_0})|}{\omega_{d-k} \eta^{d-k}} \end{aligned}$$

so that, by [AFP00, Proposition 2.101] and since $Z \cap A_{\eta_0}$ is compact and $\mathcal{H}^k$ -rectifiable

$$\liminf_{\eta \rightarrow 0^+} \mu_\eta(A) \geq \mathcal{H}^k(Z \cap A_{\eta_0}).$$

Letting $\eta_0 \rightarrow 0^+$ we get

$$\liminf_{\eta \rightarrow 0^+} \mu_\eta(A) \geq \mu(A) \quad \forall A \subset \mathbb{R}^d \text{ open}. \quad (3.17)$$

Since $\mathcal{M}^k(Z) = \mathcal{H}^k(Z)$ means that $\mu_\eta(\mathbb{R}^d) \rightarrow \mu(\mathbb{R}^d)$ as $\eta \rightarrow 0^+$, we find that, for every compact set $H \subset \mathbb{R}^d$,

$$\mu(H) = \mu(\mathbb{R}^d) - \mu(A) \geq \lim_{\eta \rightarrow 0^+} \mu_\eta(\mathbb{R}^d) - \liminf_{\eta \rightarrow 0^+} \mu_\eta(A) \geq \limsup_{\eta \rightarrow 0^+} \mu_\eta(H), \quad (3.18)$$

where we have used (3.17) with $A = \mathbb{R}^d \setminus H$. By a standard criterion for weak-star convergence of Radon measures, (3.17) and (3.18) imply that $\mu_\eta \xrightarrow{*} \mu$ in $\mathbb{R}^d$ as $\eta \rightarrow 0^+$. $\square$

4. UNIFORM LOWER DENSITY ESTIMATES

The application of Theorem 3.4 to K (where (K, E) is a generalized minimizer of $\psi(\varepsilon)$), requires proving uniform lower density estimates for $\text{cl}(K)$ (recall that K is compact relatively to Ω , not to $\mathbb{R}^{n+1}$). Now, it is a consequence of the analysis carried out in [KMS19] that there exists a radius $r_* > 0$ such that

$$\mathcal{H}^n(K \cap B_r(x)) \geq \omega_n r^n, \quad \forall x \in K, r < r_*, B_r(x) \subset\subset \Omega. \quad (4.1)$$

However, the lower density estimate in (4.1) is not sufficient to apply Theorem 3.4, because its radius of validity degenerates as x approaches $\text{cl}(K) \setminus K = \text{cl}(K) \cap \partial\Omega$. We thus need an improvement, which is provided in the following theorem.

Theorem 4.1 (Uniform lower density estimates). *If (K, E) is a generalized minimizer of $\psi(\varepsilon) = \psi(\varepsilon, W, \mathcal{C})$, then there exist $c = c(n) > 0$ and $r_0 = r_0(n, W, |\lambda|) > 0$ such that*

$$\mathcal{H}^n(K \cap B_r(x)) \geq c r^n \quad \forall x \in \text{cl}(K), \forall r < r_0. \quad (4.2)$$

Here λ is the Lagrange multiplier of (K, E) , as introduced in Theorem 1.4-(i).

The proof of Theorem 4.1 starts with the remark that the integral n -varifold V naturally associated to (K, E) has generalized mean curvature in L^∞ and it satisfies a distributional formulation of Young's law. More precisely, letting V be the n -varifold $V = \mathbf{var}(K, \theta)$ defined by K with multiplicity function

$$\theta(x) = \begin{cases} 1 & \text{if } x \in \partial^* E \\ 2 & \text{if } x \in K \setminus \partial^* E, \end{cases}$$

then

$$\|V\| = \mathcal{H}^n \llcorner (\Omega \cap \partial^* E) + 2 \mathcal{H}^n \llcorner (K \setminus \partial^* E)$$

and, by considering (1.12) in Theorem 1.4 on vector fields compactly supported in Ω , we see that V has generalized mean curvature vector $\vec{H} = \lambda 1_{\partial^* E} \nu_E$ in Ω . Actually, (1.12) says more, since it allows for vector fields not necessarily supported in Ω , provided they are tangential to $\partial\Omega$, i.e. (1.12) gives

$$\int \operatorname{div}^K X d\|V\| = \int X \cdot \vec{H} d\|V\| \quad \forall X \in C^1(\Omega; \mathbb{R}^{n+1}) \text{ with } X \cdot \nu_\Omega = 0 \text{ on } \partial\Omega. \quad (4.3)$$

The extra information conveyed in (4.3) is that, in a distributional sense, V has contact angle $\pi/2$ with $\partial\Omega$. The consequences of the validity of (4.3) have been extensively studied in the classical work of Grüter and Jost [GJ86], and their work has been recently extended to arbitrary contact angles by Kagaya and Tonegawa [KT17]. In particular, if $s_0 \in (0, \infty)$ is such that the tubular neighborhood $U_{s_0}(\partial W)$ admits a well-defined nearest point projection map $\Pi: U_{s_0}(\partial W) \rightarrow \partial W$ of class C^1 then [KT17, Theorem 3.2] ensures the existence of a constant $C = C(n, s_0)$ such that for any $x \in U_{s_0/6}(\partial W) \cap \operatorname{cl}(\Omega)$ the map

$$r \in (0, s_0/6) \mapsto \frac{\|V\|(B_r(x)) + \|V\|(\tilde{B}_r(x))}{\omega_n r^n} e^{(|\lambda|+C)r} \quad (4.4)$$

is increasing, where

$$\tilde{B}_r(x) = \{y \in \mathbb{R}^{n+1} : \tilde{y} \in B_r(x)\}, \quad \tilde{y} = \Pi(y) + (\Pi(y) - y) \quad (4.5)$$

denotes a sort of nonlinear reflection of $B_r(x)$ across ∂W . In particular, for x as above the limit

$$\sigma(x) = \lim_{r \rightarrow 0^+} \frac{\|V\|(B_r(x)) + \|V\|(\tilde{B}_r(x))}{\omega_n r^n} \quad (4.6)$$

exists for every $x \in U_{s_0/6}(\partial W) \cap \operatorname{cl}(\Omega)$, and the map $x \mapsto \sigma(x)$ is upper semicontinuous in there; see [KT17, Corollary 5.1]. The uniform density estimate (4.2) will be deduced as a consequence of the above monotonicity result, together with the following simple geometric lemma:

Lemma 4.2. *Suppose that $x \in U_{s_0}(\partial W)$, and $\rho > 0$ is such that $\operatorname{dist}(x, \partial W) \leq \rho$ and $B_\rho(x) \subset U_{s_0}(\partial W)$. Then:*

$$\tilde{B}_\rho(x) \subset B_{5\rho}(x). \quad (4.7)$$

Proof of Lemma 4.2. See [KT17, Lemma 4.2]. $\square$

Proof of Theorem 4.1. First observe that (4.2) holds with $c = \omega_n$ for all $x \in K \setminus U_{s_0/6}(\partial W)$ as soon as $r < \min\{r_*, s_0/6\}$. Therefore, we can assume that

$$x \in \operatorname{cl}(K) \cap U_{s_0/6}(\partial W). \quad (4.8)$$

Also note that for points as in (4.8) it holds $\sigma(x) \geq 1$: by upper semicontinuity of σ on $U_{s_0/6}(\partial W) \cap \operatorname{cl}(\Omega)$, we just need to show this when, in addition to (4.8), we have $x \in K$, and indeed in this case

$$\sigma(x) \geq \lim_{r \rightarrow 0^+} \frac{\mathcal{H}^n(K \cap B_r(x))}{\omega_n r^n} \geq 1$$

thanks to (4.1). Now we fix $r < r_0 = \min\{r_*, 5s_0/6\}$, and distinguish two cases depending on the validity of

$$\text{dist}(x, \partial W) > \frac{r}{5}. \quad (4.9)$$

If (4.9) holds, then by (4.1)

$$\mathcal{H}^n(K \cap B_r(x)) \geq \mathcal{H}^n(K \cap B_{r/5}(x)) \geq \omega_n \left(\frac{r}{5}\right)^n,$$

so that (4.2) holds. If $\text{dist}(x, \partial W) \leq r/5$, then, thanks to the obvious inclusion $B_{r/5}(x) \subset U_{s_0}(\partial W)$ we can apply Lemma 4.2 with $\rho = r/5$, and (4.7) yields $\tilde{B}_{r/5}(x) \subset B_r(x)$. Hence, by exploiting $\sigma(x) \geq 1$ and (4.4) we get

$$\begin{aligned} c_n r^n &\leq \sigma(x) \omega_n \left(\frac{r}{5}\right)^n \\ &\leq \left(\|V\|(B_{r/5}(x)) + \|V\|(\tilde{B}_{r/5}(x)) \right) e^{(|\lambda|+C)r/5} \\ &\leq 2 \|V\|(B_r(x)) e^{(|\lambda|+C)r_0} \leq 8 \mathcal{H}^n(K \cap B_r(x)), \end{aligned}$$

up to further decreasing r_0 . $\square$

Corollary 4.3. *Let (K, E) be a generalized minimizer of $\psi(\varepsilon)$, and let $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ be Lipschitz. Then*

$$\lim_{\eta \rightarrow 0^+} \frac{|U_\eta(f(\text{cl}(K))) \cap E|}{2\eta} = \mathcal{H}^n(f(\text{cl}(K)) \cap E) \quad (4.10)$$

whenever E is a Borel set with $\mathcal{H}^n(f(\text{cl}(K)) \cap \partial E) = 0$.

Proof. Immediate from Theorem 3.4, Proposition 3.7 and Theorem 4.1. $\square$

5. MINIMALITY WITH RESPECT TO LIPSCHITZ DEFORMATIONS

In this section we complete the first step of our strategy, by proving the Almgren minimality of the exterior collapsed set.

Theorem 5.1. *If (K, E) is a generalized minimizer of $\psi(\varepsilon)$, $B_r(x) \subset\subset \Omega \setminus \text{cl}(E)$, and $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ is a Lipschitz map with $\{f \neq \text{id}\} \subset B_r(x)$ and $f(B_r(x)) \subset B_r(x)$, then $\mathcal{H}^n(K \cap B_r(x)) \leq \mathcal{H}^n(f(K) \cap B_r(x))$.*

As explained in the introduction, an important tool in the proof is the construction of one-sided neighborhoods of K . This point is discussed in the following lemma, which is, in fact, an extension of [KMS20, Lemma 3.2] (which corresponds to the case $U = \emptyset$).

Lemma 5.2. *Let $K \subset \Omega$ be a relatively compact and $\mathcal{H}^n$ -rectifiable set, let $E \subset \Omega$ be an open set with $\Omega \cap \text{cl}(\partial^* E) = \Omega \cap \partial E \subset K$, and let $U \subset\subset \Omega \setminus \text{cl}(E)$ be an open set. Suppose that $\Sigma \subset K$ is a closed subset with empty interior relatively to K such that $K \setminus \Sigma$ is a smooth hypersurface in Ω such that there exists $\nu \in C^\infty(K \setminus \Sigma; \mathbb{S}^n)$ with $\nu(x)^\perp = T_x(K \setminus \Sigma)$ at every $x \in K \setminus \Sigma$. Set*

$$M = K \setminus (\Sigma \cup \partial E \cup \text{cl}(U)),$$

and decompose $M = M_0 \cup M_1$ by letting

$$M_0 = (K \setminus \Sigma) \setminus (\text{cl}(E) \cup \text{cl}(U)), \quad M_1 = (K \setminus \Sigma) \cap E.$$

Let $\|A_M\|(x)$ be the maximal principal curvature (in absolute value) of M at x . For given $\eta, \delta \in (0, 1)$, define a positive function $u: M \rightarrow (0, \eta]$ by setting

$$u(x) = \min \left\{ \eta, \frac{\text{dist}(x, \Sigma \cup \partial E \cup \text{cl}(U) \cup W)}{2}, \frac{\delta}{\|A_M\|(x)} \right\}, \quad (5.1)$$

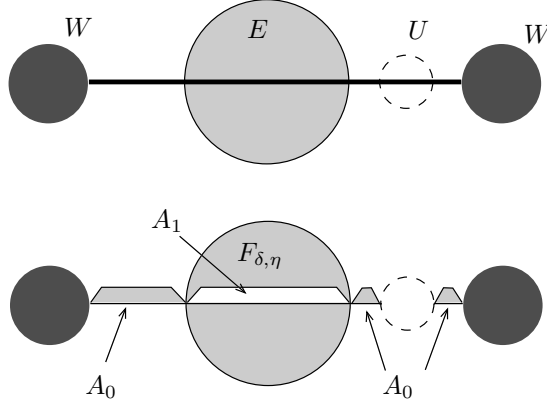


FIGURE 5.1. The construction in Lemma 5.2 gives a one-sided neighborhood of K away from $\text{cl}(E) \cup \text{cl}(U) \cup W$, thus defining an open set $F_{\delta, \eta}$ which contains $K \setminus \text{cl}(U)$ in its boundary, and which collapses onto $K \setminus \text{cl}(U)$ as $\eta \rightarrow 0^+$.

and let

$$\begin{aligned} A_0 &= \{x + t u(x) \nu(x) : x \in M_0, 0 < t < 1\}, \\ A_1 &= \{x + t u(x) \nu(x) : x \in M_1, 0 < t < 1\}, \\ F_{\delta, \eta} &= A_0 \cup (E \setminus \text{cl}(A_1)). \end{aligned}$$

Then, $F_{\delta, \eta} \subset \Omega \setminus \text{cl}(U)$ is open, $\partial F_{\delta, \eta}$ is $\mathcal{H}^n$ -rectifiable, and

$$K \setminus \text{cl}(U) \subset \Omega \cap \partial F_{\delta, \eta} \setminus \text{cl}(U), \quad (5.2)$$

$$\partial F_{\delta, \eta} \cap \partial U \subset K \cap \partial U. \quad (5.3)$$

Moreover,

$$\lim_{\eta \rightarrow 0^+} |F_{\delta, \eta} \Delta E| = 0 \quad \text{for every } \delta, \quad (5.4)$$

and

$$\begin{aligned} &\limsup_{\eta \rightarrow 0^+} \mathcal{H}^n(\Omega \cap \partial F_{\delta, \eta} \setminus \text{cl}(U)) \\ &\leq (1 + \delta)^n \left(\mathcal{H}^n(\Omega \cap \partial^* E) + 2 \mathcal{H}^n(K \setminus (\text{cl}(U) \cup \partial^* E)) \right). \end{aligned} \quad (5.5)$$

Without losing sight of the general picture, we first prove Theorem 5.1, and then take care of proving Lemma 5.2.

Proof of Theorem 5.1. Let us recall from [KMS20, Lemma 3.1], that if M is a smooth hypersurface in $\mathbb{R}^{n+1}$, then there exists a closed set $J \subset M$ with empty interior in M such that a smooth unit normal vector field to M can be defined on $M \setminus J$. Combining this fact with Theorem 1.4-(ii), we find that if (K, E) is a generalized minimizer of $\psi(\varepsilon)$, then there exists a subset³ $\Sigma \subset K$, closed and with empty interior relatively to K , such that $K \setminus \Sigma$ is a smooth *orientable* hypersurface in $\mathbb{R}^{n+1}$. We shall denote ν a smooth unit normal vector field on $K \setminus \Sigma$.

Let us fix $\rho > r$ such that $B_\rho(x) \subset \subset \Omega \setminus \text{cl}(E)$ and

$$\mathcal{H}^n(K \cap \partial B_\rho(x)) = 0. \quad (5.6)$$

Since $f(K) \cap \partial B_\rho(x) = K \cap \partial B_\rho(x)$, we can apply Corollary 4.3 to find

$$\lim_{t \rightarrow 0^+} \frac{|U_t(f(K)) \cap B_\rho(x)|}{2t} = \mathcal{H}^n(f(K) \cap B_\rho(x)).$$

³This set Σ could be much larger than the singular set of K , but is denoted with same letter used for the singular set of K since the notation should be clear from the context.

By applying the coarea formula to the distance function from $f(K)$, see e.g. [Mag12, Theorem 18.1, Remark 18.2], we find that $v(t) = |U_t(f(K)) \cap B_\rho(x)|$ satisfies

$$v(t) = \int_0^t \mathcal{H}^n(\partial(U_\eta(f(K))) \cap B_\rho(x)) d\eta,$$

and is thus absolutely continuous, with

$$v'(\eta) = \mathcal{H}^n(\partial(U_\eta(f(K))) \cap B_\rho(x)), \quad \text{for a.e. } \eta > 0;$$

moreover, again for a.e. $\eta > 0$,

$$\begin{aligned} U_\eta(f(K)) &\text{ is a set of finite perimeter} \\ \text{whose reduced boundary is } \mathcal{H}^n\text{-equivalent to } \partial[U_\eta(f(K))]. \end{aligned} \quad (5.7)$$

Therefore, for every $t > 0$ there are points of differentiability $\eta_1(t), \eta_2(t) \in (0, t)$ of v such that (5.7) holds at $\eta = \eta_1(t), \eta_2(t)$, and

$$v'(\eta_1(t)) \leq \frac{v(t)}{t} \leq v'(\eta_2(t)).$$

Picking any sequence $t_j \rightarrow 0^+$, and correspondingly setting $\eta_j = \eta_1(t_j)$, we thus find

$$\begin{aligned} \liminf_{j \rightarrow \infty} \mathcal{H}^n(\partial(U_{\eta_j}(f(K))) \cap B_\rho(x)) &\leq \liminf_{j \rightarrow \infty} \frac{|U_{t_j}(f(K)) \cap B_\rho(x)|}{t_j} \\ &= 2 \mathcal{H}^n(f(K) \cap B_\rho(x)). \end{aligned} \quad (5.8)$$

We also notice that since $\{\text{cl}(U_{\eta_j}(f(K)))\}_j$ is a decreasing sequence of sets with monotone limit $\text{cl}(f(K))$, we have

$$\lim_{j \rightarrow \infty} \mathcal{H}^n(\partial B_\rho(x) \cap \text{cl}(U_{\eta_j}(f(K)))) = \mathcal{H}^n(\partial B_\rho(x) \cap \text{cl}(f(K))) = \mathcal{H}^n(\partial B_\rho(x) \cap K) = 0, \quad (5.9)$$

again thanks to (5.6). We now pick $\delta \in (0, 1)$, and define $\{G_j\}_j$ by letting

$$G_j = (U_{\eta_j}(f(K)) \cap B_\rho(x)) \cup F_j \subset \Omega, \quad (5.10)$$

with $F_j = F_{\delta, \eta_j}$ as in Lemma 5.2 with $U = B_\rho(x)$. Since

$$\partial G_j \subset \left(\partial(U_{\eta_j}(f(K))) \cap B_\rho(x) \right) \cup \left(\text{cl}(U_{\eta_j}(f(K))) \cap \partial B_\rho(x) \right) \cup \partial F_j, \quad (5.11)$$

by (5.7) we see that ∂G_j is $\mathcal{H}^n$ -rectifiable for every j . Next, we make the following claim

$$\Omega \cap \partial G_j \text{ is } \mathcal{C}\text{-spanning } W \text{ for every } j, \quad (5.12)$$

which implies that G_j is a competitor for the problem $\psi(|G_j|)$. In this way, by

$$E \Delta G_j \subset (E \Delta F_j) \cup U_{\eta_j}(f(K)),$$

and by (5.4) we find that $|G_j| \rightarrow \varepsilon$ as $j \rightarrow \infty$, and since $\psi(\varepsilon)$ is lower semicontinuous on $(0, \infty)$, see [KMS19, Theorem 1.9], we conclude that

$$\begin{aligned} \mathcal{H}^n(\Omega \cap \partial^* E) + 2 \mathcal{H}^n(K \setminus \partial^* E) &= \psi(\varepsilon) \leq \liminf_{j \rightarrow \infty} \psi(|G_j|) \\ &\leq \liminf_{j \rightarrow \infty} \mathcal{H}^n(\Omega \cap \partial G_j). \end{aligned}$$

In turn, (5.11) implies that

$$\begin{aligned} \mathcal{H}^n(\Omega \cap \partial G_j) &\leq \mathcal{H}^n(\partial(U_{\eta_j}(f(K))) \cap B_\rho(x)) + \mathcal{H}^n(\text{cl}(U_{\eta_j}(f(K))) \cap \partial B_\rho(x)) \\ &\quad + \mathcal{H}^n(\partial F_j \cap \Omega \setminus \text{cl}(B_\rho(x))) + \mathcal{H}^n(\partial F_j \cap \partial B_\rho(x)), \end{aligned} \quad (5.13)$$

and thus, thanks to (5.6), (5.8), (5.9), (5.3) and (5.5), we have that

$$\begin{aligned} & \mathcal{H}^n(\Omega \cap \partial^* E) + 2\mathcal{H}^n(K \setminus \partial^* E) \\ & \leq 2\mathcal{H}^n(f(K) \cap B_\rho(x)) + (1 + \delta)^n \left\{ \mathcal{H}^n(\Omega \cap \partial^* E) + 2\mathcal{H}^n(K \setminus (\text{cl}(B_\rho(x)) \cup \partial^* E)) \right\}. \end{aligned}$$

By using again (5.6) and letting $\delta \rightarrow 0^+$ we deduce

$$\mathcal{H}^n(K \cap B_\rho(x)) \leq \mathcal{H}^n(f(K) \cap B_\rho(x)).$$

To complete the proof we are thus left to prove our claim (5.12).

Given $\gamma \in \mathcal{C}$ we want to show that

$$\gamma \cap \Omega \cap \partial G_j \neq \emptyset. \quad (5.14)$$

If $\gamma \cap (K \setminus \text{cl}(B_\rho(x))) \neq \emptyset$, then by (5.2) we also have $\gamma \cap (\Omega \cap \partial F_j \setminus \text{cl}(B_\rho(x))) \neq \emptyset$, and (5.14) holds. We can then suppose that $\gamma \cap (K \setminus \text{cl}(B_\rho(x))) = \emptyset$, so that, since K is $\mathcal{C}$ -spanning W , $\gamma \cap K \cap \text{cl}(B_\rho(x)) \neq \emptyset$.

If there is $x_0 \in \gamma \cap K \cap \partial B_\rho(x)$, then necessarily $x_0 \in \partial G_j$. Indeed, $G_j \cap \partial B_\rho(x) = \emptyset$ by construction, so that $x_0 \notin G_j$; on the other hand, since $\{f \neq \text{id}\} \subset\subset B_\rho(x)$, we have that $x_0 \in f(K)$, and thus $x_0 \in \text{cl}(U_{\eta_j}(f(K)) \cap B_\rho(x)) \subset \text{cl}(G_j)$.

Hence we can assume $\gamma \cap (K \setminus B_\rho(x)) = \emptyset$, and thus the existence of $x_0 \in \gamma \cap K \cap B_\rho(x)$. By [KMS19, Lemma 2.2], there exists a connected component γ_0 of $\gamma \cap \text{cl}(B_\rho(x))$ which is diffeomorphic to an interval, whose end-points p, q belong to different connected components of $\partial B_\rho(x) \setminus K$, and such that $\gamma_0 \setminus \{p, q\} \subset B_\rho(x)$. Arguing as in [DLGM17, Proof of Theorem 4, Step 3], we conclude that in fact $p = f(p)$ and $q = f(q)$ belong to the closures of distinct connected components of $B_\rho(x) \setminus f(K)$, and thus there exists $y_0 \in (\gamma_0 \setminus \{p, q\}) \cap f(K)$. Let u be the function $u(y) = \text{dist}(y, f(K) \cap B_\rho(x))$, and consider its restriction to the interval γ_0 . If $\min\{u(p), u(q)\} \leq \eta_j$, then either p or q belongs to $\partial B_\rho(x) \cap \text{cl}(U_{\eta_j}(f(K)) \cap B_\rho(x)) \subset \partial G_j$. Otherwise, both $u(p) > \eta_j$ and $u(q) > \eta_j$, whereas $u(y_0) = 0$, and thus, by the intermediate value theorem, $\gamma_0 \cap B_\rho(x) \cap \partial(U_{\eta_j}(f(K))) \neq \emptyset$. Thus $\gamma_0 \cap \partial G_j \neq \emptyset$, and the proof is complete. $\square$

Proof of Lemma 5.2. Let us recall that we have set

$$\begin{aligned} M &= K \setminus (\Sigma \cup \partial E \cup \text{cl}(U)) = M_0 \cup M_1, \\ M_0 &= M \setminus \text{cl}(E) = (K \setminus \Sigma) \setminus (\text{cl}(E) \cup \text{cl}(U)), \\ M_1 &= M \cap E = (K \setminus \Sigma) \cap E, \end{aligned}$$

and

$$A_0 = g(M_0 \times (0, 1)), \quad A_1 = g(M_1 \times (0, 1)), \quad F = F_{\delta, \eta} = A_0 \cup (E \setminus \text{cl}(A_1)),$$

where $g : M \times \mathbb{R} \rightarrow \mathbb{R}^{n+1}$ and $u : M \rightarrow (0, \eta]$ are defined by setting

$$\begin{aligned} u(x) &= \min \left\{ \eta, \frac{\text{dist}(x, \Sigma \cup \partial E \cup \text{cl}(U) \cup W)}{2}, \frac{\delta}{\|A_M\|(x)} \right\}, \\ g(x, t) &= x + t u(x) \nu(x). \end{aligned}$$

We divide the argument in two steps.

Step one: In this step we prove (5.3) as well as

$$F \text{ is open with } F \subset \Omega \setminus \text{cl}(U), \quad (5.15)$$

$$(K \setminus \text{cl}(U)) \cup \left\{ x + u(x) \nu(x) : x \in M \right\} = \Omega \cap \partial F \setminus \text{cl}(U). \quad (5.16)$$

Notice that (5.16) immediately implies (5.2), while (5.4) follows from $F \Delta E \subset A_0 \cup \text{cl}(A_1) \subset I_\eta(K)$ and the fact that, as $\eta \rightarrow 0^+$, $|I_\eta(K)| \rightarrow |K| = 0$. Therefore in step two we will only have to prove the validity of (5.5).

Since M_0 and M_1 are relatively open in M and u is positive on M , it is easily seen that A_0 and A_1 are open, and thus that F is open. Since $M \subset \Omega \setminus \text{cl}(U)$ and $u(x) < \text{dist}(x, W \cup \text{cl}(U))$ for every $x \in M$, we deduce that $A_0 = g(M_0 \times (0, 1)) \subset \Omega \setminus \text{cl}(U)$; since trivially $E \subset \Omega \setminus \text{cl}(U)$, we have proved (5.15).

As a preliminary step towards proving (5.16), we show that, for $k = 0, 1$, we have

$$M_k \cup \{x + u(x)\nu(x) : x \in M_k\} \subset \Omega \cap \partial A_k \subset K \cup \{x + u(x)\nu(x) : x \in M_k\}. \quad (5.17)$$

The first inclusion in (5.17) is due to the fact that if $y = x + s\nu(x)$ for some $x \in M$ and $|s| < 1/\|A_M\|(x)$, then x and s are uniquely determined in M and $[-1/\|A_M\|(x), 1/\|A_M\|(x)]$. The second inclusion in (5.17) follows because if $y \in \Omega \cap \partial A_k$, then y is the limit of a sequence $x_j + t_j u(x_j)\nu(x_j)$ with $t_j \in (0, 1)$, $x_j \in M_k$, $t_j \rightarrow t_0 \in [0, 1]$ and $x_j \rightarrow x_0 \in \text{cl}(M_k)$. If $x_0 \in \text{cl}(M_k) \setminus M_k \subset \Sigma \cup \partial E \cup \text{cl}(U) \cup W$, then $u(x_j) \rightarrow 0$ and therefore $y = x_0 \in K$. If $x_0 \in M_k$ then clearly $t_0 \in \{0, 1\}$: when $t_0 = 0$, then $y = x_0 \in K$; when, instead, $t_0 = 1$, then $y = x_0 + u(x_0)\nu(x_0)$ for $x_0 \in M_k$ as claimed.

We prove the inclusion $\supset$ in (5.16) by showing that actually

$$\Omega \cap \partial F \subset K \cup \{x + u(x)\nu(x) : x \in M\}. \quad (5.18)$$

Since the boundary of the union and of the intersection of two sets is contained in the union of the boundaries, and since the boundary of a set coincides with the boundary of its complement, the inclusion $\partial(\text{cl}(A_1)) \subset \partial A_1$ gives

$$\begin{aligned} \Omega \cap \partial F &\subset \Omega \cap (\partial A_0 \cup \partial[E \setminus \text{cl}(A_1)]) \subset \Omega \cap (\partial A_0 \cup \partial E \cup \partial[\mathbb{R}^{n+1} \setminus \text{cl}(A_1)]) \\ &= \Omega \cap (\partial A_0 \cup \partial E \cup \partial(\text{cl}(A_1))) \subset \Omega \cap (\partial E \cup \partial A_0 \cup \partial A_1). \end{aligned} \quad (5.19)$$

Hence, (5.18) follows from $\Omega \cap \partial E \subset K$ and (5.17).

We prove (5.3), i.e. $\partial F \cap \partial U \subset K \cap \partial U$. Indeed, $M \cap \text{cl}(U) = \emptyset$ and $u(x) < \text{dist}(x, \text{cl}(U))$ for every $x \in M$ give

$$\partial U \cap \{x + u(x)\nu(x) : x \in M\} = \emptyset,$$

so that (5.3) follows immediately from (5.18).

Finally, we complete the proof of (5.16) by showing the inclusion $\subset$. Since $\text{cl}(U) \cap \partial E = \emptyset$ and $M = K \setminus (\Sigma \cup \partial E \cup \text{cl}(U))$, this amounts to show that

$$M \cup \{x + u(x)\nu(x) : x \in M\} \subset \Omega \cap \partial F \setminus \text{cl}(U), \quad (5.20)$$

$$\Sigma \setminus (\partial E \cup \text{cl}(U)) \subset \Omega \cap \partial F \setminus \text{cl}(U), \quad (5.21)$$

$$\Omega \cap \partial E \subset \Omega \cap \partial F \setminus \text{cl}(U). \quad (5.22)$$

Proof of (5.20): Since $M_0 \cap (\text{cl}(E) \cup \text{cl}(U)) = \emptyset$, $M_1 \subset E$, and $u(x) < \text{dist}(x, \partial E \cup \text{cl}(U))$ for every $x \in M$, we find

$$g(M_0 \times [0, 1]) \cap (\text{cl}(E) \cup \text{cl}(U)) = \emptyset, \quad g(M_1 \times [0, 1]) \subset E. \quad (5.23)$$

Let us notice that, if $X, Y \subset \mathbb{R}^{n+1}$, $V \subset \mathbb{R}^{n+1}$ is open, and $X \cap V = Y \cap V$, then $V \cap \partial X = V \cap \partial Y$. We can apply this remark in the open sets $V = E$ and $V = \mathbb{R}^{n+1} \setminus \text{cl}(E)$, together with $A_0 \cap \text{cl}(E) = \emptyset$ and $A_1 \subset E$ (both consequences of (5.23)), the definition of $F = A_0 \cup (E \setminus \text{cl}(A_1))$, and $\text{cl}(U) \cap E = \emptyset$, to first deduce that

$$(\partial F) \setminus \text{cl}(E) = (\partial A_0) \setminus \text{cl}(E), \quad E \cap \partial F = E \cap \partial A_1,$$

and then that

$$((\partial A_0) \setminus (\text{cl}(E) \cup \text{cl}(U))) \cup (E \cap \partial A_1) \subset \partial F \setminus \text{cl}(U). \quad (5.24)$$

Since (5.17) and (5.23) give

$$g(M_0 \times \{0, 1\}) \subset \Omega \cap \partial A_0 \setminus (\text{cl}(E) \cup \text{cl}(U)), \quad g(M_1 \times \{0, 1\}) \subset E \cap \partial A_1, \quad (5.25)$$

we deduce (5.20) from (5.24) and (5.25).

Proof of (5.21): since $M_1 = (K \setminus \Sigma) \cap E$ and Σ has empty interior in K , we find that $\text{cl}(M_1) \cap E = K \cap E$. At the same time, $M \subset \Omega \cap \partial F \setminus \text{cl}(U)$ gives $M_1 \cap E \subset E \cap \partial F$ and thus $\text{cl}(M_1) \cap E \subset E \cap \partial F$: hence,

$$\Sigma \cap E \subset K \cap E = \text{cl}(M_1) \cap E \subset \Omega \cap \partial F \setminus \text{cl}(U).$$

Setting for the sake of brevity $T = \Omega \setminus (\text{cl}(E) \cup \text{cl}(U))$, so that T is open, we notice that $M_0 = (K \setminus \Sigma) \cap T$ implies $\Sigma \cap T \subset \Omega \cap \text{cl}(M_0) \cap T = K \cap T$, while $M \subset \Omega \cap \partial F \setminus \text{cl}(U)$ and $M_0 = M \cap T$ give $\Omega \cap \text{cl}(M_0) \cap T \subset T \cap \partial F$; hence

$$\Sigma \setminus (\text{cl}(E) \cup \text{cl}(U)) \subset K \setminus (\text{cl}(E) \cup \text{cl}(U)) \subset \Omega \cap \partial F \setminus (\text{cl}(E) \cup \text{cl}(U)).$$

By combining the last two displayed inclusions, we obtain (5.21).

Proof of (5.22): since F and E coincide in the complement of $\text{cl}(A_0) \cup \text{cl}(A_1)$, and since $\partial E \cap \text{cl}(U) = \emptyset$, we have

$$\Omega \cap \partial E \setminus (\text{cl}(A_0) \cup \text{cl}(A_1)) = \Omega \cap \partial F \setminus (\text{cl}(A_0) \cup \text{cl}(A_1) \cup \text{cl}(U)) \subset \Omega \cap \partial F \setminus \text{cl}(U).$$

Now let $y \in \Omega \cap \partial E \cap \text{cl}(A_1)$: since $A_1 = g(M_1 \times (0, 1))$ and $y \notin g(M_1 \times [0, 1])$ by (5.23), we find that y is in the closure of M_1 , and thus of M , relatively to K : thus $y \in \Omega \cap \text{cl}(M) \setminus \text{cl}(U)$; at the same time, by (5.20), we have $M \subset \Omega \cap \partial F \setminus \text{cl}(U)$ and thus $\Omega \cap \text{cl}(M) \setminus \text{cl}(U) \subset \Omega \cap \partial F \setminus \text{cl}(U)$; combining the two facts,

$$\Omega \cap \partial E \cap \text{cl}(A_1) \subset \Omega \cap \partial F \setminus \text{cl}(U).$$

We argue similarly to show that $\Omega \cap \partial E \cap \text{cl}(A_0) \subset \Omega \cap \partial F \setminus \text{cl}(U)$ and thus prove (5.22).

Step two. We prove (5.5). First, we notice that thanks to (5.16)

$$\mathcal{H}^n(\Omega \cap \partial F \setminus \text{cl}(U)) \leq \mathcal{H}^n(K \setminus \text{cl}(U)) + \mathcal{H}^n(\{x + u(x)\nu(x) : x \in M\}). \quad (5.26)$$

Since $\text{dist}(x, \Sigma \cup \partial E \cup \text{cl}(U) \cup W) > 0$ and $\|A_M\|(x) < \infty$ for every $x \in M$, we find that

$$M_\eta = \{x \in M : u(x) = \eta\} = \left\{x \in M : \|A_M\|(x) \leq \frac{\delta}{\eta}, \text{dist}(x, \Sigma \cup \partial E \cup \text{cl}(U) \cup W) \geq 2\eta\right\}$$

is monotonically increasing towards M as $\eta \rightarrow 0^+$. Moreover, $x \mapsto x + u(x)\nu(x) = x + \eta\nu(x)$ is smooth on M_η , and if κ_i are the principal curvatures of M with respect to ν ,

$$\mathcal{H}^n(\{x + u(x)\nu(x) : x \in M_\eta\}) = \int_{M_\eta} \prod_{i=1}^n (1 + \eta \kappa_i) \leq (1 + \delta)^n \mathcal{H}^n(M_\eta) \leq (1 + \delta)^n \mathcal{H}^n(M). \quad (5.27)$$

Letting $\eta \rightarrow 0^+$, $g(M_\eta \times \{1\}) = \{x + u(x)\nu(x) : x \in M_\eta\}$ is increasingly converging to $g(M \times \{1\}) = \{x + u(x)\nu(x) : x \in M\}$, so that (5.27) yields

$$\mathcal{H}^n(\{x + u(x)\nu(x) : x \in M\}) \leq (1 + \delta)^n \mathcal{H}^n(M), \quad (5.28)$$

and therefore from (5.26) we deduce

$$\limsup_{\eta \rightarrow 0^+} \mathcal{H}^n(\Omega \cap \partial F \setminus \text{cl}(U)) \leq \mathcal{H}^n(K \setminus \text{cl}(U)) + (1 + \delta)^n \mathcal{H}^n(M). \quad (5.29)$$

Finally, (5.5) follows from (5.29) once we observe that $M = K \setminus (\Sigma \cup \partial E \cup \text{cl}(U)) \subset K \setminus (\partial^* E \cup \text{cl}(U))$, so that

$$\begin{aligned} \mathcal{H}^n(K \setminus \text{cl}(U)) + \mathcal{H}^n(M) &= \mathcal{H}^n(\Omega \cap \partial^* E) + \mathcal{H}^n(K \setminus (\partial^* E \cup \text{cl}(U))) + \mathcal{H}^n(M) \\ &\leq \mathcal{H}^n(\Omega \cap \partial^* E) + 2 \mathcal{H}^n(K \setminus (\partial^* E \cup \text{cl}(U))), \end{aligned}$$

as required. $\square$

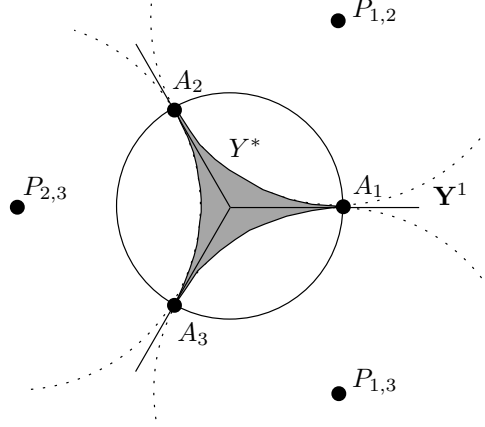


FIGURE 6.1. The construction in step one of the proof of Theorem 6.1.

6. WETTING COMPETITORS AND EXCLUSION OF POINTS OF TYPE Y

By Theorem 5.1, K is an Almgren minimal set in $\Omega \setminus \text{cl}(E)$. As we shall see in the next section, this property is compatible with K containing $(n-1)$ -dimensional submanifolds of Y -points, that is, points such that K is locally diffeomorphic to a cone of type Y in $\mathbb{R}^{n+1}$. The goal of this section is to show that, for reasons related to the specific properties of the variational problem $\psi(\varepsilon)$, such points cannot exist. As explained in detail in the next section, this bit of information will prove crucial in closing the proof of Theorem 1.5.

Theorem 6.1. *If (K, E) is a generalized minimizer of $\psi(\varepsilon)$, then there cannot be $x_0 \in K \setminus \text{cl}(E)$ such that there exist $\alpha \in (0, 1)$, a $C^{1,\alpha}$ -diffeomorphism $\Phi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ with $\Phi(0) = x_0$, and $r_0 > 0$ such that, setting $A = \Phi(B_{r_0})$,*

$$\Phi\left((Y^1 \times \mathbb{R}^{n-1}) \cap B_{r_0}\right) = A \cap K, \quad (6.1)$$

and, in addition,

$$\nabla \Phi(0) = \text{Id}, \quad \|\nabla \Phi - \text{Id}\|_{C^0(B_r)} \leq C r^\alpha, \quad \forall r < r_0. \quad (6.2)$$

Proof. The proof is achieved by showing that, up to decrease the value of r_0 , there exist a constant $c_Y = c_Y(n) > 0$ and a set $G \subset \Omega$ with

$$G \in \mathcal{E}, \quad |G| = \varepsilon, \quad \Omega \cap \partial G \text{ is } \mathcal{C}\text{-spanning } W, \quad (6.3)$$

such that

$$\mathcal{H}^n(\Omega \cap \partial G) \leq \mathcal{F}(K, E) - c_Y r_0^n. \quad (6.4)$$

The set G , defined in (6.36) below, is constructed in three stages, that we introduce as follows. We pick $x^* \in \Omega \cap \partial^* E$, so that, by Theorem 1.4, we can find an open cylinder Q^* of height and radius $r^* > 0$, centered at x^* , and with axis along $\nu_E(x^*)$, such that $E \cap Q^*$ is the subgraph of a smooth function u defined over the cross section D^* of Q^* , and such that the graph of u has mean curvature λ in the orientation induced by $\nu_E(x^*)$ (here λ is the Lagrange multiplier of (K, E)). Up to decrease r_0 and r_* , we can make sure that $A = \Phi(B_{r_0})$ and Q^* lie at positive distance, so that modification of (K, E) compactly supported in these two regions will not interact. We then argue in three stages: in the first stage (first three steps of the proof), we modify (K, E) by replacing the collapsed surface $K \cap A$ with an open set of the form $\Phi(\Delta_{r_0})$, where $\Delta_{r_0} \subset B_{r_0}$ is constructed so to achieve an $O(r_0^n)$ -gain in area, at the cost of an $O(r_0^{n+1})$ -increase in volume – this is possible, of course, only because we are assuming that x_0 is a Y -point; in the second stage (step four of the proof), we construct a one-parameter family of modifications $\{E_t\}_{t \in (0, t_0)}$ of E , all supported in Q^* , with $|E_t| = |E| - t$, and such that the area increase from ∂E to ∂E_t is

of order $O(t)$; in the final stage, we apply Lemma 5.2 to the element of $\mathcal{K}$ constructed in stage one, and then use the volume-fixing variation of stage two to create the competitor G which will eventually give the desired contradiction.

Step one: There exist positive constants v_0 and c_0 , such that for every $\delta > 0$, there is an open subset $Y_\delta^* \subset B_\delta \subset \mathbb{R}^2$ such that

$$\begin{aligned} \mathbf{Y}^1 \cap B_\delta &\subset Y_\delta^*, & \text{cl}(Y_\delta^*) \cap \partial B_\delta &= \mathbf{Y}^1 \cap \partial B_\delta, \\ |Y_\delta^*| &= v_0 \delta^2, & \mathcal{H}^1(\partial Y_\delta^*) &= 2 \mathcal{H}^1(\mathbf{Y}^1 \cap B_\delta) - c_0 \delta. \end{aligned} \quad (6.5)$$

This results from an explicit construction, see Figure 6.1. By scale invariance of the statement, we can assume that $\delta = 1$. Let $\{A_i\}_{i=1}^3 = \mathbf{Y}^1 \cap \partial B_1$, and let $\{P_{1,2}, P_{2,3}, P_{1,3}\}$ be defined as follows: for $i, j \in \{1, 2, 3\}$, $i < j$, $P_{i,j}$ is the intersection of the straight lines ℓ_i and ℓ_j tangent to ∂B_1 and passing through A_i and A_j , respectively. We also let $S_{i,j}$ be the closed disc sector centered at $P_{i,j}$ and corresponding to the arc $A_i A_j$. Finally, we define

$$Y^* = Y_1^* = B_1 \setminus \bigcup_{i < j} S_{i,j}. \quad (6.7)$$

It is easily shown that (6.5) holds with

$$v_0 = \frac{3}{2}(2\sqrt{3} - \pi), \quad c_0 = 6 - \pi\sqrt{3}. \quad (6.8)$$

Step two: We adapt to higher dimensions the construction of step one, see the set Δ_{r_0} defined in (6.12) below. We assign coordinates $x = (z, y) \in \mathbb{R}^2 \times \mathbb{R}^{n-1}$ to points $x \in \mathbb{R}^{n+1}$, so that $(0, y)$ is the component of the vector x along the spine of the cone $\mathbf{Y}^1 \times \mathbb{R}^{n-1} \subset \mathbb{R}^2 \times \mathbb{R}^{n-1}$, and $|z|$ is the distance of x from the spine. We observe that, if $\mathbf{p}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n-1}$ denotes the orthogonal projection operator onto the spine, then the *slice* of B_{r_0} with respect to $\mathbf{p}$ at $y \in \mathbb{R}^{n-1}$ is given by

$$B_{r_0} \cap \mathbf{p}^{-1}(y) = \begin{cases} (0, y) + B_{\sqrt{r_0^2 - |y|^2}}^2 & \text{if } |y| < r_0 \\ \emptyset & \text{otherwise,} \end{cases} \quad (6.9)$$

where B_ρ^2 is the disc of radius ρ in $\mathbb{R}^2 \times \{0\}$. Analogously, the slice of $(\mathbf{Y}^1 \times \mathbb{R}^{n-1}) \cap B_{r_0}$ with respect to $\mathbf{p}$ at y is

$$(\mathbf{Y}^1 \times \mathbb{R}^{n-1}) \cap B_{r_0} \cap \mathbf{p}^{-1}(y) = \begin{cases} (0, y) + \mathbf{Y}^1 \cap B_{\sqrt{r_0^2 - |y|^2}}^2 & \text{if } |y| < r_0 \\ \emptyset & \text{otherwise.} \end{cases} \quad (6.10)$$

For $\tau \in (0, 1/2)$ to be chosen later, we pick $g \in C_c^\infty([0, \infty))$ such that

$$\begin{aligned} g &\equiv \tau \quad \text{in } [0, \tau], & g &\equiv 0 \quad \text{in } [1, \infty), \\ 0 < g(t) &\leq \sqrt{1 - t^2} \quad \text{in } [0, 1], & g' &\leq 0 \text{ and } |g'| \leq 2\tau \quad \text{everywhere,} \end{aligned}$$

and set $g_{r_0}(s) = r_0 g(s/r_0)$, see Figure 6.2. Next, let U_{r_0} define the open tubular neighborhood

$$U_{r_0} = \left\{ x = (z, y) \in \mathbb{R}^{n+1} : |y| < r_0 \text{ and } |z| < g_{r_0}(|y|) \right\}, \quad (6.11)$$

and notice that $U_{r_0} \subset B_{r_0}$ by (6.9) and the properties of g . Finally, we define the set

$$\Delta_{r_0} = \left\{ x = (z, y) \in \mathbb{R}^{n+1} : |y| < r_0 \text{ and } z \in Y_{g_{r_0}(|y|)}^* \right\}. \quad (6.12)$$

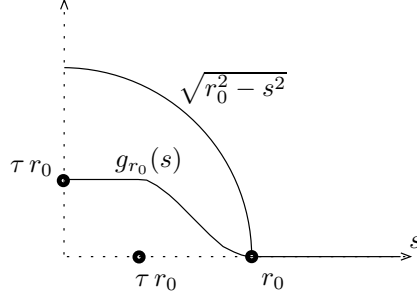


FIGURE 6.2. The damping function g_{r_0} . The set U_{r_0} is obtained by rotating the graph of g_{r_0} around the spine of $\mathbf{Y}^1 \times \mathbb{R}^{n-1}$.

We claim that Δ_{r_0} is an open subset of U_{r_0} (thus of B_{r_0}), with $\mathcal{H}^n$ -rectifiable boundary, and such that

$$(\mathbf{Y}^1 \times \mathbb{R}^{n-1}) \cap U_{r_0} \subset \Delta_{r_0}, \quad (6.13)$$

$$|\Delta_{r_0}| \leq C(n) v_0 r_0^{n+1}, \quad (6.14)$$

$$\mathcal{H}^n(\partial\Delta_{r_0}) \leq 2 \mathcal{H}^n((\mathbf{Y}^1 \times \mathbb{R}^{n-1}) \cap U_{r_0}) - C(n) c_0 r_0^n. \quad (6.15)$$

Only (6.14) and (6.15) require a detailed proof. To compute the volume of Δ_{r_0} , we apply Fubini's theorem, step one, and the definition of g_{r_0} , to get

$$|\Delta_{r_0}| = \int_{B_{r_0}^{n-1}} |Y_{g_{r_0}(|y|)}^*| dy = v_0 \int_{B_{r_0}^{n-1}} g_{r_0}(|y|)^2 dy = v_0 r_0^{n+1} \int_{B_1^{n-1}} g(|w|)^2 dw.$$

Similarly, we use the coarea formula (see e.g. [AFP00, Theorem 2.93 and Remark 2.94]) to write

$$\begin{aligned} \mathcal{H}^n(\partial\Delta_{r_0}) - 2 \mathcal{H}^n((\mathbf{Y}^1 \times \mathbb{R}^{n-1}) \cap U_{r_0}) \\ = \int_{B_{r_0}^{n-1}} dy \int_{\partial Y_{g_{r_0}(|y|)}^*} \frac{d\mathcal{H}^1(z)}{C_{n-1}(\nabla^{\partial\Delta_{r_0}} \mathbf{p}(z))} - 2 \int_{B_{r_0}^{n-1}} \mathcal{H}^1(\mathbf{Y}^1 \cap B_{g_{r_0}(|y|)}^2) dy \end{aligned} \quad (6.16)$$

where $\nabla^{\partial\Delta_{r_0}} \mathbf{p}$ is the tangential gradient of $\mathbf{p}$ along $\partial\Delta_{r_0}$, and where $C_{n-1}(L)$ is the $(n-1)$ -dimensional coarea factor of a linear map $L: \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$. Standard calculations show that, for every $y \in B_{r_0}^{n-1}$,

$$C_{n-1}(\nabla^{\partial\Delta_{r_0}} \mathbf{p}(z)) = (1 + |g'_{r_0}(|y|)|^2)^{-\frac{n-1}{2}} \quad \text{for } \mathcal{H}^1\text{-a.e. } z \in \partial Y_{g_{r_0}(|y|)}^*,$$

so that (6.16) allows to estimate

$$\begin{aligned} \mathcal{H}^n(\partial\Delta_{r_0}) &= 2 \mathcal{H}^n((\mathbf{Y}^1 \times \mathbb{R}^{n-1}) \cap U_{r_0}) \\ &\leq \int_{B_{r_0}^{n-1}} \left((1 + 4\tau^2)^{\frac{n-1}{2}} \mathcal{H}^1(\partial Y_{g_{r_0}(|y|)}^*) - 2 \mathcal{H}^1(\mathbf{Y}^1 \cap B_{g_{r_0}(|y|)}^2) \right) dy \\ &= \left((1 + 4\tau^2)^{\frac{n-1}{2}} \mathcal{H}^1(\partial Y_1^*) - 2 \mathcal{H}^1(\mathbf{Y}^1 \cap B_1^2) \right) r_0^n \int_{B_1^{n-1}} g(|w|) dw, \end{aligned}$$

and thus by (6.6) and provided τ is sufficiently small depending on n , c_0 and $\mathcal{H}^1(\mathbf{Y}^1 \cap B_1^2)$,

$$\mathcal{H}^n(\partial\Delta_{r_0}) - 2 \mathcal{H}^n((\mathbf{Y}^1 \times \mathbb{R}^{n-1}) \cap U_{r_0}) \leq -\frac{c_0}{2} r_0^n \int_{B_1^{n-1}} g(|w|) dw,$$

which gives (6.15).

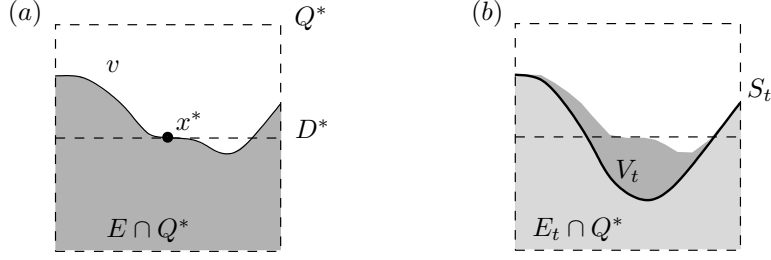


FIGURE 6.3. The volume-fixing variations constructed in step four. The surface S_t has been depicted with a bold line.

Step three: By step two and the properties of Φ , we have that $\Phi(\Delta_{r_0})$ is an open subset of $\Phi(U_{r_0}) \subset \Phi(B_{r_0}) = A$ with $\mathcal{H}^n$ -rectifiable boundary $\partial\Phi(\Delta_{r_0}) = \Phi(\partial\Delta_{r_0})$, and such that

$$K \cap \Phi(U_{r_0}) = \Phi\left((Y^1 \times \mathbb{R}^{n-1}) \cap U_{r_0}\right) \subset \Phi(\Delta_{r_0}) \quad (6.17)$$

thanks to (6.13). Moreover, up to possibly taking a smaller value for r_0 , (6.14) and (6.15), together with the area formula and (6.2), guarantee the existence of constants $v_Y = v_Y(n) > 0$ and $c_Y = c_Y(n) > 0$ such that

$$|\Phi(\Delta_{r_0})| \leq v_Y r_0^{n+1}, \quad (6.18)$$

$$\mathcal{H}^n(\partial(\Phi(\Delta_{r_0}))) - 2\mathcal{H}^n(K \cap \Phi(U_{r_0})) \leq -3c_Y r_0^n. \quad (6.19)$$

Step four: We recall the following construction from [KMS20, Proof of Theorem 2.8]; see Figure 6.3. Fix a point $x^* \in \Omega \cap \partial^* E$, and let $\nu^* = \nu_E(x^*)$. Theorem 1.4 guarantees then the existence of a radius $r^* > 0$ such that, denoting by Q^* the cylinder of center x^* , axis ν^* , height and radius r^* , and by D^* its n -dimensional cross-section passing through x^* , we have $\text{cl}(Q^*) \cap \text{cl}(\Phi(B_{r_0})) = \emptyset$ and

$$E \cap \text{cl}(Q^*) = \left\{ z + h\nu^* : z \in \text{cl}(D^*), -r^* \leq h < v(z) \right\}, \quad (6.20)$$

$$K \cap \text{cl}(Q^*) = \partial E \cap \text{cl}(Q^*) = \left\{ z + v(z)\nu^* : z \in \text{cl}(D^*) \right\}, \quad (6.21)$$

for a smooth function $v: \text{cl}(D^*) \rightarrow \mathbb{R}$ solving, for $\lambda \leq 0$ (the non-positivity of λ is not important here, but it holds thanks to the main result in [KMS20]),

$$-\text{div} \left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} \right) = \lambda \quad \text{on } D^*, \quad \max_{\text{cl}(D^*)} |v| \leq \frac{r^*}{2}. \quad (6.22)$$

We choose a smooth function $w: \text{cl}(D^*) \rightarrow \mathbb{R}$ with

$$w = 0 \quad \text{on } \partial D^*, \quad w > 0 \quad \text{on } D^*, \quad \int_{D^*} w = 1, \quad (6.23)$$

and then define, for $t > 0$, an open set V_t by setting

$$V_t = \left\{ z + h\nu^* : z \in D^*, v(z) - tw(z) < h < v(z) \right\}. \quad (6.24)$$

For t small enough (depending only on r^* and on the choice of w) we have that $V_t \subset E \cap Q^*$, with

$$\partial V_t \cap \partial Q^* = K \cap \partial Q^* = \{ z + v(z)\nu^* : z \in \partial D^* \}. \quad (6.25)$$

Furthermore, if we let S_t denote the closed set

$$S_t = \left\{ z + (v(z) - tw(z))\nu^* : z \in \text{cl}(D^*) \right\}, \quad (6.26)$$

it is easily seen that for $t < t_0$

$$|V_t| = t, \quad \mathcal{H}^n(S_t) = \mathcal{H}^n(S_t \cap \text{cl}(Q^*)) = \mathcal{H}^n(\partial E \cap \text{cl}(Q^*)) - \lambda t + O(t^2), \quad (6.27)$$

where we have used $\int_{D^*} w = 1$, $w = 0$ on ∂D^* , and (6.22). In particular, setting

$$E_t = E \setminus \text{cl}(V_t), \quad (6.28)$$

we have (see [KMS20, Equation (3.37)])

$$|E_t| = |E| - t, \quad \partial E_t \cap \text{cl}(Q^*) = S_t, \quad (6.29)$$

so that (6.27) reads

$$\mathcal{H}^n(\partial E_t \cap \text{cl}(Q^*)) = \mathcal{H}^n(\partial E \cap \text{cl}(Q^*)) + |\lambda| t + O(t^2). \quad (6.30)$$

Finally, if needed, we further reduce the value of r_0 (this time also depending on $|\lambda|$) in order to entail

$$\mathcal{H}^n(\partial E_t \cap \text{cl}(Q^*)) \leq \mathcal{H}^n(\partial E \cap \text{cl}(Q^*)) + c_{\mathbf{Y}} r_0^n \quad \forall t \leq 2v_{\mathbf{Y}} r_0^{n+1}. \quad (6.31)$$

Step five: Without loss of generality, we can assume that $r^* < \text{dist}(x^*, \text{cl}(K) \setminus \partial^* E)/2$. In this way, provided η is small enough in terms of r^* , we can enforce that

$$I_\eta(\text{cl}(K) \setminus \partial^* E) \text{ and } \text{cl}(Q^*) \text{ lie at positive distance.} \quad (6.32)$$

We thus apply the construction of Lemma 5.2 to (K, E) with the open set $U = \Phi(U_{r_0})$, and correspondingly define the function u and the sets M_0, M_1, M, A_0, A_1 . After choosing δ and η sufficiently small in terms of r_0 , we can achieve

$$(\text{cl}(A_0) \cup \text{cl}(A_1)) \cap \text{cl}(Q^*) = \emptyset, \quad |A_0| + |A_1| \leq \frac{v_{\mathbf{Y}} r_0^{n+1}}{4} \quad (6.33)$$

$$\mathcal{H}^n(\{x + u(x)\nu(x) : x \in M\}) \leq \mathcal{H}^n(M) + c_{\mathbf{Y}} r_0^n. \quad (6.34)$$

Since $\text{cl}(A_0) \cup \text{cl}(A_1) \subset I_\eta(K \setminus \partial E)$, the first condition in (6.33) is immediate from (6.32), while the second condition follows from $|I_\eta(K)| \rightarrow |\text{cl}(K)| = 0$ as $\eta \rightarrow 0^+$. Finally, (6.34) is satisfied for δ sufficiently small (in terms of r_0, n and $\mathcal{H}^n(K)$) thanks to (5.28).

Step six: We apply step four with

$$t = |\Phi(\Delta_{r_0})| + |A_0| - |A_1| \in (0, 2v_{\mathbf{Y}} r_0^{n+1}]. \quad (6.35)$$

In particular, (6.31) holds for the corresponding set E_t , and we can finally define the competitor

$$G = \Phi(\Delta_{r_0}) \cup F, \quad \text{where } F = A_0 \cup (E_t \setminus \text{cl}(A_1)). \quad (6.36)$$

We verify now that G satisfies the properties (6.3) and (6.4). First, we observe that $\Phi(\Delta_{r_0}) \subset \Phi(U_{r_0})$, whereas, by Lemma 5.2 and given that $E_t \subset E$, one has $F \subset \Omega \setminus \text{cl}(\Phi(U_{r_0}))$, so that $\Phi(\Delta_{r_0})$ and F are two *disjoint* open subsets of Ω . In particular, $G \subset \Omega$ is open and, as a consequence of (6.29) and (6.35),

$$\begin{aligned} |G| &= |\Phi(\Delta_{r_0})| + |F| = |\Phi(\Delta_{r_0})| + |A_0| + |E_t| - |\text{cl}(A_1)| \\ &= |\Phi(\Delta_{r_0})| + |A_0| - |\text{cl}(A_1)| + |E| - t \\ &= |E|. \end{aligned} \quad (6.37)$$

Since $\partial G \subset \partial[\Phi(\Delta_{r_0})] \cup \partial F$, recalling the last inclusion in (5.19) (which in the present case holds with E_t in place of E) and noticing that $\text{cl}(\Phi(\Delta_{r_0})) \subset \Omega$, we obtain

$$\Omega \cap \partial G \subset \partial\Phi(\Delta_{r_0}) \cup \left\{ \Omega \cap \left(\partial A_0 \cup \partial A_1 \cup \partial E_t \right) \right\}. \quad (6.38)$$

and, in particular, ∂G is $\mathcal{H}^n$ -rectifiable. Moreover, for $k = 0, 1$, by (5.17) and by $I_\eta(K \setminus \partial E) \cap \text{cl}(Q^*) = \emptyset$, we get

$$\Omega \cap \partial A_k \subset K \setminus (\Phi(U_{r_0}) \cup Q^*) \cup \left\{ x + u(x)\nu(x) : x \in M_k \right\}$$

while

$$\begin{aligned}\Omega \cap \partial E_t &\subset [\Omega \cap \partial E_t \cap \text{cl}(Q^*)] \cup [(\Omega \cap \partial E) \setminus \text{cl}(Q^*)] \\ &\subset S_t \cup [K \setminus (\Phi(U_{r_0}) \cup Q^*)]\end{aligned}$$

so that the $\subset$ -inclusion in the following identity

$$\Omega \cap \partial G = \partial \Phi(\Delta_{r_0}) \cup (K \setminus (\Phi(U_{r_0}) \cup Q^*)) \cup S_t \cup \left\{ x + u(x) \nu(x) : x \in M \right\} \quad (6.39)$$

follows from (6.38). To complete the proof of (6.39) we will show that

$$\partial \Phi(\Delta_{r_0}) \subset \Omega \cap \partial G, \quad (6.40)$$

$$K \cap \partial(\Phi(U_{r_0})) \subset \Omega \cap \partial G, \quad (6.41)$$

$$M \cup \{x + u(x) \nu(x) : x \in M\} \subset \Omega \cap \partial G, \quad (6.42)$$

$$\Sigma \setminus (\partial E \cup \text{cl}(\Phi(U_{r_0}))) \subset \Omega \cap \partial G, \quad (6.43)$$

$$(\Omega \cap \partial E) \setminus \text{cl}(Q^*) \subset \Omega \cap \partial G, \quad (6.44)$$

$$S_t \subset \Omega \cap \partial G. \quad (6.45)$$

Proof of (6.40): it readily follows from the fact that $\text{cl}(\Phi(\Delta_{r_0})) \subset \Omega \cap \text{cl}(G)$ together with $F \cap \text{cl}(\Phi(\Delta_{r_0})) = \emptyset$. *Proof of (6.41):* since $K \cap \partial(\Phi(U_{r_0})) \subset \Omega \setminus G$, we only have to prove that $K \cap \partial(\Phi(U_{r_0})) \subset \text{cl}(G)$. Since $K \cap \text{cl}(\Phi(U_{r_0})) = \Phi(\mathbf{Y}^1 \times \mathbb{R}^{n-1}) \cap \text{cl}(\Phi(U_{r_0}))$, any $x \in K \cap \partial(\Phi(U_{r_0}))$ is a limit of points $\Phi(z_h)$ with $z_h \in (\mathbf{Y}^1 \times \mathbb{R}^{n-1}) \cap U_{r_0} \subset \Delta_{r_0}$ by (6.13). In particular, x is a limit of points in $\Phi(\Delta_{r_0}) \subset G$. *Proof of (6.42), (6.43), and (6.44):* since $E_t \setminus \text{cl}(Q^*) = E \setminus \text{cl}(Q^*)$, the sets appearing on the left-hand sides of (6.42), (6.43), and (6.44) are all subsets of $\Omega \cap \partial F \setminus \text{cl}(\Phi(U_{r_0}))$ as a consequence of (5.20), (5.21), and (5.22), respectively. *Proof of (6.45):* By construction $G \cap Q^* = E_t \cap Q^*$ so that $Q^* \cap \partial G = Q^* \cap \partial E_t$; since $S_t \subset \text{cl}(Q^*) \cap \partial E_t$ we conclude the proof of (6.45), and thus of (6.39).

Conclusion: We first prove (6.4). Without loss of generality assume that r^* is such that $\mathcal{H}^n(\partial^* E \cup \partial Q^*) = 0$. By (6.39), (6.19), (6.31), (6.34), and the fact that $M \subset K \setminus (\partial E \cup \text{cl}(\Phi(U_{r_0})))$, we find

$$\begin{aligned}\mathcal{H}^n(\Omega \cap \partial G) &\leq \mathcal{H}^n(\partial \Phi(\Delta_{r_0})) \\ &\quad + \mathcal{H}^n((\Omega \cap \partial^* E) \setminus Q^*) + \mathcal{H}^n((K \setminus \partial^* E) \setminus \Phi(U_{r_0})) \\ &\quad + \mathcal{H}^n(S_t) + \mathcal{H}^n(\{x + u(x) \nu(x) : x \in M\}) \\ &\leq 2 \mathcal{H}^n(K \cap \Phi(U_{r_0})) - 3 c_{\mathbf{Y}} r_0^n \\ &\quad + \mathcal{H}^n((\Omega \cap \partial^* E) \setminus Q^*) + \mathcal{H}^n((K \setminus \partial^* E) \setminus \Phi(U_{r_0})) \\ &\quad + \mathcal{H}^n(\partial^* E \cap \text{cl}(Q^*)) + c_{\mathbf{Y}} r_0^n + \mathcal{H}^n(M) + c_{\mathbf{Y}} r_0^n \\ &\leq 2 \mathcal{H}^n(K \setminus \partial^* E) + \mathcal{H}^n(\Omega \cap \partial^* E) - c_{\mathbf{Y}} r_0^n,\end{aligned}$$

that is (6.4). To complete the argument we finally prove that $\Omega \cap \partial G$ is $\mathcal{C}$ -spanning W . To this aim, pick $\gamma \in \mathcal{C}$. If $\gamma \cap K \setminus (\Phi(U_{r_0}) \cup Q^*) \neq \emptyset$, then also $\gamma \cap \partial G \neq \emptyset$ by (6.39). If $\gamma \cap K \cap Q^* \neq \emptyset$, then also $\gamma \cap \partial E \cap Q^* \neq \emptyset$, and thus also $\gamma \cap S_t \neq \emptyset$ as a consequence of [KMS19, Lemma 2.3] since S_t is a diffeomorphic image of $\partial E \cap \text{cl}(Q^*)$: hence, $\gamma \cap \partial G \neq \emptyset$, again by (6.39). We can therefore assume that $\gamma \cap K \setminus \Phi(U_{r_0}) = \emptyset$, and thus, since K is $\mathcal{C}$ -spanning W , that there exists $x \in \gamma \cap K \cap \Phi(U_{r_0}) \subset \gamma \cap \Phi(\Delta_{r_0})$, where in the last inclusion we have exploited (6.17). Since $\Phi(\Delta_{r_0})$ is contractible and, as consequence of $\ell < \infty$, γ is homotopically non-trivial in Ω , γ must necessarily intersect $\mathbb{R}^{n+1} \setminus \Phi(\Delta_{r_0})$, and thus, by continuity, $\gamma \cap \partial \Phi(\Delta_{r_0}) \neq \emptyset$. Since $\partial \Phi(\Delta_{r_0}) \subset \Omega \cap \partial G$, we have completed the proof. $\square$

7. REGULARITY THEORY AND CONCLUSION OF THE PROOF OF THEOREM 1.5

7.1. Blow-ups of stationary varifolds. We say that V_0 is an **integral n -cone** in $\mathbb{R}^{n+1}$ if $V_0 = \mathbf{var}(\mathbf{C}, \theta_0)$ for a closed locally $\mathcal{H}^n$ -rectifiable cone $\mathbf{C}$ in $\mathbb{R}^{n+1}$ (so that $\lambda x \in \mathbf{C}$ for every $x \in \mathbf{C}$ and $\lambda > 0$), and a zero-homogenous multiplicity function θ_0 (so that $\theta_0(\lambda x) = \theta_0(x)$ for every $x \in \mathbf{C}$ and $\lambda > 0$). The importance of integral cones lies in the fact that if V is a stationary integral n -varifold in some open set U , $x_0 \in \text{spt } V$ and $r_j \rightarrow 0^+$ as $j \rightarrow \infty$, then, up to extracting a subsequence of r_j , there exists an integral n -cone V_0 such that

$$(\iota_{x_0, r_j})_\# V \rightharpoonup V_0,$$

in the varifold convergence (duality with $C_c^0(U \times G_n^{n+1})$), where $\iota_{x, r}(y) = (y - x)/r$ for $x, y \in \mathbb{R}^{n+1}$ and $r > 0$; moreover, V_0 is stationary in $\mathbb{R}^{n+1}$, and the collection of such limit stationary integral n -cones for V at x_0 is denoted by

$$\text{Tan}(V, x_0).$$

We recall that if $V_0 = \mathbf{var}(\mathbf{C}, \theta_0) \in \text{Tan}(V, x_0)$, then

$$\Theta_V(x_0) = \Theta_{V_0}(0) \geq \Theta_{V_0}(y), \quad \forall y \in \mathbf{C}.$$

Correspondingly, the **spine** of the integral n -cone $V_0 = \mathbf{var}(\mathbf{C}, \theta_0)$ is defined as

$$S(V_0) = \left\{ y \in \mathbb{R}^{n+1} : \Theta_{V_0}(y) = \Theta_{V_0}(0) \right\};$$

as it turns out, $S(V_0)$ is a linear space in $\mathbb{R}^{n+1}$, and it can actually be characterized as the largest linear space L of $\mathbb{R}^{n+1}$ such that V_0 is invariant by translations in L , i.e. $(\tau_v)_\# V_0 = V_0$ for every $v \in L$, where $\tau_v(y) = y + v$ for all $y \in \mathbb{R}^{n+1}$. It is easily seen that if $\dim S(V_0) = k \in \{0, \dots, n\}$ and, without loss of generality, $S(V_0) = \{0\}^{n-k+1} \times \mathbb{R}^k$, then there exist a closed $(n-k)$ -cone $\mathbf{C}_0$ in $\mathbb{R}^{n-k+1}$ and a zero-homogeneous multiplicity function ϕ_0 on $\mathbf{C}_0$ such that

$$\mathbf{C} = \mathbf{C}_0 \times \mathbb{R}^k, \quad \theta_0(z, y) = \phi_0(z) \quad \text{for } \mathcal{H}^{n-k}\text{-a.e. } z \in \mathbf{C}_0, \text{ for every } y \in \mathbb{R}^k,$$

and such that $W_0 = \mathbf{var}(\mathbf{C}_0, \phi_0)$ is a stationary integral $(n-k)$ -cone in $\mathbb{R}^{n-k+1}$ with

$$\Theta_{W_0}(0) = \Theta_{V_0}(0), \quad S(W_0) = \{0\}.$$

The concept of spine leads to defining the notion of **k -dimensional stratum** of a stationary integral n -varifold V as

$$\mathcal{S}^k(V) = \left\{ x \in \text{spt } V : \dim S(V_0) \leq k, \quad \forall V_0 \in \text{Tan}(V, x) \right\},$$

where the classical dimension reduction argument of Federer, see [Sim83, Appendix A], shows that

$$\dim_{\mathcal{H}}(\mathcal{S}^k(V)) \leq k \quad \forall k = 0, \dots, n. \quad (7.1)$$

Moreover, we have the following key result by Naber and Valtorta.

Theorem 7.1 ([NV15, Theorem 1.5]). *If V is an integral stationary n -varifold in an open set U of $\mathbb{R}^{n+1}$, then $\mathcal{S}^k(V)$ is countably k -rectifiable in U for every $k = 0, \dots, n$.*

7.2. Regularity of Almgren minimal sets and proof of Theorem 1.5. We recall that M is an Almgren minimal set in an open set $U \subset \mathbb{R}^{n+1}$ if $M \subset U$ is closed relatively to U and

$$\mathcal{H}^n(M \cap B_r(x)) \leq \mathcal{H}^n(f(M) \cap B_r(x)) \quad (7.2)$$

whenever f is a Lipschitz map with $\{f \neq \text{id}\} \subset\subset B_r(x) \subset\subset U$ and $f(B_r(x)) \subset B_r(x)$. An immediate consequence of (7.2) is that the multiplicity-one n -varifold $V = \mathbf{var}(M, 1)$ associated to M is stationary in U . The Almgren minimality of M implies that the set of tangent varifolds to V is simpler than it could be in general: indeed, varifold tangent cones

to Almgren minimal sets have multiplicity one, and their supports are Almgren minimal cones:

Theorem 7.2 ([Tay76, Corollary II.2]). *If M is an Almgren minimal set in $U \subset \mathbb{R}^{n+1}$, $x_0 \in M$, and $V_0 = \mathbf{var}(\mathbf{C}, \theta_0) \in \text{Tan}(\mathbf{var}(M, 1), x_0)$, then $\theta_0 = 1$ on $\mathbf{C}$, and $\mathbf{C}$ is an Almgren minimal cone in $\mathbb{R}^{n+1}$.*

In particular, setting

$$\text{Tan}(M, x_0) = \left\{ \mathbf{C} \subset \mathbb{R}^{n+1} : V_0 = \mathbf{var}(\mathbf{C}, 1) \in \text{Tan}(\mathbf{var}(M, 1), x_0) \right\},$$

and, correspondingly, $S(\mathbf{C}) = S(V_0)$ for every $\mathbf{C} \in \text{Tan}(M, x_0)$,

we have that

$$S^k(M) = \left\{ x_0 \in M : \dim S(\mathbf{C}) \leq k, \quad \forall \mathbf{C} \in \text{Tan}(M, x_0) \right\}$$

is countably k -rectifiable in $\mathbb{R}^{n+1}$ thanks to Theorem 7.1.

Remark 7.3 (Smoothness criterion). If $\text{Tan}(M, x_0)$ contains an n -dimensional plane, then M is a classical minimal surface in a neighborhood of M as a consequence of Allard's regularity theorem [All72] and of the fact that $V = \mathbf{var}(M, 1)$ is an integral stationary n -varifold. As a consequence, the **singular set** Σ of M in U , defined as the maximal closed subset of M such that $M \setminus \Sigma$ is a smooth minimal surface in U , can be characterized as the set of those $x_0 \in M$ such that $\text{Tan}(M, x_0)$ contains no plane.

The next important fact is that one can completely characterize Almgren minimal cones in $\mathbb{R}^2$ and $\mathbb{R}^3$:

Theorem 7.4. [Tay76, Proposition II.3] *If $\mathbf{C}$ is an Almgren minimal cone in $\mathbb{R}^2$, then, up to rotations, either $\mathbf{C} = \{0\} \times \mathbb{R}$ or $\mathbf{C} = \mathbf{Y}^1$. If $\mathbf{C}$ is an Almgren minimal cone in $\mathbb{R}^3$, then, up to rotations, either $\mathbf{C} = \{0\} \times \mathbb{R}^2$, or $\mathbf{C} = \mathbf{Y}^1 \times \mathbb{R}$, or $\mathbf{C} = \mathbf{T}^2$.*

Corollary 7.5. *If M is an Almgren minimal set in $U \subset \mathbb{R}^{n+1}$ and $\mathbf{C} \in \text{Tan}(M, x_0)$ for some $x_0 \in M$, then, up to rotations, either $\mathbf{C} = \{0\} \times \mathbb{R}^n$, or $\mathbf{C} = \mathbf{Y}^1 \times \mathbb{R}^{n-1}$, or $\mathbf{C} = \mathbf{T}^2 \times \mathbb{R}^{n-2}$ or $\dim S(\mathbf{C}) \leq n - 3$.*

Proof. One needs to notice that if $\mathbf{C} = \mathbf{C}_0 \times \mathbb{R}^k$ is an Almgren minimal cone in $\mathbb{R}^{n+1}$, then $\mathbf{C}_0$ is an Almgren minimal cone in $\mathbb{R}^{n-k+1}$, and combine this fact with Theorem 7.2 and Theorem 7.4. $\square$

If M is an Almgren minimal set in U , $\mathbf{C}$ is an Almgren minimal cone in $\mathbb{R}^{n+1}$, $\alpha \in (0, 1)$ and $x_0 \in M$, then we say that M **admits ambient parametrization of class $C^{1,\alpha}$ over $\mathbf{C}$ at x_0** , if there exist $r > 0$, an open neighborhood A of x_0 , and a $C^{1,\alpha}$ -diffeomorphism $\Phi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ such that, $\Phi(0) = x_0$, $\nabla \Phi(0) = \text{Id}$ and

$$\Phi(B_r \cap \mathbf{C}) = M \cap A. \quad (7.3)$$

The main result contained in [Tay76] can be formulated as follows:

Theorem 7.6 ([Tay76]). *If M is an Almgren minimal set in $U \subset \mathbb{R}^3$ and $x_0 \in M$, then either M is a classical minimal surface in a neighborhood of x_0 , or M admits an ambient parametrization of class $C^{1,\alpha}$ over $\mathbf{C}$ at x_0 , where, modulo isometries, $\mathbf{C} \in \{\mathbf{Y}^1 \times \mathbb{R}, \mathbf{T}^2\}$.*

Remark 7.7. The analysis of Almgren minimal sets in $\mathbb{R}^2$ is noticeably simpler, and it yields the stronger conclusions that M is locally *isometric* either to a line or to $\mathbf{Y}^1$: a detailed proof can be easily obtained, for example, by minor adaptations of [Mag12, Section 30.3].

We are finally in the position to prove Theorem 1.5.

Proof of Theorem 1.5. Let (K, E) be a generalized minimizer of $\psi(\varepsilon)$. By Theorem 5.1, $M = K \setminus \text{cl}(E)$ is an Almgren minimal set in $U = \Omega \setminus \text{cl}(E)$. By Corollary 7.5 we have that $M = R \cup \Sigma$, where R is a smooth, stable minimal hypersurface in $U \setminus \Sigma$, and Σ is a relatively closed subset of M such that if $x_0 \in \Sigma$ and $\mathbf{C} \in \text{Tan}(M, x_0)$, then either $\mathbf{C} = \mathbf{Y}^1 \times \mathbb{R}^{n-1}$ (modulo isometries), or $\dim S(\mathbf{C}) \leq n - 2$.

If $\mathbf{C} = \mathbf{Y}^1 \times \mathbb{R}^{n-1} \in \text{Tan}(M, x_0)$, then

$$V_0 = \mathbf{var}(\mathbf{C}, 1) \in \text{Tan}(V, x_0)$$

where $V = \mathbf{var}(M, 1)$ is an integral stationary n -varifold in U . By Simon's Y -regularity theorem [Sim93], see e.g. [CES17, Theorem 4.6] for a handy statement, M can be locally parameterized over $\mathbf{Y}^1 \times \mathbb{R}^{n-1}$ near x_0 , in the sense that there exist $r > 0$, an open neighborhood A of x_0 , and a homeomorphism $\Phi : (\mathbf{Y}^1 \times \mathbb{R}^{n-1}) \cap B_r \rightarrow M \cap A$ with $\Phi(0) = x_0$ and mapping the spine of $\mathbf{Y}^1 \times \mathbb{R}^{n-1}$ into $\Sigma \cap A$, such that, denoting by $\{H_i\}_{i=1}^3$ the three n -dimensional half-planes whose union gives $\mathbf{Y}^1 \times \mathbb{R}^{n-1}$, the restriction of Φ to $H_i \cap B_r$ is a $C^{1,\alpha}$ -diffeomorphism between hypersurfaces with boundary. An application of Whitney's extension theorem (which is usually mentioned without details in the literature, see e.g. the comments in [Tay76, Pag. 528] and [Sim93, Pag. 650]; we notice that a simplification of the proof of [CLM16, Theorem 3.1] gives the desired result) allows one to extend Φ into an ambient parametrization of M over $\mathbf{Y}^1 \times \mathbb{R}^{n-1}$ in a neighborhood of x_0 . However, Theorem 6.1, excludes the existence of such ambient parametrization. Therefore we conclude that $\mathbf{Y}^1 \times \mathbb{R}^{n-1}$ cannot belong modulo isometries to $\text{Tan}(M, x_0)$ for any $x_0 \in \Sigma$. As a consequence, $\dim S(\mathbf{C}) \leq n - 2$ for every $\mathbf{C} \in \text{Tan}(M, x_0)$, and thus $\Sigma = \mathcal{S}^{n-2}(M)$. By Federer's dimensional reduction argument (7.1), we conclude in particular that

$$\mathcal{H}^{n-1}(\Sigma) = 0.$$

In summary, $V = \mathbf{var}(M, 1)$ is a stationary integral n -varifold in U , whose regular part is stable thanks to (1.11), and whose singular part is $\mathcal{H}^{n-1}$ -negligible. The regularity theory of Schoen, Simon and Wickramasekera [SS81, Wic14] allows us to conclude then that Σ is empty if $1 \leq n \leq 6$, is locally finite in U if $n = 7$, and coincides with $\mathcal{S}^{n-7}(V)$ if $n \geq 8$. In particular, if $n \geq 8$ then Σ is countably $(n - 7)$ -rectifiable in U by Theorem 7.1. This completes the proof of the theorem. $\square$

We close with a few technical comments on how the regularity theory for varifolds and Almgren minimal sets has been applied in the above argument.

Remark 7.8. In the physical cases $n = 1$ and $n = 2$, which are clearly the most important ones for the soap film capillarity model, one does not need to use the full power of the regularity theory contained in [Sim93, SS81, Wic14]. Indeed, once $M = K \setminus \text{cl}(E)$ has been shown to be an Almgren minimal set in $\Omega \setminus \text{cl}(E)$, Taylor's theorem (i.e., Theorem 7.6 above) shows that if Σ is non-empty, then M admits an ambient parametrization over $\mathbf{Y}^1 \times \mathbb{R}^{n-1}$ at some of its singular points, thus triggering a contradiction with Theorem 6.1.

Remark 7.9. The following argument allows to use [SS81] in place of [Wic14] (notice that [Wic14] relies on [SS81]). Going back to the application of Corollary 7.5 to $M = K \setminus \text{cl}(E)$, and after having excluded the existence of Y points thanks to [Sim93] and Theorem 6.1, we are in the position to say that if $\mathbf{C} \in \text{Tan}(M, x_0)$, then either $\mathbf{C} = \mathbf{T}^2 \times \mathbb{R}^{n-2}$ modulo isometries or $\dim S(\mathbf{C}) \leq n - 3$. In the former case, a direct parametrization argument away from the spine of $\mathbf{C}$ (in the spirit of [CES17, Lemma 4.8]) implies the existence of Y -points near x_0 , and a contradiction with Theorem 6.1. We thus conclude that $\dim S(\mathbf{C}) \leq n - 3$ for every $\mathbf{C} \in \text{Tan}(M, x_0)$, $x_0 \in \Sigma$, and thus that $\mathcal{H}^{n-2}(\Sigma) = 0$. By [SS81], an integral stationary n -varifold V in $\mathbb{R}^{n+1}$ whose regular part is stable and whose singular

set is $\mathcal{H}^{n-2}$ -negligible is such that the singular set is empty if $1 \leq n \leq 6$, and coincides with $\mathcal{S}^{n-7}(V)$ if $n \geq 7$ (and thus it is countably $(n-7)$ -rectifiable by Naber-Valtorta).

8. LOCAL FINITENESS OF THE HAUSDORFF MEASURE OF THE SINGULAR SET

In this section we sketch the arguments needed to improve the countable $(n-7)$ -rectifiability of Σ , proved in Theorem 1.5, into local finiteness of the $(n-7)$ -dimensional Minkowski content, and thus, in particular, into local $\mathcal{H}^{n-7}$ -rectifiability; see Remark 1.6. Towards this goal, we will need to introduce the following notion of **quantitative stratification** of the singular set of a stationary integral varifold.

Let dist_{var} be a distance function of the space of n -dimensional varifolds in $B_1 \subset \mathbb{R}^{n+1}$ which induces the varifold convergence. Let V be a stationary integral n -varifold in a ball $B_r(x) \subset \mathbb{R}^{n+1}$ with $x \in \text{spt}(V)$. For any $\delta > 0$, we say that V is (k, δ) -**almost symmetric** in $B_r(x)$ if there exists a k -symmetric integral n -cone V_0 (i.e. $\dim S(V_0) \geq k$) such that

$$\text{dist}_{\text{var}}((\iota_{x,r})_{\#} V \llcorner B_1, V_0 \llcorner B_1) < \delta.$$

For $k \in \{0, \dots, n\}$ and $\delta > 0$, we define the (k, δ) -**quantitative stratum** $\mathcal{S}_{\delta}^k(V)$ by

$$\mathcal{S}_{\delta}^k(V) = \left\{ x \in \text{spt}(V) : \begin{array}{l} V \text{ is not } (k+1, \delta)\text{-almost symmetric in } B_r(x) \\ \text{for all } r > 0 \text{ such that } V \text{ is stationary in } B_r(x) \end{array} \right\}.$$

We can now recall the following theorem from [NV15]:

Theorem 8.1 (See [NV15, Theorem 1.4]). *Let $\delta, \Lambda > 0$. There exists $C_{\delta} = C(n, \Lambda, \delta) > 0$ such that if V is an integral stationary n -varifold in $B_2 \subset \mathbb{R}^{n+1}$ with $\|V\|(B_2) \leq \Lambda$ then*

$$\left| I_r(\mathcal{S}_{\delta}^k(V)) \cap B_1 \right| \leq C_{\delta} r^{n+1-k} \quad \text{for all } 0 < r < 1. \quad (8.1)$$

In particular, $\mathcal{H}^k(\mathcal{S}_{\delta}^k(V) \cap B_1) \leq C_{\delta}$. Furthermore, $\mathcal{S}_{\delta}^k(V)$ is countably k -rectifiable.

Remark 8.2. The countable k -rectifiability of $\mathcal{S}^k(V)$ claimed in Theorem 7.1 is in fact a corollary of the countable k -rectifiability of the quantitative strata $\mathcal{S}_{\delta}^k(V)$ together with the fact that

$$\mathcal{S}^k(V) = \bigcup_{\delta > 0} \mathcal{S}_{\delta}^k(V). \quad (8.2)$$

We are now in the position to show that, under the assumptions of Theorem 1.5, if $n \geq 7$, then Σ has locally finite $(n-7)$ -dimensional Minkowski content, and thus it is locally $\mathcal{H}^{n-7}$ -finite. Since we can cover any open set compactly contained in $\Omega \setminus \text{cl}(E)$ by a finite number of balls $B_{3r_*}(x_i)$ such that $B_{r_*}(x_i)$ are pairwise disjoint and $B_{9r_*}(x_i) \subset \Omega \setminus \text{cl}(E)$, we can directly focus on obtaining an upper bound on the $(n-7)$ -dimensional Minkowski content of Σ in B whenever B is an open ball with $3B \subset \Omega \setminus \text{cl}(E)$, where $3B$ denotes the concentric ball to B with three times the radius. To this end we claim that

$$\exists \delta > 0 \text{ such that } \Sigma \cap 2B \subset \mathcal{S}_{\delta}^{n-7}(V) \cap 2B.$$

Indeed, thanks to Theorem 8.1 this claim implies

$$\left| I_r(\Sigma) \cap B \right| \leq C_{\delta} r^8 \quad \text{for all } 0 < r < \text{radius}(B),$$

and thus $\mathcal{H}^{n-7}(\Sigma \cap B) \leq C_{\delta}$ for a constant $C_{\delta} = C(n, \mathcal{H}^n(K \cap 2B), \delta)$, from which the local $\mathcal{H}^{n-7}$ -finiteness of Σ follows. To prove the claim we argue by contradiction and assume the existence of a sequence $\delta_h \rightarrow 0^+$ and points $x_h \in \Sigma \cap 2B$ such that $x_h \notin \mathcal{S}_{\delta_h}^{n-7}(V)$. Assuming that $\text{radius}(B) = 1$ for simplicity, so that V is stationary in $B_1(x_h)$ for every h , the definition of quantitative strata then yields a sequence r_h of scales $0 < r_h < 1$ such that V is $(n-6, \delta_h)$ -almost symmetric in $B_{r_h}(x_h)$: in other words, there are integral n -cones

W_h with $\dim S(W_h) \geq n - 6$ such that, setting $K_h = (K - x_h)/r_h$ and $V_h = \mathbf{var}(K_h, 1)$, we have $\text{dist}_{\mathbf{var}}(V_h \llcorner B_1, W_h \llcorner B_1) \leq \delta_h$. Since the weights $\|V_h\|(B_1)$ are uniformly bounded as a consequence of the monotonicity formula, each V_h is stationary in B_1 , and $\delta_h \rightarrow 0^+$, a (not relabeled) subsequence of the varifolds $V_h \llcorner B_1$ converges, as $h \rightarrow \infty$ and in the sense of varifolds, to a stationary integral n -varifold which is the restriction to B_1 of an Almgren minimal cone $\mathbf{C}$ in $\mathbb{R}^{n+1}$ with $\dim S(\mathbf{C}) \geq n - 6$. By Remark 7.3, $\mathbf{C}$ cannot be a plane, as otherwise K would be smooth in a neighborhood of x_h for all sufficiently large h , a contradiction to $x_h \in \Sigma$. In particular, $\mathbf{C}$ is singular at the origin, and since $\dim S(\mathbf{C}) \geq n - 6$ it must be $\mathcal{H}^{n-6}(\text{Sing}(\mathbf{C})) = \infty$, if $\text{Sing}(\mathbf{C})$ denotes the set of singular points of $\mathbf{C}$. We claim that

$$\mathcal{H}^{n-1}(\text{Sing}(\mathbf{C})) = 0. \quad (8.3)$$

If this is true, then we can apply again [Wic14] and conclude that $\dim_{\mathcal{H}}(\text{Sing}(\mathbf{C})) \leq n - 7$, a contradiction. We prove (8.3) by showing that $\mathbf{C}$ cannot have points of type Y . Otherwise, there would be a point $y \in \mathbf{C} \cap B_1$ such that, modulo rotations, the (unique) tangent cone $\mathbf{C}_y$ to $\mathbf{C}$ at y is $\mathbf{Y}^1 \times \mathbb{R}^{n-1}$. Since varifold convergence of stationary integral varifolds implies Hausdorff convergence of their supports, for every $\delta > 0$ there exists $\sigma \in (0, \text{dist}(y, \partial B_1))$ such that, for all sufficiently large h ,

$$\text{hd}(\text{spt}((\iota_{y,\sigma})_{\#} V_h) \cap B_1, (\mathbf{Y}^1 \times \mathbb{R}^{n-1}) \cap B_1) \leq \delta$$

where hd denotes the Hausdorff distance. By Simon's Y -regularity theorem, K_h admits an ambient parametrization of class $C^{1,\alpha}$ over $\mathbf{Y}^1 \times \mathbb{R}^{n-1}$ in $B_{\sigma/2}(y)$, and thus, in turn, there is a point in K at which K admits an ambient parametrization of class $C^{1,\alpha}$ over $\mathbf{Y}^1 \times \mathbb{R}^{n-1}$, a contradiction to Theorem 6.1. $\square$

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