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Gapped Boundary Theories in Three Dimensions

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Abstract: We prove a theorem in 3-dimensional topological field theory: a Reshetikhin–Turaev theory admits a nonzero boundary theory iff it is a Turaev–Viro theory. The proof immediately implies a characterization of fusion categories in terms of dualizability. Our results rely on a (special case of) the cobordism hypothesis with singularities. The main theorem applies to physics, where it implies an obstruction to a gapped 3-dimensional quantum system admitting a gapped boundary theory. Appendices on bordism multicategories, on 2-dualizable categories, and on internal duals may be of independent interest.

Contents

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Quantum mechanical theories bifurcate into gapped and gapless theories. The classical notion of a local boundary condition for a partial differential equation has a quantum analog—a boundary theory. There is a basic question: Does a gapped quantum system S admit a gapped boundary theory? We formulate and prove a mathematical theorem (Theorem A in Sect. 1.4) which addresses this question for a large class of $(2+1)$-dimensional systems. The route from a gapped quantum system to the theorem goes via a low energy effective extended topological field theory, whose existence we merely assume. The absence of a gapped boundary theory implies the presence of gapless edge modes—conduction on the boundary—an important feature of quantum Hall systems, for example.

Let $F\colon \mathrm{Bord}_3^{\mathrm{fr}} \to \mathcal{C}$ be a 3-dimensional topological field theory: a homomorphism from a bordism multicategory of framed manifolds to a symmetric monoidal 3-category $\mathcal{C}$. We impose hypotheses on $\mathcal{C}$, F to model Reshetikhin–Turaev theories [RT1,RT2,T], whose key invariant is a modular fusion category C, the value of F on the bounding framed circle. Theorem A asserts that if F admits a nonzero boundary theory, then C is the Drinfeld center of a fusion[1] category Φ. With extra assumptions on the codomain $\mathcal{C}$, we conclude (Theorem A′ in Sect. 1.4) that the entire theory F is isomorphic to the Turaev–Viro theory T_Φ based on Φ. Conversely, given a fusion category Φ, the Turaev–Viro theory T_Φ has a nonzero boundary theory built from the (regular) left Φ-module Φ. The class of C in the Witt group [DMNO], which is nonzero when C is not

[1] We do not assume that a fusion category has a simple unit: our ‘fusion’ is [EGNO]’s ‘multifusion’.

the Drinfeld center of a fusion category, is almost a complete obstruction to the existence of a nonzero boundary theory; see Remark 1.24(4).

A corollary of our proof is a characterization of fusion categories (Theorem B in Sect. 1.5). Let $\mathrm{Cat}_{\mathbb{C}}$ be the symmetric monoidal 2-category of finitely cocomplete $\mathbb{C}$-linear categories and right exact $\mathbb{C}$-linear functors, and let $\mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$ be the Morita 3-category of tensor categories. (In this paper 'tensor category' means 'algebra object in $\mathrm{Cat}_{\mathbb{C}}$'.) Then $\Psi \in \mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$ is a fusion category iff Ψ is 3-dualizable and the regular left Ψ-module is 2-dualizable. The forward direction ('only if') is proved in [DSS].

A key feature of our approach is the use of fully extended field theories. Naive analogs of Theorem A fail in the traditional context of (1, 2, 3)-theories; see Remark 1.27.

Here is a brief outline of the paper. Section 1 contains background and the statements of the main theorems. In Sect. 2 we discuss preliminaries about bordism multicategories and algebras in multicategories. The proofs of the main theorems are deferred to Sect. 3. The application to gapped quantum systems is the subject of Sect. 4. We provide several appendices with background material of interest independent of our main theorems. "Appendix A" contains a detailed definition of objects and morphisms in bordism multicategories. In "Appendix B" we prove a characterization of finite semisimple abelian categories in the 2-category $\mathrm{Cat}_{\mathbb{C}}$. "Appendix C" proves a criterion for internal duals in tensor categories, developed in a more general context. "Appendix D" proves the complete reducibility of fusion categories, which is implicit in [EGNO]: a fusion category is Morita equivalent to a direct sum of fusion categories with simple unit.

Papers related to the problems considered here include [KK, KS, FSV, Le, KZ].

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1. Mathematical Background and Statement of Main Theorem

1.1. RT theories and TV theories. In the late 1980s Witten [W1] and Reshetikhin–Turaev [RT1, RT2] introduced new invariants of closed 3-manifolds and generalizations of the Jones invariants of knots. Witten's starting point is the classical Chern–Simons invariant, which he feeds into the physicists' path integral, whereas Reshetikhin–Turaev begin with an intricate algebraic structure: a quantum group. Later, quantum groups were replaced by modular fusion categories [T], which were originally introduced[2] in the context of 2-dimensional conformal field theory [MS]. These disparate approaches are reconciled in *extended* topological field theory [F1]. In modern terms [L1] this extended field theory is a symmetric monoidal functor

$$F_{\langle 1,2,3\rangle}\colon \mathrm{Bord}^{\mathrm{fr}}_{\langle 1,2,3\rangle} \longrightarrow \mathrm{Cat}_{\mathbb{C}} \tag{1.1}$$

with domain the 2-category[3] of 3-framed[4] 1-, 2-, and 3-dimensional bordisms; the codomain is a certain 2-category of complex linear categories; see Definition 1.7. The

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[2] The version in [MS] uses a central charge in $\mathbb{Q}/24\mathbb{Z}$, whereas the version standard in mathematics, which we use, only has a central charge in $\mathbb{Q}/8\mathbb{Z}$.

[3] In this paper we use discrete categories: for example $\mathrm{Cat}_{\mathbb{C}}$ is a (2, 2)-category (as opposed to a more general $(\infty, 2)$-category). Some of our exposition in this section applies to (∞, n)-categories though we just write 'n-categories'.

[4] A *3-framing* of a 3-manifold is a global parallelism, a trivialization of its tangent bundle. For a manifold M of dimension $k < 3$ it is a trivialization of the inflated tangent bundle $\underline{\mathbb{R}^{3-k}} \oplus TM \to M$.

value of $F_{\langle 1,2,3\rangle}$ on the bounding 3-framed circle S^1_b is the modular fusion category that defines the theory. We call (1.1) a *Reshetikhin–Turaev* (RT) theory.

Remark 1.2. The RT theories in the original references factor through the bordism 2-category of manifolds equipped with a (w_1, p_1)-structure [BHMV], that is, an orientation and a trivialization of the first Pontrjagin class p_1. There is a unique isomorphism class of (w_1, p_1)-structures on a circle, so no distinction between bounding and non-bounding circles. "Spin Chern–Simons theories" require a trivialization of the second Stiefel–Whitney class w_2 as well. For those theories the codomain should include $\mathbb{Z}/2\mathbb{Z}$-gradings; see Remark 1.28.

A *fully extended* topological field theory has domain $\mathrm{Bord}_3^{\mathrm{fr}} = \mathrm{Bord}^{\mathrm{fr}}_{\langle 0,1,2,3\rangle}$, the 3-category of 3-framed bordisms of dimension ≤ 3. There is no canonical codomain for these theories, so for now we posit an arbitrary symmetric monoidal 3-category $\mathcal{C}$. The *cobordism hypothesis*—conjectured by Baez–Dolan [BD], proved by Hopkins–Lurie in 2 dimensions and by Lurie [L1] in all dimensions; see also [AF]—asserts that a fully extended theory

$$F\colon \mathrm{Bord}_3^{\mathrm{fr}} \longrightarrow \mathcal{C} \tag{1.3}$$

is determined by its value $F(+)$ on a positively oriented 3-framed point. Furthermore, a *3-dualizable* object of $\mathcal{C}$ determines a unique theory (1.3), up to a contractible space of choices. A symmetric monoidal 3-category $\mathcal{C}$ has a *fully dualizable part* $\mathcal{C}^{\mathrm{fd}} \subset \mathcal{C}$ whose objects are 3-dualizable and whose morphisms have all adjoints. We say $\mathcal{C}$ *has duals* if $\mathcal{C} = \mathcal{C}^{\mathrm{fd}}$. A functor (1.3) factors through $\mathcal{C}^{\mathrm{fd}}$. Given a general RT theory (1.1) it is still an open problem to construct $\mathcal{C}$ and an extension (1.3), or even better to construct a single $\mathcal{C}$ which works for all RT theories. (However, see [He] for a special case in the framework of bicommutant categories.)

There is a subclass of RT theories, the *Turaev–Viro (TV) theories* [TV], which are fully extended. Let Fus be the symmetric monoidal 3-category whose objects are *fusion* categories; see Definition 1.10. We remark that throughout this paper we take $\mathbb{C}$ as the ground field.

Theorem 1.4 (Douglas–Schommer-Pries–Snyder [DSS]). Fus *has duals, i.e.,* $\mathrm{Fus} = \mathrm{Fus}^{\mathrm{fd}}$.

In particular, a fusion category Φ is 3-dualizable in Fus. The cobordism hypothesis implies that there is a fully extended topological field theory

$$T_\Phi\colon \mathrm{Bord}_3^{\mathrm{fr}} \longrightarrow \mathrm{Fus}, \tag{1.5}$$

unique up to equivalence, whose value on a chosen framed point + is $T_\Phi(+) = \Phi$. A TV theory is a fully extended theory with codomain Fus; its truncation to $\mathrm{Bord}^{\mathrm{fr}}_{\langle 1,2,3\rangle}$ is an RT theory. Examples include 3-dimensional gauge theory for a finite group G, in which case Φ is the fusion category of finite rank complex vector bundles over G with convolution product; there is also a version twisted by a cocycle for a class in $H^3(G;\mathbb{Q}/\mathbb{Z})$, as in [DW]. Special toral Chern–Simons theories are also TV theories.

Remark 1.6. The original state sum construction [TV] is quite different from the construction with the cobordism hypothesis, but nevertheless we use 'Turaev–Viro theory' to identify this class of topological field theories. Also, the construction in [TV] is for unoriented manifolds and a (2, 3)-theory, whereas we use framed manifolds and a fully extended (0, 1, 2, 3)-theory.

1.2. Definitions and terminology. The definitions and terminology for abelian categories are standard; see [EGNO, §1] for example. For tensor categories there is tremendous variation in the literature, so we spell out our usage here. The term 'modular fusion category' is standard; see [EGNO, §8.13], for example.

Definition 1.7. (i) $\mathrm{Cat}_{\mathbb{C}}$ is the symmetric monoidal 2-category defined as follows.[5] Its objects are finitely cocomplete $\mathbb{C}$-linear categories. 1-morphisms in $\mathrm{Cat}_{\mathbb{C}}$ are right exact $\mathbb{C}$-linear functors—functors that preserve finite colimits—and 2-morphisms are natural transformations. The symmetric monoidal structure is the Deligne–Kelly tensor product $\boxtimes$; see [K, D, Fr].

(ii) A *tensor category* is an algebra object in $\mathrm{Cat}_{\mathbb{C}}$.

(iii) $\mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$ is the symmetric monoidal 3-category defined as follows. Its objects are tensor categories. A 1-morphism $M\colon A \to B$ is an object $M \in \mathrm{Cat}_{\mathbb{C}}$ equipped with the structure of a (B, A)-bimodule category. A 2-morphism $M' \to M$ is a 1-morphism in $\mathrm{Cat}_{\mathbb{C}}$ which respects the bimodule structure. A 3-morphism is a natural transformation of functors.

(iv) $\mathrm{Alg}_2(\mathrm{Cat}_{\mathbb{C}})$ is the symmetric monoidal 4-category defined as follows. Its objects are braided tensor categories, a 1-morphism $M\colon A \to B$ is an object $M \in \mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$ equipped with the compatible structure of a (B, A)-bimodule category, etc. (See [BJS, Definition-Proposition 1.2].)

Remark 1.8. (1) We do not assume that a tensor category has internal duals (rigidity). See "Appendix C" for a discussion of internal duals in tensor categories. Also, we do not assume that the tensor unit of a tensor category is a simple object.

(2) The algebraic theory of tensor categories is exposited in the text Etingof et al. [EGNO] (where rigidity and simple unit are included in the definition of 'tensor category'). The theory of Morita higher categories, such as $\mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$ and $\mathrm{Alg}_2(\mathrm{Cat}_{\mathbb{C}})$, is developed by Haugseng [H], Johnson et al. [JS], Gwilliam and Scheimbauer [GS], among others. Douglas et al. [DSS] define a 3-category of tensor categories which is a subcategory of $\mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$; in particular, they assume rigidity. Tensor categories and braided tensor categories in an infinite setting are explored in [BJS], and in an ∞-setting in [L2].

(3) The symmetric monoidal structure on $\mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$ is Deligne–Kelly tensor product, as in $\mathrm{Cat}_{\mathbb{C}}$. Composition of 1-morphisms in $\mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$ is the *relative* Deligne–Kelly tensor product: tensor product of module categories over a tensor category. Its existence is discussed in [BZBJ, Remark 3.2.1] and [JS, Example 8.10].

(4) For a 3-category $\mathcal{C}$ set $\Omega\mathcal{C} = \mathrm{End}_{\mathcal{C}}(1)$, the endomorphism 2-category of the tensor unit object. There is a canonical identification

$$\Omega\,\mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}}) = \mathrm{Cat}_{\mathbb{C}}\,. \tag{1.9}$$

(5) A modular fusion category is invertible as an object of $\mathrm{Alg}_2(\mathrm{Cat}_{\mathbb{C}})$; see [BJSS].

The 3-category Fus of fusion categories is introduced in [DSS].

Definition 1.10. (i) FSCat is the full subcategory of $\mathrm{Cat}_{\mathbb{C}}$ whose objects are finite semisimple abelian categories.

[5] The symbol 'Rex' is sometimes used in place of '$\mathrm{Cat}_{\mathbb{C}}$'; it emphasizes the right exactness of 1-morphisms.

(ii) A *fusion category* is a finite semisimple rigid tensor category.

(iii) Fus is the symmetric monoidal 3-category subcategory of $\mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$ whose objects are fusion categories and whose 1-morphisms are finite semisimple abelian bimodule categories.

Remark 1.11. (1) $\mathrm{FSCat} \subset \mathrm{Cat}_{\mathbb{C}}$ is closed under Deligne–Kelly tensor product [Fr, §5].
(2) We do not assume that a fusion category has simple unit. ([EGNO] use 'multifusion' for Definition 1.10(ii) and reserve 'fusion' for the case of a simple unit.)
(3) The loop category of Fus is

$$\Omega\,\mathrm{Fus} = \mathrm{FSCat}\,. \tag{1.12}$$

A companion result to Theorem 1.4 asserts that the symmetric monoidal 2-category FSCat has duals. We also need the following result, which we prove in "Appendix B".

Theorem 1.13. *If $C \in \mathrm{Cat}_{\mathbb{C}}$ is 2-dualizable, then C is finite semisimple abelian.*

There are many variations of this theorem, such as [Se, Ti]; see [BDSV, Appendix] for a survey.

1.3. Relations between RT and TV theories. Let

$$T\colon \mathrm{Bord}_3^{\mathrm{fr}} \longrightarrow \mathrm{Fus} \tag{1.14}$$

be a TV theory. Then the associated modular fusion category $T(S^1_b)$, the value of T on the bounding 3-framed circle, is the *Drinfeld center* of $T(+)$. It is the modular fusion category of the associated RT theory, which is the truncation

$$T_{\langle 1,2,3\rangle}\colon \mathrm{Bord}^{\mathrm{fr}}_{\langle 1,2,3\rangle} \longrightarrow \mathrm{Cat}_{\mathbb{C}}\,. \tag{1.15}$$

Also, the value $T(S^1_n)$ on the nonbounding 3-framed circle is the *Drinfeld cocenter*[6] of $T(+)$, a module category over $T(S^1_b)$. (Notice that $T(S^1_n)$ is equivalent to $T(S^1_b)$ if $T(+)$ is spherical.) See [DSS, §3.2.2] for an exposition. For a general RT theory (1.1) there does not exist a fusion category whose Drinfeld double is the modular fusion category $F_{\langle 1,2,3\rangle}(S^1_b)$.

Remark 1.16. The double $|F|^2 = FF^\vee$ of an RT theory F is the truncation of a TV theory, as we now explain. ($F^\vee$ is the dual theory to F in the symmetric monoidal category of theories.) Let B be a modular fusion category. Suppose $F\colon \mathrm{Bord}_3^{\mathrm{fr}} \to \mathcal{C}$ is an extension of an RT theory (1.1) with $F(S^1_b) = B$, where we assume the hypotheses on $\mathcal{C}$ in Theorem A below. Use the cobordism hypothesis to define theories

$$F^\vee, |F|^2\colon \mathrm{Bord}_3^{\mathrm{fr}} \longrightarrow \mathcal{C} \tag{1.17}$$

which are characterized by

$$\begin{aligned} F^\vee(+) &= F(+)^\vee \\ |F|^2(+) &= F(+) \otimes F(+)^\vee = F(S^0). \end{aligned} \tag{1.18}$$

[6] Objects of the Drinfeld cocenter of a fusion category Φ are pairs (x, γ) in which x is an object of Φ and the functor $\gamma\colon x \otimes - \to - \otimes x^{**}$ is a twisted half-braiding.

Here $F(+)^\vee$ is the dual object to $F(+) \in \mathcal{C}$. The theory $F^\vee$ may be defined as the composition of F with the involution $\mathrm{Bord}_3^{\mathrm{fr}} \to \mathrm{Bord}_3^{\mathrm{fr}}$ which reverses the first "arrow of time"; see Sect. 2.1.5. Using this description identify the braided fusion category $F^\vee(S_b^1)$ with B^{rev}, which is the same underlying fusion category B equipped with the inverse braiding. It follows that $|F|^2(S_b^1) \cong B \boxtimes B^{\mathrm{rev}}$, which by [EGNO, Proposition 8.20.12] is braided tensor equivalent to the Drinfeld center of B.

1.4. Existence of boundary theories. Let

$$F\colon \mathrm{Bord}_3^{\mathrm{fr}} \longrightarrow \mathcal{C} \tag{1.19}$$

be a 3-dimensional 3-framed topological field theory, as in (1.3). Lurie [L1, Example 4.3.22] defines an extended bordism 3-category $\mathrm{Bord}_{3,\partial}^{\mathrm{fr}}$ and an inclusion $\mathrm{Bord}_3^{\mathrm{fr}} \to \mathrm{Bord}_{3,\partial}^{\mathrm{fr}}$. (See Sect. A.3 for a definition of objects in $\mathrm{Bord}_{3,\partial}^{\mathrm{fr}}$.) A *boundary theory for F* is an extension

$$\widetilde{F}\colon \mathrm{Bord}_{3,\partial}^{\mathrm{fr}} \to \mathcal{C} \tag{1.20}$$

of F to a symmetric monoidal functor. The cobordism hypothesis with singularities implies that $\widetilde{F}$ is determined by a 3-dualizable object $F(+) \in \mathcal{C}$ together with a 2-dualizable 1-morphism $1 \to F(+)$. To isolate the data of the boundary theory, let

$$\tau_{\le 2}F\colon \mathrm{Bord}_2^{\mathrm{fr}} \longrightarrow \mathcal{C} \tag{1.21}$$

be the truncation of F to *2-framed* 0-, 1-, and 2-dimensional bordisms. It is a *once categorified topological field theory*. Then the data of a boundary theory for F is a natural transformation[7]

$$\beta\colon 1 \longrightarrow \tau_{\le 2}F \tag{1.22}$$

of symmetric monoidal functors on $\mathrm{Bord}_2^{\mathrm{fr}}$. More precisely, β is an *oplax* natural transformation in the sense of Johnson-Freyd and Scheimbauer [JS]. They apply the cobordism hypothesis with singularities [L1, §4.3] to prove that the data of an extended theory (1.20) is equivalent to the data of the pair consisting of (1.19) and (1.22); see [JS, Theorem 7.15]. Furthermore, that data is determined by the values $F(+)$ and $\beta(+)\colon 1 \to F(+)$ on a point, which satisfy maximal dualizability constraints. We discuss natural transformations in pictorial terms in Sect. 2.1.7.

Our main result is the following.

Theorem A. *Let $\mathcal{C}$ be a symmetric monoidal 3-category whose fully dualizable part $\mathcal{C}^{\mathrm{fd}}$ contains the 3-category* Fus *of fusion categories as a full subcategory. Let $F\colon \mathrm{Bord}_3^{\mathrm{fr}} \to \mathcal{C}$ be a 3-framed topological field theory such that*

(a) *$F(S^0)$ is isomorphic in $\mathcal{C}$ to a fusion category, and*
(b) *$F(S_b^1)$ is invertible as an object in the 4-category $\mathrm{Alg}_2(\Omega\mathcal{C})$ of braided tensor categories.*

Assume F extends to $\widetilde{F}\colon \mathrm{Bord}_{3,\partial}^{\mathrm{fr}} \to \mathcal{C}$ such that the associated boundary theory $\beta\colon 1 \to \tau_{\le 2}F$ is nonzero. Then $F(S_b^1)$ is braided tensor equivalent to the Drinfeld center of a fusion category Φ, which may be taken to be $\mathrm{End}_{\mathcal{C}^{\mathrm{fd}}}\big(\beta(+)\big)$.

[7] There are also boundary theories $\tau_{\le 2}F \to 1$. They are extension of F to a variation of $\mathrm{Bord}_{3,\partial}^{\mathrm{fr}}$.

The hypothesis that Fus is a full subcategory of $\mathcal{C}^{\text{fd}}$ ensures that Theorem A applies to TV theories. That hypothesis implies that

$$\Omega\mathcal{C}^{\text{fd}} = \text{FSCat}; \tag{1.23}$$

see (1.9). Since FSCat $\subset \text{Cat}_{\mathbb{C}}$, the $\langle 1, 2, 3\rangle$ truncation $F_{\langle 1,2,3\rangle}$ of F has the form (1.1). Hypotheses (a) and (b) express that $F_{\langle 1,2,3\rangle}$ is an RT theory. A modular fusion category is invertible as an object in the 4-category of braided tensor categories [S-P,BJSS]; hypothesis (b) captures this central feature of RT theories. It remains to explain what it means that β is *nonzero*. Observe that the value of β on a closed 1-manifold is an object in a finite semisimple complex linear abelian category, and the value of β on a closed 2-manifold is a vector in a finite dimensional complex vector space. We require that β take a nonzero value on some nonempty closed 1- or 2-manifold.

Remark 1.24. (1) The unit in our fusion category Φ may not be simple. We prove (Corollary D.2) that Φ is Morita equivalent to a fusion category Φ_0 with simple unit. The category Φ is canonical in terms of $\widetilde{F} = (F, \beta)$, whereas Φ_0 is only determined up to Morita equivalence.

(2) The relationship between $F(S^0)$ and Φ is explained in Lemma 3.23 (in which Ξ is a fusion category isomorphic to $F(S^0)$).

(3) Theorem A gives an obstruction to a nonzero topological boundary theory: the modular fusion category $F(S^1_b)$ must be the Drinfeld center of a fusion category. There is a simpler obstruction, even if we only assume F is a (2,3)-theory: the central charge c—which is defined modulo 24—must vanish. To see this, let Y be a closed 3-framed surface of genus $g \geq 3$. Then π_0 of the automorphism group of Y is a central extension of the mapping class group by $\mathbb{Z}$, and the central $\mathbb{Z}$ acts[8] on the state space $F(Y)$ by the character $e^{2\pi ic/12}$. If β is a nonzero boundary theory, then $\beta(Y) \in F(Y)$ is an invariant vector, which we may suppose is nonzero for some g, and so the center must act trivially.

(4) The modular fusion category $F(S^1_b)$ does not determine the theory F completely [BK]. For example, the E_8 Chern–Simons theory at level 1 is invertible and the modular fusion category is trivial, whereas the theory is nontrivial—its central charge is 8 mod 24—and it is not a Turaev–Viro theory. Therefore, by Theorem A′ below, it does not admit a nonzero boundary theory.

(5) At first glance it might seem that the hypotheses of Theorem A, which lead to (1.23), are too restrictive: we might rather have nonsemisimple categories be possible values of $F_{\langle 1,2,3\rangle}$. However, this is ruled out by Theorem 1.13 and a dimensional reduction argument, such as in the proof of Lemma 3.4.

(6) A converse to Theorem A follows from the cobordism hypothesis. Namely, if Φ is a fusion category, then the Turaev-Viro theory T_Φ defined in (1.5) has a canonical boundary theory built from the regular left Φ-module Φ, as in Theorem A′ below.

(7) In forthcoming work with Claudia Scheimbauer [FST], we construct an extension of any RT theory which satisfies the hypotheses of Theorem A.

We prove Theorem A in Sect. 3.2.

Following a suggestion of Theo Johnson-Freyd, we can improve Theorem A by making a slightly stronger assumption on $\mathcal{C}$. We spell out that assumption in the following

[8] See [MR] for the case of (w_1, p_1)-theories (as in Remark 1.2), where the generator of the appropriate centrally extended mapping class group acts as $e^{2\pi ic/24}$. A framing trivializes $(w_1, w_2, p_1/2)$; the extra factor of 2 explains the discrepancy between $e^{2\pi ic/24}$ here and $e^{2\pi ic/12}$ in the text.

definition, which expresses the co-completeness of $\mathcal{C}$ under very special finite colimits coming from tensor products.

Definition 1.25. Let $\mathcal{C}$ be a symmetric monoidal 3-category whose fully dualizable part $\mathcal{C}^{\mathrm{fd}}$ contains the 3-category Fus of fusion categories as a full subcategory. Then $\mathcal{C}$ is *fusion tensor cocomplete* if the following holds for all objects $x, y, z \in \mathcal{C}$:

(a) for every triple (Φ, M, N) consisting of a fusion category Φ, viewed as an algebra object in $\mathrm{End}_{\mathcal{C}}(1)$; a left Φ-module M in $\mathcal{C}(x, y)$; and a right Φ-module N in $\mathrm{End}_{\mathcal{C}}(1)$, the relative tensor product $N \boxtimes_{\Phi} M$ exists as a colimit of the bar resolution;
(b) for all $H \in \mathcal{C}(y, z)$ the natural map $(H \circ M) \boxtimes_{\Phi} N \to H \circ (M \boxtimes_{\Phi} N)$ is an equivalence;
(c) for all $K \in \mathcal{C}(z, x)$ the natural map $M \boxtimes_{\Phi} (N \circ K) \to (M \boxtimes_{\Phi} N) \circ K$ is an equivalence.

Theorem A′. *In the context of Theorem A assume in addition that $\mathcal{C}$ is fusion tensor cocomplete. Then F is isomorphic to the Turaev-Viro theory T_{Φ}, and the boundary theory is determined by the regular left module category Φ.*

We prove Theorem A′ in Sect. 3.3.

Remark 1.26. A more obvious hypothesis on $\mathcal{C}$—namely that (i) for all $x, y \in \mathcal{C}$ the 2-category $\mathcal{C}(x, y)$ of 1-morphisms is finitely cocomplete, and (ii) composition of 1-morphisms in $\mathcal{C}$ is finitely cocontinuous—is too strong for many applications. (We thank Sam Raskin for this observation.)

Remark 1.27. If F is isomorphic to the tensor unit theory, then for any Turaev–Viro representative T_{Φ} the fusion category Φ is endomorphisms of a finite semisimple abelian category M. Under the isomorphism $T_{\Phi} \xrightarrow{\sim} 1$, executed via the Morita trivialization of $\mathrm{End}(M)$, the regular left Φ-module goes over to the finite semisimple abelian category M. Note that the $(1, 2)$ part of the $(0, 1, 2)$-theory based on M assigns a semisimple commutative algebra to S^1_b.

If from the beginning we work with $(1, 2, 3)$-theories (1.1), then the conclusion of Theorem A′ can fail. For example, consider a $(1, 2)$-theory with $\beta(S^1_b) = \mathbb{C}[x]/(x^2)$, a nonsemisimple commutative algebra. This theory is not extendable to a (0,1,2)-theory with values in $\Omega\,\mathrm{Fus} = \mathrm{FSCat}$, so does not arise as in previous paragraph.

Remark 1.28. There is a generalization of RT theories (1.1) with codomain a 2-category of "super" complex linear categories. Some developments in the theory of these categories—which are either enriched over the category $\mathrm{sVect}_{\mathbb{C}}$ of super vector spaces or are a module category over $\mathrm{sVect}_{\mathbb{C}}$—especially for fusion supercategories, may be found in [GK,BE,U]. Our main theorems generalize to allow supercategories in the codomain [FT].

1.5. A characterization of fusion categories. En route to proving Theorem A, we prove the following characterization of fusion categories.

Theorem B. *Let $\Psi \in \mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$ be a tensor category. Then Ψ is a fusion category iff*

(i) *Ψ is 3-dualizable in $\mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$, and*
(ii) *Ψ as a left Ψ-module is 2-dualizable as a 1-morphism in $\mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$.*

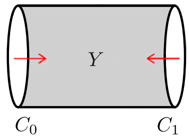


Fig. 1. A 2-morphism $C_0 \amalg C_1 \xrightarrow{Y} \emptyset^1$ in Bord_2

The forward direction, proved in [DSS], is stated as Theorem 1.4. We prove the converse in Sect. 3.1.

Remark 1.29. [BJS] Let $A = \mathbb{C}[x]/(x^2)$ be the non-semisimple algebra of dual numbers. The tensor category Ψ of finite dimensional A-A bimodules is Morita equivalent to $\mathrm{Vect}_{\mathbb{C}}$, so is 3-dualizable, but it is not a fusion category: for example, it does not have internal duals. This example illustrates that 'fusion' is not a Morita invariant notion. The dualizability in Theorem B(i) is Morita invariant, whereas the regular module in Theorem B(ii) is not. For example, under the Morita equivalence which sends Ψ to $\mathrm{Vect}_{\mathbb{C}}$, the regular left module ${}_{\Psi}\Psi$ is sent to the linear category of A-modules, which is not the regular left module over $\mathrm{Vect}_{\mathbb{C}}$. We regard Theorem A as a Morita invariant variant of Theorem B.

2. Preliminaries

2.1. Bordism n-categories. As a preliminary, we recall features of bordism multicategories and explain how they are encoded in the pictures we draw. In "Appendix A" we give a formal and precise account valid in all dimensions; the heuristic exposition here is focused on the low dimensional cases of interest. See [BM,CS,AF] for complete constructions of the bordism multicategory.

2.1.1. Arrows of time Begin with Bord_2, the 2-category of unoriented bordisms of dimension $\leq$ 2. An endomorphism $\emptyset^0 \xrightarrow{C} \emptyset^0$ of the empty 0-manifold is a closed 1-manifold. In the bordism 2-category its tangent bundle is stabilized to the rank 2 vector bundle $\underline{\mathbb{R}} \oplus TC \to C$, where $\underline{\mathbb{R}} \to C$ is the trivial bundle of rank 1 with its canonical orientation. This orientation is called an "arrow of time". In a 1-morphism $C_0 \amalg C_1 \xrightarrow{Y} \emptyset^1$, such as the one depicted in Fig. 1, the arrows of time distinguish incoming and outgoing boundary components. An object P in Bord_2 is a finite set of points with inflated tangent bundle $\mathbb{R} \oplus \mathbb{R} \to P$. Figure 2 is a 1-morphism $P_0 \amalg P_1 \xrightarrow{C} \emptyset^0$. Note that the manifolds P_0, P_1 have two ordered arrows of time; the ordering is depicted in our figures by the number of arrowheads. (In our conventions the indexing is by codimension, so the single-headed arrow has index -1 and the double-headed arrow has index -2.) The single-headed arrow of time is constrained to be compatible with the single arrow of time of C; that is, corresponding trivial summands augmenting the tangent bundle are identified at ∂C. The single-headed arrow of time in Fig. 2 carries no information—it evokes the standard orientation of the trivial real line bundle over C—whereas the double-headed arrows of time in Fig. 2 distinguish incoming and outgoing boundary components. By contrast, the arrows of time in Fig. 5 do carry information; they depict the standard basis of $\mathbb{R} \oplus \mathbb{R}$, so give meaning to the depicted framings.

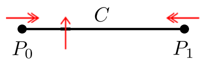


Fig. 2. A 1-morphism $P_0 \amalg P_1 \xrightarrow{C} \emptyset^0$ in Bord_2

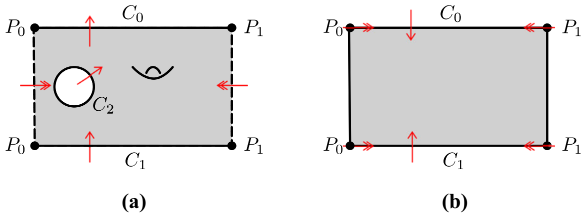


Fig. 3. **a** A legal 2-morphism and **b** not a 2-morphism

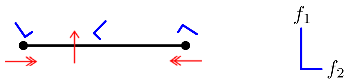


Fig. 4. A 2-framed 1-morphism

2.1.2. Constancy data A 2-morphism in Bord_2 is a compact 2-manifold Y with corners and arrows of time together with *constancy data* [CS, Definition 5.1]. Namely, if $Y = \mathring{Y}_0 \amalg \mathring{Y}_{-1} \amalg \mathring{Y}_{-2}$ is the partition into boundaries and corners, so $\dim \mathring{Y}_{-k} = 2 - k$, then there is a free involution specified on $\mathring{Y}_{-2}$ with quotient P and an embedding

$$[0, 1] \times P \hookrightarrow \overline{Y_{-1}} \tag{2.1}$$

such that the arrows of time are constant along the image of $[0, 1] \times \{p\}$ for all $p \in P$. Thus Fig. 3a is legal: the dashed vertical edges comprise the image of the constancy embedding (2.1). The constancy gives rise to an interpretation of Fig. 3a as a 2-morphism

$$P_0 \amalg P_1 \underset{C_1 \amalg C_2}{\overset{C_0}{\Uparrow Y}} \emptyset^0 \ , \tag{2.2}$$

essentially by collapsing the dashed edges. On the other hand, Fig. 3b is not allowed because there is no embedding $[0, 1] \times \{P_i\} \hookrightarrow Y_{-1}$ for which the specified arrows of time are constant.

A formal specification of constancy data is (A.10), a consequence of Definition A.6.

2.1.3. Tangential structures The main point to emphasize is that in a bordism n-category of manifolds of dimension $\leq n$, the tangential structure is on the *stabilized* rank n tangent bundle. In particular, Fig. 4 is a valid 1-morphism in $\mathrm{Bord}_2^{\mathrm{fr}}$, the bordism 2-category of *2-framed* manifolds. In the figure the arrows of time are notated as earlier. The 2-frame f_1, f_2 is depicted as a long line segment (f_1) followed by a short line segment (f_2). There is no relationship imposed between the arrows of time and the tangential structure (2-framing).

Remark 2.3. The relative positions of the framing and the arrows of time does have significance, for example in Fig. 5.

Fig. 5. The standard points

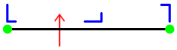


Fig. 6. A 1-morphism in $\mathrm{Bord}^{\mathrm{fr}}_{2,\partial}$

2.1.4. Two conventions We depict objects and morphisms in $\mathrm{Bord}^{\mathrm{fr}}_{2}$ as manifolds with corners embedded in the Euclidean plane—the plane of the paper/screen—together with arrows of time, constancy data, and orthonormal 2-framings. The first convention is that we identify two such which are related by either a translation or a translation composed with a reflection about either a horizontal or a vertical line. (If any such identification exists it is unique, since it preserves the 2-framings.) Take the standard points $+, -$ to be those depicted in Fig. 5. Although our pictures lie in $\mathrm{Bord}^{\mathrm{fr}}_{2}$, our arguments in the proof are for $\mathrm{Bord}^{\mathrm{fr}}_{3}$. (Only in the proof of Lemma 3.6 do we use truly 3-dimensional pictures.) The second convention, then, is that we embed $\mathrm{Bord}^{\mathrm{fr}}_{2}$ in $\mathrm{Bord}^{\mathrm{fr}}_{3}$ by adding a trivial line bundle[9] to the (inflated) rank 2 tangent bundle of each k-morphism in $\mathrm{Bord}^{\mathrm{fr}}_{2}$. Furthermore, we inflate orthonormal 2-framings to orthonormal 3-framings by prepending the standard basis f_0 of the trivial line bundle to the 2-framing f_1, f_2. In the pictures we regard f_0 and the aligned arrow of time as pointing *into* the paper/screen. (One should then slide the indices $f_0, f_1, f_2 \rightsquigarrow f_1, f_2, f_3$ to normal positions, but we will not do so in the text or figures.)

2.1.5. Duals and adjoints A topological bordism n-category or (∞, n)-category has all duals and adjoints. Duals are formed by reversing an arrow of time. This can be done at any depth, which reflects the $O(1)^{\times n}$-action discussed in [L1, Remark 4.4.10] and [BS-P, §4]. For example, the two standard points in $\mathrm{Bord}^{\mathrm{fr}}_{2}$, depicted in Fig. 5, are dual by reversing the double-headed arrow of time. Right and left adjoints of morphisms are constructed by a more complicated prescription which we specify in Sect. A.2.5.

2.1.6. Coloring with a boundary theory We now briefly describe a bordism 2-category $\mathrm{Bord}^{\mathrm{fr}}_{2,\partial}$ into which $\mathrm{Bord}^{\mathrm{fr}}_{2}$ embeds. (Use the second convention of Sect. 2.1.4 to extend objects, 1-morphisms, and 2-morphisms to the 3-category $\mathrm{Bord}^{\mathrm{fr}}_{3,\partial}$.) A sketch of this construction appears in [L1, Example 4.3.22]; details for $\mathrm{Bord}^{\mathrm{fr}}_{n,\partial}$, $n \in \mathbb{Z}^{\geq 1}$, are provided in Sect. A.3.

The 0-morphisms in $\mathrm{Bord}^{\mathrm{fr}}_{2,\partial}$ are the same as 0-morphisms in $\mathrm{Bord}^{\mathrm{fr}}_{2}$: a finite set of 2-framed points equipped with arrows of time. There are new 1-morphisms, such as the one depicted in Fig. 6. The two boundary points are colored and the extra arrow of time at colored boundary points is replaced by the conditions that the frame vector f_1 be tangent to the colored boundary and that f_2 be an inward normal.[10] Effectively, we have a 1-framing at those points. There are, of course, new 2-morphisms, such as the one

[9] For remarks on conventions, see Remark A.1. We index by codimension, so for $\mathrm{Bord}^{\mathrm{fr}}_{3}$ the indices are $-1, -2. - 3$. We might have used 'f_{-1}, f_{-2}, f_{-3}' for the framings, but that would have been a step too far.

[10] This is appropriate for a boundary theory $1 \to \tau_{\leq 2}F$; for a boundary theory $\tau_{\leq 2}F \to 1$ we require that f_2 be an outward normal.

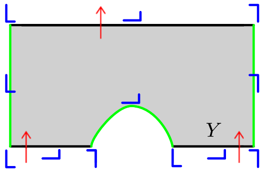


Fig. 7. A 2-morphism in $\mathrm{Bord}^{\mathrm{fr}}_{2,\partial}$

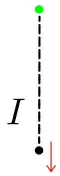


Fig. 8. The new picture I

depicted in Fig. 7. In terms of the partition $Y = \mathring{Y}_0 \amalg \mathring{Y}_{-1} \amalg \mathring{Y}_{-2}$, there is a submanifold with boundary $\mathring{B}_{-1} \amalg \mathring{B}_{-2} \subset \mathring{Y}_{-1} \amalg \mathring{Y}_{-2}$ that is colored with the boundary condition; in Fig. 7 we have $\mathring{B}_{-2} = \mathring{Y}_{-2}$, and $\mathring{Y}_{-1} \backslash \mathring{B}_{-1}$ consists of three open line segments. There are no arrows of time on $\mathring{B}_{-1}$. The constancy data (2.1), vacuous for Fig. 7, is as in Sect. 2.1.2 with $\mathring{Y}_{-2} \backslash \mathring{B}_{-2}$ replacing $\mathring{Y}_{-2}$. (Fig. 14 illustrates the constancy data.)

Remark 2.4. A boundary theory (1.22) for an n-dimensional theory is essentially an $(n-1)$-dimensional theory. This explains why at a boundary component "colored" with a boundary theory there is one fewer arrow of time and a constraint on the tangential structure.

Remark 2.5. There does not exist a 2-morphism in $\mathrm{Bord}^{\mathrm{fr}}_2$ from which Fig. 7 is obtained by coloring a subset of the boundary (with inward arrow of time).

Remark 2.6. Intuitively, the colored boundary components include a time direction—they are timelike—which motivates our convention about which frame vector is constrained; see Remark A.1.

2.1.7. Natural transformations A boundary theory β of a topological field theory F is a natural transformation of functors out of a bordism category; see (1.22). The pair $\widetilde{F} = (F, \beta)$ is best encoded as a functor out of $\mathrm{Bord}_{n,\partial}$, as in Sect. 2.1.6. In this section we introduce new "pictures" which encode the idea of a natural transformation, and then too an algorithm for converting them to morphisms in $\mathrm{Bord}_{n,\partial}$. We do not use these new pictures in the sequel, which justifies the sparse provisional account here of a single example.

Remark 2.7. We leave open the question of whether there is a single bordism category which includes $\mathrm{Bord}_{n,\partial}$ as well as the new pictures.

Introduce the manifold $I = [0, 1]$, embellished as in Fig. 8. We view it as 1-framed, with framing vector aligned with the arrow of time. Let X be a k-morphism in $\mathrm{Bord}^{\mathrm{fr}}_2$, $k \in \{0, 1, 2\}$, and consider the Cartesian product $I \times X$. For example, in Fig. 9a we have the 1-morphism $X = \mathrm{id}_+$, the identity map on the + point. Using our numbering

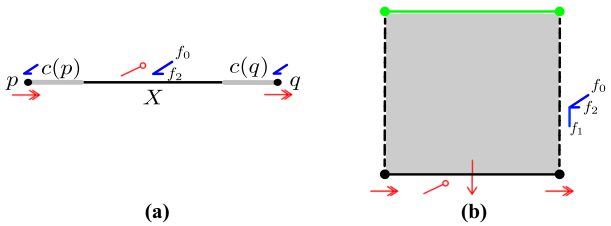


Fig. 9. **a** id_+, **b** $I \times X$

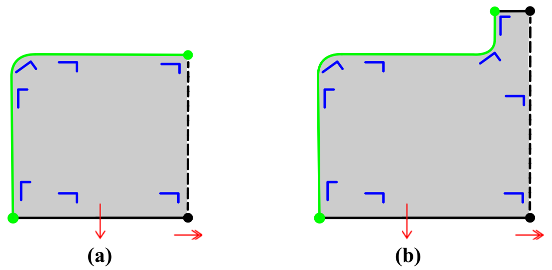


Fig. 10. Two maneuvers: Fig. 9b $\rightsquigarrow$ a 2-morphism in $\mathrm{Bord}^{\mathrm{fr}}_{3,\partial}$

convention $0, 1, 2$ in $\mathrm{Bord}^{\mathrm{fr}}_3$—which replaces the more uniform convention $-1, -2, -3$ used in "Appendix A"—we choose to number the directions in $\mathrm{Bord}^{\mathrm{fr}}_2$ as $0, 2$ and the direction in I as 1. Thus in Fig. 9a the frame vector f_0 and 0-arrow of time point into the paper/screen. The frame vector f_2 and 2-arrow of time point to the right. The Cartesian product $I \times X$ is Fig. 8b, with arrows of time and constant 3-framing as indicated.

We now proceed in three steps to convert Fig. 8b to a 2-morphism in $\mathrm{Bord}^{\mathrm{fr}}_{3,\partial}$.

(1) Rotate the frame through angle $\pi/2$ in the f_1–f_2 plane: send $f_2 \to f_1$ and $f_1 \to -f_2$. Use our standard pictorial representation of the frame vectors f_1, f_2 and omit the pictorial representation of f_0; it points into the paper/screen everywhere in subsequent pictures.
(2) Now convert to a 2-morphism in $\mathrm{Bord}^{\mathrm{fr}}_{3,\partial}$. Let $c(\partial X) = c(p) \amalg c(q)$ be a collar neighborhood of the boundary, as in Fig. 9a. For the incoming boundary point p, "pull" the colored boundary in Fig. 9b "down" through $c(p) \times I$ and rotate the framing accordingly, as depicted in Fig. 10a. Let $I' = [1, 2]$, viewed as glued "above" $I = [0, 1]$. For the outgoing boundary point q, "pull" the colored boundary in Fig. 9b "up" through $c(q) \times I'$ and rotate the framing accordingly, as depicted in Fig. 10b. There is a new edge which connects to the dotted edge. The arrows of time on the new edge and on the new vertex are fixed by the constancy condition along the dotted edge. The result can be redrawn in a more standard form; see Fig. 11.
(3) Finally, in a tubular neighborhood of the outgoing dotted boundary, rotate the frame through angle $\pi/2$ in the f_1-f_2 plane by the inverse of the rotation in step (1): send $f_1 \to f_2$ and $f_2 \to -f_1$. The result is depicted in Fig. 12.

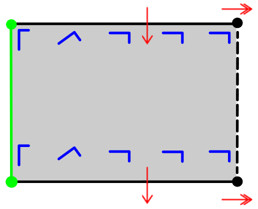
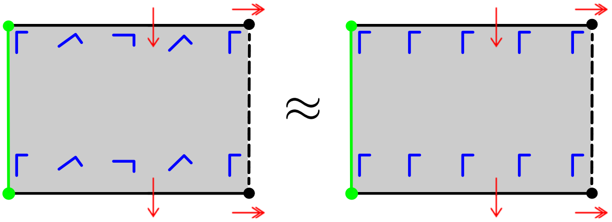


Fig. 11. The 2-morphism in Fig. 10b redrawn



(a) **(b)**

Fig. 12. **a** The third maneuver, **b** the 2-morphism id_{b_+} in $\mathrm{Bord}^{\mathrm{fr}}_{3,\partial}$

The 2-morphism represented by Fig. 9b is derived from the diagram

$$\begin{array}{ccc} \emptyset^0 & \xrightarrow{\mathrm{id}_{\emptyset^0}} & \emptyset^0 \\ {\scriptstyle b_+}\downarrow & \swarrow\!\!\!\!\swarrow & \downarrow{\scriptstyle b_+} \\ + & \xrightarrow[\mathrm{id}_+]{} & + \end{array} \tag{2.8}$$

which is a map $b_+ \circ \mathrm{id}_{\emptyset^0} \Longrightarrow \mathrm{id}_+ \circ b_+$, namely the identity map $\mathrm{id}_{b_+} : b_+ \Longrightarrow b_+$. This is precisely the 2-morphism depicted in Fig. 11.

Remark 2.9. It is instructive to carry this out for the coevaluation c and the evaluation e.

2.2. *A higher categorical preliminary.*

2.2.1. Internal homs Let $\mathcal{M}$ be a (weak) 2-category. A 1-morphism $x \xrightarrow{f} y$ in $\mathcal{M}$ defines a functor $\mathcal{M}(z, x) \xrightarrow{f\circ -} \mathcal{M}(z, y)$ for any $z \in \mathcal{M}$. We call its right adjoint, if it exists, the *right internal hom* functor:

$$\mathcal{M}(z, x) \underset{\underline{\mathrm{Hom}}^R(f,-)}{\overset{f\circ -}{\rightleftarrows}} \mathcal{M}(z, y)\ . \tag{2.10}$$

The *left internal hom* is defined as a right adjoint for right composition for any $w \in \mathcal{M}$:

$$\mathcal{M}(y, w) \underset{\underline{\mathrm{Hom}}^L(f,-)}{\overset{-\circ f}{\rightleftarrows}} \mathcal{M}(x, w)\ . \tag{2.11}$$

See [W] and [MaSi, §16] for discussions.[11]

If $\mathcal{M}$ is a symmetric monoidal 2-category, and the 1-morphism f has right and left adjoints f^R, f^L in $\mathcal{M}$, then there are natural isomorphisms

$$\begin{aligned} \underline{\mathrm{Hom}}^R(f,g) &\cong f^R \circ g \in \mathcal{M}(z,x), \qquad g\colon z \to y, \\ \underline{\mathrm{Hom}}^L(f,h) &\cong h \circ f^L \in \mathcal{M}(y,w), \qquad h\colon x \to w. \end{aligned} \tag{2.12}$$

If $h = g = f$, hence $z = x$ and $w = y$, then we use the notations

$$\begin{aligned} \underline{\mathrm{End}}^R(f) &:= \underline{\mathrm{Hom}}^R(f,f) \cong f^R \circ f \in \mathcal{M}(x,x), \\ \underline{\mathrm{End}}^L(f) &:= \underline{\mathrm{Hom}}^L(f,f) \cong f \circ f^L \in \mathcal{M}(y,y). \end{aligned} \tag{2.13}$$

The 1-morphism $\underline{\mathrm{End}}^R(f)$ is an algebra object in the 1-category $\mathcal{M}(x,x)$, and $\underline{\mathrm{End}}^L(f)$ is an algebra object in $\mathcal{M}(y,y)$. The unit of $\underline{\mathrm{End}}^R(f)$ is the unit $1 \to f^R \circ f$ of the adjunction, and the multiplication uses the counit $f \circ f^R \to 1$ of the adjunction:

$$\underline{\mathrm{End}}^R(f) \circ \underline{\mathrm{End}}^R(f) = f^R \circ (f \circ f^R) \circ f \longrightarrow f^R \circ f = \underline{\mathrm{End}}^R(f). \tag{2.14}$$

The formulas for $\underline{\mathrm{End}}^L(f)$ are similar.

Remark 2.15. This discussion applies to n-categories, $n \geq 3$, by taking 2-categorical slices.

Remark 2.16. A symmetric monoidal functor preserves internal homs, internal endomorphisms, and the composition laws.

2.2.2. Internal homs in Fus *and* $\mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$ Recall from Definition 1.7 that Fus is a symmetric monoidal 3-category whose objects are fusion categories. Our convention, different than some references, is that a (B,A)-module category M is a 1-morphism $M\colon A \to B$ in Fus. This convention renders tensor products and compositions in the same order. The following results are generalized in [BJS, §5.1].

Proposition 2.17 [DSS, §3.2.1]. *Let $A, B \in$ Fus and $M\colon A \to B$ a finite semisimple (B,A)-bimodule category. Then M has right and left adjoints, and we can take them to be*

$$\begin{aligned} M^R &= \mathrm{Hom}_B(M,B), \\ M^L &= \mathrm{Hom}_A(M,A). \end{aligned} \tag{2.18}$$

Corollary 2.19. *If also $C, D \in$ Fus and $N\colon C \to B$, $P\colon A \to D$ are finite semisimple bimodule categories, then*

$$\begin{aligned} \underline{\mathrm{Hom}}^R(M,N) &= \mathrm{Hom}_B(M,N), \\ \underline{\mathrm{Hom}}^L(M,P) &= \mathrm{Hom}_A(M,P). \end{aligned} \tag{2.20}$$

In particular,

$$\begin{aligned} \underline{\mathrm{End}}^R(M) &= \mathrm{End}_B(M), \\ \underline{\mathrm{End}}^L(M) &= \mathrm{End}_A(M). \end{aligned} \tag{2.21}$$

[11] We thank Emily Riehl for correspondence and for pointing us to [W]. The nonstandard terminology and notation are our responsibility. The name 'right internal hom' is apt if $z = x$, and 'left internal hom' is apt if $w = y$.

Furthermore, the algebra structure on $\underline{\mathrm{End}}^R(M)$ *(and* $\underline{\mathrm{End}}^L(M)$*) is composition of module functor endomorphisms.*

Proof. All but the final assertion follow from Proposition 2.17 and [DSS, Proposition 2.4.10]. For the algebra structure on $\underline{\mathrm{End}}^R(M)$, the unit $1 \to M^R \circ M$ is the (A, A)-bimodule map

$$A \longrightarrow \mathrm{Hom}_B(M, B) \boxtimes_B M \cong \mathrm{End}_B(M) \tag{2.22}$$

which maps $a \in A$ to right multiplication by a; in particular $1 \in A$ maps to the identity endomorphism of M. Since the counit $M \circ M^R \to 1$ is the evaluation map, the multiplication (2.14) on $\underline{\mathrm{End}}^R(M)$

$$\begin{aligned}\mathrm{End}_B(M) \boxtimes_A \mathrm{End}_B(M) &\cong \mathrm{Hom}_B(M, B) \boxtimes_B \big(M \boxtimes_A \mathrm{Hom}_B(M, B)\big) \boxtimes_B M \\ &\longrightarrow \mathrm{Hom}_B(M, B) \boxtimes_B B \boxtimes_B M \cong \mathrm{End}_B(M)\end{aligned} \tag{2.23}$$

is the usual composition of endomorphisms of M. □

We prove a partial converse to Proposition 2.17 in $\mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$, which for convenience we state for *right* adjoints only.

Proposition 2.24. *Suppose* $A, B \in \mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$ *and* $M : A \to B$ *has a right adjoint* N*. Then* $N \cong \mathrm{Hom}_B(M, B)$ *as* (A, B)*-bimodule categories.*

Proof. The adjunction can be expressed as isomorphisms

$$\mathrm{Hom}_A(X, N \boxtimes_B Y) \xrightarrow{\;\sim\;} \mathrm{Hom}_B(M \boxtimes_A X, Y) \tag{2.25}$$

which are functorial in left A-module categories X and right A-module categories Y. Choose $X = A$ and $Y = B$ to obtain the desired isomorphism $N \xrightarrow{\sim} \mathrm{Hom}_B(M, B)$. The isomorphism intertwines the right A-action on X and the right B-action on Y. □

3. Proofs

3.1. Proof of Theorem B. To begin, assume given a symmetric monoidal 3-category $\mathcal{C}$ such that $\Omega\mathcal{C}^{\mathrm{fd}} \subset \mathrm{Cat}_{\mathbb{C}}$ and a symmetric monoidal functor $\widetilde{F} : \mathrm{Bord}^{\mathrm{fr}}_{3,\partial} \to \mathcal{C}$. The restriction of $\widetilde{F}$ to $\mathrm{Bord}^{\mathrm{fr}}_3$ is denoted F, and we define β as in (1.22).

Diagrams for the extended theory $\widetilde{F} : \mathrm{Bord}^{\mathrm{fr}}_{3,\partial} \to \mathcal{C}$, are drawn according to the rules of Sect. 2.1.6; see Sect. A.3 for more detail. Figure 13 depicts a 1-morphism $b_+ : \emptyset^0 \to +$ in $\mathrm{Bord}^{\mathrm{fr}}_{2,\partial} \subset \mathrm{Bord}^{\mathrm{fr}}_{3,\partial}$ and its right adjoint $b_+^R : + \to \emptyset^0$, the latter constructed according to Sect. A.2.5. The $\widetilde{F}$-image of b_+ is $\beta(+) : 1 \to F(+)$. The cobordism hypothesis with singularities [L1, §4.3] implies that the right adjoint boundary theory $\beta^R : \tau_{\leq 2} F \to 1$ is determined by $F(+)$ and $\beta^R(+) := \widetilde{F}(b_+^R)$, hence the verification that β^R is right adjoint to β proceeds by producing a unit and counit (Fig. 14) for an adjunction between b_+ and b_+^R in $\mathrm{Bord}^{\mathrm{fr}}_{2,\partial}$, and then using their $\widetilde{F}$-images to exhibit $\beta^R(+)$ as the right adjoint of $\beta(+)$ in $\mathcal{C}$.

Definition 3.1. Set

$$\Phi = \widetilde{F}\big(\underline{\mathrm{End}}^R(b_+)\big) \in \Omega\mathcal{C}^{\mathrm{fd}} \subset \mathrm{Cat}_{\mathbb{C}}\,. \tag{3.2}$$

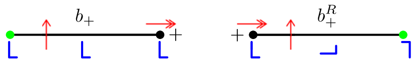


Fig. 13. The 1-morphisms whose $\widetilde{F}$-images are $\beta(+)$ and $\beta^R(+)$

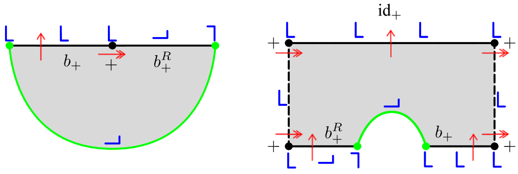


Fig. 14. The unit $1 \to b_+^R \circ b_+$ and counit $b_+ \circ b_+^R \to 1$

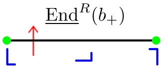


Fig. 15. The category Φ is the $\widetilde{F}$-image of $\underline{\mathrm{End}}^R(b_+) = b_+^R \circ b_+$

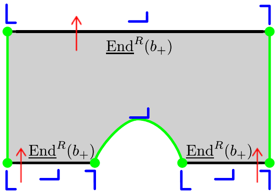


Fig. 16. The monoidal structure of Φ is the $\widetilde{F}$-image of this 2-morphism

See Fig. 15 for a depiction of $\underline{\mathrm{End}}^R(b_+)$. Also, note we can write $\Phi = \underline{\mathrm{End}}^R\big(\beta(+)\big)$. By virtue of being $\underline{\mathrm{End}}^R$ of a 1-morphism, Φ is an algebra object in $\mathrm{Cat}_{\mathbb{C}}$, i.e., Φ is a tensor category. The composition law (2.14) is the $\widetilde{F}$-image of the 2-morphism depicted in Fig. 16.

Proposition 3.3. *Φ is a fusion category.*

The proof of Proposition 3.3 is broken up into Lemmas 3.4 and 3.6.

Lemma 3.4. *Φ is a finite semisimple abelian category.*

The stronger hypothesis of Theorem A, that $\mathrm{Fus} \subset \mathcal{C}^{\mathrm{fd}}$ is a *full* subcategory, immediately implies Lemma 3.4 in view of (1.23). Under the hypotheses of Theorem B we use the following argument.

Proof. The 1-morphism in $\mathrm{Bord}_{2,\partial}^{\mathrm{fr}}$ depicted in Fig. 17 has left boundary colored with β and right boundary colored with β^R. It evaluates under (F, β, β^R) to Φ. Define the *dimensional reduction*

$$\overline{F}\colon \mathrm{Bord}_2^{\mathrm{fr}} \longrightarrow \mathrm{FSCat} \tag{3.5}$$

of F as the theory whose value on any object or morphism in $\mathrm{Bord}_2^{\mathrm{fr}}$ is the value of F on its Cartesian product with the 1-morphism $\emptyset^0 \to \emptyset^0$ in Fig. 17. Since the 2-framing of the latter is induced from a 1-framing, the Cartesian product is naturally equipped with a 3-framing. The lemma now follows since $\overline{F}(+)$ is a 2-dualizable category, hence is finite semisimple abelian by Theorem 1.13. ◻

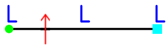


Fig. 17. A 1-framed bordism with boundary theories β on the left and β^R on the right

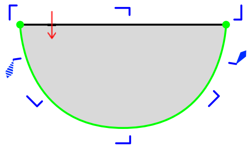


Fig. 18. The right adjoint to the unit in Fig. 14

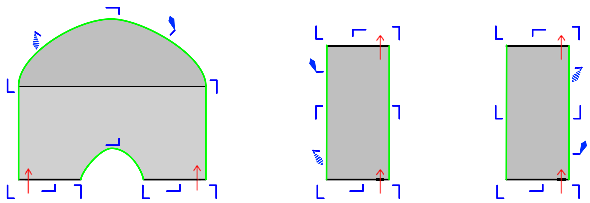


Fig. 19. Multiplication followed by the counit and its duals

Lemma 3.6. Φ *is a rigid monoidal category.*

That is, Φ has internal left and right duals [EGNO, §2.10]. See "Appendix C" for a discussion and generalization of rigidity.

Proof. As a corollary of Lemma 3.4, the dual $\Phi^\vee$ to Φ in FSCat is its opposite category. We prove rigidity by verifying the hypotheses of Theorem C.1.

The unit of Φ is the $\widetilde{F}$-image of the first 2-morphism in Fig. 14. Its right adjoint has $\widetilde{F}$-image the counit ε of (C.3). We implement the prescription of Sect. A.2.5 to compute it. The main concern is the 3-framing which results on the boundary of the hemidisk; it necessarily extends to the interior, and since $\pi_2 SO_3 = 0$ that extension is unique up to isotopy. The result is illustrated in Fig. 18. Recall (Sect. 2.1.6) that the frame vectors are labeled[12] f_0, f_1, f_2; that f_1 is depicted as long, f_2 as short; and that in all previous pictures, such as Fig. 14, the vectors f_1, f_2 lie in the plane of the paper/screen and f_0 is perpendicular to that plane and points into the paper/screen. Now, in Fig. 18, the vectors f_0, f_1 rotate in the plane perpendicular to f_2 as we descend from the incoming boundary. The dashed line in indicates that f_1 points into the paper/screen; the solid wedge in indicates that f_1 points out of the paper/screen. Compose this right adjoint with the multiplication depicted in Fig. 16 to compute the 2-morphism in $\mathrm{Bord}_3^{\mathrm{fr}}$ whose $\widetilde{F}$-image is the pairing B of "Appendix C". It and the 2-morphisms obtained from it by duality (C.5) are depicted in Fig. 19. The Frobenius condition of Definition C.7 is satisfied since the latter two 2-morphisms are invertible in $\mathrm{Bord}_3^{\mathrm{fr}}$.

The right adjoint bordism to Fig. 16, computed following Sect. A.2.5, is depicted in Fig. 20; its $\widetilde{F}$-image is the comultiplication Δ on Φ. Again it suffices to compute the framing on the boundary. Notice that the half-turns in the framing cancel on the vertical colored edges, whereas they cohere into a full turn on the colored half-circle.

[12] The numbering $0, 1, 2$ corresponds to the numbering $-1, -2, -3$ by codimension utilized in "Appendix A".

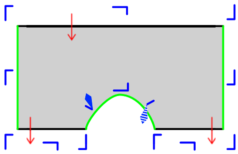


Fig. 20. The right adjoint to Fig. 16

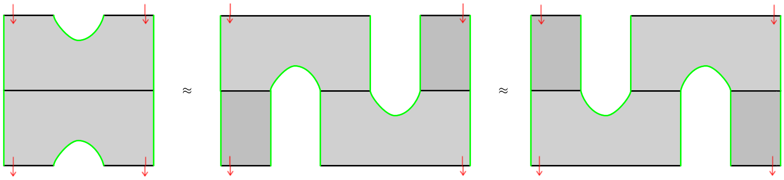


Fig. 21. Comultiplication is a bimodule map

Fig. 22. Duality of + and −: evaluation and coevaluation

The second condition in Theorem C.1, that the comultiplication is a bimodule map, follows immediately from Fig. 21. □

Remark 3.7. In the context of Theorem A, identify the category Φ with its collection of objects $\mathrm{Hom}_{\mathrm{Fus}}(1, \Phi) = \mathrm{Hom}_{\mathcal{C}\mathrm{fd}}(1, \Phi)$. By duality,

$$\mathrm{Hom}_{\mathcal{C}\mathrm{fd}}(1, \Phi) \cong \mathrm{Hom}_{\mathcal{C}\mathrm{fd}}\big(1, \beta^R(+) \circ \beta(+)\big) \cong \mathrm{Hom}_{\mathcal{C}\mathrm{fd}}\big(\beta(+), \beta(+)\big), \tag{3.8}$$

which explains the last statement in Theorem A.

Proof of Theorem B. As pointed out earlier, the forward direction follows from Theorem 1.4. For the converse, first apply the cobordism hypothesis to construct $F\colon \mathrm{Bord}_3^{\mathrm{fr}} \to \mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$ with $F(+) = \Psi$, and then apply the cobordism hypothesis with singularities to construct an extension $\widetilde{F}\colon \mathrm{Bord}_{3,\partial}^{\mathrm{fr}} \to \mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$ whose associated boundary theory $\beta\colon 1 \to \tau_{\le 2}F$ has $\beta(+) = {}_{\Psi}\Psi$, the regular left Ψ-module. The category defined in Definition 3.1 is $\Phi = \underline{\mathrm{End}}^R({}_{\Psi}\Psi)$. By Proposition 2.24 the right adjoint to ${}_{\Psi}\Psi$ is $\mathrm{Hom}_{\Psi}(\Psi, \Psi)$, which may be identified with Ψ_{Ψ}, the regular right Ψ-module. Hence $\Phi = \underline{\mathrm{End}}^R({}_{\Psi}\Psi) \cong \Psi \boxtimes_{\Psi} \Psi \cong \Psi$. Conclude using Proposition 3.3. □

3.2. Proof of Theorem A. The + point and − point (Fig. 5) are duals in $\mathrm{Bord}_2^{\mathrm{fr}}$. Choose duality data as the evaluation $e\colon + \amalg - \to \emptyset^0$ and coevaluation $c\colon \emptyset^0 \to - \amalg +$ 1-morphisms depicted in Fig. 22. One of the "S-diagrams"

$$\Big(+ \xrightarrow{\mathrm{id}\otimes c} + \amalg - \amalg + \xrightarrow{e\otimes \mathrm{id}} +\Big) \xrightarrow{\sim} \Big(+ \xrightarrow{\mathrm{id}} +\Big) \tag{3.9}$$

that proves that $(-, c, e)$ are duality data for + is the 2-morphism of Fig. 23.

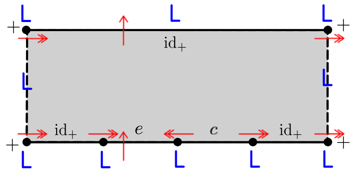


Fig. 23. The S-diagram (3.9)

Fig. 24. The right and left adjoints to e

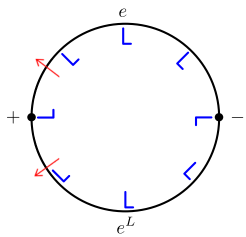


Fig. 25. The isomorphism $S^1_b \cong e \circ e^L$ in $\mathrm{Bord}_2^{\mathrm{fr}}$

The 1-morphism e has right and left adjoints $e^R, e^L\colon \emptyset^0 \to -\amalg +$ depicted in Fig. 24. Their construction follows the general prescription in Sect. A.2.5; see especially Fig. 30. In $\mathrm{Bord}_2^{\mathrm{fr}}$ they are distinct and distinct from coevaluation: $e^R \neq c \neq e^L$. In $\mathrm{Bord}_3^{\mathrm{fr}}$ we have $e^L \cong e^R$ since $\pi_1\,\mathrm{SO}_3 \cong \mathbb{Z}/2\mathbb{Z}$ with generator a full rotation of the frame f_0, f_1, f_2 in the f_1-f_2 plane. In $\mathrm{Bord}_3^{\mathrm{fr}}$ we have an isomorphism

$$S^1_b \cong e \circ e^L = \underline{\mathrm{End}}^L(e) \tag{3.10}$$

illustrated in Fig. 25.

Remark 3.11. In $\mathrm{Bord}_3^{\mathrm{fr}}$ the nonbounding 3-framed circle S^1_n satisfies the isomorphism

$$S^1_n \cong e \circ c. \tag{3.12}$$

Define the 2-framed 0-sphere as $S^0 = +\amalg -$. Note $e\colon S^0 \to \emptyset^0$.

Proposition 3.13. *Let F satisfy the hypotheses of Theorem A. Let Ξ be a fusion category which is isomorphic to $F(S^0)$. Then Ξ is Morita equivalent to $F(S^1_b)$ as fusion categories.*

Recall that a fusion category is *indecomposable* if it is not a nontrivial direct sum. As a preliminary we prove the following.

Lemma 3.14. *Ξ is an indecomposable fusion category.*

Proof. As in Remark 1.16 introduce the double theory

$$|F|^2\colon \mathrm{Bord}_3^{\mathrm{fr}} \longrightarrow \mathrm{Fus} \tag{3.15}$$

characterized by $|F|^2(+) = \Xi \cong F(S^0) = F(+) \otimes F(-) \cong F(+) \otimes F(+)^\vee$. Then by Hypothesis (b) of Theorem A, deduce that $|F|^2(S^1_b) \cong F(S^1_b) \boxtimes F(S^1_b)^{\mathrm{rev}}$ is invertible

Fig. 26. The bordisms b_- and $b_\pm$

as a braided tensor category. (Recall that the reverse of a braided tensor category is the same underlying tensor category equipped with the inverse braiding.) On the other hand, $|F|^2(S^1_b)$ is the Drinfeld center of $|F|^2(+) \cong \Xi$. Since the Drinfeld center of the direct sum of tensor categories is the direct sum of the Drinfeld centers, and a nontrivial direct sum is not invertible, it follows that Ξ is indecomposable. $\square$

Proof of Proposition 3.13. Define

$$M := \Xi \xrightarrow{\sim} F(S^0) \xrightarrow{F(e)} 1, \tag{3.16}$$

where $1 = F(\emptyset) = \mathrm{Vect}_{\mathbb{C}} \in \mathrm{Fus} \subset \mathcal{C}$ is the tensor unit. Since $\mathrm{Fus} \subset \mathcal{C}$ is a *full* subcategory, M is a 1-morphism in Fus. By (3.10) and (2.21) we have the categorical equivalences

$$\begin{aligned} F(S^1_b) &\simeq_{\mathrm{cat}} F\big(\underline{\mathrm{End}}^L(e)\big) \\ &\simeq_{\mathrm{cat}} \underline{\mathrm{End}}^L\big(F(e)\big) \\ &\simeq_{\mathrm{cat}} \mathrm{End}_{\Xi}(M). \end{aligned} \tag{3.17}$$

The last assertion of Corollary 2.19 implies that (3.17) is an equivalence of tensor categories.

To conclude the proof of Proposition 3.13 we must show that M is a *faithful* right Ξ-module. According to the remark after [EGNO, Definition 7.12.9], this follows from Lemma 3.14 since $F(e)$ is nonzero by virtue of being part of the duality data between $F(+)$ and $F(-)$. $\square$

Since Φ is a fusion category, as in (1.5) the cobordism hypothesis produces

$$T_\Phi\colon \mathrm{Bord}_3^{\mathrm{fr}} \longrightarrow \mathrm{Fus}, \tag{3.18}$$

a theory of Turaev–Viro type with $T_\Phi(+) = \Phi$. Then $T_\Phi(S^1_b)$ is the Drinfeld center

$$Z(\Phi) = \mathrm{End}_{T_\Phi(S^0)}(\Phi), \tag{3.19}$$

where $T_\Phi(S^0) = T_\Phi(+) \boxtimes T_\Phi(-) \simeq \Phi \boxtimes \Phi^{\mathrm{mo}}$. (Here Φ^{mo} is the monoidal opposite to the monoidal category Φ, which is its dual in Fus.) We identify

$$T_\Phi \cong \underline{\mathrm{End}}^R(\beta) \tag{3.20}$$

as algebra objects in the 2-category of symmetric monoidal functors $\mathrm{Bord}_2^{\mathrm{fr}} \to \Omega\mathcal{C} = \mathrm{FSCat}$.

Let $b_-\colon \emptyset \to -$ be the morphism dual to b_+^R in $\mathrm{Bord}_{3,\partial}^{\mathrm{fr}}$, obtained by reversing the double-headed arrow; see Sect. 2.1.5 and Fig. 26. Let $b_\pm = b_+ \amalg b_-$. Then the proof of [JS, Proposition 7.10] implies that $\beta(-) \cong \widetilde{F}(b_-)$. Define the composition

$$N\colon 1 \xrightarrow{\widetilde{F}(b_\pm)} F(S^0) \xrightarrow{\sim} \Xi. \tag{3.21}$$

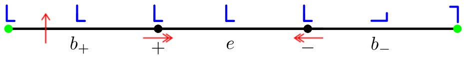


Fig. 27. The equivalence (3.25)

Notice $\widetilde{F}(b_\pm) = \beta(S^0)$. As in the proof of Proposition 3.13, N is a 1-morphism in Fus. Apply (3.20) and (2.21) to deduce an equivalence of tensor categories

$$T_\Phi(S^0) \cong \underline{\mathrm{End}}^R(N) \cong \mathrm{End}_\Xi(N). \tag{3.22}$$

Lemma 3.23. *The left Ξ-module category N provides a Morita equivalence $\Xi \xrightarrow{\sim} T_\Phi(S^0)$.*

Proof. As in the proof of Proposition 3.13, it suffices to show that N is nonzero. But $\beta(S^0) = \beta(+) \otimes \beta(-)$, so if $N = 0$ then so too $\beta(+) = 0$, and then the cobordism hypothesis would imply $\beta = 0$, which contradicts the hypothesis in Theorem A that β is nonzero. □

Lemma 3.24. *There is an equivalence of categories*

$$\Phi \simeq_{\mathrm{cat}} M \boxtimes_\Xi N. \tag{3.25}$$

Proof. The 1-morphisms b_+^R and $e \circ b_-$ are equivalent in $\mathrm{Bord}^{\mathrm{fr}}_{3,\partial}$, as follows from Sect. A.2.5 or directly by inspection of Figs. 13 and 27. Hence $b_+^R \circ b_+$ is equivalent to $e \circ (b_- \amalg b_+)$; compare Figs. 15 and 27. Apply $\widetilde{F}$ to deduce (3.25). □

Since the $\big(\Xi, T_\Phi(S^0)\big)$-bimodule N is invertible—by Lemma 3.23 it induces a Morita equivalence—from Lemma 3.24 we deduce that tensoring with id_N induces a tensor equivalence

$$\alpha\colon \mathrm{End}_\Xi(M) \xrightarrow{\boxtimes\, \mathrm{id}_N} \mathrm{End}_{T_\Phi(S^0)}\big(\Phi\big). \tag{3.26}$$

By (3.17) and (3.19) this is a tensor equivalence

$$\alpha\colon F(S^1_b) \longrightarrow T_\Phi(S^1_b), \tag{3.27}$$

since $T_\Phi(S^1_b)$ is the Drinfeld center of Φ. To complete the proof of Theorem A we need the following.

Lemma 3.28. *α is a braided equivalence.*

Proof. We have already proved that α is a tensor equivalence; it remains to verify the *condition* that α preserve the braiding. Lemma 3.23 states that the $\big(\Xi, T_\Phi(S^0)\big)$-bimodule N induces an isomorphism[13] $\Xi \to T_\Phi(S^0)$ in Fus. By the cobordism hypothesis this induces an isomorphism

$$\theta\colon F^d \xrightarrow{\sim} T^d_\Phi \tag{3.29}$$

[13] Our conventions are that N is a 1-morphism $T_\Phi(S^0) \to \Xi$, but in this proof we use categories of *right* modules rather than left modules, and so the convention applies oppositely.

of topological field theories $\mathrm{Bord}_3^{\mathrm{fr}} \to \mathrm{Fus}$, where $F^d(+) = \Xi$ and $T_\Phi^d(+) = T_\Phi(S^0) \cong \Phi \boxtimes \Phi^{\mathrm{mo}}$. ($F^d$ is essentially the double theory of Remark 1.16.) Let $\widetilde{F}^d \colon \mathrm{Bord}_{3,\partial}^{\mathrm{fr}} \to \mathrm{Fus}$ be the extension of F^d with boundary theory β^d characterized by

$$\beta^d(+) \colon 1 \xrightarrow{\beta(S^0)} F(S^0) \xrightarrow{\sim} \Xi. \tag{3.30}$$

Repeat the arguments above for $\widetilde{F}^d$: introduce $\Xi^d = \Xi \boxtimes \Xi^{\mathrm{mo}}$, the right Ξ^d-module $M^d = F^d(e)$, the left Ξ^d-module $N^d = \beta^d(S^0)$, and the tensor equivalence

$$\alpha^d \colon F^d(S_b^1) \xrightarrow{\boxtimes\, \mathrm{id}_{N^d}} T_\Phi^d(S_b^1). \tag{3.31}$$

We have $M^d = M \boxtimes M'$ and $N^d = N \boxtimes N'$, where the primed Ξ^{mo}-modules are computed in the theory whose value on + is $F(-)$. Hence in the doubled theories (3.25) becomes

$$\Phi \boxtimes \Phi^{\mathrm{mo}} \simeq_{\mathrm{cat}} (M \boxtimes M') \boxtimes_{\Xi \boxtimes \Xi^{\mathrm{mo}}} (N \boxtimes N'), \tag{3.32}$$

and α^d tensors with $\mathrm{id}_{N \boxtimes N'}$. Since tensoring with N is the isomorphism (3.29) of theories, it follows that the induced map on Drinfeld centers is (3.31), i.e., $\alpha^d = \theta(S_b^1)$. Therefore, α^d is a *braided* tensor functor, and then so too is its restriction to

$$F(S_b^1) = F(S_b^1) \boxtimes 1 \subset F(S_b^1) \boxtimes F(S_b^1)^{\mathrm{rev}} \subset F^d(S_b^1). \tag{3.33}$$

This completes the proof of Lemma 3.28, and so too of Theorem A. □

3.3. Proof of Theorem A′. For the remainder of this section we put in force the stronger hypotheses that $\mathcal{C}$ is fusion tensor cocomplete. This allows the following relative composition in terms of the notation of Definition 1.25. If $H \in \mathcal{C}(y, z)$ is a right Φ-module, then define

$$H \circ_\Phi M := (H \circ M) \boxtimes_{\Phi \boxtimes \Phi^{\mathrm{mo}}} \Phi. \tag{3.34}$$

Proof of Theorem A′. By the preceding it suffices to prove that F is isomorphic to T_Φ, and by the cobordism hypothesis it suffices to construct an isomorphism $F(+) \to \Phi$ in $\mathcal{C}$. Consider

$$\Phi \xrightarrow{\Phi_\Phi} 1 \xrightarrow{\beta(+)} F(+), \tag{3.35}$$

where Φ_Φ is the regular right Φ-module. The fusion category $\Phi = \beta(+)^R \circ \beta(+)$ acts on Φ_Φ on the left and on $\beta(+)$ on the right. Define $g \colon \Phi \to F(+)$ and $h \colon F(+) \to \Phi$ as

$$g = \beta(+) \circ_\Phi \Phi_\Phi \tag{3.36}$$

$$h = {}_\Phi\Phi \circ_\Phi \beta(+)^R. \tag{3.37}$$

We claim that g and h are inverse isomorphisms. First,

$$\begin{aligned} h \circ g &= {}_\Phi\Phi \circ_\Phi \beta(+)^R \circ \beta(+) \circ_\Phi \Phi_\Phi \\ &= {}_\Phi\Phi \circ_\Phi \Phi \circ_\Phi \Phi_\Phi \\ &= {}_\Phi\Phi_\Phi \end{aligned} \tag{3.38}$$

is the (Φ, Φ)-bimodule which represents id_Φ. In the other direction,

$$\begin{aligned} g \circ h &= \beta(+) \circ_\Phi \Phi_\Phi \circ {}_\Phi\Phi \circ_\Phi \beta(+)^R \\ &= \beta(+) \circ_\Phi \beta(+)^R \end{aligned} \tag{3.39}$$

as an endomorphism of $F(+)$. Dualize $\beta(+)^R$ to transpose $g \circ h$ to a 1-morphism

$$(g \circ h)^{\mathrm{T}}\colon 1 \longrightarrow F(+) \otimes F(-) = F(S^0), \tag{3.40}$$

and since the dual to $\beta(+)^R$ is $\beta(-)$ we find

$$(g \circ h)^{\mathrm{T}} = \beta(S^0) \boxtimes_{\Phi \boxtimes \Phi^{\mathrm{mo}}} \Phi. \tag{3.41}$$

Now $g \circ h = \mathrm{id}_{F(+)}$ if and only if $(g \circ h)^{\mathrm{T}}$ is the coevaluation of a duality pairing between $F(+)$ and $F(-)$. Recall the coevaluation 1-morphism c in $\mathrm{Bord}_3^{\mathrm{fr}}$ (Fig. 22) which in the theory T_Φ evaluates to $T_\Phi(c) = \Phi\colon 1 \to \Phi \boxtimes \Phi^{\mathrm{mo}}$. Also, $\beta(S^0) = \widetilde{F}(b_\pm)$ is essentially the module N in (3.21). Thus rewrite $(g \circ h)^{\mathrm{T}}$ as the composition

$$(g \circ h)^{\mathrm{T}}\colon 1 \xrightarrow{T_\Phi(c)} T_\Phi(S^0) \xrightarrow{N} \Xi \xrightarrow{\sim} F(S^0). \tag{3.42}$$

Note Lemma 3.23 implies that $N\colon T_\Phi(S^0) \to \Xi$ is an isomorphism. Therefore, (3.41) is the desired coevaluation map and so $g \circ h = \mathrm{id}_{F(+)}$. □

4. Application to Physics

A quantum mechanical system S is *gapped* if its minimum energy is an eigenvalue of finite multiplicity of the Hamiltonian, assumed bounded below, and is an isolated point of the spectrum. This notion generalizes to a relativistic quantum field theory if we understand ‘spectrum’ to mean the spectrum of representations of the translation group of Minkowski spacetime. A basic question:

$$\text{Does a gapped system } S \text{ admit a gapped boundary theory?} \tag{4.1}$$

We argue heuristically that Theorems A and A′ gives an obstruction for certain (2 + 1)-dimensional systems. We remark that the chiral WZW model is a gapless boundary theory for Chern-Simons theory [W2], so at least for these systems a *gapless* boundary theory exists.

We reduce (4.1) to a question in topological field theory by application of the following two heuristic physics principles:

(1) the phase of a quantum system is determined by its low energy behavior;
(2) the low energy physics of a *gapped* quantum system is well-approximated by a topological* field theory.

For now we ignore the ‘*’ in ‘topological*’. Principle (1) seems incontrovertible, though unproved, whereas (2) is more problematic. For example, certain “fracton” lattice systems seem to have no continuum limit as a standard field theory. Nonetheless, (2) appears to hold in many important cases; we simply assume it here. Applying these principles to both the bulk and boundary systems, the general problem (4.1) reduces to a question in topological field theory: Does a topological field theory F admit a nonzero boundary theory β? If not, then the answer to (4.1) is ‘no’. If the topological field theory F does admit a nonzero boundary theory β, then we need a converse to (2) to construct a gapped boundary theory.

Remark 4.2. We suspect that the answer to (4.1) depends only on the *phase* of S, that is, its path component in a putative moduli stack of gapped systems.

We now explain the '*' in 'topological*' by means of an example that is a main focus of interest. The starting point is a quantum field theory, though one can imagine a lattice model in its place. Namely, let S be $(2+1)$-dimensional Yang–Mills theory with a nondegenerate Chern–Simons term. The latter gives the gauge field a mass, which means that the system is gapped. Its low energy physics is thought to be well-approximated by a pure Chern–Simons theory Γ. Observe that S in its Wick-rotated form is a theory of manifolds equipped with an orientation and Riemannian metric. In other words, it is a functor on a *geometric* bordism category of oriented Riemannian manifolds. The naive expectation is that Γ is a functor on the same bordism category, and this is the case. In fact, as discussed by Witten [W1, §2], the dependence on the Riemannian metric is mild: "locally" Γ is the tensor product of a *topological* field theory F and an *invertible* non-topological field theory α_c, where $c \in \mathbb{R}$ is the central charge. More precisely, the pullback of Γ to the bordism category of 3-framed Riemannian manifolds splits as $\Gamma \cong F \otimes \alpha_c$. The theory F is topological—it does not depend on a Riemannian metric. It is an example of an RT theory as described in Sect. 1. The invertible dependence of Γ on the metric through α_c is the '*' in 'topological*'.

The invertible theory α_1 descends to a theory ${\alpha_1}^{\mathrm{SO}}$ with domain the bordism category of *oriented* Riemannian manifolds. Its partition function on a closed oriented Riemannian 3-manifold X is the exponentiated η-invariant $\exp(2\pi i\eta_X/2)$, where $\eta_X \in \mathbb{R}/2\mathbb{Z}$ is the secondary invariant associated to the signature operator [APS]. The deformation class of ${\alpha_1}^{\mathrm{SO}}$ is a generator of the abelian group $[MTSO, \Sigma^4 I\mathbb{Z}]$ of invertible theories, and at least conjecturally it can be constructed using generalized differential cohomology. (We refer to [FH,F2] for notation and details.) The deformation class of the lift α_1 of ${\alpha_1}^{\mathrm{SO}}$ to 3-framed manifolds vanishes, since

$$[MTSO, \Sigma^4 I\mathbb{Z}] \longrightarrow [S^0, \Sigma^4 I\mathbb{Z}] \tag{4.3}$$

is the zero map. In terms of the differential cohomology construction, the equivalence class of α_1 belongs to the subgroup of topologically trivial theories, so is defined by a universal 3-form: one-third the "gravitational Chern–Simons term". Then for any $c \in \mathbb{R}$, the family $\alpha_{tc},\ 0 \le t \le 1$, is an explicit deformation of the trivial theory to α_c. Put differently, it is a "nonflat trivialization" $\beta\colon 1 \xrightarrow{\cong} \tau_{\le 2}\alpha_c$ of the truncation $\tau_{\le 2}\alpha_c$.[14] In other words, α_c is equipped with a boundary theory; compare (1.22). Therefore, topological* boundary theories for Γ correspond to topological boundary theories for F.

Theorems A and A′ give an obstruction to the existence of a nonzero topological boundary theory for F: the theory F must be of Turaev–Viro type. If not, then the heuristics in this section suggest that there are no gapped boundary theories for Yang–Mills plus Chern–Simons, nor for a lattice system meant to represent the same phase. It would be interesting to construct a gapped boundary theory for Yang–Mills plus Chern–Simons in case F is of Turaev–Viro type.

Remark 4.4. One implicit assumption in Principle (2) is that a gapped quantum system exhibits relativistic invariance in the long-range approximation. The Wick-rotated manifestation is the fact that the domain bordism category is made from manifolds whose tangential structure does not break O_3 further than the subgroup SO_3. In particular, a

[14] An example of a nonflat trivialization is a not-necessarily-flat section of a circle bundle with connection. The notion of nonflat trivialization should be part of an axiomatization of *families* of field theories.

3-framing breaks relativistic invariance. Here the 3-framing is introduced to isolate the metric dependence of Γ to the invertible theory α_c; the physically relevant theory has SO_3-invariance.

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Appendix A. Bordism Multicategories

In this appendix we give the precise definitions behind the descriptions in Sect. 2.1 and the pictures throughout Sects. 2 and 3. Complete constructions of the bordism multicategory appear in [CS, AF] among other references. In these approaches an object or morphism is equipped with a *global* map to a cube or stratified ball, and this data is used to define composition laws. Our limited goal here is to define objects and morphisms in Bord_n with minimal data *localized at the boundary*; they too lead to composition laws, though we do not pursue the latter.[15] Underlying a bordism is a *manifold with corners*, so we begin with a quick review in Sect. A.1. Then in Sect. A.2 we specify the additional data required for a morphism in the bordism multicategory. A similar discussion is in [CS, §8.1], based in part on [La]. We incorporate "colored boundaries"—morphisms in $\mathrm{Bord}_{n,\partial}$—in Sect. A.3.

Remark A.1 (Some conventions). In this paper we use *topological* bordism multicategories, but we take inspiration from *geometric* bordism multicategories. In a geometric bordism category—the domain of a *non-topological* field theory—a k-morphism is a k-dimensional compact manifold X with corners which comes equipped with embeddings

$$X^{(0)} \supset X^{(1)} \supset \cdots \supset X^{(n-k)} = X, \tag{A.2}$$

where $\dim X^{(i)} = n - i$ and $X^{(0)}, \ldots, X^{(n-k-1)}$ are *germs* of smooth manifolds. The successive normal line bundles are oriented: these orientations are "arrows of time". The trivial line bundles in the stabilization (A.14) are what remains of this structure in the *topological* bordism category, and their standard orientations are the remnants of the arrows of time into the germs.[16] Our indexing of these trivial line bundles is $-1, -2, \ldots, -(n-k)$, reading from left to right in (A.14) and use the standard orientations.

A word about 'time' and 'space'. In topological field theory, which is modeled on *Wick-rotated* field theory, there is no notion of time versus space: the passage from Lorentz geometry to Euclidean geometry discards the unique time dimension in favor of an additional space dimension. Still, the codimension 1 boundary of an n-manifold in Bord_n plays the role of a spatial slice, hence its normal bundle can reasonably be said to represent time, for example as depicted by the arrows in Fig. 1. In higher codimension, for example the double-headed arrows in Fig. 2, the interpretation as a "time" is only figurative.

Our convention is to order line decompositions of the inflated tangent bundle by codimension from the top dimension of the theory, in order of increasing codimension,

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[15] Nor do we specify collaring data which would give a smooth structure on compositions.

[16] We could instead specify a completion of TX to a flag: $TX \subset E^{(n-k+1)} \subset \cdots \subset E^{(n)}$ and orient the successive quotient line bundles, but we opt for the simpler stabilization (A.14).

and we use the labels $-1, -2, \dots, -n$ for the summands. (See Sects. A.2.2, A.2.3.) *Heuristically*, the first direction is "temporal" and the remaining $n-1$ directions are "spatial".

In the main text we embed $\mathrm{Bord}_2^{\mathrm{fr}} \to \mathrm{Bord}_3^{\mathrm{fr}}$, which facilitates the pictures; see Sect. A.2.6. As far as we know, this does not correspond to anything in physics; it is a convenient mathematical device. The direction we add is "temporal" in our conventions, but again that choice has no physical meaning. By contrast, dimensional reduction—say, along a circle—is effected via a map $\mathrm{Bord}_{n-1} \xrightarrow{\times S^1} \Omega\mathrm{Bord}_n$, and this Cartesian product map is easily checked to be compatible with our labeling conventions.

In Sect. A.3 we bring in "colored" boundaries. They model boundaries in space, not boundaries in time, and so the transverse direction is "spatial"; see (A.35).

A.1. Manifolds with corners. There are several definitions and a long history of the subject of manifolds with corners, both of which are reviewed in Joyce [J]. He develops the theory in detail, and we defer to his paper and the references therein for details.

Fix $k \in \mathbb{Z}^{\geq 0}$. A neighborhood of a point in a smooth k-manifold is modeled by an open set in real affine space $\mathbb{A}^k$. Similarly, a neighborhood of a point in a k-manifold with corners is modeled by an open set in

$$\mathbb{A}^k_{\leq 0} = \Big\{(x^1, \dots, x^k) \in \mathbb{A}^k : x^i \leq 0\Big\}. \tag{A.3}$$

The usual notions of chart and atlas generalize accordingly. A point $x = (x^1, \dots, x^k) \in \mathbb{A}^k_{\leq 0}$ has *depth* $j \in \mathbb{Z}^{\geq 0}$ if precisely j of its coordinates vanish. The depth is invariant under diffeomorphism of open sets in $\mathbb{A}^k_{\leq 0}$, so is a well-defined function

$$\text{depth}\colon M \longrightarrow \mathbb{Z}^{\geq 0} \tag{A.4}$$

on a manifold M with corners. For $j \in \{0, \dots, k\}$, let $\mathring{M}_{-j} \subset M$ denote the $(k-j)$-manifold of points in M of depth j, and let $M_{-j} \subset M$ be the closure of $\mathring{M}_{-j}$. If the maximum value of (A.4) is $d \in \{0, \dots, k\}$, we say M is a *manifold with corners of depth* $\leq d$ and we call d the *depth* of M. If $d = 1$, then M is a *manifold with boundary*. There is a canonical filtering and partition

$$\begin{aligned} M &= M_0 \supset M_{-1} \supset \cdots \supset M_{-d} \\ &= \mathring{M}_0 \amalg \mathring{M}_{-1} \amalg \cdots \amalg \mathring{M}_{-d} \end{aligned} \tag{A.5}$$

A *face* of M is the closure of a component of $\mathring{M}_{-1}$.

The tangent space $T_m M$ to M at $m \in M$ is a k-dimensional real vector space. If m has depth j, then there are j transverse hyperplanes $H_1, \dots, H_j \subset T_m M$ and orientations of the lines $T_m M/H_i$: the positively oriented direction leads out of M.

Variant definitions of 'manifold with corners' include global constraints and/or data in addition to the local normal form. For example, one might require that every point of depth j lie in j distinct faces. The bigon in Fig. 28 satisfies this condition, whereas the teardrop does not. There are more stringent possible global specifications; see [J, Remark 2.11] and the references therein. The extra data we introduce in Sect. A.2 to define a morphism in a bordism multicategory endows the underlying manifold with corners with the data/constraints to be of these more restricted types.

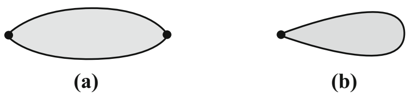


Fig. 28. A bigon (**a**) and a teardrop (**b**)

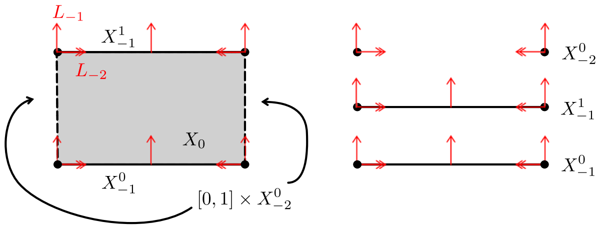


Fig. 29. A 2-morphism of depth 2

There are two distinct notions of the boundary of a manifold with corners. For our purposes we define $\partial M = M_{-1}$ as the closed subset of points of positive depth. This is not generally a manifold with corners, as Fig. 28 illustrates. However, there is a "blow up" which surjects onto ∂M and which is a manifold with corners; see [J, Definition 2.6].

A.2. k-morphisms in Bord$_n$.

A.2.1. The definition Fix $n \in \mathbb{Z}^{>0}$. For $k \in \{0, \dots, n\}$ we specify the data of a k-morphism in Bord$_n$. (For $k = 0$ it is an object in Bord$_n$.) Tangential structures are introduced in Sect. A.2.4.

Definition A.6. Fix n, k as above and suppose $d \in \{0, \dots, k\}$. Let X be a compact k-dimensional manifold with corners of depth $\le d$. The data of a *k-morphism of depth d* on X are:

(i) if $d \ge 1$, closed $(k-d)$-manifolds X^0_{-d}, X^1_{-d}, not both empty;
(ii) if $d \ge 2$, recursively for $j = d-1, d-2, \dots, 1$ compact $(k-j)$-manifolds X^0_{-j}, X^1_{-j} with corners of depth $\le d-j$ equipped with diffeomorphisms

$$\varphi^\delta_{-j}\colon X^0_{-(j+1)} \;\cup\; [0,1]\times\left\{X^0_{-(j+2)} \amalg X^1_{-(j+2)}\right\} \;\cup\; X^1_{-(j+1)} \longrightarrow \partial(X^\delta_{-j}), \quad \delta \in \{0,1\}, \tag{A.7}$$

where the unions are along $\{0\}\times\{X^0_{-(j+2)} \amalg X^1_{-(j+2)}\}$ and $\{1\}\times\{X^0_{-(j+2)} \amalg X^1_{-(j+2)}\}$, respectively;
(iii) if $d \ge 1$, a diffeomorphism

$$\varphi_0\colon X^0_{-1} \;\cup\; [0,1]\times\left\{X^0_{-2} \amalg X^1_{-2}\right\} \;\cup\; X^1_{-1} \longrightarrow \partial X, \tag{A.8}$$

where the unions are along $\{0\}\times\{X^0_{-2} \amalg X^1_{-2}\}$ and $\{1\}\times\{X^0_{-2} \amalg X^1_{-2}\}$, respectively.

Remark A.9. (1) See Fig. 29 for an example of a 2-morphism of depth 2. In that example X^0_{-2} consists of two points, $X^1_{-2} = \emptyset^0$ is the empty 0-manifold, and $X^0_{-1} \approx X^1_{-1}$ are closed intervals.

(2) To interpret (A.7) for $j = d - 1$, set $X^{\delta}_{-(d+1)} = \emptyset$, $\delta \in \{0, 1\}$.
(3) In the categorical interpretation, X is a k-morphism with source and target the empty i-manifold $\emptyset^i$, $i \in \{0, \dots, k-d-1\}$; and source and target i-morphisms $X^0_{-(k-i)}$, $X^1_{-(k-i)}$, respectively, $i \in \{k-d, \dots, k-1\}$.
(4) If $d \geq 1$, the embeddings $\varphi^{\delta}_{-j}, \varphi_0$, $j \in \{1, \dots, d-1\}$, $\delta \in \{0, 1\}$, combine to embeddings

$$\psi^{\delta}_{-j}\colon [0,1]^{j-1} \times X^{\delta}_{-j} \longrightarrow \partial X, \qquad j \in \{1, \dots, d\}, \quad \delta \in \{0,1\}. \tag{A.10}$$

For $j \in \{1, \dots, d-1\}$ let $\mathring{\psi}^{\delta}_{-j}$ denote the restriction of ψ^{δ}_{-j} to $[0,1]^{j-1} \times \mathring{X}^{\delta}_{-j}$, and set $\mathring{\psi}^{\delta}_{d} = \psi^{\delta}_{d}$. Then ∂X is the disjoint union of the images of $\mathring{\psi}^{\delta}_{-j}$, $j \in \{1, \dots, d\}$, $\delta \in \{0, 1\}$. Heuristically, the bordism is "constant" on $\mathring{\psi}^{\delta}_{-j}\big([0,1]^{j-1} \times \{x\}\big)$, $x \in \mathring{X}^{\delta}_{-j}$.
(5) The pictures in Sects. 2 and 3 are of 1- and 2-morphisms of various depths. The images of the embeddings (A.10) for $j = 2$ are depicted as dashed edges, as described in Sect. 2.1.2.

A.2.2. The tangent filtration The structure described in Definition A.6 has a tangential implication. Namely, let X be a k-morphism of depth d, and suppose $x \in \partial X$. Choose the unique j, δ and $t^1, \dots, t^{j-1} \in [0,1]$, $\mathring{x} \in \mathring{X}^{\delta}_{-j}$ such that $x = \mathring{\psi}^{\delta}_{-j}(t^1, \dots, t^{j-1}; \mathring{x})$. Then T_xX has a decreasing filtration

$$T_xX = T_{x,0}X \supset T_{x,-1}X \supset \cdots \supset T_{x,-j}X = T_{\mathring{x}}\mathring{X}_{-j} \tag{A.11}$$

in which

$$T_{x,-i}X = d\mathring{\psi}^{\delta}_{-j}\big(0^{i-1} \oplus \mathbb{R}^{j-i} \oplus T_{\mathring{x}}\mathring{X}_{-j}\big), \qquad i \in \{1, \dots, j-1\}. \tag{A.12}$$

The associated graded is a sum of real lines, which we number by codimension *in the theory*,[17] i.e., count down from n:

$$L_{x,-(n-k+1)} \oplus \cdots \oplus L_{x,-(n-k+j)}. \tag{A.13}$$

Orient $L_{x,-(n-k+j)}$ so that the positive direction leads into X if $\delta = 0$ (incoming/source morphism) and leads out of X if $\delta = 1$ (outgoing/target morphism). Orient $L_{x,-(n-k+i)}$, $i \in \{1, \dots, j-1\}$, so that the positive direction points towards increasing t^{j-i}. The orientations are constant over the image of $\mathring{\psi}^{\delta}_{-j}$. Moreover, Definition A.6(ii) and (iii) ensure that the orientations are consistent as we move among the images of the various $\mathring{\psi}^{\delta}_{-j}$.

A.2.3. The inflated tangent bundle of a k-morphism Define the "inflated tangent bundle"

$$\widetilde{T}X = \underbrace{\underline{\mathbb{R}} \oplus \cdots \oplus \underline{\mathbb{R}}}_{n-k \text{ times}} \oplus TX \longrightarrow X, \tag{A.14}$$

where $\underline{\mathbb{R}} \to X$ is the constant line bundle with fiber $\mathbb{R}$. The orientations of the line bundles in (A.13) and the standard orientations on the $n-k$ trivial line bundles in (A.14) are the "arrows of time" discussed in Sect. 2.1.1. We label them by codimension: $-1, -2, \dots, -(n-k)$.

[17] By contrast, subscripts in X^{δ}_{-j} and $T_{x,-i}X$ are codimensions *in* X, so count down from $k = \dim X$.

Remark A.15. (1) For the 2-morphisms of various depths depicted in Fig. 29 and in Sects. 2, 3, in Bord_2 the single-headed arrows correspond to codimension $i = 1$ and the double-headed arrows correspond to codimension $i = 2$. In Bord_3 the codimensions should be shifted to $i = 2$ and $i = 3$.

(2) Fix $i \in \{1, \ldots, n\}$. Then the i^{th} duality of the $O(1)^{\times n}$-action discussed in Sect. 2.1.5 acts trivially if $i \leq n - k$ and exchanges X^0_{n-k-i} and X^1_{n-k-i} if $i > n - k$. If there is a tangential structure (Sect. A.2.4), then for $i \leq n - k$ the tangential structure is pulled back under the reflection in the inflated tangent bundle (A.14) which reverses the sign on the summand $\underline{\mathbb{R}}_{-i}$.

(3) In Fig. 29 the arrows of time on X^0_{-2}, drawn on the upper right of the figure, carry no meaning; they merely embed the trivial lines in (A.14) in the plane of the figure. Similarly for the single-headed arrow of time in X^δ_{-1}, $\delta \in \{0, 1\}$ on the right hand side of Fig. 29.

(4) Each X^δ_{-j}, $j \in \{1, \ldots, d\}$, $\delta \in \{0, 1\}$, has the structure of a $(k - j)$-morphism Y of depth $d - j$ with $Y^\epsilon_{-i} = X^\epsilon_{-(j+i)}$, $i \in \{1, \ldots, d - j\}$, $\epsilon \in \{0, 1\}$.

A.2.4. Tangential structures Let $\rho_n \colon \mathcal{X}_n \to B\mathrm{GL}_n\mathbb{R}$ be a continuous map. The choice of classifying map $X \to B\mathrm{GL}_n\mathbb{R}$ for the inflated tangent bundle $\widetilde{T}X \to X$ of a k-dimensional manifold X with corners, $k \leq n$, is a contractible choice we assume given. Then a *tangential structure of type ρ_n on X* is a lift of that classifying map to $\mathcal{X}_n$. We can use rigid models instead, such as for orientations, spin structures, or n-framings. An isomorphism $X' \to X$ of manifolds with corners is a diffeomorphism $\Phi \colon X' \to X$ together with a linear isomorphism $\widetilde{T}X' \to \Phi^*\widetilde{T}X$ and a homotopy of the tangential structure on X' to the pullback of the tangential structure on X. If rigid models are employed, the homotopy may be replaced by a more rigid alternative, which may be a combination of conditions and data.

There is a variant of Definition A.6 for tangential structures of type ρ_n: each manifold X^δ_{-j} with corners is equipped with a tangential structure of type ρ_n and the diffeomorphisms $\varphi^\delta_{-j}, \varphi_0$ are lifted to isomorphisms in the sense of the previous paragraph. The tangential structure on $[0, 1] \times Y$ is taken to be that on Y, extended to be constant along the $[0, 1]$-direction. An isomorphism $\Phi \colon X' \to X$ of k-morphisms of depth d is an isomorphism $X' \to X$ of manifolds with corners and tangential structures, and a collection of isomorphisms $(X')^\delta_{-j} \to X^\delta_{-j}$ of manifolds with corners and tangential structures, compatible with $(\varphi')^\delta_{-j}, \varphi^\delta_{-j}$ and φ'_0, φ_0.

A.2.5. Duals and adjoints An object in Bord_n is a finite set of points X; the tangential structure, if present, is on the trivial vector bundle $X \times \mathbb{R}^n$. The dual object $X^\vee$ consists of the same data, but with tangential structure pulled back via reflection $(\xi^1, \ldots, \xi^{n-1}, \xi^n) \mapsto (\xi^1, \ldots, \xi^{n-1}, -\xi^n)$ on $\mathbb{R}^n$. Evaluation and coevaluation morphisms are constructed from $[0, 1] \times X$; see Fig. 22. The dual of a k-morphism is constructed by exchanging X^0_{-k} and X^1_{-k}. More generally, a closed k-manifold X is an object in $\Omega^k \mathrm{Bord}_n$. Its dual has tangential structure pulled back along reflection in $\underline{\mathbb{R}}_{-(n-k)}$ in the inflated tangent bundle.

If $1 \leq k \leq n - 1$, then a k-morphism X in Bord_n has both a right adjoint X^R and a left adjoint X^L. For our purposes in this paper, we restrict to k-morphisms of depth 1:

manifold with boundary (no corners). We now specify data[18] for these objects X^A, where $A = R$ for the right adjoint and $A = L$ for the left adjoint. Let $X^A = X$ as a manifold with boundary. Reverse the arrows of time on the codimension one strata: set $(X^A)^{\delta}_{-1} = X^{1-\delta}_{-1}$ for $\delta \in \{0, 1\}$. Construct a diffeomorphisms φ^A_0 from the corresponding diffeomorphism in the data of X. For unoriented bordisms that is a complete specification of X^A; in particular, right and left adjoints agree. If a tangential structure is present, define a tangential structure on X^A according to the following procedure. Choose collar neighborhoods

$$\begin{aligned} c(X^A)^0_{-1} &\approx \ [0,1) \ \times (X^A)^0_{-1} \\ c(X^A)^1_{-1} &\approx (-1,0] \times (X^A)^1_{-1} \end{aligned} \tag{A.16}$$

and let t be the coordinate in the intervals $[0, 1), (-1, 0]$. In the collar the tangent bundle splits off a trivial line bundle:

$$T\big(c(X^A)^{\delta}_{-1}\big) \cong \underline{\mathbb{R}}_{-(n-k+1)} \ \oplus \ T(X^A)^{\delta}_{-1}. \tag{A.17}$$

Orient the summand $\underline{\mathbb{R}}_{-(n-k+1)}$ according to the *opposite* of the orientation of $L_{-(n-k+1)}$ in (A.13), with which it is identified at $t = 0$. In other words, orient it according to the arrow of time in X^A. Let $\underline{\mathbb{R}}_{-(n-k)}$ denote the trivial summand in the inflated tangent bundle of $(X^A)^{\delta}_{-1}$ that corresponds to codimension $n - k$. Let

$$V = \underline{\mathbb{R}}_{-(n-k)} \ \oplus \ \underline{\mathbb{R}}_{-(n-k+1)} \tag{A.18}$$

with its direct sum orientation; V is a direct summand of the inflated tangent bundle $\widetilde{T}\big(c(X^A)^{\delta}_{-1}\big)$ in the collar. Transport the tangential structure from X to X^A as follows. At $t = 0$ transport via the hyperplane reflection $\mathrm{id} \oplus -\mathrm{id}$ on (A.17): flip the sign on $\underline{\mathbb{R}}_{-(n-k+1)}$. Moving in the collars (A.16) in X^A from $t = 0$ to $t = (-1)^{\delta}/2$ transport via a path of rotations in V which begins at id_V and ends at $-\mathrm{id}_V$ and turns[19]

$$\begin{Bmatrix} \text{clockwise} \\ \text{counterclockwise} \end{Bmatrix} \text{ according as } A = \begin{Bmatrix} R \\ L \end{Bmatrix}. \tag{A.19}$$

For $|t| \geq 1/2$ in the collar and also outside the collar, transport the tangential structure from X to X^A via the hyperplane reflection in the extended tangent bundle which flips the sign on $\underline{\mathbb{R}}_{-(n-k)}$.

Figure 30, a reworking of Fig. 22, illustrates the right and left adjoints of the evaluation map $e\colon +\amalg- \to \emptyset^0$ in $\mathrm{Bord}_2^{\mathrm{fr}}$. In these figures the single-headed red arrows indicate the positive direction in the summand $\underline{\mathbb{R}}_{-2}$, which is necessary for the framing to have meaning in the figure; the double-headed red arrows indicate the orientation of L_{-2}, determined by whether a boundary component is incoming or outgoing. The counterclockwise versus clockwise specification (A.19) can be checked in four adjunctions: e^A as an adjoint of e and e as an adjoint of e^A, each for $A = R, L$.

Remark A.20. A useful isomorphic representative of X^A is obtained via the identity diffeomorphism of X^A, lifted to the inflated tangent bundle $\widetilde{T}X^A$ as the hyperplane reflection which is $-\mathrm{id}$ on $\underline{\mathbb{R}}_{-(n-k)}$. This is illustrated by the diffeomorphisms in Fig. 30, under which both the framings and arrows of time have been transported.

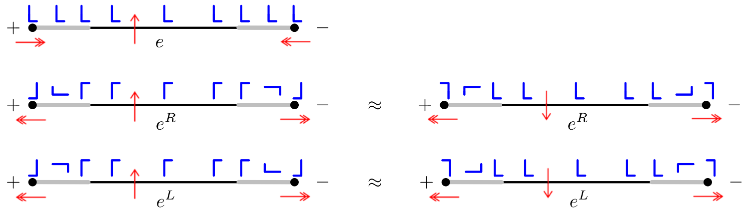


Fig. 30. Right and left adjoints of evaluation in $\mathrm{Bord}_2^{\mathrm{fr}}$

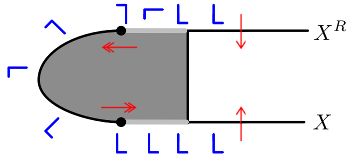


Fig. 31. The first move towards a counit $X \circ X^R \to \mathrm{id}$

The units and counits of the adjunctions may be constructed in two stages, which we now sketch. The first step for the counit $X \circ X^R \to \mathrm{id}$ is illustrated in Fig. 31. Glue $(X^R)^1_{-1}$ to $X^0_{-1} = (X^R)^1_{-1}$ by adjoining a cylinder; then the tangential structures are such that we can push along a 2-disk in the direction of the vector space V in (A.18) to construct a $(k+1)$-dimensional bordism which eliminates the cylinder and the collared neighborhoods of those boundary components. For the second stage, choose a Morse function $f\colon X^R \setminus c(X^R)^1_{-1} \to [0, 1)$ with $f^{-1}(0) = (X^R)^0_{-1}$, and use $2 - f$ as a Morse function on $X \backslash c(X^0_{-1})$. Do surgeries to cancel corresponding critical points and so produce the desired $(k + 1)$-dimensional bordism to the identity k-morphism (cylinder) on $(X^R)^0_{-1}$. See Fig. 14 for an example of a unit and counit, though with trivial second stage.

A.2.6. The inclusion $\mathrm{Bord}_{n-1} \to \mathrm{Bord}_n$ If $k \in \{0, \dots, n-1\}$ and $d \in \{0, \dots, k\}$, then a k-morphism of depth d in Bord_{n-1} is also a k-morphism of depth d in Bord_n; see Definition A.6. The inflated tangent bundle (A.14) in Bord_n has an extra direction, of course. Note that the inclusion $\mathrm{Bord}_{n-1} \to \mathrm{Bord}_n$ increases codimensions in the theory (from the top dimension) by 1. We can define general maps of tangential structures from Bord_{n-1} to Bord_n, as in Sect. A.3.3 below. For the map $\mathrm{Bord}^{\mathrm{fr}}_{n-1} \to \mathrm{Bord}^{\mathrm{fr}}_n$ of framed bordism relevant to this paper, if X is an $(n-1)$-framed k-morphism in Bord_{n-1}, then the induced n-framing on X regarded as a k-morphism in Bord_n appends the standard basis vector on the additional summand $\underline{\mathbb{R}}$ in (A.14). It has label -1.

Remark A.21. The + point in $\mathrm{Bord}^{\mathrm{fr}}_\ell$, $\ell \in \{n-1, n\}$, is the manifold $X = \mathrm{pt}$ with the standard framing on

$$\widetilde{T}X = \underbrace{\underline{\mathbb{R}} \oplus \cdots \oplus \underline{\mathbb{R}}}_{\ell times}. \tag{A.22}$$

[18] The triple consisting of X^A, a unit, and a counit, is unique up to unique isomorphism in an appropriate 2-category truncation of Bord_n. Here we define X^A and only give an indication of the construction of the unit and counit of the adjunctions.

[19] Counterclockwise rotation turns the positive direction in the first summand of (A.18) towards the positive direction in the second summand.

The $-$ point, defined to be dual to the $+$ point in $\mathrm{Bord}_\ell^{\mathrm{fr}}$, has the opposite framing on the *last* summand in (A.22). (The summands are ordered by increasing codimension, so are labeled $-1, -2, \dots, -\ell$.) Then under the inclusion $\mathrm{Bord}_{n-1}^{\mathrm{fr}} \to \mathrm{Bord}_n^{\mathrm{fr}}$ we have $+ \mapsto +$ and $- \mapsto -$: since the extra direction has label -1, so is *prepended* to (A.22), the last of the $n-1$ directions in $\mathrm{Bord}_{n-1}^{\mathrm{fr}}$ maps to the last of the n directions in $\mathrm{Bord}_n^{\mathrm{fr}}$.

Remark A.23. More generally, our conventions about duals and adjoints are preserved under the map $\mathrm{Bord}_{n-1} \to \mathrm{Bord}_n$. This justifies computing adjoints in $\mathrm{Bord}_2^{\mathrm{fr}}$, as in Fig. 13, and using the result as the adjoint in $\mathrm{Bord}_3^{\mathrm{fr}}$. (In these pictures we work in the bordism categories with colored boundaries, where the same holds.)

A.3. k-morphisms in $\mathrm{Bord}_{n,\partial}$.

A.3.1. The definition The bordism multicategory with boundary theory $\mathrm{Bord}_{n,\partial}$ is an extension of Bord_n. The boundary ∂X of a k-morphism of depth d has a distinguished subset B_{-1}, the "colored" subset of Sect. 2.1.6. There are many variations of this construction, which for example allow for multiple boundary theories and domain walls. (We use two boundary theories in the proof of Lemma 3.4.)

Definition A.24. Fix $n \in \mathbb{Z}^{>0}$, $k \in \{0, \dots, n\}$ and $d \in \{0, \dots, k\}$. Let X be a compact k-dimensional manifold with corners of depth $\le d$. The data of a *k-morphism of depth d in* $\mathrm{Bord}_{n,\partial}$ on X are:

(i) if $d \ge 2$, closed $(k-d)$-manifolds $X^0_{-d}, X^1_{-d}, B^0_{-d}, B^1_{-d}$, not all empty;
(ii) if $d \ge 3$, recursively for $j = d-1, d-2, \dots, 2$ compact $(k-j)$-manifolds $X^0_{-j}, X^1_{-j}, B^0_{-j}, B^1_{-j}$ with corners of depth $\le d-j$ equipped with diffeomorphisms

$$\begin{aligned} \varphi^\delta_{-j}\colon X^0_{-(j+1)} \;\cup\; [0,1] \times \left\{X^0_{-(j+2)} \amalg X^1_{-(j+2)}\right\} \;\cup\; X^1_{-(j+1)} \cup\; B^\delta_{-(j+1)} \longrightarrow \partial(X^\delta_{-j}),\\ \beta^\delta_{-j}\colon B^0_{-(j+1)} \;\cup\; [0,1] \times \left\{B^0_{-(j+2)} \amalg B^1_{-(j+2)}\right\} \;\cup\; B^1_{-(j+1)} \longrightarrow \partial(B^\delta_{-j}), \end{aligned} \tag{A.25}$$

for $\delta \in \{0,1\}$;
(iii) if $d \ge 1$, compact $(k-1)$-manifolds $X^0_{-1}, X^1_{-1}, B_{-1}$ with corners of depth $\le d-1$ equipped with diffeomorphisms

$$\begin{aligned} \varphi^\delta_{-1}\colon X^0_{-2} \;\cup\; [0,1] \times \left\{X^0_{-3} \amalg X^1_{-3}\right\} \;\cup\; X^1_{-2} \cup\; B^\delta_{-2} \longrightarrow \partial(X^\delta_{-1})\\ \beta_{-1}\colon B^0_{-2} \;\cup\; [0,1] \times \left\{B^0_{-3} \amalg B^1_{-3}\right\} \;\cup\; B^1_{-2} \longrightarrow \partial(B_{-1}), \end{aligned} \tag{A.26}$$

for $\delta \in \{0,1\}$;
(iv) if $d \ge 1$, a diffeomorphism

$$\varphi_0\colon X^0_{-1} \;\cup\; [0,1] \times \left\{X^0_{-2} \amalg X^1_{-2}\right\} \;\cup\; X^1_{-1} \;\cup\; B_{-1} \longrightarrow \partial X. \tag{A.27}$$

Examples are depicted in Fig. 32.

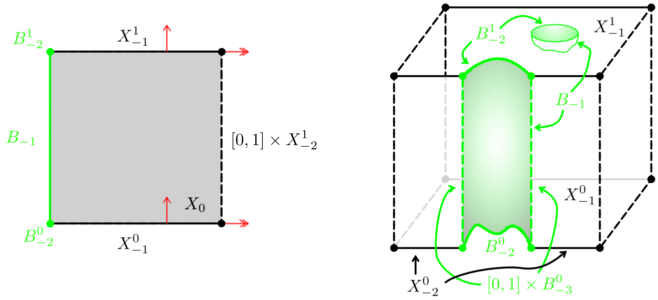


Fig. 32. A 2-morphism of depth 2 and a 3-morphism of depth 3

Remark A.28. As in Remark A.9(2), set $X^\delta_{-(d+1)} = B^\delta_{-(d+1)} = B^\delta_{-(d+2)} = \emptyset$, $\delta \in \{0, 1\}$. The categorical interpretation of the bordism described in Remark A.9(3) is still valid. The embeddings (A.10) still exist, but now the embeddings $\mathring{\psi}^\delta_{-j}$ do not cover ∂X. Rather, the embeddings $\beta^\delta_{-j}, \beta_{-1}$, $j \in \{2, \dots, d-1\}$, $\delta \in \{0, 1\}$, combine to embeddings

$$\gamma^\delta_{-j} \colon [0,1]^{j-2} \times B^\delta_{-j} \longrightarrow B_{-1}, \qquad j \in \{2, \dots, d\}, \quad \delta \in \{0, 1\}. \tag{A.29}$$

Let $\mathring{\gamma}^\delta_{-j}$ be the restriction of γ^δ_{-j} to $[0,1]^{j-2} \times \mathring{B}^\delta_{-j}$. Then ∂X is the disjoint union of three sets: (i) the images of $\mathring{\psi}^\delta_{-j}$, $j \in \{1, \dots, d\}$, $\delta \in \{0, 1\}$; (ii) the images of $\mathring{\gamma}^\delta_{-j}$, $j \in \{2, \dots, d-1\}$, $\delta \in \{0, 1\}$; and (iii) the $(k-1)$-manifold $\mathring{B}_{-1}$.

A.3.2. The tangent filtration For points $x \in \partial X$ in the image of one of the $\mathring{\psi}^\delta_{-j}$, the filtration (A.11) and orientation of the lines (A.13) apply. If $x \in \partial X$ lies in the image of some $\mathring{\gamma}^\delta_{-j}$, choose $t^1, \dots, t^{j-2} \in [0, 1]$, $\mathring{b} \in \mathring{B}^\delta_{-j}$ such that $x = \mathring{\gamma}^\delta_{-j}(t^1, \dots, t^{j-2}; \mathring{b})$. Then $T_x X$ has a decreasing filtration

$$T_x X = T_{x,0} X \supset T_{x,-1} X \supset \cdots \supset T_{x,-(j-1)} X \supset T_{x,-j} X = T_{\mathring{x}} \mathring{B}_{-j} \tag{A.30}$$

in which

$$\begin{aligned} T_{x,-1} X &= T_x B_{-1}, \\ T_{x,-i} X &= d\mathring{\gamma}^\delta_{-j}\big(0^{i-2} \oplus \mathbb{R}^{j-i} \oplus T_{\mathring{x}} \mathring{B}_{-j}\big), \qquad i \in \{2, \dots, j\}. \end{aligned} \tag{A.31}$$

The associated graded is a sum of real lines

$$L_{x,-(n-k+1)} \oplus \cdots \oplus L_{x,-(n-k+j)}. \tag{A.32}$$

Orient $L_{x,-(n-k+j)}$ so that the positive direction leads into X. (This is for a boundary theory $1 \to \tau_{\le 2} F$; for boundary theories $\tau_{\le 2} F \to 1$ choose the opposite orientation.) Orient $L_{x,-(n-k+j-1)}$ so that the positive direction leads into B_{-1} if $\delta = 0$ and leads out of B_{-1} if $\delta = 1$. Orient $L_{x,-(n-k+j)}$, $i \in \{1, \dots, j-2\}$, so that the positive direction points towards increasing t^{j-1-i}. These orientations—arrows of time—can be omitted (as in Sects. 2 and 3) since they can be deduced from the arrows of time on the rest of ∂X.

A.3.3. Tangential structures, duals, and adjoints The distinguished boundary $B_{-1} \subset X$ has a tangential structure of rank $n-1$ whereas the "bulk" $X \backslash B_{-1}$ has a tangential structure of rank n. The former is allowed to be "different" than the latter—for example, we may have a spin boundary theory of an oriented theory. Following Sect. A.2.4, let $\rho_n\colon \mathcal{X}_n \to B\mathrm{GL}_n\mathbb{R}$ be the bulk tangential structure. A *boundary tangential structure* consists of (i) a rank $n-1$ tangential structure $\rho_{n-1}\colon \mathcal{X}_{n-1} \to B\mathrm{GL}_{n-1}\mathbb{R}$; (ii) an inclusion $\mathrm{GL}_{n-1}\mathbb{R} \hookrightarrow \mathrm{GL}_n\mathbb{R}$; and (iii) a map $\phi\colon \mathcal{X}_{n-1} \to \mathcal{X}_n$ such that the diagram

$$\begin{array}{ccc} \mathcal{X}_{n-1} & \xrightarrow{\phi} & \mathcal{X}_n \\ {\scriptstyle\rho_{n-1}}\downarrow & & \downarrow{\scriptstyle\rho_n} \\ B\mathrm{GL}_{n-1}\mathbb{R} & \longrightarrow & B\mathrm{GL}_n\mathbb{R} \end{array} \tag{A.33}$$

commutes. Up to homotopy, (ii) is an element of $\mathrm{O}_n / \mathrm{O}_{n-1} \approx S^{n-1}$, a choice of unit vector in $\mathbb{R}^n$.

Remark A.34. In this paper ρ_n represents n-framings and ρ_{n-1} represents $(n-1)$-framings. Concretely, fix a separable infinite dimensional real Hilbert space $\mathcal{H}$, define the contractible Stiefel manifold $\mathrm{St}_n = \{\text{isometric embeddings } \mathbb{R}^n \to \mathcal{H}\}$, and let the Grassmannian $\mathrm{Gr}_n = \mathrm{St}_n / \mathrm{O}_n$ be the quotient of the Stiefel manifold by the natural right O_n-action. Then $\mathrm{Gr}_n \simeq B\mathrm{GL}_n\mathbb{R}$. Our convention in Sect. 2.1.6 uses the embedding $\mathrm{O}_{n-1} \hookrightarrow \mathrm{O}_n$ induced from the inclusion[20]

$$\begin{aligned} \mathbb{R}^{n-1} &\longrightarrow \mathbb{R}^n \\ (\xi^1, \dots, \xi^{n-1}) &\longmapsto (\xi^1, \dots, \xi^{n-1}, 0) \end{aligned} \tag{A.35}$$

Let $\widetilde{\mathrm{Gr}}_{n-1} = \mathrm{St}_n / \mathrm{O}_{n-1}$. A point of $\widetilde{\mathrm{Gr}}_{n-1}$ is an $(n-1)$-dimensional subspace $W \subset \mathcal{H}$ together with a unit vector $\nu \in W^{\perp}$; the map to the usual Grassmannian Gr_{n-1} which forgets ν is a homotopy equivalence. Let $\rho_{n-1}\colon \mathrm{St}_n \to \widetilde{\mathrm{Gr}}_{n-1}$ be the quotient map and $\phi\colon \mathrm{St}_n \to \mathrm{St}_n$ the identity map.

The constructions of duals and adjoints in Sect. A.2.5 carry over without modification; the colored boundary components are unchanged when forming duals and adjoints.

Appendix B. Semisimplicity of 2-Dualizable Categories

In this appendix we write a proof of the following folk result, stated in the body of the paper as Theorem 1.13 and stated here as Theorem B.1. The theorem concerns 2-dualizable objects in $\mathrm{Cat}_{\mathbb{C}}$; see [BDSV,BJSS] for related variants.

Theorem B.1. *If C is 2-dualizable in $\mathrm{Cat}_{\mathbb{C}}$, then C is finite semisimple abelian, and $C^\vee$ may be identified with C^{op} in such a way that the duality pairing $\langle\,|\,\rangle : C^\vee \times C \to \mathrm{Vect}$ is identified with* Hom*:*

$$\langle x^{\mathrm{op}} | y \rangle = \mathrm{Hom}_C(x, y).$$

The proof is broken up into three lemmas, which we state after introducing the following.

[20] The extra direction at a colored boundary point is "spatial" in the sense of Remark A.1. The choice (A.35) is made so as to be stable under the inclusion $\mathrm{Bord}^{\mathrm{fr}}_{n-1} \to \mathrm{Bord}^{\mathrm{fr}}_n$.

Definition B.2. A $\mathbb{C}$-linear category C is Hom-*finite* if all of its Hom spaces are finite dimensional.

Lemma B.3. *Let C be 1-dualizable in* $\mathrm{Cat}_{\mathbb{C}}$*, with C and $C^\vee$ both* Hom*-finite. There is then an equivalence of linear categories $C^\vee \equiv C^{\mathrm{op}}$ with the duality pairing being*

$$x^{\mathrm{op}} \times y \mapsto \mathrm{Hom}_C(y, x)^\vee.$$

Remark B.4. We do not know ab initio that C^{op} has cokernels; this is a consequence of the Lemma. Thus the proof is executed in the world of all $\mathbb{C}$-linear categories and not within $\mathrm{Cat}_{\mathbb{C}}$, whose objects are finitely cocomplete. In particular, C and C^{op} have kernels and cokernels (i.e. are pre-abelian). One can also show that they must be balanced (all monic epimorphisms are isomorphisms), but we do not know if they must be abelian.

Lemma B.5. *Let C be 2-dualizable in* $\mathrm{Cat}_{\mathbb{C}}$*. Then C and $C^\vee$ are Hom-finite.*

Lemma B.6. *Under the assumptions of Theorem* B.1*, the functor* $\mathrm{Hom}_C \colon C^{\mathrm{op}} \times C \to \mathrm{Vect}$ *is bi-exact.*

In particular, all epimorphisms and monomorphisms are split.

We briefly defer the proofs of Lemmas B.3, B.5, and B.6 in favor of the following.

Proof of Theorem B.1. Lemmata B.3 and B.5 imply that C has kernels and cokernels, and their splitting, from Lemma B.6, implies that C is abelian. Semisimplicity follows from Hom-finiteness and Lemma B.6. Decompose objects using nonzero noninvertible endomorphisms until their endomorphism algebras become division rings. Finiteness of the number of simple isomorphism classes is enforced by 2-dualizability: for example, isomorphism classes of simple objects label a basis of the Hochschild homology space, and the latter is finite dimensional. The remainder of Theorem B.1 requires an identification of the Hom pairing with the vector space dual of its opposite, which is immediate from semisimplicity. □

Now to the lemmas.

Proof of Lemma B.3. We construct a left/right adjoint pair of linear functors

$$L : C^{\mathrm{op}} \rightleftarrows C^\vee : R$$

which we prove to be inverse equivalences. In fact, R is the opposite of the $C^\vee$-counterpart of L, so that $(R^{\mathrm{op}}, L^{\mathrm{op}})$ is the corresponding pair of functors if we start with the category $C^\vee$ instead of C.

Define L as the functor $x^{\mathrm{op}} \to \check{x}$, where given $x \in C$, we define $\check{x} \in C^\vee$ by

$$\langle \check{x} | y \rangle = \mathrm{Hom}_C(y, x)^\vee, \qquad y \in C.$$

Note that $\check{x}$ is a right exact functional on C, so it defines an object of $C^\vee = \mathrm{Hom}_{\mathrm{Cat}_{\mathbb{C}}}(C, \mathrm{Vect}_{\mathbb{C}})$. (Moreover, the assignment $x^{\mathrm{op}} \mapsto \check{x}$ is right exact, although this does not mean much before C^{op} is shown to have cokernels.)

As advertised, R sends $\eta \in C^\vee$ to the object $\check{\eta}^{\mathrm{op}}$ of C^{op}, where $\check{\eta} \in C$ is defined by

$$\langle \xi | \check{\eta} \rangle = \mathrm{Hom}_{C^\vee}(\xi, \eta)^\vee, \qquad \xi \in C^\vee.$$

Now for $x \in C$, $\eta \in C^\vee$ we have the desired adjunction

$$\mathrm{Hom}_{C^{\mathrm{op}}}(x^{\mathrm{op}}, R\eta) = \mathrm{Hom}_C(\check{\eta}, x) = \langle \check{x} | \check{\eta} \rangle^\vee = \mathrm{Hom}_{C^\vee}(\check{x}, \eta) = \mathrm{Hom}_{C^\vee}(Lx^{\mathrm{op}}, \eta). \tag{B.7}$$

The Yoneda embedding asserts that L is fully faithful; a formal consequence is that the adjunction unit $\mathrm{Id}_{C^{\mathrm{op}}} \to R \circ L$ is an isomorphism. Similarly, R^{op}, and therefore R, is also fully faithful, so the evaluation $L \circ R \to \mathrm{Id}_{C^\vee}$ is an isomorphism as well. □

Proof of Lemma B.5. Denote by $\Delta\colon \mathrm{Vect} \to C^\vee \boxtimes C$ the coevaluation of the duality, and let S_C be the Serre automorphism. The second of the adjunctions

$$\Delta^L = \langle\,|\,\rangle \circ (\mathrm{Id} \boxtimes S_C), \quad \Delta^R = \langle\,|\,\rangle \circ (\mathrm{Id} \boxtimes S_C^{-1}), \tag{B.8}$$

combined with the right exactness of $\langle\,|\,\rangle$, shows that the functor

$$\mathrm{Hom}_{C^\vee \boxtimes C}(\Delta(\mathbf{1}); \_) : C^\vee \boxtimes C \to \mathrm{Vect} \tag{B.9}$$

is also right exact. Since $\Delta(\mathbf{1})$ is the quotient of a product $\Xi \boxtimes X \in C^\vee \boxtimes C$—as is any object in $C^\vee \boxtimes C$—the right exactness of (B.9) implies that $\Delta(\mathbf{1})$ must therefore be a direct summand of $\Xi \boxtimes X$, i.e., the image of a projector P in $\mathrm{End}(\Xi \boxtimes X)$. For all $\xi \in C^\vee$ and $x \in C$, this P induces *finite-rank* projectors on all spaces $\mathrm{Hom}(\Xi \boxtimes X; \xi \boxtimes x)$ and $\mathrm{Hom}(\xi \boxtimes x; \Xi \boxtimes X)$, because the respective images are the finite-dimensional spaces

$$\mathrm{Hom}(\Delta(\mathbf{1}); \xi \boxtimes x) = \langle \xi | S_C^{-1} x\rangle, \quad \mathrm{Hom}(\xi \boxtimes x; \Delta(\mathbf{1})) = \langle \xi | S_C x\rangle^\vee.$$

Given now $x, y \in C$, let's compute $\mathrm{Hom}_C(x; y)$ via the Zorro diagram, where we denote by *tr* the transposition in the two variables:

$$\begin{aligned}\mathrm{Hom}_C(x; y) &= \mathrm{Hom}_C\Big(\big[\langle\,|\,\rangle^{tr} \boxtimes \mathrm{Id}\big]\big[x \boxtimes \Delta(\mathbf{1})\big]; y\Big)\\ &= \mathrm{Hom}_{C\boxtimes C^\vee \boxtimes C}\Big(S_C^{-1} x \boxtimes \Delta(\mathbf{1}); \Delta(\mathbf{1})^{tr} \boxtimes y\Big)\end{aligned}$$

In the last step, we have used the first adjunction in (B.8). The last Hom space is the common image of the two commuting projectors acting by pre- and post-composition with P on the space

$$\begin{aligned}&\mathrm{Hom}_{C\boxtimes C^\vee\boxtimes C}(x \boxtimes \Xi \boxtimes X; X \boxtimes \Xi \boxtimes y)\\ &\quad = \mathrm{Hom}_C(x; X) \otimes \mathrm{Hom}_{C^\vee}(\Xi; \Xi) \otimes \mathrm{Hom}_C(X; y)\\ &\quad := U \otimes V \otimes W.\end{aligned}$$

Post-composition with P acts on $U \otimes V$ (and as the identity on W), and its finite rank implies that the image is contained in $F \otimes V \otimes W$, for some finite-dimensional $F \subset U$. But pre-composition by P now acts with finite rank on $V \otimes W$, which proves that $\mathrm{Hom}_C(x, y)$ is finite dimensional.

The Hom-finiteness of $C^\vee$ is proved by a similar argument. □

Proof of Lemma B.6. Let $\Delta\colon \mathrm{Vect} \to C^\vee \boxtimes C$ be the unit for duality. Its right adjoint Δ^R satisfies

$$\Delta^R(X) = \mathrm{Hom}_{C^\vee\boxtimes C}\big(\Delta(\mathbf{1}), X\big), \qquad X \in C^\vee \boxtimes C,$$

which implies Δ^R is left exact. It is also right exact, being a 1-morphism internal to $\mathrm{Cat}_{\mathbb{C}}$. Recall too the formula (B.8) for Δ^R. Now the structural functor $C^\vee \times C \to C^\vee \boxtimes C$, $\xi \times x \mapsto \xi \boxtimes x$, is bi-exact. Following it with Δ^R leads to the bi-exact functor from $C^{\mathrm{op}} \times C \to \mathrm{Vect}$

$$x^{\mathrm{op}} \times y \to \langle \check{x} | S_C^{-1} y\rangle = \mathrm{Hom}(S_C^{-1} y, x)^\vee,$$

which proves the bi-exactness of Hom. □

Appendix C. Internal Duals

We describe here an abstract notion of internal duals, generalizing from a tensor category (Definition C.14) to an algebra object in a 2-category (Theorem C.18). In particular, we show that our TFT F with nonzero boundary condition β leads to a fusion category $\Phi = \underline{\mathrm{End}}^R\big(\beta(+)\big)$ (Definition 3.1). Since our knowledge of Φ comes from TFT calculus, we must avoid unpictorial internal structures (for example, the use of contravariant functors such as $x \mapsto x^*$) in describing internal duality. The main application is

Theorem C.1. *A tensor category whose underlying category is dual to its opposite category and which satisfies the Frobenius condition of Definition C.7 and the bimodule property of Proposition C.13, has internal left and right duals.*

For our Φ, these conditions are checked in Lemma 3.6.

We refer to [BJS] for another discussion of rigidity and dualizability.

Remark C.2. In the setting of TFT, the conditions separate neatly into a Frobenius-bimodule condition and an adjunction condition, reflecting two different geometric properties of a TFT with boundary generated by an algebra object and its regular boundary conditions. The logic of our application to Φ compels a different path; we will return to the more natural statements in a future paper.

Let Φ, $\Phi^\vee$ be a dual pair of objects in a symmetric monoidal 2-category $(\mathcal{M}, \boxtimes)$. We are mostly interested in the *categorical case* when Φ, $\Phi^\vee$ are an opposite couple of linear categories paired by Hom, and can even restrict to semisimple categories, but the algebra below is agnostic about that, unless we explicitly flag it. It is convenient to denote the duality pairing $\Phi^\vee \boxtimes \Phi \to \mathbf{1}$ by writing $\langle \xi \mid y \rangle$, as in the categorical case, when $y \in \Phi$, $\xi \in \Phi^\vee = \Phi^{\mathrm{op}}$ (the opposite category). This convention is symmetric under simultaneous swapping of the arguments and of Φ with $\Phi^\vee$. When checking identities, conversion to the formalism of arrows is straightforward.[21] Equalities stand for canonical isomorphisms of 1-morphisms.

Assume given an E_1 structure on Φ, with strict unit $\eta\colon \mathbf{1} \to \Phi$ and multiplication $\nabla\colon \Phi \boxtimes \Phi \to \Phi$. When Φ is a category and when no confusion ensues, we also write $x \cdot y$ for $\nabla(x, y)$ and 1 for the tensor unit. The dual object $\Phi^\vee$ is a Φ-Φ bimodule. This bimodule is invertible, if Φ is 2-dualizable as an algebra object, and represents then the *Serre autofunctor* of the (category of modules over the) E_1 object Φ.

We shall not adopt the a priori assumption of 2-dualizability here; however, we will require that η and ∇ have right adjoints $\varepsilon\colon \Phi \to \mathbf{1}$ and $\Delta\colon \Phi \to \Phi \boxtimes \Phi$. This condition is always met in the categorical case, with explicit formulas for ε and for the dual functor $\Delta^\vee\colon \Phi^\vee \boxtimes \Phi^\vee \to \Phi^\vee$:

$$\varepsilon(z) = \mathrm{Hom}_\Phi(1, z); \quad \Delta^\vee(x^{\mathrm{op}} \boxtimes y^{\mathrm{op}}) = \nabla(x, y)^{\mathrm{op}}, \quad \text{for } x, y, z \in \Phi. \tag{C.3}$$

With this structure, $\Phi^\vee$ becomes a tensor category with unit $1^{\mathrm{op}} = \varepsilon^\vee(1)$. More generally, the dual object $\Phi^\vee$ is an E_1 object with the same features as Φ: the dual arrow $\nabla^\vee$ defines a comultiplication which is right adjoint to the multiplication $\Delta^\vee$, and the latter has unit $\varepsilon^\vee$, with right adjoint $\eta^\vee$.

Remark C.4. This interpretation of the dual right adjoint of ∇ holds for any functor $\varphi : X \to Y$ between categories which are in duality with their opposites: namely, $\varphi^{\mathrm{op}}\colon X^{\mathrm{op}} \to Y^{\mathrm{op}}$ is $\varphi^{\mathrm{op}} = (\varphi^R)^\vee = \big(\varphi^\vee\big)^L$. In particular, adjoints exist. Recall also that,

[21] At any rate, we can reduce to the categorical case by passing to the functors on $\mathcal{M}$ represented by Φ, $\Phi^\vee$.

when X, Y are 2-dualizable, with (additive) Serre automorphisms S_X^+, S_Y^+, the left and right adjoints of φ are related by $S_Y^+ \circ \varphi^L = \varphi^R \circ S_X^+$. Commuting duals with adjoints will therefore bring out additive Serre functors.

Define now the pairing $B\colon \Phi \boxtimes \Phi \to \mathbf{1}$ as $B = \varepsilon \circ \nabla$. When Φ is a category, $B(x, y) = \mathrm{Hom}_\Phi(1, x \cdot y)$, for two general objects x, y. From B, we define a dual pair of functors, by dualizing separately with respect to each variable:

$$f, f^\vee : \Phi \to \Phi^\vee, \quad f(x) := B(x, -), \quad f^\vee(y) := B(-, y) \tag{C.5}$$

Proposition C.6. *f is a right, and $f^\vee$ a left Φ-module morphism.*

Proof. $\langle f(x \cdot y) \,|\, z\rangle = B\,(x \cdot y, z) = \varepsilon(x \cdot y \cdot z) = B(x, y \cdot z) = \langle f(x) \,|\, y \cdot z\rangle = \langle f(x).y \,|\, z\rangle$, so that $f(x \cdot y) = f(x).y$, and similarly for $f^\vee$. □

Definition C.7. We say that Φ satisfies the *(non-symmetric) adjoint Frobenius condition* when B is a perfect pairing: that is, f and $f^\vee$ are isomorphisms. If so, we define the Serre automorphism of Φ as $S^\otimes = (f^\vee)^{-1} \circ f$.

Proposition C.8. *Assume that Φ satisfies the Frobenius condition. The following natural isomorphisms apply:*

(i) *$B(x, y) = B(y, S^\otimes x)$. In particular, symmetry of B is equivalent to a trivialization of $S^\otimes$.*
(ii) *$f \circ \eta = f^\vee \circ \eta = \varepsilon^\vee$. For a category, $f(1) = f^\vee(1) = 1^{op}$.*
(iii) *As functors $\Phi \to \Phi^\vee$, we have $\varepsilon^\vee.(\_) = S^\otimes(\_).\varepsilon^\vee$. In the categorical case, $1^{op}.x = S^\otimes(x).1^{op}$.*
(iv) *$S^\otimes$ is naturally a tensor automorphism of Φ, and twisting the Φ-action by $S^\otimes$ induces the Serre autofunctor $M \mapsto \Phi^\vee \boxtimes_\Phi M$ on the 2-category of left Φ-modules.*

Remark C.9. Promoting $S^\otimes$ to a tensor functor means equipping it with isomorphisms, compatible with the associativity and unit laws on Φ,

$$S^\otimes \circ \nabla \cong \nabla \circ (S^\otimes \times S^\otimes), \quad S^\otimes \circ \eta \cong \eta;$$

while not evident from the expression $(f^\vee)^{-1} \circ f$, they do follow from Parts (i)–(iii), as in the proof below. On the other hand, reduction of $\Phi^\vee \boxtimes_\Phi \_$ to a tensor automorphism of Φ is a formal consequence of the isomorphy of $f^\vee$.

Proof. Parts (i)–(iii) are immediate from the properties of $f, f^\vee, B$; thus,

$$\begin{aligned} B(x, y) = \langle f(x) \,|\, y\rangle = \langle f^\vee \circ S^\otimes(x) \,|\, y\rangle = \langle f(y) \,|\, S^\otimes(x)\rangle \qquad &\text{for (i)},\\ \langle f(1) \,|\, x\rangle = B(1, x) = \varepsilon(x) = B(x, 1) = \langle f^\vee(1) \,|\, x\rangle \qquad &\text{for (ii)},\\ S^\otimes(x).1^{op} = f^\vee\left(S^\otimes(x)\right) = f(x) = 1^{op}.x \qquad &\text{for (iii)}. \end{aligned}$$

Multiplicativity of $S^\otimes$ now follows:

$$\begin{aligned} f^\vee\left(S^\otimes(xy)\right) &= S^\otimes(xy).1^{op} = 1^{op}.xy = S^\otimes(x).1^{op}.y = S^\otimes(x) \cdot S^\otimes(y).1^{op}\\ &= f^\vee\left(S^\otimes(x) \cdot S^\otimes(y)\right), \end{aligned}$$

using categorical notation for simplicity. To complete (iv), consider the following diagram of right Φ-modules, with left multiplication in the bottom row:

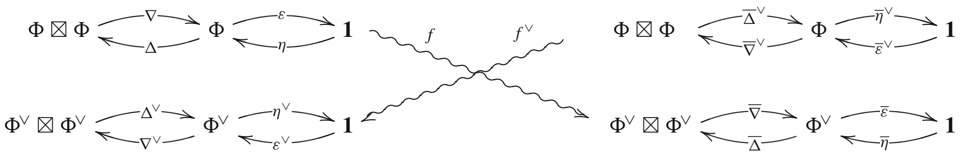
$$\begin{array}{ccc} \Phi \boxtimes \Phi & \xrightarrow{\nabla} & \Phi \\ \downarrow{\scriptstyle S^{\otimes}\boxtimes f} & & \downarrow{\scriptstyle f} \\ \Phi \boxtimes \Phi^{\vee} & \longrightarrow & \Phi^{\vee} \end{array}$$

We claim this commutes naturally. Assuming this, let us interpret the diagram: the right vertical arrow f gives an isomorphism of the identity with the Serre autofunctor on Φ-modules, while the left arrow exhibits the necessary intertwining twist by $S^{\otimes}$ in the left Φ-action.

Exploiting the right Φ-module structure, it suffices to check commutativity on $\Phi \boxtimes \eta$, when this becomes the isomorphism $S^{\otimes}(x).f(1) = f(1).x = f(x)$, from (ii) and (iii). □

Remark C.10. The Serre functor $S^{\otimes}$ above need not agree with the additive Serre automorphism S^{+}_{Φ} of Remark C.4, which is independently defined whenever the object $\Phi \in \mathcal{M}$ is 2-dualizable. However, the two will agree for a fusion category Φ, because of its 3-dualizability. See also Remark C.19 below for a general relation between the two.

The isomorphisms $f, f^{\vee}$ allow us to transport the structure tensors $\eta, \nabla, \Delta, \varepsilon$ to a matching structure on $\Phi^{\vee}$, denoted by overbars. Choosing either f or $f^{\vee}$ results in isomorphic structures on $\Phi^{\vee}$, because all structure tensors commute with $S^{\otimes}$. Dualizing them gives a new structure $\bar{\varepsilon}^{\vee}, \bar{\Delta}^{\vee}, \bar{\nabla}^{\vee}, \bar{\eta}^{\vee}$ on Φ. We get the following diagram, in which the bottom row maps are related to the top row maps by duality and adjunction using uniform rules, $\varepsilon = \eta^{R}$, $\Delta = \nabla^{R}$, $\bar{\varepsilon} = \bar{\eta}^{R}$, $\bar{\Delta} = \bar{\nabla}^{R}$ and all ensuing relations:



(C.11)

The dual corners are related by the morphism $f^{\vee}$. Because $\bar{\eta} = f \circ \eta$, etc., we find from Proposition C.8.ii that

Proposition C.12. *In the diagram above, units and traces match in each row:* $\bar{\eta} = \varepsilon^{\vee}$, $\bar{\varepsilon} = \eta^{\vee}$. □

Proposition C.13. *Under the Frobenius assumption, the following conditions are equivalent:*

(i) *The coproduct Δ is a Φ-Φ bimodule map (for the outer Φ-actions on the two Φ-factors).*
(ii) *The multiplication $\Delta^{\vee}$ is a Φ-Φ bimodule map (for the* inner *Φ-actions on the two $\Phi^{\vee}$-factors).*
(iii) *The two structures on Φ in the top row of* (C.11) *are transpose-isomorphic.*
(iv) *The two structures on $\Phi^{\vee}$ in the bottom row of* (C.11) *are transpose-isomorphic.*

Proof. Parts (i) and (ii) are equivalent by duality, (iii) and (iv) are so via the isomorphisms induced by f. Note further that the diagonal arrow is compatible with the Φ-Φ bimodule structures: on the right, because f is a right module map, an on the left, because we could equally well have used the left module isomorphism $f^\vee$ instead. In light of the matching units, which are free generators of $\Phi^\vee$ over Φ, the bimodule condition determines the multiplication maps and forces the agreement of the remaining structure maps on each row. □

Definition C.14. When Φ is a category, the internal right and left duals ${}^*x, x^*$ of an object $x \in \Phi$ are the objects characterized (up to unique isomorphism, if they exist) by the functorial (in y, z) identities

$$\mathrm{Hom}(x \cdot y, z) = \mathrm{Hom}(y, x^* \cdot z), \quad \mathrm{Hom}(y \cdot x, z) = \mathrm{Hom}(y, z \cdot {}^*x). \tag{C.15}$$

It turns out that the conditions in (C.13) force the existence of internal duals and their expression in terms of $f^\vee$ and f. To see this, we first give an abstract formulation.

Dualizing the product ∇ in the second argument gives the *left multiplication map* λ : $\Phi \to \Phi \boxtimes \Phi^\vee$. In the categorical case, $\lambda(x)$ represents the left multiplication by $x \in \Phi$. Similarly, for the first argument we get the right multiplication map $\rho\colon \Phi \to \Phi \boxtimes \Phi^\vee$. Repeating this for $\Delta^\vee$ leads to the two maps $\lambda', \rho'\colon \Phi^\vee \to \Phi^\vee \boxtimes \Phi$. In the abusive but readable argument notation, with Greek arguments living in $\Phi^\vee$,

$$\langle \lambda'(\xi_1) \mid y \boxtimes \xi_2\rangle = \langle \Delta^\vee(\xi_1, \xi_2) \mid y\rangle, \quad \langle \rho'(\xi_2) \mid y \boxtimes \xi_1\rangle = \langle \Delta^\vee(\xi_1, \xi_2) \mid y\rangle. \tag{C.16}$$

The maps λ', ρ' will be the abstract versions of the 'tensoring with duals'

$$x^{op} \mapsto \big(z \mapsto x^* \cdot z\big), \quad x^{op} \mapsto \big(z \mapsto z \cdot {}^*x\big).$$

Remark C.17. λ', ρ' are related to the op-conjugates $\lambda^{op}, \rho^{op} : \Phi^\vee \to \Phi^\vee \boxtimes \Phi$ as follows:

$$\lambda' \cong (\mathrm{Id} \boxtimes S_\Phi^{+\,-1}) \circ \lambda^{op}, \quad \rho' \cong (\mathrm{Id} \boxtimes S_\Phi^{+\,-1}) \circ \rho^{op}$$

The source of the additive Serre correction S_Φ^+ is described in Remark C.4.

Denote by τ the symmetry $\Phi \boxtimes \Phi^\vee \to \Phi^\vee \boxtimes \Phi$.

Theorem C.18. *The equivalent conditions of Proposition C.13 are also equivalent to:*

(i) $\lambda' \cong \tau \circ \lambda \circ f^{-1}$.
(ii) $\rho' \cong \tau \circ \rho \circ (f^\vee)^{-1}$.
(iii) *In the categorical case: Φ has internal left and right duals.*

Proof. We check the two sides by pairing against a triple of arguments $(\xi_1, y, \xi_2) \in \Phi^\vee \times \Phi \times \Phi^\vee$, leaving to the reader the unenviable task of convert this to identities between morphisms, duals and adjoints. Having written out the left sides in (C.16) above, we start with the right side of (i):

$$\begin{aligned}\langle \tau \circ \lambda \circ f^{-1}(\xi_1) \mid y \boxtimes \xi_2\rangle = \langle \lambda \circ f^{-1}(\xi_1) \mid \xi_2 \boxtimes y\rangle &= \langle \xi_2 \mid f^{-1}(\xi_1) \cdot y\rangle \\ &= \langle \xi_2 . f^{-1}(\xi_1) \mid y\rangle\end{aligned}$$

where the middle line is the definition of λ, while dot represents the right multiplication action of $f^{-1}\xi_1$ upon $\xi_2 \in \Phi^\vee$. Agreement with λ' is then equivalent to

$$\xi_2 . f^{-1}(\xi_1) = \Delta^\vee(\xi_1, \xi_2);$$

but using the right module property of f, we have

$$\xi_2.f^{-1}(\xi_1) = f\left[f^{-1}(\xi_2)\cdot f^{-1}(\xi_1)\right],$$

thus reaching Condition (iv) in Proposition C.13.

Similarly, for the right side of (ii),

$$\begin{aligned}\langle \tau\circ\rho\circ(f^{\vee})^{-1}(\xi_2)\,|\,y\boxtimes\xi_1\rangle &= \langle \rho\circ(f^{\vee})^{-1}(\xi_2)\,|\,\xi_1\boxtimes y\rangle = \langle \xi_1\,|\,y\cdot(f^{\vee})^{-1}(\xi_2)\rangle \\ &= \langle (f^{\vee})^{-1}(\xi_2).\xi_1\,|\,y\rangle\end{aligned}$$

and identity (ii) is equivalent to

$$(f^{\vee})^{-1}(\xi_2).\xi_1 = \Delta^{\vee}(\xi_1,\xi_2),$$

which follows as before, this time from the left-module property of $f^{\vee}$.

Finally, for Part (iii) we must convert the identities into the recognizable form (C.14). For this, we let $\xi_{1,2}$ be opposites of objects $x_{1,2}\in\Phi$; then, $\Delta^{\vee}(\xi_1,\xi_2)$ is the opposite object to $x_1\cdot x_2$, and we can rewrite

$$\begin{aligned}\langle \xi_2\,|\,f^{-1}(\xi_1)\cdot y\rangle &= \mathrm{Hom}_{\Phi}\left(x_2, f^{-1}(\xi_1)\cdot y\right),\\ \langle \xi_1\,|\,y\cdot(f^{\vee})^{-1}(\xi_2)\rangle &= \mathrm{Hom}_{\Phi}\left(x_1, y\cdot(f^{\vee})^{-1}(\xi_2)\right),\\ \langle \Delta^{\vee}(\xi_1,\xi_2)\,|\,y\rangle &= \mathrm{Hom}_{\Phi}\left(x_1\cdot x_2, y\right)\end{aligned}$$

exhibiting $f^{-1}(\xi_1)$ as x_1^* and $(f^{\vee})^{-1}(\xi_2)$ as ${}^*x_2$ in Definition C.14. □

Remark C.19. Under the assumptions of (C.13), and if, in addition, Φ is 2-dualizable, one can prove [FT] that the additive Serre functor S^+_{Φ} is related to $S^{\otimes}$:

$$S^+(x\cdot y) = S^{\otimes}(x)\cdot S^+(y) = S^+(x)\cdot S^{\otimes^{-1}}(y). \tag{C.20}$$

In particular, we have

$$S^+(x) = S^{\otimes}(x)\cdot S^+(1) = S^+(1)\cdot S^{\otimes^{-1}}(x),$$

and, as the functor S^+ is invertible, $S^+(1)$ must be a unit. In the categorical case, $S^{\otimes}(x) = x^{**}$, and the relations follow by applying Serre duality to the adjunction relations in Definition C.14.

One instance of (C.20) is when $S^+ = S^{\otimes} = S^{\otimes^{-1}}$, which happens in the case of fusion categories [DSS], but that is not the only option. Thus, if Φ is the derived category of bounded complexes of coherent sheaves on a projective manifold with the obvious internal duals, the multiplicative Serre functor $S^{\otimes}$ is the identity, while the functor S^+ is tensoring with the canonical line bundle of X in degree $(-\dim X)$.

Appendix D. Complete Reducibility of Fusion Categories

A fusion category whose unit is simple cannot be decomposed as a direct sum, even after passing to a Morita equivalent model: otherwise, we would split the unit. The following converse follows easily from several statements in [EGNO], but we give a complete proof, at the price of rehashing some basic facts. Throughout, Φ will denote a fusion category.

Theorem D.1 (Complete Reducibility). *Φ is Morita equivalent to a direct sum of fusion categories with simple unit.*

Corollary D.2. *Φ is Morita equivalent to a fusion category Φ_0 with simple unit if and only if the Drinfeld center of Φ is invertible.*

We prove Corollary D.2 at the end of the "Appendix".

Remark D.3. A closely related statement is used in [DMNO, Remark 5.2]: if Φ is an indecomposable fusion category, then there exists a fusion category Φ' with simple unit and a braided equivalence $Z(\Phi') \xrightarrow{\simeq} Z(\Phi)$ of the Drinfeld centers. The proof is based on Lemma 3.24 and Corollary 3.35 in [EO].

We break up the proof of Theorem D.1 into small steps. Let $\mathbf{1} = \sum_i p_i$ be the decomposition of the unit of Φ into simple objects. Call an object x *self-adjoint* if it is isomorphic with x^*.

Lemma D.4. *Each p_i is a self-adjoint projector: $p_i^* \cong p_i$, $p_i^2 = p_i$, $\mathrm{End}(p_i) = \mathbb{C}$. In addition, $p_i p_j = 0$ if $i \neq j$.*

Proof. We have $p_i = p_i \cdot \mathbf{1} = \sum_j p_i p_j$, so $p_i p_j = 0$ except for a single j, when it equals p_i. On the other hand, $p_j = \mathbf{1} \cdot p_j = \sum_k p_k p_j$, but the sum contains $p_i p_j = p_i$, so $p_i = p_j$, proving the multiplicative claims. Further, $\mathbf{1}^* = \mathbf{1}$, and $p_i^* p_i \neq 0$ because $\mathrm{Hom}(\mathbf{1}, p_i^* p_i) \cong \mathrm{End}(p_i) \neq 0$, so we must have $p_i^* \cong p_i$. □

Lemma D.4 gives a "matrix decomposition" of Φ as

$$\Phi \cong \bigoplus_{i,j} p_i \cdot \Phi \cdot p_j =: \bigoplus_{i,j} \Phi_{ij},$$

with fusion categories Φ_{ii} having simple units p_i on the diagonal, Φ_{ii}-Φ_{jj} bimodule categories Φ_{ij} (identified with Φ_{ji}^{op} under internal duality), and multiplication compatible with matrix calculus:

$$\Phi_{ij} \times \Phi_{jl} \to \Phi_{il}, \quad \Phi_{ij} \cdot \Phi_{kl} = 0 \text{ if } j \neq k.$$

The equivalence classes of indices generated by the condition $\Phi_{ij} \neq 0$ gives a direct sum decomposition of Φ, matching the block-decomposition of the matrix. Call Φ *indecomposable* if a single block is present. We claim that an indecomposable Φ is Morita equivalent to any of its diagonal entries, selecting Φ_{11} for our argument.

The equivalence is induced by the first row and the first column of Φ: the Φ_{11}-Φ bimodule $R := \bigoplus_i \Phi_{1i}$ and the Φ-Φ_{11} bimodule $C := \bigoplus_j \Phi_{j1}$. We check it in the following two lemmata.

Lemma D.5. *The multiplication map $R \boxtimes_\Phi C \to \Phi_{11}$ is an equivalence of Φ_{11}-Φ_{11} bimodule categories.*

Proof. We have $\Phi \boxtimes_{\Phi} \Phi = \Phi$, and splitting the left factor Φ into its rows R_i and the right factor into its columns C_j gives a direct sum decomposition of Φ as $R_i \boxtimes_{\Phi} C_j$. Examining the action of the projectors p_k, on R_i on the left and on C_j on the right, identifies this with the Φ_{ij} decomposition of Φ. □

Lemma D.6. *The multiplication map* $\mu: C \boxtimes_{\Phi_{11}} R \to \Phi$ *is an equivalence of* Φ-Φ *bimodule categories.*

Lemma D.6 concludes the proof of Theorem D.1.

The proof of this direction requires some preliminary facts.

Lemma D.7 (Linearity of adjoints). *The adjoints* φ^L, φ^R *of an* Φ*-linear map* $\varphi : M \to N$ *between right or left finite semisimple*[22] Φ*-module categories have a natural* Φ*-linear structure.*

Proof. Choosing left modules and the right adjoint, we write a functorial isomorphism

$$\mathrm{Hom}_M\left(m, \varphi^R(x.n)\right) = \mathrm{Hom}_M\left(m, x.\varphi^R(n)\right)$$

by rewriting the left side as

$$\begin{aligned}\mathrm{Hom}_N\left(\varphi(m), x.n\right) &= \mathrm{Hom}_N\left({}^*x.\varphi(m), n\right) = \mathrm{Hom}_M\left(\varphi({}^*x.m), n\right)\\ &= \mathrm{Hom}_M\left({}^*x.m, \varphi^R(n)\right)\end{aligned}$$

and finish by moving x back to the right. The other cases are similar. □

Lemma D.8. *If* $N = \Phi$ *above, then* $\varphi \circ \varphi^L(\mathbf{1})$ *is self-adjoint in* Φ.

Proof. Let $h(x, y) := \dim \mathrm{Hom}(x, y)$. In our semisimple case, $h(x, y) = h(y, x)$, as we are only counting multiplicities of simple objects. Moreover, x is determined up to isomorphism by its multiplicities, so x is self-adjoint iff $h(x, y) = h(x^*, y)$ for all y; the latter is also $h(x, y^*)$. We show this for $x = \varphi \circ \varphi^L(\mathbf{1})$:

$$\begin{aligned}h(x, y^*) &= h(y \cdot x, \mathbf{1}) = h(\varphi \circ \varphi^L(y), \mathbf{1}) = h(\mathbf{1}, \varphi \circ \varphi^L(y))\\ &= h(\varphi^L(\mathbf{1}), \varphi^L(y)) = h(\varphi^L(y), \varphi^L(\mathbf{1})) = h(y, \varphi \circ \varphi^L(\mathbf{1})) = h(x, y).\end{aligned}$$

□

Lemma D.9. *Every self-adjoint projector* ϖ *is isomorphic to a sum of distinct* p_i *from Lemma D.4, i.e., is a direct summand of the unit 1.*

Proof. Let $\varpi = p + x$, where p collects all the p_i appearing in ϖ. Writing the relation $\varpi^2 \cong \varpi$ as

$$p + x \cong p^2 + p \cdot x + x \cdot p + x^2,$$

we see that each p_i appears at most once, otherwise its multiplicity in p^2 exceeds the one in p. Moreover, an isomorphism $x \cong x^*$ gives an identification $\mathrm{Hom}(\mathbf{1}, x^2) = \mathrm{End}(x)$, while $\mathrm{Hom}(\mathbf{1}, x) = 0$ by assumption; comparing left and right sides shows that $\mathrm{End}(x) = 0$ and therefore $x = 0$. □

[22] We only use semisimplicity here to ensure the existence of adjoints.

Proof of Lemma D.6. Writing B for the Φ–Φ bimodule category $C \boxtimes_{\Phi_{11}} R$, Lemma D.5 gives an equivalence $B \boxtimes_\Phi B \cong B$, which is μ-compatible with the identification $\Phi \boxtimes_\Phi \Phi = \Phi$. The left adjoint μ^L is also a bimodule map, by Lemma D.7, and because $\mu \boxtimes_\Phi \mu \cong \mu$ and $\boxtimes_\Phi$ is composition of 1-morphisms in the 3-category Fus, we obtain an equivalence $\mu^L \boxtimes_\Phi \mu^L \cong \mu^L$. Then, $\mu \circ \mu^L$ is an idempotent bimodule endomorphism of Φ, since $\circ$ and $\boxtimes_\Phi$ commute. It is the multiplication by the object $p := \mu \circ \mu^L(\mathbf{1})$—on the left, or on the right—which must then be a projector in Φ. Moreover, p is self-adjoint by Lemma D.8. Lemma D.9 identifies it as a sum of p_i. If $p \ncong \mathbf{1}$, it would split the image $\Phi \cdot p \cong p \cdot \Phi$ as a block of Φ, contradicting indecomposability.

It follows that $\mu \circ \mu^L \cong \mathrm{Id}_\Phi$, splitting B into Φ and a complementary bimodule. But the relation $B \boxtimes_\Phi B \cong B$ can only hold if this complement is zero, so μ is an equivalence. □

Proof of Corollary D.2. First, Morita equivalent fusion categories have braided tensor equivalent Drinfeld centers [EGNO, §8.12]. If Φ_0 is a fusion category with simple unit, then its Drinfeld center $Z(\Phi_0)$ is nondegenerate [EGNO, §8.20]. Therefore, by [S-P, BJSS] the Drinfeld center is invertible. Conversely, by Theorem D.1 any fusion category is Morita equivalent to a finite direct sum of fusion categories with simple unit. Then, as in the proof of Lemma 3.14, the Drinfeld center of a direct sum is the direct sum of the Drinfeld centers, and if the Drinfeld center is invertible, it follows that the direct sum has a single summand. □

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