GLOBAL SOLUTIONS TO MULTI-DIMENSIONAL TOPOLOGICAL EULER ALIGNMENT SYSTEMS

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Abstract. We present a systematic approach to regularity theory of the multi-dimensional Euler alignment systems with topological diffusion introduced in [35]. While these systems exhibit flocking behavior emerging from purely local communication, bearing direct relevance to empirical field studies, global and even local well-posedness has proved to be a major challenge in multi-dimensional settings due to the presence of topological effects. In this paper we reveal two important classes of global smooth solutions – parallel shear flocks with incompressible velocity and stationary density profile, and nearly aligned flocks with close to constant velocity field but arbitrary density distribution. Existence of such classes is established via an efficient continuation criterion requiring control only on the Lipschitz norm of state quantities, which makes it accessible to the applications of fractional parabolic theory. The criterion presents a major improvement over the existing result of [28], and is proved with the use of quartic paraproduct estimates.

1. Introduction

One of the major problems of the mathematical theory of collective behavior is to understand how global phenomena emerge from local interactions between agents. In the context of alignment dynamics such questions were addressed already in the seminal works of Cucker and Smale [10, 11] and studied extensively in [4, 7, 17, 18, 26, 27, 31]. The underlying mechanisms that lie behind most rigorous results in this direction require some type of connectivity of the flock either through the assumption of strong communication at long range (fat-tail kernels) or, if the communication is short range, the graph connectivity at that range, see [13, 36] for other conditional results. Alignment models can be roughly classified into two categories – *metric* ones based on communication kernels which depend only on the Euclidean distance between agents, $\phi(x,y) = \phi(x-y)$, and *topological* ones that use local density of the crowd as a measure of distance. For instance, if

$$d(x,y) = \left(\int_{\Omega(x,y)} \rho(\xi,t)\, \mathrm{d}\xi \right)^{1/n},$$

where $\Omega(x,y)$ is a domain connecting x and y, then a topological model would incorporate d into its communication protocol, $\phi(x,y) = \phi(x-y, d(x,y))$, making it actively dependent upon the evolving density ρ. The underlying principle behind topological models mirrors empirical observations of the actual flock behavior described in the StarFlag project [1, 6, 8]. The probe horizon of a given bird x is determined by the K-nearest neighbors within its detection range. Thus, in thicker crowds the communication radius gets smaller than in thinner ones. In mathematical literature prototypical example of an agent-based topological system was introduced in the work of Haskovec [20], where $\Omega(x,y)$ was assumed to be the ball centered at x of radius $|x-y|$. Kinetic models based on the K-nearest neighbor rule were studied by Blanchet and Degond [2, 3]. In these cases the connectivity of the flock remains an essential assumption to achieve a collective outcome of the system.

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We will be concerned with a macroscopic model given by the hydrodynamic Euler alignment system derived from the agent-based Cucker-Smale system in the work of Ha and Tadmor [19]:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ u_t + u \cdot \nabla u = \displaystyle\int_{\mathbb{T}^n} \phi(x,y)(u(y,t) - u(x,t))\rho(y,t)\,\mathrm{d}y. \end{cases} \tag{1}$$

Here, ϕ stands for the communication protocol, ρ the density of the crowd and u is the velocity field. The choice of the periodic domain $\mathbb{T}^n$ is motivated by avoiding the obvious examples of disconnected flocks dispersing at infinity, see [31], and by focusing on dynamics "in the bulk". In this context we seek to establish alignment of solutions, i.e. vanishing of the velocity variations

$$\mathcal{A}(t) = \max_{x,y \in \mathbb{T}^n} |u(x,t) - u(y,t)| \to 0, \quad t \to \infty.$$

For metric models with smooth fat tail communication, $\phi = \phi(|x-y|)$, $\int_0^\infty \phi(r)\,\mathrm{d}r = \infty$, such a result goes back to the seminal Cucker and Smale works, [10, 11], and has been extended to kinetic and macroscopic systems thereafter, see [31] for detailed exposition and references. For large flocks (1) with purely local communication, the mechanism of exchange of information at long range is not available. So, one resorts to rely on *hydrodynamic connectivity* expressed by a lower bound on the density $\rho(x,t) \geq \overline{\rho}(t)$. The standard methods based on spectral analysis require, roughly, $\rho(\cdot,t) \gtrsim \frac{1}{\sqrt{1+t}}$, see [31, 35, 36]. Such a bound is not known to hold a priori for general non-vacuous solutions, except for the case of global singular metric models in 1D, or under certain threshold conditions in the smooth kernel case, see [14, 32, 33, 34].

In [35] a new class of local topological models was introduced. The domain is assumed to be symmetric $\Omega(x,y) = \Omega(y,x)$, and obtained by translation and dilation of a basic region $\Omega_0 = \Omega(0,\mathbf{e}_1)$. The basic region is assumed to have smooth boundary everywhere but at x, y and fits within the intersection of cones of opening less than π at x and y. Then $\phi(x,y)$ is defined to be a symmetric singular kernel of total degree $n+\alpha$ and with the topological component gauged by a parameter $\tau > 0$,

$$\phi(x,y) = \frac{h(x-y)}{|x-y|^{n+\alpha-\tau} d^\tau(x,y)}, \quad 0 < \alpha < 2. \tag{2}$$

Here, $h = h(r)$ is a smooth radial bump function satisfying

$$\lambda \mathbb{1}_{r<r_0/2} \leq h(r) \leq \Lambda \mathbb{1}_{r<r_0}.$$

Note that for the metric case $\tau = 0$ the action of the kernel is that of the classical (short-range) fractional Laplacian. When $\tau > n$ the power of the density on the bottom supersedes the one on the top in the alignment force (1) creating a mechanism similar to *fast diffusion*.

The main result of [35] states that with the implementation of fast topological diffusion the assumption on connectivity can be weakened. Specifically, if $\tau \geq n$ and $\rho(\cdot,t) \gtrsim \frac{1}{1+t}$ for all $t \geq 0$ one has

$$\mathcal{A}(t) \lesssim \frac{1}{\sqrt{\ln(1+t)}}.$$

Moreover, it is proved that the connectivity holds automatically for any solution with a non-vacuous data in 1D. Thus, the topological model offers an obvious improvement over the metric one.

This result prompted investigation into regularity properties of the system (1)-(2). In fact, for the metric singular models, $\tau = 0$, in 1D the theory was developed earlier in the trilogy of papers [32, 33, 34]. It covers the full range $0 < \alpha < 2$ and with the use of parabolic regularization techniques establishes global well-posedness for classical solutions in

$$u \in H^{m+1}, \quad \rho \in H^{m+\alpha}, \quad m \geq 3. \tag{3}$$

Independently, Do et al. [14], implemented the modulus of continuity approach of Kiselev-Nazarov-Volberg, [23], for the range $0 < \alpha < 1$.

The major advantage of the 1D case over higher dimensions is the presence of an additional conservation law

$$\begin{gathered} e = u_x + \mathcal{L}_\phi \rho, \quad e_t + (u\rho)_x = 0, \\ \mathcal{L}_\phi f(x) = p.v. \int_{\mathbb{T}^1} \phi(x,y)(f(y) - f(x))\, \mathrm{d}y. \end{gathered} \tag{4}$$

This law facilitates a priori control over the higher order norms via the bound $\nabla^k e \lesssim \nabla^k \rho$, which is crucial in establishing propagation of regularity.

In multiple dimensions, the e-equation gains a spectral product term on the right hand side

$$e = \nabla \cdot u + \mathcal{L}_\phi \rho, \quad e_t + \nabla \cdot (u\rho) = (\nabla \cdot u)^2 - \mathrm{Tr}(\nabla u)^2, \tag{5}$$

destroying the conservative structure of the 1D case. Still, several classes of global smooth solutions have been identified. Those include small spectral gap data in 2D by He and Tadmor [21], unidirectional flocks [24], and nearly aligned flocks [12, 30]. Moreover, an effective continuation criterion that requires only control on the gradients $\|\nabla \rho\|_{L^\infty([0,T)\times\mathbb{T}^n)}$ and $\|\nabla u\|_{L^\infty([0,T)\times\mathbb{T}^n)}$ was proved in [24].

For topological systems in dimension 1, the same conservation law (4) holds. Although the kernel $\phi(x,y) = \phi(x-y, d(x,y))$ now depends actively on the density, a similar theory to the metric case was developed in [35]. The main technical complexity lies in understanding regularization properties of the diffusion $\mathcal{L}_\phi$ under a limited a priori smoothness of the kernel.

The multi-dimensional system presents the ultimate challenge even in the context of local theory and has remained largely unexplored. The e-equation in this case includes both the spectral term and an uncanceled topological component,

$$e_t + \nabla \cdot (ue) = (\nabla \cdot u)^2 - \mathrm{Tr}(\nabla u)^2 + \mathcal{T}[\rho, u], \tag{6}$$

where $\mathcal{T}[\rho, u]$ is given by

$$\mathcal{T}[\rho, u] = \mathcal{L}_{\phi_t}(\rho) + \mathcal{L}_{\nabla\phi\cdot}(\rho u), \tag{7}$$

see Section 3.2 for details. This extra component results in a derivative overload on the density which prevents closing the Sobolev bounds classically. Nevertheless, local well-posedness in class (3) for a large $m \in \mathbb{N}$ was established in [28] for the full range $0 < \alpha < 2$ via fractional estimates and using the special structure of $\mathcal{T}$. In order to apply this result to construct global extensions for possibly special classes of solutions an effective continuation criterion is necessary. The result of [28] already comes with a "free" criterion: as long as $u, \rho \in C^2$ the initial regularity class propagates. With more effort the technique can be pushed to reduce the requirement to $u, \rho \in C^\gamma$, where $\gamma = 1 + \varepsilon$ for $\alpha \leq 1$, and $\gamma = \alpha + \varepsilon$ for $\alpha > 1$. However, the parabolic regularization results relevant to the model, see for example [15, 22, 29, 35], provide uniform bounds in a Hölder class $C^{1+\delta}$, at best, for an indeterminate small parameter $\delta > 0$. This is not enough to apply the criterion. So, even though local solutions were constructed in [28] the developed technology was not sufficient to reveal any non-trivial classes of global solutions.

In this work we aim to fill the gaps left in the regularity theory of topological systems (1)-(2) in multi-dimensional settings, and bring its state of the art closer to what is known for metric ones. Our first, and most technical result provides the proper continuation criterion in terms of gradients of u and ρ.

Theorem 1.1. *Let $(u, \rho) \in L^\infty_{\mathrm{loc}}([0,T); H^{m+1} \times H^{m+\alpha})$ be a local non-vacuous solution to (1)-(2) such that*

$$\|\nabla \rho\|_{L^\infty([0,T)\times\mathbb{T}^n)} + \|\nabla u\|_{L^\infty([0,T)\times\mathbb{T}^n)} < \infty. \tag{8}$$

Then the solution can be extended beyond the interval $[0,T]$.

Our basic approach is similar to [28] – to establish a bound of type

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_m \leq C(\|\nabla\rho\|_\infty, \|\nabla u\|_\infty) Y_m,$$

for a grand quantity Y_m which controls the needed Sobolev norms,

$$Y_m \sim \|u\|^2_{\dot{H}^{m+1}} + \|\rho\|^2_{H^{m+\alpha}}.$$

But to extract the sharp gradient components, as opposed to more straightforward second C^2 gradients, our methodology will be much different in two technical aspects. First, we establish new sets of estimates for multilinear singular integral operators, see Lemmas 2.2, 2.3. These allow us to extract ∇u and $\nabla\rho$ in all subcritical terms that appear in handling the topological ingredients in both the e and momentum equations. We also establish sharp coercivity bounds for the topological diffusion operator

$$\|\mathcal{L}_\phi \rho\|_{H^m} \sim \|\rho\|_{H^{m+\alpha}},$$

up to a polynomial factor depending only on the gradients as well. The second aspect has to do with the critical terms that emerge from the topological component $\mathcal{T}$ of the e-equation in the form of quartic products. To achieve control on such terms under the criterion assumption (8) we employ paraproduct estimates that are inspired by the proof of the positive side of the Onsager conjecture for incompressible Euler equation [9], see [5] for the full overview of this subject.

As intended, Theorem 1.1 applies to reveal new classes of global smooth solutions, which we discuss in Section 4. First, is the class of *parallel shear flocks*. It is similar to the unidirectional solutions described in [24] but with incompressible velocity field

$$u = (U(x_2, \dots, x_n, t), 0, \dots, 0).$$

In this case the density is smooth, stationary, and independent of x_1 as well, $\rho = \rho_0(x_2, \dots, x_n)$. The velocity U will be shown to satisfy a fractional parabolic equation which falls under the range of known regularity results of [22, 29]. As a result U gains a uniform Schauder bound in class $C^{1+\gamma}$. Hence, the criterion applies to provide global extension, see Section 4.1.

Second is the class of *nearly aligned flocks* formerly discovered for metric models in [30]. These are solutions with initial velocity amplitude $\mathcal{A}_0$ inversely proportional to the size of the Sobolev norms of the data and lower bound on the density, see Theorem 4.2 for precise formulation. The solutions may have large density profiles, so the class can be viewed as partially small data. We prove that all flocks with nearly aligned data exist globally in time and settle exponentially fast to a *flocking state*, i.e. a traveling wave with constant velocity and smooth density profile

$$u \to \bar{u}, \quad \rho \to \rho_\infty(x - t\bar{u}).$$

Although the results of this present work brings the state of the regularity theory for topological models (1) essentially to the same level of development as for metric ones, it has to be noted that a general global well-posendess theory for both classes remains an outstanding open problem.

2. Preliminaries

In this section we collect all basic properties of the system as well as recall analytical tools, notations, and conventions that will be used in later sections.

2.1. **Notation.** First, we denote all L^p-norms by $\|\cdot\|_p$ for short. The notation $A \lesssim B$ means $A \leq CB$, where C depends only upon absolute constants or a priori bounded quantities such as $\|\nabla\rho\|_{L^\infty([0,T)\times\mathbb{T}^n)}$ and $\|\nabla u\|_{L^\infty([0,T)\times\mathbb{T}^n)}$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

We denote finite differences by

$$\delta_z f(x) = f(x+z) - f(x), \qquad \delta_z^2 f(x) = f(x+z) + f(x-z) - 2f(x).$$

2.2. **Sobolev spaces.** For $1 \leq p < \infty$ and $0 < s < 1$ we adopt the use of the Gagliardo-Sobolevskii fractional Sobolev spaces

$$\|f\|_{W^{s,p}}^p = \|f\|_p^p + \int_{\mathbb{T}^{2n}} \frac{|\delta_z f(x)|^p}{|z|^{n+sp}} h(z) \, \mathrm{d}z \, \mathrm{d}x.$$

For the upper range $1 < s < 2$ one has to use the next Taylor term:

$$\|f\|_{W^{s,p}}^p = \|f\|_p^p + \int_{\mathbb{T}^{2n}} \frac{|\delta_z f(x) - z\nabla f(x)|^p}{|z|^{n+sp}} h(z) \, \mathrm{d}z \, \mathrm{d}x.$$

And for the extended range $0 < s < 2$ including the integer value $s = 1$ one can define the Sobolev space using second finite difference:

$$\|f\|_{W^{s,p}}^p = \|f\|_p^p + \int_{\mathbb{T}^{2n}} \frac{|\delta_z^2 f(x)|^p}{|z|^{n+sp}} h(z) \, \mathrm{d}z \, \mathrm{d}x.$$

The L^2-based spaces will be denoted by $H^s = W^{2,s}$.

In the course of the proof we encounter finite differences with respect to a parameter ξ depending on z and the communication domain at question. Let us recall from [28] the change of variables

$$\int_{\Omega(0,z)} f(\xi) \, \mathrm{d}\xi = |z|^{n-1} \int_{\partial\Omega_0} f(|z| U_z \theta) \, \mathrm{d}\theta,$$

where $\Omega_0 = \Omega(0, \mathbf{e_1})$ is the basic communication domain connecting the origin with the first basis vector $\mathbf{e_1}$, and $U_z : \mathbb{R}^n \to \mathbb{R}^n$ is a unitary transformation sending $\mathbf{e_1}$ to $z/|z|$, and hence $\Omega_0 \to \Omega(0, z)$. We often keep the same notation $\xi = |z| U_z \theta$ for the variable of integration keeping in mind that $\xi = \xi(z, \theta)$. We have the following inequality for any function ξ satisfying $|\xi| \leq |z|$:

$$\int_{\mathbb{T}^{2n}} \frac{|f(x+\xi) - f(x)|^2}{|z|^{n+2s}} h(z) \, \mathrm{d}z \, \mathrm{d}x \lesssim \|f\|_{H^s}^2, \quad 0 < s < 1. \tag{9}$$

Indeed, by the Parseval identity,

$$\int_{\mathbb{T}^{2n}} \frac{|f(x+\xi) - f(x)|^2}{|z|^{n+2s}} h(z) \, \mathrm{d}z \, \mathrm{d}x = \sum_{k \in \mathbb{Z}^n} |\hat{f}(k)|^2 \int_{\mathbb{T}^n} |e^{i\xi \cdot k} - 1|^2 \frac{h(z)}{|z|^{n+2s}} \, \mathrm{d}z.$$

Given that $|e^{i\xi\cdot k} - 1|^2 \lesssim \min\{1, |z|^2 |k|^2\}$, the integral in z is bounded by $|k|^{2s}$ and the result follows.

2.3. **Basic properties of the system. Local well-posedness.** A detailed discussion of the properties of the system (1)–(2) is presented in [35]. We recall a few that are needed for our future analysis. First, any smooth solution obeys the maximum principle

$$\mathcal{A}(t) \leq \mathcal{A}_0.$$

The system is invariant under Galilean transformation

$$x \to x - t\bar{u}, \quad u \to u - \bar{u}.$$

Due to continuity and symmetry of the kernel, solutions preserve mass and momentum

$$M = \int_{\mathbb{T}^n} \rho \, \mathrm{d}x, \quad P = \int_{\mathbb{T}^n} \rho u \, \mathrm{d}x.$$

In view of the above the mean velocity $\bar{u} = P/M$ is preserved, and we can assume that $\bar{u} = 0$ by modding it out.

We have the following energy law for smooth solutions

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} \int_{\mathbb{T}^n} \rho |u|^2 \, \mathrm{d}x, \\ \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E} &= - \int_{\mathbb{T}^{2n}} \rho(x)\rho(y) |u(x) - u(y)|^2 \phi(x,y) \, \mathrm{d}y \, \mathrm{d}x. \end{aligned} \tag{10}$$

Finally, we recall the local well-posedness result proved in [28].

Theorem 2.1. *Let $0 < \alpha < 2$ and $\tau \geq 0$. For any initial data $u_0 \in H^{m+1}(\mathbb{T}^n)$, $\rho_0 \in H^{m+\alpha}(\mathbb{T}^n)$, $m \geq m(\alpha, n)$, with no vacuum $\rho_0(x) > 0$ there exists a unique non-vacuous solution to the system (1)-(2) on a time interval $[0, T)$ where T depends on the initial conditions, in the class*

$$u \in C_w([0,T), H^{m+1}) \cap L^2([0,T), H^{m+1+\frac{\alpha}{2}}),$$
$$\rho \in C_w([0,T), H^{m+\alpha}). \tag{11}$$

Here, C_w stands for weakly continuous functions.

2.4. **Gagliardo-Nirenberg inequalities.** For a function $f \in H^{s+1} \cap W^{1,q}$, $s \geq 0$, recall the classical Gagliardo-Nirenberg inequalities

$$\|f\|_{W^{j+1,p}} \leq \|f\|_{H^{s+1}}^{\theta} \|\nabla f\|_q^{1-\theta},$$

where

$$\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{2} - \frac{s}{n}\right)\theta + \frac{1-\theta}{q}, \qquad \frac{j}{s} \leq \theta \leq 1. \tag{12}$$

We will be interested in placing the smallest possible power $\theta = \frac{j}{s}$ onto the highest norm H^{s+1} without care about the resulting q, because eventually we simply replace $\|\nabla f\|_q \leq \|\nabla f\|_\infty$, which under our assumption will always be bounded a priori. However, one still needs to ensure that such a q exists within the allowed range $1 \leq q \leq \infty$. For this purpose let us set $\theta = \frac{j}{s}$ in (12) and obtain

$$\frac{1}{p} = \frac{j}{2s} + \frac{1}{q}\left(1 - \frac{j}{s}\right).$$

Consequently, such a q exists if and only if

$$\frac{j}{2s} \leq \frac{1}{p} \leq 1 - \frac{j}{2s}, \tag{13}$$

and we have (adopting the convention for $\lesssim$)

$$\|f\|_{W^{j+1,p}} \lesssim \|f\|_{H^{s+1}}^{\frac{j}{s}}. \tag{14}$$

In all the situations we encounter, $p \geq 2$, so the right hand side of (13) will be automatically satisfied.

2.5. **Paraproducts.** In the product estimates of the e-equation we will utilize paraproduct decompositions. The classical Littlewood-Paley decomposition is given by the series

$$f = \sum_{q=0}^{\infty} f_q,$$

where f_q denotes the Littlewood-Paley projection onto the qth dyadic shell in Fourier space, see [16]. For any $q \in \mathbb{N}$ we also denote

$$f_{<q} = \sum_{p<q} f_p, \qquad f_{\sim q} = \sum_{q-2 \leq p \leq q+2} f_p.$$

Let us denote the frequency parameters by $\lambda_q = 2^q$. Recall that

$$\|f\|_{H^s}^2 \sim \sum_q \lambda_q^{2s} \|f_q\|_2^2, \qquad s \geq 0.$$

Any triple product can be decomposed into the Bony paraproduct formula:

$$\langle f, g, h \rangle := \int_{\mathbb{T}^n} fgh \, \mathrm{d}x = LHH + HLH + HHL,$$

where

$$LHH = \sum_q \sum_{p>q-1} \langle f_q, g_{\sim p}, h_{\sim p}\rangle,$$
$$HLH = \sum_q \langle f_q, g_{<q}, h_{\sim q}\rangle,$$
$$HHL = \sum_q \langle f_q, g_{\sim q}, h_{<q}\rangle.$$

We will encounter further decompositions into quartic paraproducts if one of the terms is a product of two functions, $h = h'h''$. For that purpose we note two identities

$$(h'h'')_{<q} = (h'_{<q+2}h''_{<q+2})_{<q} + \sum_{r>q+1} (h'_{\sim r}h''_{\sim r})_{<q}, \tag{15}$$

$$(h'h'')_{\sim q} = (h'_{<q}\, h''_{\sim q})_{\sim q} + (h'_{\sim q}\, h''_{<q})_{\sim q} + \sum_{r>q-2} (h'_{\sim r}\, h''_{\sim r})_{\sim q}. \tag{16}$$

Finally, we recall the classical commutator estimate which we will use repeatedly,

$$\|\partial^m(fg) - f\partial^m g\|_2 \le \|\nabla f\|_\infty \|g\|_{\dot H^{m-1}} + \|f\|_{\dot H^m}\|g\|_\infty, \tag{17}$$

and the product formula

$$\|\partial^m(fg)\|_2 \le \|f\|_{H^m}\|g\|_\infty + \|f\|_\infty\|g\|_{H^m}, \tag{18}$$

both of which can be easily obtained via the Bony decomposition.

2.6. **The Faa di Bruno formula.** We will make repeated use of the Faa di Bruno expansion formula for a multiple derivative of a composite function

$$\partial^P h(g) = \sum_{\mathbf{j}} \frac{P!}{j_1!1!^{j_1} j_2!2!^{j_2} \ldots j_P!P!^{j_P}} h^{(j_1+\ldots+j_P)}(g) \prod_{k=1}^{P} \left(\partial^k g\right)^{j_k},$$

where the sum is over all P-tuples of non-negative integers $\mathbf{j} = (j_1, \ldots, j_P)$ satisfying

$$1j_1 + 2j_2 + \ldots + Pj_P = P.$$

More often we will not need to know the breakdown of repeated derivatives in the product as long as the total order adds up to P:

$$\partial^P h(g) = \sum_{\mathbf{j}} C_{\mathbf{j}} h^{|\mathbf{j}|}(g) \prod_{i=1}^{|\mathbf{j}|} \partial^{k_i} g, \qquad k_1 + \cdots + k_{|\mathbf{j}|} = P. \tag{19}$$

2.7. **Subcritical product estimates.** In what follows we encounter many subcritical terms which take on the standard forms described in the lemmas below.

Lemma 2.2. *Consider the singular integral*

$$I(x) = \int_{\mathbb{T}^n} |\partial^{l_1} g_1(x+\eta_1) \ldots \partial^{l_M} g_M(x+\eta_M)||\delta_{\xi_1}\partial^{k_1} f_1(x) \ldots \delta_{\xi_N}\partial^{k_N} f_N(x)| \frac{h(z)}{|z|^{n+\alpha-2+N}}\, dz,$$

where $|\xi_i(z)| \le |z|$, $\eta_j = \eta_j(z)$, *and*

$$l_1 + \ldots + l_M + k_1 + \ldots + k_N \le m, \quad N, M \ge 0. \tag{20}$$

Then

$$\|I\|_2 \le C \prod_{j=1}^{M} \|g_j\|_{H^{m+\alpha}}^{\alpha_j} \times \prod_{i=1}^{N} \|f_i\|_{H^{m+\alpha}}^{\beta_i},$$

for some $\alpha_j, \beta_i \geq 0$,

$$\alpha_1 + \ldots + \alpha_M + \beta_1 + \ldots + \beta_N < 1,$$

where C *depends only on the Lipschitz norms of* g_j*'s and* f_i*'s.*

Proof. We can assume without loss of generality that all $k_i > 0$. Indeed, for those that are equal to zero, we replace the finite difference by the gradient thereby lowering N, still satisfying (20).

If after this $N = M = 0$, then the lemma is trivial. Otherwise, if we still have $\alpha - 2 + N < 0$, then the singularity is integrable, and we estimate

$$|I(x)|^2 \leq C \int_{\mathbb{T}^n} |\partial^{l_1} g_1(x+\eta_1) \ldots \partial^{l_M} g_M(x+\eta_M)|^2 |\delta_{\xi_1} \partial^{k_1} f_1(x) \ldots \delta_{\xi_N} \partial^{k_N} f_N(x)|^2 \frac{h(z)}{|z|^{n+\alpha-2+N}} \,\mathrm{d}z.$$

Let us define

$$p_i = \frac{2m}{k_i}, \qquad q_j = \frac{2m}{l_j}.$$

Then by the Hölder ineqiality,

$$\|I\|_2 \leq \|g_1\|_{W^{l_1,q_1}} \ldots \|g_M\|_{W^{l_M,q_M}} \times \|f_1\|_{W^{k_1,p_1}} \ldots \|f_N\|_{W^{k_N,p_N}}.$$

Applying the Gagliardo-Nirenberg inequality (14) we obtain

$$\|I\|_2 \leq \|g_1\|_{H^{m+\alpha}}^{\alpha_1} \ldots \|g_M\|_{H^{m+\alpha}}^{\alpha_M} \times \|f_1\|_{H^{m+\alpha}}^{\beta_1} \ldots \|f_N\|_{H^{m+\alpha}}^{\beta_N},$$

where

$$\alpha_j = \frac{l_j - 1}{m+\alpha-1}, \quad \beta_i = \frac{k_i - 1}{m+\alpha-1}.$$

These exponents add up to $\frac{m-N-M}{m+\alpha-1} < 1$, as desired, because $N + M \geq 1$.

If $\alpha - 2 + N \geq 0$, let us fix a small $\delta > 0$ to be determined later and define

$$s = \frac{\alpha - 2 + N + 2\delta}{N}.$$

If δ is small enough this exponent satisfies $0 < s < 1$. Let us now distribute the singularity as follows

$$|I(x)| = \int_{\mathbb{T}^n} \frac{|\delta_{\xi_1} \partial^{k_1} f_1(x)|}{|z|^{\frac{n}{p_1}+s}} \ldots \frac{|\delta_{\xi_N} \partial^{k_N} f_N(x)|}{|z|^{\frac{n}{p_N}+s}} \frac{|\partial^{l_1} g_1(x+\eta_1)|}{|z|^{\frac{n}{q_1}-\frac{\delta}{M}}} \ldots \frac{|\partial^{l_M} g_M(x+\eta_M)|}{|z|^{\frac{n}{q_M}-\frac{\delta}{M}}} \frac{h(z)}{|z|^{\frac{n}{2}-\delta}} \,\mathrm{d}z,$$

and as before apply the Hölder inequality,

$$\begin{aligned}|I(x)| \leq &\left(\int_{\mathbb{T}^n} \frac{|\delta_{\xi_1} \partial^{k_1} f_1(x)|^{p_1}}{|z|^{n+sp_1}} h(z)\,\mathrm{d}z\right)^{\frac{1}{p_1}} \ldots \left(\int_{\mathbb{T}^n} \frac{|\delta_{\xi_N} \partial^{k_N} f_N(x)|^{p_N}}{|z|^{n+sp_N}} h(z)\,\mathrm{d}z\right)^{\frac{1}{p_N}} \\ &\times \left(\int_{\mathbb{T}^n} \frac{|\partial^{l_1} g_1(x+\eta_1)|^{q_1}}{|z|^{n-\delta q_1}} h(z)\,\mathrm{d}z\right)^{\frac{1}{q_1}} \ldots \left(\int_{\mathbb{T}^n} \frac{|\partial^{l_M} g_M(x+\eta_M)|^{q_M}}{|z|^{n-\delta q_M}} h(z)\,\mathrm{d}z\right)^{\frac{1}{q_M}}.\end{aligned}$$

Thus,

$$\begin{aligned}\|I\|_2 &\leq \|g_1\|_{W^{l_1,q_1}} \ldots \|g_M\|_{W^{l_M,q_M}} \times \|f_1\|_{W^{k_1+s,p_1}} \ldots \|f_N\|_{W^{k_N+s,p_N}} \\ &\lesssim \|g_1\|_{H^{m+\alpha}}^{\alpha_1} \ldots \|g_M\|_{H^{m+\alpha}}^{\alpha_M} \|f_1\|_{H^{m+\alpha}}^{\beta_1} \ldots \|f_N\|_{H^{m+\alpha}}^{\beta_N},\end{aligned}$$

where

$$\alpha_j = \frac{l_j - 1}{m+\alpha-1}, \quad \beta_i = \frac{k_i - 1 + s}{m+\alpha-1}.$$

Clearly, $\alpha_1 + \cdots + \beta_N < 1$, as desired. □

We can prove a similar estimate for a more singular integral if one of the integrands can absorb $m + 1 + \frac{\alpha}{2}$ derivatives.

Lemma 2.3. *Consider the singular integral*

$$II(x) = \int_{\mathbb{T}^n} |\partial^{l_1} g_1(x+\eta_1)\dots\partial^{l_M} g_M(x+\eta_M)||\delta_{\xi_1}\partial^{k_1} f_1(x)\dots\delta_{\xi_N}\partial^{k_N} f_N(x)| \times |\delta_z \partial^d u(x)| \frac{h(z)}{|z|^{n+\alpha-1+N}}\, dz,$$

where $|\xi_i(z)| \le |z|$, $\eta_j = \eta_j(z)$, *and*

$$l_1 + \dots + l_M + k_1 + \dots + k_N + d \le m+1, \quad N \ge 0, \quad d \ge 1.$$

Then

$$\|I\|_2 \le C \|u\|^{\gamma}_{H^{m+1+\frac{\alpha}{2}}} \prod_{j=1}^{M} \|g_j\|^{\alpha_j}_{H^{m+\alpha}} \times \prod_{i=1}^{N} \|f_i\|^{\beta_i}_{H^{m+\alpha}},$$

for some $\alpha_j, \beta_i \ge 0$,

$$\alpha_1 + \dots + \alpha_M + \beta_1 + \dots + \beta_N + \gamma < 1,$$

where C *depends only on the Lipschitz norms of* u, g_j*'s and* f_i*'s.*

Proof. As before, we can assume without loss of generality that all $k_i > 0$, and $\alpha - 1 + N \ge 0$.

Let us fix two small parameters $\delta' \ll \delta$ so that $\alpha + \delta < \frac{\alpha}{2} + 1$, and define

$$s = \frac{\alpha - 1 + N + 2\delta'}{N+1}, \quad p_i = \frac{2(m+1)}{k_i}, \quad q_j = \frac{2(m+1)}{l_j}, \quad r = \frac{2(m+1)}{d}.$$

Let us now distribute the singularity as follows

$$|II(x)| = \int_{\mathbb{T}^n} \frac{|\partial^{l_1} g_1(x+\eta_1)|}{|z|^{\frac{n}{q_1}-\frac{\delta'}{M}}} \dots \frac{|\partial^{l_M} g_M(x+\eta_M)|}{|z|^{\frac{n}{q_M}-\frac{\delta'}{M}}} \frac{|\delta_{\xi_1}\partial^{k_1} f_1(x)|}{|z|^{\frac{n}{p_1}+s}} \dots \dots \frac{|\delta_{\xi_N}\partial^{k_N} f_N(x)|}{|z|^{\frac{n}{p_N}+s}} \frac{|\delta_z \partial^d u(x)|}{|z|^{\frac{n}{r}+s}} \frac{h(z)}{|z|^{\frac{n}{2}-\delta'}}\, \mathrm{d}z,$$

and as before apply the Hölder inequality,

$$\begin{aligned} |II(x)| \le & \left(\int_{\mathbb{T}^n} \frac{|\partial^{l_1} g_1(x+\eta_1)|^{q_1}}{|z|^{n-\delta q_1}} h(z)\,\mathrm{d}z\right)^{\frac{1}{q_1}} \dots \left(\int_{\mathbb{T}^n} \frac{|\partial^{l_M} g_M(x+\eta_M)|^{q_M}}{|z|^{n-\delta q_M}} h(z)\,\mathrm{d}z\right)^{\frac{1}{q_M}} \\ & \times \left(\int_{\mathbb{T}^n} \frac{|\delta_{\xi_1}\partial^{k_1} f_1(x)|^{p_1}}{|z|^{n+sp_1}} h(z)\,\mathrm{d}z\right)^{\frac{1}{p_1}} \dots \left(\int_{\mathbb{T}^n} \frac{|\delta_{\xi_N}\partial^{k_N} f_N(x)|^{p_N}}{|z|^{n+sp_N}} h(z)\,\mathrm{d}z\right)^{\frac{1}{p_N}} \\ & \times \left(\int_{\mathbb{T}^n} \frac{|\delta_z \partial^d u(x)|^{r}}{|z|^{n+sr}} h(z)\,\mathrm{d}z\right)^{\frac{1}{r}}. \end{aligned}$$

Thus,

$$\begin{aligned} \|II\|_2 &\le \|g_1\|_{W^{l_1,q_1}} \dots \|g_M\|_{W^{l_M,q_M}} \times \|f_1\|_{W^{k_1+s,p_1}} \dots \|f_N\|_{W^{k_N+s,p_N}} \|u\|_{W^{d+s,r}} \\ &\lesssim \|g_1\|^{\alpha_1}_{H^{m+\alpha}} \dots \|g_M\|^{\alpha_M}_{H^{m+\alpha}} \|f_1\|^{\beta_1}_{H^{m+\alpha}} \dots \|f_N\|^{\beta_N}_{H^{m+\alpha}} \|u\|^{\gamma}_{H^{m+\alpha+\delta}}, \end{aligned}$$

where

$$\alpha_j = \frac{l_j - 1}{m+\alpha-1}, \quad \beta_i = \frac{k_i - 1 + s}{m+\alpha-1}, \quad \gamma = \frac{d-1+s}{m+\alpha-1+\delta}.$$

The sum of all exponents is less than

$$\frac{m-N}{m+\alpha-1} + s\left(\frac{N}{m+\alpha-1} + \frac{1}{m+\alpha-1+\delta}\right).$$

It remains to notice that if δ' were 0 then the above expression would be strictly less than 1. So, by continuity we can pick a small $\delta' > 0$ for which the sum is still < 1. Thus, $\alpha_1 + \cdots + \beta_N + \gamma < 1$, as desired. □

2.8. **Coercivity of the topological diffusion.** The last tool we will need in the proof of the continuation criterion is the coercivity estimate for the topological diffusion

$$\mathcal{L}_\phi f(x) = p.v. \int_{\mathbb{T}^n} \delta_z f(x) \phi(x, x+z) \, \mathrm{d}z.$$

It states a very much intuitive fact that $\mathcal{L}_\phi$ acts as a derivative of order α. In view of the highly non-linear dependence on the density in the kernel ϕ this fact requires a separate treatment. An estimate of this sort was already established in [28, 35], however the dependence of residual constants was not traced sharply to the gradients of ρ, which is important for our particular application. In this section we present a much different and shorter proof based on Lemma 2.2.

Proposition 2.4. *For any $\rho \in H^{m+\alpha}$, $0 < \alpha < 2$, we have the following estimates*

$$\|\mathcal{L}_\phi \rho\|_{\dot{H}^m} \leq 2\underline{\rho}^{-\tau/n} \|\rho\|_{\dot{H}^{m+\alpha}} + C_1,$$
$$\|\mathcal{L}_\phi \rho\|_{\dot{H}^m} \geq \frac{1}{2} \overline{\rho}^{-\tau/n} \|\rho\|_{\dot{H}^{m+\alpha}} - C_2,$$

where C_1, C_2 are constants which depend only on $\underline{\rho}, \overline{\rho}$, and $\|\nabla \rho\|_\infty$.

Proof. We start by "freezing the coefficients" in the topological part of the kernel:

$$\mathcal{L}_\phi \rho = \rho^{-\tau/n} \Lambda_\alpha \rho + \mathcal{R}\rho,$$

where

$$\mathcal{R}\rho = \int_{\mathbb{T}^n} \delta_z \rho \, R_z \frac{h(z)}{|z|^{n+\alpha}} \, \mathrm{d}z,$$
$$R_z = \frac{1}{\left[\fint_{\Omega(0,z)} \rho(x+\xi) \, \mathrm{d}\xi \right]^{\frac{\tau}{n}}} - \frac{1}{\rho^{\frac{\tau}{n}}(x)},$$

and Λ_α represents the pure fractional Laplacian with cutoff h.

Then

$$\|\partial^m \mathcal{L}_\phi \rho\|_2 \leq \|\rho^{-\tau/n} \partial^m \Lambda_\alpha \rho\|_2 + \|\partial^m (\rho^{-\tau/n} \Lambda_\alpha \rho) - \rho^{-\tau/n} \partial^m \Lambda_\alpha \rho\|_2 + \|\partial^m \mathcal{R}\rho\|_2,$$
$$\|\partial^m \mathcal{L}_\phi \rho\|_2 \geq \|\rho^{-\tau/n} \partial^m \Lambda_\alpha \rho\|_2 - \|\partial^m (\rho^{-\tau/n} \Lambda_\alpha \rho) - \rho^{-\tau/n} \partial^m \Lambda_\alpha \rho\|_2 - \|\partial^m \mathcal{R}\rho\|_2.$$

Clearly,

$$\|\rho^{-\tau/n} \partial^m \Lambda_\alpha \rho\|_2 \sim \|\rho\|_{H^{m+\alpha}}.$$

It remains to show that all the other terms are of smaller order. Let us start with the commutator. We have by (17),

$$\|\partial^m (\rho^{-\tau/n} \Lambda_\alpha \rho) - \rho^{-\tau/n} \partial^m \Lambda_\alpha \rho\|_2 \leq \|\nabla \rho^{-\tau/n}\|_\infty \|\rho\|_{H^{m+\alpha-1}} + \|\rho^{-\tau/n}\|_{H^m} \|\Lambda_\alpha \rho\|_\infty.$$

Then by interpolation,

$$\lesssim \varepsilon \|\rho\|_{H^{m+\alpha}} + C + \|\rho\|_{H^m} \|\rho\|_{W^{\alpha+\delta,\infty}}.$$

Applying Gagliardo-Nirenberg inequalities to the last term we further obtain

$$\lesssim \varepsilon \|\rho\|_{H^{m+\alpha}} + C + \|\rho\|_{H^{m+\alpha}}^{\frac{m-1}{m+\alpha-1} + \frac{2(\alpha+\delta-1)}{2(m+\alpha-1)-n}}.$$

One can check that the last exponent is strictly less than 1 if $\delta > 0$ is small enough. So, the whole term is

$$\lesssim \varepsilon \|\rho\|_{H^{m+\alpha}} + C.$$

Let us now turn to the remainder term $\partial^m \mathcal{R}$. Its Leibnitz expansion consists of terms

$$\int_{\mathbb{T}^n} \delta_z \partial^l \rho \, \partial^{m-l} R_z \frac{h(z)}{|z|^{n+\alpha}} \, \mathrm{d}z. \tag{21}$$

We use the following representation for R_z

$$R_z = \fint_{\Omega_0} \delta_\xi \rho(x) \, \mathrm{d}\theta \int_0^1 \left(\lambda \rho(x) + (1-\lambda) \fint_{\Omega_0} \rho(x+\xi) \, \mathrm{d}\theta \right)^{-1-\tau/n} \mathrm{d}\lambda.$$

Applying $m-l$ derivatives to R_z and using Leibnitz and Faa di Bruno formula (19) we can see that the expansion will consist of terms (taking the communication integrals outside)

$$\delta_\xi \partial^{m-l-p} \rho(x) \prod_{i=1}^{|\mathbf{j}|} \partial^{k_i} \rho(x+\xi_i),$$

up to a bounded function depending on λ, ρ, and where all $|\xi_i| \leq |z|$, and $k_1 + \cdots + k_{|\mathbf{j}|} = p$. Thus, the integral (21) will consist of terms bounded by

$$\int_{\mathbb{T}^n} |\delta_z \partial^l \rho| |\delta_\xi \partial^{m-l-p} \rho(x)| \left| \prod_{i=1}^{|\mathbf{j}|} \partial^{k_i} \rho(x+\xi_i) \right| \frac{h(z)}{|z|^{n+\alpha}} \, \mathrm{d}z.$$

We can see that these integrals fall under the scope of Lemma 2.2 with $N = 2$. This proves that the entire residual term is estimated by

$$\|\partial^m \mathcal{R} \rho\|_2 \lesssim \|\rho\|_{H^{m+\alpha}}^\theta, \qquad \theta < 1.$$

The generalized Young's inequality finishes the proof. ∎

2.9. **Sobolev norm of $\rho^{-\tau/n}$.** The last technical ingredient is the Sobolev bound on the power function of the density.

Lemma 2.5. *We have*

$$\|\rho^{-\tau/n}\|_{H^{m+\alpha}} \lesssim \|\rho\|_{H^{m+\alpha}} + C(\|\nabla \rho\|_\infty).$$

Proof. Without loss of generality we can assume that $\alpha < 1$, for otherwise we simply replace m by $m+1$ and α by $\alpha - 1$. Forming the finite difference we have

$$\delta_z \partial^m \rho^{-\tau/n}(x) = \partial^m (\delta_z \rho^{-\tau/n}(x)).$$

Using that

$$\delta_z \rho^{-\tau/n}(x) = \delta_z \rho(x) \int_0^1 (\lambda \rho(x+z) + (1-\lambda)\rho(x))^{-1-\tau/n} \, \mathrm{d}\lambda,$$

we distribute the m derivatives to obtain terms

$$\partial^l \delta_z \rho(x) \int_0^1 \partial^{m-l} (\lambda \rho(x+z) + (1-\lambda)\rho(x))^{-1-\tau/n} \, \mathrm{d}\lambda.$$

Using the Faa di Bruno expansion in the latter, we obtain terms that are bounded by

$$|\partial^l \delta_z \rho(x)| \prod_{i=1}^{|\mathbf{j}|} |\partial^{k_i} \rho(x+\xi_i)|,$$

where $\xi = 0$ or $\xi = z$, and $k_1 + \cdots + k_{|\mathbf{j}|} = m - l$. So, the norm $\|\rho^{-\tau/n}\|_{H^{m+\alpha}}^2$ is bounded by the terms

$$\int_{\mathbb{T}^{2n}} |\partial^l \delta_z \rho(x)|^2 \prod_{i=1}^{|\mathbf{j}|} |\partial^{k_i} \rho(x+\xi_i)|^2 \frac{\mathrm{d}z}{|z|^{n+2\alpha}}.$$

If $|\mathbf{j}| = 0$, then $l = m$, and this gives the classical Sobolev norm of $H^{m+\alpha}$. Otherwise, if $|\mathbf{j}| \geq 1$, applying the Hölder inequality, we obtain

$$\leq \|\rho\|_{W^{l+\alpha+\delta,q}} \prod_{i=1}^{|\mathbf{j}|} \|\rho\|_{W^{k_i,p_i}},$$

where

$$q = \frac{m}{l}, \qquad p_i = \frac{m}{k_i}.$$

And by the Gagliardo-Nirenberg inequality (14), this is bounded by $\|\rho\|^{\gamma}_{H^{m+\alpha}}$ with $\gamma < 1$. This finishes the proof. □

3. The continuation criterion

In this section we prove Theorem 1.1. Instead of working with the momentum-mass system directly, we replace the density with the e-quantity given by

$$e = \nabla \cdot u + \mathcal{L}_\phi \rho.$$

This strategy is prompted by the fact that placing $m + \alpha$ derivatives directly on the continuity equation creates a derivative overload on the velocity, which comes with the order of $m + \alpha + 1$, higher than even the order of dissipation. The equation for e, instead, is capable to handle this issue due to cancelation of the highest order terms. Since the natural order of regularity of e is m we thus define the grand quantity

$$Y_m = \|u\|^2_{\dot{H}^{m+1}} + \|e\|^2_{\dot{H}^m}.$$

First, let us note that directly from the continuity equation we have a control over the lower and upper bounds on the density:

$$\underline{\rho} = \min \rho, \qquad \overline{\rho} = \max \rho.$$

Differentiating at the maximum and the minimum points we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\rho} \leq \|\nabla u\|_\infty \overline{\rho},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{\rho}^{-1} \leq \|\nabla u\|_\infty \underline{\rho}^{-1}.$$

By virtue of the assumption (8) these two are uniformly bounded on the interval $[0, T)$. Consequently, the coercivity bounds of Proposition 2.4 imply that

$$Y_m \sim \|u\|^2_{\dot{H}^{m+1}} + \|\rho\|^2_{H^{m+\alpha}}.$$

Thus controlling Y_m we control the solution in the needed class.

The theorem will follow by the Grönwall inequality if we establish

$$\frac{\mathrm{d}}{\mathrm{d}t} Y_m \leq C(\underline{\rho}, \overline{\rho}, \|\nabla \rho\|_\infty, \|\nabla u\|_\infty) Y_m,$$

on the interval of regularity $[0, T)$. This will be the main goal of the next two sections.

3.1. Estimates on the velocity equation.

The goal of this section is to establish the bound

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|^2_{\dot{H}^m} \leq C Y_m - c_0 \|u\|^2_{H^{m+1+\frac{\alpha}{2}}},$$

for $c_0 > 0$, where C depends on all the norms we already control uniformly on $[0, T)$. Let us rewrite the velocity equation as

$$u_t + u \cdot \nabla u = \mathcal{C}_\phi(u, \rho),$$
$$\mathcal{C}_\phi(u, \rho)(x) = \int_{\mathbb{T}^n} \phi(x, x+z) \delta_z u(x) \rho(x+z) \, \mathrm{d}z = \mathcal{L}_\phi(u\rho) - u \mathcal{L}_\phi \rho.$$

Let us apply ∂^{m+1} and test with $\partial^{m+1}u$. We have

$$\partial_t \|u\|^2_{\dot H^{m+1}} = -\int_{\mathbb{T}^n} \partial^{m+1}(u\cdot\nabla u)\cdot\partial^{m+1}u \,\mathrm{d}x + \int_{\mathbb{T}^n} \partial^{m+1}\mathcal{C}_\phi(u,\rho)\cdot\partial^{m+1}u\,\mathrm{d}x.$$

The transport term is estimated using the classical commutator estimate

$$\partial^{m+1}(u\cdot\nabla u)\cdot\partial^{m+1}u = u\cdot\nabla(\partial^{m+1}u)\cdot\partial^{m+1}u + [\partial^{m+1},u]\nabla u\cdot\partial^{m+1}u.$$

Then

$$\int_{\mathbb{T}^n} u\cdot\nabla(\partial^{m+1}u)\cdot\partial^{m+1}u\,\mathrm{d}x = -\frac{1}{2}\int_{\mathbb{T}^n}(\nabla\cdot u)|\partial^{m+1}u|^2\,\mathrm{d}x \le \|\nabla u\|_\infty\|u\|^2_{\dot H^{m+1}},$$

and using (17) for $f=u$, $g=\nabla u$, we obtain

$$\int_{\mathbb{T}^n} |[\partial^{m+1},u]\nabla u\cdot\partial^{m+1}u|\,\mathrm{d}x \le \|\nabla u\|_\infty\|u\|^2_{\dot H^{m+1}}.$$

Thus,

$$\partial_t\|u\|^2_{\dot H^{m+1}} \lesssim Y_m + \int_{\mathbb{T}^n}\partial^{m+1}\mathcal{C}_\phi(u,\rho)\cdot\partial^{m+1}u\,\mathrm{d}x.$$

In the rest of the argument we focus on estimating the commutator term. So, we expand by the product rule

$$\partial^{m+1}\mathcal{C}_\phi(u,\rho) = \sum_{k=k_1+k_2=0}^{m+1}\frac{(m+1)!}{k_1!k_2!(m+1-k)!}\mathcal{C}_{\partial^{m+1-k}\phi}(\partial^{k_1}u,\partial^{k_2}\rho).$$

Various terms in this expansion will be estimated differently. One special end-point case provides the necessary dissipation.

3.1.1. *Case $k_1 = m+1$.* We symmetrize to obtain

$$\begin{aligned}\int_{\mathbb{T}^n}\mathcal{C}_\phi(\partial^{m+1}u,\rho)\cdot\partial^{m+1}u\,\mathrm{d}x &= -\frac{1}{2}\int_{\mathbb{T}^{2n}}\rho(x)|\delta_z\partial^{m+1}u(x)|^2\phi(x,x+z)\,\mathrm{d}z\,\mathrm{d}x\\ &\quad+\frac{1}{2}\int_{\mathbb{T}^{2n}}\delta_z\rho(x)\delta_z\partial^{m+1}u(x)\partial^{m+1}u(x)\phi(x,x+z)\,\mathrm{d}z\,\mathrm{d}x\\ &\le -c\|u\|^2_{\dot H^{m+1+\frac{\alpha}{2}}} + \int_{\mathbb{T}^n}|\partial^{m+1}u(x)|\int_{\mathbb{T}^n}|\delta_z\partial^{m+1}u(x)|\frac{h(z)}{|z|^{n+\alpha-1}}\,\mathrm{d}z\,\mathrm{d}x.\end{aligned}$$

In the last term the inner integral falls under the scope of Lemma 2.3 with $d=m+1$, $N=0$. So, it applies together with the generalized Young inequality to yield the bound by $CY_m+\varepsilon\|u\|^2_{\dot H^{m+1+\frac{\alpha}{2}}}$.

3.1.2. *Case $k_2 = m+1$.* The other extreme case is when all derivatives fall on the density in the numerator. This causes a derivative overload on ρ at least when $\alpha<1$. We therefore apply the following relaxation argument:

$$\int_{\mathbb{T}^n}\mathcal{C}_\phi(u,\partial^{m+1}\rho)\cdot\partial^{m+1}u\,\mathrm{d}x = \int_{\mathbb{T}^{2n}}\phi(x,x+z)\delta_z u(x)\partial^{m+1}\rho(x+z)\partial^{m+1}u(x)\,\mathrm{d}z\,\mathrm{d}x.$$

Observe that

$$\partial^{m+1}\rho(x+z) = \partial_z\partial_x^m\rho(x+z) = \partial_z(\partial_x^m\rho(x+z)-\partial_x^m\rho(x)) = \partial_z\delta_z\partial^m\rho(x).$$

Let us integrate by parts in z:

$$\begin{aligned}\int_{\mathbb{T}^n}\mathcal{C}_\phi(u,\partial^{m+1}\rho)\cdot\partial^{m+1}u\,\mathrm{d}x &= \int_{\mathbb{T}^{2n}}\phi(x,x+z)\partial u(x+z)\delta_z\partial^m\rho(x)\partial^{m+1}u(x)\,\mathrm{d}z\,\mathrm{d}x\\ &\quad+\int_{\mathbb{T}^{2n}}\partial_z\phi(x,x+z)\delta_z u(x)\delta_z\partial^m\rho(x)\partial^{m+1}u(x)\,\mathrm{d}z\,\mathrm{d}x := J_1+J_2.\end{aligned}$$

Symmetrizing in J_1 we further split

$$J_1 = \int_{\mathbb{T}^n} \left(\int_{\mathbb{T}^n} \delta_z \partial u(x) \delta_z \partial^m \rho(x) \phi \, \mathrm{d}z \right) \partial^{m+1} u(x) \, \mathrm{d}x - \int_{\mathbb{T}^n} \left(\int_{\mathbb{T}^n} \delta_z \partial^m \rho(x) \delta_z \partial^{m+1} u(x) \phi \, \mathrm{d}z \right) \partial u(x) \, \mathrm{d}x$$
$$:= J_{11} + J_{12}.$$

The inner integral of J_{11} falls under the scope of Lemma 2.3 with $d = 1$, $N = 1$. We thus bound it by $CY_m + \varepsilon \|u\|^2_{\dot{H}^{m+1+\frac{\alpha}{2}}}$ as we did earlier. As for J_{12} the $\partial u(x)$ is simply bounded a priori, so we obtain

$$J_{12} \lesssim \int_{\mathbb{T}^{2n}} \frac{|\delta_z \partial^m \rho(x)|}{|z|^{\frac{n}{2}+\frac{\alpha}{2}}} \frac{|\delta_z \partial^{m+1} u(x)|}{|z|^{\frac{n}{2}+\frac{\alpha}{2}}} \, \mathrm{d}z \, \mathrm{d}x \leq \|\rho\|_{H^{m+\frac{\alpha}{2}}} \|u\|_{H^{m+1+\frac{\alpha}{2}}} \leq CY_m + \varepsilon \|u\|^2_{\dot{H}^{m+1+\frac{\alpha}{2}}}. \tag{22}$$

As to J_2, let us first observe that $\partial_z \phi(x, x+z) = \psi(x, x+z)$ is antisymmetric, $\psi(x, y) = -\psi(y, x)$. Consequently, performing symmetrization we obtain

$$J_2 = \frac{1}{2} \int_{\mathbb{T}^{2n}} \partial_z \phi(x, x+z) \delta_z u(x) \delta_z \partial^m \rho(x) \delta_z \partial^{m+1} u(x) \, \mathrm{d}z \, \mathrm{d}x.$$

Since

$$\partial_z \phi(x, x+z) = -\frac{(n+\alpha-\tau) h(z) z_i}{|z|^{n+\alpha+2-\tau} d^\tau(x, x+z)} + h(z) \frac{\partial_z \int_{\Omega(x,x+z)} \rho(\xi) \, \mathrm{d}\xi}{|z|^{n+\alpha-\tau} d^{\tau+n}(x, x+z)} + \frac{\partial_z h(z)}{|z|^{n+\alpha-\tau} d^\tau(x, x+z)},$$

and noticing that

$$\left| \partial_z \int_{\Omega(x,x+z)} \rho(\xi) \, \mathrm{d}\xi \right| \leq \|\rho\|_\infty |z|^{n-1},$$

we can see that

$$|\partial_z \phi(x, x+z)| \lesssim \frac{\mathbb{1}_{|z|<2r_0}}{|z|^{n+\alpha+1}}.$$

The one derivative loss is compensated by $|\delta_z u(x)| \leq |z| \|\nabla u\|_\infty$. With this at hand we estimate

$$J_2 \lesssim \int_{\mathbb{T}^{2n}} \frac{|\delta_z \partial^m \rho(x) \delta_z \partial^{m+1} u(x)|}{|z|^{n+\alpha}} \, \mathrm{d}z \, \mathrm{d}x,$$

and we are back to (22).

3.1.3. *All other cases.* The bulk of the other terms can be estimated simultaneously. We start by the standard symmetrization:

$$\int_{\mathbb{T}^n} \mathcal{C}_{\partial^{m+1-k}\phi}(\partial^{k_1} u, \partial^{k_2} \rho) \cdot \partial^{m+1} u \, \mathrm{d}x = \frac{1}{2} \int_{\mathbb{T}^{2n}} \delta_z \partial^{k_1} u(x) \delta_z \partial^{k_2} \rho(x) \partial^{m+1} u(x) \partial^{m+1-k} \phi(x, x+z) \, \mathrm{d}z \, \mathrm{d}x$$
$$+ \frac{1}{2} \int_{\mathbb{T}^{2n}} \delta_z \partial^{k_1} u(x) \partial^{k_2} \rho(x) \delta_z \partial^{m+1} u(x) \partial^{m+1-k} \phi(x, x+z) \, \mathrm{d}z \, \mathrm{d}x$$
$$= J_1 + J_2.$$

The term J_1 is subcritical. To see that we use the Faa di Bruno expansion (19) for the kernel $\partial^{m+1-k} \phi(x, x+z)$. Each term takes the form

$$\int_{\mathbb{T}^{2n}} \delta_z \partial^{k_1} u(x) \delta_z \partial^{k_2} \rho(x) \partial^{m+1} u(x) \frac{h(z)}{|z|^{n+\alpha-\tau}} \frac{\prod_{i=1}^{|\mathbf{j}|} \int_{\Omega(x,x+z)} \partial^{l_i} \rho(\xi) d\xi}{d^{\tau+|\mathbf{j}|n}(x, x+z)} \, \mathrm{d}z \, \mathrm{d}x,$$

where $l_1 + \cdots + l_{|\mathbf{j}|} = m + 1 - k$. Integrating by parts in each of the topological domains and reducing to the basic $\partial \Omega_0$ we obtain

$$\int_{\Omega(x,x+z)} \partial^l \rho(\xi) \, \mathrm{d}\xi = |z|^{n-1} \int_{\partial \Omega_0} \delta_\xi \partial^{l-1} \rho(x) \, \mathrm{d}\theta,$$

where $\xi = \xi(\theta, z)$, $|\xi| \le |z|$. Moving all the communication integration outside, we obtain a family of terms

$$\int_{\mathbb{T}^n} \partial^{m+1} u(x) \int_{\mathbb{T}^n} \delta_z \partial^{k_1} u(x) \delta_z \partial^{k_2} \rho(x) \prod_{i=1}^{|\mathbf{j}|} \delta_{\xi_i} \partial^{l_i - 1} \rho(x) \frac{h(z)}{|z|^{n+\alpha-\tau}} \frac{|z|^{(n-1)|\mathbf{j}|}}{d^{\tau+|\mathbf{j}|n}(x, x+z)} \,\mathrm{d}z \,\mathrm{d}x.$$

Here the interior integral is bounded by

$$I(x) = \int_{\mathbb{T}^n} |\delta_z \partial^{k_1} u(x) \delta_z \partial^{k_2} \rho(x)| \prod_{i=1}^{|\mathbf{j}|} |\delta_{\xi_i} \partial^{l_i-1} \rho(x)| \frac{h(z)}{|z|^{n+\alpha+|\mathbf{j}|}} \,\mathrm{d}z.$$

The total order of derivatives here is $m+1-|\mathbf{j}|$. So, if $|\mathbf{j}| \ge 1$, then this term falls under Lemma 2.2 to produce the necessary estimates in the same fashion as previously. If $|\mathbf{j}| = 0$, this corresponds to the situation when no derivatives fall on the kernel, and we have $k_1 + k_2 = m+1$. Then we are dealing with the terms of type

$$II(x) = \int_{\mathbb{T}^n} |\delta_z \partial^{k_1} u(x) \delta_z \partial^{k_2} \rho(x)| \frac{h(z)}{|z|^{n+\alpha}} \,\mathrm{d}z.$$

Since $k_2 < m+1$, $k_1 \ge 1$, these fall under Lemma 2.3 with $d = k_1$ and $N = 1$ to conclude the estimate.

Turning to J_2 we can see that the Faa di Bruno expansion of the kernel produces terms of type

$$\int_{\mathbb{T}^{2n}} \delta_z \partial^{k_1} u(x) \partial^{k_2} \rho(x) \delta_z \partial^{m+1} u(x) \frac{\prod_{i=1}^{|\mathbf{j}|} \int_{\Omega(x,x+z)} \partial^{l_i} \rho(\xi) d\xi}{d^{\tau+|\mathbf{j}|n}(x, x+z)} \frac{h(z)}{|z|^{n+\alpha-\tau}} \,\mathrm{d}z \,\mathrm{d}x,$$

which, moving the communication integrals outside, are bounded by,

$$\int_{\mathbb{T}^{2n}} |\delta_z \partial^{k_1} u(x) \partial^{k_2} \rho(x) \delta_z \partial^{m+1} u(x)| \prod_{i=1}^{|\mathbf{j}|} |\partial^{l_i} \rho(x+\xi_i)| \frac{h(z)}{|z|^{n+\alpha}} \,\mathrm{d}z \,\mathrm{d}x.$$

Denoting

$$p_i = \frac{2(m+1)}{l_i}, \qquad q_1 = \frac{2(m+1)}{k_1}, \qquad q_2 = \frac{2(m+1)}{k_2},$$

and distributing the α-exponent between the terms accordingly,

$$\int_{\mathbb{T}^{2n}} \frac{|\delta_z \partial^{k_1} u(x)|}{|z|^{\frac{n}{q_1}+\frac{\alpha}{2}+\delta}} \frac{|\partial^{k_2} \rho(x)|}{|z|^{\frac{n}{q_2}-\frac{\delta}{|\mathbf{j}|+1}}} \frac{|\delta_z \partial^{m+1} u(x)|}{|z|^{\frac{n}{2}+\frac{\alpha}{2}}} \prod_{i=1}^{|\mathbf{j}|} \frac{|\partial^{l_i} \rho(x+\xi_i)|}{|z|^{\frac{n}{p_i}-\frac{\delta}{|\mathbf{j}|+1}}} h(z) \,\mathrm{d}z \,\mathrm{d}x,$$

we apply the Hölder inequality to obtain

$$\le \|u\|_{H^{m+1+\alpha/2}} \|u\|_{W^{k_1+\frac{\alpha}{2}+\delta, q_1}} \|\rho\|_{W^{k_2,q_2}} \prod_{i=1}^{|\mathbf{j}|} \|\rho\|_{W^{l_i,p_i}}.$$

By the Gagliardo-Nirenberg inequality (14) applied to each term except the first one we obtain

$$\lesssim \|u\|_{H^{m+1+\alpha/2}} \|u\|^{\gamma_1}_{H^{m+\alpha}} \|\rho\|^{\gamma_2}_{H^{m+\alpha}} \prod_{i=1}^{|\mathbf{j}|} \|\rho\|^{\beta_i}_{H^{m+\alpha}},$$

where

$$\gamma_1 = \frac{k_1 + \frac{\alpha}{2} - 1 + \delta}{m+\alpha-1}, \quad \gamma_2 = \frac{k_2 - 1}{m+\alpha-1}, \quad \beta_i = \frac{l_i - 1}{m+\alpha-1}.$$

The sum of all these exponents is equal to $\frac{m+\frac{\alpha}{2}-1-|\mathbf{j}|+\delta}{m+\alpha-1} < 1$ provided δ is small enough. Application of the generalized Young inequality, and noting that $\|u\|^{\gamma_1}_{H^{m+\alpha}} \le \|u\|^{\gamma_1}_{H^{m+1+\frac{\alpha}{2}}}$, produces

$$\lesssim CY_m + \varepsilon \|u\|^2_{\dot{H}^{m+1+\frac{\alpha}{2}}}.$$

3.2. **Estimates on the e-equation.** The goal of this section is to show the bound

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2_{\dot{H}^m} \leq CY_m + \varepsilon \|u\|^2_{H^{m+1+\frac{\alpha}{2}}},$$

for any $\varepsilon > 0$, where C depends on all the norms we already control uniformly on $[0, T)$, and on ε.

Taking the divergence of the momentum equation and using the continuity equation we obtain the following equation on e,

$$e_t + \nabla \cdot (ue) = (\nabla \cdot u)^2 - \mathrm{Tr}(\nabla u)^2 + \mathcal{T}[\rho, u], \tag{23}$$

where

$$\mathcal{T}[\rho, u] = \partial_t(\mathcal{L}_\phi(\rho)) + \nabla \cdot \mathcal{L}_\phi(\rho u).$$

Let us take a closer look the the topological term $\mathcal{T}$ and work out a more explicit formula for it. We have

$$\mathcal{T}[\rho, u] = \mathcal{L}_\phi(\rho_t) + \mathcal{L}_{\phi_t}(\rho) + \mathcal{L}_\phi(\nabla \cdot (\rho u)) + \mathcal{L}_{\nabla\phi\cdot}(\rho u).$$

The first and third terms obviously cancel by the continuity equation. For the rest we have

$$\begin{aligned}
\mathcal{L}_{\phi_t}(\rho) &= -\frac{\tau}{n}\int_{\mathbb{T}^n} \frac{\int_{\Omega(x,x+z)} \rho_t(\xi)\,\mathrm{d}\xi}{d^{\tau+n}(x, x+z)} \delta_z \rho(x) \frac{h(z)}{|z|^{n+\alpha-\tau}}\,\mathrm{d}z \\
&= \frac{\tau}{n}\int_{\mathbb{T}^n} \frac{\int_{\Omega(x,x+z)} \nabla \cdot (\rho u)(\xi)\,\mathrm{d}\xi}{d^{\tau+n}(x, x+z)} \delta_z \rho(x) \frac{h(z)}{|z|^{n+\alpha-\tau}}\,\mathrm{d}z, \\
\mathcal{L}_{\nabla\phi\cdot}(\rho u) &= \int_{\mathbb{T}^n} \nabla\phi(x, x+z) \cdot \delta_z(\rho u)(x)\,\mathrm{d}z \\
&= -\frac{\tau}{n}\int_{\mathbb{T}^n} \frac{h(z)}{|z|^{n+\alpha-\tau}} \frac{\int_{\Omega(x,x+z)} \nabla\rho(\xi)\,\mathrm{d}\xi}{d^{\tau+n}(x, x+z)} \cdot \delta_z(\rho u)(x)\,\mathrm{d}z.
\end{aligned}$$

We arrive at

$$\mathcal{T}[\rho, u] = \frac{\tau}{n}\int_{\mathbb{T}^n}\int_{\Omega(0,z)} [\nabla \cdot (\rho u)(x+\xi)\delta_z\rho(x) - \nabla\rho(x+\xi) \cdot \delta_z(\rho u)(x)] \frac{h(z)}{|z|^{n+\alpha-\tau} d^{\tau+n}(x, x+z)}\,\mathrm{d}\xi\,\mathrm{d}z.$$

Let us now compute the energy equation for the H^m-norm:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2_{\dot{H}^m} = -\partial^m e\, \partial^m (u \cdot \nabla e + e\nabla \cdot u) + \partial^m e\, \partial^m \left[(\nabla \cdot u)^2 - \mathrm{Tr}(\nabla u)^2\right] + \partial^m e\, \partial^m \mathcal{T}[\rho, u].$$

Estimating the last term will be the main technical component of this section. So, let us make a few quick comments as to the remaining terms. The transport term becomes

$$\partial^m e\,(u \cdot \nabla\partial^m e) + \partial^m e\,[\partial^m(u \cdot \nabla e) - u \cdot \nabla\partial^m e] + \partial^m e\,\partial^m(e\nabla \cdot u).$$

In the first term we integrate by parts and estimate

$$|\partial^m e u \cdot \nabla\partial^m e| \leq \|e\|^2_{\dot{H}^m}\|\nabla u\|_\infty \lesssim Y_m.$$

For the next term we use the commutator estimate (17) to obtain

$$|\partial^m e[\partial^m(u \cdot \nabla e) - u \cdot \nabla\partial^m e]| \leq \|\nabla u\|_\infty \|e\|^2_{\dot{H}^m} + \|e\|_{\dot{H}^m}\|u\|_{\dot{H}^m}\|\nabla e\|_\infty.$$

Using the Gagliardo-Nirenberg inequality we estimate the latter term as

$$\|e\|_{\dot{H}^m}\|u\|_{\dot{H}^m}\|\nabla e\|_\infty \leq \|e\|_{\dot{H}^m}\|u\|^{\theta_1}_{\dot{H}^{m+1}}\|\nabla u\|^{1-\theta_1}_\infty \|e\|^{\theta_2}_{\dot{H}^m}\|e\|^{1-\theta_2}_\infty,$$

where $\theta_1 = \frac{2(m-1)-n}{2m-n}$ and $\theta_2 = \frac{2}{2m-n}$. The two exponents add up to 1, so by the generalized Young inequality,

$$\leq (\|e\|^2_{\dot{H}^m} + \|u\|^2_{\dot{H}^{m+1}})(\|e\|_\infty + \|\nabla u\|_\infty) \leq (\|e\|_\infty + \|\nabla u\|_\infty)Y_m.$$

Next term in the e-equation is estimated by the product formula (18). So, we have

$$|\partial^m e\partial^m(e\nabla \cdot u)| \leq \|e\|^2_{\dot{H}^m}\|\nabla u\|_\infty + \|e\|_{\dot{H}^m}\|e\|_\infty\|u\|_{\dot{H}^{m+1}} \leq (\|e\|_\infty + \|\nabla u\|_\infty)Y_m.$$

Finally,

$$|\partial^m e[(\nabla\cdot u)^2 - \mathrm{Tr}(\nabla u)^2]| \le \|e\|_{\dot H^m}\|u\|_{\dot H^{m+1}}\|\nabla u\|_\infty \le \|\nabla u\|_\infty Y_m.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2_{\dot H^m} \le (\|e\|_\infty + \|\nabla u\|_\infty)Y_m + \partial^m e\,\partial^m \mathcal{T}[\rho,u].$$

Let us address the issues related with the norm $\|e\|_\infty$. For the range $0<\alpha<1$ the norm is uniformly bounded by a straightforward application of the representation $e = \nabla\cdot u - \mathcal{L}_\phi(\rho)$,

$$\|e\|_\infty \lesssim \|\nabla u\|_\infty + \|\Lambda^\alpha \rho\|_\infty \le \|\nabla u\|_\infty + \|\nabla\rho\|_\infty.$$

So, in this case $\|e\|_\infty$ is a priori bounded.

To complete the full range $(1\le\alpha<2)$ a more subtle estimate is required. Coming back to the transport terms, we just need to consider a more precise computation for the expression:

$$\|e\|_{\dot H^m}\left(\|u\|_{\dot H^m}\|\nabla e\|_\infty + \|e\|_\infty\|u\|_{\dot H^{m+1}}\right). \tag{24}$$

Now, to take advantage of dissipation, we apply the following Gagliardo-Nirenberg inequalities to the terms of the last factor:

$$\|u\|_{\dot H^m} \lesssim \|u\|^{\frac{2(m-1)}{2m+\alpha}}_{\dot H^{m+1+\frac{\alpha}{2}}}, \qquad \|\nabla e\|_\infty \le \|e\|^{\theta_1}_{\dot H^m}\|\Lambda^{-1}e\|^{1-\theta_1}_{\dot W^{2-(\alpha+\varepsilon),\infty}},$$

$$\|u\|_{\dot H^{m+1}} \lesssim \|u\|^{\frac{2m}{2m+\alpha}}_{\dot H^{m+1+\frac{\alpha}{2}}}, \qquad \|e\|_\infty \le \|e\|^{\theta_2}_{\dot H^m}\|\Lambda^{-1}e\|^{1-\theta_2}_{\dot W^{1-(\frac{\alpha}{2}-\varepsilon),\infty}},$$

where

$$\theta_1 = \frac{\alpha+\varepsilon}{m+1-(2-(\alpha+\varepsilon))-\frac{n}{2}}, \qquad \theta_2 = \frac{\frac{\alpha}{2}-\varepsilon}{m+\frac{\alpha}{2}-\varepsilon-\frac{n}{2}}.$$

By generalized Young inequality, we further obtain that (24) can be bounded by

$$\le \varepsilon\|u\|^2_{\dot H^{m+1+\frac{\alpha}{2}}} + C_\varepsilon\, p_N(|\nabla u|_\infty, \|\Lambda^{-1}e\|_{W^{2-(\alpha+\varepsilon),\infty}})\|e\|^{\max\{(1+\theta_1)q_1,(1+\theta_2)q_2\}}_{\dot H^m},$$

with q_1, q_2 conjugate exponents of

$$p_1 = \frac{2m+\alpha}{m-1}, \qquad p_2 = \frac{2m+\alpha}{m}.$$

We obtain $(1+\theta_1)q_1 < 2$ as long as $m > 1 + \left(\frac{2+\alpha}{2-\alpha}\right)\frac{n}{2}$ and $(1+\theta_2)q_2<2$ as long as $m > \frac{n\alpha}{4\varepsilon}$. In addition, as we need that $\frac{\alpha}{2}+\varepsilon<1$ we impose the smallness conditon $2\varepsilon < 2-\alpha$, which give us the required bound for the exponent, i.e. $\max\{(1+\theta_1)q_1,(1+\theta_2)q_2\}<2$ if

$$m > 1 + \left(\frac{2+\alpha}{2-\alpha}\right)\frac{n}{2}.$$

Using again the definition of the e-quantity we have

$$\|\Lambda^{-1}e\|_{W^{2-(\alpha+\varepsilon),\infty}} = \|\Lambda^{-1}\nabla u\|_{W^{2-(\alpha+\varepsilon),\infty}} + \|\Lambda^{-1}\mathcal{L}_\phi\rho\|_{W^{2-(\alpha+\varepsilon),\infty}},$$

where the first term is trivially bounded by $\|\nabla u\|_\infty$ thanks to the fact that $2-(\alpha+\varepsilon)<1$. For the last one we need to work a litle bit.

To finish with the transport terms we apply the next result.

Lemma 3.1. *For $1\le\alpha<2$ and $\varepsilon>0$ such that $\alpha+\varepsilon<2$ the following bound holds:*

$$\|\Lambda^{-1}\mathcal{L}_\phi\rho\|_{W^{2-(\alpha+\varepsilon),\infty}} \lesssim \|\nabla\rho\|_\infty,$$

where $\lesssim$ means up to a factor of $\underline{\rho},\overline{\rho}$.

Proof. The idea of the proof is just to use the smoother properties of the Riesz potential Λ^{-1} and $\Lambda^{1-(\alpha+\varepsilon)}$. In first place, applying interpolation we have

$$\|\Lambda^{-1}\mathcal{L}_\phi\rho\|_{W^{2-(\alpha+\varepsilon),\infty}} \leq \|\Lambda^{-1}\mathcal{L}_\phi\rho\|_\infty + \|\Lambda^{1-(\alpha+\varepsilon)}\mathcal{L}_\phi\rho\|_\infty.$$

Since $-1 < 1-(\alpha+\varepsilon) < 0$, we focus our attention on the last one. The other follows similar ideas. By definition of Riesz potential we have the expression

$$\Lambda^{1-(\alpha+\varepsilon)}\mathcal{L}_\phi\rho(x) = \int_{\mathbb{T}^n} \frac{\mathcal{L}_\phi\rho(x+y)}{|y|^{n-(1-(\alpha+\varepsilon))}}\,\mathrm{d}y = \int_{\mathbb{T}^{2n}} \frac{\rho(x+y)-\rho(x+z)}{|y|^{n-(1-(\alpha+\varepsilon))}|y-z|^{n+\alpha-\tau}} \frac{h(|y-z|)}{d_\rho^\tau(x+y,x+z)}\,\mathrm{d}z\,\mathrm{d}y.$$

Let us get rid of the dependence on x in the topological kernel by freezing the coefficients as we did previously,

$$\begin{aligned}\Lambda^{1-(\alpha+\varepsilon)}\mathcal{L}_\phi\rho(x) &= \rho^{-\frac{\tau}{n}}(x)\int_{\mathbb{T}^{2n}} \frac{\rho(x+y)-\rho(x+z)}{|y|^{n-(1-(\alpha+\varepsilon))}|y-z|^{n+\alpha}} h(|y-z|)\,\mathrm{d}z\,\mathrm{d}y \\ &+ \int_{\mathbb{T}^{2n}} \frac{\rho(x+y)-\rho(x+z)}{|y|^{n-(1-(\alpha+\varepsilon))}|y-z|^{n+\alpha}} h(|y-z|) \times \left(\frac{1}{\left[\fint_{\Omega(y,z)} \rho(x+\xi)\,\mathrm{d}\xi\right]^{\frac{\tau}{n}}} - \frac{1}{\rho^{\frac{\tau}{n}}(x)} \right)\mathrm{d}z\,\mathrm{d}y.\end{aligned}$$

For sake of brevity, due to extra cancelation of the last factor which give us $\lesssim |y-z|\|\nabla\rho\|_\infty$, we focus only in the first term, which is the most singular one. To overcome this issue we apply integration by parts and the fact that $|y-z|^{-(n+\alpha)} \approx \nabla_z\cdot(y-z)|y-z|^{-(n+\alpha)}$, which gives us

$$\begin{aligned}\int_{\mathbb{T}^{2n}} \frac{\rho(x+y)-\rho(x+z)}{|y|^{n-(1-(\alpha+\varepsilon))}|y-z|^{n+\alpha}} h(|y-z|)\,\mathrm{d}z\,\mathrm{d}y &\approx \int_{\mathbb{T}^{2n}} \frac{\nabla\rho(x+z)}{|y|^{n-(1-(\alpha+\varepsilon))}|y-z|^{n+\alpha-1}} h(|y-z|)\,\mathrm{d}z\,\mathrm{d}y \\ &\approx \|\nabla\rho\|_\infty \int_{\mathbb{T}^{2n}} \frac{1}{|z+w|^{n-(1-(\alpha+\varepsilon))}|w|^{n+\alpha-1}}\,\mathrm{d}w\,\mathrm{d}z.\end{aligned}$$

Integrating first in the variable z and then in w, the last double integral is bounded by an universal constant and consequently we have proved our goal. □

In conclusion, we have proved the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e\|^2_{\dot{H}^m} \leq (\|\nabla u\|_\infty + \|\nabla\rho\|_\infty)Y_m + \partial^m e\,\partial^m \mathcal{T}[\rho,u].$$

We now focus solely on the topological term. First let us derive a form of $\mathcal{T}$ that is most suitable to our analysis. Integrating by parts inside the communication integrals we obtain

$$\begin{aligned}&\int_{\Omega(0,z)} [\nabla\cdot(\rho u)(x+\xi)\delta_z\rho(x) - \nabla\rho(x+\xi)\cdot\delta_z(\rho u)(x)]\,\mathrm{d}\xi \\ &\qquad = \int_{\partial\Omega(0,z)} [(\rho u)(x+\xi)\delta_z\rho(x) - \rho(x+\xi)\delta_z(\rho u)(x)]\cdot\nu_\xi\,\mathrm{d}\xi,\end{aligned}$$

using cancelation of the integral of a constant,

$$= \int_{\partial\Omega(0,z)} [\delta_\xi(\rho u)(x)\delta_z\rho(x) - \delta_\xi\rho(x)\delta_z(\rho u)(x)]\cdot\nu_\xi\,\mathrm{d}\xi,$$

and adding and subtracting cross-difference terms,

$$= \int_{\partial\Omega(0,z)} [\delta_\xi\rho\delta_z\rho\delta_\xi u - \delta_\xi\rho\delta_z\rho\delta_z u + \rho\delta_z\rho\delta_\xi u - \rho\delta_\xi\rho\delta_z u]\cdot\nu_\xi\,\mathrm{d}\xi.$$

Using unitary transformation of $\Omega(0,z)$ to the basic domain we obtain the following representation (here $\xi = \xi(\theta, z) = |z| U_z \theta$)

$$\begin{aligned}
\mathcal{T} &= \mathcal{T}_1 - \mathcal{T}_2 + \mathcal{T}_3 - \mathcal{T}_4,\\
\mathcal{T}_1 &= \frac{\tau}{n} \int_{\partial\Omega_0} \int_{\mathbb{T}^n} \delta_\xi \rho \delta_z \rho \delta_\xi u \frac{h(z) U_z \nu_\theta}{|z|^{1+\alpha-\tau} d^{\tau+n}(x, x+z)} \, \mathrm{d}z \, \mathrm{d}\theta,\\
\mathcal{T}_2 &= \frac{\tau}{n} \int_{\partial\Omega_0} \int_{\mathbb{T}^n} \delta_\xi \rho \delta_z \rho \delta_z u \frac{h(z) U_z \nu_\theta}{|z|^{1+\alpha-\tau} d^{\tau+n}(x, x+z)} \, \mathrm{d}z \, \mathrm{d}\theta,\\
\mathcal{T}_3 &= \frac{\tau}{n} \int_{\partial\Omega_0} \int_{\mathbb{T}^n} \rho \delta_z \rho \delta_\xi u \frac{h(z) U_z \nu_\theta}{|z|^{1+\alpha-\tau} d^{\tau+n}(x, x+z)} \, \mathrm{d}z \, \mathrm{d}\theta,\\
\mathcal{T}_4 &= \frac{\tau}{n} \int_{\partial\Omega_0} \int_{\mathbb{T}^n} \rho \delta_\xi \rho \delta_z u \frac{h(z) U_z \nu_\theta}{|z|^{1+\alpha-\tau} d^{\tau+n}(x, x+z)} \, \mathrm{d}z \, \mathrm{d}\theta.
\end{aligned}$$

It will help us cut the estimates in half by simply noticing that the pairs $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3, \mathcal{T}_4$ are completely similar, with the only difference in appearance of z vs ξ inside the finite differences. Since for any θ, $|\xi| \leq |z|$, and the bound (9) asserts that we can replace z with ξ to get access to the same Sobolev norms uniformly in θ, the analysis of these terms will be completely identical. So, in what follows we only focus on $\mathcal{T}_1$ and $\mathcal{T}_3$.

Looking further into the structure of $\mathcal{T}_1$ and $\mathcal{T}_3$ one observes that $\mathcal{T}_1$ is distinctly easier in the sense that all the three state quantities appear in the form of finite differences, which allows to dissolve the singularity in z among all three. Term $\mathcal{T}_3$ on the other hand has only two quantities in finite difference form, while one density appears straight. For this reason the singularity presents itself relatively stronger. We will handle this term with the help of paraproduct estimates.

3.2.1. *Estimates on* $\mathcal{T}_1, \mathcal{T}_2$. Each term in the Leibnitz and Faa di Bruno expansion of $\partial^m \mathcal{T}_1[u, \rho]$ takes form

$$\int_{\mathbb{T}^n} \partial^l (\delta_\xi \rho \delta_z \rho \delta_\xi u) \prod_{i=1}^{|\mathbf{j}|} \int_{\Omega(x,x+z)} \partial^{k_i} \rho(\xi) \, \mathrm{d}\xi \frac{h(z)}{|z|^{1+\alpha-\tau} d^{\tau+n+|\mathbf{j}|n}(x, x+z)} \, \mathrm{d}z,$$

where $k_1 + \cdots + k_{|\mathbf{j}|} = m - l$. Integrating by parts in each of the topological domains and reducing to the basic $\partial\Omega_0$ we further obtain

$$\int_{\mathbb{T}^n} \partial^l (\delta_\xi \rho \delta_z \rho \delta_\xi u) \prod_{i=1}^{|\mathbf{j}|} \int_{\partial\Omega_0} \delta_{\xi(\theta,z)} \partial^{k_i - 1} \rho(x) \, \mathrm{d}\theta \frac{h(z)}{|z|^{1+\alpha-\tau-(n-1)|\mathbf{j}|} d^{\tau+n+|\mathbf{j}|n}(x, x+z)} \, \mathrm{d}z.$$

Taking integration over communication domains outside we arrive at the term

$$\int_{\mathbb{T}^n} \partial^l (\delta_\xi \rho \delta_z \rho \delta_\xi u) \prod_{i=1}^{|\mathbf{j}|} \delta_{\xi_i} \partial^{k_i - 1} \rho(x) \frac{h(z)}{|z|^{1+\alpha-\tau-(n-1)|\mathbf{j}|} d^{\tau+n+|\mathbf{j}|n}(x, x+z)} \, \mathrm{d}z.$$

Bounding the d-distance trivially we obtain the singular integral

$$I = \int_{\mathbb{T}^n} |\partial^l (\delta_\xi \rho \delta_z \rho \delta_\xi u)| \prod_{i=1}^{|\mathbf{j}|} |\delta_{\xi_i} \partial^{k_i - 1} \rho(x)| \frac{h(z)}{|z|^{n+1+\alpha+|\mathbf{j}|}(x, x+z)} \, \mathrm{d}z.$$

Now, notice that the total order of derivatives in the numerator is $m - |\mathbf{j}| \leq m$, the total number of terms on the top is $N = 3 + |\mathbf{j}|$, and the order of singularity is $n + \alpha - 2 + N$. So, we are entirely under the scope of Lemma 2.2, which gives the bound by

$$\|I\|_2 \lesssim \|\rho\|_{H^{m+\alpha}}^{\theta_1} \|u\|_{H^{m+\alpha}}^{\theta_2}, \qquad \theta_1 + \theta_2 < 1.$$

Noticing that $m+\alpha < m+1+\frac{\alpha}{2}$, for any $\varepsilon>0$ by the generalized Young inequality, we obtain

$$\left| \int_{\mathbb{T}^n} \partial^m e \, \partial^m \mathcal{T}_1[u,\rho] \, \mathrm{d}x \right| \leq C_\varepsilon Y_m + \varepsilon \|u\|^2_{H^{m+1+\frac{\alpha}{2}}}.$$

Clearly, the estimate for $\mathcal{T}_2$ is entirely similar.

3.2.2. *Estimates on $\mathcal{T}_3, \mathcal{T}_4$.* The two terms are similar, so let us focus on $\mathcal{T}_3$.

Proposition 3.2. *For any $\varepsilon>0$ there is a constant $C_\varepsilon>0$ such that*

$$\left| \int_{\mathbb{T}^n} \partial^m e \, \partial^m \mathcal{T}_3[u,\rho] \, dx \right| \leq C_\varepsilon Y_m + \varepsilon \|u\|^2_{H^{m+1+\frac{\alpha}{2}}}. \tag{25}$$

Before we apply ∂^m on $\mathcal{T}_3$, let us freeze the coefficients,

$$\begin{aligned}
\mathcal{T}_3 &= \mathcal{T}_{31} + \mathcal{T}_{32}, \\
\mathcal{T}_{31} &= \rho^{-\tau/n}(x) \int_{\mathbb{T}^n} \delta_z \rho \, \delta_\xi u \frac{h(z) U_z \nu_\theta}{|z|^{n+1+\alpha}} \, \mathrm{d}z, \\
\mathcal{T}_{32} &= \int_{\mathbb{T}^n} (\rho \, \delta_z \rho \, \delta_\xi u) R_z \frac{h(z) U_z \nu_\theta}{|z|^{n+1+\alpha}} \, \mathrm{d}z.
\end{aligned}$$

Let us first analyze $\partial^m \mathcal{T}_{32}$ as this term will turn out to be subcritical and fall under the scope of Lemma 2.2.

Lemma 3.3. *We have*

$$\|\partial^m \mathcal{T}_{32}\|_2 \leq \|\rho\|^{\theta_1}_{H^{m+\alpha}} \|u\|^{\theta_2}_{H^{m+\alpha}}, \qquad \theta_1+\theta_2<1.$$

Proof. The Leibnitz expansion consists of terms

$$\int_{\mathbb{T}^n} \partial^l (\rho \, \delta_z \rho \, \delta_\xi u) \partial^{m-l} R_z \frac{h(z) U_z \nu_\theta}{|z|^{n+1+\alpha}} \, \mathrm{d}z.$$

We can see that they are completely analogous to the terms we encountered in the proof of Proposition 2.4. The only difference is that N is smaller by one in this case. Hence, Lemma 2.2 applies to finish the proof. □

We now turn to the critical term $\mathcal{T}_{31}$.

Because we have only two finite differences available, in order to get access to higher regularity of ρ and u at the first step we symmetrize in z using oddness of the map $z \to U_z$:

$$\begin{aligned}
\mathcal{T}_{31} &= \mathcal{T}_{311} - \mathcal{T}_{312}, \\
\mathcal{T}_{311} &= \rho^{-\tau/n} \int_{\mathbb{T}^n} \delta_z^2 \rho \, \delta_\xi u \, \frac{h(z) U_z \nu_\theta}{|z|^{n+1+\alpha}} \, \mathrm{d}z, \\
\mathcal{T}_{312} &= \rho^{-\tau/n} \int_{\mathbb{T}^n} \delta_{-z} \rho \, \delta_\xi^2 u \, \frac{h(z) U_z \nu_\theta}{|z|^{n+1+\alpha}} \, \mathrm{d}z.
\end{aligned}$$

The two integrals are similar and have a common form

$$T = \int_{\mathbb{T}^n} g \; \delta_\circ^2 g' \, \delta_\circ g'' \, \frac{h(z) U_z \nu_\theta}{|z|^{n+1+\alpha}} \, \mathrm{d}z,$$

where $\circ = z, -z$, or ξ.

Lemma 3.4. *We have*

$$\int_{\mathbb{T}^n} f \, \partial^m T \, dx \leq C \|f\|_2 (\|g\|_{H^{m+\alpha}} + \|g'\|_{H^{m+\alpha}} + \|g''\|_{H^{m+\alpha}}),$$

where C depends only on the gradients of g, g', g''.

Applying this lemma the terms above and replacing the $m+\alpha$ integrability for u with $m+1+\frac{\alpha}{2}$ we obtain the necessary bound (25) by invoking Lemma 2.5.

Proof. Moving ∂^m back onto f we will apply the Bony decomposition to the triple $(\partial^m f, g, \delta_\circ g' \delta_\circ g'')$, treating the product $\delta_\circ g' \delta_\circ g''$ as a single piece, then splitting it into further paraproducts using (15), (16). This results in a number of terms according to interacting low (L), medium (M), and high (H) frequencies:

$$\int_{\mathbb{T}^n} f \partial^m T \,\mathrm{d}x = \int_{\mathbb{T}^n} (\text{paraproducts}) \times \frac{h(z) U_z \nu_\theta}{|z|^{n+1+\alpha}} \,\mathrm{d}z,$$

where "paraproducts" consists of the following terms

$$\begin{aligned}
LMHH &= \sum_q \sum_{p>q-1} \sum_{r>p-2} \langle \partial^m f_q, g_{\sim p}, (\delta_\circ^2 g'_{\sim r}\, \delta_\circ g''_{\sim r})_{\sim p} \rangle, \\
MHLH &= \sum_q \sum_{p>q-1} \langle \partial^m f_q, g_{\sim p}, (\delta_\circ^2 g'_{<p}\, \delta_\circ g''_{\sim p})_{\sim p} \rangle, \\
MHHL &= \sum_q \sum_{p>q-1} \langle \partial^m f_q, g_{\sim p}, (\delta_\circ^2 g'_{\sim p}\, \delta_\circ g''_{<p})_{\sim p} \rangle, \\
MLHH &= \sum_q \sum_{r>q-2} \langle \partial^m f_q, g_{<q}, (\delta_\circ^2 g'_{\sim r}\, \delta_\circ g''_{\sim r})_{\sim q} \rangle, \\
HLLH &= \sum_q \langle \partial^m f_q, g_{<q}, (\delta_\circ^2 g'_{<q}\, \delta_\circ g''_{\sim q})_{\sim q} \rangle, \\
HLHL &= \sum_q \sum_{r>q-2} \langle \partial^m f_q, g_{<q}, (\delta_\circ^2 g'_{\sim q}\, \delta_\circ g''_{<q})_{\sim q} \rangle, \\
HHLL &= \sum_q \langle \partial^m f_q, g_{\sim q}, (\delta_\circ^2 g'_{<q+2}\, \delta_\circ g''_{<q+2})_{<q} \rangle, \\
LLHH &= \sum_q \sum_{r>q+2} \langle \partial^m f_q, g_{\sim q}, (\delta_\circ^2 g'_{\sim r}\, \delta_\circ g''_{\sim r})_{<q} \rangle.
\end{aligned}$$

The general strategy will be to split the z-integral into short range $|z| < 1/\lambda$ and long range $|z| > 1/\lambda$ where λ is the highest frequency of the components at hand. In the short range we use all the available gradients of g', g'', which is only one.

Let us start with the low-medium-high-high term,

$$\begin{aligned}
\int_{\mathbb{T}^n} |LMHH| \frac{h(z)}{|z|^{n+1+\alpha}} \,\mathrm{d}z \leq& \sum_q \sum_{p>q-1} \sum_{r>p-2} \int_{|z|<1/\lambda_r} |\langle \partial^m f_q, g_{\sim p}, (\delta_\circ^2 g'_{\sim r}\, \delta_\circ g''_{\sim r})_{\sim p} \rangle| \frac{h(z)}{|z|^{n+1+\alpha}} \,\mathrm{d}z \\
&+ \sum_q \sum_{p>q-1} \sum_{r>p-2} \int_{|z|>1/\lambda_r} |\langle \partial^m f_q, g_{\sim p}, (\delta_\circ^2 g'_{\sim r}\, \delta_\circ g''_{\sim r})_{\sim p} \rangle| \frac{h(z)}{|z|^{n+1+\alpha}} \,\mathrm{d}z \\
\leq& \sum_q \sum_{p>q-1} \sum_{r>p-2} \int_{|z|<1/\lambda_r} \lambda_q^m \|f_q\|_2 \|g_{\sim p}\|_\infty \|\nabla^2 g'_{\sim r}\|_2 \|\nabla g''_{\sim r}\|_\infty \frac{h(z)}{|z|^{n+\alpha-2}} \,\mathrm{d}z \\
&+ \sum_q \sum_{p>q-1} \sum_{r>p-2} \int_{|z|>1/\lambda_r} \lambda_q^m \|f_q\|_2 \|g_{\sim p}\|_\infty \|g'_{\sim r}\|_2 \|\nabla g''_{\sim r}\|_\infty \frac{h(z)}{|z|^{n+\alpha}} \,\mathrm{d}z.
\end{aligned}$$

The first integral results in the term

$$\|\nabla^2 g'_{\sim r}\|_2 \|\nabla g''_{\sim r}\|_\infty \lambda_r^{\alpha-2} \lesssim \lambda_r^{-m} \|\nabla^{m+\alpha} g'_{\sim r}\|_2,$$

and the second results in a similar term,

$$\|g'_{\sim r}\|_2 \|\nabla g''_{\sim r}\|_\infty \lambda_r^{\alpha} \lesssim \lambda_r^{-m} \|\nabla^{m+\alpha} g'_{\sim r}\|_2.$$

We continue,

$$\lesssim \|g'\|_{H^{m+\alpha}} \sum_q \sum_{p>q-1} \frac{\lambda_q^m}{\lambda_p^m} \|f_q\|_2 \|g_{\sim p}\|_\infty \lesssim \|g'\|_{H^{m+\alpha}} \|f\|_2.$$

In all the remaining seven terms we proceed similarly with a few modifications. Next up is MHLH,

$$\begin{aligned}
\int_{\mathbb{T}^n} |MHLH| \frac{h(z)}{|z|^{n+1+\alpha}} \, \mathrm{d}z &\leq \sum_q \sum_{p>q-1} \int_{|z|<1/\lambda_p} |\langle \partial^m f_q, g_{\sim p}, \, (\delta_\circ^2 g'_{<p} \, \delta_\circ g''_{\sim p})_{\sim p} \rangle| \frac{h(z)}{|z|^{n+1+\alpha}} \, \mathrm{d}z \\
&+ \sum_q \sum_{p>q-1} \int_{|z|>1/\lambda_p} |\langle \partial^m f_q, g_{\sim p}, \, (\delta_\circ^2 g'_{<p} \, \delta_\circ g''_{\sim p})_{\sim p} \rangle| \frac{h(z)}{|z|^{n+1+\alpha}} \, \mathrm{d}z \\
&\leq \sum_q \sum_{p>q-1} \lambda_q^m \|f_q\|_2 \|g_{\sim p}\|_\infty \|\nabla^2 g'_{<p}\|_\infty \|\nabla g''_{\sim p}\|_2 \lambda_p^{\alpha-2} \\
&+ \sum_q \sum_{p>q-1} \lambda_q^m \|f_q\|_2 \|g_{\sim p}\|_\infty \|\nabla g'_{<p}\|_\infty \|g''_{\sim p}\|_2 \lambda_p^{\alpha} \\
&\leq \sum_q \sum_{p>q-1} \lambda_q^m \|f_q\|_2 \lambda_p^{-1} \|\nabla g_{\sim p}\|_\infty \|\nabla g'_{<p}\|_\infty \|\nabla^{m+\alpha} g''_{\sim p}\|_2 \lambda_p^{-m} \\
&\lesssim \|g''\|_{H^{m+\alpha}} \sum_q \lambda_q^m \|f_q\|_2 \lambda_q^{-m-1} \leq \|g''\|_{H^{m+\alpha}} \|f\|_2.
\end{aligned}$$

Next, in a similar manner,

$$\begin{aligned}
\int_{\mathbb{T}^n} |MHHL| \frac{h(z)}{|z|^{n+1+\alpha}} \, \mathrm{d}z &\leq \sum_q \sum_{p>q-1} \lambda_q^m \|f_q\|_2 \|g_{\sim p}\|_\infty \|\nabla^2 g'_{\sim p}\|_2 \|\nabla g''_{<p}\|_\infty \lambda_p^{\alpha-2} \\
&+ \sum_q \sum_{p>q-1} \lambda_q^m \|f_q\|_2 \|g_{\sim p}\|_\infty \|g'_{\sim p}\|_2 \|\nabla g''_{<p}\|_\infty \lambda_p^{\alpha} \\
&\leq \sum_q \sum_{p>q-1} \lambda_q^m \|f_q\|_2 \lambda_p^{-1} \|\nabla g_{\sim p}\|_\infty \|\nabla^{m+\alpha} g'_{\sim p}\|_2 \lambda_p^{-m} \lesssim \|g'\|_{H^{m+\alpha}} \|f\|_2.
\end{aligned}$$

Next,

$$
\begin{aligned}
\int_{\mathbb{T}^n} |MLHH| \frac{h(z)}{|z|^{n+1+\alpha}}\, \mathrm{d}z &\leq \sum_q \sum_{r>q-2} \int_{|z|<1/\lambda_r} |\langle \partial^m f_q, g_{<q}, \, (\delta_\circ^2 g'_{\sim r}\, \delta_\circ g''_{\sim r})_{\sim q}\rangle| \frac{h(z)}{|z|^{n+1+\alpha}}\, \mathrm{d}z \\
&+ \sum_q \sum_{r>q-2} \int_{|z|>1/\lambda_r} |\langle \partial^m f_q, g_{<q}, \, (\delta_\circ^2 g'_{\sim r}\, \delta_\circ g''_{\sim r})_{\sim q}\rangle| \frac{h(z)}{|z|^{n+1+\alpha}}\, \mathrm{d}z \\
&\leq \sum_q \sum_{r>q-2} \lambda_q^m \|f_q\|_2 \|g_{<q}\|_\infty \|\nabla^2 g'_{\sim r}\|_2 \|\nabla g''_{\sim r}\|_\infty \lambda_r^{\alpha-2} \\
&+ \sum_q \sum_{r>q-2} \lambda_q^m \|f_q\|_2 \|g_{<q}\|_\infty \|g'_{\sim r}\|_2 \|\nabla g''_{\sim r}\|_\infty \lambda_r^{\alpha} \\
&\lesssim \sum_q \sum_{r>q-2} \|f_q\|_2 \frac{\lambda_q^m}{\lambda_r^m} \|\nabla^{m+\alpha} g'_{\sim r}\|_2 \lesssim \sum_r \|\nabla^{m+\alpha} g'_{\sim r}\|_2 \sum_{q<r+2} \frac{\lambda_q^m}{\lambda_r^m} \|f_q\|_2 \\
&\lesssim \|g'\|_{H^{m+\alpha}} \left(\sum_r \left(\sum_{q<r+2} \frac{\lambda_q^m}{\lambda_r^m} \|f_q\|_2 \right)^2 \right)^{1/2} \\
&\leq \|g'\|_{H^{m+\alpha}} \left(\sum_r \sum_{q<r+2} \frac{\lambda_q^m}{\lambda_r^m} \|f_q\|_2^2 \right)^{1/2} \leq \|g'\|_{H^{m+\alpha}} \|f\|_2.
\end{aligned}
$$

In the next HLLH and HLHL terms we split relative to the scale $1/\lambda_q$ to obtain

$$
\begin{aligned}
\int_{\mathbb{T}^n} |HLLH| \frac{h(z)}{|z|^{n+1+\alpha}}\, \mathrm{d}z &\leq \sum_q \lambda_q^m \|f_q\|_2 \|g_{<q}\|_\infty \|\nabla^2 g'_{<q}\|_\infty \|\nabla g''_{\sim q}\|_2 \lambda_q^{\alpha-2} \\
&+ \sum_q \lambda_q^m \|f_q\|_2 \|g_{<q}\|_\infty \|\nabla g'_{<q}\|_\infty \|g''_{\sim q}\|_2 \lambda_q^{\alpha} \lesssim \sum_q \|f_q\|_2 \|\nabla^{m+\alpha} g''_{\sim q}\|_2 \leq \|f\|_2 \|g''\|_{H^{m+\alpha}}.
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathbb{T}^n} |HLHL| \frac{h(z)}{|z|^{n+1+\alpha}}\, \mathrm{d}z &\leq \sum_q \lambda_q^m \|f_q\|_2 \|g_{<q}\|_\infty \|\nabla^2 g'_{\sim q}\|_2 \|\nabla g''_{<q}\|_\infty \lambda_q^{\alpha-2} \\
&+ \sum_q \lambda_q^m \|f_q\|_2 \|g_{<q}\|_\infty \|g'_{\sim q}\|_2 \|\nabla g''_{<q}\|_\infty \lambda_q^{\alpha} \lesssim \sum_q \|f_q\|_2 \|\nabla^{m+\alpha} g'_{\sim q}\|_2 \leq \|f\|_2 \|g'\|_{H^{m+\alpha}}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
\int_{\mathbb{T}^n} |HHLL| \frac{h(z)}{|z|^{n+1+\alpha}}\, \mathrm{d}z &\leq \sum_q \int_{|z|<1/\lambda_q} |\langle \partial^m f_q, g_{\sim q}, \, (\delta_\circ^2 g'_{<q+2}\, \delta_\circ g''_{<q+2})_{<q}\rangle| \frac{h(z)}{|z|^{n+1+\alpha}}\, \mathrm{d}z \\
&+ \sum_q \int_{|z|>1/\lambda_q} |\langle \partial^m f_q, g_{\sim q}, \, (\delta_\circ^2 g'_{<q+2}\, \delta_\circ g''_{<q+2})_{<q}\rangle| \frac{h(z)}{|z|^{n+1+\alpha}}\, \mathrm{d}z \\
&\leq \sum_q \lambda_q^m \|f_q\|_2 \|g_{\sim q}\|_2 \|\nabla^2 g'_{<q+2}\|_\infty \|\nabla g''_{<q+2}\|_\infty \lambda_q^{\alpha-2} \\
&+ \sum_q \lambda_q^m \|f_q\|_2 \|g_{\sim q}\|_2 \|g'_{<q+2}\|_\infty \|\nabla g''_{<q+2}\|_\infty \lambda_q^{\alpha} \\
&\lesssim \sum_q \|f_q\|_2 \|\nabla^{m+\alpha} g_{\sim q}\|_2 \leq \|f\|_2 \|g\|_{H^{m+\alpha}}.
\end{aligned}
$$

Next,

$$\begin{aligned}\int_{\mathbb{T}^n} |LLHH| \frac{h(z)}{|z|^{n+1+\alpha}} \,\mathrm{d}z \leq& \sum_q \sum_{r>q+2} \int_{|z|<1/\lambda_r} |\langle \partial^m f_q, g_{\sim q}, (\delta_\circ^2 g'_{\sim r}\, \delta_\circ g''_{\sim r})_{<q}\rangle| \frac{h(z)}{|z|^{n+1+\alpha}} \,\mathrm{d}z \\ &+ \sum_q \sum_{r>q+2} \int_{|z|>1/\lambda_r} |\langle \partial^m f_q, g_{\sim q}, (\delta_\circ^2 g'_{\sim r}\, \delta_\circ g''_{\sim r})_{<q}\rangle| \frac{h(z)}{|z|^{n+1+\alpha}} \,\mathrm{d}z \\ &\leq \sum_q \sum_{r>q+2} \lambda_q^m \|f_q\|_2 \|g_{\sim q}\|_\infty \|\nabla^2 g'_{\sim r}\|_2 \|\nabla g''_{\sim r}\|_\infty \lambda_r^{\alpha-2} \\ &+ \sum_q \sum_{r>q+2} \lambda_q^m \|f_q\|_2 \|g_{\sim q}\|_\infty \|g'_{\sim r}\|_2 \|\nabla g''_{\sim r}\|_\infty \lambda_r^{\alpha} \\ &\leq \sum_q \sum_{r>q+2} \lambda_q^m \|f_q\|_2 \|\nabla^{m+\alpha} g'_{\sim r}\|_2 \lambda_r^{-m},\end{aligned}$$

and we finish as in the MLHH case.

□

4. Global solutions

4.1. **Parallel shear flocks.** One immediate application of the continuation criterion would be to parallel shear flocks. Up to rotation these are given by velocities independent of x_1,

$$u = (U(x_2, \dots, x_n, t), 0, \dots, 0), \qquad \rho = \rho_0(x_2, \dots, x_n). \tag{26}$$

In this case the density is stationary $\partial_t \rho = 0$, and so is the e-quantity. The momentum equation takes the form of pure diffusion

$$U_t(x) = \int_{\mathbb{T}^n} \phi(x, x+z) \rho_0(x+z) \delta_z U(x) \,\mathrm{d}z. \tag{27}$$

Taking partial derivative ∂ with respect to any variable results in

$$\partial U_t = \int_{\mathbb{T}^n} \phi(x, x+z) \rho_0(x+z) \delta_z \partial U(x) \,\mathrm{d}z + \int_{\mathbb{T}^n} \partial_x[\phi(x, x+z)\rho_0(x+z)] \delta_z U(x) \,\mathrm{d}z.$$

If $\rho_0 \in H^{m+\alpha}$ and $\rho_0(x) \geq \underline{\rho} > 0$, then $\partial_x[\phi(x, x+z)\rho_0(x+z)]$ is still a kernel of singularity $n + \alpha$. Thus, if $\alpha < 1$, the last integral is bounded by $C\|\nabla U\|_\infty$. Evaluating the above at the point of maximum of ∂U and summing over all partials gives the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla U\|_\infty \leq C \|\nabla U\|_\infty.$$

It is therefore a priori bounded and the criterion applies.

To handle the range $1 \leq \alpha < 2$, we note that (27) fall under a general class of fractional diffusion equations

$$w_t(x) = \int_{\mathbb{T}^n} K(x, z, t) \delta_z w(x) \,\mathrm{d}z.$$

Typical regularity results for such equations requires either the assumption of evenness of K in z or symmetry in $x, y = x + z$. In our settings, the kernel is given by

$$K = \frac{h(z)}{|z|^{n+\alpha}} \kappa(x, x+z), \qquad \text{with} \qquad \kappa(x, x+z) = \frac{\rho_0(x+z)}{\left(\fint_{\Omega(x,x+z)} \rho_0(\xi) \,\mathrm{d}\xi \right)^{\frac{\tau}{n}}},$$

which satisfies neither of the above requirements. However, freezing the coefficients and representing the kernel as a sum of the main even and residual parts where

$$F(x, z) = \frac{h(z)}{|z|^{n+1}} \kappa(x, x), \qquad G(x, z) = \frac{h(z)}{|z|^{n+1}} \left(\kappa(x, x+z) - \kappa(x, x) \right),$$

fulfills the hypotheses of Theorem 1.1 of [22] in the particular case $\alpha = 1$, which provides Schauder estimates

$$\|U\|_{C^{1+\gamma}(\mathbb{T}^n\times[T/2,T))} \lesssim \|U\|_{L^\infty(\mathbb{T}^n\times[0,T))}.$$

Since the right hand side is uniformly bounded due to the maximum principle, this fulfills the continuation criterion and the proof is complete for $\alpha = 1$.

For the subcritical case $\alpha > 1$ we can adopt [29, Theorem 8.1]. To get the model to satisfy its assumptions, we use a cut-off function $\chi(z)$ and small $\varepsilon > 0$ to break the residual term G further into inner singular part

$$G_\varepsilon = \frac{h(z)\chi(z/\varepsilon)}{|z|^{n+\alpha}} \left[\frac{\rho_0(x+z)}{\left(\fint_{\Omega(0,z)} \rho_0(x+\xi)\,\mathrm{d}\xi\right)^{\frac{\tau}{n}}} - \frac{\rho_0(x)}{\rho_0(x)^{\frac{\tau}{n}}} \right],$$

and the outer regular part

$$H_\varepsilon = \frac{h(z)(1-\chi(z/\varepsilon))}{|z|^{n+\alpha}} \left[\frac{\rho_0(x+z)}{\left(\fint_{\Omega(0,z)} \rho_0(x+\xi)\,\mathrm{d}\xi\right)^{\frac{\tau}{n}}} - \frac{\rho_0(x)}{\rho_0(x)^{\frac{\tau}{n}}} \right].$$

The integral

$$f = \int H_\varepsilon(x,z)\delta_z U(x)\,\mathrm{d}z,$$

contributes with a bounded source to the equation, while the principal kernel

$$K_\varepsilon = F + G_\varepsilon,$$

for small $\varepsilon > 0$ satisfies all the assumptions (A1)–(A4) of [29].

For $\alpha > 1$ it is necessary to use the next Taylor term in the definition of the finite difference, so we add and subtract it in the singular integral to produce an extra drift:

$$U_t + b\cdot\nabla U = \int K_\varepsilon(x,z)[\delta_z U(x) - z\cdot\nabla U(x)]\,\mathrm{d}z + f, \tag{28}$$

where

$$b(x) = -\int_{\mathbb{T}^n} G_\varepsilon(x,z) z\,\mathrm{d}z.$$

The latter is no longer a singular integral and hence b is bounded. This fulfills all the assumptions of [29, Theorem 8.1] pertaining to the equation (28), and the $C^{1+\gamma}$-continuity of u follows. We have proved the following theorem.

Theorem 4.1. *For any $U_0, \rho_0 \in H^{m+1}\times H^{m+\alpha}$, $0<\alpha<2$, there exists a unique global solution to the equation (1)–(2) in the form of a parallel shear flock (26) which belongs to the same class.*

4.2. **Nearly aligned flocks.** In this section we reveal another class of global solutions with nearly aligned velocity field. We denote homogeneous Hölder norms by

$$[f]_k = \|\nabla^k f\|_\infty.$$

Theorem 4.2. *There exists an $R_0 > 0$ and $N\in\mathbb{N}$ dependent only on the parameters of the system and m such that if $R > R_0$ and the initial condition satisfies*

$$Y_m(0) + \underline{\rho}_0^{-1} + \overline{\rho}_0 \le R, \qquad \mathcal{A}_0 \le \frac{1}{R^N},$$

then there exists a global unique solution to (1) starting from such initial condition. Moreover, such solution will align exponentially fast,

$$\mathcal{A}(t) \le \frac{1}{R^{N-b}} e^{-\frac{c}{R^a}t},$$

where $a, b > 0$ depend only on the parameters of the system, and flock to a smooth traveling density profile ρ_∞:

$$\rho \to \rho_\infty(x - t\bar{u}). \tag{29}$$

Proof. We assume without loss of generality that the momentum vanishes, $P = 0$.

Let us observe that since $\|u(t)\|_{H^{m+1}}$ on a given time interval controls $[\rho]_1$ via the continuity equation, the local solution to (1) according to our criterion can be extended up to the critical time $t = t^*$, at which

$$\|u(t^*)\|_{H^{m+1}} = 2R,$$

for the first time, i.e.

$$\|u(t)\|_{H^{m+1}} < 2R, \qquad t < t^*.$$

Our goal will be to show that such time t^* never happens, and thus the solution is global.

Lemma 4.3. *On the same time interval $[0, t^*]$ we have*

$$\frac{1}{2R} \leq \underline{\rho} \leq \overline{\rho} \leq 2R. \tag{30}$$

Proof. Let t^{**} be the first time when one of these inequalities fails. We would like to show that $t^{**} > t^*$. If not, let us make a preliminary alignment estimate on the shorter time interval $[0, t^{**}]$.

Recall the energy law (10). Note that

$$\phi(x, y) \geq \frac{\lambda}{M^{\tau/n}} \mathbb{1}_{|x-y|<r_0}.$$

So, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leq -\frac{c}{R^{2+\frac{\tau}{n}}} \int_{|x-y|<r_0} |u(x) - u(y)|^2 \,\mathrm{d}y \,\mathrm{d}x.$$

Applying [25, Lemma 2.1], we further continue

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leq -\frac{c}{R^{2+\frac{\tau}{n}}} \int_{\mathbb{T}^n} |u(x) - \bar{u}|^2 \,\mathrm{d}x,$$

where $\bar{u}$ is the usual average of u. Note that it may not be 0 despite vanishing momentum. Let us reinsert the density noting that $\rho/R \leq 2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leq -\frac{c}{R^{3+\frac{\tau}{n}}} \int_{\mathbb{T}^n} \rho(x)|u(x) - \bar{u}|^2 \,\mathrm{d}x.$$

Expanding the square and using vanishing of the momentum we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \leq -\frac{c}{R^{3+\frac{\tau}{n}}} \int_{\mathbb{T}^n} \rho(x)|u(x)|^2 \,\mathrm{d}x = -\frac{c}{R^{3+\frac{\tau}{n}}}\mathcal{E}.$$

Thus, on the time interval $[0, t^{**}]$ we have

$$\mathcal{E}(t) \leq \mathcal{E}_0 e^{-\frac{c}{R^a}t}, \qquad a = 3 + \frac{\tau}{n}.$$

Let us note that initial energy is bounded by (again using the vanishing momentum)

$$\mathcal{E}_0 \leq \int_{\mathbb{T}^{2n}} \rho_0(x)\rho_0(y)|u_0(x) - u_0(y)|^2 \,\mathrm{d}y \,\mathrm{d}x \leq \mathcal{A}_0 M^2 \leq CR^{2-N}.$$

So,

$$\mathcal{E}(t) \leq CR^{2-N} e^{-\frac{c}{R^a}t}.$$

Now we estimate the decay of the amplitude $\mathcal{A}$ itself. Let us pick one coordinate of u and evaluate at a point of its maximum x_+:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}u(x_+) &\leq \frac{c}{R^{\tau/n}}\int_{|z|<r_0}\rho(x_+ + z)[u(x_+ + z) - u(x_+)]\,\mathrm{d}z \leq \frac{c}{R^{\tau/n}}M^{1/2}\mathcal{E}^{1/2} - \frac{c}{R^{\tau/n}}u(x_+)\\ &\leq CR^{5/2-\tau/n-N}e^{-\frac{c}{R^a}t} - \frac{c}{R^{\tau/n}}u(x_+).\end{aligned}$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}u(x_-) \geq -CR^{5/2-\tau/n-N}e^{-\frac{c}{R^a}t} + \frac{c}{R^{\tau/n}}u(x_-).$$

Subtracting the two,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A} \leq CR^{5/2-\tau/n-N}e^{-\frac{c}{R^a}t} - \frac{c}{R^{\tau/n}}\mathcal{A}.$$

By the Grönwall inquality,

$$\mathcal{A}(t) \leq \mathcal{A}_0 e^{-\frac{c}{R^{\tau/n}}t} + CR^{5/2-\tau/n-N}te^{-\frac{c}{R^a}t} \leq \frac{1}{R^N}e^{-\frac{c}{R^a}t} + CR^{5/2-\tau/n-N+a}e^{-\frac{c}{R^a}t}.$$

Thus,

$$\mathcal{A}(t) \leq \frac{1}{R^{N-b}}e^{-\frac{c}{R^a}t}, \tag{31}$$

where b depends only on n,τ. By interpolation,

$$\|\nabla u\|_\infty \leq \mathcal{A}^\theta \|u\|_{H^{m+1}}^{1-\theta} \leq \frac{1}{R^{\theta(N-b)+\theta-1}}e^{-\frac{\theta c}{R^a}t}.$$

Integrating the continuity equation along characteristics, we obtain

$$\overline{\rho} \leq \|\rho_0\|_\infty \exp\left\{\int_0^t \|\nabla u\|_\infty \,\mathrm{d}s\right\} \leq R\exp\left\{\frac{1}{\theta c R^{\theta(N-b)+\theta-1-a}}\right\},$$

and similarly,

$$\underline{\rho} \geq \frac{1}{R}\exp\left\{-\frac{1}{\theta c R^{\theta(N-b)+\theta-1-a}}\right\}.$$

Clearly, if R and N are large enough the exponential is < 2. This leads to a contradiction with the definition of t^{**}. □

From this point on we will denote by $\mathcal{N}(t)$ any "negligent" quantity which has a bound of the form

$$\mathcal{N}(t) \leq \frac{C}{R^{\theta N}}e^{-\frac{c}{R^a}t},$$

where $C, c > 0$, and $0 < \theta < 1$ and $a > 0$ depend only on τ, n, α, m, the parameters of the system. We observe the identities

$$R^b\mathcal{N} \sim \mathcal{N}, \quad \mathcal{N}^\lambda \sim \mathcal{N}, \quad \text{etc.}$$

As a consequence of the proof of the lemma we have shown that as long as (30) holds, the estimate on the amplitude (31) holds. As a result, by interpolation, we have similar exponential bounds in Hölder classes,

$$[u]_1, [u]_2, [u]_3 \leq \mathcal{N},$$

and as a consequence, from the continuity equation,

$$[\rho]_1, [\rho]_2 \leq 2R,$$

for all $t < t^*$, provided N is large enough. As a further consequence, we obtain

$$\|e\|_\infty \leq [u]_1 + \|\mathcal{L}_\phi \rho\|_\infty.$$

We appeal to [35, Lemma B.1] (with $r = 1$ and $\gamma = 0$) to conclude

$$\|\mathcal{L}_\phi \rho\|_\infty \leq R^b([\rho]_2 + \|\rho\|_\infty + [\rho]_1^2) \leq R^b,$$

where b depends only on the parameters of the system. Thus,

$$\|e(t)\|_\infty \leq CR^b, \qquad t < t^*. \tag{32}$$

Now, if we look back at the e-equation, we can see that all the transport terms contain a power of $[u]_1$, and with (32) can be estimated by $\leq \mathcal{N} Y_m$. At the same time the topological terms $\mathcal{T}_1, \mathcal{T}_2$ will include a power of $[u]_1$ as a result of application of Lemma 2.2,

$$\|\mathcal{T}_{1,2}\| \leq \mathcal{N} \|\rho\|_{H^{m+\alpha}}^{\theta_1} \|u\|_{H^{m+\alpha}}^{\theta_2} \leq \mathcal{N} Y_m + \mathcal{N} + \frac{\varepsilon}{R^{1+\tau/n}} \|u\|^2_{H^{m+1+\frac{\alpha}{2}}}.$$

Similarly, all paraproduct estimates of Lemma 3.4 for $\mathcal{T}_3, \mathcal{T}_4$ will include either $\|u\|_{H^{m+\alpha}}$ or $\|\rho\|_{H^{m+\alpha}}$ coupled with a power of $[u]_1$. Again, in the latter case this results in a factor of $\mathcal{N}$, while in the former case, by interpolation

$$\|u\|_{H^{m+\alpha}} \leq \mathcal{N} \|u\|^\theta_{H^{m+1+\frac{\alpha}{2}}}.$$

In summary we have a factor of $\mathcal{N}$ to appear in the main term of the e-equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|e\|^2_{H^m} \leq \mathcal{N} Y_m + \mathcal{N} + \frac{\varepsilon}{R^{1+\tau/n}} \|u\|^2_{H^{m+1+\frac{\alpha}{2}}}.$$

Examining the u-equation in a similar manner we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|^2_{H^{m+1}} \leq \mathcal{N} Y_m + \mathcal{N} - \frac{c_0}{R^{1+\tau/n}} \|u\|^2_{H^{m+1+\frac{\alpha}{2}}}. \tag{33}$$

Adding the two together, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} (Y_m + 1) \leq \mathcal{N}(Y_m + 1) - \frac{c_0}{2R^{1+\tau/n}} \|u\|^2_{H^{m+1+\frac{\alpha}{2}}}.$$

Ignoring the dissipation term for a moment we conclude by integration that

$$(Y_m + 1)(t^*) \leq (R+1) e^{R^{-\theta N}} \leq 2R,$$

if N and R are large enough. Thus,

$$Y_m(t^*) \leq 3R.$$

Plugging this back into (33) we conclude that at the critical time t^*,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u\|^2_{H^{m+1}} \leq 3\mathcal{N} R + \mathcal{N} - c_0 R^{1-\tau/n} \leq 4R^{1-\theta N} - c_0 R^{1-\tau/n} < 0,$$

if again R and N are chosen large enough. This is a contradiction with the definition of t^*.

The flock convergence (29) follows immediately from the continuity equation and exponential decay of all norms of u up to H^{m+1}. □

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