Erratum: Towards an understanding of ramified extensions of structured ring spectra

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The original version of this article unfortunately contained a mistake in a calculation that resulted in erroneous statements in Lemma 5.1 and Theorem 5.2. We are grateful to Eva Höning who discovered the mistake.

LEMMA 0·1 (replaces Lemma 5.1). *There is an isomorphism of graded commutative augmented ku_*-algebras*

$$(ku \wedge_{ko} ku)_* \cong ku_*[s]/(s^2 - su)$$

with $|s| = 2$, where the ku_*-algebra structure on $(ku \wedge_{ko} ku)_*$ is from the left and the augmentation is given by the multiplication $m\colon ku \wedge_{ko} ku \to ku$ and by $s \mapsto 0$.

Proof. Smashing Wood's cofiber sequence $ko \xrightarrow{\iota} ku \xrightarrow{\mathrm{J}} \Sigma^2 ko$ with ku (from the left) over ko gives a split exact sequence (with unit isomorphism $ku \cong ku \wedge_{ko} ko$ suppressed)

$$\begin{tikzcd} 0 \arrow[r] & \pi_* ku \arrow[r, "(1\wedge\iota)_*"] & \pi_*(ku \wedge_{ko} ku) \arrow[l, bend right, "m_*"'] \arrow[r, "(1\wedge\mathrm{J})_*"] & \pi_* \Sigma^2 ku \arrow[r] & 0. \end{tikzcd}$$

Let $u \in \pi_2 ku$ be the generator with $\mathrm{J}_*(u) = \Sigma^2 2 \in \pi_2 \Sigma^2 ko \cong \mathbb{Z}$. Let u_l and u_r be the images of u in $\pi_2(ku \wedge_{ko} ku)$ induced by the left and right inclusion of ku in $ku \wedge_{ko} ku$. If s is the unique element in $\pi_2(ku \wedge_{ko} ku)$ with $(1 \wedge \mathrm{J})_* s = -\Sigma^2 1$ and $m_* s = 0$, then $(1 \wedge \mathrm{J})_*(u_r + 2s) = 1 \cdot \mathrm{J}_* u - 2 = 0$ and $m_*(u_r + 2s) = u$. Since also $(1 \wedge \mathrm{J})_* u_l = 0$ and $m_* u_l = u$ we must have $u_r + 2s = u_l$.

In $ku_* \otimes_{ko_*} ku_*$, and hence also in $\pi_4(ku \wedge_{ko} ku)$, we have that $2u_r^2 - 2u_l^2 = 0$. As

$\pi_*(ku \wedge_{ko} ku)$ is torsion free, we get $u_r^2 - u_l^2 = 0$ and therefore

$$u_r^2 - u_l^2 = (u_l - 2s)^2 - u_l^2 = 4s^2 - 4su_l = 0.$$

Again, since there is no torsion, this yields $s^2 - su_l = 0$. □

THEOREM 0·2 (replaces Theorem 5.2). *The Tor spectral sequence*

$$E^2_{*,*} = \mathsf{Tor}^{(ku \wedge_{ko} ku)_*}_{*,*}(ku_*, ku_*) \Rightarrow \mathsf{THH}^{ko}_*(ku)$$

collapses at the E^2*-page and* $\mathsf{THH}^{ko}_*(ku)$ *is a square zero extension of* ku_**:*

$$\mathsf{THH}^{ko}_*(ku) \cong ku_* \rtimes (ku_*/u)\{y_0, y_1, \ldots\}$$

with $|y_j| = (1 + |u|)(2j + 1) = 3(2j + 1)$.

Proof. Lemma 0·1 implies that the E^2-term of the Tor spectral sequence is

$$E^2_{*,*} = \mathsf{Tor}^{(ku \wedge_{ko} ku)_*}_{*,*}(ku_*, ku_*) = \mathsf{Tor}^{ku_*[s]/(s^2 - su)}_{*,*}(ku_*, ku_*).$$

We have a periodic free resolution of ku_* as a module over $ku_*[s]/(s^2 - su)$

$$\ldots \xrightarrow{s} \Sigma^4 ku_*[s]/(s^2 - su) \xrightarrow{s-u} \Sigma^2 ku_*[s]/(s^2 - su) \xrightarrow{s} ku_*[s]/(s^2 - su).$$

Tensoring this down to ku_* yields

$$\ldots \xrightarrow{0} \Sigma^4 ku_* \xrightarrow{-u} \Sigma^2 ku_* \xrightarrow{0} ku_*.$$

As ku_* splits off $\mathsf{THH}^{ko}_*(ku)$, the zero column has to survive and cannot be hit by differentials and hence all differentials are trivial.

For the E^∞-term we therefore get $E^\infty_{0,*} \cong ku_*$, $E^\infty_{2j,*} = 0$ for $j > 0$, and $E^\infty_{2j+1,*} \cong (ku_*/u)\{y_j\}$ for y_j in bidegree $(2j + 1, 4j + 2)$ if $j > 0$. So even total degrees occur only in $E^\infty_{0,*}$ and odd total degrees occur only in at most one bidegree and we do not need to worry about additive extensions. As y_j corresponds to $\Sigma^{4j+2}1 \in \Sigma^{4j+2}ku_*$, the action of $u^i \in ku_*$ on y_j is $u^i y_j$ and this is trivial for $i \geqslant 1$ in $(ku_*/u)\{y_j\}$ so $u^i y_j$ is zero in $E^\infty_{2j+1,4j+2}$. But $E^\infty_{2j+1,4j+2}$ has all the elements of total degree $6j + 3$ in the entire E^∞-term, so in fact the element in $\mathsf{THH}^{ko}_*(ku)$ that y_j represents is killed by multiplication by u^i for any $i \geqslant 1$. Thus we have no nontrivial products of the u^i, $i \geqslant 1$, and the odd dimensional elements.

Since the elements of $\mathsf{THH}^{ko}_*(ku)$ represented by the y_i are all in odd degrees, if there were nonzero products among them they would have to be elements in $E^\infty_{0,*} \cong ku_*$. But elements in ku_* are not killed by multiplying by u, whereas the elements represented by the y_j are. So there can be no such nontrivial products. □