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Principal spectral theory and asynchronous exponential growth for age-structured models with nonlocal diffusion of Neumann type

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Abstract

In this paper we study the principal spectral theory and asynchronous exponential growth for age-structured models with nonlocal diffusion of Neumann type. First, we provide two general sufficient conditions to guarantee existence of the principal eigenvalue of the age-structured operator with nonlocal diffusion. Then we show that such conditions are also enough to ensure that the semigroup generated by solutions of the age-structured model with nonlocal diffusion exhibits asynchronous exponential growth. Compared with previous studies, we prove that the semigroup is essentially compact instead of eventually compact, where the latter is usually obtained by showing the compactness of solution trajectories. Next, following the technique developed in Vo (Principal spectral theory of time-periodic nonlocal dispersal operators of Neumann type. arXiv:1911.06119, 2019), we overcome the difficulty that the principal eigenvalue of a nonlocal Neumann operator is not monotone with respect to the domain and obtain some limit properties of the principal eigenvalue with respect to the diffusion rate and diffusion range. Finally, we establish the strong maximum principle for the age-structured operator with nonlocal diffusion.

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Contents

1 Introduction

In this paper we study the following age-structured model with nonlocal diffusion and Neumann boundary condition:

$$\begin{cases} \frac{\partial u(t,a,x)}{\partial t}+\frac{\partial u(t,a,x)}{\partial a} \\ = D\int_\Omega J(x-y)(u(t,a,y)-u(t,a,x))dy- & \mu(a,x)u(t,a,x), \\ & (t,a,x)\in(0,\infty)\times(0,a^+)\times\overline{\Omega}, \\ u(t,0,x)=\int_0^{a^+}\beta(a,x)u(t,a,x)da, & (t,x)\in(0,\infty)\times\overline{\Omega}, \\ u(0,a,x)=u_0(a,x), & (a,x)\in(0,a^+)\times\overline{\Omega}, \end{cases} \tag{1.1}$$

where $u(t,a,x)$ denotes the density of a population at time t of age $a\in[0,a^+]$ at location $x\in\overline{\Omega}$, in which $a^+<\infty$ represents the maximum age and $\Omega\subset\mathbb{R}^N$ is a bounded and convex domain with smooth boundary, $D>0$ is the diffusion rate. The nonlocal diffusion kernel $J\in C^1(\mathbb{R}^N)$ is nonnegative and supported in $B(0,r)$ for some $r>0$, and satisfies $J(0)>0$ and $\int_{\mathbb{R}^N}J(x)dx=1$, where $B(0,r)\subset\mathbb{R}^N$ is the open ball centered at 0 with radius r. We remark that the nonlocal diffusion operator in (1.1) corresponds to an elliptic operator with Neumann boundary condition. We assume that the birth rate $\beta(a,x)$ and death rate $\mu(a,x)$ are positive and belong to $C^{0,1}([0,a^+]\times\overline{\Omega})$ for the convenience to study the spectrum later, where $C^{0,1}$ denotes the continuity with respect to the first variable and continuous differentiability with respect to the second variable. Define

$$\underline{\mu}(a):=\inf_{x\in\overline{\Omega}}\mu(a,x),\quad \overline{\mu}(a):=\sup_{x\in\overline{\Omega}}\mu(a,x),$$
$$\underline{\beta}(a):=\inf_{x\in\overline{\Omega}}\beta(a,x),\quad \overline{\beta}(a):=\sup_{x\in\overline{\Omega}}\beta(a,x).$$

For scalar linear and nonlinear age-structured equations with nonlocal diffusion of Dirichlet type, recently we (Kang and Ruan [24,25], Kang et al. [26]) developed some basic theories including the semigroup of linear operators, asymptotic behavior, spectral theory, asynchronous exponential growth, strong maximum principle, global dynamics, etc.

In the first part of this paper, we continue to study the principal spectral theory for the age-structured model (1.1) with nonlocal diffusion of Neumann type based on our previous work (Kang and Ruan [24]). More precisely, we are interested in the following eigenvalue problem obtained from (1.1):

$$\begin{cases} \frac{\partial u(a,x)}{\partial a} = \frac{D}{\sigma^m} \int_\Omega J_\sigma(x-y)(u(a,y)-u(a,x))dy - \mu(a,x)u(a,x) - \lambda u(a,x), & a \in (0,a^+), x \in \overline{\Omega}, \\ u(0,x) = \int_0^{a^+} \beta(a,x)u(a,x)da, & x \in \overline{\Omega}, \end{cases} \tag{1.2}$$

where $\sigma > 0$ is the diffusion range and $m > 0$ is the cost parameter with $J_\sigma(x) := \frac{1}{\sigma^N} J\left(\frac{x}{\sigma}\right)$ for $x \in \mathbb{R}^N$. Note that (1.2) is with kernel scaling , thus a little bit different from (1.1) without kernel scaling. In fact, the eigenvalue problem associated with (1.1) is a specific case of (1.2); i.e. $\sigma = 1$.

Now let us first briefly recall some history of principal spectral theory of nonlocal diffusion operators. Berestycki et al. [8] introduced the concept of generalized principal eigenvalue for second-order elliptic operators in general domains. Coville [11] studied existence of the principal eigenvalue and gave a non-locally-integrable condition based on the generalized Krein-Rutman theorem. Berestycki et al. [7] further studied the problem in both bounded and unbounded domains and investigated the asymptotic behavior of generalized principal eigenvalue on the diffusion rate. See also Brasseur et al. [10], Coville and Hamel [12], García-Melián and Rossi [19], Li et al. [28], Yang et al. [52], and the references cited therein. On the other hand, Shen and Xie [39] and Rawal and Shen [36] investigated the existence of the principal eigenvalue for autonomous and time periodic cases respectively, where they gave sufficient and necessary conditions for both cases by using the idea of perturbation of positive operators, but they required that the operator has dense domain and generates a positive semigroup of contractions, which seems to be restrictive and in general not satisfied in our case. See also Bao and Shen [6], Liang et al. [29] and Liu et al. [30]. Combining these two directions, recently Shen and Vo [40] and Su et al. [41] discussed the asymptotic behavior of generalized principal eigenvalue on the diffusion rate in the time-periodic case. Kang and Ruan [24] combined their treatment of the nonautonomous case and the theory of resolvent positive operators with their perturbations to deal with age-structured models with nonlocal diffusion of Dirichlet type. Most recently, Vo [47] proved some important limits of the principal eigenvalue for nonlocal operator of Neumann type with respect to the parameters. In the first part of this paper, based on the technique developed in Vo [47], we study the principal spectral theory for age-structured models with nonlocal diffusion of Neumann type and prove some limit properties of the principal eigenvalue with respect to the diffusion rate and diffusion range. Moreover, with the definition of essential compactness in hands, we will observe a key fact that the sufficient conditions which we present for the existence of the principal eigenvalue of an operator from the theory of resolvent positive operators with their perturbations are equivalent to that in some sense obtained from the generalized Krein-Rutman theorem (see Edmunds et al. [16], Nussbuam [34] or Zhang [53]). Furthermore, we improve some limiting properties for $m \in [0, 2)$ that were established in Vo [47] for $m = 0$.

In the second part of this paper, we study asynchronous (balanced) exponential growth of model (1.1). Asynchronous exponential growth is one of the most important properties in population dynamics since it is observed in many reproducing populations before the impacts of crowding and resource limitation take hold. Sharpe and Lotka [38] were the first to study asynchronous exponential growth in age-structured populations. Feller [18] was the first to give a rigorous proof of asynchronous exponential growth in age-structured population dynamics. On the one hand it was recognized that the idea of asynchronous exponential growth can be described in the framework of strongly continuous semigroups of bounded linear operators in Banach spaces, see for example, Diekmann et al. [14], Greiner [20], Greiner and Nagel [21], Greiner et al. [22], Webb [50], and the references cited therein. Webb [49] provided a new proof of the Sharpe-Lotka Theorem by using the theory of semigroups of operators in Banach spaces. Thieme [43] characterized strong and uniform approach to asynchronous exponential growth and Thieme [44] derived conditions for the positively perturbed semigroups to have asynchronous exponential growth. Gyllenberg and Webb [23] considered asynchronous exponential growth of semigroups of nonlinear operators. On the other hand, many researchers have studied asynchronous exponential growth in various structured biological models, see for example, Arino et al. [2,3], Bai and Xu [4], Banasiak et al. [5], Bernard and Gabriel [9], Dyson et al. [15], Farkas [17], Piazzera and Tonetto [35], Webb and Grabosch [51], and the references cited therein.

We would like to mention that asynchronous exponential growth in age-structured models was studied by Webb [50] and was generalized by Thieme [44] to age-structured models with Laplace diffusion. Here we investigate asynchronous exponential growth in age-structured models with nonlocal diffusion (1.1) which is not included in [44]. In fact, we have studied asynchronous exponential growth in such a type of equations in Kang and Ruan [25], where a nonlocal boundary condition was assumed to make the semigroup generated by solutions to be eventually compact and further exhibit asynchronous exponential growth. Here we find that the previous two general sufficient conditions that ensure existence of the principal eigenvalue are also just enough to guarantee the semigroup to be essentially compact (rather than eventually compact as before) and to exhibit asynchronous exponential growth without additional assumptions on the boundary condition as in [25]. Moreover, we would like to mention that the property of asynchronous exponential growth also occurs under the Dirichlet boundary condition.

The paper is organized as follows. In Sect. 2, we first introduce the theory of resolvent positive operators and asynchronous exponential growth. Then we recall a few important theorems that will be used later. In Sect. 3, we establish the basic theory including necessary lemmas and propositions for proving the main results later. In Sect. 4, we show the main theorem and provide two easily verifiable sufficient conditions for the existence of the principal eigenvalue and asynchronous exponential growth. In Sect. 5, we derive a formula for the asynchronous exponential growth. In Sect. 6, we study the effects of diffusion rate and diffusion range on the generalized principal eigenvalue. In Sect. 7, we establish the strong maximum principle which is of fundamental importance and independent interest. The paper ends with a brief discussion in Sect. 8.

Finally, we would like to mention that the conditions that J has a compact support and Ω is bounded can be relaxed. For the principal spectral theory, we only need Ω to be bounded without requiring that J has a compact support. However, in order to study the limiting properties of principal eigenvalues, J needs to be compactly supported. In addition, the condition that Ω is bounded can even be removed if one only defines the generalized principal eigenvalue, see Berestycki et al. [7]. Here to give a unified presentation of the results, we assumed both of them.

2 Preliminaries

In this section we present some preliminary notation and results on positive operators and asynchronous exponential growth.

2.1 Positive operators

Let E be a real or complex Banach space. A nonempty closed subset E_+ is called a cone if the following hold: (1) $E_+ + E_+ \subset E_+$; (2) $\lambda E_+ \subset E_+$ for $\lambda \geq 0$; and (3) $E_+ \cap (-E_+) = \{0\}$. Let $\mathring{E}_+$ be the interior of E_+, $\partial E_+ = E_+ \setminus \mathring{E}_+$ the boundary of E_+, and $\dot{E}_+ = E_+ \setminus \{0\}$.

Let us define the order in E such that $x \leq y$ if and only if $y - x \in E_+$, $x < y$ if and only if $y - x \in \dot{E}_+$, and $x \ll y$ if and only if $y - x \in \mathring{E}$. The cone E_+ is said to be total if the set $\{\psi - \phi : \psi, \phi \in E_+\}$ is dense in E. If a cone has a nonempty interior $\mathring{E}_+$, we call it a solid cone. Obviously, if $\mathring{E}_+ \neq \emptyset$, then E_+ is total. The dual cone E_+^* is the subset of E^* consisting of all positive linear functionals on E; that is, $f \in E_+^*$ if and only if $\langle f, \psi \rangle \geq 0$ for all $\psi \in E_+$. $f \in E_+^*$ is said to be strictly positive if $\langle f, \psi \rangle > 0$ for all $\psi \in E_+ \setminus \{0\}$. The cone E_+ is called generating if $E = E_+ - E_+$ and is called normal if the associated norm on E is semimonotone; that is, there exists a constant $\delta > 0$ such that $0 \leq f \leq g$ implies $\|f\| \leq \delta \|g\|$.

Let $B(E)$ be the collection of all bounded linear operators from E to E. $T \in B(E)$ is said to be positive if $T : E_+ \to E_+$ and $T \in B(E)$ is said to be strongly positive if $T : \dot{E}_+ \to \mathring{E}_+$. Let $\sigma(T)$ and $\sigma_e(T)$ denote the spectrum and essential spectrum of $T \in B(E)$ respectively, whose radius are denoted by $r(T)$ and $r_e(T)$, respectively. Let us recall the following strong version of the generalized Krein-Rutman theorem.

Theorem 2.1 (Zhang [53, Theorem 1.3]) Let E be a Banach space having a cone $E_+ \subset X$ with $\mathring{E}_+ \neq \emptyset$ and $T \in B(E)$ be a strongly positive operator with $r(T) > r_e(T)$. Then $r(T)$ is an algebraically simple eigenvalue of T with an eigenvector $x \in \mathring{E}_+$ and $|\lambda| < r(T)$ for any other eigenvalue of T.

2.2 Resolvent positive operators

Now we recall some results about resolvent positive operators, the readers can refer to Thieme [44–46] and Webb [50] for details. Let Z denote a Banach space and Z_+ be a closed convex cone that is normal and generating. Assume that $C : Z_1 \to Z$ is a positive linear operator defined on a linear subspace Z_1 of Z, which means that $Cx \in Z_+$ for all $x \in Z_1 \cap Z_+$ and C is not the 0 operator.

Definition 2.2 A closed operator A in Z is said to be resolvent positive if the resolvent set of A, $\rho(A)$, contains a ray (ω, ∞) and $(\lambda I - A)^{-1}$ is a positive operator (i.e., it maps Z_+ into itself) for all $\lambda > \omega$.

Definition 2.3 We define the spectral bound of a closed operator A by

$$s(A) = \sup\{\mathrm{Re}\lambda \in \mathbb{R}; \lambda \in \sigma(A)\},$$

the real spectral bound of A by

$$s_{\mathbb{R}}(A) = \sup\{\lambda \in \mathbb{R}; \lambda \in \sigma(A)\},$$

and the spectral radius of A by

$$r(A) = \sup\{|\lambda|; \lambda \in \sigma(A)\}.$$

Definition 2.4 A semigroup $\{S(t)\}_{t\geq 0}$ is said to be essentially compact if its essential growth bound $\omega_1(S)$ is strictly smaller than its growth bound $\omega(S)$, where the growth bound and essential growth bound are defined respectively as follows:

$$\omega(S) := \lim_{t\to\infty} \frac{\log\|S(t)\|}{t}, \quad \omega_1(S) := \lim_{t\to\infty} \frac{\log \alpha[S(t)]}{t}, \tag{2.1}$$

and α denotes the measure of noncompactness, which is defined as follows:

$$\alpha[L] = \inf\{\epsilon > 0, L(\mathbb{B}) \text{ can be covered by a finite number of balls of radius } \leq \epsilon\},$$

where L is a closed linear operator in Z and $\mathbb{B}$ is the unit ball of Z.

By the formulas

$$r_e(S(t)) = e^{\omega_1(S)t}, \quad r(S(t)) = e^{\omega(S)t},$$

we can see that equivalently $r_e(S(t))$ (the essential spectral radius of $S(t)$) is strictly smaller than $r(S(t))$ (the spectral radius of $S(t)$) for one (actually for all) $t > 0$.

If B is a resolvent positive operator and $C : D(B) \to Z$ is a positive linear operator, then $A = B + C$ is called a positive perturbation of B. If $B + C$ is a positive perturbation of B and $\lambda > s(B)$, then $C(\lambda I - B)^{-1}$ is automatically bounded (without

C being necessarily closed). This is a consequence of Z_+ being normal and generating (De Pagter [13, A.2.11]).

Denote the part of A in $\overline{D(A)}$ by A_0 and the part of B in $\overline{D(B)}$ by B_0, respectively. Let A_0 and B_0 generate positive C_0-semigroups $\{S(t)\}_{t\geq 0}$ and $\{T(t)\}_{t\geq 0}$, respectively. If A and B are resolvent positive, then by Thieme [44, Proposition 2.4] we have

$$s(A) = s(A_0) = \omega(S), \quad s(B) = s(B_0) = \omega(T).$$

Theorem 2.5 (Thieme [45, Theorem 3.5]) Let the cone Z_+ be normal and generating and A be a resolvent positive operator in Z. Then $s(A) = s_{\mathbb{R}}(A) < \infty$ and $s(A) \in \sigma(A)$ whenever $s(A) > -\infty$. Moreover, there is a constant $c > 0$ such that

$$\left\|(\lambda I - A)^{-1}\right\| \leq c\left\|(\mathrm{Re}\lambda I - A)^{-1}\right\| \quad \text{whenever} \quad \mathrm{Re}\lambda > s(A).$$

Now define

$$F_\lambda = C(\lambda I - B)^{-1}, \quad \lambda > s(B). \tag{2.2}$$

Definition 2.6 The operator $C : D(B) \to Z$ is called a compact perturbator of B and $A = B + C$ a compact perturbation of B if

$$(\lambda I - B)^{-1} F_\lambda : \overline{D(B)} \to \overline{D(B)} \quad \text{is compact for some} \quad \lambda > s(B)$$

and

$$(\lambda I - B)^{-1} (F_\lambda)^2 : Z \to Z \quad \text{is compact for some} \quad \lambda > s(B).$$

C is called an essentially compact perturbator of B and $A = B + C$ an essentially compact perturbation of B if there is some $n \in \mathbb{N}$ such that $(\lambda I - B)^{-1}(F_\lambda)^n$ is compact for all $\lambda > s(B)$.

Theorem 2.7 (Thieme [46, Theorem 3.6]) Let Z be an ordered Banach space with a normal and generating cone Z_+ and let $A = B + C$ be a positive perturbation of B. Then $r(F_\lambda)$ is a decreasing convex function of $\lambda > s(B)$ and exactly one of the following three cases holds:

- (i) if $r(F_\lambda) \geq 1$ for all $\lambda > s(B)$, then A is not resolvent positive;
- (ii) if $r(F_\lambda) < 1$ for all $\lambda > s(B)$, then A is resolvent positive and $s(A) = s(B)$;
- (iii) if there exists $\nu > \lambda > s(B)$ such that $r(F_\nu) < 1 \leq r(F_\lambda)$, then A is resolvent positive and $s(B) < s(A) < \infty$; further $s = s(A)$ is characterized by $r(F_s) = 1$.

Theorem 2.8 (Thieme [44, Theorems 3.4 and 4.9]) If C is a compact perturbator of B, then $S(t) - T(t)$ is a compact operator for $t \geq 0$. Moreover, if $\omega(T) < \omega(S)$, then $\{S(t)\}_{t\geq 0}$ is an essentially compact semigroup.

Theorem 2.9 (Thieme [45, Theorems 4.7 and 4.9]) Assume that C is an essentially compact perturbator of B. Moreover assume that there exists $\lambda_2 > \lambda_1 > s(B)$ such that $r(F_{\lambda_1}) \geq 1 > r(F_{\lambda_2})$. Then $s(B) < s(A)$ and the following statements hold:

(i) $s(A)$ is an eigenvalue of A associated with positive eigenvectors of A and A^*, has finite algebraic multiplicity, and is a pole of the resolvent of A. If C is a compact perturbator of B, then all spectral values λ of A with $\text{Re}\lambda \in (s(B), s(A)]$ are poles of the resolvent of A and are eigenvalues of A with finite algebraic multiplicity;
(ii) 1 is an eigenvalue of $F_{s(A)}$ and is associated with an eigenvector $w \in Z$ of $F_{s(A)}$ such that $(\lambda I - B)^{-1} w \in Z_+$ and with an eigenvector $v^* \in Z_+^*$ of $F_{s(A)}^*$. Actually $s(A)$ is the largest $\lambda \in \mathbb{R}$ for which 1 is an eigenvalue of F_λ.

Moreover, if Z is a Banach lattice and there exists a fixed point of F_s^* in Z_+^* that is conditionally strictly positive, then the following statements hold:

(iii) $s = s(A)$ is associated with a positive eigenvector v of A such that $w = (s(A)I - B)v$ is a positive fixed point of $F_{s(A)}$;
(iv) s is the only eigenvalue of A associated with a positive eigenvector.

2.3 Asynchronous exponential growth

Next we recall the formal definition of asynchronous exponential growth, which is an important property on the asymptotic behavior of operator semigroups.

Definition 2.10 We say that a C_0-semigroup $\{S(t)\}_{t \geq 0}$ in a Banach space Z has asynchronous exponential growth with intrinsic growth constant $\lambda_0 \in \mathbb{R}$ if there exists a non-zero finite rank operator P on Z such that

$$\lim_{t \to \infty} e^{-\lambda_0 t} S(t) = P,$$

where the limit is in the operator norm topology.

Definition 2.11 Let $\{F_\lambda\}_{\lambda > s(B)}$ be a positive resolvent output family for the operator B. A vector $x \in X_+$ is said to be conditionally strictly positive if the following holds:

(a) If $x^* \in Z_+^*$ and $F_\lambda^* x^* \neq 0$ for some (and then for all) $\lambda > s(B)$, then $\langle x, x^* \rangle > 0$.

Similarly we say that a functional $x^* \in Z_+^*$ is conditionally strictly positive if the following holds:

(b) If $x \in Z_+$ and $F_\lambda x \neq 0$ for some (and then for all) $\lambda > s(B)$, then $\langle x, x^* \rangle > 0$.

In addition, $\{F_\lambda\}_{\lambda > s(B)}$ is said to be conditionally strictly positive if the following holds:

(c) If $x \in Z_+$, $x^* \in Z_+^*$ and $F_\lambda x \neq 0$, $F_\lambda^* x^* \neq 0$ for some (and then for all) $\lambda > s(B)$, then there exist some $n \in \mathbb{N}$ and some $\lambda > s(B)$ such that $\langle F_\lambda^n x, x^* \rangle > 0$.

Theorem 2.12 (Thieme [44, Theorem 4.13]) Assume that $\{S(t)\}_{t\geq 0}$ generated by A is an essentially compact semigroup. Let the resolvent output family $\{F_\lambda\}_{\lambda>s(B)}$ for the operator A be conditionally strictly positive. Then $\{S(t)\}_{t\geq 0}$ exhibits asynchronous exponential growth. In particular, there exists positive eigenvectors v of A and v^* of A^* associated with $s(A)$ such that $\langle v, v^*\rangle = 1$ and

$$\left\| e^{-s(A)t}S(t) - v\otimes v^* \right\| \to 0 \;\text{ as }\; t\to\infty,$$

where $v\otimes v^*$ is the projection defined by $(v\otimes v^*)x = \langle x, v^*\rangle v$.

The following theorem provides sufficient and necessary conditions for a strongly continuous semigroup to have asynchronous exponential growth, which was proved by Webb [50] (see also Magal and Ruan [32, Theorem 4.6.2]).

Theorem 2.13 (Webb [50, Proposition 2.3]) Let $\{S(t)\}_{t\geq 0}$ be a strongly continuous semigroup of bounded linear operators on a Banach space X with infinitesimal generator A. Then $\{S(t)\}_{t\geq 0}$ has asynchronous exponential growth with intrinsic growth constant $\lambda_0 \in \mathbb{R}$ if and only if

(i) $\omega_1(A) < \lambda_0$;
(ii) $\lambda_0 = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$;
(iii) λ_0 is a simple pole of $(\lambda I - A)^{-1}$,

where $\omega_1(A)$ denotes the essential growth bound of A which is defined by (2.1).

2.4 Abstract setting

In this subsection we introduce our working spaces. Let X be an ordered Banach space that represents distributions of a population $u(a,\cdot)$ with respect to a spatial structure differing from the age a. Since we consider nonlocal diffusion, let X be a Banach space such as $C(\overline{\Omega})$ or $L^1(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a bounded and convex domain. It is easy to see that X has a normal and generating cone $X_+ = \{f \in X : f \geq 0\}$. In order to make the operator $\mathcal{A}$ defined in (2.3) contain the initial integral boundary condition, we define the following function spaces

$$\begin{aligned}
\mathcal{X} &= X \times L^1((0,a^+), X),\\
\mathcal{X}_0 &= \{0\} \times L^1((0,a^+), X),\\
\mathcal{X}_0^+ &= \{0\} \times L^1_+((0,a^+), X)\\
&= \{0\} \times \{u \in L^1((0,a^+), X) : u(a,x) \geq 0, (a,x) \in (0,a^+) \times \overline{\Omega}\},
\end{aligned}$$

and define an operator

$$\mathcal{A} = \mathcal{B} + \mathcal{C} \quad \text{with domain} \quad D(\mathcal{A}) = \{0\} \times W^{1,1}((0,a^+), X), \tag{2.3}$$

where $W^{1,1}$ represents the weak differentiability in a and that the derivative also belongs to L^1, with

$$\begin{aligned} \mathcal{B}(0,f) &= \left(-f(0,\cdot),\ -f' + L_{\sigma,m} f\right), \\ \mathcal{C}(0,f) &= \left(\int_0^{a^+} \beta(a,\cdot) f(a,\cdot) da,\ 0\right),\quad (0,f) \in D(\mathcal{A}), \end{aligned} \tag{2.4}$$

in which $f' := \frac{\partial f}{\partial a}$ and

$$\begin{aligned} & L_{\sigma,m}[f](a,x) \\ & = \frac{D}{\sigma^m} \int_\Omega J_\sigma(x-y)(f(a,y) - f(a,x))dy - \mu(a,x) f(a,x), \quad f \in L^1((0,a^+),X). \end{aligned} \tag{2.5}$$

Note that $\mathcal{X}_0$ is a Banach space with a positive cone $\mathcal{X}_0^+$ which is normal and generating. $\mathcal{X}_0$ can be identified with $L^1((0,a^+),X)$ in an obvious way. Define $\mathcal{A}_0$ to be the part of $\mathcal{A}$ in $\mathcal{X}_0$ with

$$D(\mathcal{A}_0) = \{(0,f); \mathcal{A}f \in \mathcal{X}_0\}.$$

Then $(0,f) \in D(\mathcal{A}_0)$ implies that $f(0,\cdot) = \int_0^{a^+} \beta(a,\cdot) f(a,\cdot) da$, the boundary condition in (1.2). Let $\{S(t)\}_{t\geq 0}$ be the C_0-semigroup generated by $\mathcal{A}_0$, the part of $\mathcal{A}$ in $\mathcal{X}_0$, see [25] for the existence of $\{S(t)\}_{t\geq 0}$ (Note that the proof is identical with nonlocal diffusion of Dirichlet type being replaced by Neumann type.); that is, $u(t,a,x) = S(t)u_0(a,x)$ is the solution of (1.1).

Moreover, define the nonlocal operator of Neumann type as

$$L^0_{\sigma,m}[f](a,x) = \frac{D}{\sigma^m} \int_\Omega J_\sigma(x-y)(f(a,y) - f(a,x))dy, \quad f \in L^1((0,a^+),X)$$

and denote

$$\kappa(x) := \int_\Omega J(x-y)dy,\ \ \kappa_\sigma(x) := \int_\Omega J_\sigma(x-y)dy. \tag{2.6}$$

It is obvious that $0 < \kappa(x) \leq 1$ and $\lim\limits_{\sigma\to\infty} \kappa_\sigma(x) = 0$. Observe that by standard Sobolev embedding we have

$$W^{1,1}((0,a^+),X) \hookrightarrow C([0,a^+],X). \tag{2.7}$$

This will enable us to define the strong maximum principle and sub/super-solutions of (1.2) (see Andreu-Vaillo et al. [1]) and, in particular, to study the limiting properties in Sect. 6 in the appropriate sense by noting that the functions are defined in $D(\mathcal{A})$.

Finally, we would like to mention that the Cauchy problem (1.1) has been investigated in an abstract setting by using the theory of integrated semigroups, see Thieme [44,46] and, in particular, Magal and Ruan [31] in a more general framework where the operators are neither densely-defined nor of Hille-Yosida types, for example in L^p spaces ($p \geq 1$), see Remark 3.2.

3 Principal spectral theory

In this section we establish some lemmas and propositions that will be used to show the main results in next section. We would like to mention that all the results in this section are parallel to those for nonlocal diffusion of Dirichlet type obtained previously in Kang and Ruan [24, Section 3]. Here for completeness we provide all proofs including necessary modifications.

We first consider the kernel J without scaling for convenience (since the principal spectral theory are the same for scaling cases); i.e., $\mathcal{A} = \mathcal{B} + \mathcal{C}$ with

$$\mathcal{B}(0, f) = \left(-f(0,\cdot), \quad -f' + Lf\right),$$
$$\mathcal{C}(0, f) = \left(\int_0^{a^+} \beta(a,\cdot)f(a,\cdot)da, \quad 0\right), \quad (0, f) \in D(\mathcal{A}),$$

where

$$L[f](a,x) = D\int_\Omega J(x-y)(f(a,y) - f(a,x))dy - \mu(a,x)f(a,x),$$
$$f \in L^1((0,a^+), X). \tag{3.1}$$

Definition 3.1 The principal spectrum point of $\mathcal{A}$ is defined by $\lambda_1(\mathcal{A}) = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(\mathcal{A})\}$. If $\lambda_1(\mathcal{A})$ is an isolated eigenvalue of $\mathcal{A}$ with a positive eigenfunction in $D(\mathcal{A})$, then it is called the principal eigenvalue of $\mathcal{A}$.

Note that $\lambda_1(\mathcal{A}) = s(\mathcal{A})$, where $s(\mathcal{A})$ denotes the spectral bound of $\mathcal{A}$. Define

$$\mathcal{B}_1(0, f) = \left(-f(0,\cdot), -f' - (D\kappa(\cdot) + \mu)f\right),$$
$$\mathcal{B}_2(0, f) = \left(0, D\int_\Omega J(\cdot - y)f(a,y)dy\right), (0, f) \in D(\mathcal{A}). \tag{3.2}$$

Remark 3.2 If the operator $\mathcal{B}$ is decomposed as follows:

$$\mathcal{B}_1'(0, f) = \left(-f(0,\cdot), -f'\right),$$
$$\mathcal{B}_2'(0, f) = \left(0, D\int_\Omega J(\cdot - y)f(a,y)dy - (Dk(\cdot) + \mu)f\right), (0, f) \in D(\mathcal{A}),$$

then the results in Thieme [44,46] for L^1 spaces and in Magal and Ruan [31] for L^p spaces ($p \geq 1$) can be applied directly to obtain the well-posedness of the Cauchy problem (1.1). Nevertheless, we shall follow our decomposition defined in (3.2) to study the principal spectral theory which is necessary for our purpose and, in particular, to obtain the existence of the principal eigenvalue and to investigate the limiting properties in the following.

One can see that $\mathcal{B} = \mathcal{B}_1 + \mathcal{B}_2$. Observe that if $\alpha \in \mathbb{C}$ such that $(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}$ exists, then

$$(\mathcal{B}_2 + \mathcal{B}_1 + \mathcal{C})u = \alpha u$$

has nontrivial solutions in $\mathcal{X}_0 \oplus i\mathcal{X}_0$ is equivalent to

$$\mathcal{B}_2(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}v = v$$

has nontrivial solutions in $\mathcal{X} \oplus i\mathcal{X}$, where

$$\mathcal{X}_0 \oplus i\mathcal{X}_0 = \{u + iv | u, v \in \mathcal{X}_0\}, \quad \mathcal{X} \oplus i\mathcal{X} = \{u + iv | u, v \in \mathcal{X}\}.$$

Proposition 3.3 The resolvent operator $(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}$ exists when $\mathrm{Re}\alpha > \alpha^{**}$ with $\alpha^{**} \in \mathbb{R}$ satisfying

$$r(\mathcal{G}_{\alpha^{**}}) = r\left(\int_0^{a^+} \beta(a, \cdot)e^{-(\alpha^{**}+D\kappa(\cdot))a}\Pi(0, a, \cdot)da\right) = 1, \tag{3.3}$$

in which

$$\Pi(\gamma, a, \cdot) := e^{-\int_\gamma^a \mu(s, \cdot)ds} \tag{3.4}$$

and $\mathcal{G}_\alpha : X \to X$ is a linear bounded operator defined by

$$[\mathcal{G}_\alpha g](x) = \int_0^{a^+} \beta(a, x)e^{-(\alpha+D\kappa(x))a}\Pi(0, a, x)g(x)da, \quad g \in X, \tag{3.5}$$

where κ is defined in (2.6). Moreover, $\mathcal{B}_1 + \mathcal{C}$ is a resolvent positive operator. In addition, $s(\mathcal{B}_1 + \mathcal{C}) = \alpha^{**}$ and α^{**} also satisfies the following equation

$$\max_{x \in \overline{\Omega}} \int_0^{a^+} \beta(a, x)e^{-(\alpha^{**}+D\kappa(x))a}\Pi(0, a, x)da = 1. \tag{3.6}$$

Proof Writing the resolvent equation $(\alpha I - \mathcal{B}_1 - \mathcal{C})(0, \phi) = (\eta, \psi) \in \mathcal{X}$ explicitly, we obtain

$$\begin{cases} \frac{\partial \phi(a,x)}{\partial a} = -(\alpha + D\kappa(x) + \mu(a,x))\phi(a,x) + \psi(a,x), & (a,x) \in (0,a^+) \times \overline{\Omega} \\ \phi(0,x) = \int_0^{a^+} \beta(a,x)\phi(a,x)da + \eta(x), & x \in \overline{\Omega}. \end{cases} \tag{3.7}$$

Solving the equation, we have

$$\begin{aligned} \phi(a,x) &= e^{-(\alpha+D\kappa(x))a}\Pi(0,a,x)\phi(0,x) \\ &\quad + \int_0^a e^{-(\alpha+D\kappa(x))(a-\gamma)}\Pi(\gamma,a,x)\psi(\gamma,x)d\gamma, \end{aligned} \tag{3.8}$$

where $\Pi(\gamma,a,x) = e^{-\int_\gamma^a \mu(s,x)ds}$, and accordingly

$$\begin{aligned} &\phi(0,x) - \int_0^{a^+} \beta(a,x)e^{-(\alpha+D\kappa(x))a}\Pi(0,a,x)\phi(0,x)da \\ &\quad = \int_0^{a^+} \beta(a,x) \int_0^a e^{-(\alpha+D\kappa(x))(a-\gamma)}\Pi(\gamma,a,x)\psi(\gamma,x)d\gamma da + \eta(x), \end{aligned}$$

which is equivalent to

$$(I - \mathcal{G}_\alpha)\phi(0,x) = \int_0^{a^+} \beta(a,x) \int_0^a e^{-(\alpha+D\kappa(x))(a-\gamma)}\Pi(\gamma,a,x)\psi(\gamma,x)d\gamma da + \eta(x), \tag{3.9}$$

in which $\mathcal{G}_\alpha$ is given in (3.5). Thus if $1 \in \rho(\mathcal{G}_\alpha)$, then

$$\begin{aligned} \phi(0,x) = (I - \mathcal{G}_\alpha)^{-1} \Bigg[\int_0^{a^+} \beta(a,x) \\ \int_0^a e^{-(\alpha+D\kappa(x))(a-\gamma)}\Pi(\gamma,a,x)\psi(\gamma,x)d\gamma da + \eta(x) \Bigg], \end{aligned} \tag{3.10}$$

which implies that

$$\begin{aligned} \phi(a,x) &= e^{-(\alpha+D\kappa(x))a}\Pi(0,a,x)(I - \mathcal{G}_\alpha)^{-1}\Big[\int_0^{a^+} \beta(s,x) \int_0^s e^{-(\alpha+D\kappa(x))(s-\gamma)}\Pi(\gamma,s,x)\psi(\gamma,x)d\gamma ds \\ &\quad + \eta(x) \Big] + \int_0^a e^{-(\alpha+D\kappa(x))(a-\gamma)}\Pi(\gamma,a,x)\psi(\gamma,x)d\gamma. \end{aligned} \tag{3.11}$$

It follows that $\alpha \in \rho(\mathcal{B}_1 + \mathcal{C})$ and thus $(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}$ exists. Now the problem becomes to find such an α such that $1 \in \rho(\mathcal{G}_\alpha)$. By assumptions on β and μ, we have

$$\mathcal{G}_\alpha g \geq \int_0^{a^+} \underline{\beta}(a)e^{-(\alpha+D)a}\widetilde{\Pi}(0,a)dag, \quad g \in X, \tag{3.12}$$

where $\widetilde{\Pi}(\gamma, a) := e^{-\int_\gamma^a \overline{\mu}(s)ds}$ and the fact that $0 < \kappa(x) \leq 1$ for any $x \in \overline{\Omega}$ was used. Now define

$$\mathcal{H}_\alpha := \int_0^{a^+} \underline{\beta}(a) e^{-(\alpha+D)a} \widetilde{\Pi}(0, a) da.$$

Then it follows from (3.12) that $\mathcal{G}_\alpha \geq \mathcal{H}_\alpha$ in the sense of positive operators (actually $\mathcal{H}_\alpha$ is a function of α) and that $r(\mathcal{G}_\alpha)$ is a strictly decreasing continuous function with respect to α, see Kang and Ruan [26, Lemmas 3.3-3.4]. The classical theory of age-structured models implies that there is a unique $\alpha^* \in \mathbb{R}$ such that

$$\int_0^{a^+} \underline{\beta}(a) e^{-(\alpha^*+D)a} \widetilde{\Pi}(0, a) da = 1$$

i.e. $\mathcal{H}_{\alpha^*} = 1$. Now by using the theory of positive operators, we have $r(\mathcal{G}_{\alpha^*}) \geq r(\mathcal{H}_{\alpha^*}) = \mathcal{H}_{\alpha^*} = 1$ and there exists a unique $\alpha^{**} \in \mathbb{R}$ satisfying $r(\mathcal{G}_{\alpha^{**}}) = 1$. Note that for any $\alpha \in \mathbb{C}$, when $\text{Re}\alpha > \alpha^{**}$ we have $r(\mathcal{G}_{\text{Re}\alpha}) < r(\mathcal{G}_{\alpha^{**}}) = 1$ and $(I - \mathcal{G}_{\text{Re}\alpha})^{-1}$ exists. It follows that $\alpha \in \rho(\mathcal{B}_1 + \mathcal{C})$ when $\text{Re}\alpha > \alpha^{**}$, which implies that $\rho(\mathcal{B}_1 + \mathcal{C})$ contains a ray (α^{**}, ∞) and $(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}$ is a positive operator for all $\alpha > \alpha^{**}$ by (3.11). Hence, $\mathcal{B}_1 + \mathcal{C}$ is a resolvent positive operator. Moreover, α^{**} is larger than all other real spectral values in $\sigma(\mathcal{B}_1 + \mathcal{C})$. It implies that $\alpha^{**} = s_{\mathbb{R}}(\mathcal{B}_1 + \mathcal{C})$. Now since $\mathcal{X}_0$ is a Banach space with a normal and generating cone $\mathcal{X}_0^+$ and $s(\mathcal{B}_1 + \mathcal{C}) \geq \alpha^{**} > -\infty$ due to $\alpha^{**} \in \sigma(\mathcal{B}_1 + \mathcal{C})$, we can conclude from Theorem 2.5 that $s(\mathcal{B}_1 + \mathcal{C}) = s_{\mathbb{R}}(\mathcal{B}_1 + \mathcal{C}) = \alpha^{**}$.

Notice that $\mathcal{G}_\alpha$ is actually a positive multiplication operator in X. We can determine the spectral radius $r(\mathcal{G}_\alpha)$ of $\mathcal{G}_\alpha$ as follows:

$$r(\mathcal{G}_\alpha) = \max_{x \in \overline{\Omega}} \int_0^{a^+} \beta(a, x) e^{-(\alpha+D\kappa(x))a} \Pi(0, a, x) da.$$

Hence, α^{**} satisfies (3.6). Denote

$$\alpha_{\min} := \min_{x \in \overline{\Omega}} \int_0^{a^+} \beta(a, x) e^{-(\alpha+D\kappa(x))a} \Pi(0, a, x) da.$$

We can see from Liang et al. [29, Proposition 2.7] that $\sigma_e(\mathcal{G}_\alpha) = \sigma(\mathcal{G}_\alpha) = \cup_{x \in \overline{\Omega}} \sigma(G_\alpha(x)) = [\alpha_{\min}, r(\mathcal{G}_\alpha)]$, where

$$G_\alpha(x) = \int_0^{a^+} \beta(a, x) e^{-(\alpha+D\kappa(x))a} \Pi(0, a, x) da, \tag{3.13}$$

and $\sigma_e(A)$ represents the essential spectrum of A. □

Next consider the following equation corresponding to the age-structured model without nonlocal diffusion:

$$\begin{cases} \frac{\partial u(a,x)}{\partial a} = -(\alpha + D\kappa(x))u(a,x) - \mu(a,x)u(a,x), & (a,x) \in (0,a^+) \times \overline{\Omega}, \\ u(0,x) = \int_0^{a^+} \beta(a,x)u(a,x)da, & x \in \overline{\Omega}. \end{cases} \tag{3.14}$$

The solvability of such an equation is the key in constructing sub- and super-solutions later. Before proceeding, let us recall an important theorem on the global implicit function theorem which we will use in the proof of the following proposition.

Theorem 3.4 (Global implicit function theorem, Sandberg [37, Theorem 1]) Let S_1, S_2 and W be normed vector linear spaces and let U and V denote nonempty subsets of S_1 and S_2 respectively, such that U is open in S_1 and V is open in S_2. Let 0_W be the zero element of W. Let $\{V_i\}$ be any family of compact subsets of V such that for each compact subset $\hat{V}$ of V, there is an $V_k \in \{V_i\}$ such that $\hat{V} \subset V_k$, and similarly, let $\{U_i\}$ denote any collection of compact subsets of U with the property that for any compact set $\hat{U}$ in U, there is $U_k \in \{U_i\}$ such that $\hat{U} \subset U_k$.

Now assume that V is convex and that f is a continuous map from $U \times V$ to W. Then there is a unique $g : U \to V$ such that $f(g(y), y) = 0_W$ for all $y \in V$, and g is continuous, if and only if

(i) for some $y_0 \in V$, there is exactly one $x_0 \in U$ such that $f(x_0, y_0) = 0_W$;
(ii) f is locally solvable for x;
(iii) for each $V_k \in \{V_i\}$, there is a $U_k \in \{U_i\}$ such that $y \in V_k, x \in U$ and $f(x, y) = 0_W$ imply that $x \in U_k$.

Proposition 3.5 There exists a continuous function $\alpha : \overline{\Omega} \to \mathbb{R}$ such that equation (3.14) has positive solutions and

$$\int_0^{a^+} \beta(a,x)e^{-(\alpha(x)+D\kappa(x))a}\Pi(0,a,x)da = 1, \quad \forall x \in \overline{\Omega},$$

where κ and Π are defined in (2.6) and (3.4) respectively. Moreover, $\alpha(x) \leq \alpha^{**}$ for all $x \in \overline{\Omega}$.

Proof To prove the proposition, we shall verify that the three hypothesis (i), (ii) and (iii) in Theorem 3.4 are satisfied. Solving (3.14) explicitly, we obtain a formal positive solution

$$u(a,x) = e^{-(\alpha+D\kappa(x))a}\Pi(0,a,x)u(0,x)$$

provided $u(0,x) > 0$. Then plugging it into the integral initial condition, we obtain after canceling $u(0,x)$ that

$$\int_0^{a^+} \beta(a,x)e^{-(\alpha+D\kappa(x))a}\Pi(0,a,x)da = 1.$$

Now define

$$G(\alpha,x) := G_\alpha(x) = \int_0^{a^+} \beta(a,x)e^{-(\alpha+D\kappa(x))a}\Pi(0,a,x)da. \tag{3.15}$$

We can verify that $G : \mathbb{R} \times \overline{\Omega} \to (0,\infty)$ is a continuously differentiable function with respect to α and x due to the continuous differentiability of β and μ. Moreover,

$$\frac{\partial G}{\partial \alpha} = -\int_0^{a^+} \beta(a,x)ae^{-(\alpha+D\kappa(x))a}\Pi(0,a,x)da < 0, \quad \forall x \in \overline{\Omega}, \tag{3.16}$$

$$\begin{aligned}\frac{\partial G}{\partial x_i} &= \int_0^{a^+} \frac{\partial \beta(a,x)}{\partial x_i} e^{-(\alpha+D\kappa(x))a}\Pi(0,a,x)da \\ &\quad - \int_0^{a^+}\int_0^{a} \beta(a,x)e^{-(\alpha+D\kappa(x))a}\frac{\partial \mu(s,x)}{\partial x_i}\Pi(0,a,x)dsda \\ &\quad - \int_0^{a^+} \beta(a,x)aD\frac{\partial \kappa(x)}{\partial x_i}e^{-(\alpha+D\kappa(x))a}\Pi(0,a,x)da, \quad i=1,\ldots,N.\end{aligned} \tag{3.17}$$

It follows by implicit function theorem that (ii) G is locally solvable for $x \in \overline{\Omega}$ due to (3.16); i.e. for each $(\alpha_0, x_0) \in O_G := \{(\alpha,x) \in \mathbb{R} \times \overline{\Omega} : G(\alpha,x) = 1\}$, there are open neighborhoods N_{α_0} and N_{x_0} of α_0 and x_0 respectively, and a unique continuously differentiable map α of N_{x_0} into N_{α_0} such that for $x \in N_{x_0}, \alpha = \alpha(x)$ is the unique solution in N_{α_0} of $G(\alpha,x) = 1$.

Next, let $\{V_i\}$ be any family of compact subsets of $\overline{\Omega}$ such that for each compact subset $\hat{V}$ of $\overline{\Omega}$, there is a subset $V_k \in \{V_i\}$ such that $\hat{V} \subset V_k$. Similarly, let $\{U_i\}$ denote any collection of compact subsets of $\mathbb{R}$ with the property that for any compact set $\hat{U}$ in $\mathbb{R}$, there is a subset $U_k \in \{U_i\}$ such that $\hat{U} \subset U_k$. Note that due to the fact that $\mu, \beta \in C^{0,1}([0,a^+] \times \overline{\Omega})$, we have from (3.16) and (3.17) that

$$\left|\frac{\partial \alpha}{\partial x}\right| = \left|\frac{\partial G}{\partial x}\right| \Big/ \left|\frac{\partial G}{\partial \alpha}\right| \le \text{Constant}, \quad \forall x \in \overline{\Omega},$$

where $\left|\frac{\partial \alpha}{\partial x}\right| \left(\left|\frac{\partial G}{\partial x}\right| \text{respectively}\right)$ denotes the length of vector $\frac{\partial \alpha}{\partial x} = \left(\frac{\partial \alpha}{\partial x_1}, \ldots, \frac{\partial \alpha}{\partial x_N}\right)$ ($\frac{\partial G}{\partial x} = \left(\frac{\partial G}{\partial x_1}, \ldots, \frac{\partial G}{\partial x_N}\right)$ respectively) in the usual sense. Now the mean value theorem implies that we can extend continuously α up to the boundary of N_{x_0}. In fact if any sequence $\{x_j\} \in N_{x_0}$ converges to $b \in \partial N_{x_0}$, then

$$|\alpha(x_k) - \alpha(x_l)| \le \left|\frac{\partial \alpha(\xi)}{\partial x}\right| |x_k - x_l|$$

for some ξ depending on x_k and x_l, which implies that $\{\alpha(x_k)\}$ is a Cauchy sequence, where x_k and x_l are two points in $\{x_j\} \in N_{x_0}$. Thus $\{\alpha(x_k)\}$ converges to $\alpha(b)$ by the continuity of α. Hence, by the above argument we have (iii) for each $V_k \in \{V_i\}$, there is a $U_k \in \{U_i\}$ such that $x \in V_k, \alpha \in \mathbb{R}$ and $G(\alpha, x) = 1$ imply that $\alpha \in U_k$.

Finally, since $\frac{\partial G}{\partial \alpha} < 0$ for all $x \in \overline{\Omega}$, then (i) holds for some $x_0 \in \overline{\Omega}$, there is exactly one α_0 such that $G(\alpha_0, x_0) = 1$. Actually, (i) implies that the extension of α is unique.

Now we have verified that the three hypotheses (i), (ii) and (iii) in Theorem 3.4 are satisfied. It follows that we have a unique $\alpha : \overline{\Omega} \to \mathbb{R}$ such that $G(\alpha(x), x) = 1$ for all $x \in \overline{\Omega}$ and α is a continuous function. Moreover, it follows from (3.6) that $\alpha(x) \leq \alpha^{**}$. In fact, $\alpha^{**} = \max_{x\in\overline{\Omega}} \alpha(x)$. This completes the proof of the proposition. □

Remark 3.6 Note that we split $\mathcal{A}$ into $\mathcal{A} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{C}$ and studied the spectral bound of $\mathcal{B}_1 + \mathcal{C}$; i.e. α^{**} in (3.6) (an algebraic equation) which is easily and explicitly obtained compared with that in Thieme [44,46], where $\mathcal{A}$ was decomposed into $\mathcal{A} = \mathcal{B} + \mathcal{C}$ and the spectral bound of $\mathcal{B}$ was obtained by an operator equation since it contains the spatial diffusion.

Now assume that $\mathcal{B}_1^0 + \mathcal{C}^0$, the part of $\mathcal{B}_1 + \mathcal{C}$ in $\mathcal{X}_0$, generates a positive C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{X}_0$. Since $\mathcal{B}_1 + \mathcal{C}$ is resolvent positive, by Thieme [44, Proposition 2.4] we know that $s(\mathcal{B}_1 + \mathcal{C}) = s(\mathcal{B}_1^0 + \mathcal{C}^0) = \omega(T)$ when $X = L^1(\Omega)$, since now $\mathcal{X}$ is an abstract L space. Next, we provide a lower bound for $(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}$ when $\text{Re}\alpha > \alpha^{**}$.

Proposition 3.7 For any $\text{Re}\alpha > \alpha^{**}$, the resolvent operator $(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1} : \mathcal{X} \to \mathcal{X}_0$ has the estimate for any $\psi \in L^1_+((0, a^+), X)$ with $\psi(a, x) \equiv \psi(x)$,

$$\left((\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}(0, \psi)\right)(a, x) \geq \frac{M(\alpha, D)}{1 - G_\alpha(x)}(0, \psi)(x), \quad (a, x) \in [0, a^+] \times \overline{\Omega},$$

where $M(\alpha, D) > 0$ will be determined in the proof.

Proof Define

$$I_1(\alpha, D, x) := \int_0^{a^+} \beta(a, x) \int_0^a e^{-(\alpha + D\kappa(x))(a-\gamma)} \Pi(\gamma, a, x) d\gamma da,$$
$$I_2(\alpha, D, a, x) = e^{-(\alpha + D\kappa(x))a} \Pi(0, a, x).$$

Then by (3.11), we can see that

$$\phi(a, x) \geq (1 - G_\alpha(x))^{-1} \min_{[0,a^+]\times\overline{\Omega}} I_2(\alpha, D, \cdot, \cdot) I_1(\alpha, D, \cdot)\psi(x)$$

for any $\psi(a, x) \equiv \psi(x)$. Thus $M(\alpha, D)$ is given by

$$M(\alpha, D) := \min_{[0,a^+]\times\overline{\Omega}} I_2(\alpha, D, \cdot, \cdot) I_1(\alpha, D, \cdot).$$

The result follows. □

Next we consider the following evolution equation

$$\begin{cases} \frac{\partial u(a,x)}{\partial a} = D \int_\Omega J(x-y)(u(a,y)-u(a,x))dy \\ \qquad\qquad -\mu(a,x)u(a,x), \quad (a,x) \in (0,a^+) \times \overline{\Omega}, \\ u(\tau,x) = \phi(x) \in X. \end{cases} \tag{3.18}$$

Define an evolution family $\{\mathcal{U}(\tau,a)\}_{0\le\tau\le a\le a^+}$ associated with (3.18); that is, the solution $u(a,x)$ of (3.18) can be written as

$$u(a,x) = \mathcal{U}(\tau,a)\phi(x). \tag{3.19}$$

The existence of such an evolution family $\{\mathcal{U}(\tau,a)\}_{0\le\tau\le a\le a^+}$ is guaranteed by results in Andreu-Vaillo et al. [1]. Moreover, it is positive in X.

Proposition 3.8 The operator $\mathcal{A}$ is resolvent positive and $s(\mathcal{A}) = \lambda_0$, where λ_0 satisfies

$$r(\mathcal{M}_{\lambda_0}) = r\left(\int_0^{a^+} \beta(a,\cdot)e^{-\lambda_0 a}\mathcal{U}(0,a)da\right) = 1, \tag{3.20}$$

in which $U(0,a)$ is defined in (3.19) and for each $\lambda \in \mathbb{C}$, $\mathcal{M}_\lambda : X \to X$ is a linear bounded operator defined by

$$[\mathcal{M}_\lambda\phi](x) = \int_0^{a^+} \beta(a,x)e^{-\lambda a}\mathcal{U}(0,a)\phi(x)da, \quad \forall\phi \in X.$$

Proof The proof can be found in [26, Theorem 3.6] or [24, Proposition 3.7], just noting that nonlocal diffusion of Dirichlet type is replaced by Neumann type and there also exists a principal eigenvalue equalling to zero for the nonlocal operator of Neumann type associated with a positive constant eigenfunction. Thus we omit it here. We would like to recall the solution of the resolvent equation $(\lambda I - \mathcal{A})^{-1}(\vartheta,\varphi)$ in the following, which will be used later:

$$\begin{aligned} &[(\lambda I - \mathcal{A})^{-1}(\vartheta,\varphi)](a,x) \\ &= \Big(0,\ \ e^{-\lambda a}\mathcal{U}(0,a)(I-\mathcal{M}_\lambda)^{-1}\left[\int_0^{a^+}\beta(s,x)\int_0^s e^{-\lambda(s-\gamma)}\mathcal{U}(\gamma,s)\varphi(\gamma,x)d\gamma ds + \vartheta(x)\right] \\ &\quad + \int_0^a e^{-\lambda(a-\gamma)}\mathcal{U}(\gamma,a)\varphi(\gamma,x)d\gamma\Big). \end{aligned} \tag{3.21}$$

This completes the proof. □

Moreover, by Thieme [44, Proposition 2.4], when $X = L^1(\Omega)$ we have $s(\mathcal{A}) = s(\mathcal{A}_0) = \omega(S)$ since $\mathcal{A}$ is resolvent positive. Next, we have the following lemma in characterizing the relation between the evolution system family $\{\mathcal{U}(\tau, a)\}_{0\le\tau\le a\le a^+}$ and $\{e^{-D\kappa(x)(a-\tau)}\Pi(\tau, a, x)\}_{0\le\tau\le a\le a^+}$.

Lemma 3.9 We have $e^{-D\kappa(x)a}\Pi(0, a, x) \ll \mathcal{U}(0, a)$ in X, where κ is defined in (2.6); i.e.,

$$0 < \phi \in C(\overline{\Omega}) \Rightarrow e^{-D\kappa(x)a}\Pi(0, a, x)\phi(x) < \mathcal{U}(0, a)\phi(x), \quad \forall x \in \overline{\Omega}.$$

Proof Let $u_1(a, x) = e^{-D\kappa(x)a}\Pi(0, a, x)\phi(x)$ and $u_2(a, x) = \mathcal{U}(0, a)\phi(x)$ be the solutions of the following equations

$$\begin{cases} \frac{\partial u_1(a,x)}{\partial a} = -D\kappa(x)u_1(a, x) - \mu(a, x)u_1(a, x), & (a, x) \in (0, a^+) \times \overline{\Omega} \\ u_1(0, x) = \phi(x), & x \in \overline{\Omega} \end{cases} \tag{3.22}$$

and

$$\begin{cases} \frac{\partial u_2(a,x)}{\partial a} = D\int_\Omega J(x-y)(u_2(a,y)-u_2(a,x))dy - \mu(a,x)u_2(a,x), & (a,x)\in(0,a^+)\times\overline{\Omega} \\ u_2(0,x) = \phi(x), & x \in \overline{\Omega}, \end{cases} \tag{3.23}$$

respectively. Consider the difference of (3.22) and (3.23) in the following

$$\begin{cases} \frac{\partial u(a,x)}{\partial a} = D\int_\Omega J(x-y)u(a,y)dy, & (a, x) \in (0, a^+) \times \overline{\Omega} \\ u(0, x) = \phi(x), & x \in \overline{\Omega}. \end{cases} \tag{3.24}$$

Define

$$(Kv)(x) = D\int_\Omega J(x-y)v(y)dy, \quad \forall v \in C(\overline{\Omega}).$$

Then the solution $u(a, x)$ of (3.24) can be written as

$$u = e^{Ka}\phi = \phi + aK\phi + \frac{a^2K^2\phi}{2!} + \cdots + \frac{a^nK^n\phi}{n!} + \cdots.$$

Let $x_0 \in \overline{\Omega}$ be such that $\phi(x_0) > 0$. Then by the fact that $\phi \in C(\overline{\Omega})$, there is a constant $\eta > 0$ such that $\phi(x) > 0$ for $x \in B(x_0, \eta) \cap \overline{\Omega}$. This implies that

$$(K\phi)(x) = D\int_\Omega J(x-y)\phi(y)dy > 0 \quad \text{for} \quad x \in B(x_0, r+\eta) \cap \overline{\Omega}.$$

Then

$$(K^n\phi)(x) > 0 \quad \text{for} \quad x \in B(x_0, nr+\eta) \cap \overline{\Omega}.$$

It then follows that $e^{Ka}\phi \gg 0$ for $a > 0$. □

Remark 3.10 The result in Lemma 3.9 remains valid when $X = L^1(\Omega)$. From Lemma 3.9, comparing (3.3) with (3.20) plus their monotonicity with respect to α and λ, we can see that $s(\mathcal{A}) \geq s(\mathcal{B}_1 + \mathcal{C})$. In fact, this can also be obtained by the fact that $\mathcal{A}$ is resolvent positive from Proposition 3.8 and Theorem 2.7 since case (i) was ruled out. But we cannot obtain the strict relation, i.e. $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$, even if $e^{-D\kappa(x)a}\Pi(0,a,x) \ll \mathcal{U}(0,a)$ holds, because α^{**} and λ_0 are obtained by taking the spectral radius of the operators to be equal to 1, where a limit process occurs in which the strict relation may not be preserved. However, if $r(\mathcal{G}_\alpha)$ and $r(\mathcal{M}_\lambda)$ are eigenvalues of $\mathcal{G}_\alpha$ and $\mathcal{M}_\lambda$ respectively, we could obtain the strict relation, see Marek [33, Theorem 4.3] which is the Frobenius theory for positive operators.

Proposition 3.11 $\mathcal{B}_2(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}$ is a compact operator in $\mathcal{X} \oplus i\mathcal{X}$ when Re$\alpha > \alpha^{**}$, where $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{C}$ are defined in (3.2) and (2.4) respectively.

Proof By (3.11) we have for Re$\alpha > \alpha^{**}$ that

$$\begin{aligned}&\left[\mathcal{B}_2(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}(\eta,\psi)\right](a,x)\\&= \Bigg(0, D\int_\Omega J(x-y)e^{-(\alpha+D\kappa(y))a}\Pi(0,a,y)(I-\mathcal{G}_\alpha)^{-1}\Big[\int_0^{a^+}\beta(s,y)\int_0^s e^{-(\alpha+D\kappa(y))(s-\gamma)}\Pi(\gamma,s,y)\\&\quad\times\psi(\gamma,y)d\gamma ds + \eta(y)\Big]dy + D\int_\Omega J(x-y)\int_0^a e^{-(\alpha+D\kappa(y))(a-\gamma)}\Pi(\gamma,a,y)\psi(\gamma,y)d\gamma dy\Bigg). \quad (3.25)\end{aligned}$$

It then follows that for any bounded subset $E \subset \mathcal{X} \oplus i\mathcal{X}$, $\mathcal{B}_2(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}E$ is a relatively compact subset of $\mathcal{X} \oplus i\mathcal{X}$ by Aubin-Lions Lemma. In fact, from (3.25) one can see by the fact $J \in C^1(\mathbb{R})$ that the second component of $\mathcal{B}_2(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}(\eta,\psi)$ belongs to $W^{1,1}((0,a^+), C^1(\overline{\Omega}))$. Due to Arzelà-Ascoli Theorem, $C^1(\overline{\Omega})$ is compactly embedded into $C(\overline{\Omega})$. Thus $W^{1,1}((0,a^+), C^1(\overline{\Omega}))$ is compactly embedded into $L^1((0,a^+), X)$ by Aubin-Lions Lemma when $X = C(\overline{\Omega})$ or $X = L^1(\Omega)$. Hence $\mathcal{B}_2(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}$ is compact in $\mathcal{X} \oplus i\mathcal{X}$. □

Corollary 3.12 $\mathcal{B}_2$ is a compact perturbator of $\mathcal{B}_1 + \mathcal{C}$ and $\mathcal{A} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{C}$ a compact perturbation of $\mathcal{B}_1 + \mathcal{C}$.

Proof $(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}\mathcal{B}_2(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}$ is compact for some $\alpha > s(\mathcal{B}_1 + \mathcal{C})$ since $\mathcal{B}_2(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}$ is compact by Proposition 3.11. □

We next give a proposition to characterize the relation between the eigenvalues of $\mathcal{M}_\lambda$ to those of $\mathcal{A} = \mathcal{B} + \mathcal{C}$, also see Kang and Ruan [25] or Walker [48].

Proposition 3.13 Let $\lambda \in \mathbb{C}$ and let $m \in \mathbb{N} \setminus \{0\}$. Then $\lambda \in \sigma_p(\mathcal{A})$ with geometric multiplicity m if and only if $1 \in \sigma_p(\mathcal{M}_\lambda)$ with geometric multiplicity m, where $\sigma_p(A)$ denotes the point spectrum of A.

Proof Let $\lambda \in \mathbb{C}$. Suppose that $\lambda \in \sigma_p(\mathcal{A})$ has geometric multiplicity m so that there are m linearly independent elements

$$\left(0,\phi_1\right),\ldots,\left(0,\phi_m\right) \in D(\mathcal{A}) \text{ with } (\lambda I-\mathcal{A})\left(0,\phi_j\right)=(0,0) \text{ for } j=1,\ldots,m.$$

Then by solving the eigenvalue problem as above, we obtain

$$\phi_j(a,\cdot)=e^{-\lambda a}\mathcal{U}(0,a)\phi_j(0,\cdot) \quad \text{with} \quad \phi_j(0,\cdot)=\mathcal{M}_\lambda\phi_j(0,\cdot).$$

Hence, $\phi_1(0,\cdot),\ldots,\phi_m(0,\cdot)$ are necessarily linearly independent eigenvectors of $\mathcal{M}_\lambda$ corresponding to the eigenvalue 1. Now suppose that $1 \in \sigma_p(\mathcal{M}_\lambda)$ has geometric multiplicity m so that there are linearly independent $\psi_1,\ldots,\psi_m \in X$ with $\mathcal{M}_\lambda\psi_j = \psi_j$ for $j=1,\ldots,m$. Setting $(0,\phi_j)=\left(0,e^{-\lambda a}\mathcal{U}(0,a)\psi_j\right) \in \mathcal{X}_0$ and noting that for $j=1,\ldots,m$, we have

$$\frac{\partial\phi_j}{\partial a}+\lambda\phi_j-L\phi_j=0, \quad \int_0^{a^+}\beta(a,\cdot)\phi_j(a,\cdot)da=\mathcal{M}_\lambda\psi_j=\psi_j=\phi_j(0,\cdot),$$

which are equivalent to

$$\mathcal{A}(0,\phi_j)=\lambda(0,\phi_j) \quad \text{and} \quad (0,\phi_j)\in D(\mathcal{A}).$$

Thus $\lambda \in \sigma_p(\mathcal{A})$. If $\alpha_1,\ldots,\alpha_m$ are any scalars, the unique solvability of the Cauchy problem

$$\frac{\partial\phi}{\partial a}+\lambda\phi-L\phi=0, \quad \phi(0,x)=\sum_{j=1}^m\alpha_j\psi_j$$

ensures that $(0,\phi_1),\ldots,(0,\phi_m)$ are linearly independent. This completes the proof. □

4 Main theorems

In this section, we state and prove the main theorems of this paper which address the existence of the principal eigenvalue and the property of asynchronous exponential growth simultaneously.

Theorem 4.1 Assume that $s(\mathcal{A}) > s(\mathcal{B}_1+\mathcal{C})$, then $\lambda_1(\mathcal{A})=s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$. Moreover, $\{S(t)\}_{t\ge 0}$ exhibits asynchronous exponential growth. Conversely, if λ is an eigenvalue of $\mathcal{A}$ with a eigenfunction $(0,\phi(a,x))$ with ϕ being positive, then $\lambda=s(\mathcal{A})$.

Proof Define

$$\mathcal{F}_\lambda=\mathcal{B}_2(\lambda I-\mathcal{B}_1-\mathcal{C})^{-1}, \quad \mathrm{Re}\lambda>\alpha^{**}. \tag{4.1}$$

Note that $\mathcal{A} = \mathcal{B}_1 + \mathcal{C} + \mathcal{B}_2$ is a compact perturbation of $\mathcal{B}_1 + \mathcal{C}$ by Corollary 3.12. First, we use Theorem 2.9 to prove that $\lambda_1(\mathcal{A}) = s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$. We know that $\mathcal{A}$ is resolvent positive by Proposition 3.8. It implies that case (i) in Theorem 2.7 is ruled out. Secondly, by the assumption $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$ we know that only case (iii) in Theorem 2.7 will occur; otherwise $s(\mathcal{A}) = s(\mathcal{B}_1 + \mathcal{C})$, which is a contradiction if case (ii) in Theorem 2.7 would happen. Hence, there exists $\lambda_2 > \lambda_1 > s(\mathcal{B}_1 + \mathcal{C})$ such that $r(\mathcal{F}_{\lambda_1}) \geq 1 > r(\mathcal{F}_{\lambda_2})$. Now the hypothesis in Theorem 2.9 holds, then $s(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a positive eigenfunction, has finite algebraic multiplicity, and is a pole of the resolvent of $\mathcal{A}$. It follows that $\lambda_1(\mathcal{A}) = s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$.

Next, we show that $\{S(t)\}_{t\geq 0}$ exhibits asynchronous exponential growth under the assumption when $X = L^1(\Omega)$. Observing $\omega(S) = s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C}) = \omega(T)$, it follows that $\{S(t)\}_{t\geq 0}$ is an essentially compact semigroup by Theorem 2.8. In addition, it can be seen that the resolvent output family $\mathcal{F}_\lambda$ is conditionally strictly positive regarding to (3.25) when $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$. In fact, first observe that $\mathcal{F}_\lambda$ maps $\mathcal{X}$ into $\mathcal{X}_0 = \{0\} \times L^1((0, a^+), L^1(\Omega))$, then we introduce the restriction of $\mathcal{F}_\lambda$ to $\mathcal{X}_0$ and the associated operator L_λ in $Q := L^1((0, a^+), L^1(\Omega))$ (see (3.25)),

$$\begin{aligned}[L_\lambda \psi](a, x) = D \int_\Omega J(x - y) e^{-(\lambda + D\kappa(y))a} \Pi(0, a, y) [(I - \mathcal{G}_\lambda)^{-1} g(\psi)](y) dy \\ + D \int_\Omega J(x - y) \int_0^a e^{-(\lambda + D\kappa(y))(a-\gamma)} \Pi(\gamma, a, y) \psi(\gamma, y) d\gamma dy,\end{aligned} \tag{4.2}$$

where

$$g(\psi)(y) := \int_0^{a^+} \beta(s, y) \int_0^s e^{-(\lambda + D\kappa(y))(s-\gamma)} \Pi(\gamma, s, y) \psi(\gamma, y) d\gamma ds.$$

We use L_λ for both the operators in Q and the operator in $\mathcal{X}_0 = \{0\} \times Q$. Next for any $\psi \in Q_+$ with $L_\lambda \psi \neq 0$, there exists some $(a_0, x_0) \in [0, a^+] \times \overline{\Omega}$, such that

$$\begin{aligned}[L_\lambda \psi](a, x) \geq D\epsilon \min\{e^{-\lambda a^+}, 1\} e^{-Da^+} \\ \times \int_{B(x_0, r/2) \cap \Omega} \Pi(0, a^+, y) [(I - \mathcal{G}_\lambda)^{-1} g(\psi)](y) dy > 0,\end{aligned}$$

for all $(a, x) \in [0, a^+] \times B(x_0, r/2) \cap \Omega$, since $J(x) > \epsilon > 0$ in $B(0, r)$ with some $r > 0$ due to $J(0) > 0$ and $(I - \mathcal{G}_\lambda)^{-1}$ and g are positive due to the positivity of β. Now by the argument similar with Lemma 3.9, one can show that

$$[L_\lambda^n \psi](a, x) > 0, \ \forall (a, x) \in [0, a^+] \times B(x_0, nr/2) \cap \Omega.$$

On the other hand, for any $\psi^* \in Q_+^*$ with $L_\lambda^* \psi^* \neq 0$, one can also similarly obtain that there exists some subset E of $[0, a^+] \times \overline{\Omega}$ with positive measure such that $\psi^*(a, x) > 0$ in E, otherwise $L_\lambda^* \psi^* = 0$. Observe that when n is large enough, one has $E \cap \left([0,a^+] \times B(x_0, nr/2) \cap \Omega\right) \neq \emptyset$. It follows that

$$\langle L_\lambda^n \psi, \psi^* \rangle > 0, \tag{4.3}$$

which implies that L_λ is conditionally strictly positive and so is $\mathcal{F}_\lambda$.

Now Theorem 2.12 implies that $\{S(t)\}_{t\geq 0}$ exhibits asynchronous exponential growth. In particular, there exists positive eigenvectors v of $\mathcal{A}$ and v^* of $\mathcal{A}^*$ associated with $s(\mathcal{A})$ such that $\langle v, v^* \rangle = 1$ and

$$\left\| e^{-s(\mathcal{A})t} S(t) - v \otimes v^* \right\| \to 0, \quad t \to \infty,$$

where $v \otimes v^*$ is the projection defined by $(v \otimes v^*)x = \langle x, v^* \rangle v$. Observe now that we have obtained the existence of principal eigenvalue $s(\mathcal{A})$ associated with two positive eigenfunctions respectively in $L^1((0,a^+), C(\overline{\Omega}))$ and $L^1((0,a^+), L^1(\Omega))$. Furthermore, we have also verified that $\{\mathcal{F}_\lambda\}_{\lambda > s(\mathcal{B}_1+\mathcal{C})}$ is conditionally strictly positive in $L^1((0,a^+), L^1(\Omega))$, it follows from Theorem 4.9 from Thieme [45] that $s(\mathcal{A})$ is a first order pole of the resolvent of $\mathcal{A}$ and that the eigenspace of $\mathcal{A}$ associated with $s(\mathcal{A})$ is one dimensional. These facts conclude that the two principal eigenfunction are the same since $C(\overline{\Omega}) \subset L^1(\Omega)$.

Conversely, if $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathcal{A}$ associated to an eigenfunction $(0, \phi(a,x))$ with ϕ being positive, we prove that $s(\mathcal{A}) = \lambda$. Let $\{S(t)\}_{t\geq 0}$ be the C_0-semigroup generated by $\mathcal{A}_0$, the part of $\mathcal{A}$ in $\mathcal{X}_0$, see Kang and Ruan [25] for the existence of $\{S(t)\}_{t\geq 0}$; that is $u(t,a,x) = S(t)u_0(a,x)$ is the solution of (1.1). By direct computation, we have $S(t)\phi(a,x) = e^{\lambda t}\phi(a,x)$. Since $\phi(a,x) > 0, \forall (a,x) \in [0,a^+] \times \overline{\Omega}$, for any $u_0 \in C_+([0,a^+], C(\overline{\Omega})) \subset L^1_+((0,a^+), L^1(\Omega))$ with

$$u_0(a,x) \leq M_0 \phi(a,x), \quad \forall (a,x) \in [0,a^+] \times \overline{\Omega},$$

where $M_0 = \frac{\|u_0\|}{\min_{[0,a^+]\times\overline{\Omega}} \phi(a,x)}$, it follows from the comparison principle for (1.1) (see [24, Lemma 8.2]) that

$$S(t)u_0 \leq M_0 S(t)\phi = M_0 e^{\lambda t}\phi, \quad \forall t > 0.$$

This, noting that $C_+([0,a^+], L^1(\Omega))$ is dense in $L^1_+((0,a^+), C(\overline{\Omega}))$, together with Thieme [44, Theorem 5.4] and Thieme [46, Theorem 6.2] which state that $\omega(S) = s(\mathcal{A}_0) = s(\mathcal{A})$, implies that $s(\mathcal{A}) = \lambda$, where $\omega(S)$ represents the growth bound of $\{S(t)\}_{t\geq 0}$. □

Corollary 4.2 The inequality $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$ holds if and only if there is $\lambda^* > s(\mathcal{B}_1 + \mathcal{C})$ such that $r(\mathcal{F}_{\lambda^*}) \geq 1$, where $\mathcal{F}_\lambda$ is defined in (4.1).

Proof If there exists $\lambda^* > s(\mathcal{B}_1 + \mathcal{C})$ such that $r(\mathcal{F}_{\lambda^*}) \geq 1$, then case (iii) of Theorem 2.7 will occur, which implies that $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$, because we can alway find ϑ large enough such that $r(\mathcal{F}_\vartheta) < 1$ according to (3.25). Conversely, if $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$, by the same argument in Theorem 4.1, we have the desired result. □

In applications, the condition $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$ is hard to check, so it is desirable to find easily verifiable conditions to ensure that $\lambda_1(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$ and $\{S(t)\}_{t\geq 0}$ exhibits asynchronous exponential growth. This leads us to the following results on the existence of the principal eigenvalue of $\mathcal{A}$ and the property of asynchronous exponential growth of $\{S(t)\}_{t\geq 0}$ in this section.

Theorem 4.3 (Existence of the principal eigenvalue and asynchronous exponential growth - I) Assume that for every $\alpha > \alpha^{**}$,

$$\frac{1}{1-G_\alpha} \notin L^1_{loc}(\overline{\Omega}), \tag{4.3}$$

then $\lambda_1(\mathcal{A}) = s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$ and $\{S(t)\}_{t\geq 0}$ exhibits asynchronous exponential growth, where

$$G_\alpha(x) = G(\alpha, x) = \int_0^{a^+} \beta(a,x)e^{-(\alpha + D\kappa(x))a}\Pi(0,a,x)da,$$

which is defined in (3.15).

Proof The idea of the proof below traced back to Shen and Vo [40]. For completeness and reader's convenience, we include some necessary modifications and provide a detailed proof.

By contradiction, assume that $\lambda_1(\mathcal{A}) = s(\mathcal{A})$ is not the principal eigenvalue of $\mathcal{A}$, by the contrapositive statement of Theorem 4.1, $s(\mathcal{A}) \leq s(\mathcal{B}_1 + \mathcal{C})$ (in fact, by Remark 3.10, one can get a stronger result $s(\mathcal{A}) = s(\mathcal{B}_1 + \mathcal{C})$). It follows by Corollary 4.2 that

$$r(\mathcal{F}_\alpha) = r(\mathcal{B}_2(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}) < 1, \quad \forall \, \mathrm{Re}\alpha > s(\mathcal{B}_1 + \mathcal{C}). \tag{4.4}$$

We can see from (3.11) that the operator $(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}$ has monotonicity in the sense that

$$\begin{aligned}&(u_1, u_2), (v_1, v_2) \in \mathcal{X} \text{ with } (u_1, u_2) \geq (v_1, v_2)\\ &\quad \Rightarrow (\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}(u_1, u_2) \geq (\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}(v_1, v_2),\end{aligned}$$

where $(u_1, u_2) \geq (v_1, v_2)$ represents $u_1 \geq v_1, u_2 \geq v_2$.

Now Proposition 3.7 implies that

$$((\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}(0,1))(a,x) \geq \left(0, \quad \frac{M(\alpha, D)}{1 - G_\alpha(x)}\right) \geq (0,0), \quad (a,x) \in [0,a^+] \times \overline{\Omega}.$$

Note that in the following estimates, we will focus on the second component of $(\alpha I - \mathcal{B}_1 - \mathcal{C})^{-1}(0,1)$ since the first component is always zero. Thus for the convenience of notation, we will only write down the second component without ambiguity. Now applying $\mathcal{B}_2$ to both sides of the above estimate, we find that

$$(\mathcal{B}_2(\alpha I-\mathcal{B}_1-\mathcal{C})^{-1}(0,1))(a,x) = D\int_\Omega J(x-y)((\alpha I-\mathcal{B}_1-\mathcal{C})^{-1}(0,1))(a,y)dy,$$
$$\geq \int_\Omega J(x-y)\frac{DM(\alpha,D)}{1-G_\alpha(y)}dy, \quad (a,x)\in[0,a^+]\times\overline{\Omega}. \tag{4.5}$$

By the monotonicity of $(\alpha I-\mathcal{B}_1-\mathcal{C})^{-1}$, (4.5) and Proposition 3.7, we find for each $(a,x)\in[0,a^+]\times\overline{\Omega}$ that

$$\begin{aligned}&\left((\alpha I-\mathcal{B}_1-\mathcal{C})^{-1}\mathcal{B}_2(\alpha I-\mathcal{B}_1-\mathcal{C})^{-1}(0,1)\right)(a,x)\\ &=\left((\alpha I-\mathcal{B}_1-\mathcal{C})^{-1}\int_\Omega J(\cdot-y)\frac{DM(\alpha,D)}{1-G_\alpha(y)}dy\right)(a,x),\\ &\geq \frac{M(\alpha,D)}{1-G_\alpha(x)}\int_\Omega J(x-y)\frac{DM(\alpha,D)}{1-G_\alpha(y)}dy, \quad (a,x)\in[0,a^+]\times\overline{\Omega}.\end{aligned} \tag{4.6}$$

Applying $\mathcal{B}_2$ to both sides of the above estimate again, we have

$$\begin{aligned}&\left((\mathcal{B}_2(\alpha I-\mathcal{B}_1-\mathcal{C})^{-1})^2(0,1)\right)(a,x)\\ &\geq \int_\Omega J(x-y)\frac{DM(\alpha,D)}{1-G_\alpha(y)}\int_\Omega J(y-z)\frac{DM(\alpha,D)}{1-G_\alpha(z)}dzdy.\end{aligned} \tag{4.7}$$

Repeating the above procedure, we find for each $(a,x_0)\in[0,a^+]\times\overline{\Omega}$ the following estimate

$$\begin{aligned}&\left((\mathcal{B}_2(\alpha I-\mathcal{B}_1-\mathcal{C})^{-1})^n(0,1)\right)(a,x_0)\\ &\geq \int_\Omega\cdots\int_\Omega\prod_{m=1}^n\left[J(x_{m-1}-x_m)\frac{DM(\alpha,D)}{1-G_\alpha(x_m)}\right]dx_n\cdots dx_1.\end{aligned}$$

As a result,

$$\begin{aligned}\left\|(\mathcal{B}_2(\alpha I-\mathcal{B}_1-\mathcal{C})^{-1})^n\right\| &\geq \max_{(a,x_0)\in[0,a^+]\times\overline{\Omega}}\left((\mathcal{B}_2(\alpha I-\mathcal{B}_1-\mathcal{C})^{-1})^n(0,1)\right)(a,x_0)\\ &\geq \max_{x_0\in\overline{\Omega}}\int_\Omega\cdots\int_\Omega\prod_{m=1}^n\left[J(x_{m-1}-x_m)\frac{DM(\alpha,D)}{1-G_\alpha(x_m)}\right]dx_n\cdots dx_1,\end{aligned}$$

which implies that for any $x_0\in\overline{\Omega}$ and $\delta>0$,

$$\begin{aligned}&\left\|(\mathcal{B}_2(\alpha I-\mathcal{B}_1-\mathcal{C})^{-1})^n\right\|\\ &\geq \int_{\Omega\cap B(x_0,\delta)}\cdots\int_{\Omega\cap B(x_0,\delta)}\prod_{m=1}^n\left[J(x_{m-1}-x_m)\frac{DM(\alpha,D)}{1-G_\alpha(x_m)}\right]dx_n\cdots dx_1\end{aligned}$$

$$\geq \left[\inf_{x\in\Omega\cap B(x_0,\delta)} \int_{\Omega\cap B(x_0,\delta)} J(x-y) \frac{DM(\alpha, D)}{1-G_\alpha(y)} dy \right]^n, \tag{4.8}$$

where $B(x_0, \delta)$ is an open ball in $\mathbb{R}^N$ centered at x_0 with radius δ. We can use (4.4) and Gelfand's formula for the spectral radius of a bounded linear operator to find that

$$1 \geq \inf_{x\in\Omega\cap B(x_0,\delta)} \int_{\Omega\cap B(x_0,\delta)} J(x-y) \frac{DM(\alpha, D)}{1-G_\alpha(y)} dy := I(x_0, \delta, \alpha, D) \tag{4.9}$$

for all $x_0 \in \overline{\Omega}, \delta > 0$ and $\mathrm{Re}\alpha > s(\mathcal{B}_1 + \mathcal{C})$.

Since J is continuous and $J(0) > 0$, there exist $\delta_* > 0$ and $c_* > 0$ such that $J \geq c_*$ on $B(0, \delta_*)$, an open ball in $\mathbb{R}^N$ centered at 0 with radius δ_*. Hence,

$$\begin{aligned} I(x_0, \delta, \alpha, D) &\geq \inf_{x\in\Omega\cap B(x_0,\delta)} \int_{\Omega\cap B(x_0,\delta)\cap B(x,\delta_*)} J(x-y) \frac{DM(\alpha, D)}{1-G_\alpha(y)} dy \\ &\geq c_* \inf_{x\in\Omega\cap B(x_0,\delta)} \int_{\Omega\cap B(x_0,\delta)\cap B(x,\delta_*)} \frac{DM(\alpha, D)}{1-G_\alpha(y)} dy \\ &= c_* \int_{\Omega\cap B(x_0,\delta)} \frac{DM(\alpha, D)}{1-G_\alpha(y)} dy \end{aligned} \tag{4.10}$$

provided $2\delta \leq \delta_*$ so that $B(x_0, \delta) \subset B(x, \delta_*)$ whenever $x \in \overline{B(x_0, \delta)}$. In particular, for any $x_0 \in \overline{\Omega}$ and $\mathrm{Re}\alpha > s(\mathcal{B}_1 + \mathcal{C})$,

$$I(x_0, \delta_*/2, \alpha, D) \geq c_* \int_{\Omega\cap B(x_0,\delta_*/2)} \frac{DM(\alpha, D)}{1-G_\alpha(y)} dy. \tag{4.11}$$

Since $\frac{1}{1-G_\alpha} \notin L^1_{loc}(\overline{\Omega})$, there exists $x_* \in \overline{\Omega}$ such that

$$\frac{1}{1-G_\alpha} \notin L^1\left(\overline{\Omega} \cap B(x_*, \delta_*/2)\right),$$

which implies the existence of some small enough $\epsilon_* \in (0, 1)$ such that

$$c_* \int_{\Omega\cap B(x_*,\delta_*/2)} \frac{DM(\alpha, D)}{1-G_\alpha(y)+\epsilon} dy \geq 2$$

for all $\epsilon \in (0, \epsilon_*]$. In particular, $I(x_*, \delta_*/2, \alpha, D) \geq 2$, which contradicts (4.9). □

Corollary 4.4 Assume that $\mu(a, x), \beta(a, x)$ and $J(x)$ are C^N in x, there is some $x_0 \in \mathrm{Int}(\Omega)$ satisfying that $G_\alpha(x_0) = \max_{x\in\overline{\Omega}} G_\alpha(x) = 1$ and the partial derivatives of $G_\alpha(x)$ up to order $N-1$ at x_0 are zero, then $\lambda_1(\mathcal{A}) = s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$ and $\{S(t)\}_{t\geq 0}$ exhibits asynchronous exponential growth.

Proof We shall follow the proof of Theorem 4.3 by contradiction. Let $x_0 \in \Omega$ be such that $G_\alpha(x_0) = 1$. Also without loss of generality, we may assume that $x_0 \in \mathrm{Int}(\Omega)$. Since the partial derivatives of $G_\alpha(x)$ up to order $N-1$ at x_0 are zero, there is $M > 0$ such that

$$G_\alpha(x_0) - G_\alpha(y) \leq M\|x_0 - y\|^N \quad \text{for } y \in \mathbb{R}^N.$$

Then following the arguments of Theorem 4.3, we have

$$I(x_0, \delta_*/2, \alpha, D) \geq c_* \int_{\Omega \cap B(x_0, \delta_*/2)} \frac{DM(\alpha, D)}{M\|x_0 - y\|^N} dy,$$

see (4.11). Note that $\int_{\Omega \cap B(x_0,\delta_*/2)} \frac{DM(\alpha,D)}{M\|x_0-y\|^N} dy = \infty$. This, together with the arguments in Theorem 4.3, yields a contradiction. It follows that the desired result is concluded. □

Next, we give another nonlocally-integrable condition similar to (4.3) to check the existence of the principal eigenvalue of $\mathcal{A}$ and asynchronous exponential growth of $\{S(t)\}_{t\geq 0}$.

Theorem 4.5 (Existence of the principal eigenvalue and asynchronous exponential growth - II) Assume that for every $\zeta > \alpha^{**}$,

$$\frac{1}{\zeta - \alpha} \notin L^1_{loc}(\overline{\Omega}), \tag{4.12}$$

then $\lambda_1(\mathcal{A}) = s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$ and $\{S(t)\}_{t\geq 0}$ exhibits asynchronous exponential growth, where $\alpha(x)$ is defined in Proposition 3.5.

Proof The idea of the proof below came from Liang et al. [29, Lemma 3.8] or Bao and Shen [6, Proposition 3.1]. For completeness and reader's convenience, we provide a detailed and modified proof.

By the assumption on the kernel J, there exist $r > 0$ and $c_0 > 0$ such that $J(x - y) > c_0$ for all $x, y \in \overline{\Omega}$ with $|x - y| < r$. By Proposition 3.5 and classical theory of age-structured models, for each $x \in \overline{\Omega}$, $\mathcal{B}_1^x + \mathcal{C}^x$ possesses a strongly positive eigenfunction

$$(0, [E(x)](a)) := (0, e^{-(\alpha + D\kappa(x))a}\Pi(0, a, x)\phi(x))$$

corresponding to eigenvalue $\alpha(x)$, where $\phi(x)$ is an arbitrary positive nontrivial fixed point of $G_\alpha(x)$ and $\mathcal{B}_1^x + \mathcal{C}^x$ is defined in (3.2) with upper script representing each fixed $x \in \overline{\Omega}$. It then follows from Kato [27, Section IV.3.5] that $E(x)$ is continuous in $x \in \overline{\Omega}$. Without loss of generality, we assume that

$$\max_{(a,x)\in\times[0,a^+]\times\overline{\Omega}} [E(x)](a) = 1.$$

Next let $c_1 = \min_{(a,x)\in[0,a^+]\times\overline{\Omega}}[E(x)](a)$. Since $(\zeta-\alpha)^{-1} \notin L^1_{loc}(\overline{\Omega})$, we can choose some $\delta > 0$ and $x_1 \in \Omega$ such that $B(x_1,\delta) \subset B(x_1, 2\delta) \subset \Omega$,

$$\int_{B(x_1,\delta)} \frac{1}{\zeta-\alpha(x)} dx \geq 2(Dc_0c_1)^{-1},$$

and $3\delta < r$, where $B(x,r)$ is the ball centered at x with radius r. Let $p(x)$ be a continuous function on $\overline{\Omega}$ defined by

$$p(x) = \begin{cases} 1, & x \in B(x_1,\delta), \\ 0, & x \in \overline{\Omega} \setminus B(x_1, 2\delta) \end{cases} \tag{4.13}$$

and $[\hat{E}(x)](a) := \hat{E}(a,x) := p(x)[E(x)](a), \forall (a,x) \in [0,a^+] \times \overline{\Omega}$. It then follows that for any $(a,x) \in [0,a^+] \times \overline{\Omega} \setminus B(x_1, 2\delta)$, we have

$$\int_\Omega J(x-y)\frac{dy}{\zeta-\alpha(y)}\hat{E}(a,y) \geq 0.$$

For any $(a,x) \in [0,a^+] \times B(x_1,2\delta)$, we see that

$$\begin{aligned} &\int_\Omega J(x-y)\frac{dy}{\zeta-\alpha(y)}\hat{E}(a,y) \\ &\geq \int_{B(x_1,\delta)} J(x-y)\frac{dy}{\zeta-\alpha(y)}[E(y)](a) \\ &\geq 2c_0c_1(Dc_0c_1)^{-1} \geq 2D^{-1}\hat{E}(a,x). \end{aligned} \tag{4.14}$$

Note that

$$\begin{aligned} [(\zeta I - \mathcal{B}_1 - \mathcal{C})^{-1}(0,\hat{E})](x) &= (\zeta I - \mathcal{B}_1^x - \mathcal{C}^x)^{-1}(0,[\hat{E}(x)]) \\ &= (\zeta - \alpha(x))^{-1}(0,[\hat{E}(x)]) \end{aligned} \tag{4.15}$$

for all $x \in \overline{\Omega}$. It then follows that

$$\mathcal{F}_\zeta(0,\hat{E}) = \mathcal{B}_2(\zeta I - \mathcal{B}_1 - \mathcal{C})^{-1}(0,\hat{E}) \geq 2(0,\hat{E}) > (0,\hat{E}). \tag{4.16}$$

Thus, there exists $\zeta > s(\mathcal{B}_1 + \mathcal{C})$ such that $r(\mathcal{F}_\zeta) > 1$. Then by Corollary 4.2, it follows that $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$, which implies the desired result by Theorem 4.1. □

Again parallel to Corollary 4.4, we have the following corollary.

Corollary 4.6 Assume that $\mu(a,x), \beta(a,x)$ and $J(x)$ are C^N in x, there is some $x_0 \in \text{Int}(\Omega)$ satisfying that $\alpha(x_0) = \max_{x\in\overline{\Omega}} \alpha(x)$ and the partial derivatives of $\alpha(x)$ up to order $N-1$ at x_0 are zero, then $\lambda_1(\mathcal{A}) = s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$ and $\{S(t)\}_{t\geq 0}$ exhibits asynchronous exponential growth.

Remark 4.7 (a) Observe that the criterion for the existence of the principal eigenvalue that we provided in (4.3) and (4.12) are reasonable and comparable with the ones obtained for other nonlocal problems, for instance, see Coville [11] who employed generalized Krein-Rutman Theorem to obtain existence of the principal eigenvalue of a nonlocal diffusion operator. In fact, for our case (4.3) and (4.12) imply $s(\mathcal{A}) > s(\mathcal{B}_1 + \mathcal{C})$. It follows from the proof of Theorem 4.1 that $\{S(t)\}_{t\geq 0}$ is an essentially compact semigroup which implies that $r(\mathcal{A}) > r_e(\mathcal{A})$. Then by generalized Krein-Rutman Theorem (Theorem 2.1) we can also conclude existence of the principal eigenvalue. This shows the equivalence in using the theory of resolvent positive operators with their perturbations and using generalized Krein-Rutman Theorem to obtain existence of the principal eigenvalue. In addition, such criteria in (4.3) and (4.12) are sharp in the sense that if they are not satisfied, $\mathcal{A}$ admits no principal eigenvalue, see a counterexample in [24] for details.

(b) We would like to mention again that such sufficient conditions (4.3) and (4.12) are also valid for age-structured models with nonlocal diffusion of Dirichlet type to obtain the existence of the principal eigenvalue and asynchronous exponentially growth (see [24] in which we only discussed the existence of the principal eigenvalue).

5 Formula of asynchronous exponential growth

In this section, we derive a formula for the projection $P_{\lambda_0} := v \otimes v^* : \mathcal{X}_0 \to \ker(\mathcal{A} - \lambda_0 I)$ inspired by Walker [48], where $\lambda_0 = s(\mathcal{A})$.

Note that $\lambda_0 = s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$ when the assumptions in Theorems 4.3 or 4.5 are satisfied. Since $\{S(t)\}_{t\geq 0}$ exhibits asynchronous exponential growth, $\lambda_0 = s(\mathcal{A})$ is a simple eigenvalue of $\mathcal{A}$ by Theorem 2.13-(iii), which implies that $1 \in \sigma_p(\mathcal{M}_{\lambda_0})$ with geometric multiplicity 1. It follows that there is a positive element $\Phi_0 \in X$ such that

$$\ker(I - \mathcal{M}_{\lambda_0}) = \text{span}\{\Phi_0\} \text{ and } \ker(\mathcal{A} - \lambda_0 I) = \text{span}\left\{\begin{pmatrix} 0 \\ e^{-\lambda_0 a}\mathcal{U}(0,a)\Phi_0 \end{pmatrix}\right\}.$$

Let $\phi \in Q := L^1((0,a^+), L^1(\Omega))$ be fixed and let $c(\phi) \in \mathbb{R}$ be such that

$$P_{\lambda_0}\begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ c(\phi)e^{-\lambda_0 a}\mathcal{U}(0,a)\Phi_0 \end{pmatrix}.$$

Note that we only need to find the second component of $P_{\lambda_0}\begin{pmatrix} 0 \\ \phi \end{pmatrix}$ since the first one is always zero. Thus in the following we will write $P_{\lambda_0}\phi = c(\phi)e^{-\lambda_0 a}\mathcal{U}(0,a)\Phi_0$ without ambiguity. Recall that λ_0 is a simple pole of the resolvent $(\mathcal{A} - \lambda I)^{-1}$. Denote

$$(H_\lambda \phi)(a) := \int_0^a e^{-\lambda(a-\sigma)}\mathcal{U}(\sigma,a)\phi(\sigma,\cdot)d\sigma.$$

Then $H_\lambda \phi$ is holomorphic in λ and it follows from (3.21) and residue theorem that

$$P_{\lambda_0}\phi = \lim_{\lambda\to\lambda_0} (\lambda-\lambda_0)e^{-\lambda a}\mathcal{U}(0,a)(I-\mathcal{M}_\lambda)^{-1}W_\lambda\phi,$$

where

$$W_\lambda\phi = \int_0^{a^+} \beta(a,\cdot)\int_0^a e^{-\lambda(a-\sigma)}\mathcal{U}(\sigma,a)\phi(\sigma,\cdot)d\sigma da.$$

Let $w' \in X'$ be a positive eigenfunctional of the dual operator $\mathcal{M}'_{\lambda_0}$ of $\mathcal{M}_{\lambda_0}$ corresponding to the eigenvalue $r(\mathcal{M}_{\lambda_0}) = 1$. Then for $f' \in Q'$ defined by

$$\langle f', \psi\rangle := \left\langle w', \int_0^{a^+} \beta(a,\cdot)\psi(a,\cdot)da\right\rangle, \quad \psi \in Q,$$

we have due to $\mathcal{M}'_{\lambda_0}w' = w'$ that

$$\begin{aligned} c(\phi)\langle w',\Phi_0\rangle = \langle f', P_{\lambda_0}\phi\rangle &= \lim_{\lambda\to\lambda_0}\langle f', (\lambda-\lambda_0)e^{-\lambda a}\mathcal{U}(0,a)(I-\mathcal{M}_\lambda)^{-1}W_\lambda\phi\rangle \\ &= \lim_{\lambda\to\lambda_0}\langle w', (\lambda-\lambda_0)(I-(I-\mathcal{M}_\lambda))(I-\mathcal{M}_\lambda)^{-1}W_\lambda\phi\rangle \\ &= \lim_{\lambda\to\lambda_0}\langle w', (\lambda-\lambda_0)(I-\mathcal{M}_\lambda)^{-1}W_\lambda\phi\rangle. \end{aligned}$$

Decompose $W_\lambda\phi$ as

$$W_\lambda\phi = d(W_\lambda\phi)\Phi_0 \oplus (I-\mathcal{M}_{\lambda_0})g(W_\lambda\phi). \tag{5.1}$$

According to the decomposition $X = \mathbb{R}\cdot\Phi_0 \oplus \mathrm{rg}(I-\mathcal{M}_{\lambda_0})$, it follows that

$$\lim_{\lambda\to\lambda_0}\langle w', (\lambda-\lambda_0)(I-\mathcal{M}_\lambda)^{-1}W_\lambda\phi\rangle = d(W_{\lambda_0}\phi)\lim_{\lambda\to\lambda_0}\langle w', (\lambda-\lambda_0)(I-\mathcal{M}_\lambda)^{-1}\Phi_0\rangle$$

due to the continuity of $\mathcal{M}_\lambda$ in λ. But it follows from (5.1) that

$$\langle w', W_{\lambda_0}\phi\rangle = d(W_{\lambda_0}\phi)\langle w', \Phi_0\rangle$$

since $\mathcal{M}'_{\lambda_0}w' = w'$, whence $d(W_{\lambda_0}\phi) = \xi\langle w', W_{\lambda_0}\phi\rangle$ with $\xi^{-1} = \langle w', \Phi_0\rangle$. Similarly, decomposing

$$Y_\lambda := (\lambda-\lambda_0)(I-\mathcal{M}_\lambda)^{-1}\Phi_0,$$

we find that

$$\lim_{\lambda\to\lambda_0}\langle w', Y_\lambda\rangle = \left(\lim_{\lambda\to\lambda_0} d(Y_\lambda)\right)\langle w', \Phi_0\rangle.$$

Based on these observations, we derive that

$$c(\phi)\langle w', \Phi_0\rangle = C_0\langle w', W_{\lambda_0}\phi\rangle\langle w', \Phi_0\rangle$$

for some constant C_0. Consequently,

$$P_{\lambda_0}\phi = C_0\langle w', W_{\lambda_0}\phi\rangle e^{-\lambda_0 a}\mathcal{U}(0,a)\Phi_0.$$

Since P_{λ_0} is a projection; i.e., $P_{\lambda_0}^2 = P_{\lambda_0}$, the constant C_0 can be easily computed and we obtain the following result.

Proposition 5.1 Under the assumptions of Theorems 4.3 or 4.5, the projection $P_{\lambda_0} := v \otimes v^*$ is given by

$$P_{\lambda_0}\phi = \frac{\langle w', W_{\lambda_0}\phi\rangle}{\langle w', \int_0^{a^+} a\beta(a,\cdot)e^{-\lambda_0 a}\mathcal{U}(0,a)\Phi_0 da\rangle}\, e^{-\lambda_0 a}\mathcal{U}(0,a)\Phi_0 \tag{5.2}$$

for $\phi \in Q$, where

$$W_{\lambda_0}\phi = \int_0^{a^+} \beta(a,\cdot)\int_0^a e^{-\lambda_0(a-\sigma)}\mathcal{U}(\sigma,a)\phi(\sigma,\cdot)d\sigma da$$

and $w' \in X'$ is a positive eigenfunctional of the dual operator $\mathcal{M}'_{\lambda_0}$ of $\mathcal{M}_{\lambda_0}$ corresponding to the eigenvalue $r(\mathcal{M}_{\lambda_0}) = 1$.

6 Limiting properties

In this section we study the effects of diffusion rate on the principal spectrum point $\lambda_1(\mathcal{A})$ of $\mathcal{A}$. Following the idea from Berestycki et al. [7,8], we introduce the following definition.

Definition 6.1 Define the generalized principal eigenvalue by

$$\begin{cases} \lambda_p(\mathcal{A}) := \sup\{\lambda \in \mathbb{R} : \exists\,(0,\phi) \in D(\mathcal{A}) \cap \mathcal{X}_0^{++} \\ \qquad\qquad \text{s.t. } (-\mathcal{A}+\lambda)(0,\phi) \le (0,0) \text{ in } [0,a^+]\times\overline{\Omega}\}, \\ \lambda'_p(\mathcal{A}) := \inf\{\lambda \in \mathbb{R} : \exists\,(0,\phi) \in D(\mathcal{A}) \cap \mathcal{X}_0^{++} \\ \qquad\qquad \text{s.t. } (-\mathcal{A}+\lambda)(0,\phi) \ge (0,0) \text{ in } [0,a^+]\times\overline{\Omega}\}, \end{cases} \tag{6.1}$$

where $\mathcal{X}_0^{++} = \{0\}\times\{u \in C([0,a^+]\times\overline{\Omega}) : u(a,x) > 0, (a,x) \in [0,a^+]\times\overline{\Omega}\}$. We would like to mention that the sets in Definition 6.1 are nonempty (see the proof of Theorem 6.5 in the following). This idea has been widely used to prove the existence and asymptotic behavior of the principal eigenvalue with respect to diffusion rate, see Coville [11], Li et al. [28] and Su et al. [42] for nonlocal diffusion equations, Shen and Vo [40] and Su et al. [41] for time periodic nonlocal diffusion equations.

As Shen and Vo [40] pointed out for the time periodic case, we emphasize that the parabolic-type operator $\mathcal{A}$ containing ∂_a is not self-adjoint, so we do not have the usual $L^2(\Omega)$ variational formula for the principal eigenvalue $\lambda_1(\mathcal{A})$. The generalized principal eigenvalues $\lambda_p(\mathcal{A})$ and $\lambda'_p(\mathcal{A})$ defined in (6.1) are helpful in addressing this issue.

6.1 Without scaling

In this subsection first we study the diffusion kernel without scaling; i.e., L defined in (3.1).

Proposition 6.2 $\lambda_1(\mathcal{A}) = \lambda_p(\mathcal{A}) = \lambda'_p(\mathcal{A})$ if $\lambda_1(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$.

Proof First we prove that $\lambda_1 = \lambda_p$. Since $\lambda_1(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$, there exists $(0, \phi_1) \in D(\mathcal{A}) \cap \mathcal{X}_0^{++}$ such that

$$\mathcal{A}(0, \phi_1) - \lambda_1(0, \phi_1) = (0, 0) \quad \text{in} \quad [0, a^+] \times \overline{\Omega}. \tag{6.2}$$

Since $\inf_{[0,a^+]\times\overline{\Omega}} \phi_1 > 0$, we have $\lambda_1 \leq \lambda_p$. Suppose by contradiction that $\lambda_1 < \lambda_p$. From the definition of λ_p, there are $\lambda \in (\lambda_1, \lambda_p)$ and $(0, \phi) \in D(\mathcal{A}) \cap \mathcal{X}_0^{++}$ such that

$$-\mathcal{A}(0, \phi) + \lambda(0, \phi) \leq (0, 0) \quad \text{in} \quad [0, a^+] \times \overline{\Omega} \tag{6.3}$$

that is,

$$\begin{cases} \frac{\partial \phi(a,x)}{\partial a} - D\int_\Omega J(x-y)(\phi(a,y) - \phi(a,x))dy + \mu(a,x)\phi + \lambda\phi \leq 0, \\ \phi(0,x) - \int_0^{a^+} \beta(a,x)\phi(a,x)da \leq 0. \end{cases} \tag{6.4}$$

Now solving the first inequality in (6.4), we obtain

$$\phi(a, \cdot) \leq e^{-\lambda a} \mathcal{U}(0, a)\phi(0, \cdot).$$

Plugging it into the second inequality in (6.4), we have

$$\phi(0, \cdot) \leq \int_0^{a^+} \beta(a, \cdot) e^{-\lambda a} \mathcal{U}(0, a)\phi(0, \cdot)da. \tag{6.5}$$

It follows that $\mathcal{M}_\lambda \phi(0, \cdot) \geq \phi(0, \cdot)$, which implies that $r(\mathcal{M}_\lambda) \geq 1$. But we know that λ_1 is the principal eigenvalue of $\mathcal{A}$, then by Proposition 3.13, we have $r(\mathcal{M}_{\lambda_1}) = 1$. Since $\lambda \to r(\mathcal{M}_\lambda)$ is strictly decreasing following a similar argument as in Proposition 3.3 or [24, Proposition 3.7], one has $\lambda_1 \geq \lambda$. This contradiction leads to $\lambda_1 = \lambda_p$.

Next we prove $\lambda_1 = \lambda'_p$. Obviously, $\lambda_1 \geq \lambda'_p$. Assume that $\lambda_1 > \lambda'_p$. There are $\tilde{\lambda} \in (\lambda'_p, \lambda_1)$ and $(0, \tilde{\phi}) \in D(\mathcal{A}) \cap \mathcal{X}_0^{++}$ such that $-\mathcal{A}(0, \tilde{\phi}) + \tilde{\lambda}(0, \tilde{\phi}) \geq (0, 0)$. By reversing the above inequalities, we have the desired conclusion by using a similar argument as above. □

Next we recall a lemma from Vo [47] on a Poincaré-type inequality of the operator $\mathcal{K} : L^2(\Omega) \to L^2(\Omega)$ defined by

$$\mathcal{K}[f](x) = -\int_\Omega J(x-y)[f(y) - f(x)]dy, \quad x \in \Omega.$$

Lemma 6.3 (Vo [47, Lemma 3.2]) Assume that J is symmetric with respect to each component. Then

$$\int_\Omega \mathcal{K}[f](x) f(x) dx = \frac{1}{2}\int_\Omega \int_\Omega J(x-y)[f(y) - f(x)]^2 dy dx,$$

and there exists $C > 0$ such that

$$\int_\Omega \mathcal{K}[f](x) f(x) dx \geq C \int_\Omega f^2(x) dx$$

for all $f \in L^2(\Omega)$ with $\int_\Omega f(x)dx = 0$.

Lemma 6.4 $\lambda_1 \leq \lambda_1(\mathcal{A}) \leq \lambda^1$ for all $D > 0$, where λ_1 and λ^1 are defined in (6.10) and (6.7), respectively.

Proof Let $\Psi^1(a)$ be the positive solution of the following age-structured equation (note that the existence is guaranteed by the theory of age-structured models)

$$\begin{cases} \frac{\partial \Psi^1(a)}{\partial a} = -(\lambda^1 + \underline{\mu}(a))\Psi^1(a), \\ \Psi^1(0) = \int_0^{a^+} \overline{\beta}(a)\Psi^1(a)da, \end{cases} \tag{6.6}$$

where λ^1 satisfies

$$\int_0^{a^+} \overline{\beta}(a) e^{-\lambda^1 a} e^{-\int_0^a \underline{\mu}(s)ds} da = 1. \tag{6.7}$$

Then $(0, \Psi^1) \in \mathcal{X}_0^{++} \cap D(\mathcal{A})$ and it is easy to compute that

$$\begin{aligned} &-\mathcal{A}(0, \Psi^1) + \lambda^1(0, \Psi^1) \\ &= \begin{pmatrix} \Psi^1(0) - \int_0^{a^+} \beta(a,x)\Psi^1(a)da, \frac{\partial \Psi^1(a)}{\partial a} - D\int_\Omega J(x-y) \\ (\Psi^1(a) - \Psi^1(a))dy + \mu(a,x)\Psi^1 + \lambda^1 \Psi^1 \end{pmatrix} \\ &= \left(\int_0^{a^+} (\overline{\beta}(a) - \beta(a,x))\Psi^1(a)da, (\mu(a,x) - \underline{\mu}(a))\Psi^1\right) \\ &\geq \begin{pmatrix} 0, & 0 \end{pmatrix}. \end{aligned} \tag{6.8}$$

It follows by Proposition 6.2 that $\lambda_1(\mathcal{A}) = \lambda_p'(\mathcal{A}) \leq \lambda^1$.

Similarly, consider the following equation with a positive solution $\Psi_1(a)$:

$$\begin{cases} \frac{\partial \Psi_1(a)}{\partial a} = -(\lambda_1 + \overline{\mu}(a))\Psi_1(a), \\ \Psi_1(0) = \int_0^{a^+} \underline{\beta}(a)\Psi_1(a)da, \end{cases} \tag{6.9}$$

where λ_1 satisfies

$$\int_0^{a^+} \underline{\beta}(a)e^{-\lambda_1 a}e^{-\int_0^a \overline{\mu}(s)ds}da = 1. \tag{6.10}$$

Then similar computation yields $-\mathcal{A}(0,\Psi_1) + \lambda_1(0,\Psi_1) \leq (0,0)$, which implies that $\lambda_1(\mathcal{A}) = \lambda_p(\mathcal{A}) \geq \lambda_1$. Thus the conclusion is proven. □

Now we give the main theorem about the effects of diffusion rate on $\lambda_1(\mathcal{A})$. We write $\lambda_1^D(\mathcal{A})$ for $\lambda_1(\mathcal{A})$ to emphasize the dependence on D.

Theorem 6.5 Assume that $\lambda_1^D(\mathcal{A}) = s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$, then the function $D \to \lambda_1^D(\mathcal{A})$ is continuous on $(0,\infty)$ and satisfies

(i) $\lim_{D\to 0^+} \lambda_1^D(\mathcal{A}) = s(\mathcal{B}_1^0 + \mathcal{C})$, where

$$\mathcal{B}_1^0(0,f) := \left(-f(0,\cdot), \quad -f' - \mu f\right), \quad (0,f) \in D(\mathcal{A});$$

(ii) In addition, if $\beta(a,x) \equiv \beta(a)$ and $\|\beta\|^2_{L^2(0,a^+)} > 2\lambda_1$ with λ_1 defined in (6.10), then $\lim_{D\to\infty} \lambda_1^D(\mathcal{A}) = \lambda_0$, where λ_0 satisfies the following equation

$$\int_0^{a^+} \beta(a)e^{-\lambda_0 a}e^{-\frac{1}{|\Omega|}\int_0^a\int_\Omega \mu(s,x)dxds}da = 1. \tag{6.11}$$

Proof Since $\lambda_1^D(\mathcal{A})$ is an isolated eigenvalue, the continuity of $D \to \lambda_1^D(\mathcal{A})$ follows from the classical perturbation theory (see Kato [27, Section IV.3.5]).

(i) For the limits, we first claim that for every $\epsilon > 0$, there exists $D_\epsilon > 0$ such that

$$s(\mathcal{B}_1^0 + \mathcal{C}) - \epsilon \leq \lambda_1^D(\mathcal{A}) \leq s(\mathcal{B}_1^0 + \mathcal{C}) + \epsilon, \quad \forall D \in (0, D_\epsilon). \tag{6.12}$$

Denote $\vartheta = s(\mathcal{B}_1^0 + \mathcal{C})$. Consider the following equation

$$\begin{cases} \frac{\partial \phi(a,x)}{\partial a} = -(\alpha(x) + \mu(a,x))\phi(a,x), & (a,x) \in (0,a^+) \times \overline{\Omega}, \\ \phi(0,x) = \int_0^{a^+} \beta(a,x)\phi(a,x)da, & x \in \overline{\Omega}. \end{cases} \tag{6.13}$$

By Proposition 3.5, we know that (6.13) has a solution $\phi(a,x) = e^{-\alpha(x)a}\Pi(0,a,x)\phi(0,x) \in C^{1,0}_{++}([0,a^+] \times \overline{\Omega})$ if the initial data $\phi(0,x)$ is continuous, positive and bounded, where $C^{1,0}([0,a^+]\times\overline{\Omega})$, which represents the space of functions which have continuous differentiability with respect to $a\in[0,a^+]$ and continuity with respect to $x\in\overline{\Omega}$, the double plus sign $_{++}$ denotes the strict positive cone. Thus, $(0,\phi)\in D(\mathcal{A})\cap\mathcal{X}_0^{++}$. Moreover, it can be checked that

$$\begin{aligned}&-\mathcal{A}(0,\phi)+(\vartheta+\epsilon)(0,\phi)\\&=\Bigg(\phi(0,x)-\int_0^{a^+}\beta(a,x)\phi(a,x)da,\frac{\partial\phi(a,x)}{\partial a}-D\int_\Omega J(x-y)(\phi(a,y)\\&\quad-\phi(a,x))dy+\mu(a,x)\phi+(\vartheta+\epsilon)\phi\Bigg).\end{aligned}$$

Since $\min_{[0,a^+]\times\overline{\Omega}}\phi>0$ and $\max_{[0,a^+]\times\overline{\Omega}}\phi<\infty$, it is straightforward to check that for each $\epsilon>0$, there exists $D_{1\epsilon}>0$ such that for each $D\in(0,D_{1\epsilon})$, we have

$$\begin{aligned}&\frac{\partial\phi(a,x)}{\partial a}-D\int_\Omega J(x-y)(\phi(a,y)-\phi(a,x))dy+\mu(a,x)\phi+(\vartheta+\epsilon)\phi\\&=-D\int_\Omega J(x-y)(\phi(a,y)-\phi(a,x))dy+(\vartheta-\alpha(x))\phi+\epsilon\phi\\&\geq-D\int_\Omega J(x-y)(\phi(a,y)-\phi(a,x))dy+\epsilon\phi\\&\geq 0,\end{aligned}\tag{6.14}$$

where we used $\vartheta\geq\alpha(x)$ from Proposition 3.5 in which $D=0$. It then follows that $-\mathcal{A}(0,\phi)+(\vartheta+\epsilon)(0,\phi)\geq(0,0)$, which by the definition of $\lambda_p'(\mathcal{A})$ implies that

$$\lambda_1^D(\mathcal{A})=\lambda_p'(\mathcal{A})\leq s(\mathcal{B}_1^0+\mathcal{C})+\epsilon.\tag{6.15}$$

Next, from Proposition 3.3, we know that $s(\mathcal{B}_1^0+\mathcal{C})=\alpha_1$ and $s(\mathcal{B}_1+\mathcal{C})=\alpha^{**}$, respectively, which satisfy

$$\begin{aligned}&\max_{x\in\overline{\Omega}}\int_0^{a^+}\beta(a,x)e^{-\alpha_1 a}\Pi(0,a,x)da=1,\\&\max_{x\in\overline{\Omega}}\int_0^{a^+}\beta(a,x)e^{-(\alpha^{**}+D\kappa(x))a}\Pi(0,a,x)da=1.\end{aligned}\tag{6.16}$$

It follows that $\alpha^{**}\uparrow\alpha_1$ as $D\to 0^+$. Then for the previous same $\epsilon>0$, there exists $D_{2\epsilon}>0$, such that for each $D\in(0,D_{2\epsilon})$, we have

$$s(\mathcal{B}_1^0+\mathcal{C})=\alpha_1\leq\alpha^{**}+\epsilon=s(\mathcal{B}_1+\mathcal{C})+\epsilon.\tag{6.17}$$

Now combing with (6.15), we have by Remark 3.10 that for any $\epsilon > 0$, there exists $D_\epsilon = \min\{D_{1\epsilon}, D_{2\epsilon}\}$ such that for each $D \in (0, D_\epsilon)$,

$$s(\mathcal{B}_1^0 + \mathcal{C}) - \epsilon \le s(\mathcal{B}_1 + \mathcal{C}) \le s(\mathcal{A}) = \lambda_1^D(\mathcal{A}) \le s(\mathcal{B}_1^0 + \mathcal{C}) + \epsilon.$$

Setting $D \to 0^+$, we find that

$$s(\mathcal{B}_1^0 + \mathcal{C}) - \epsilon \le \liminf_{D\to 0^+} \lambda_1^D(\mathcal{A}) \le \limsup_{D\to 0^+} \lambda_1^D(\mathcal{A}) \le s(\mathcal{B}_1^0 + \mathcal{C}) + \epsilon, \quad \forall \epsilon > 0,$$

which leads to $\lambda_1^D(\mathcal{A}) \to s(\mathcal{B}_1^0 + \mathcal{C})$ as $D \to 0^+$.

(ii) Finally, we prove the other limit $\lambda_1^D(\mathcal{A}) \to \lambda_0$ as $D \to \infty$. Assume that $(0, \phi) \in \mathcal{X}_0^{++} \cap D(\mathcal{A})$ is the principal eigenfunction associated with eigenvalue $\lambda_D = \lambda_1^D(\mathcal{A})$; i.e. $(\lambda_D, (0, \phi))$ as an eigen-pair satisfying (6.4) (with $\le$ replaced by $=$). Multiplying the first equation of (6.4) by ϕ and integrating the resulting equation over $[0, a^+] \times \Omega$, we find from the normalization $\|\phi\|_{L^2((0,a^+)\times\Omega)} = 1$ that

$$\begin{aligned}
&D \int_0^{a^+} \int_\Omega \left\{ \int_\Omega J(x-y)[\phi(a,y) - \phi(a,x)]dy \right\} \phi(a,x)dxda \\
&- \int_0^{a^+} \int_\Omega \mu(a,x)\phi^2(a,x)dxda + \frac{1}{2}\int_\Omega \left\{ \int_0^{a^+} \beta(a)\phi(a,x)da \right\}^2 dx \\
&- \frac{1}{2}\left\|\phi(a^+,\cdot)\right\|_{L^2(\Omega)}^2 - \lambda_D = 0. \qquad (6.18)
\end{aligned}$$

By the symmetry of J, we have (see Vo [47, Lemma 3.2])

$$\begin{aligned}
&\int_0^{a^+} \left\{ \int_\Omega \int_\Omega J(x-y)[\phi(a,y) - \phi(a,x)]^2 dydx \right\} da \\
&\quad = -2\int_0^{a^+} \int_\Omega \left\{ \int_\Omega J(x-y)[\phi(a,y) - \phi(a,x)]dy \right\} \phi(a,x)dxda. \qquad (6.19)
\end{aligned}$$

It then follows from (6.18) that

$$\begin{aligned}
&-\frac{D}{2}\int_0^{a^+} \int_\Omega \left\{ \int_\Omega J(x-y)[\phi(a,y) - \phi(a,x)]^2 dy \right\} dxda - \int_0^{a^+} \int_\Omega \mu(a,x)\phi^2(a,x)dxda \\
&+\frac{1}{2}\int_\Omega \left\{ \int_0^{a^+} \beta(a)\phi(a,x)da \right\}^2 dx - \frac{1}{2}\left\|\phi(a^+,\cdot)\right\|_{L^2(\Omega)}^2 - \lambda_D = 0. \qquad (6.20)
\end{aligned}$$

Since $\mu(a, x)$ and $\beta(a, x)$ are bounded and $\{\lambda_D\}_{D\gg 1}$ is bounded by Lemma 6.4, there exists $C = C(\beta, \lambda_1) > 0$ by Hölder's inequality and the assumption of theorem such that

$$\begin{aligned}
&\frac{D}{2}\int_0^{a^+}\int_\Omega\left\{\int_\Omega J(x-y)[\phi(a,y)-\phi(a,x)]^2dy\right\}dxda\\
&\leq \frac{1}{2}\|\beta\|^2_{L^2(0,a^+)}-\lambda_D-\frac{1}{2}\left\|\phi(a^+,\cdot)\right\|^2_{L^2(\Omega)}-\min_{[0,a^+]\times\overline{\Omega}}\mu(a,x)\\
&\leq \frac{1}{2}\|\beta\|^2_{L^2(0,a^+)}-\lambda_1:=C. \qquad (6.21)
\end{aligned}$$

Define $\overline{\phi}(a)=\frac{1}{|\Omega|}\int_\Omega\phi(a,x)dx$ for $a\in[0,a^+]$ and set $\psi=\phi-\overline{\phi}$. Applying Lemma 6.3, we have from (6.21) that

$$\begin{aligned}
&\int_0^{a^+}\int_\Omega\mathcal{K}[\psi(a,\cdot)](x)\psi(a,x)dxda\\
&=\frac{1}{2}\int_0^{a^+}\left\{\int_\Omega\int_\Omega J(x-y)[\psi(a,y)-\psi(a,x)]^2dydx\right\}da\\
&=\frac{1}{2}\int_0^{a^+}\left\{\int_\Omega\int_\Omega J(x-y)[\phi(a,y)-\phi(a,x)]^2dydx\right\}da\\
&\leq\frac{C}{D}.
\end{aligned}$$

Since $\int_\Omega\psi(a,x)dx=0$ for all $a\in[0,a^+]$, we can apply Lemma 6.3 to derive

$$\int_\Omega\psi^2(a,x)dx\leq C_1\int_\Omega\mathcal{K}[\psi(a,\cdot)](x)\psi(a,x)dx,\quad\forall a\in[0,a^+],$$

for some $C_1>0$. Hence,

$$\int_0^{a^+}\int_\Omega\psi^2(a,x)dxda\leq\frac{C_1C}{D}. \qquad (6.22)$$

Integrating the first equation of (6.4) (with $\leq$ replaced by $=$) over Ω and dividing the resulting equation by $|\Omega|$, we find

$$\begin{aligned}
\frac{\partial\overline{\phi}}{\partial a}&=\frac{1}{|\Omega|}\int_\Omega\left\{D\int_\Omega J(x-y)[\phi(a,y)-\phi(a,x)]dy-\mu(a,x)\phi(a,x)-\lambda_D\phi(a,x)\right\}dx\\
&=-\lambda_D\overline{\phi}-\frac{1}{|\Omega|}\int_\Omega\mu(a,x)\phi(a,x)dx. \qquad (6.23)
\end{aligned}$$

Setting $\hat{\mu}(a)=\frac{1}{|\Omega|}\int_\Omega\mu(a,x)dx$, we have

$$\begin{aligned}
\frac{\partial\overline{\phi}}{\partial a}+[\hat{\mu}(a)+\lambda_D]\overline{\phi}&=-\frac{1}{|\Omega|}\int_\Omega\mu(a,x)[\phi(a,x)-\overline{\phi}(a)]dx\\
&=-\frac{1}{|\Omega|}\int_\Omega\mu(a,x)\psi(a,x)dx. \qquad (6.24)
\end{aligned}$$

It follows from the variation of constants formula that

$$\overline{\phi}(a) = \overline{\phi}(0)e^{-\int_0^a(\hat{\mu}(s)+\lambda_D)ds} - \frac{1}{|\Omega|}\int_0^a e^{-\int_\tau^a(\hat{\mu}(s)+\lambda_D)ds} \times \int_\Omega \mu(\tau,x)\psi(\tau,x)dxd\tau, \quad a \in [0,a^+].$$

Since $\mu(a,x)$ and $\{\lambda_D\}_{D\gg 1}$ are bounded, we deduce from (6.22) and Hölder's inequality that

$$\overline{\phi}(a) = \overline{\phi}(0)e^{-\int_0^a[\hat{\mu}(s)+\lambda_D]ds} + O\left(\frac{1}{\sqrt{D}}\right), \quad \forall a \in [0,a^+], \tag{6.25}$$

for all $D \gg 1$. Now plugging (6.25) into the initial boundary condition, we obtain

$$\overline{\phi}(0) = \int_0^{a^+}\beta(a)\overline{\phi}(a)da = \int_0^{a^+}\beta(a)\overline{\phi}(0)e^{-\int_0^a[\hat{\mu}(s)+\lambda_D]ds}da + O\left(\frac{1}{\sqrt{D}}\right)\int_0^{a^+}\beta(a)da.$$

Since $\beta(a)$ is also bounded, after cancellation of $\overline{\phi}(0)$ in both sides, we have

$$1 = \int_0^{a^+}\beta(a)e^{-\lambda_D a}e^{-\frac{1}{|\Omega|}\int_0^a\int_\Omega \mu(s,x)dxds}da + O\left(\frac{1}{\sqrt{D}}\right)$$

for all $D \gg 1$. The reason that $\overline{\phi}(0) \nrightarrow 0$ is given in Vo [47, Theorem A(3)], we omit it here. Now letting $D \to \infty$, we obtain

$$\int_0^{a^+}\beta(a)e^{-\lambda_D a}e^{-\frac{1}{|\Omega|}\int_0^a\int_\Omega \mu(s,x)dxds}da = 1,$$

which leads to $\lambda_D = \lambda_0$ defined in (6.11) by the strict monotonicity of $H(\lambda)$ with respect to λ, where

$$H(\lambda) := \int_0^{a^+}\beta(a)e^{-\lambda a}e^{-\frac{1}{|\Omega|}\int_0^a\int_\Omega \mu(s,x)dxds}da.$$

This completes the proof. □

Remark 6.6 It is worthwhile to point out that $s(\mathcal{B}_1^0 + \mathcal{C})$ can be explicitly characterized by using α_1 as shown in (6.16). Moreover, it is interesting and open to investigate the limit $\lim_{D\to\infty}\lambda_1^D(\mathcal{A})$ without the additional assumption that $\beta(a,x) \equiv \beta(a)$ in (ii). We conjecture that $\lim_{D\to\infty}\lambda_1^D(\mathcal{A}) = \lambda_1$, where λ_1 satisfies the following equation

$$\frac{1}{|\Omega|}\int_0^{a^+}\int_\Omega \beta(a,x)e^{-\lambda_1 a}e^{-\frac{1}{|\Omega|}\int_0^a\int_\Omega \mu(s,x)dxds}dxda = 1.$$

We leave it for future consideration.

Theorem 6.7 If $\mu(a,x) = \mu_1(a) + \mu_2(x)$ with $\mu_2(\cdot) \not\equiv$ constant, $\beta(a,x) \equiv \beta(a)$, and suppose that J is symmetric with respect to each component and in addition the operator

$$v \to D\left[\int_\Omega J(\cdot - y)(v(y) - v(\cdot))dy\right] - \mu_2 v : X \to X$$

admits a principal eigenvalue, then $D \to \lambda_1^D(\mathcal{A})$ is strictly decreasing.

Proof We write $\mathcal{A} = \mathcal{T} + L_\Omega$, where

$$\mathcal{T}(0,\phi) = \left(-\phi(0) + \int_0^{a^+} \beta(a)\phi(a)da, \quad -\phi' - \mu_1\phi\right), \quad \phi \in W^{1,1}([0,a^+]),$$

$$L_\Omega(0,v) = \left(0, \quad D\left[\int_\Omega J(\cdot - y)(v(y) - v(\cdot))dy\right] - \mu_2 v\right), \quad v \in X.$$

Let $(\lambda_1^D(L_\Omega), (0, v_\Omega))$ be the principal eigenpair of $-L_\Omega$. Then by using the same argument as in Shen and Xie [39, Theorem 2.2(1)], we know that $D \to \lambda_1^D(L_\Omega)$ is strictly increasing. Now define $(0,\phi_1)$ to be the solution of the characteristic equation $\mathcal{T}(0,\phi) = \lambda_1(0,\phi)$ (note that the existence of (λ_1, ϕ_1) is guaranteed by the theory of age-structured models). It follows that $\lambda_1^D(\mathcal{A}) = -\lambda_1^D(L_\Omega) + \lambda_1$ is the principal eigenvalue of $\mathcal{A}$ with the principal eigenfunction $\left(0, v_\Omega(x)\phi_1(a)\right)$. As $D \to \lambda_1^D(L_\Omega)$ is strictly increasing, so $D \to \lambda_1^D(\mathcal{A})$ is strictly decreasing. □

6.2 With scaling

In this subsection we investigate the diffusion kernel with scaling; i.e., $L_{\sigma,m}$ defined in (2.5). First we give a proposition to address the effects of μ and β on the principal eigenvalue. Write $\mathcal{A}_{\sigma,m} = \mathcal{B}_{\sigma,m} + \mathcal{C}$ for $\mathcal{A} = \mathcal{B} + \mathcal{C}$ to highlight the dependence on σ and m. Also use $\mathcal{B}^\mu_{\sigma,m}$ and $\mathcal{C}^\beta$ for $\mathcal{B}$ and $\mathcal{C}$ to represent the dependence on μ and β respectively.

Proposition 6.8 Let $m \geq 0$ and $\sigma > 0$. We have the following statements:

(i) Assume that $\lambda_1(\mathcal{B}_{\sigma,m} + \mathcal{C}^\beta)$ and $\lambda_1(\mathcal{B}^\mu_{\sigma,m} + \mathcal{C})$ are the principal eigenvalues of $\mathcal{B}_{\sigma,m} + \mathcal{C}^\beta$ and $\mathcal{B}^\mu_{\sigma,m} + \mathcal{C}$ respectively, then $\lambda_1(\mathcal{B}_{\sigma,m} + \mathcal{C}^\beta)$ is non-decreasing with respect to β and $\lambda_1(\mathcal{B}^\mu_{\sigma,m} + \mathcal{C})$ is non-increasing with respect to μ;

(ii) Moreover, $\lambda_1(\mathcal{B}^\mu_{\sigma,m} + \mathcal{C})$ is Lipschitz continuous with respect to μ in $C([0,a^+], X)$ if $\lambda_1(\mathcal{A}_{\sigma,m})$ is the principal eigenvalue of $\mathcal{A}_{\sigma,m}$. More precisely,

$$|\lambda_1(\mathcal{B}^{\mu_1}_{\sigma,m} + \mathcal{C}) - \lambda_1(\mathcal{B}^{\mu_2}_{\sigma,m} + \mathcal{C})| \leq \|\mu_1 - \mu_2\|_{C([0,a^+],X)}$$

for any $\mu_1, \mu_2 \in C([0,a^+], X)$;

The proof is almost identical to [24, Proposition 5.6], thus we omit it. Note that we do not have the monotonicity dependence of $\lambda_1(\mathcal{A})$ on Ω due to the Neumann boundary condition, which is different from the Dirichlet boundary condition.

Theorem 6.9 Let $\mathcal{B}_1$ be defined by (3.2) and

$$\mathcal{B}_1^0(0,f) := \Big(-f(0,\cdot), \quad -f' - \mu f\Big), \quad (0,f) \in D(\mathcal{A}).$$

Assume that $\lambda_1(\mathcal{A}_{\sigma,m}) = s(\mathcal{A}_{\sigma,m})$ is the principal eigenvalue of $\mathcal{A}_{\sigma,m}$, then the following statements hold:

(i) For each $m \geq 0$, there holds

$$\lim_{\sigma\to\infty} \lambda_1(\mathcal{A}_{\sigma,m}) = s(\mathcal{B}_1^0 + \mathcal{C}); \tag{6.26}$$

(ii) If $m \in [0,2)$ and $\beta, \mu \in C^{1,4}([0,a^+] \times \overline{\Omega})$, in addition, assume that J is symmetric with respect to each component, then

$$\lim_{\sigma\to 0^+} \lambda_1(\mathcal{A}_{\sigma,m}) = s(\mathcal{B}_1^0 + \mathcal{C}).$$

Before proving Theorem 6.9, we make the following remark.

Remark 6.10 Observing Theorem 6.9-(ii), we obtained a better result for $m \in [0,2)$ compared with Theorem B-(2) only for $m = 0$ in Vo [47]. In fact, such results when $m \in [0,2)$ for nonlocal operators of Neumann type were also obtained by Su et al. [41], where they compared the principal eigenvalue of time-periodic nonlocal diffusion operators with autonomous ones (without time derivative) to get the desired estimates. However, their method is not valid for our case, since we do not have an autonomous operator. We shall follow and improve the estimates of Theorem B-(2) in Vo [47].

Proof Note that Proposition 6.2 holds for $\mathcal{A}_{\sigma,m}$. (i) Let us consider $m > 0$. By using a similar argument as in (6.17) in Theorem 6.5 via replacing D by $\frac{D}{\sigma^m}$, we find that for any $\epsilon > 0$, there exists $\sigma_0 \gg 1$ such that for each $\sigma > \sigma_0$ there holds

$$s(\mathcal{B}_1^0 + \mathcal{C}) - \epsilon \leq s(\mathcal{B}_1 + \mathcal{C}).$$

It follows from Remark 3.10 that

$$\liminf_{\sigma\to\infty} \lambda_1(\mathcal{A}_{\sigma,m}) \geq s(\mathcal{B}_1 + \mathcal{C}) \geq s(\mathcal{B}_1^0 + \mathcal{C}) - \epsilon. \tag{6.27}$$

Let us still consider the equation in (6.13) with a solution $\phi(a,x) \in C^{1,0}_{++}([0,a^+] \times \overline{\Omega})$, under initial data $\phi(0,x)$, which is continuous, positive and bounded with $\vartheta = s(\mathcal{B}_1^0 + \mathcal{C})$. For the previous same $\epsilon > 0$, we see that for each $(a,x) \in [0,a^+] \times \overline{\Omega}$,

$$-\mathcal{A}_{\sigma,m}(0,\phi)+(\vartheta+\epsilon)(0,\phi)$$
$$=\left(\phi(0,x)-\int_0^{a^+}\beta(a,x)\phi(a,x)da, \frac{\partial\phi(a,x)}{\partial a}-\frac{D}{\sigma^m}\int_\Omega J_\sigma(x-y)(\phi(a,y)\right.$$
$$\left.-\phi(a,x))dy+\mu(a,x)\phi+(\vartheta+\epsilon)\phi\right)$$

and

$$\begin{aligned}&\frac{\partial\phi(a,x)}{\partial a}-\frac{D}{\sigma^m}\int_\Omega J_\sigma(x-y)(\phi(a,y)-\phi(a,x))dy+\mu(a,x)\phi+(\vartheta+\epsilon)\phi\\&=-\frac{D}{\sigma^m}\int_\Omega J_\sigma(x-y)(\phi(a,y)-\phi(a,x))dy+\epsilon\phi+(\vartheta-\alpha(x))\phi\\&\geq-\frac{D}{\sigma^m}\int_\Omega J_\sigma(x-y)(\phi(a,y)-\phi(a,x))dy+\epsilon\phi.\end{aligned}\tag{6.28}$$

Since $\min_{[0,a^+]\times\overline{\Omega}}\phi>0,\ \max_{[0,a^+]\times\overline{\Omega}}\phi<\infty$, and

$$\left\|\frac{D}{\sigma^m}\int_\Omega J_\sigma(\cdot-y)(\phi(a,y)-\phi(a,\cdot))dy\right\|_X\to 0\quad\text{as}\quad\sigma\to\infty,$$

there is $\sigma_\epsilon>0$ such that (6.28)≥ 0 for all $\sigma\geq\sigma_\epsilon$. It then follows that $-\mathcal{A}_{\sigma,m}(0,\phi)+(\vartheta+\epsilon)(0,\phi)\geq(0,0)$ which by the definition of $\lambda_p'(\mathcal{A}_{\sigma,m})$ implies that

$$\lambda_1(\mathcal{A}_{\sigma,m})=\lambda_p'(\mathcal{A}_{\sigma,m})\leq s(\mathcal{B}_1^0+\mathcal{C})+\epsilon.$$

The arbitrariness of ϵ with (6.27) then yields (i) for $m>0$.

Now we consider $m=0$. Note that $\kappa_\sigma(x)\to 0$ as $\sigma\to\infty$. It follows by the same argument in (6.17) that for any $\epsilon>0$, there exists $\sigma_1\gg 1$, such that for any $\sigma>\sigma_1$ there holds

$$s(\mathcal{B}_1^0+\mathcal{C})-\epsilon\leq s(\mathcal{B}_1+\mathcal{C}).$$

Then by Remark 3.10 we have

$$\liminf_{\sigma\to\infty}\lambda_1(\mathcal{A}_{\sigma,m})\geq s(\mathcal{B}_1+\mathcal{C})\geq s(\mathcal{B}_1^0+\mathcal{C})-\epsilon.\tag{6.29}$$

Next note that

$$\left\|D\int_\Omega J_\sigma(\cdot-y)(\phi(a,y)-\phi(a,\cdot))dy\right\|_X\to 0\quad\text{as}\quad\sigma\to\infty.$$

Thus by using the same argument as in (6.28), we obtain

$$\lambda_1(\mathcal{A}_{\sigma,m})=\lambda_p'(\mathcal{A}_{\sigma,m})\leq s(\mathcal{B}_1^0+\mathcal{C})+\epsilon.$$

The arbitrariness of ϵ with (6.29) yields (i) for $m = 0$.

Now we prove (ii). For $m \in [0, 2)$, let ϕ be the solution of (6.13) with a $C^4(\overline{\Omega})$ function $\phi_0 > 0$ as its initial data which is positive and bounded. And we can normalize ϕ_0 such that $\sup_{[0,a^+]\times\overline{\Omega}} \phi = 1$. Since $\mu, \beta \in C^{1,4}([0, a^+] \times \overline{\Omega})$, by the global implicit function Theorem 3.4 and Proposition 3.5, we have $\alpha \in C^4(\overline{\Omega})$ which implies that ϕ also belongs to $C^{1,4}([0, a^+] \times \overline{\Omega})$. For any $\epsilon > 0$ and $(a, x) \in [0, a^+] \times \overline{\Omega}$, we have

$$\begin{aligned}
&\frac{\partial \phi(a,x)}{\partial a} - \frac{D}{\sigma^m}\int_\Omega J_\sigma(x-y)(\phi(a,y)-\phi(a,x))dy + \mu(a,x)\phi + (\vartheta+\epsilon)\phi \\
&\geq -\frac{D}{\sigma^m}\int_\Omega J_\sigma(x-y)(\phi(a,y)-\phi(a,x))dy + \epsilon\phi \\
&\geq -\frac{D}{\sigma^m}\left[\int_\Omega \frac{1}{\sigma^N} J\left(\frac{x-y}{\sigma}\right)(\phi(a,y)-\phi(a,x))dy\right] + \epsilon\phi \\
&= -\frac{D}{\sigma^m}\int_{\frac{\Omega-x}{\sigma}} J(z)[\phi(a,x+\sigma z)-\phi(a,x)]dz + \epsilon\phi. \qquad (6.30)
\end{aligned}$$

For σ small enough, say $\sigma \leq \sigma_1$, we obtain $\mathrm{supp} J \subset \frac{\Omega - x}{\sigma}$ for all $x \in \Omega$. Thus by Taylor expansion and the symmetry of J, we obtain that

$$\begin{aligned}
&-\frac{D}{\sigma^m}\int_{\frac{\Omega-x}{\sigma}} J(z)[\phi(a,x+\sigma z)-\phi(a,x)]dz + \epsilon\phi \\
&= -\frac{D}{\sigma^m}\int_{\mathbb{R}^N} J(z)[\phi(a,x+\sigma z)-\phi(a,x)]dz + \epsilon\phi \\
&= -\frac{D}{\sigma^m}\int_{\mathbb{R}^N} J(z)\left[\partial_x\phi(a,x)(\sigma z) + \frac{1}{2}(\sigma z)^T\partial_x^2\phi(a,x)(\sigma z) + o(\sigma^2)\right]dz + \epsilon\phi \\
&= -\frac{D\sigma^{2-m}}{2}\int_{\mathbb{R}^N} J(z)z^T\partial_x^2\phi(a,x)zdz + o(\sigma^{2-m}) + \epsilon\phi. \qquad (6.31)
\end{aligned}$$

It follows that there exists $0 < \sigma_\epsilon < \sigma_1$ such that

$$\frac{\partial \phi(a,x)}{\partial a} - \frac{D}{\sigma^m}\int_\Omega J_\sigma(x-y)(\phi(a,y)-\phi(a,x))dy + \mu(a,x)\phi + (\vartheta+\epsilon)\phi \geq 0$$

for all $\sigma \leq \sigma_\epsilon$. Using the definition of $\lambda_1(\mathcal{A}_{\sigma,m})$, we have $\lambda_1(\mathcal{A}_{\sigma,m}) \leq \lambda_p'(\mathcal{A}_{\sigma,m}) \leq s(\mathcal{B}_1^0 + \mathcal{C}) + \epsilon$. Hence,

$$\limsup_{\sigma\to 0^+} \lambda_1(\mathcal{A}_{\sigma,m}) \leq s(\mathcal{B}_1^0 + \mathcal{C}) + \epsilon.$$

The arbitrariness of ϵ implies that

$$\limsup_{\sigma\to 0^+} \lambda_1(\mathcal{A}_{\sigma,m}) \leq s(\mathcal{B}_1^0 + \mathcal{C}).$$

Now we prove the inverse inequality

$$\liminf_{\sigma\to 0^+} \lambda_1(\mathcal{A}_{\sigma,m}) \geq s(\mathcal{B}_1^0 + \mathcal{C}), \tag{6.32}$$

where

$$\mathcal{A}_{\sigma,m}(0,\psi) = \Bigg(-\psi(0,x) + \int_0^{a^+} \beta(a,x)\psi(a,x)da,$$
$$-\frac{\partial\psi(a,x)}{\partial a} + \frac{D}{\sigma^m}\int_\Omega J_\sigma(x-y)(\psi(a,y)-\psi(a,x))dy - \mu(a,x)\psi(a,x)\Bigg).$$

For any $\epsilon > 0$, there exists an open ball B_ϵ of radius ϵ such that $\alpha(x) + \epsilon \geq s(\mathcal{B}_1^0 + \mathcal{C}) := \vartheta$ in $B_\epsilon \cap \Omega$, where $\alpha(x)$ is from Proposition 3.5 for $D = 0$ and $s(\mathcal{B}_1^0 + \mathcal{C})$ corresponds to the value α_1 in (6.16). Let $\tilde{\phi}_\epsilon \in C^{1,4}([0,a^+] \times \mathbb{R}^N)$ be nonnegative and satisfy

$$\tilde{\phi}_\epsilon = \phi \text{ in } [0,a^+] \times \overline{B}_\epsilon, \quad \tilde{\phi}_\epsilon = 0, \text{ in } [0,a^+] \times (\mathbb{R}^N \setminus B_{2\epsilon}) \text{ and } \sup_{[0,a^+]\times\mathbb{R}^N} \tilde{\phi}_\epsilon \leq \sup_{[0,a^+]\times\mathbb{R}^N} \phi = 1.$$

It is obvious that $\tilde{\phi}(a,\cdot) \in C^4(\mathbb{R}^N)$ for each $a \in [0,a^+]$. Let $\tilde{\lambda}_p(\mu, A^O_{\sigma,m})$ be the principal eigenvalue of the operator $A^O_{\sigma,m}$, where $A^O_{\sigma,m}$ is defined as follows:

$$A^O_{\sigma,m}(0,\psi) = \Bigg(-\psi(0,x) + \int_0^{a^+} \beta(a,x)\psi(a,x)da,$$
$$-\frac{\partial\psi(a,x)}{\partial a} + \frac{D}{\sigma^m}\int_O J_\sigma(x-y)\psi(a,y)dy - \frac{D}{\sigma^m}\psi(a,x) - \mu(a,x)\psi(a,x)\Bigg).$$

Note that it is an operator of Dirichlet type. Then we have for $(a,x) \in [0,a^+] \times \overline{B}_\epsilon$ that

$$-A^{B_\epsilon}_{\sigma,m}(0,\phi) + \left(\vartheta - \epsilon - \frac{1}{|\ln\epsilon|}\right)(0,\phi) := (I_3, I_4),$$

where

$$I_3 = \phi(0,x) - \int_0^{a^+} \beta(a,x)\phi(a,x)da = 0$$

and

$$\begin{aligned}
I_4 &= \frac{\partial \phi(a,x)}{\partial a} - \frac{D}{\sigma^m}\left[\int_{B_\epsilon} J_\sigma(x-y)\phi(a,y)dy - \phi(a,x)\right] \\
&\quad + \left[\mu(a,x) + \vartheta - \epsilon - \frac{1}{|\ln\epsilon|}\right]\phi(a,x) \\
&= -\frac{D}{\sigma^m}\left[\int_{B_\epsilon} J_\sigma(x-y)\phi(a,y)dy - \phi(a,x)\right] + \left[-\alpha(x) + \vartheta - \epsilon - \frac{1}{|\ln\epsilon|}\right]\phi(a,x) \\
&\le -\frac{D}{\sigma^m}\left[\int_{B_\epsilon} J_\sigma(x-y)\phi(a,y)dy - \phi(a,x)\right] - \frac{1}{|\ln\epsilon|}\phi(a,x) \\
&= -\frac{D}{\sigma^m}\left[\int_{\mathbb{R}^N} J_\sigma(x-y)\tilde\phi_\epsilon(a,y)dy - \tilde\phi_\epsilon(a,x) - \int_{B_{2\epsilon}\setminus B_\epsilon} J_\sigma(x-y)\tilde\phi_\epsilon(a,y)dy\right] \\
&\quad - \frac{1}{|\ln\epsilon|}\phi(a,x).
\end{aligned}$$

Now by the argument in Shen and Vo [40, Theorem D(2)] after choosing $\epsilon = \sigma^k$ with $k = \frac{m+2N}{N}$, we have for $0 < \sigma \ll 1$ that

$$-A^{B_{\sigma^k}}_{\sigma,m}(0,\phi) + \left(\vartheta - \sigma^k - \frac{1}{|\ln(\sigma^k)|}\right)(0,\phi) \le (0,0), \quad \text{in} \quad [0,a^+]\times \overline{B}_{\sigma^k}.$$

Then by Proposition 6.2, we have

$$\tilde\lambda'_p(\mu, A^{B_{\sigma^k}}_{\sigma,m}) = \tilde\lambda_p(\mu, A^{B_{\sigma^k}}_{\sigma,m}) \ge s(\mathcal{B}^0_1 + \mathcal{C}) - \sigma^k - \frac{1}{|\ln(\sigma^k)|}.$$

Proposition 5.6 (iii) in Kang and Ruan [24] yields that $\tilde\lambda_p(\mu, A^{\Omega}_{\sigma,m}) \ge \tilde\lambda_p(\mu, A^{B_{\sigma^k}}_{\sigma,m})$ and thus

$$\tilde\lambda_p(\mu, A^{\Omega}_{\sigma,m}) \ge s(\mathcal{B}^0_1 + \mathcal{C}) - \sigma^k - \frac{1}{|\ln(\sigma^k)|}. \tag{6.33}$$

Let $-\tilde\mu^\sigma(a,x) = -\mu(a,x) + \frac{D}{\sigma^m} - \frac{D}{\sigma^m}\int_{\frac{\Omega-x}{\sigma}} J(z)dz$. Obviously, for a sufficient small σ, one has $\int_{\frac{\Omega-x}{\sigma}} J(z)dz = 1$, which implies that

$$\lim_{\sigma\to 0}\left\|\tilde\mu^\sigma - \mu\right\|_{C([0,a^+],X)} = 0, \tag{6.34}$$

and we derive, by Proposition 5.6 (ii) in [24], that

$$|\tilde\lambda_p(\tilde\mu^\sigma, A^{\Omega}_{\sigma,m}) - \tilde\lambda_p(\mu, A^{\Omega}_{\sigma,m})| \le \left\|\tilde\mu^\sigma - \mu\right\|_{C([0,a^+],X)}, \tag{6.35}$$

where $\tilde{\lambda}_p(\tilde{\mu}^\sigma, A^\Omega_{\sigma,m})$, for σ small enough, is the principal eigenvalue of the operator defined by

$$A^\Omega_{\sigma,m}(0,\psi) = \Bigg(-\psi(0,x)+\int_0^{a^+}\beta(a,x)\psi(a,x)da,$$
$$-\frac{\partial\psi(a,x)}{\partial a} + \frac{D}{\sigma^m}\int_\Omega J_\sigma(x-y)(\psi(a,y)-\psi(a,x))dy - \mu(a,x)\psi(a,x)\Bigg).$$

Combining (6.33), (6.34) and (6.35), we take the limit as $\sigma \to 0^+$ and get the desired inequality

$$\liminf_{\sigma\to 0^+}\lambda_1(\mathcal{A}_{\sigma,m}) = \liminf_{\sigma\to 0^+}\tilde{\lambda}_p(\tilde{\mu}^\sigma, A^\Omega_{\sigma,m}) \geq s(\mathcal{B}^0_1+\mathcal{C}),$$

which proves (6.32). □

7 Strong maximum principle

In this section by using the sign of the principal spectrum point $\lambda_1(\mathcal{A})$ we establish the strong maximum principle for the operator $\mathcal{A}$ with L defined in (3.1) without scaling.

Definition 7.1 (Strong Maximum Principle) We say that $\mathcal{A}$ admits the strong maximum principle if for any function $(0,u)\in D(\mathcal{A})$ satisfying

$$\begin{cases}\mathcal{A}(0,u)\leq(0,0) & \text{in } [0,a^+]\times\Omega,\\ (0,u)\geq(0,0) & \text{in } [0,a^+]\times\partial\Omega,\end{cases} \tag{7.1}$$

there must hold $u>0$ in $[0,a^+]\times\Omega$ unless $u\equiv 0$ in $[0,a^+]\times\Omega$.

Theorem 7.2 Assume that $\lambda_1(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$, then $\mathcal{A}$ admits the strong maximum principle if and only if $\lambda_1(\mathcal{A})<0$.

Proof If $\lambda_1 := \lambda_1(\mathcal{A}) = s(\mathcal{A})$ is the principal eigenvalue of $\mathcal{A}$ associated with a positive eigenfunction $\phi\in\mathcal{X}_0^{++}\cap D(\mathcal{A})$, then

$$\mathcal{A}(0,\phi)-\lambda_1(0,\phi)=(0,0);$$

that is

$$\begin{cases}-\frac{\partial\phi}{\partial a}+D\int_\Omega J(x-y)(\phi(a,y)-\phi(a,x))dy-\mu(a,x)\phi-\lambda_1\phi=0,\\ \phi(0,x)-\int_0^{a^+}\beta(a,x)\phi(a,x)da=0.\end{cases} \tag{7.2}$$

For the sufficiency, that is $\lambda_1<0$ implies the strong maximum principle, let $(0,u)\in D(\mathcal{A})$ be nonzero and satisfy (7.1). Assume by contradiction that there exists

$(a_0, x_0) \in [0, a^+] \times \Omega$ such that $u(a_0, x_0) = \min_{[0,a^+]\times\Omega} u \leq 0$. Then consider the set

$$\Gamma := \{\epsilon \in \mathbb{R} : u + \epsilon\phi \geq 0 \text{ in } [0, a^+] \times \Omega\}.$$

Denote by $\epsilon_0 = \min \Gamma$ and $\psi = u + \epsilon_0\phi$. It is clear that $\epsilon_0 \geq 0$ by the assumption that $u(a_0, x_0) \leq 0$ and the fact that $\psi \geq 0$. Now if $\epsilon_0 > 0$, by simple computations, we have

$$\begin{cases} \frac{\partial\psi}{\partial a} - D\int_\Omega J(x-y)(\psi(a,y) - \psi(a,x))dy \\ \qquad +\mu(a,x)\psi \geq -\epsilon_0\lambda_1\phi > 0, & (a,x) \in (0,a^+) \times \overline{\Omega}, \\ \psi(0,x) \geq \int_0^{a^+} \beta(a,x)\psi(a,x)da, & x \in \overline{\Omega}. \end{cases} \tag{7.3}$$

That is,

$$\begin{cases} \frac{\partial\psi}{\partial a} > D\int_\Omega J(x-y)(\psi(a,y) - \psi(a,x))dy - \mu(a,x)\psi, & (a,x) \in (0,a^+) \times \overline{\Omega}, \\ \psi(0,x) \geq \int_0^{a^+} \beta(a,x)\psi(a,x)da, & x \in \overline{\Omega}. \end{cases} \tag{7.4}$$

It follows from the first inequality in (7.4) that $\psi(a,x) > \mathcal{U}(0,a)\psi(0,x) \geq 0$ for $(a,x) \in (0,a^+] \times \Omega$. Plugging it into the second inequality, we have $\psi(0,x) > 0$, which implies that ψ is strictly positive in $[0,a^+] \times \Omega$. This contradicts the fact that ϵ_0 is the infimum of Γ.

If $\epsilon_0 = 0$, it follows that $u(a_0, x_0) = 0$. Then if $a_0 > 0$, $\frac{\partial u(a_0,x_0)}{\partial a} \leq 0$, which implies that

$$0 \geq \frac{\partial u(a_0,x_0)}{\partial a} \geq D\int_\Omega J(x_0-y)(u(a_0,y) - u(a_0,x_0))dy - \mu(a_0,x_0)u(a_0,x_0) > 0. \tag{7.5}$$

This contradicts again that ϵ_0 is the infimum of Γ. If $a_0 = 0$, from the integral boundary condition, we have

$$\int_0^{a^+} \beta(a,x_0)u(a,x_0)da \leq u(0,x_0) = 0,$$

which by the positivity of β and $u \geq 0$ implies that $u(a,x_0) = 0$ for all $a \in [0,a^+]$. Then integrating (7.5) from 0 to a^+ at $x = x_0$, we still have the contradiction as above. Hence $u > 0$ in $[0,a^+] \times \Omega$, which concludes the desired result.

For the necessity, that is, strong maximum principle implies $\lambda_1 < 0$, the proof is similar to that of Vo [47, Theorem C] and is omitted here. □

8 Discussion

In this paper, we studied principal spectral theory and asynchronous exponential growth for age-structured models with nonlocal diffusion of Neumann type. First, we gave two sufficient conditions to guarantee existence of the principal eigenvalue of age-structured operators with nonlocal diffusion. Then such conditions were also used to show that the semigroup generated by the solutions to age-structured models with nonlocal diffusion is essentially compact and exhibits asynchronous exponential growth. We would like to mention that, to our best knowledge, it is the first time that explicit and easily verifiable sufficient conditions are given to guarantee that the semigroup exhibits asynchronous exponential growth, without additional compactness assumption and without proving the compactness of solution trajectories (which implies the eventual compactness of the semigroup), in particular, compared with the results in Thieme [44] in which it was assumed that the evolution family associated with spatial diffusion (for example Laplace diffusion) is compact. Moreover, such conditions are also valid for age-structured models with nonlocal diffusion of Dirichlet type to exhibit asynchronous exponential growth.

Next, by employing the generalized principal eigenvalue, we investigated the limiting properties of the principal eigenvalue with respect to the diffusion rate D and diffusion range σ. We improved some estimates in Vo [47] for a wide range of $m \in [0, 2)$ instead of $m = 0$. Finally, we established strong maximum principle for age-structured models with nonlocal diffusion of Neumann type. We would like to mention that we also used such principal spectral theory to investigate the global dynamics and asymptotic behavior of steady states with respect to diffusion rate and range for an age-structured model with nonlocal diffusion of Dirichlet type and nonlinearity on the birth rate or death rate in [24]. In fact, the global dynamics and asymptotic behavior of steady states with respect to diffusion rate and range of Neumann type are similar to those for Dirichlet type, thus we omit them here. The interested readers can refer to [24] for details.

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