Connectivity for Quantum Graphs

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Abstract

In Quantum Information Theory there is a construction for quantum channels, appropriately called a quantum graph, that generalizes the confusability graph construction for classical channels in classical information theory. In this paper a definition of connectedness for quantum graphs is provided, which generalizes the classical definition. It is shown that several examples of well-known quantum graphs (quantum Hamming cubes and quantum expanders) are connected. A quantum version of a particular case of the classical tree-packing theorem from Graph Theory is also proved. Generalizations for the related notions of k-connectedness and of orthogonal representation are also proposed for quantum graphs, and it is shown that orthogonal representations have the same implications for connectedness as they do in the classical case.

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1. Introduction

In classical zero-error information theory, one is interested in the accurate transmission and recovery of messages through a noisy channel. Typically these messages are transmitted one letter of the alphabet at a time and properties of the transmission needed to ensure an accurate reading of the message (such as repetition of a sent letter) are determined from the noise of the channel. To model this sort of scenario, we consider finite sets V and W that represent the input and output alphabets, respectively. A *classical channel*

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consists of choosing for each input $v \in V$ a probability distribution over W, specifying how v might be read after transmission through the channel; this represents the noise of the channel. The accuracy of a sent message boils down to how likely two different input letters might be transmitted and then received as the same output. Thus, a natural graph-theoretical construction that we can associate to a channel as above is the graph having elements in V as vertices and where $u, v \in V$ are connected by an edge if there is positive probability that u and v could be transmitted and received as the same output. This graph is called the *confusability graph* of the channel, and it is not hard to see that every (finite) graph (with all possible loops) can be realized as the confusability graph of some channel. In this way, there is a rich interplay between graph theory and information theory.

The purpose of this paper is to study the connectivity of the analogue of confusability graph that arises naturally from *quantum* information theory (see [6]). To better motivate the definition of a quantum channel, observe first that a classical channel as described in the previous paragraph is canonically associated with a linear map $\mathbb{R}^V \to \mathbb{R}^W$: For each $v \in V$, the vector having a 1 in the v-th position and zeroes everywhere else gets mapped to the probability density associated to v, and this map is then extended linearly. Observe that this linear map is positive (that is, it sends nonnegative vectors to nonnegative vectors) and moreover it maps probability densities to probability densities. In quantum information theory, the role of a probability density is played by a quantum state, that is, a positive semidefinite matrix with trace 1. A *quantum channel* is then represented by a linear map $\Phi\colon M_n \to M_m$ between spaces of matrices with complex entries, which is trace-preserving and *completely positive*; the latter term means that not only is the map Φ positive (i.e., it maps positive semidefinite matrices to positive semidefinite matrices), but also the same is true whenever we take the tensor product of Φ with the identity mapping on M_k for each $k \in \mathbb{N}$. By Choi's theorem ([3]), since $\Phi\colon M_n \to M_m$ is completely positive there exist matrices $K_1, K_2, \dots K_N \in M_{m,n}$ such that $\Phi(\rho) = \sum_{i=1}^N K_i \rho K_i^\dagger$ for all matrices $\rho \in M_n$, where the dagger denotes the conjugate transpose of a matrix. In the quantum setting, two transmitted states ρ and ψ are distinguishable from each other if their images are orthogonal, and this may be seen to be equivalent to the condition that $\rho A \psi = 0$ for all $A \in \operatorname{span}\{K_i^\dagger K_j\}_{1 \le i,j \le N}$ [6]. For this reason, and by analogy to the classical setting, $\operatorname{span}\{K_i^\dagger K_j\}_{1 \le i,j \le N}$ is called the *quantum confusability graph* associated to Φ. It is easy to see that a quantum confusability graph is an *operator system*, that is, a linear space of matrices with complex entries which is closed under taking adjoints and contains the identity matrix (since Φ is trace-preserving, $\sum_{i=1}^{N} K_i^\dagger K_i$ is the identity matrix), and in fact every operator system can be realized as the quantum confusability graph of some quantum channel [5, 4]. With the motivation given above, and despite several other strong contenders for the title, we follow [19] in using the terminology *quantum graph* rather than operator system to emphasize the graph-theoretical flavor of our investigations. Indeed, even without the connection to quantum information theory, there is already good justification for doing this ([10]).

It is our hope to expand the toolbox available to quantum information theorists by discovering the limits of what methods can be transferred from the well-understood classical graph theory setting into the quantum one; results of this nature have already appeared in works such as [6, 16, 14, 20, 11, 9, 21]. There are many important classical graph-theoretical concepts that deserve investigation, and if any of these have a good quantum analogue, it can be reasonably expected that they possess a utility similar to their classical counterparts. One of the most fundamental of these concepts is the notion of connectivity. We provide natural definitions of quantum connectedness and quantum connectivity for quantum graphs that generalize the classical ones, and explore what extensions/analogues of classical connectivity theorems hold true in the quantum setting.

2. Notation

We denote the space of all k by n matrices with complex entries by $M_{k,n}$, or by M_n if $k = n$. We let $X^\dagger$ denote the Hermitian adjoint of a matrix $X \in M_{k,n}$ and let $\|X\|$ denote the operator norm of X , so that $\|X\|^2$ is the largest eigenvalue of $X^\dagger X$. We equip M_n with the inner product given by $\langle X, Y\rangle = \mathrm{tr}(X^\dagger Y)$, where $\mathrm{tr}(Z)$ is the trace of a matrix $Z \in M_n$. We write I_n (or simply I) for the identity matrix in M_n. A projection is $P \in M_n$ such that $P = P^2 = P^\dagger$, and a nontrivial projection is a projection which is neither zero nor I_n. We use Dirac's bra-ket notation: $|u\rangle \in \mathbb{C}^n = M_{n,1}$ is a vector, $\langle u| = |u\rangle^\dagger \in M_{1,n}$ is its adjoint (a linear form), $\langle u|v\rangle$ is the standard Hilbert space inner product (linear in the second argument) of u and v, and $|v\rangle \langle u| \in M_n$ is the corresponding rank-one operator defined by $|v\rangle \langle u| (|w\rangle) = \langle u|w\rangle |v\rangle$. The list $(|e_k\rangle)_{k=1}^n$ will always denote the standard basis of $\mathbb{C}^n$. The cardinality of a set S is denoted by $|S|$. For $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, 2, \dots, n\}$.

By a *quantum graph* on M_n we mean an operator system: A linear subspace of M_n which is closed under taking adjoints and contains the identity matrix. To any classical graph G with vertex set $[n]$ we associate the quantum graph

$$\mathcal{S}_G = \operatorname{span}\left\{ |e_i\rangle \langle e_j| \mid i = j \text{ or } i \text{ is adjacent to } j \right\} \subseteq M_n.$$

Given two quantum graphs $\mathcal{U}, \mathcal{V} \subseteq M_n$, by their product we mean

$$\mathcal{U}\mathcal{V} = \operatorname{span}\left\{UV \mid U \in \mathcal{U}, V \in \mathcal{V}\right\},$$

and we define $\mathcal{U}^m$ for $m \in \mathbb{N} \cup \{0\}$ recursively by

$$\mathcal{U}^0 = \mathbb{C}I_n, \qquad \mathcal{U}^{k+1} = \mathcal{U}^k\mathcal{U}.$$

Note that the product of a quantum graph with itself is also a quantum graph, and more generally $\mathcal{U}^m$ is a quantum graph when $\mathcal{U}$ is a quantum graph.

To emphasize the distinction between quantum and non-quantum graphs, we use the adjective *classical* when we are talking about a combinatorial graph. We use the notation $i \sim j$ to indicate that two vertices i and j are adjacent in a classical graph.

3. Connectedness

In this section, we define what it means for a quantum graph to be "connected" and show some equivalences that highlight the similarity to classical connectedness, including a quantum analogue of the base case of the classical tree-packing theorem. In particular, we show that a classical graph is connected if and only if its associated quantum graph is connected. Philosophically, the "vertices" in a quantum graph correspond to rank one projections, and collections of vertices correspond to possibly higher rank projections. Because of this, the main obstacle for directly adapting a classical graph concept to quantum graph theory is that we should require such concepts to be coordinate-free. Indeed, if an orthonormal basis is fixed, there are natural classical graphs that can be associated to any quantum graph such that collections of vertices correspond to projections whose images align with the basis. We will show that for a connected quantum graph, any choice of orthonormal basis will give rise to a connected classical graph.

The following definition of connectedness is based on the intuition that in a connected graph, there is a path between any two vertices.

Definition 3.1. *A quantum graph $\mathcal{S} \subseteq M_n$ is* connected *if there exists $m \in \mathbb{N}$ such that $\mathcal{S}^m = M_n$. A quantum graph which is not connected will be called* disconnected*.*

Example 3.2 (The quantum hamming cube is connected)**.** *The quantum Hamming cube [10, Defn. 3.7] is the quantum graph*

$$\mathcal{C}_n = \operatorname{span}\left\{ \bigotimes_{i=1}^{n} A_i \mid A_i \in M_2, \textit{all but one of the } A_i \textit{ are equal to } I_2 \right\} \subseteq M_{2^n}.$$

Notice that $\mathcal{C}_n^n$ contains

$$\operatorname{span}\left\{ \bigotimes_{i=1}^{n} A_i \mid A_i \in M_2 \right\} = M_{2^n},$$

so $\mathcal{C}_n$ is connected.

Another intuitive condition that we could have used to motivate a definition of connectedness is that in a disconnected classical graph, there is always a partition of the set of vertices into two nonempty pieces such that the two pieces have no edge between them. The next theorem shows that the quantum analogue of this condition is equivalent to our definition of connectedness. We thank an anonymous referee for providing the simplified proof below.

Theorem 3.3. *Let $\mathcal{S} \subseteq M_n$ be a quantum graph. Then $\mathcal{S}$ is disconnected if and only if there exists a nontrivial projection $P \in M_n$ such that $P\mathcal{S}(I_n - P) = \{0\}$.*

Proof. Since $(\mathcal{S}^m)_{m=1}^{\infty}$ is an increasing sequence of subspaces of the finite-dimensional space M_n, there exists $m_0 \in \mathbb{N}$ such that $\bigcup_{m=1}^{\infty} \mathcal{S}^m = \mathcal{S}^{m_0}$, and observe that this is the C^*-subalgebra of M_n generated by $\mathcal{S}$. Note that the commutant

$$\mathcal{S}' = \left\{ B \in M_n \mid AB = BA \text{ for all } A \in \mathcal{S} \right\}$$

of $\mathcal{S}$ coincides with the commutant of $\mathcal{S}^{m_0}$, and therefore it follows from the von Neumann double commutant theorem that $\mathcal{S}$ is connected if and only of $\mathcal{S}' = \mathbb{C} \cdot I_n$.

Thus, if $\mathcal{S}$ is disconnected there exists a nontrivial projection $P \in \mathcal{S}'$. This implies that for each $A \in \mathcal{S}$ we have $PA = AP = AP^2 = PAP$, from where $P\mathcal{S}(I_n - P) = \{0\}$.

Conversely, suppose there exists a nontrivial projection $P \in M_n$ such that $P\mathcal{S}(I_n - P) = \{0\}$. Since $\mathcal{S}$ is closed under taking adjoints, it follows that $(I_n - P)\mathcal{S}P = \{0\}$. Thus for each $A \in \mathcal{S}$ we have $PA = PAP = AP$, so $P \in \mathcal{S}'$ and thus $\mathcal{S}$ is disconnected. □

As a consequence of Theorem 3.3, quantum connectedness generalizes classical connectedness.

Corollary 3.4. *Let G be a classical graph with vertex set $[n]$ and associated quantum graph $\mathcal{S}_G$. Then G is connected if and only if $\mathcal{S}_G$ is connected.*

Proof. Suppose G is connected. Then for each $i, j \in [n]$, there is a path $(p_k)_{k=1}^m$ in G such that $p_1 = i$, $p_m = j$, and $m \leq n$. But this means $|e_{p_k}\rangle \langle e_{p_{k+1}}| \in \mathcal{S}_G$ for all $1 \leq k \leq m-1$ and so $|e_i\rangle \langle e_j| = \prod_{k=1}^{m-1} |e_{p_k}\rangle \langle e_{p_{k+1}}| \in \mathcal{S}_G^m \subseteq \mathcal{S}_G^n$. As $\{|e_i\rangle \langle e_j|\}_{1 \leq i,j \leq n}$ forms a basis for M_n, this implies $\mathcal{S}_G^n = M_n$, and so $\mathcal{S}_G$ is connected.

On the other hand, suppose G is disconnected. Then $[n]$ can be partitioned into two nonempty sets K and L that are not connected to each other by any edge in G. This implies that $|e_i\rangle \langle e_j|$ and $|e_j\rangle \langle e_i|$ are orthogonal to $\mathcal{S}_G$ whenever $i \in K$ and $j \in L$. Thus, if $P = \sum_{j \in K} |e_j\rangle \langle e_j|$ is the orthogonal projection onto $\operatorname{span}\{e_j\}_{j \in K}$, then $P\mathcal{S}_G(I_n - P) = \{0\}$. And so, by Theorem 3.3, $\mathcal{S}_G$ is disconnected. □

Classical expander graphs are graphs which are sparse but have strong connectivity properties, and have found applications both in Computer Science and in Mathematics; see the survey [8]. Various authors, e.g. [2, 7, 1], have studied a quantum analog of an expander graph which is a quantum channel as defined in the introduction. As another consequence of Theorem 3.3 we will now show that the quantum graphs associated to these quantum expanders are connected.

Definition 3.5. *(a) Let $\Phi\colon M_n \to M_n$ be a completely positive and trace preserving (CPTP) unital map, i.e., satisfying $\Phi(I_n) = I_n$. For $0 < \varepsilon <$*

1, *we say that* Φ has an ε-spectral gap *if for all* $X \in M_n$ *we have*

$$\left\|\Phi(X) - \tfrac{1}{n}\operatorname{tr}(X)I\right\|_{\mathrm{HS}} \leq (1-\varepsilon)\left\|X - \tfrac{1}{n}\operatorname{tr}(X)I\right\|_{\mathrm{HS}},$$

where $\|\cdot\|_{\mathrm{HS}}$ *denotes the Hilbert-Schmidt norm.*

(b) *We say that a map* $\Phi\colon M_n \to M_n$ *is a* d-regular ε-quantum expander *if there exist unitaries* $U_1, \ldots, U_d \in M_n$ *such that* $\Phi(X) = \frac{1}{d}\sum_{j=1}^{d} U_j X U_j^\dagger$ *for each* $X \in M_n$*, and* Φ *has an* ε*-spectral gap.*

The following is just a restatement of [17, Lemma 20], which can be described as a quantum Cheeger inequality. Below, $\Phi^\dagger$ is the conjugate of Φ with respect to the Hilbert-Schmidt inner product.

Lemma 3.6. *Let* $\Phi\colon M_n \to M_n$ *be a CPTP unital map with an* ε*-spectral gap. Then* $h \geq (1-\varepsilon)/2$*, where*

$$h = \min_{0<\operatorname{rank}(P)\leq n/2} \frac{\operatorname{tr}\left((I-P)\Phi^\dagger\Phi(P)\right)}{\operatorname{tr}(P)}$$

and the minimum is taken over projections P*.*

Let us remark that the expression appearing in the numerator in the preceding Lemma can be nicely understood in terms of the inner product associated to the Hilbert-Schmidt norm, namely $\langle A, B\rangle = \operatorname{tr}(B^\dagger A)$. Indeed,

$$\begin{aligned}\operatorname{tr}\left((I-P)\Phi^\dagger\Phi(P)\right) &= \operatorname{tr}\left((I-P)^\dagger\Phi^\dagger\Phi(P)\right)\\ &= \langle \Phi^\dagger\Phi(P), I-P\rangle = \langle \Phi(P), \Phi(I-P)\rangle.\end{aligned}$$

Thus, Lemma 3.6 says that Φ is mapping orthogonal pairs $P, I-P$ to nonorthogonal pairs in a uniform way. We now show that the quantum graphs associated to quantum expanders are connected.

Proposition 3.7. *Let* $\Phi\colon M_n \to M_n$ *be a* d*-regular* ε*-quantum expander. Then its associated quantum graph is connected.*

Proof. Let $U_1, \ldots, U_d \in M_n$ be unitaries such that $\Phi(X) = \frac{1}{d}\sum_{j=1}^{d} U_j X U_j^\dagger$ for each $X \in M_n$. Suppose that the associated quantum graph $\mathcal{S} = \operatorname{span}\{U_j^\dagger U_i \mid 1 \leq i, j \leq d\}$ is disconnected. By Theorem 3.3, there exists a nontrivial projection $P \in M_n$ such that $P\mathcal{S}(I-P) = \{0\}$. Without loss of generality, we may assume $0 < \operatorname{rank}(P) \leq n/2$. From $P\mathcal{S}(I-P) = \{0\}$ we get $PU_j^\dagger U_i(I-P) = 0$ for all $1 \leq i,j \leq n$, and thus

$$\langle \Phi(P), \Phi(I-P) \rangle = \frac{1}{d^2} \sum_{i,j=1}^{d} \langle U_j P U_j^\dagger, U_i(I-P)U_i^\dagger \rangle$$

$$= \frac{1}{d^2} \sum_{i,j=1}^{d} \operatorname{tr}\big(U_i(I-P)U_i^\dagger U_j P U_j^\dagger\big) = \frac{1}{d^2} \sum_{i,j=1}^{d} \operatorname{tr}\big(U_i^\dagger U_j P U_j^\dagger U_i(I-P)\big) = 0,$$

contradicting Lemma 3.6. □

Observe that a different way of stating Theorem 3.3 is the following: A quantum graph $\mathcal{S} \subseteq M_n$ is connected if and only if whenever P_1, P_2 are nontrivial disjoint projections adding up to I_n, we have $\dim[P_1\mathcal{S}P_2] + \dim[P_2\mathcal{S}P_1] \geq 2$. This suggests a quantum version of the following particular case of the tree-packing theorem of Tutte [18] and Nash-Williams [13]: A classical graph contains a spanning tree (i.e., it is connected) if and only if every partition $\mathscr{P}$ of its vertex set has at least $|\mathscr{P}| - 1$ cross-edges (that is, edges joining two vertices that belong to different pieces of the partition).

Theorem 3.8. *A quantum graph $\mathcal{S} \subseteq M_n$ is connected if and only if*

$$\sum_{i \neq j} \dim\big[P_j \mathcal{S} P_i\big] \geq 2(m-1)$$

whenever $P_1, \ldots, P_m$ are nontrivial disjoint projections adding up to the identity.

Proof. Suppose first that $P \in M_n$ is a nontrivial projection such that

$$\dim\big[P\mathcal{S}(I_n - P)\big] + \dim\big[(I_n - P)\mathcal{S}P\big] \geq 2.$$

Then $\dim\big[P\mathcal{S}(I_n-P)\big] = \dim\big[(I_n-P)\mathcal{S}P\big] \geq 1$, because $\mathcal{S}$ is closed under taking adjoints, and so $P\mathcal{S}(I_n - P) \neq \{0\}$. By Theorem 3.3, $\mathcal{S}$ is connected.

Now assume that $\mathcal{S}$ is connected, and let $P_1, \ldots, P_m$ be nontrivial disjoint projections adding up to the identity. Define a classical graph G on $[m]$ via $i \sim j$ if and only if $P_i\mathcal{S}P_j \neq \{0\}$. We claim that G is a connected classical graph. Otherwise, we can partition $[m]$ into disjoint nonempty subsets A and B such that $i \in A$ and $j \in B$ implies $P_i\mathcal{S}P_j = \{0\}$. But this would imply

$$\Big(\sum_{i \in A} P_i\Big)\mathcal{S}\Big(\sum_{j \in B} P_j\Big) = \{0\},$$

contradicting the fact that $\mathcal{S}$ is connected, by Theorem 3.3. Since G is connected it must have at least $m-1$ edges, which implies $\sum_{i<j} \dim \left[P_i \mathcal{S} P_j\right] \geq m-1$. $\square$

For any classical graph G, there is a canonical quantum graph $\mathcal{S}_G$ associated to G. To go in the other direction and associate a classical graph to a given quantum graph, an orthonormal basis (o.n.b) for $\mathbb{C}^n$ must first be chosen. If $v = (|v_k\rangle)_{k=1}^n$ is an (ordered) o.n.b for $\mathbb{C}^n$, then one of the most natural classical graphs we can associate to a quantum graph $\mathcal{S}$ with respect to v is the graph $C_v(\mathcal{S})$ with vertex set $[n]$, where $i, j \in [n]$ are adjacent exactly when $\langle v_i| A |v_j\rangle \neq 0$ for some $A \in \mathcal{S}$. We call $C_v(\mathcal{S})$ the *confusability graph of* $\mathcal{S}$ *with respect to* v (note that our terminology does not agree with that of [9]). It is not hard to see that if v is the standard basis, then $C_v(\mathcal{S}_G) = G$ for any classical graph G, and it is this property that informs our choice of graph construction. We have already seen that quantum connectedness is a generalization of classical connectedness. Even so, the following proposition allows us to rephrase quantum connectedness in terms of classical connectedness.

Proposition 3.9. *Let $\mathcal{S} \subseteq M_n$ be a quantum graph. Then $\mathcal{S}$ is connected if and only if $C_v(\mathcal{S})$ is connected for every o.n.b. $v = (|v_k\rangle)_{k=1}^n$ of $\mathbb{C}^n$.*

Proof. Suppose $\mathcal{S}$ is disconnected, so that by Theorem 3.3 there exists a nontrivial projection $P \in M_n$ such that $P\mathcal{S}(I_n - P) = \{0\}$. Let $v = (|v_k\rangle)_{k=1}^n$ be an o.n.b. of $\mathbb{C}^n$ such that for some $1 \leq m < n$, $(|v_k\rangle)_{k=1}^m$ is an o.n.b. for the range of P, and hence $(|v_k\rangle)_{k=m+1}^n$ is an o.n.b for the range of $I_n - P$. For each $1 \leq i \leq m$, $m+1 \leq j \leq n$, and $A \in \mathcal{S}$ we then have $0 = |v_i\rangle \langle v_i| A |v_j\rangle \langle v_j|$, which implies $\langle v_i|A|v_j\rangle = 0$ and therefore $i \not\sim j$ in $C_v(\mathcal{S})$, showing that $C_v(\mathcal{S})$ is disconnected.

Suppose now that there exists $v = (|v_k\rangle)_{k=1}^n$ an o.n.b. of $\mathbb{C}^n$ such that $C_v(\mathcal{S})$ is disconnected. Let K, L partition $[n]$ into disjoint nonempty sets such that for all $i \in K$ and $j \in L$ we have $i \not\sim j$ in $C_v(\mathcal{S})$, that is, $\langle v_i|A|v_j\rangle = 0$ for all $A \in \mathcal{S}$. Set $P = \sum_{i \in K} |v_i\rangle \langle v_i|$, so that $I_n - P = \sum_{j \in L} |v_j\rangle \langle v_j|$ and thus, for every $A \in \mathcal{S}$,

$$PA(I_n - P) = \sum_{i \in K, j \in L} |v_i\rangle \langle v_i| A |v_i\rangle \langle v_i| = 0,$$

which implies that $\mathcal{S}$ is disconnected, by Theorem 3.3. $\square$

4. *k*-connectedness

In the previous section, we defined a notion of connectedness for quantum graphs that generalizes the notion of connectedness for classical graphs. In this section, we provide a measure for the amount of connectedness a quantum graph has by way of a quantum analogue of connectivity. In the classical case, the connectivity of a graph G is the number of vertices that would have to be removed from G to create a graph that either is disconnected or contains a single vertex. This idea can be mimicked for quantum graphs once one considers how to properly define a notion of creating a "subgraph" by "removal of vertices".

In other words, what is needed is a notion of *restriction*: Given a quantum graph and a "subset of vertices", we would like to define the "subgraph" obtained when we restrict our attention to the given subset. This has already been considered by Weaver in the more general setting of quantum relations [19, Sec. 3], and we adopt the same definition.

Concretely, given a quantum graph $\mathcal{S} \subseteq M_n$ and a projection $P \in M_n$, we consider $P\mathcal{S}P$ to be a subgraph of $\mathcal{S}$ restricted to $M_{\mathrm{rank}(P)} \cong PM_nP$ (it is easy to check that $P\mathcal{S}P$ is indeed a quantum graph). We are led to the following definitions.

Definition 4.1. *Let $\mathcal{S} \subseteq M_n$ be a quantum graph. A nontrivial projection $P \in M_n$ is called a* separator *of $\mathcal{S}$ if $(I_n - P)\mathcal{S}(I_n - P)$ is either disconnected (viewed as a subspace of $M_{n-\mathrm{rank}(P)}$) or 1-dimensional.*

Remark 4.2. *By Definition 3.1, a projection P such that* $\mathrm{rank}(P) < n-1$ *is a separator for a quantum graph $\mathcal{S} \subseteq M_n$ if and only if there is no $m \in \mathbb{N}$ such that*

$$\big((I_n - P)\mathcal{S}(I_n - P)\big)^m = (I_n - P)M_n(I_n - P).$$

Theorem 3.3 provides another characterization: A projection P such that $\mathrm{rank}(P) < n-1$ *is a separator if and only if there exist nontrivial projections Q_1 and Q_2, disjoint from each other and from P, such that $Q_1 + Q_2 = I_n - P$ and $Q_1\mathcal{S}Q_2 = \{0\}$.*

We will use whichever property of separator is most useful in what is to follow.

Definition 4.3. *Let $k \in \mathbb{N}$. A quantum graph $\mathcal{S} \subseteq M_n$ is called k-*connected *if every separator for $\mathcal{S}$ has rank at least k.*

In particular, a quantum graph $\mathcal{S}$ on M_n is connected if and only if either $\mathcal{S}$ is 1-connected or $\mathcal{S} = M_n$.

Let us now prove that this quantum notion of k-connectedness generalizes the classical one.

Proposition 4.4. *Let G be a classical graph with vertex set $[n]$ and associated quantum graph $\mathcal{S}_G$, and let $k \in \mathbb{N}$. Then G is k-connected if and only if $\mathcal{S}_G$ is k-connected.*

Proof. Suppose $\mathcal{S}_G$ is k-connected. If G is a complete graph, then G is k-connected. So suppose G is not a complete graph. Let $\{p_i\}_{i=1}^m \subseteq [n]$ be a vertex cut of G that induces a disconnected subgraph of G. Then $P = \sum_{i=1}^m |p_i\rangle \langle p_i|$ is a separator of rank m for $\mathcal{S}_G$. Thus $m \geq k$, which implies that G is k-connected.

Suppose now that G is k-connected. If every separator of $\mathcal{S}_G$ has rank at least $n-1$, then $\mathcal{S}_G$ is k-connected by definition. So suppose there is a separator $P \in M_n$ such that $\operatorname{rank}(P) < n-1$. Then there exist nontrivial disjoint projections $Q_1, Q_2 \in M_n$, also disjoint from P, such that $I_n = P + Q_1 + Q_2$ and $Q_1 \mathcal{S}_G Q_2 = \{0\}$. Let $(|v_i\rangle)_{i=1}^n$ be an orthonormal basis for $\mathbb{C}^n$ which consists of the union of some orthonormal bases for the ranges of P, Q_1, and Q_2. By permuting the indices if necessary, it follows from [9, Lemma 13] (which in turn is [15, Lemma 7.28]) that we can assume $\langle v_i | e_i \rangle \neq 0$ for each $1 \leq i \leq n$. Let K, L_1, L_2 be disjoint subsets of $[n]$ such that L_1, L_2 are nonempty and $K \cup L_1 \cup L_2 = [n]$, and such that

$$P = \sum_{i \in K} |v_i\rangle \langle v_i| , \quad Q_1 = \sum_{i \in L_1} |v_i\rangle \langle v_i| , \quad Q_2 = \sum_{i \in L_2} |v_i\rangle \langle v_i| .$$

Notice that if $k \sim l$ in G (i.e., $|e_k\rangle \langle e_l| \in \mathcal{S}_G$), then

$$0 = Q_1 |e_k\rangle \langle e_l| Q_2 = \sum_{i \in L_1, j \in L_2} |v_i\rangle \langle v_i | e_k\rangle \langle e_l | v_j\rangle \langle v_j| ,$$

and thus $\langle v_i | e_k\rangle \langle e_l | v_j\rangle = 0$ for each $i \in L_1$ and $j \in L_2$. But for $i \in L_1$ and $j \in L_2$ we have $\langle v_i | e_i\rangle \langle e_j | v_j\rangle \neq 0$, which means $i \not\sim j$ in G. Since G is k-connected, this implies $k \leq |K| = \operatorname{rank}(P)$, showing that $\mathcal{S}_G$ is k-connected. □

Just as in the case of connectedness (see Proposition 3.9), k-connectedness of a quantum graph is equivalent to the k-connectedness of all its confusability graphs.

Proposition 4.5. *Let $\mathcal{S} \subseteq M_n$ be a quantum graph, and $k \in \mathbb{N}$. Then $\mathcal{S}$ is k-connected if and only if $C_v(\mathcal{S})$ is k-connected for every o.n.b. $v = (|v_i\rangle)_{i=1}^n$ of $\mathbb{C}^n$.*

Proof. Suppose $\mathcal{S}$ is k-connected, and let $(|v_i\rangle)_{i=1}^n$ be an orthonormal basis for $\mathbb{C}^n$. If $C_v(\mathcal{S})$ is a complete graph, then $C_v(\mathcal{S})$ is k-connected. So suppose $C_v(\mathcal{S})$ is not a complete graph. Let K, L_1, L_2 be disjoint subsets of $[n]$ such that L_1, L_2 are nonempty and $K \cup L_1 \cup L_2 = [n]$, and such that there are no edges in $C_v(\mathcal{S})$ between L_1 and L_2. Notice that this means for each $i \in L_1$, $j \in L_2$ and $A \in \mathcal{S}$ we have $\langle v_i|A|v_j\rangle = 0$. Define

$$P = \sum_{i \in K} |v_i\rangle \langle v_i|, \quad Q_1 = \sum_{i \in L_1} |v_i\rangle \langle v_i|, \quad Q_2 = \sum_{i \in L_2} |v_i\rangle \langle v_i|.$$

Notice that for each $A \in \mathcal{S}$

$$Q_1 A Q_2 = \sum_{i \in L_1, j \in L_2} |v_i\rangle \langle v_i|A|v_j\rangle \langle v_j| = 0.$$

Therefore P is a separator for $\mathcal{S}$, so $k \leq \text{rank}(P) = |K|$, showing that $C_v(\mathcal{S})$ is k-connected.

Assume now that $C_v(\mathcal{S})$ is k-connected for every o.n.b. v of $\mathbb{C}^n$. If every separator of $\mathcal{S}_G$ has rank at least $n-1$, then $\mathcal{S}_G$ is k-connected by definition. So suppose there is a separator $P \in M_n$ such that $\text{rank}(P) < n-1$. Then there exist nontrivial disjoint projections $Q_1, Q_2 \in M_n$, also disjoint from P, such that $I_n = P + Q_1 + Q_2$ and $Q_1 \mathcal{S} Q_2 = \{0\}$. Let $v = (|v_i\rangle)_{i=1}^n$ be an orthonormal basis for $\mathbb{C}^n$ which consists of the union of some orthonormal bases for the ranges of P, Q_1 and Q_2, and let K, L_1, L_2 be disjoint subsets of $[n]$ such that L_1, L_2 are nonempty and $K \cup L_1 \cup L_2 = [n]$, and such that

$$P = \sum_{i \in K} |v_i\rangle \langle v_i|, \quad Q_1 = \sum_{i \in L_1} |v_i\rangle \langle v_i|, \quad Q_2 = \sum_{i \in L_2} |v_i\rangle \langle v_i|.$$

Notice that for each $A \in \mathcal{S}$ we have

$$0 = Q_1 A Q_2 = \sum_{i \in L_1, j \in L_2} |v_i\rangle \langle v_i|A|v_j\rangle \langle v_j|,$$

which implies that for each $i \in L_1$, $j \in L_2$ and $A \in \mathcal{S}$ we have $\langle v_i|A|v_j\rangle = 0$. But this means that there are no edges in $C_v(\mathcal{S})$ between L_1 and L_2, so by the k-connectivity of $C_v(\mathcal{S})$ we conclude $k \leq |K| = \text{rank}(P)$ and therefore $\mathcal{S}$ is k-connected. □

In the classical setting, a graph on n vertices is $(n-1)$-connected if and only if it is complete. In the quantum setting this is no longer true, but we can still characterize the maximally connected quantum graphs.

Proposition 4.6. *Let $\mathcal{S} \subseteq M_n$ be a quantum graph. Then $\mathcal{S}$ is $(n-1)$-connected if and only if $A\mathcal{S}B \neq \{0\}$ for every $A, B \in M_n \setminus \{0\}$.*

Proof. Suppose that $A\mathcal{S}B \neq \{0\}$ for every $A, B \in M_n \setminus \{0\}$. It follows from Remark 4.2 that $\mathcal{S}$ does not admit a separator of rank strictly smaller than $n-1$, and therefore $\mathcal{S}$ is $(n-1)$-connected. Suppose, on the contrary, that there exist $A, B \in M_n \setminus \{0\}$ such that $A\mathcal{S}B = \{0\}$. Let v be a unit vector in the range of B and u a unit vector in the range of $A^\dagger$. Then $I_n - |u\rangle\langle u| - |v\rangle\langle v|$ is a separator for $\mathcal{S}$ with rank $n-2$, and therefore $\mathcal{S}$ is not $(n-1)$-connected. $\square$

It is not difficult to produce examples of quantum graphs contained in M_n satisfying the condition in the previous Proposition without being all of M_n. An example for $n = 2$ is provided at the beginning of [19, Sec. 4], and more generally one can consider

$$\operatorname{span}\left\{I_n, |e_i\rangle\langle e_j| \mid 1 \leq i, j \leq n, i \neq j\right\} \subsetneq M_n,$$

which plays an important role, for instance, in [11].

5. Orthogonal representations

With a definition of k-connectedness that generalizes the classical notion, the next order of business is to find sufficient conditions for a quantum graph to be k-connected. One motivation for such a condition comes from the classical realm in the form of orthogonal representations of graphs (see [12]). Recall that for a classical graph $G = (V, E)$, an *orthogonal representation* is an assignment $f\colon V \to \mathbb{R}^d$ or $f\colon V \to \mathbb{C}^d$ such that for every $i, j \in V$ with $i \neq j$,

$$i \not\sim j \quad \Rightarrow \quad f(i) \perp f(j).$$

An orthogonal representation f of $G = (V, E)$ is said to be in *general position* if for any $U \subseteq V$ such that $|U| = d$, the vectors in $\{f(i)\}_{i \in U}$ are linearly independent. A weaker condition is to require only that the vectors representing the vertices nonadjacent to any fixed vertex are linearly independent. For brevity, we will say that such a representation is in *locally general position.*

The relationship between these notions and connectivity is given by Theorem 1.1' in [12]:

Theorem 5.1. *If G is a classical graph with n vertices, then the following are equivalent:*

(a) G is $(n-d)$-connected.

(b) G has an orthogonal representation in $\mathbb{R}^d$ in general position.

(c) G has an orthogonal representation in $\mathbb{R}^d$ in locally general position.

Our desire is to find a condition such as (b) or (c) in the above theorem that will imply some amount of connectivity for a quantum graph. We start by considering what it should mean for a quantum graph to be "orthogonally represented", motivated by orthogonality-preserving notions such as the concept of order zero maps. Recall the following definition.

Definition 5.2. *Let A, B be C^*-algebras.*

(a) Two elements $a, b \in A$ are called orthogonal*, denoted $a \perp b$, if $0 = ab = ba = a^*b = ab^*$ (where a^*, b^* are the C^*-algebraic adjoints of a, b in A, respectively). Note that if a and b are self adjoint, this reduces simply to $ab = 0$.*

(b) A completely positive map $\phi\colon A \to B$ is said to be order zero *if $\phi(a) \perp \phi(b)$ whenever $a \perp b$.*

Order zero maps are known to have a nice structure, see [23, Thm. 1.2] and [22, Thm. 2.3]. Note that whenever P, Q are projections in M_n, the condition that $P \perp Q$ is equivalent to the condition that $P\mathcal{S}Q = Q\mathcal{S}P = \{0\}$, where $\mathcal{S}$ is the quantum graph $\mathbb{C} \cdot I_n$. This can intuitively be read as stating that P and Q are not connected to each other by any "edge" in the quantum graph $\mathbb{C} \cdot I_n$, because of what happens in the classical case: If G is a classical graph with vertex set $[n]$, and we consider subsets $K, L \subseteq [n]$ and the corresponding projections $P = \sum_{i \in K} |e_i\rangle \langle e_i|$, $\quad Q = \sum_{i \in L} |e_i\rangle \langle e_i|$, then $P\mathcal{S}_G Q = \{0\}$ if and only if K and L are disjoint and not connected by any edge in G. We are led by analogy to the following definition.

Definition 5.3. *Let $\mathcal{S} \subseteq M_n$ be a quantum graph. A completely positive map $\phi\colon M_n \to M_d$ is said to be an* orthogonal representation *of $\mathcal{S}$ if $\phi(P) \perp \phi(Q)$ for any projections $P, Q \in M_n$ such that $P\mathcal{S}Q = \{0\}$.*

Note that if $\mathcal{S} \subseteq M_n$ is a quantum graph, then the identity map $\mathrm{Id}_n\colon M_n \to M_n$ is trivially an orthogonal representation of $\mathcal{S}$. Definition 5.3 is justified by the following two propositions.

Proposition 5.4. *Let $G = (V, E)$ be a classical graph with n vertices and let $f\colon V \to \mathbb{C}^d$ be an orthogonal representation of G. Then the completely positive map $\phi\colon M_n \to M_d$ defined by*

$$\phi(X) = \sum_i |f(i)\rangle \langle e_i| X |e_i\rangle \langle f(i)| \qquad \text{for all } X \in M_n$$

is an orthogonal representation of $\mathcal{S}_G$.

Proof. Let $P, Q \in M_n$ be projections such that $P\mathcal{S}Q = \{0\}$. By definition

$$\phi(P)\phi(Q) = \sum_{i,j} |f(i)\rangle \langle e_i| P |e_i\rangle \langle f(i)|f(j)\rangle \langle e_j| Q |e_j\rangle \langle f(j)|,$$

and since $\langle f(i)|f(j)\rangle = 0$ whenever $i \neq j$ and $i \not\sim j$, this reduces to

$$\begin{aligned}\phi(P)\phi(Q) &= \sum_{\substack{i,j \\ i=j \text{ or } i\sim j}} |f(i)\rangle \langle e_i| P |e_i\rangle \langle f(i)|f(j)\rangle \langle e_j| Q |e_j\rangle \langle f(j)| \\ &= \sum_{\substack{i,j \\ i=j \text{ or } i\sim j}} \langle f(i)|f(j)\rangle |f(i)\rangle \langle e_i| P |e_i\rangle \langle e_j| Q |e_j\rangle \langle f(j)|.\end{aligned}$$

But when $i = j$ or $i \sim j$, we have $P|e_i\rangle\langle e_j|Q = 0$, and therefore $\phi(P)\phi(Q) = 0$. From $P = P^\dagger$ and $Q = Q^\dagger$, taking adjoints in the definition of ϕ immediately yields $\phi(P)^\dagger = \phi(P)$ and $\phi(Q)^\dagger = \phi(Q)$, so $\phi(P) \perp \phi(Q)$. □

Proposition 5.5. *Let $G = (V, E)$ be a classical graph with n vertices and let $\phi\colon M_n \to M_d$ be an orthogonal representation of $\mathcal{S}_G$. For each $i \in [n]$, let v_i be a vector in the range of $\phi(|e_i\rangle\langle e_i|)$. Then the map $f\colon V \to \mathbb{C}^d$ defined by $f(i) = v_i$ is an orthogonal representation of G.*

Proof. Pick any $i, j \in V$ with $i \neq j$ and $i \not\sim j$. Since $\mathcal{S}_G = \mathrm{span}\{|e_k\rangle\langle e_\ell| \mid k \sim \ell \text{ or } k = \ell\}$, note that $|e_i\rangle\langle e_i| \mathcal{S}_G |e_j\rangle\langle e_j| = \{0\}$. As ϕ is an orthogonal representation, this implies $\phi(|e_i\rangle\langle e_i|) \perp \phi(|e_j\rangle\langle e_j|)$ in the C^*-algebra M_d. But then Definition 5.3 implies $f(i) \perp f(j)$ in the Hilbert space $\mathbb{C}^d$, by the way f was defined. Therefore f is an orthogonal representation of G. □

Remark 5.6. *The same proof as above also shows that an orthogonal representation of a quantum graph $\mathcal{S} \subseteq M_n$ induces a natural complex-valued orthogonal representation of $C_v(\mathcal{S})$ for each orthonormal basis v of $\mathbb{C}^n$.*

Orthogonal representations of quantum graphs are already present in the quantum information literature. In fact, essentially the same proof as that of Proposition 5.4 shows that if $\phi\colon M_n \to M_d$ is a quantum channel (i.e., a completely positive and trace-preserving map), then ϕ is an orthogonal representation for its associated quantum confusability graph $\mathcal{S}_\phi$. More generally, the notions of quantum (sub-)complexity of a quantum graph $\mathcal{S} \subseteq M_n$ from [11] involve considering completely positive and trace-preserving maps $\psi\colon M_n \to M_d$ whose associated quantum confusability graphs $\mathcal{S}_\psi$ are contained in $\mathcal{S}$, which by the above means that such ψ are orthogonal representations for $\mathcal{S}$.

We have already observed that projections are analogues to collections of vertices when viewing quantum graphs as analogues of classical graphs. As was the case for connectedness, this viewpoint leads to a potential candidate for a quantum definition of what it means for an orthogonal representation to be in locally general position.

Suppose Q is a rank one projection in M_n. If viewed as a "quantum vertex" of some quantum graph $\mathcal{S} \subseteq M_n$, then we also view another projection P as a "collection of vertices nonadjacent to Q" if $P\mathcal{S}Q = \{0\}$. And in this case, $\operatorname{rank}(P)$ is viewed as the "number of vertices in P". By analogy to the classical definition, an orthogonal representation ϕ for $\mathcal{S}$ should preserve the rank of P if it is to be viewed as being in "locally general position".

Definition 5.7. *Let $\mathcal{S} \subseteq M_n$ be a quantum graph, and $\phi\colon M_n \to M_d$ an orthogonal representation of $\mathcal{S}$. We say that ϕ is in* locally general position *if, for any fixed nonzero projection $Q \in M_n$, $\operatorname{rank}(\phi(P)) \geq \operatorname{rank}(P)$ whenever $P \in M_n$ is a projection such that $Q\mathcal{S}P = \{0\}$.*

Observe that in the definition above, it suffices to check the inequality for all rank one projections Q. Our definition is justified by the following proposition.

Proposition 5.8. *Let $G = (V, E)$ be a classical graph with n vertices and let $f\colon V \to \mathbb{C}^d$ be an orthogonal representation of G in locally general position. Let $\phi\colon M_n \to M_d$ be the associated quantum orthogonal representation of $\mathcal{S}_G$,* *i.e., the mapping $\phi \colon M_n \to M_d$ given by*

$$\phi(X) = \sum_i |f(i)\rangle \langle e_i| X |e_i\rangle \langle f(i)| \quad \textit{for all } X \in M_n.$$

Then ϕ is in locally general position.

Proof. Fix a rank one projection $Q \in M_n$ and suppose $P \in M_n$ is a projection such that $Q\mathcal{S}_G P = \{0\}$. Let $|u\rangle \in \mathbb{C}^n$ be a unit vector such that $Q = |u\rangle\langle u|$, and observe that $|u\rangle$ is orthogonal to the range of P. Let $v = \{|v_j\rangle\}_{j=1}^n$ be an orthonormal basis of $\mathbb{C}^n$ aligned with P. By permuting the basis if necessary, we can assume that $\langle e_i|v_i\rangle \neq 0$ for each $1 \leq i \leq n$ [9, Lemma 13], [15, Lemma 7.28]. Let $i_0 \in [n]$ be such that $\langle u|e_{i_0}\rangle \neq 0$.

Let $J \subseteq [n]$ be such that $P = \sum_{j\in J} |v_j\rangle\langle v_j|$, noting that $|J| = \operatorname{rank}(P)$, and for each $j \in J$, $\langle e_j| P |e_j\rangle \neq 0$, so that in particular $\langle e_j| P \neq 0$. Observe that for each $j \in J$, we must have $i_0 \not\sim j$ and $i_0 \neq j$, since otherwise we would have

$$Q |e_{i_0}\rangle \langle e_j| P = \langle u|e_{i_0}\rangle |u\rangle \langle e_j| P \neq 0,$$

a contradiction.

Now, for any vector $|x\rangle \in \mathbb{C}^d$,

$$\begin{aligned}
\phi(P)|x\rangle = 0 &\Rightarrow \langle x| \phi(P) |x\rangle = 0 \\
&\Rightarrow \sum_i \langle x|f(i)\rangle \langle e_i| P |e_i\rangle \langle f(i)|x\rangle = 0 \\
&\Rightarrow \sum_i |\langle x|f(i)\rangle|^2 \langle e_i| P |e_i\rangle = 0 \\
&\Rightarrow \langle x|f(i)\rangle \cdot \langle e_i| P |e_i\rangle = 0 \text{ for every } i \in [n] \\
&\Rightarrow \langle x|f(j)\rangle = 0 \text{ for every } j \in J.
\end{aligned}$$

That is,

$$\ker\big(\phi(P)\big) \subseteq \big(\operatorname{span}\{f(j) \mid j \in J\}\big)^{\perp}.$$

Because f is in locally general position, and the indices in J correspond to vertices in G which are not adjacent to i_0, the dimension of $\operatorname{span}\{f(j) \mid j \in J\}$ is exactly $|J|$ and $|J| \leq d$. Therefore

$$\dim\Big(\ker\big(\phi(P)\big)\Big) \leq d - \operatorname{rank}(P),$$

from where

$$\operatorname{rank}\big(\phi(P)\big) \geq \operatorname{rank}(P).$$

That is, ϕ is in locally general position. □

Finally, we arrive at the main result of this section, which shows that some amount of connectivity of a quantum graph can be inferred from the existence of an orthogonal representation in locally general position, in analogy to the classical result.

Proposition 5.9. *Let $\mathcal{S} \subseteq M_n$ be a quantum graph, and suppose there exists an orthogonal representation $\phi\colon M_n \to M_d$ of $\mathcal{S}$ in locally general position. Then $\mathcal{S}$ is $(n-d)$-connected.*

Proof. If every separator of $\mathcal{S}$ has rank greater than or equal to $n-1$, then $\mathcal{S}$ is $(n-1)$-connected and so also $(n-d)$-connected. So suppose there is a separator P for $\mathcal{S}$ such that $\mathrm{rank}(P) < n-1$, so that there exist nontrivial disjoint projections Q_1, Q_2, also disjoint from P, such that $I_n = P + Q_1 + Q_2$ and $Q_1 \mathcal{S} Q_2 = \{0\}$. Since ϕ is an orthogonal representation of $\mathcal{S}$, $\phi(Q_1) \perp \phi(Q_2)$ in the C^*-algebra M_d, and thus

$$d \geq \mathrm{rank}(\phi(Q_1)) + \mathrm{rank}(\phi(Q_2)).$$

And since ϕ is in locally general position, it follows that $\mathrm{rank}(\phi(Q_j)) \geq \mathrm{rank}(Q_j)$, so

$$d \geq \mathrm{rank}(Q_1) + \mathrm{rank}(Q_2) = n - \mathrm{rank}(P),$$

and so $\mathrm{rank}(P) \geq n-d$. Therefore $\mathcal{S}$ is $(n-d)$-connected. □

It would be really interesting to know whether the opposite implication holds, that is, whether a certain amount of connectivity implies the existence of an orthogonal representation in locally general position of the appropriate size. We point out that this does hold in the case of maximal connectivity: If $\mathcal{S} \subseteq M_n$ is $(n-1)$-connected, it follows from Proposition 4.6 that the trace $\mathrm{tr}\colon M_n \to M_1 = \mathbb{C}$ is an orthogonal representation in locally general position for $\mathcal{S}$ (because the required conditions are vacuously satisfied).

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