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RELAXATION OSCILLATIONS AND THE ENTRY-EXIT FUNCTION IN MULTIDIMENSIONAL SLOW-FAST SYSTEMS*

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Abstract. For a slow-fast system of the form $\dot{p} = \epsilon f(p, z, \epsilon) + h(p, z, \epsilon)$, $\dot{z} = g(p, z, \epsilon)$ for $(p, z) \in \mathbb{R}^n \times \mathbb{R}^m$, we consider the scenario that the system has invariant sets $M_i = \{(p, z) : z = z_i\}$, $1 \leq i \leq N$, linked by a singular closed orbit formed by trajectories of the limiting slow and fast systems. Assuming that the stability of M_i changes along the slow trajectories at certain turning points, we derive criteria for the existence and stability of relaxation oscillations for the slow-fast system. Our approach is based on a generalization of the entry-exit relation to systems with multidimensional fast variables. We then apply our criteria to several predator-prey systems with rapid ecological evolutionary dynamics to show the existence of relaxation oscillations in these models.

Key words. slow-fast system, relaxation oscillation, entry-exit function, delay of stability loss, turning point, geometric singular perturbation theory

AMS subject classifications. 34C26, 92D25

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1. Introduction.

We consider a system of ordinary differential equations for $(p, z) \in \mathbb{R}^n \times \mathbb{R}^m$ of the form

$$\begin{aligned} \dot{p} &= \epsilon f(p, z, \epsilon) + h(p, z, \epsilon), \\ \dot{z} &= g(p, z, \epsilon), \end{aligned} \tag{1.1}$$

where $\cdot$ denotes $\frac{d}{dt}$, the functions f, g, and h are smooth, and $\epsilon > 0$ is a parameter. When h is identically zero, the system reduces to the standard slow-fast systems in Fenichel [18]. Note that slow-fast systems in nonstandard forms can be locally converted to the standard form near normally hyperbolic critical manifolds (see, e.g., Wechselberger [60, Lemma 3.6]). This more general setting of singularly perturbed problems provides different global return mechanisms which induce different types of relaxation-type behavior not observed in the standard setting.

In the scenario that g and h both vanish on some level sets $M_i = \{(p, z) : z = z_i\}$ for $\epsilon \in [0, \epsilon_0]$, $i = 1, 2, \ldots, N$, where $z_i \in \mathbb{R}^m$ and $\epsilon_0 > 0$ are constants, each M_i is invariant under (1.1) since $\dot{z} = 0$. System (1.1) restricted on M_i becomes

$$p' = f(p, z_i, \epsilon), \quad z = z_i, \tag{1.2}$$

where $'$ denotes $\frac{d}{d\tau}$ with $\tau = \epsilon t$. Hence system (1.1) has two *distinguished limits*: the *limiting fast system*

$$\dot{p} = h(p, z, 0), \quad \dot{z} = g(p, z, 0), \tag{1.3}$$

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obtained by setting $\epsilon = 0$ in system (1.1), and the *limiting slow system*

$$p' = f(p, z_i, 0), \quad z = z_i, \tag{1.4}$$

obtained by setting $\epsilon = 0$ in (1.2). When there are trajectories γ_i of (1.3) and trajectories $\sigma_i \subset M_i$ of (1.4) such that

$$\gamma_1 \cup \sigma_1 \cup \gamma_2 \cup \sigma_2 \cup \cdots \cup \gamma_N \cup \sigma_N \tag{1.5}$$

forms a closed configuration, in the spirit of *geometric singular perturbation theory* (GSPT) (see, e.g., Fenichel [18], Jones [32] and Kuehn [38]), there is potentially a periodic orbit of (1.1) near configuration (1.5) for all small $\epsilon > 0$. However, in the case that σ_i contains *turning points*, at which the stability of M_i changes, the so-called *entry-exit function* is needed to determine whether there are trajectories of (1.1) near the singular orbit. The classical entry-exit function was defined for system (1.1) with p being a one-dimensional variable (see de Maesschalck [13], de Maesschalck and Schecter [15], Hsu [26], Wang and Zhang [59] and references therein). The entry-exit function can be traced back to Benoît [4] and is called the *way-in way-out function* in Diener [16]. This phenomenon, that the landing and jumping points satisfy the entry-exit function, has been called *bifurcation delay* in Benoît [5], *Pontryagin delay* in Mishchenko et al. [46], and *delay of instability* in Liu [43]. In the present paper we generalize the entry-exit function (see section 2.2) for system (1.1) with a multidimensional variable p. Using our generalized entry-exit function, we provide criteria under which periodic orbits near the singular orbit exist. Note that if such periodic orbits exist, they must form a *relaxation oscillation* because for a trajectory of (1.1) to travel along the vicinity of γ_i (where h and g are nonvanishing) and σ_i (where $|h| \ll \epsilon$ and $|g| \ll \epsilon$), respectively, the time lengths need to be of orders $O(1)$ and $O(1/\epsilon)$.

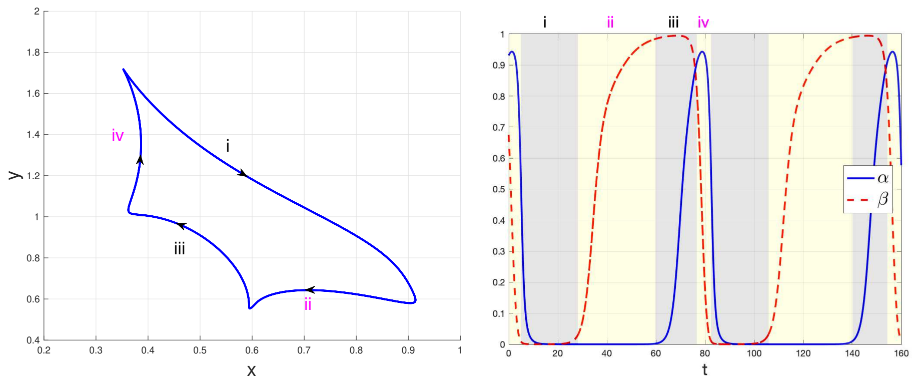
Our motivation is to understand the mechanism of *rapid regime shifts* in ecological systems.

Example 1. One example is trait oscillations exhibited in an eco-evolutionary system proposed by Cortez and Weitz [12]. The system takes the following form:

$$\begin{aligned} x' &= F(x, \alpha) - G(x, y, \alpha, \beta), \\ y' &= H(x, y, \alpha, \beta) - D(y, \beta), \\ \epsilon\,\alpha' &= \alpha(1-\alpha)\frac{\partial}{\partial \alpha}\left(\frac{x'}{x}\right), \\ \epsilon\,\beta' &= \beta(1-\beta)\frac{\partial}{\partial \beta}\left(\frac{y'}{y}\right), \end{aligned} \tag{1.6}$$

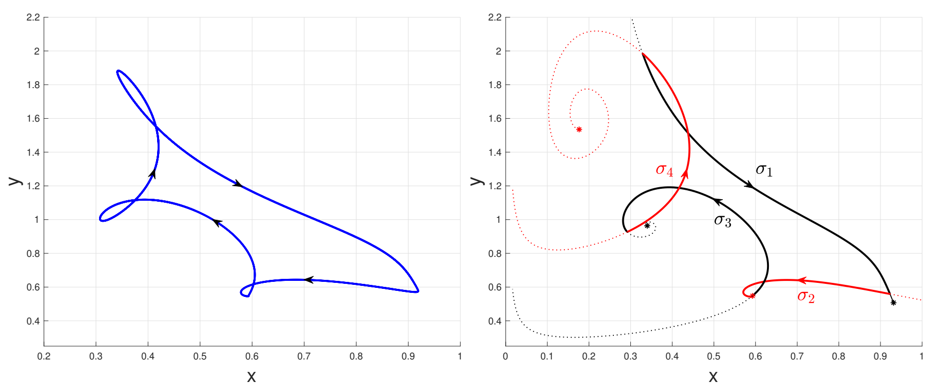
where $x(t)$ and $y(t)$ are the prey and predator densities, respectively, and $\alpha(t)$ and $\beta(t)$ are the average trait values of the prey and predators, respectively, at time t. The functions F and H are related to the growth rates of the prey and predators, respectively, G is related to the encounter rate, and D is related to the death rate of predators. The equations of α and β were derived from the assumption that the adaptive change in the trait follows fitness-gradient dynamics (see Abrams, Matsuda, and Harada [1]), i.e., the rate of change of the mean trait value is proportional to the fitness gradient of an individual with this mean trait value. In Cortez and Weitz [12], numerical evidences of periodic orbits oscillating between the level sets, for $(\alpha, \beta) = (0,0), (0,1), (1,1)$ and $(1,0)$, were provided for certain functional responses.

A simulation of a periodic orbit with data from that paper is shown in Figure 1. Note that system (1.6) is an example of system (1.1) with $h = 0$ and the variables (x, y) and (α, β) playing the roles of p and z, respectively. Applying one of our criteria (Theorem 2.6) in section 4.3, besides confirming the existence of periodic orbits, we will determine the limiting configuration (see Figure 2) of the periodic orbits as $\epsilon \to 0$.



(a) (b)

FIG. 1. *A periodic orbit for system* (1.6) *with* $\epsilon = 0.25$. (a) *On the* (x, y)*-plane the trajectory can roughly be split into four segments.* (b) *The value of* α *remains close to* 0 *along segments* i *and* ii *and becomes close to* 1 *in segments* iii *and* iv. *The value of* β *is close to* 0 *in segments* i *and* iv *and is close to* 1 *in segments* ii *and* iii.



(a) (b)

FIG. 2. (a) *A periodic orbit for system* (1.6) *with* $\epsilon = 0.10$. (b) *A singular closed orbit which consists of trajectories of limiting subsystems.*

Example 2. Another example, proposed by Cortez and Ellner [10], is a predator-prey system with rapid prey evolution:

$$
\begin{aligned}
x' &= x(\alpha + r - kx) - \frac{xy(a\alpha^2 + b\alpha + c)}{1 + x}, \\
y' &= \frac{xy(a\alpha^2 + b\alpha + c)}{1 + x} - dy, \qquad (1.7) \\
\epsilon\,\alpha' &= \alpha(1 - \alpha)\left(1 - \frac{y(2a\alpha + b)}{1 + x}\right) \equiv \alpha(1 - \alpha)E(x, y, \alpha),
\end{aligned}
$$

which can be regarded as a special case of (1.6) with β being constant. Periodic orbits that travel back and forth between the manifolds M_0 and M_1 corresponding to $\alpha = 0$ and $\alpha = 1$, respectively, were discovered numerically by Cortez and Ellner [10] (see Figure 3 for a simulation with data from that paper). Note that the sign of $E(x, y, \alpha)$, where $\alpha = 0$ (resp., $\alpha = 1$), determines whether M_0 (resp., M_1) is attracting or repelling at that point. It was indicated in [10] that if the trait oscillation occurs, at the landing and jumping points on each M_i the value of E has opposite signs. Note that system (1.7) is an example of system (1.1) with $h = 0$ and the variables (x, y) and α playing the roles of p and z, respectively. In section 4.1, applying our criterion (Theorem 2.5) we will determine two pairs of landing and jumping points, $A_1, B_1 \in M_0$ and $A_2, B_2 \in M_1$, by the equations

$$\int_{\sigma_1} E(x, y, 0)\, dt = \int_{\sigma_2} E(x, y, 1)\, dt = 0, \tag{1.8}$$

where σ_1 is a trajectory on M_0 connecting A_1 and B_1, and σ_2 is a trajectory on M_1 connecting A_2 and B_2 (see Figure 3). The derivation of (1.8) is based on the entry-exit functions on M_i. Also we will prove that the corresponding periodic orbits are locally orbitally asymptotically stable.

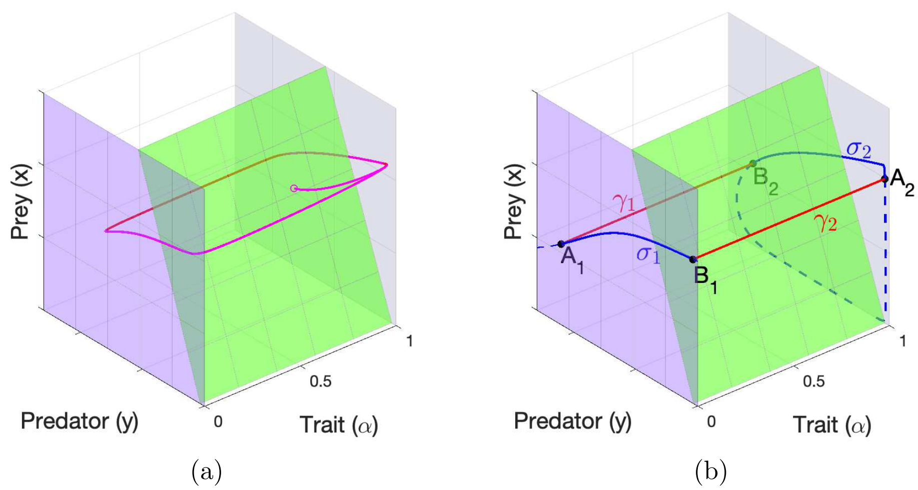


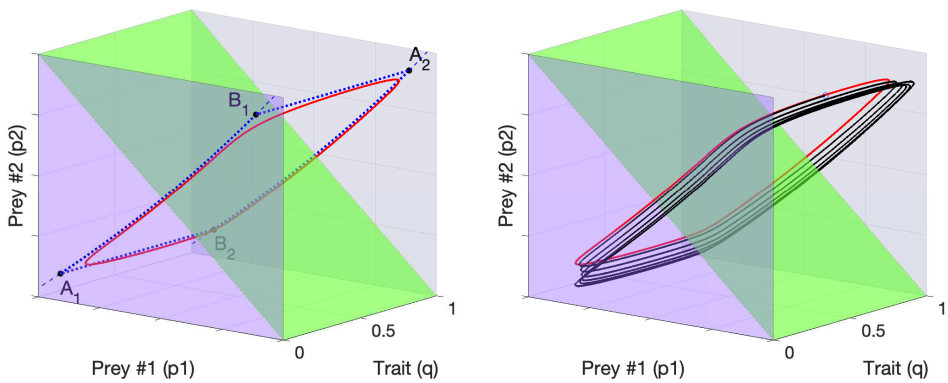
FIG. 3. (a) *The trajectory of* (1.7) *with* $\epsilon = 0.1$ *and initial data* $(x, y, \alpha) = (10, 0.5, 0.5)$ *converges to a periodic orbit.* (b) *A singular configuration consists of trajectories of limiting subsystems and is locally uniquely determined by* (1.8).

Example 3. The third example is a 1-predator-2-prey system with rapid prey evolution proposed by Piltz et al. [49]:

$$\begin{aligned} p_1' &= r_1 p_1 - q f_1(p_1) z, \\ p_2' &= r_2 p_2 - (1-q) f_2(p_2) z, \\ z' &= c_1 q f_1(p_1) z + c_2 (1-q) f_2(p_2) z - m z, \\ \epsilon\, q' &= q(1-q)\big(c_1 f_1(p_1) - c_2 f_2(p_2)\big), \end{aligned} \tag{1.9}$$

where p_1 and p_2 are population densities of two prey species, z is the population density of predators, and q is the mean trait value of predators. The equation of q'

is analogous to the equation of α' in (1.6). A two-parameter family of closed singular configurations formed by trajectories of limiting slow and fast systems of (1.9) has been derived in Piltz et al. [49]. Note that system (1.9) is an example of system (1.1) with $h = 0$ and the variables (p_1, p_2, z) and q playing the roles of p and z, respectively. In section 4.2, using our criterion (Theorem 2.5) we will prove that there is a locally unique closed singular configuration that admits periodic orbits (see Figure 4(a)). Moreover, with parameters adapted from that paper, by computing the linearization of the singular transition maps we will prove that the periodic orbits are orbitally unstable (see Figure 4(b)) for all small $\epsilon > 0$.



(a) (b)

FIG. 4. (a) *A periodic orbit for* (1.9) *(red solid curve) with* $\epsilon = 0.01$ *is close to the singular configuration (blue dotted curve) with vertices* A_i *and* B_i*.* (b) *A trajectory for* (1.9) *with* $\epsilon = 0.01$ *and initial value (black open circle) close to the periodic orbit leaves the vicinity of the periodic orbit as time evolves, which suggests that the periodic orbit is unstable.*

The rapid evolution model, i.e., system (1.6) with $0 < \epsilon \ll 1$, has been studied by Cortez [6, 7, 8, 9], Cortez and Ellner [10], Cortez and Patel [11], Cortez and Weitz [12], and Haney and Siepielski [21]. System (1.6) with slow evolution, i.e., $\epsilon \gg 1$, has been studied by Khibnik and Kondrashov [34] and Shen, Hsu, and Yang [55]. Transient behaviors, which are related to regime shifts in ecological systems, have been studied by Hastings [22], Wysham and Hastings [61], and Hastings et al. [23]. Model (1.9) is a continuous version of the piecewise-smooth model in Piltz, Porter, and Maini [48]. A comparison of the numerical solutions of (1.9) with real data was given in Piltz, Veerman, and Maini [48].

Example 4. In section 4.4, we consider the planar system studied by Hsu and Wolkowicz [28]:

$$\dot{a} = \epsilon\, F(a, b, \epsilon) + b\, H(a, b, \epsilon), \qquad \dot{b} = b\, G(a, b, \epsilon). \tag{1.10}$$

The a-axis is a critical manifold for the limiting fast system of (1.10). Note that the variables a and b in system (1.10) play the roles of p and z, respectively, in (1.1). For singular closed orbits of this system, a criterion on the existence and stability of corresponding relaxation oscillations was derived in Hsu and Wolkowicz [28], which generalizes the criterion in Hsu [27]. In the present paper, we provide an alternative proof of that result. The derivations in those papers were based on the asymptotic expansion of Floquet exponents for system (1.10) with $\epsilon > 0$. Here we will analyze the transition maps for the limiting slow and fast systems with $\epsilon = 0$ directly, which provides a better understanding of the slow-fast feature in the system.

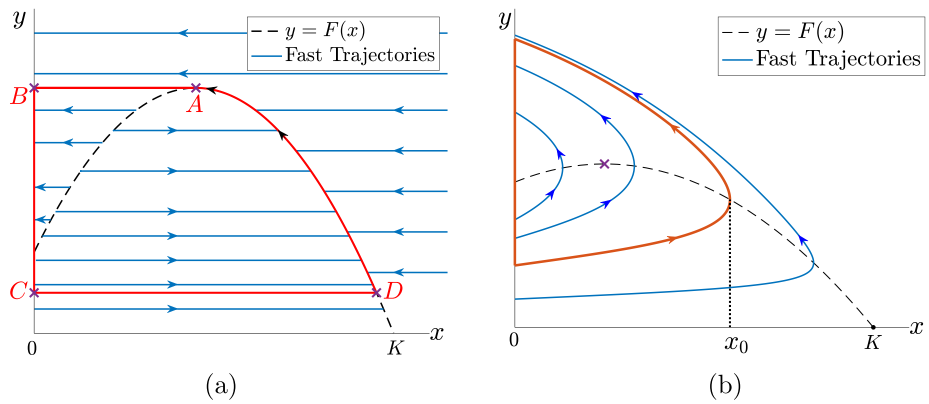


FIG. 5. (a) *For system* (1.11) *with* $\epsilon = 0$*, all trajectories are horizontal. The closed loop* $\overline{ABCD}$ *starting from the maximum point* A *of the nullcline* $y = F(x)$ *is the limiting configuration of the relaxation oscillation as* $\epsilon \to 0$. (b) *For system* (1.12) *with* $\epsilon = 0$*, there is a family of heteroclinic orbits connecting points on the* y*-axis. For a certain value* x_0*, the closed loop formed by the heteroclinic orbit passing* $(x_0, F(x_0))$ *and a portion of the* y*-axis is the limiting configuration.*

The main contribution of this study is to provide criteria on the existence and stability of relaxation oscillations near slow-fast trajectories passing through turning points in multidimensional systems away from fold points (i.e., singular points of the slow flow). Our criteria are generalizations of those given by Hsu [27] and Hsu and Wolkowicz [28], where planar systems (i.e., system (1.1) with $n = m = 1$) were considered. Relaxation oscillations in planar or three-dimensional systems with similar settings were also investigated by Hsu and Shi [24], Huzak [29], Jardón-Kojakhmetov et al. [31], Li et al. [40], Shan [54], and Shen, Hsu, and Yang [55]. A recent work by Ai and Yi [3] generalized the results in Example 4 for a class of systems including (1.10) using a different approach based on the construction of invariant regions.

We would like to point out that the absence of fold points makes our study significantly different from existing theories in the literature involving folds points. Here we use two examples to emphasize the effect of fold points. A well known example of relaxation oscillation involving fold points, besides the van der Pol oscillator, is the predator-prey system studied by Rinaldi and Muratori [50], which can be written as

$$\dot{x} = p(x)\big(F(x) - y\big), \quad \dot{y} = \epsilon\big(c\,p(x) - d\big)y, \tag{1.11}$$

where c, d, and ϵ are positive constants and functions $p(x)$ and $F(x)$ satisfy $p(0) = 0$, $p'(0) > 0$, $p(x) > 0$ for $x > 0$, and, for some constant $K > 0$, $F(x) > 0$ for $x \in [0, K)$ and $F(K) = 0$. Assuming that $F(x)$ takes certain forms, it was proved in [50] that, as $\epsilon \to 0$, there is a family of periodic orbits that converges to a closed loop formed by trajectories of the limiting systems of (1.11). The trajectories forming the closed loop have four end points, which we label as A, B, C, and D in Figure 5(a). A crucial feature for the existence of relaxation oscillations for this system is that the flow of (1.11) in the vicinity of the segment from D to A is exponentially contracting as $\epsilon \to 0$ (see Krupa and Szmolyan [36, Theorem 2.1]). In contrast, for the predator-prey system studied by Hsu [27], which can be written as

$$\dot{x} = p(x)\big(F(x) - y\big), \quad \dot{y} = \big(c\,p(x) - \epsilon\big)y, \tag{1.12}$$

the limiting fast system of (1.12) has a family of heteroclinic orbits connecting points on the y-axis, and it was proved that, under the assumption that $F(x)$ has a single interior maximum point $\hat{x}$ in $(0, K)$, as $\epsilon \to 0$ system (1.12) has a family of relaxation oscillations that converges to a closed loop passing through $(x_0, F(x_0))$ for a unique $x_0 \in (\hat{x}, K)$ (see Figure 5(b)). Different from the former example (1.11), where the Poincaré map along the singular configuration is exponentially attracting as $\epsilon \to 0$, the limiting Poincaré map in the later example has full rank due to the absence of fold points. In fact, for system (1.12) the exponential attraction and repelling forces are balanced through the passage of the slow trajectory containing the turning point $(x, y) = (0, F(0))$, so the limiting Poincaré map does not contract exponentially.

Relaxation oscillations involving both turning points and fold points, different from the context in this present work, for planar or three-dimensional systems have been studied by Ai and Sadhu [2], de Maesschalck, Dumortier, and Roussarie [14], Ghazaryan, Manukian, and Schecter [20], Li and Zhu [39], Liu, Xiao, and Yi [44], and Szmolyan and Wechselberger [57] (for which the proof also holds for multidimensional systems, as indicated in the proof of [60, Proposition 5.1]). Our work is complementary to those results since our singular orbits are away from fold points. Relaxation oscillations in multidimensional slow-fast systems without turning points have been studied by Soto-Treviño [56]. Boundary value problems for slow-fast systems have been studied by Lin [42] and Tin, Kopell, and Jones [58].

The proofs of our criteria were based on a generalization of methods in Hsu [25, 26] for studying the dynamics along the passage between entry and exit points. The idea is to apply a sequence of transformations on system (1.1). Each successive transform in the sequence is obtained simply by appending or dropping an auxiliary variable. This approach is a variation of the classical blow-up method developed by Dumortier and Roussarie [17] and Krupa and Szmolyan [36, 37], where the equation $\epsilon' = 0$ is appended to the system, but all succeeding transformations are homeomorphisms.

The classical blow-up method has been applied extensively to study various problems, including Gasser, Szmolyan, and Wächtler [19], Iuorio, Popović, and Szmolyan [30], Kosiuk and Szmolyan [35], Manukian and Schecter [45], Schecter [51], and Schecter and Szmolyan [53]. We do not claim that our method can be applied to those problems.

This paper is organized as follows. In section 2, we state our criteria for the existence and stability of relaxation oscillations. Proofs of the criteria are given in section 3. In section 4 we apply our criteria to the three models described in section 1. Some computable formulas for verifying the conditions of our criteria numerically are shown and derived in the appendix.

2. Main theorems.

Before stating the general theorem, in section 2.1 we present two special cases in order to motivate the definitions in the subsequent sections. Assumptions needed for our main results are stated in section 2.2. The criteria for the existence of relaxation oscillations are split into sections 2.3–2.5, from single to multiple dimensional fast variables.

2.1. Special cases.

Having systems (1.6) and (1.7) in mind, we present two basic forms of our main theorems, which are stated in later sections.

THEOREM 2.1. *Consider a system for* $(p, \alpha) \in \mathbb{R}^n \times \mathbb{R}$ *of the form*

$$\begin{aligned} p' &= F(p, \alpha, \epsilon), \\ \epsilon\, \alpha' &= (\alpha - \alpha_{\min})(\alpha_{\max} - \alpha) E(p, \alpha, \epsilon), \end{aligned} \tag{2.1}$$

where F and E are smooth functions, and $\alpha_{\min} < \alpha_{\max}$ are constants. Assume that there are points $A_1, A_2 \in \mathbb{R}^n$ satisfying the following conditions (see Figure 3(b)*):*

(R.i) *There exist $A_1, A_2 \in \mathbb{R}^n$, $T_1 > 0$, and $T_2 > 0$ such that*

$$\Phi_1(A_1, T_1) = A_2 \quad \textit{and} \quad \Phi_2(A_2, T_2) = A_1,$$

where Φ_1 and Φ_2 are the solution operators for $p' = F(p, \alpha, 0)$ with $\alpha = \alpha_{\min}$ and $\alpha_{\max}$, respectively.

(R.ii) $E(A_1, \alpha_{\min}, 0), E(A_2, \alpha_{\max}, 0) < 0$ *and* $E(A_2, \alpha_{\min}, 0), E(A_1, \alpha_{\max}, 0) > 0.$

(R.iii) *There hold*

$$\int_0^s E\big(\Phi_1(A_1, t), \alpha_{\min}, 0\big)\, dt \begin{cases} < 0 & \textit{for } 0 < s < T_1, \\ = 0 & \textit{for } s = T_1 \end{cases}$$

and

$$\int_0^s E\big(\Phi_2(A_2, t), \alpha_{\max}, 0\big)\, dt \begin{cases} < 0 & \textit{for } 0 < s < T_2, \\ = 0 & \textit{for } s = T_2. \end{cases}$$

Let Q_1 and Q_2 be functions defined in neighborhoods of A_1 and A_2, respectively, such that $Q_1(A_1) = A_2$, $Q_2(A_2) = A_1$, the point $Q_i(A)$ lies in the forward trajectory of A along the flow of Φ_i, and

$$\int_{\ell_1(A, Q_1(A))} E(p(t), \alpha_{\min}, 0)\, dt = 0 \quad \textit{and} \quad \int_{\ell_2(A, Q_2(A))} E(p(t), \alpha_{\max}, 0)\, dt = 0,$$

where $\ell_i(A, B)$ is the trajectory from A to B following Φ_i and $p(t) = \Phi_i(A, t)$. If the function $P = Q_2 \circ Q_1$ satisfies

$$\det(DP(A_1) - \mathrm{I}_n) \neq 0,$$

where $DP(A)$ is the Jacobian matrix of P at A and I_n is the identify matrix of rank n, then the configuration $(\ell_1(A_1, A_2) \times \{\alpha_{\min}\}) \cup (\ell_2(A_2, A_1) \times \{\alpha_{\max}\})$ admits a relaxation oscillation. Furthermore, the corresponding periodic orbits are orbitally asymptotically stable if $r(DP(A_1)) < 1$ and orbitally unstable if $r(DP(A_1)) > 1$, where $r(DP(A))$ is the spectral radius of $DP(A)$.

Note that system (1.7) satisfies (2.1) with $p = (x, y)$, $\alpha_{\min} = 0$, and $\alpha_{\max} = 1$. Theorem 2.1 is a special case of Theorems 2.5 in section 2.3.

For treating system (1.6), we have the following result.

THEOREM 2.2. *Consider a system for $(p, \alpha, \beta) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ of the form*

$$\begin{aligned} p' &= F(p, \alpha, \beta, \epsilon), \\ \epsilon\, \alpha' &= \alpha(1 - \alpha) E_1(p, \alpha, \beta, \epsilon), \\ \epsilon\, \beta' &= \beta(1 - \beta) E_2(p, \alpha, \beta, \epsilon), \end{aligned} \tag{2.2}$$

where F, E_1, and E_2 are smooth functions. Let M_1, M_2, M_3, and M_4 be the level sets of $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$ with $(\alpha, \beta) = (0, 0)$, $(0, 1)$, $(1, 1)$, and $(1, 0)$, respectively. For convenience, we also denote $M_i = M_{(\alpha_i, \beta_i)}$, where (α_i, β_i) is the constant value (α, β) on M_i (see Figure 6). Assume that there are points A_1, A_2, A_3, and A_4 in $\mathbb{R}^n$ such that the following conditions hold (in the manner that $A_{k+4} = A_k$, $J_{k+4} = J_k$, etc.).

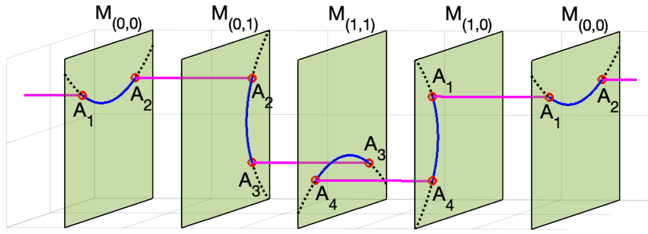


FIG. 6. *Each $M_{(\alpha_i,\beta_i)}$ is the level set of (α_i,β_i). Solid and dotted curves on $M_{(\alpha_i,\beta_i)}$ are trajectories $p' = F(p,\alpha_i,\beta_i,0)$. The curves between $M_{(\alpha_i,\beta_i)}$ and $M_{(\alpha_{i+1},\beta_{i+1})}$ are line segments.*

(S.i) *A_{i+1} lies in the forward trajectory of A_i along the flow of $p' = F(p,\alpha_i,\beta_i,0)$ for all $i \in \{1,2,3,4\}$.*

(S.ii) *$E_1(A_1,0,0,0)$, $E_2(A_2,0,1,0)$, $E_1(A_3,1,1,0)$, $E_2(A_4,1,0,0) < 0$ and $E_2(A_2,0,0,0)$, $E_1(A_3,0,1,0)$, $E_2(A_4,1,1,0)$, $E_1(A_1,1,0,0) > 0$.*

(S.iii) *With $(J_1,J_2,J_3,J_4) = (1,2,1,2)$, for each $i \in \{1,2,3,4\}$ if we denote $\ell_i(A,B)$ the trajectory of the flow of $p' = F(p,\alpha_i,\beta_i,0)$ from A to B, then for any $B \in \ell_{i-1}(A_{i-1},A_i) \setminus \{A_{i-1}\}$,*

$$\int_{\ell_{i-1}(A_{i-1},B)} E_{J_{i+1}}(p(t),\alpha_{i-1},\beta_{i-1},0)\,dt < 0,$$

where $p(t)$ is the parametrization of the curve of integration along the flow, and if we define

$$\bar{\zeta}_i = -\int_{\ell_{i-1}(A_{i-1},A_i)} E_{J_{i+1}}(p(t),\alpha_{i-1},\beta_{i-1},0)\,dt, \tag{2.3}$$

then

$$-\bar{\zeta}_i + \int_{\ell_i(A_i,B)} E_{J_{i+1}}(p(t),\alpha_i,\beta_i,0)\,dt \begin{cases} < 0 & \text{if } B \in \ell_i(A_i,A_{i+1}) \setminus \{A_{i+1}\}, \\ = 0 & \text{if } B = A_{i+1}. \end{cases}$$

Let $\widehat{Q}_i(A,\zeta)$ be the function implicitly defined by $\widehat{Q}_i(A_i,\bar{\zeta}_i) = (A_{i+1},\bar{\zeta}_{i+1})$ and that, denoting $\widehat{Q}_i(A,\zeta) = (\widehat{A},\widehat{\zeta})$, $\widehat{A}$ is a point in the forward trajectory of A along the flow of $p' = F(p,\alpha_i,\beta_i,0)$ satisfying

$$-\zeta + \int_{\ell_i(A,\widehat{A})} E_{J_{i+1}}(p(t),\alpha_i,\beta_i,0)\,dt = 0,$$

and $\widehat{\zeta}$ is the number defined to be

$$\widehat{\zeta} = -\int_{\ell_i(A,\widehat{A})} E_{J_i}(p(t),\alpha_i,\beta_i,0)\,dt. \tag{2.4}$$

Let $\widehat{P} = \widehat{Q}_4 \circ \widehat{Q}_3 \circ \widehat{Q}_2 \circ \widehat{Q}_1$. If

$$\det(D\widehat{P}(A_1,\zeta_1) - \mathrm{I}_{n+1}) \neq 0,$$

then the configuration $\bigcup_{i=1}^4 (\ell_i(A_i,A_{i+1}) \times \{(\alpha_i,\beta_i)\})$ admits a relaxation oscillation. Furthermore, the corresponding periodic orbits are orbitally asymptotically stable if $r(D\widehat{P}(A_1,\zeta_1)) < 1$ and orbitally unstable if $r(D\widehat{P}(A_1,\zeta_1)) > 1$.

Note that (2.3) means

$$\bar{\zeta}_1 = -\int_{\ell_4(A_4,A_1)} E_2(p(t),1,0,0)\,dt, \quad \bar{\zeta}_2 = -\int_{\ell_1(A_1,A_2)} E_1(p(t),0,0,0)\,dt \tag{2.5}$$

and

$$\bar{\zeta}_3 = -\int_{\ell_2(A_2,A_3)} E_2(p(t),0,1,0)\,dt, \quad \bar{\zeta}_4 = -\int_{\ell_3(A_3,A_4)} E_1(p(t),1,1,0)\,dt. \tag{2.6}$$

Theorem 2.2 is a special case of Theorem 2.6 in section 2.4. In section 2.2 we show how systems (2.1) and (2.2) satisfy the conditions in the general theorems.

2.2. Assumptions. Let N be a fixed positive integer. Throughout this paper we adopt the notion that $A_i = A_{i+N}$ for any integer i and any object A. For any vector z in $\mathbb{R}^m$, we denote $z^{(j)}$ the jth component of z. We denote $\{\mathsf{e}_1, \mathsf{e}_2, \dots, \mathsf{e}_m\}$ the standard basis of $\mathbb{R}^m$.

Assumption 1. For each $j = 1, 2, \dots, m$, there exist $-\infty \le z^{(j)}_{\min} < z^{(j)}_{\max} \le \infty$ such that for all sufficiently small $\epsilon \ge 0$,

$$h(p,z,\epsilon) = 0 \quad \text{and} \quad g^{(j)}(p,z,\epsilon) = 0$$

whenever $z^{(j)} = z^{(j)}_{\min}$ or $z = z^{(j)}_{\max}$.

Note that system (2.1) satisfies Assumption 1 with $z = z^{(1)} = \alpha$. System (2.2) satisfies the assumption with $z = (z^{(1)}, z^{(2)}) = (\alpha, \beta)$ and $(z^{(j)}_{\min}, z^{(j)}_{\max}) = (0,1)$ for $j = 1, 2$.

Assumption 2. For each $i = 1, 2, \dots, N$, where N is a positive integer, there exist $A_i, B_i \in \mathbb{R}^n$, $J_i \in \{1, 2, \dots, m\}$,

$$z_i \in \{z^{(1)}_{\min}, z^{(1)}_{\max}\} \times \{z^{(2)}_{\min}, z^{(2)}_{\max}\} \times \cdots \times \{z^{(m)}_{\min}, z^{(m)}_{\max}\} \quad \text{with } |z_i| < \infty,$$

and smooth functions $p_i : \mathbb{R} \to \mathbb{R}^n$ and $q_i : \mathbb{R} \to \mathbb{R}$ such that q_i is nonconstant and the curve

$$\gamma_i(t) = \big(p_i(t), z_i + q_i(t)\mathsf{e}_{J_i}\big), \ \ -\infty < t < \infty,$$

is a heteroclinic orbit of (1.3) that connects (B_{i-1}, z_{i-1}) and (A_i, z_i). In addition, for each $j = 1, 2, \dots, m$, there exists $i \in \{1, 2, \dots, N\}$ such that $J_i = j$.

Since the limiting fast system of (2.1) leaves (x, y) values constant, system (2.1) satisfies Assumption 2 with $B_1 = A_2$, $B_2 = A_1$, $\gamma_1(t)$ connecting $(A_1, \alpha_{\max})$ to $(A_1, \alpha_{\min})$ and $\gamma_2(t)$ connecting $(A_2, \alpha_{\min})$ to $(A_2, \alpha_{\max})$. Similarly, system (2.2) satisfies Assumption 2 with $B_i = A_{i+1}$, $p_i(t) = 0$ and $q_i(t)$ being scalar functions mapping onto either interval $(0,1)$ or $(-1,0)$.

The expression of the heteroclinic orbit in Assumption 2 implies that z_i differs from z_{i+1} at no more than one component. Note that we do not exclude the possibility that $z_i = z_{i+1}$.

The assumption of the existence of i such that $J_i = j$ means that each component $z^{(j)}$ of (p, z) must be nonconstant along at least one γ_i. If it is not the case, then we can treat $z^{(j)}$ as a constant and replace the equation of $\dot{z}^{(j)}$ in (1.1) by $\dot{z}^{(j)} = 0$ because the space $\{(p,z) : z^{(j)} = z^{(j)}_{\min} \text{ or } z^{(j)}_{\max}\}$ is invariant under (1.1) by Assumption 1.

We define $M_i = \{(p,z) : p \in \mathbb{R}^n, z = z_i\}$ for $i = 1, 2, \dots, N$. Then Assumption 1 implies that M_i is invariant under (1.1) for all sufficiently small $\epsilon > 0$. The restriction of (1.1) on M_i is (1.4). We denote the solution operator of (1.4) by Φ_i.

Assumption 3. For each $i = 1, 2, \dots, N$, $f_i(A_i, z_i, 0) \neq 0$ and there exists $\tau_i > 0$ such that $\Phi_i(A_i, \tau_i) = B_i$.

In systems (2.1) and (2.2), since each A_{i+1} lies in a trajectory of the corresponding limiting systems passing through A_i, Assumption 3 is satisfied with $B_i = A_{i+1}$.

Denote $\sigma_i = \Phi_i(A_i, [0, \tau_i]) \times \{z_i\}$. Then by Assumptions 2–3 the configuration (1.5) forms a closed orbit. The idea of GSPT is that trajectories of the full system can potentially be obtained by perturbing a union of trajectories of the limiting systems. Limiting systems (1.3) and (1.4) provide a family of uncountably many loops. Our goal is to establish a criterion for the existence of a locally unique periodic orbit near this singular closed orbit.

We impose the following nondegeneracy condition.

Assumption 4. For $i = 1, 2, \dots, N$,

$$\frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(A_i, z_i, 0) < 0 \quad \text{and} \quad \frac{\partial g^{(J_{i+1})}}{\partial z^{(J_{i+1})}}(B_i, z_i, 0) > 0.$$

Assumption 4 corresponds to (R.ii) and (S.ii) in Theorems 2.1 and 2.2, respectively, for systems (2.1) and (2.2).

Remark 2.3. By Assumption 1, the linearization of (1.3) at any point (p, z_i) in M_i has the Jacobian matrix

$$\begin{pmatrix} 0_{n\times n} & * \\ 0_{m\times n} & \operatorname{diag}\left(\frac{\partial g^{(1)}}{\partial z^{(1)}}, \dots, \frac{\partial g^{(m)}}{\partial z^{(m)}}\right) \end{pmatrix},$$

where the partial derivatives are evaluated at $(p, z_i, 0)$. In the case that $m = 1$, the inequalities in Assumption 4 imply that M_i is normally hyperbolic at (A_i, z_i) and (B_i, z_i) and that there is a turning point on M_i between these two points.

In the case that $m = 1$, where z and g are scalar, the classical entry-exit relation for (1.1) between A_i and B_i can be expressed by

$$\int_0^s \frac{\partial g}{\partial z}\big(\Phi_i(A_i, t), z_i, 0\big)\, dt \begin{cases} < 0 & \text{if } 0 < s < \tau_i, \\ = 0 & \text{if } s = \tau_i. \end{cases} \tag{2.7}$$

Under (2.7) and Assumption 4, in some neighborhood $\mathscr{A}_i$ of A_i in $\mathbb{R}^n$ we can implicitly define $T_i : \mathscr{A}_i \to (0, \infty)$ by $T_i(A_i) = \tau_i$ and

$$\int_0^{T_i(p)} \frac{\partial g}{\partial z}\big(\Phi_i(p, t), z_i, 0\big)\, dt = 0. \tag{2.8}$$

The entry-exit function is then defined by

$$Q_i(p) = \Phi_i(p, T_i(p)). \tag{2.9}$$

Each pair of points (p, z_i) and $(Q_i(p), z_i)$, where $p \in \mathscr{A}_i$, is a pair of landing and jumping points on M_i.

For the general case that $m \geq 1$, we first introduce some notation. Let J_i, where $i = 1, 2, \ldots, N$, be the numbers defined in Assumption 1. For each $j = 1, 2, \ldots, m$, let

$$I_j = \max\{i \in \{0, -1, -2, \ldots, -(N-1)\} : J_i = j\}.$$

This means that I_j is the largest nonpositive i for which the value of $z^{(j)}$ changes along the trajectory γ_i. By Assumption 2, each I_j is well-defined and is finite. We define

$$\zeta_i^{(j)} = -\sum_{k=I_j}^{i-1} \left(\int_0^{\tau_k} \frac{\partial g^{(j)}}{\partial z^{(j)}} \big(\Phi_k(A_k, t), z_k, 0\big)\, dt \right) \tag{2.10}$$

for $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, m$. Also we denote $\zeta_i = (\zeta_i^{(1)}, \ldots, \zeta_i^{(m)})$. The following assumption is a generalization of (2.7).

Assumption 5. For each $i \in \{1, 2, \ldots, N\}$, $j \in \{1, 2, \ldots, m\}$ and $s \in (0, \tau_i]$,

$$-\zeta_i^{(j)} + \int_0^s \frac{\partial g^{(j)}}{\partial z^{(j)}} \big(\Phi_i(A_i, t), z_i, 0\big)\, dt \begin{cases} = 0 & \text{if } j = J_{i+1} \text{ and } s = \tau_i, \\ < 0 & \text{otherwise.} \end{cases}$$

Assumption 5 corresponds to (R.iii) and (S.iii) in Theorems 2.1 and 2.2, respectively, for systems (2.1) and (2.2). More specifically, for system (2.1) both Assumptions 5 and (R.iii) are equivalent to (2.7). In the settings of Theorem 2.2 for system (2.2), we have $I_1 = -1$ and $I_2 = 0$ (because $J_{-1} = J_3 = 1$ and $J_0 = J_4 = 2$), so (2.10) under Assumption 5 with $(J_1, J_2, J_3, J_4) = (1, 2, 1, 2)$ gives

$$\zeta_1 = (0, \bar{\zeta}_1), \quad \zeta_2 = (\bar{\zeta}_2, 0), \quad \zeta_3 = (0, \bar{\zeta}_3), \quad \text{and} \quad \zeta_4 = (\bar{\zeta}_4, 0),$$

where $\bar{\zeta}_1$, $\bar{\zeta}_2$, $\bar{\zeta}_3$, and $\bar{\zeta}_4$ are the numbers defined by (2.3), or (2.5)–(2.6), in Theorem 2.2.

For each $i = 1, 2, \ldots, N$, we consider the system

$$\begin{aligned} &\frac{d}{d\tau} p = f(p, z_i, 0), \\ &\frac{d}{d\tau} \zeta^{(j)} = \frac{\partial g^{(j)}}{\partial z^{(j)}}(p, z_i, 0), \quad j = 1, 2, \ldots, m. \end{aligned} \tag{2.11}$$

Let

$$\Lambda_i = \left\{ \zeta \in \mathbb{R}^m : \left|\zeta - \zeta_i\right| < \delta,\ \zeta^{(J_i)} = \zeta_i^{(J_i)} \right\}, \tag{2.12}$$

where $\delta > 0$. Let $\widehat{\Phi}_i$ be the solution operator for (2.11). From Assumption 4, by shrinking $\mathscr{A}_i$ and δ if necessary, we can define $\widehat{T}_i(p, \zeta)$ on $\mathscr{A}_i \times \Lambda_i$ implicitly by $\widehat{T}_i(A_i, \zeta_i) = 0$ and

$$-\zeta^{(J_{i+1})} + \int_0^{\widehat{T}_i(p,\zeta)} \frac{\partial g^{(J_{i+1})}}{\partial z^{(J_{i+1})}} (\Phi_i(p, t), z_i, 0)\, dt = 0. \tag{2.13}$$

Finally, we define the generalized entry-exit function $\widehat{Q}_i(p, \zeta)$ on $\mathscr{A}_i \times \Lambda_i$ by

$$\widehat{Q}_i(p, \zeta) = \widehat{\Phi}_i\big((p, \zeta), \widehat{T}_i(p, \zeta)\big). \tag{2.14}$$

Note that $\widehat{T}_i(p, \zeta_i) = T_i(p)$ and therefore $\widehat{Q}_i(p, \zeta_i) = (Q_i(p), \zeta_{i+1})$ for all $p \in \mathscr{A}_i$. In particular, $\widehat{Q}_i(A_i, \zeta_i) = (B_i, \zeta_{i+1})$.

Remark 2.4. In the case that $m = 1$, we have $\zeta_i^{(j)} = 0$ for all i and j, so Assumption 5 is reduced to the classical entry-exit relation (2.7), and $\widehat{Q}_i$ defined by (2.13)–(2.14) coincides with Q_i defined by (2.8)–(2.9).

2.3. Systems in the standard form with a single fast variable. For the case where the fast variable has simple dynamics, namely, $h = 0$ in (1.1), the system is in the standard form of geometric singular perturbation theory in Fenichel [18]. First we state our results for system (1.1) with $n \geq 1$, $m = 1$, and $h = 0$, which can be applied to study models (1.7) and (1.9). These restrictions mean that the system has a single variable and that the slow variable is steady in the fast system (1.3).

Since the slow variable is steady in the fast system (1.3) in the case that $h = 0$, the function p_i in Assumption 2 is constant for each $i = 1, 2, \ldots, N$. Hence $B_i = A_{i+1}$ for each i, where B_i and A_{i+1} and the points given in Assumption 2. Since $Q_i(A_i) = B_i$, where Q_i is defined in (2.9), it follows that $Q_i(A_i) = A_{i+1}$. Let

$$P = Q_N \circ \cdots \circ Q_2 \circ Q_1. \tag{2.15}$$

Then $P(A_1) = A_1$ and P maps a neighborhood of A_1 in $\mathscr{A}_1$ into $\mathscr{A}_1$.

Our first main result is as follows.

THEOREM 2.5. *Suppose that Assumptions* 1–5 *hold for system* (1.1) *with* $m = 1$ *and* $h = 0$. *Let* P *be defined by* (2.15). *If*

$$\det(DP(A_1) - \mathrm{I}_n) \neq 0,$$

then the configuration (1.5) *admits a relaxation oscillation. Furthermore, the corresponding periodic orbits are orbitally asymptotically stable if* $r(DP(A_1)) < 1$ *and orbitally unstable if* $r(DP(A_1)) > 1$.

The proof of the theorem is shown in section 3.1, and the computation formula of the Jacobian matrix is given in the appendix.

2.4. Systems in the standard form. System (1.1) with $n, m \geq 1$ and $h = 0$ can be applied to (1.6). For this case, we introduce the following notation.

Under the assumption that $h = 0$, we have $B_i = A_{i+1}$ as in section 2.3. Since $\widehat{Q}_i(A_i, \zeta_i) = (B_i, \zeta_{i+1})$, where $\widehat{Q}_i$ is defined by (2.14), it follows that $\widehat{Q}_i(A_i, \zeta_i) = (A_{i+1}, \zeta_{i+1})$. Let

$$\widehat{P} = \widehat{Q}_N \circ \cdots \circ \widehat{Q}_2 \circ \widehat{Q}_1. \tag{2.16}$$

Then $\widehat{P}(A_1, \zeta_1) = (A_1, \zeta_1)$ and $\widehat{P}$ maps a neighborhood of (A_1, ζ_1) in $\mathscr{A}_1 \times \Lambda_1$ into $\mathscr{A}_1 \times \Lambda_1$. Our second result is as follows.

THEOREM 2.6. *Suppose that Assumptions* 1–5 *hold for system* (1.1) *with* $h = 0$. *Let* $\widehat{P}$ *be defined by* (2.16). *If*

$$\det(D\widehat{P}(A_1, \zeta_1) - \mathrm{I}_{n+m-1}) \neq 0,$$

where $D\widehat{P}$ *is the Jacobian matrix with respect to the standard coordinate of* $\mathscr{A}_1 \times \Lambda_1$, *then the configuration* (1.5) *admits a relaxation oscillation. Furthermore, the corresponding periodic orbits are orbitally asymptotically stable if* $r(DP(A_1)) < 1$ *and orbitally unstable if* $r(DP(A_1)) > 1$.

Theorem 2.6 is resulted from a more general theorem, Theorem 2.7, stated below.

2.5. Systems with multiple slow and fast variables. Now we consider system (1.1) with general h, which can be used to treat system (1.10).

For $i = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, m$, let

$$\omega_i^{(j)} = \begin{cases} 1 & \text{if } z_i^{(j)} = z_{\min}^{(j)}, \\ -1 & \text{if } z_i^{(j)} = z_{\max}^{(j)}, \end{cases} \tag{2.17}$$

where $z_{\min}^{(j)}$ and $z_{\max}^{(j)}$ are the numbers given in Assumption 1. Let

$$\phi_i(q) = \begin{cases} \dfrac{\omega_i^{(J_i)}}{q - z_i^{(J_i)}} & \text{if } z_{i-1}^{(J_i)} = z_{i-1}^{(J_i)}, \\ \dfrac{\omega_i^{(J_i)}}{q - z_i^{(J_i)}} \; \dfrac{\omega_{i-1}^{(J_i)}}{q - z_{i-1}^{(J_i)}} & \text{if } z_i^{(J_i)} \neq z_{i-1}^{(J_i)}, \end{cases}$$

where J_i is the index defined in Assumption 2. By the construction of ϕ_i, we have $\phi_i(z^{(J_i)}) > 0$ for all (p, z) on γ_i, where γ_i is the trajectory defined in Assumption 2.

Define functions g_i and h_i of $(p, q) \in \mathbb{R}^N \times \mathbb{R}$ by

$$(g_i, h_i)(p, q) = \phi_i(q) \left[(g^{(J_i)}, h)(p, z_{i-1} + q\mathsf{e}_{J_i}, 0) \right] \quad \text{for } q \neq z_i^{(J_i)}, z_{i-1}^{(J_i)}. \tag{2.18}$$

That is, g_i and h_i are rescaled values of $g^{(J_i)}$ and h, respectively, along γ_i. By Assumption 1, (g_i, h_i) can be continuously extended at $q = z_i^{(J_i)}$ and $z_{i-1}^{(J_i)}$, which correspond to $(p, z) = (B_{i-1}, z_{i-1})$ and (A_i, z_i), where A_i and B_{i-1} are points in $\mathbb{R}^n$ introduced in Assumption 2. We identify (g_i, h_i) with its continuous extension. Thus $g_i(B_{i-1}, z_{i-1}^{(J_i)})$ and $g_i(A_i, z_i^{(J_i)})$ are multiples of $\frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(B_{i-1}, z_{i-1}, 0)$ and $\frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(A_i, z_i, 0)$, respectively, by nonzero constants. By Assumption 4, it follows that $g_i(B_{i-1}, z_{i-1}^{(J_i)}) \neq 0$ and $g_i(A_i, z_i^{(J_i)}) \neq 0$.

Note that the functions p_i and q_i in Assumption 2 satisfy that $\{(p_i, q_i)(t)) : t \in \mathbb{R}\}$ is a trajectory of the system

$$\dot{p} = h_i(p, q), \quad \dot{q} = g_i(p, q), \tag{2.19}$$

which connects $(B_{i-1}, z_{i-1}^{(J_i)})$ and $(A_i, z_i^{(J_i)})$. Since $g_i(B_{i-1}, z_{i-1}^{(J_i)}) \neq 0$ and $g_i(A_i, z_i^{(J_i)}) \neq 0$, there exists a neighborhood $\mathscr{B}_{i-1}$ of B_{i-1} such that we can define $\pi_i : \mathscr{B}_{i-1} \to \mathscr{A}_i$ implicitly by the fact that

$$\left(p, z_{i-1}^{(J_i)}\right) \text{ and } \left(\pi_i(p), z_i^{(J_i)}\right) \text{ are connected by a trajectory of (2.19).} \tag{2.20}$$

Let $\pi_i \times \mathrm{id}$ be the map from $\mathscr{B}_{i-1} \times \Lambda_i$ to $\mathscr{A}_i \times \Lambda_i$ given by $(\pi_i \times \mathrm{id})(p, \zeta) = (\pi_i(p), \zeta)$. Define

$$\widetilde{P} = (\pi_N \times \mathrm{id}) \circ \widehat{Q}_N \circ (\pi_N \times \mathrm{id}) \circ \cdots \circ \widehat{Q}_2 \circ (\pi_2 \times \mathrm{id}) \circ \widehat{Q}_1. \tag{2.21}$$

Theorem 2.7. *Suppose that Assumptions* 1–5 *hold for system* (1.1). *Let* $\widetilde{P}$ *be defined by* (2.21). *If*

$$\det(D\widetilde{P}(A_1, \zeta_1) - \mathrm{I}_{n+m-1}) \neq 0,$$

where $D\widetilde{P}$ is the Jacobian matrix with respect to the standard coordinate of $\mathscr{A}_1 \times \Lambda_1$, then the configuration (1.5) *admits a relaxation oscillation. Furthermore, the corresponding periodic orbits are orbitally asymptotically stable if $r(DP(A_1)) < 1$ and orbitally unstable if $r(DP(A_1)) > 1$.*

The proof of the theorem is given in section 3.2, and the computation formula of the Jacobian matrix is given in the appendix.

3. Proofs of the main theorems.

Note that Theorem 2.7 is a generalization of Theorems 2.6 and 2.5. While Theorem 2.7 can be proved without relying on the results of the other theorems, for clarity we prove Theorem 2.5 first in section 3.1 and then prove the general Theorem 2.7 in section 3.2.

3.1. Proof of Theorem 2.5.

In this section we assume $m = 1$ and $h = 0$ in system (1.1). With $h = 0$ in the limiting system (1.3) of system (1.1), the curve γ_i given in Assumption 2 can be written as $\gamma_i = \{(A_i, q_i(t)\}$, where $A_i \in \mathbb{R}^n$ is given in Assumption 2 and q_i satisfies $q_i(-\infty) = z_{i-1}$ and $q_i(+\infty) = z_i$. Since q_i is a nonconstant function by assumption, we can choose a point $(A_i, z_{0i}) \in \gamma_i$ at which $\dot{q}_i \neq 0$. Let Γ_i be a cross section of γ_i at (A_i, z_{0i}) of the form

$$\Gamma_i = \{(p, z) : |p - A_i| < \delta_0,\ z = z_{0i}\}, \tag{3.1}$$

where $\delta_0 > 0$ is to be determined. Our strategy is to track trajectories that evolve from Γ_i along the flow (1.1) and reach Γ_{i+1} near the configuration $\gamma_i \cup \sigma_i \cup \gamma_{i+1}$. We set a cross section Σ_i of σ_i and analyze the dynamics between Γ_i and Σ_i (see Figure 7). By symmetry, the dynamics between Σ_i and Γ_{i+1} can also be treated. Near Γ_i we will use the original coordinates (p, z); near Σ_i we will use the coordinates (p, ζ), where ζ is a blow-up variable for z to be defined later; and near (A_i, z_i) we will use the coordinates (p, z, ζ) to connect the other two coordinates. We will choose two cross sections, $\mathscr{A}_i^{\text{in}}$ and $\mathscr{A}_i^{\text{out}}$, near (A_i, z_i) to analyze the transition map from Γ_i to Σ_i.

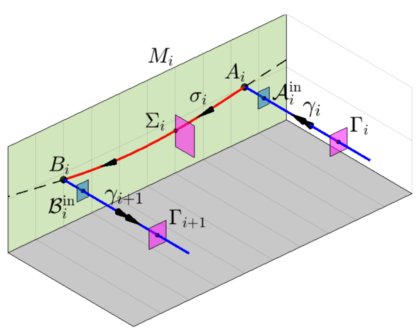


FIG. 7. *The transition map from Γ_i to Γ_{i+1} can be split into transition maps between Γ_i, Σ_i, and Γ_{i+1}. By symmetry, the dynamics between Σ_i and Γ_{i+1} are similar to that between Γ_i and Σ_i.*

Here we give heuristic reasonings for the use of these charts. If we start from a cross section $\mathscr{A}_i^{\text{in}}$ (of dimension n) of γ_i in the original (p, z)-coordinates, following the limiting fast flow this cross section projects onto a subset $\mathscr{A}_i \subset M_i$ of dimension n (see Figure 8). If we evolve $\mathscr{A}_i$ along the limiting slow flow on M_i, then the evolved manifold still has the same dimension n as $\mathscr{A}_i$. This means that some information is missing. The remedy is to introduce a blow-up variable ζ, which is obtained essentially by setting $\zeta = \epsilon \ln(1/|z - z_i|)$. The image of $\mathscr{A}_i^{\text{in}}$ in the ϵ-dependent

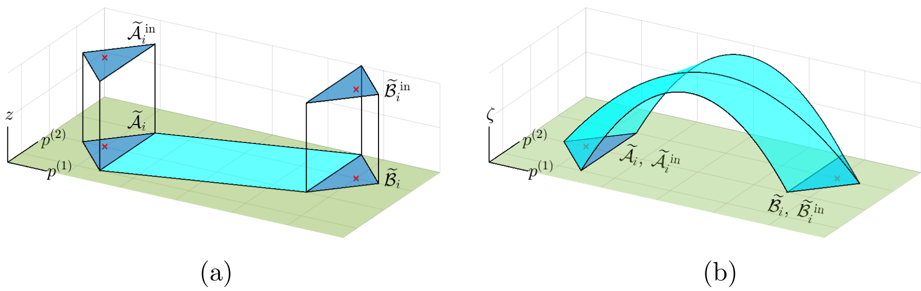


FIG. 8. *$\widetilde{\mathscr{A}_i}^{\text{in}}$ is the image of $\mathscr{A}_i^{\text{in}}$ in the (p, z, ζ)-space with ϵ-dependent coordinates. $\widetilde{\mathscr{A}_i}$ is the projection of $\widetilde{\mathscr{A}_i}^{\text{in}}$ on M_i along the limiting fast system.* (a) *In the (p, z)-space, the image of the manifold evolved from $\widetilde{\mathscr{A}_i}$ and the image of $\widetilde{\mathscr{A}_i}$ itself both have dimension n.* (b) *In the (p, ζ)-space with $\epsilon = 0$, the image of the evolved manifold has dimension $n + 1$.*

(p, z, ζ)-coordinates, denoted by $\widetilde{\mathscr{A}_i}^{\text{in}}$, has the limit $\mathscr{A}_i^{\text{in}} \times \{0\}$ as $\epsilon \to 0$. The images of $\widetilde{\mathscr{A}_i}^{\text{in}}$ and $\widetilde{\mathscr{A}_i}$ in the (p, ζ)-space with $\epsilon = 0$ both lie in the space $\{\zeta = 0\}$. Following the extended limiting slow flow (associated to $\widehat{Q}_i$ defined in section 2.2), the manifold evolved from the image of $\widetilde{\mathscr{A}_i}$ in the (p, ζ)-space has the full dimension $n + 1$. Therefore, our approach consists of the following three steps:

1. Use the original (p, z)-coordinates to track the transition from the cross section Γ_i of γ_i to another cross section $\mathscr{A}_i^{\text{in}}$ that is closer to M_i.
2. Use the (p, z, ζ)-coordinates to track the manifold evolved from $\mathscr{A}_i^{\text{in}}$. (This corresponds to Proposition 3.2.)
3. Use the (p, ζ)-coordinates to track trajectories in the vicinity of M_i. (This corresponds to Proposition 3.1.)

We refer interested readers to Hsu [26] for a

We define a set of transforms between various coordinates. Analogous to the notation κ_{jk} in Krupa and Szmolyan [36], we use the notation $\kappa_{\epsilon i}^{(jk)}$ to denote an ϵ-dependent transform from the kth space to the jth space in the vicinity of M_i (explicit fomulas are given in later paragraphs). In particular, these transforms satisfy that $\kappa_{\epsilon i}^{(kj)} \circ \kappa_{\epsilon i}^{(jk)}$ is the identity map and that $\kappa_{\epsilon i}^{(jk)} \circ \kappa_{\epsilon i}^{(kl)} = \kappa_{\epsilon i}^{(jl)}$ whenever they are defined. A list of symbols used in this section is given in Table 1.

TABLE 1
Notation in section 3.1.

Variables	Charts	Objects
$(p, z) \in \Omega$ $= \mathbb{R}^n \times (z_{\min}, z_{\max})$	$\kappa_{\epsilon i}^{(12)}(p, z, \zeta) = (p, z)$ $\kappa_{\epsilon i}^{(13)}(p, \zeta) = (p, z)$	Ω, Γ_i
$p \in \mathbb{R}^m$		$\mathscr{A}_i$, $\mathscr{B}_i$
$(p, z, \zeta) \in \Omega \times \mathbb{R}$	$\kappa_{\epsilon i}^{(21)}(p, z) = (p, z, \zeta)$ $\kappa_{\epsilon i}^{(23)}(p, \zeta) = (p, z, \zeta)$	$\widetilde{\mathscr{A}_i}$, $\widetilde{\mathscr{A}_i}^{\text{in}}$, $\widetilde{\mathscr{A}_i}^{\text{out}}$
$(p, \zeta) \in \mathbb{R}^n \times \mathbb{R}$	$\kappa_{\epsilon i}^{(31)}(p, z) = (p, \zeta)$ $\kappa_{\epsilon i}^{(32)}(p, z, \zeta) = (p, \zeta)$	$\widehat{\mathscr{A}_i}^{\text{out}}$, $\widehat{\Sigma}_i$

Let ω_i, $1 \leq i \leq N$, be the numbers defined in (2.17) for $m = 1$, which means that $\omega_i = \omega_i^{(1)}$. By Assumption 4, in a neighborhood of (A_i, z_i), for $\delta_1 > 0$ sufficiently small, there is a unique point $(A_i, z_i + \omega_i\delta_1)$ that lies in the curve γ_i. Here $\mathbb{B}(p, r)$ is the open ball centered at p with radius r. Let

$$\mathscr{A}_i^{\text{in}} = \{(p, z) : p \in \mathbb{B}(A_i, \delta_2), z = z_i + \omega_i\delta_1\}, \tag{3.2}$$

where δ_1 and δ_2 are positive constants to be determined. By shrinking δ_0 in the definition of Γ_i in (3.1) if necessary, under the assumption that $h = 0$ the transition map $\Pi_{\epsilon\Gamma_i}^{\mathscr{A}_i^{\text{in}}}$ of the flow of system (1.3) from Γ_i to $\mathscr{A}_i^{\text{in}}$ is well-defined and satisfies

$$\Pi_{\epsilon\Gamma_i}^{\mathscr{A}_i^{\text{in}}}(p, z) = \Pi_{0\Gamma_i}^{\mathscr{A}_i^{\text{in}}}(p, z) = (p, z_i + \omega_i\delta_1) \tag{3.3}$$

for all small $\epsilon \geq 0$.

Next we investigate the dynamics near σ_i. Let $\Omega = \mathbb{R}^n \times (z_{\min}, z_{\max})$. We define an ϵ-dependent chart $\kappa_{\epsilon i}^{(31)}$ on Ω by

$$\kappa_{\epsilon i}^{(31)}(p, z) = (p, \zeta) \quad \text{with} \quad \zeta = \epsilon \ln\left(\frac{\omega_i}{z - z_i}\right).$$

In this chart system (1.1) with $h = 0$ is converted to

$$\begin{aligned} p' &= f(p, z, \epsilon), \\ \zeta' &= -\omega_i \frac{g(p, z, \epsilon)}{z - z_i}, \quad \text{where } z = z_i + \epsilon\,\omega_i \exp(-\zeta_i/\epsilon). \end{aligned} \tag{3.4}$$

Formally, the limiting system of (3.4) as $\epsilon \to 0$ with $z = z_i + o(\epsilon)$ is

$$\begin{aligned} p' &= f(p, z_i, 0), \\ \zeta' &= -\omega_i \frac{\partial g}{\partial z}(p, z_i, 0). \end{aligned} \tag{3.5}$$

Let $\widehat{\Phi}_i$ to be the solution operator of (3.5). Let

$$\mathscr{A}_i = \mathbb{B}(A_i, \delta_4) \tag{3.6}$$

and

$$\widehat{\mathscr{A}}_i^{\text{out}} = \widehat{\Phi}_i(\mathscr{A}_i \times \{0\}, \delta_3), \tag{3.7}$$

where $\delta_3 > 0$ and $\delta_4 > 0$ are constants to be determined. Let $\widehat{\sigma}_i(\tau) = \widehat{\Phi}_i((A_i, \zeta_i), \tau)$, $0 \leq \tau \leq T_i$. Let $\widehat{\Sigma}_i$ be a cross section of the curve $\widehat{\sigma}_i$ at $\widehat{\sigma}_i(T_i/2)$ in $\mathbb{R}^n \times \mathbb{R}$. We denote $\Pi_{0\widehat{\mathscr{A}}_i^{\text{out}}}^{\widehat{\Sigma}}$ the transition map from $\widehat{\mathscr{A}}_i^{\text{out}}$ to $\widehat{\Sigma}_i$ following the flow of (3.5).

PROPOSITION 3.1. *Let $\mathscr{A}_i$ and $\widehat{\mathscr{A}}_i^{\text{out}}$ be defined by* (3.6) *and* (3.7)*, respectively. For any fixed $\delta_3 > 0$, if $\delta_4 > 0$ is sufficiently small, then the transition map $\Pi_{\epsilon\widehat{\mathscr{A}}_i^{\text{out}}}^{\widehat{\Sigma}}$ from $\widehat{\mathscr{A}}_i^{\text{out}}$ to $\widehat{\Sigma}_i$ for system* (3.4) *is well-defined for all small $\epsilon > 0$. Moreover, $\Pi_{\epsilon\widehat{\mathscr{A}}_i^{\text{out}}}^{\widehat{\Sigma}}$ is $O(\epsilon)$-close to $\Pi_{0\widehat{\mathscr{A}}_i^{\text{out}}}^{\widehat{\Sigma}}$ in the $C^1(\widehat{\mathscr{A}}_i^{\text{out}})$-norm as $\epsilon \to 0$.*

Proof. Let Σ be the image of $\widehat{\Sigma}$ via the projection $(p, \zeta) \mapsto p$. Since the trajectory σ_i of (1.4) connects A_i and Σ_i, we can choose $\Delta > 0$ such that the transition map from $\mathscr{A}_i$ to Σ_i whenever $\delta_4 > 0$ is sufficiently small.

Note that the p-component of $\widehat{\Phi}_i(A_i,\tau)$ equals $\sigma_i(\tau)=\Phi_i(A_i,\tau)$ in Assumption 3. Also note that Assumption 5 gives

$$\inf\left\{\zeta : (p,\zeta)\in\widehat{\Phi}_i((A_i,0),\tau),\ \tau\in[\delta_3,\tau_i-\delta_3]\right\}>0.$$

Therefore, by decreasing Δ if necessary, for $\mathscr{A}_i$ defined by (3.6) with $\delta_3\in(0,\Delta)$,

$$\inf\left\{\zeta : (p,\zeta)\in\widehat{\Phi}_i((p_0,0),\tau),\ p_0\in\mathscr{A}_i,\ \tau\in[\delta_3,\tau_i-\delta_3]\right\}>C \tag{3.8}$$

for some $C>0$. Substituting (3.8) into (3.4), we have

$$\begin{aligned} p' &= f(p,z_i,0)+O\big(\epsilon+e^{-C/\epsilon}/\epsilon\big),\\ \zeta' &= -\omega_i\frac{\partial g}{\partial z}(p,z_i,0)+O(\epsilon). \end{aligned} \tag{3.9}$$

Hence (3.4) is a regular perturbation of (3.5) in a neighborhood of the set

$$\{\widehat{\Phi}(x,\tau) : x\in\widehat{\mathscr{A}}_i^{\,\mathrm{out}},\tau\in[0,\tau_i-2\delta_3]\}.$$

Therefore, by regular perturbation theory, $\Pi^{\widehat{\Sigma}}_{\epsilon\widehat{\mathscr{A}}_i^{\mathrm{out}}}$ is well-defined for small $\epsilon>0$ and is $O(\epsilon)$-close to $\Pi^{\widehat{\Sigma}}_{0\widehat{\mathscr{A}}_i^{\mathrm{out}}}$ in the $C^1(\widehat{\mathscr{A}}_i^{\,\mathrm{out}})$-norm as $\epsilon\to 0$. ∎

Finally we investigate the dynamics near the union of γ_i and σ_i. We define

$$\kappa^{(21)}_{\epsilon i}(p,z)=(p,z,\zeta)\quad\text{with}\quad \zeta=\epsilon\ln\left(\frac{\omega_i}{z-z_i}\right)\quad\text{for}\quad (p,z)\in\Omega,\ \epsilon\geq 0.$$

Note that $\kappa^{(21)}_{\epsilon i}(p,z)=(p,z,\zeta)$ can be obtained by appending z to $\kappa^{(31)}_{\epsilon i}(p,z)=(p,\zeta)$. The transformation $\kappa^{(21)}_{\epsilon i}$ converts system (1.1) with $h=0$ to

$$\begin{aligned} \dot p &= \epsilon f(p,z,\epsilon),\\ \dot z &= g(p,z,\epsilon),\\ \dot\zeta &= -\epsilon\,\omega_i\,\frac{g(p,z,\epsilon)}{z-z_i}. \end{aligned} \tag{3.10}$$

We define

$$\kappa^{(21)}_{\epsilon i}(p,z,\zeta)=(p,z)$$

and

$$\widetilde{\mathscr{A}}^{\,\mathrm{in}}_{\epsilon i}=\kappa^{(12)}_{\epsilon i}(\mathscr{A}_i^{\,\mathrm{in}})\quad\text{for }\epsilon\geq 0, \tag{3.11}$$

which means that

$$\widetilde{\mathscr{A}}^{\,\mathrm{in}}_{\epsilon i}=\left\{(p,z,\zeta) : p\in\mathbb{B}(p^{\mathrm{in}}_{0i},\delta_2),\ z=z_i+\omega_i\delta_1,\ \zeta=\epsilon\ln\delta_1\right\}.$$

Note that $\kappa^{(21)}_{0i}(p,z)=(p,z,0)$ for all $(p,z)\in\mathscr{A}_i^{\,\mathrm{in}}$.

Taking $\epsilon\to 0$ in (3.10) leads to system (1.3) companioned with $\dot\zeta=0$. By Assumptions 2 and 4, the projection

$$\Pi^{\mathscr{A}_i}_{0\,\mathscr{A}_i^{\mathrm{in}}} : \mathscr{A}_i^{\,\mathrm{in}}\to\mathscr{A}_i\times\{z_i\}$$

following the flow of (1.3) is well-defined and is a local homeomorphism. We define $\Pi^{\widetilde{\mathscr{A}}_{0i}}_{0\,\widetilde{\mathscr{A}}_{0i}^{\text{in}}} = \Pi^{\mathscr{A}_i}_{0\,\mathscr{A}_i^{\text{in}}} \times \text{id}$, which means

$$\Pi^{\widetilde{\mathscr{A}}_{0i}}_{0\,\widetilde{\mathscr{A}}_{0i}^{\text{in}}}(p, z, \zeta_i) = \left(\Pi^{\mathscr{A}_i}_{0\,\mathscr{A}_i}(p, z), \zeta_i\right).$$

In the slow time variable $\tau = \epsilon t$, taking $\epsilon \to 0$ in (3.10) with $z = z_i + o(\epsilon)$ leads to (3.5) appended by the equation $z = z_i$. We define $\widetilde{\Phi}_i\big((p, z_i, \zeta), \tau\big)$ on $\widetilde{\mathscr{A}}_{0i} \times [0, \tau_i]$ to be the image of $\widehat{\Phi}\big((p, \zeta), \tau\big)$ in the space $\{(p, z, \zeta) : z = z_i\}$. Also we define

$$\kappa^{(23)}_{\epsilon i}(p, \zeta) = (p, z, \zeta) \quad \text{with } z = z_i + \omega_i\, e^{\zeta/\epsilon}$$

and

$$\widetilde{\mathscr{A}}_{\epsilon i}^{\text{out}} = \kappa^{(23)}_{\epsilon i}\left(\widehat{\mathscr{A}}_i^{\text{out}}\right) \quad \text{for } \epsilon > 0. \tag{3.12}$$

Note that $\Pi^{\widetilde{\mathscr{A}}_{0i}^{\text{out}}}_{0\,\widetilde{\mathscr{A}}_{0i}} = \widetilde{\Phi}_i(\cdot, \delta_3)$ by (3.7).

Proposition 3.2. *There exists $\Delta > 0$ such that the following assertions hold. Let $\widetilde{\mathscr{A}}_i^{\text{in}}$ and $\widetilde{\mathscr{A}}_i^{\text{out}}$ be defined by* (3.11) *and* (3.12) *with $\delta_j < \Delta$, $j = 1, 2, 3$. Then for all sufficiently small $\delta_4 > 0$, the transition map $\Pi^{\widetilde{\mathscr{A}}_{\epsilon i}^{\text{out}}}_{\epsilon\,\widetilde{\mathscr{A}}_{\epsilon i}^{\text{in}}}$ from $\widetilde{\mathscr{A}}_{\epsilon i}^{\text{in}}$ to $\widetilde{\mathscr{A}}_{\epsilon i}^{\text{out}}$ following the flow of* (3.10) *is well-defined for all small $\epsilon > 0$. Moreover,*

$$\left\| \Pi^{\widetilde{\mathscr{A}}_{\epsilon i}^{\text{out}}}_{\epsilon\,\widetilde{\mathscr{A}}_{\epsilon i}^{\text{in}}} \circ \kappa^{(21)}_{\epsilon i} - \Pi^{\widetilde{\mathscr{A}}_{0i}^{\text{out}}}_{0\,\widetilde{\mathscr{A}}_{0i}} \circ \Pi^{\widetilde{\mathscr{A}}_{0i}}_{0\,\widetilde{\mathscr{A}}_{0i}^{\text{in}}} \circ \kappa^{(21)}_{0i} \right\|_{C^1(\mathscr{A}_i^{\text{in}})} = O(\epsilon) \tag{3.13}$$

as $\epsilon \to 0$.

A schematic diagram representing Proposition 3.2 is shown in Figure 9. The significance in estimate (3.13) is that the transition map $\Pi^{\mathscr{A}_{\epsilon i}^{\text{out}}}_{\epsilon\,\mathscr{A}_{\epsilon i}^{\text{in}}}$ can be approximated by the composition function of $\Pi^{\widetilde{\mathscr{A}}_{0i}^{\text{out}}}_{0\,\widetilde{\mathscr{A}}_{0i}}$ and $\Pi^{\widetilde{\mathscr{A}}_{0i}}_{0\,\widetilde{\mathscr{A}}_{0i}^{\text{in}}}$, which are determined only by the limiting systems.

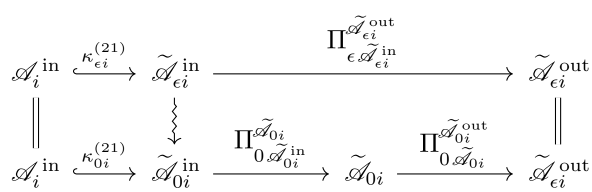


Fig. 9. *A schematic diagram representing Proposition* 3.2. *Here $\hookrightarrow$ indicates injection and $\rightsquigarrow$ indicates the limit as $\epsilon \to 0$. The transition map from $\widetilde{\mathscr{A}}_{\epsilon i}^{\text{in}}$ to $\widetilde{\mathscr{A}}_{\epsilon i}^{\text{out}}$ along* (3.10) *is approximated by the composition function of transition maps for the limiting systems.*

To prove Proposition 3.2, we consider in general a system for $(a, b) \in \mathbb{R}^n \times \mathbb{R}$, $n \geq 1$, of the form

$$\begin{aligned} \dot{a} &= \epsilon F(a, b, \epsilon) + b\, H(a, b, \epsilon), \\ \dot{b} &= b\, G(a, b, \epsilon), \end{aligned} \tag{3.14}$$

where $\cdot$ denotes $\frac{d}{dt}$, and F, G, and H are smooth functions. Note that the expression (3.14) is identical to (1.10), but the variable a is a vector in (3.14) and is a scalar in

(1.10). In this section we will only use system (3.14) with $H = 0$ but will consider general H for convenience in next section.

The limiting fast system, obtained by setting $\epsilon \to 0$ in (3.14) is

$$\begin{aligned} \dot{a} &= b\,H(a,b,0), \\ \dot{b} &= b\,G(a,b,0), \end{aligned} \tag{3.15}$$

which has a line of equilibria $\{b = 0\}$. The limiting slow system on $\{b = 0\}$ is

$$a' = F(a,0,0). \tag{3.16}$$

The following is a variation of the exchange lemma in Jones and Tin [33] and Schecter [52].

LEMMA 3.3. *Consider system* (3.14) *for* $(a,b) \in \mathbb{R}^n \times \mathbb{R}$. *Assume that for some* $\bar{a} \in \mathbb{R}^n$ *satisfies that* $G(\bar{a},0,0) < 0$ *and that the point* $(\bar{a},0)$ *is the omega limit point of a trajectory* γ *of system* (3.15). *Then there exists* $\Delta > 0$ *such that the following assertions hold:*

Suppose that $\{\mathscr{A}_\epsilon^{\text{in}}\}_{\epsilon\in[0,\epsilon_0]}$ *is a smooth family of* ℓ*-dimensional manifolds,* $0 \le \ell \le N$*, that intersects* γ *at a point in* $\mathbb{B}((\bar{a},0),\Delta)$ *and satisfies that*

(H1) $\mathscr{A}_0^{\text{in}}$ *is nontangential to the flow of* (3.15),

and the projection $\Lambda \subset \mathbb{R}^n$ *of* $\mathscr{A}_0^{\text{in}}$ *along the flow of system* (3.15) *satisfies that*

(H2) $\bar{a} \in \Lambda$ *and* Λ *is compact and is nontangential to the flow of* (3.16).

Let Φ *the solution operator for the system* (3.16). *Let* $\iota_\epsilon : K \to \mathscr{A}_\epsilon^{\text{in}}$ *be a smooth parameterization of* $\mathscr{A}_\epsilon^{\text{in}}$ *for* $\epsilon \in [0,\epsilon_0]$*, where* K *is an* ℓ*-dimensional manifold. Let* $\bar{x} \in \mathscr{A}_0^{\text{in}} \cap \gamma$ *be the preimage of* $\bar{a}$ *along* (3.15) *and* $\bar{k} \in K$ *be the preimage of* $\bar{x}$ *by* ι_0. *If* $\tau_1 > 0$ *satisfies that*

(H3) *the trajectory* $\sigma = \Phi(\bar{a},[0,\tau_1])$ *lies in* $\mathbb{B}(\bar{a},\Delta)$ *and is rectifiable and not self-intersecting,*

and $\mathscr{A}^{\text{out}}$ *is an* n*-dimensional manifold that intersects transversally at an interior point of* $\sigma \times \{0\}$ *in* $\mathbb{R}^n \times \mathbb{R}$*, then there is an open neighborhood* V *of* $\bar{k}$ *in* K *such that the transition map* $\Pi_{\epsilon\,\mathscr{A}_\epsilon^{\text{in}}}^{\mathscr{A}^{\text{out}}}$ *from* $\iota_\epsilon(V) \subset \mathscr{A}_\epsilon^{\text{in}}$ *to* $\mathscr{A}^{\text{out}}$ *following the flow of* (3.14) *is well defined for all sufficiently small* $\epsilon > 0$. *Moreover,*

$$\left\| \Pi_{\epsilon\,\mathscr{A}_\epsilon^{\text{in}}}^{\mathscr{A}^{\text{out}}} \circ \iota_\epsilon - \Pi_{0\Lambda}^{\mathscr{A}^{\text{out}}} \circ \Pi_{0\,\mathscr{A}_0^{\text{in}}}^{\Lambda} \circ \iota_0 \right\|_{C^1(V)} = O(\epsilon) \tag{3.17}$$

as $\epsilon \to 0$*, where* $\Pi_{0\,\mathscr{A}_0}^{\Lambda}$ *is the transition map from* $\mathscr{A}_0$ *to* Λ *along the flow of* (3.15)*, and* $\Pi_{0\Lambda}^{\mathscr{A}^{\text{out}}}$ *is the transition map from* Λ *to* $\mathscr{A}^{\text{out}} \cap \{b = 0\}$ *along the flow of* (3.16).

Proof of Lemma 3.3. Using a Fenichel type coordinate (see Jones [32]), in the open ball $\mathbb{B}(0,2\Delta)$ in the (a,b)-space, for sufficiently small $\Delta > 0$ we can choose an ϵ-dependent change of variables $(a,b) \mapsto (\tilde{a},\tilde{b})$ with

$$(\tilde{a},\tilde{b})\big|_{b=0} = (a,0)$$

such that system (3.14) is converted to

$$\begin{aligned} \dot{\tilde{a}} &= \epsilon \tilde{F}(\tilde{a},\epsilon), \\ \dot{\tilde{b}} &= \tilde{b}\,\tilde{G}(\tilde{a},\tilde{b},\epsilon). \end{aligned} \tag{3.18}$$

We will drop the tildes in the rest of the proof. We write

$$\mathscr{A}_\epsilon^{\text{in}} = \{(a,b) : a \in \Lambda, b = \beta_\epsilon(a)\}.$$

Since $\mathscr{A}^{\text{out}}$ intersects σ transversally, for some neighborhood U of $\bar{a}$ in $\mathbb{R}^n$, we can write

$$\Pi_{0\Lambda}^{\mathscr{A}^{\text{out}}}(a) = \Phi(a, T_0(a)) \quad \forall\, a \in \Lambda \cap U,$$

where T_0 is a smooth function with $\tau_- < T_0 < \tau_+$ for some $\tau_-, \tau_+ \in (0, \tau_1)$. To prove (3.17), it suffices to show that

$$\left\| \Pi_{\epsilon \mathscr{A}_\epsilon^{\text{in}}}^{\mathscr{A}^{\text{out}}}(a, \beta_\epsilon(a)) - \big(\Phi(a, T_0(a)), 0\big) \right\|_{C^1(\Lambda \cap U)} = O(\epsilon) \tag{3.19}$$

as $\epsilon \to 0$. Let $(a_\epsilon, b_\epsilon)(t; a_0)$ be the solution of (3.18) at time t with initial data $(a_0, \beta_\epsilon(a_0))$. Define

$$(a_{\epsilon 1}, b_{\epsilon 1})(a_0, \tau) = (a_\epsilon, b_\epsilon)(\tau/\epsilon; a_0) \quad \text{for } \ a_0 \in \Lambda_1, \tau \in [\tau_-, \tau_+]. \tag{3.20}$$

By the general exchange lemma (see Schecter [52]),

$$\big\|(a_{\epsilon 1}, b_{\epsilon 1})(a_0, \tau) - \big(\Phi(a_0, \tau), 0\big)\big\|_{C^1(\Lambda_1 \times [\tau_-, \tau_+])} = O(\epsilon) \tag{3.21}$$

as $\epsilon \to 0$. Since the graph of $\big(\Phi(a_0, \tau), 0\big)$ is transversal to $\mathscr{A}^{\text{out}}$, it follows from the implicit function theorem that there exists a function $T_\epsilon(a_0)$ defined for all small $\epsilon > 0$ such that

$$\|T_\epsilon - T_0\|_{C^1(\Lambda \cap U)} = O(\epsilon) \tag{3.22}$$

and

$$(a_{\epsilon 1}, b_{\epsilon 1})(a_0, T_\epsilon(a_0)) \in \mathscr{A}^{\text{out}} \quad \forall\, a_0 \in \Lambda \cap U.$$

Note that the last relation means that

$$\Pi_{\epsilon \mathscr{A}^{\text{in}}}^{\mathscr{A}^{\text{out}}}(a_0, \delta) = (a_{\epsilon 1}, b_{\epsilon 1})(a_0, T_\epsilon(a_0)). \tag{3.23}$$

From (3.21), (3.22), and (3.23) we then obtain (3.19). □

Proof of Proposition 3.2. Setting $s = z - z_i$ in system (3.10) yields

$$\begin{aligned} \dot{p} &= \epsilon f(p, z_i + s, 0) + O(|(\epsilon, s)|^2), \\ \dot{s} &= s \frac{\partial g}{\partial z}(p, z_i, 0) + O(|(\epsilon, s)|^2), \\ \dot{\zeta} &= \epsilon \frac{\partial g}{\partial z}(p, z_i, 0) + O(|(\epsilon, s)|^2) \end{aligned}$$

as $(\epsilon, s) \to 0$. Note that system (3.14) with $H = 0$ can be written as

$$\begin{aligned} \dot{a} &= \epsilon F(a, b, 0) + O(|(\epsilon, b)|^2), \\ \dot{b} &= b\, G(a, b, 0) + O(|(\epsilon, b)|^2). \end{aligned}$$

Since $\frac{\partial g}{\partial z}(A_i, z_i, 0) < 0$ by Assumption 5, applying Lemma 3.3 with $b = s$, $a = (p, \zeta)$, $F = (f, \partial g/\partial z)$, and $G = \partial g/\partial z$, we obtain (3.13). □

We denote $\Pi_{0\Gamma_i}^{\mathscr{A}_i}$ the transition map from Γ_i to $\mathscr{A}_i \times \{z_i\}$ along the flow of (1.3) and $\Pi_{0\widehat{\mathscr{A}}_i}^{\widehat{\Sigma}_i}$ the transition map from $\widehat{\mathscr{A}}_i$ to $\widehat{\Sigma}_i$ along the flow of (3.5).

Proposition 3.4. *There exist $\delta_j > 0$, $0 \le j \le 4$, such that if Γ_i, $\mathscr{A}_i$, Σ_i are defined in the preceding paragraphs, then the transition map $\Pi^{\Sigma_i}_{\epsilon\Gamma_i}$ from Γ_i to Σ_i following the flow of* (1.1) *is well-defined for all small $\epsilon > 0$, and*

$$\left\| \kappa^{(31)}_{\epsilon i} \circ \Pi^{\Sigma_i}_{\epsilon\Gamma_i} - \Pi^{\widehat{\Sigma}_i}_{0\widehat{\mathscr{A}}_i} \circ \kappa^{(31)}_{0i} \circ \Pi^{\mathscr{A}_i}_{0\Gamma_i} \right\|_{C^1(\Gamma_i)} = O(\epsilon) \tag{3.24}$$

as $\epsilon \to 0$.

Proof of Proposition 3.4. First we fix constants δ_1, δ_2, and δ_3 in $(0, \Delta)$, where Δ is the number given in in Propositions 3.2. Then we choose positive constants δ_0 and δ_4, such that (3.3) and the results in Proposition 3.1 hold. Then

$$\begin{aligned}\Pi^{\Sigma_i}_{\epsilon\Gamma_i} &= \Pi^{\Sigma_i}_{\epsilon\,\mathscr{A}_i^{\text{out}}} \circ \Pi^{\mathscr{A}_i^{\text{out}}}_{\epsilon\,\mathscr{A}_i^{\text{in}}} \circ \Pi^{\mathscr{A}_i^{\text{in}}}_{\epsilon\Gamma_i} \\ &= \left(\kappa^{(13)}_{\epsilon i} \circ \Pi^{\widehat{\Sigma}_i}_{\epsilon\,\widehat{\mathscr{A}}_i^{\text{out}}} \circ \kappa^{(31)}_{\epsilon i}\right) \circ \left(\kappa^{(12)}_{\epsilon i} \circ \Pi^{\widetilde{\mathscr{A}}_{\epsilon i}^{\text{out}}}_{\epsilon\,\widetilde{\mathscr{A}}_{\epsilon i}^{\text{in}}} \circ \kappa^{(21)}_{\epsilon i}\right) \circ \Pi^{\mathscr{A}_i^{\text{in}}}_{\epsilon\Gamma_i}.\end{aligned}$$

From (3.3) and Propositions 3.1 and 3.2, it follows that

$$\begin{aligned}\Pi^{\Sigma_i}_{\epsilon\Gamma_i} &= \left(\kappa^{(13)}_{0i} \circ \Pi^{\widehat{\Sigma}_i}_{0\,\widehat{\mathscr{A}}_i^{\text{out}}} \circ \kappa^{(31)}_{0i}\right) \circ \left(\kappa^{(12)}_{0i} \circ \Pi^{\widetilde{\mathscr{A}}_i^{\text{out}}}_{0\,\widetilde{\mathscr{A}}_i} \circ \Pi^{\widetilde{\mathscr{A}}_{0i}}_{0\,\widetilde{\mathscr{A}}_{0i}^{\text{in}}} \circ \kappa^{(21)}_{0i}\right) + O(\epsilon) \\ &= \kappa^{(13)}_{0i} \circ \left(\Pi^{\widehat{\Sigma}_i}_{0\,\widehat{\mathscr{A}}_i^{\text{out}}} \circ \kappa^{(32)}_{0i} \circ \Pi^{\widetilde{\mathscr{A}}_i^{\text{out}}}_{0\,\widetilde{\mathscr{A}}_i}\right) \circ \left(\Pi^{\widetilde{\mathscr{A}}_{0i}}_{0\,\widetilde{\mathscr{A}}_{0i}^{\text{in}}} \circ \kappa^{(21)}_{0i} \circ \Pi^{\mathscr{A}_i^{\text{in}}}_{0\Gamma_i}\right) + O(\epsilon).\end{aligned}$$

Since

$$\Pi^{\widehat{\Sigma}_i}_{0\,\widehat{\mathscr{A}}_i^{\text{out}}} \circ \kappa^{(32)}_{0i} \circ \Pi^{\widetilde{\mathscr{A}}_i^{\text{out}}}_{0\,\widetilde{\mathscr{A}}_i} = \Pi^{\widehat{\Sigma}_i}_{0\,\widehat{\mathscr{A}}_i} \circ \kappa^{(23)}_{0i}$$

and

$$\Pi^{\widetilde{\mathscr{A}}_{0i}}_{0\,\widetilde{\mathscr{A}}_{0i}^{\text{in}}} \circ \kappa^{(21)}_{0i} \circ \Pi^{\mathscr{A}_i^{\text{in}}}_{0\Gamma_i} = \kappa^{(21)}_{0i} \circ \Pi^{\mathscr{A}_i}_{0\Gamma_i},$$

it follows that

$$\begin{aligned}\Pi^{\Sigma_i}_{\epsilon\Gamma_i} &= \kappa^{(13)}_{0i} \circ \left(\Pi^{\widehat{\Sigma}_i}_{0\,\widehat{\mathscr{A}}_i} \circ \kappa^{(32)}_{0i}\right) \circ \left(\kappa^{(21)}_{0i} \circ \Pi^{\mathscr{A}_i}_{0\Gamma_i}\right) + O(\epsilon) \\ &= \kappa^{(13)}_{0i} \circ \Pi^{\widehat{\Sigma}_i}_{0\,\widehat{\mathscr{A}}_i} \circ \kappa^{(31)}_{0i} \circ \Pi^{\mathscr{A}_i}_{0\Gamma_i} + O(\epsilon).\end{aligned}$$

Applying both sides of equation by $\kappa^{(31)}_{0i}$ yields (3.24). ∎

Proof of Theorem 2.5. By a reversal of the time variable and applying Proposition 3.4, we obtain

$$\left\| \kappa^{(31)}_{\epsilon i} \circ \Pi^{\Sigma_i}_{\epsilon\Gamma_{i+1}} - \Pi^{\widehat{\Sigma}_i}_{0\widehat{\mathscr{B}}_i} \circ \kappa^{(31)}_{0i} \circ \Pi^{\mathscr{B}_i}_{0\Gamma_{i+1}} \right\|_{C^1(\Gamma_{i+1})} = O(\epsilon).$$

Taking the inverse of the mappings in the last equation, we obtain

$$\left\| \Pi^{\Sigma_i}_{\epsilon\Gamma_i} \circ \kappa^{(13)}_{\epsilon i} - \Pi^{\Gamma_{i+1}}_{0\mathscr{B}_i} \circ \kappa^{(13)}_{0i} \circ \Pi^{\widehat{\mathscr{B}}_i}_{0\widehat{\Sigma}_i} \right\|_{C^1(\widehat{\Sigma}_i)} = O(\epsilon). \tag{3.25}$$

By (3.24) and (3.25), it follows that

$$\begin{aligned}\Pi^{\Gamma_{i+1}}_{\epsilon\Gamma_i} &= \left(\Pi^{\Sigma_i}_{\epsilon\Gamma_i} \circ \kappa^{(13)}_{i\epsilon}\right) \circ \left(\kappa^{(31)}_{i\epsilon} \circ \Pi^{\Sigma_i}_{\epsilon\Gamma_i}\right) \\ &= \left(\Pi^{\Gamma_{i+1}}_{0\mathscr{B}_i} \circ \kappa^{(13)}_{0i} \circ \Pi^{\widehat{\mathscr{B}}_i}_{0\widehat{\Sigma}_i}\right) \circ \left(\Pi^{\widehat{\Sigma}_i}_{0\,\widehat{\mathscr{A}}_i} \circ \kappa^{(31)}_{0i} \circ \Pi^{\mathscr{A}_i}_{0\Gamma_i}\right) + O(\epsilon) \\ &= \Pi^{\Gamma_{i+1}}_{0\mathscr{B}_i} \circ \kappa^{(13)}_{0i} \circ \Pi^{\widehat{\mathscr{B}}_i}_{0\,\widehat{\mathscr{A}}_i} \circ \kappa^{(31)}_{0i} \circ \Pi^{\mathscr{A}_i}_{0\Gamma_i} + O(\epsilon).\end{aligned} \tag{3.26}$$

Define $\varrho(p,z)=p$. Since we assumed $h=0$ in (1.3), it follows that

$$\varrho \circ \Pi_{\epsilon\mathscr{B}_i}^{\mathscr{A}_{i+1}}(p,z)=p \quad \forall\,(p,z)\in\mathscr{B}_i.$$

Hence (3.26) implies that

$$\varrho \circ \Pi_{\epsilon\Gamma_i}^{\Gamma_{i+1}} = Q_i + O(\epsilon), \tag{3.27}$$

where Q_i is defined in (2.9). (Here and in the rest of the proof we identify Γ_i with $\mathbb{R}^n$ since the z-coordinate is constant on each Γ_i.) Let

$$P_\epsilon = \Pi_{\epsilon\Gamma_N}^{\Gamma_1}\circ\cdots\circ\Pi_{\epsilon\Gamma_2}^{\Gamma_3}\circ\Pi_{\epsilon\Gamma_1}^{\Gamma_2}.$$

Then

$$\varrho\circ P_\epsilon = Q_N\circ\cdots\circ Q_2\circ Q_1 + O(\epsilon) = P + O(\epsilon),$$

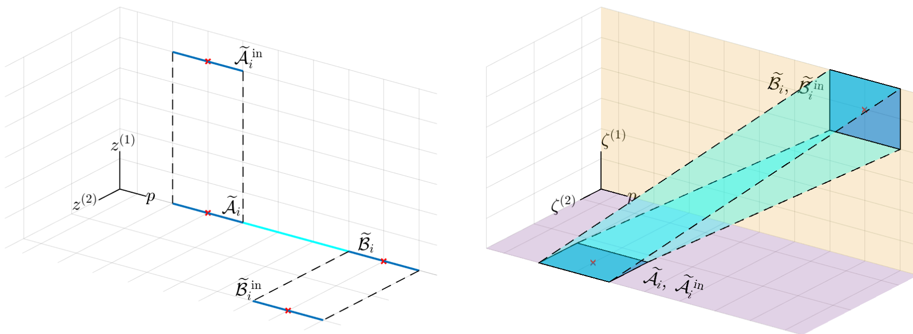
where P is defined by (2.15). Since the z-component on Γ_1 is a constant, we conclude that

$$\det(P_\epsilon - \mathrm{id}) = \det(DP - \mathrm{id}) + O(\epsilon).$$

Since we assumed $\det(DP(A)-\mathrm{I}_n)\neq 0$, it follows that $\det(P_\epsilon-\mathrm{id})\neq 0$ for all small $\epsilon>0$. Consequently, for all small $\epsilon>0$ there exists a locally unique fixed point $p_\epsilon\in\Gamma_1$ of P_ϵ. Then the trajectory passing through (p_ϵ, z_{01}) is a periodic orbit of system (1.1), where z_{01} is given in (3.1) with $i=1$, and $p_\epsilon\to A_1$ as $\epsilon\to 0$. If $r(DP(A_1))<1$ (resp., $r(DP(A_1))>1$), then P_ϵ is a contraction (resp., expansion), and hence the periodic orbit is orbitally asymptotically stable (resp., unstable). This proves the theorem. ∎

3.2. Proof of Theorem 2.7. The approach in this section is to generalize the proof of Theorem 2.5. First we give a heuristic explanation of our approach. To show the idea, we assume that $z=(z^{(1)},z^{(2)})$ (i.e., $m=2$) and that, for some index i, $J_i=1$ and $J_{i+1}=2$ (see Figure 10(a)). Since $J_i\neq 2$, we have $z_{i-1}^{(2)}=z_i^{(2)}$, so the value $|z^{(2)}-z_i^{(2)}|$ is expected to remain small during the transition from M_{i-1} to M_i, which suggests that the blow-up variable $\zeta^{(2)}=\epsilon\ln(1/|z^{(2)}-z_i^{(2)}|)$ remains valid throughout this transition. Hence, we transform $\mathscr{A}_i^{\mathrm{in}}$ from the $(p,z^{(1)},z^{(2)})$-space into the $(p,z^{(1)},\zeta^{(1)},\zeta^{(2)})$-space, which has ϵ-dependent coordinates. The image of $\mathscr{A}_i^{\mathrm{in}}$ in this blow-up space is denoted by $\widetilde{\mathscr{A}_i}^{\mathrm{in}}$. Since $z^{(1)}$ is away from the value $z_i^{(1)}$ on $\mathscr{A}_i^{\mathrm{in}}$, in the limit $\epsilon=0$ the $\zeta^{(1)}$-coordinate equals 0 on $\widetilde{\mathscr{A}_i}^{\mathrm{in}}$ (see Figure 10(b)). The fact that $\zeta^{(2)}$ is defined in the vicinity of the union of M_{i-1} and M_i implies that the limiting value of $\zeta^{(2)}$ on $\widetilde{\mathscr{A}_i}^{\mathrm{in}}$ as $\epsilon\to 0$ is $\zeta_i^{(2)}=\zeta_{i-1}^{(2)}-\int_{\sigma_{i-1}}\frac{\partial g^{(2)}}{\partial z^{(2)}}\,dt$, as defined in (2.10). We denote $\widetilde{A}_i$ the projection of $\widetilde{A}_i^{\mathrm{in}}$ along the extended fast system into M_i. Then the manifold evolved from $\widetilde{A}_i$ with $\epsilon=0$ has constant $z_i^{(1)}$-coordination. Hence, we drop the $z_i^{(1)}$-coordination and adapt the $(p,\zeta^{(1)},\zeta^{(2)})$-coordinates in the vicinity of M_i to ensure that the evolved manifold has the full dimension. Therefore, the treatment of M_i consists of the following three steps:

1. Use the $(p,z^{(1)},\zeta^{(2)})$-coordinates to track the transition from Γ_i to another manifold $\mathscr{A}_i^{\mathrm{in}}$ that is closer to M_i. (This corresponds to Proposition 3.5.)
2. Use the $(p,z^{(1)},\zeta^{(1)},\zeta^{(2)})$-coordinates to track the manifold evolved from $\mathscr{A}_i^{\mathrm{in}}$. (This corresponds to Proposition 3.7.)
3. Use the $(p,\zeta^{(1)},\zeta^{(2)})$-coordinates to track trajectories in the vicinity of M_i. (This corresponds to Proposition 3.6.)



(a) (b)

FIG. 10. (a) *In the (p,z)-space, with $\epsilon = 0$ the image of $\widehat{\mathscr{A}_i}^{\rm in}$ (dark segment with $z^{(1)} > 0$) and the image of the manifold evolved (light segment in the p-axis) from $\widehat{\mathscr{A}_i}$ both have dimension n.* (b) *In the (p,ζ)-space with $\epsilon = 0$, the image of $\widehat{\mathscr{A}_i}^{\rm in}$ has dimension $n+m-1$ and the image of the evolved manifold has dimension $n+m$.*

TABLE 2
Notation in section 3.2.

Variables	Charts	Objects
$(p,z) \in \Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ with $z^{(j)} \in (z^{(j)}_{\rm min}, z^{(j)}_{\rm max})$	$\kappa^{(01)}_{\epsilon i}(p,q,\widehat{\zeta}) = (p,z)$	Ω, $\overline{\Gamma}_i$
$p \in \mathbb{R}^m$		$\mathscr{A}_i$, $\mathscr{B}_i$
$(p,q,\widehat{\zeta}) \in \mathbb{R}^n \times \left(z^{(J_i)}_{\rm min}, z^{(J_i)}_{\rm max}\right) \times \mathbb{R}^{m-1}$	$\kappa^{(10)}_{\epsilon i}(p,z) = (p,q,\widehat{\zeta})$ $\kappa^{(13)}_{\epsilon i}(p,\zeta) = (p,z,\widehat{\zeta})$	Γ_i, $\mathscr{A}_i^{\rm in}$, $\mathscr{A}_i^{\rm out}$
$(p,q,\zeta) \in \mathbb{R}^n \times \left(z^{(J_i)}_{\rm min}, z^{(J_i)}_{\rm max}\right) \times \mathbb{R}^m$	$\kappa^{(21)}_{\epsilon i}(p,q,\widehat{\zeta}) = (p,q,\zeta)$ $\kappa^{(23)}_{\epsilon i}(p,\zeta) = (p,q,\zeta)$	$\widetilde{\mathscr{A}}_i$, $\widetilde{\mathscr{A}}_i^{\rm in}$, $\widetilde{\mathscr{A}}_i^{\rm out}$
$(p,\zeta) \in \mathbb{R}^n \times \mathbb{R}^m$	$\kappa^{(30)}_{\epsilon i}(p,z) = (p,\zeta)$ $\kappa^{(31)}_{\epsilon i}(p,q,\widehat{\zeta}) = (p,\zeta)$	$\widehat{\mathscr{A}}_i^{\rm out}$, $\widehat{\Sigma}_i$

As illustrated by Figure 7, we will set a cross section Σ_i of σ_i and analyze the dynamics between Γ_i and Σ_i. By symmetry, the dynamics between Σ_i and Γ_{i+1} can also be obtained. Near Γ_i we will use the original coordinates $(p,q,\widehat{\zeta})$, where $q = z^{(J_i)}$, and $\widehat{\zeta}$, to be defined later, is a blow-up variable of all but the J_ith components of z. Near Σ_i we will use the coordinates (p,ζ), where ζ is a blow-up variable for z to be defined later; near A_i we will use the coordinates (p,q,ζ) to connect the other two coordinates. Some notation to be used is listed in Table 2. Let

$$\Omega = \mathbb{R}^n \times \left(z^{(1)}_{\rm min}, z^{(1)}_{\rm max}\right) \times \cdots \times \left(z^{(N)}_{\rm min}, z^{(N)}_{\rm max}\right) \subset \mathbb{R}^n \times \mathbb{R}^m,$$

where $z^{(j)}_{\rm min}$ and $z^{(j)}_{\rm max}$ are the numbers given in Assumption 1. Motivated by the classical blow-up method, we define the ϵ-dependent chart on Ω by

$$\kappa^{(10)}_{\epsilon i}(p,z) = (p, z^{(J_i)}, \widehat{\zeta}) \quad \text{with} \quad \widehat{\zeta}^{(j)} = \begin{cases} \zeta_i^{(J_i)} & \text{if } j = J_i, \\ \epsilon \ln \frac{\omega_i^{(j)}}{z^{(j)} - z_i^{(j)}} & \text{if } j \neq J_i, \end{cases}$$

where J_i is the index in Assumption 2. On the curve $(p_i(t), q_i(t)) \subset \mathbb{R}^m \times \mathbb{R}$ in

Assumption 2, since $q_i(t)$ is nonconstant, we can choose a point (p_{0i}, q_{0i}) at which $q_i'(t) \neq 0$. Let

$$\Gamma_i = \left\{(p,q,\widehat{\zeta}) \in \mathbb{R}^n \times \mathbb{R} \times \Lambda_i : |p - p_{0i}| < \delta_0,\ q = q_{0i},\ |\widehat{\zeta} - \zeta_i| < \delta_0\right\}, \tag{3.28}$$

where $\delta_0 > 0$ is to be determined. Let $\overline{\Gamma}_i = \kappa_{\epsilon 1}^{(10)}(\Gamma_i)$. Our strategy is to track the transition map from Γ_i to Γ_{i+1} in the $(p,q,\widehat{\zeta})$-space to find a fixed point of a composition map from Γ_1 to Γ_1 and then convert it back via $\kappa_{\epsilon 1}^{(01)}$ to obtain a periodic orbit passing through $\overline{\Gamma}_1$ in the (p,z)-space.

Let

$$\mathscr{A}_i^{\text{in}} = \{(p,q,\widehat{\zeta}) : p \in \mathbb{B}(p_i^{\text{in}}, \delta_2),\ q = z_i^{(J_i)} + \omega_i \delta_1,\ |\widehat{\zeta} - \widehat{\zeta}_i| < \delta_2\}, \tag{3.29}$$

where δ_1 and δ_2 are positive constants to be determined.

PROPOSITION 3.5. *Let Γ_i and $\mathscr{A}_i^{\text{in}}$ be defined by* (3.28) *and* (3.29)*, respectively. For any fixed $\delta_1 > 0$ and $\delta_2 > 0$, if $\delta_0 > 0$ is sufficiently small, then the transition map $\Pi_{\epsilon\Gamma_i}^{\mathscr{A}_i^{\text{in}}}$ from Γ_i to $\mathscr{A}_i^{\text{in}}$ following the flow of* (1.1) *is well-defined for all small $\epsilon \geq 0$ and is $O(\epsilon)$-close to $\Pi_{0\Gamma_i}^{\mathscr{A}_i^{\text{in}}}$ in the $C^1(\Gamma_i)$-norm as $\epsilon \to 0$.*

Proof. Chart $\kappa_{\epsilon i}^{(10)}$ converts system (1.1) to

$$\begin{aligned}
&\dot{p} = \epsilon f(p,z,\epsilon) + h(p,z,\epsilon),\\
&\dot{q} = g^{(J_i)}(p,z,\epsilon),\\
&\dot{\widehat{\zeta}}^{(j)} = -\epsilon\, \frac{g^{(j)}(p,z,\epsilon)}{z^{(j)} - z_i^{(j)}}, \quad j \in \{1,2,\dots,m\} \setminus \{J_i\},\\
&\text{with} \quad z^{(J_i)} = q \quad \text{and} \quad z^{(j)} = z_i^{(j)} + \omega_i^{(j)} \exp(-\widehat{\zeta}^{(j)}/\epsilon) \text{ for } j \neq J_i.
\end{aligned} \tag{3.30}$$

By Assumption 5, all components of $\widehat{\zeta}_i \in \Lambda_i$ are bounded away from zero. Therefore, for each $j \in \{1,2,\dots,m\} \setminus \{J_i\}$,

$$z_i^{(j)} + \omega_i^{(j)} \exp(-\widehat{\zeta}^{(j)}/\epsilon) \to z_i^{(j)} \quad \text{as } \epsilon \to 0,$$

which implies that

$$\frac{g^{(j)}(p,z,0)}{z^{(j)} - z_i^{(j)}} \to \frac{\partial g^{(j)}}{\partial z^{(j)}}(p, z_{i-1} + q\, \mathsf{e}_{J_i}, 0) \quad \text{as } \epsilon \to 0.$$

Hence the expression of $\dot{\widehat{\zeta}}^{(j)}$ in (3.30) tends to zero as $\epsilon \to 0$. Consequently, (3.30) is a regular perturbation of the system

$$\begin{aligned}
&\dot{p} = h(p, z_{i-1} + q\, \mathsf{e}_{J_{i-1}}, 0),\\
&\dot{q} = g^{(J_i)}(p, z_{i-1} + q\, \mathsf{e}_{J_{i-1}}, 0),\\
&\dot{\widehat{\zeta}}^{(j)} = 0, \quad j \in \{1,2,\dots,m\} \setminus \{J_i\}.
\end{aligned} \tag{3.31}$$

Hence $\Pi_{\epsilon\Gamma_i}^{\mathscr{A}_i^{\text{in}}}$ is well-defined and is $O(\epsilon)$ C^1-close to $\Pi_{0\Gamma_i}^{\mathscr{A}_i^{\text{in}}}$ as $\epsilon \to 0$. ∎

We define charts $\kappa_{\epsilon i}^{(30)}$ for $(p,z)\in\Omega$ by

$$\kappa_{\epsilon i}^{(30)}(p,z) = (p,\zeta)$$
$$\text{with} \quad \zeta^{(j)} = \epsilon \ln \frac{\omega_i^{(j)}}{z^{(j)} - z_i^{(j)}} \quad \text{for } j = 1,2\ldots,m.$$

In this chart system (1.1) is converted to

$$\begin{aligned} &\frac{d}{d\tau}p = f(p,z,\epsilon) + h(p,z,\epsilon)/\epsilon, \\ &\frac{d}{d\tau}\zeta^{(j)} = \frac{-g^{(j)}(p,z,\epsilon)}{z^{(j)} - z_i^{(j)}}, \quad j = 1,2,\ldots,m, \\ &\quad \text{with} \quad z^{(j)} = z_i^{(j)} + \omega_i^{(j)}\exp(-\zeta^{(j)}/\epsilon) \ \text{ for } j = 1,2,\ldots,m. \end{aligned} \tag{3.32}$$

Let $\widehat{\Phi}_i$ be the solution operator of

$$\begin{aligned} &\frac{d}{d\tau}p = f(p,z_i,0), \\ &\frac{d}{d\tau}\zeta^{(j)} = \frac{-\partial g^{(j)}}{\partial z^{(j)}}(p,z_i,0) \quad \text{for } j = 1,2,\ldots,m. \end{aligned} \tag{3.33}$$

Let $\mathscr{A}_i$ and $\mathscr{A}_i^{\text{in}}$ be defined by (3.6) and (3.2), respectively. We define

$$\widehat{\mathscr{A}}_i = \mathscr{A}_i \times \Lambda_i \quad \text{and} \quad \widehat{\mathscr{A}}_i^{\text{out}} = \widehat{\Phi}_i(\widehat{\mathscr{A}}_i, \delta_3), \tag{3.34}$$

where $\delta_3 > 0$ is a constant to be determined. Let $\widehat{\sigma}_i(\tau) = \widehat{\Phi}_i((A_i,\zeta_i),\tau)$, $0 \le \tau \le T_i$. Let $\widehat{\Sigma}_i$ be a cross section of the curve $\widehat{\sigma}_i$ at $\widehat{\sigma}_i(\tau_i/2)$. We denote $\Pi^{\widehat{\Sigma}}_{0\widehat{\mathscr{A}}_i^{\text{out}}}$ the transition map from $\widehat{\mathscr{A}}_i^{\text{out}}$ to $\widehat{\Sigma}_i$ following the flow of (3.5).

Proposition 3.6. *Let $\mathscr{A}_i$ and $\widehat{\mathscr{A}}_i^{\text{out}}$ be defined by* (3.6) *and* (3.34), *respectively. For any fixed $\delta_3 > 0$, if $\delta_4 > 0$ is sufficiently small, then the transition map $\Pi^{\widehat{\Sigma}}_{\epsilon\widehat{\mathscr{A}}_i^{\text{out}}}$ from $\widehat{\mathscr{A}}_i^{\text{out}}$ to $\widehat{\Sigma}_i$ for system* (3.32) *is well-defined for all small $\epsilon > 0$. Moreover, $\Pi^{\widehat{\Sigma}}_{\epsilon\widehat{\mathscr{A}}_i^{\text{out}}}$ is $O(\epsilon)$-close to $\Pi^{\widehat{\Sigma}}_{0\widehat{\mathscr{A}}_i^{\text{out}}}$ in the $C^1(\widehat{\mathscr{A}}_i^{\text{out}})$-norm as $\epsilon \to 0$.*

Proof. By Assumption 5, we have

$$\inf\left\{\zeta^{(j)} : (p,\zeta) = \widehat{\sigma}_i(\tau), \tau \in [\delta_3, \tau_i - \delta_3], j = 1,2,\ldots,m\right\} > C$$

for some $C > 0$. Therefore, similar to the proof of Proposition 3.5, system (3.32) is a regular perturbation of (3.33), and the desired result follows. ∎

Define chart $\kappa_{\epsilon i}^{(20)}$ for $(p,z)\in\Omega$ by

$$\kappa_{\epsilon i}^{(20)}(p,z) = (p,q,\zeta)$$
$$\text{with } \ q = z^{(J_i)} \ \text{ and } \ z^{(j)} = z_i^{(j)} + \omega_i^{(j)}\exp(-\widehat{\zeta}^{(j)}/\epsilon) \ \text{ for } j = 1,2\ldots,m.$$

This chart converts system (1.1) to

$$
\begin{aligned}
&\dot p = \epsilon f(p,z,\epsilon) + h(p,z,\epsilon),\\
&\dot q = g^{(J_i)}(p,z,\epsilon),\\
&\zeta^{(j)} = \epsilon\,\frac{-g^{(j)}(p,z,\epsilon)}{z^{(j)}-z_i^{(j)}}, \quad j=1,2,\dots,m,\\
&\text{with}\quad z^{(j)} = z_i^{(j)} + \omega_i^{(j)}\exp(-\widehat\zeta^{(j)}/\epsilon).
\end{aligned} \tag{3.35}
$$

Here we temporarily ignore the relation $z^{(J_{i-1})} = z_{i-1}^{(J_{i-1})} + q$. Formally, the limiting slow system of (3.35) at $z = z_i$ is

$$
\begin{aligned}
&\frac{d}{d\tau}p = f(p,z_i 0),\\
&\frac{d}{d\tau}q = 0,\\
&\frac{d}{d\tau}\zeta^{(j)} = \frac{-\partial g^{(j)}}{\partial z^{(j)}}(p,z_i,0), \quad j=1,2,\dots,m.
\end{aligned} \tag{3.36}
$$

Denote $\widetilde\Phi_i$ the solution operator for (3.36). Let $\mathscr{A}_i^{\text{in}}$ and $\widehat{\mathscr{A}}_i^{\text{out}}$ be the sets defined by (3.2) and (3.34). We define chart $\kappa_{\epsilon i}^{(21)} : \mathbb{R}^n \times (z_{\min}^{(J_i)}, z_{\max}^{(J_i)}) \times \mathbb{R}^{m-1} \to \mathbb{R}^n \times (z_{\min}^{(J_i)}, z_{\max}^{(J_i)}) \times \mathbb{R}^m$ by

$$
\kappa_{\epsilon i}^{(21)}(p,q,\widehat\zeta) = (p,q,\zeta) \quad \text{with}\ \ \zeta^{(j)} = \begin{cases} \epsilon\ln\frac{\omega_i^{(J_i)}}{q-z_i^{(J_i)}} & \text{if } j=J_i,\\ \widehat\zeta^{(J_i)} & \text{if } j\neq J_i,\end{cases}
$$

and chart $\kappa_{\epsilon i}^{(23)} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m$ by

$$
\kappa_{\epsilon i}^{(23)}(p,\zeta) = (p,q,\zeta) \quad \text{with}\ \ q = \omega_i^{(J_i)}\exp(-\zeta^{(J_i)}/\epsilon),
$$

and we define sets

$$
\widetilde{\mathscr{A}}_{\epsilon i}^{\text{in}} = \kappa_{\epsilon i}^{(21)}\big(\mathscr{A}_i^{\text{in}}\big), \quad \widetilde{\mathscr{A}}_{\epsilon i}^{\text{out}} = \kappa_{\epsilon i}^{(23)}\big(\widehat{\mathscr{A}}_i^{\text{out}}\big) \quad \text{for } \epsilon \ge 0. \tag{3.37}
$$

Note that

$$
\Pi_{0\widetilde{\mathscr{A}}_{0i}}^{\widetilde{\mathscr{A}}_{0i}^{\text{out}}} = \widetilde\Phi_i(\cdot,\delta_3).
$$

Proposition 3.7. *There exists $\Delta > 0$ such that the following assertions hold. Let $\widetilde{\mathscr{A}}_i^{\text{in}}$ and $\widetilde{\mathscr{A}}_i^{\text{out}}$ be defined by (3.37), (3.29), and (3.34) with $\delta_j < \Delta$, $j = 1,2,3$. Then for all sufficiently small $\delta_4 > 0$, the transition map $\Pi_{\epsilon\widetilde{\mathscr{A}}_{\epsilon i}^{\text{in}}}^{\widetilde{\mathscr{A}}_{\epsilon i}^{\text{out}}}$ from $\widetilde{\mathscr{A}}_{\epsilon i}^{\text{in}}$ to $\widetilde{\mathscr{A}}_{\epsilon i}^{\text{out}}$ following the flow of (3.35) is well-defined for all small $\epsilon > 0$. Moreover,*

$$
\Big\| \Pi_{\epsilon\widetilde{\mathscr{A}}_{\epsilon i}^{\text{in}}}^{\widetilde{\mathscr{A}}_{\epsilon i}^{\text{out}}} \circ \kappa_{\epsilon i}^{(21)} - \Pi_{0\widetilde{\mathscr{A}}_{0i}}^{\widetilde{\mathscr{A}}_{0i}^{\text{out}}} \circ \Pi_{0\widetilde{\mathscr{A}}_{0i}^{\text{in}}}^{\widetilde{\mathscr{A}}_{0i}} \circ \kappa_{0i}^{(21)} \Big\|_{C^1(\mathscr{A}_i^{\text{in}})} = O(\epsilon). \tag{3.38}
$$

Proof. Note that we have $z^{(J_{i-1})} = z_{i-1}^{(J_{i-1})} + q$ when converting (1.1) to (3.35). Let $s = q - z_i^{(J-1)} + z_{i-1}^{(J_{i-1})}$. By Assumption 1, (3.35) can be written as

$$\begin{aligned}
\dot{p} &= \epsilon f(p, z_i, 0) + h(p, z_i + s\,\mathsf{e}_{J_{i-1}}, 0) + O(|(\epsilon, s)|^2),\\
\dot{s} &= g(p, z_i + s\,\mathsf{e}_{J_{i-1}}, 0) + O(|(\epsilon, s)|^2),\\
\dot{\zeta}^{(j)} &= -\epsilon \frac{\partial g^{(j)}}{\partial z^{(j)}}(p, z_i, 0) + O(|(\epsilon, s)|^2)
\end{aligned}$$

as $(\epsilon, s) \to 0$. Since $\frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(A_i, z_i, 0) < 0$ by Assumption 5, applying Lemma 3.3 with $b = s$ and $a = (p, \zeta)$ we obtain (3.38). ∎

We denote $\Pi_{0\Gamma_i}^{\mathscr{A}_i}$ the transition map from Γ_i to $\mathscr{A}_i \times \{z_i^{(J_i)}\} \times \{\widehat{\zeta}_i\}$ along the flow of (3.31) and $\Pi_{0\widehat{\mathscr{A}}_i}^{\widehat{\Sigma}_i}$ the transition map from $0\ \widehat{\mathscr{A}}_i$ to $\widehat{\Sigma}_i$ along the flow of (3.33). We define chart $\kappa_{\epsilon i}^{(31)} : \mathbb{R}^n \times (z_{\min}^{(J_i)}, z_{\max}^{(J_i)}) \times \mathbb{R}^{m-1} \to \mathbb{R}^n \times \mathbb{R}^m$ by

$$\kappa_{\epsilon i}^{(31)}(p, q, \widehat{\zeta}) = (p, \zeta) \quad \text{with} \;\; \zeta^{(j)} = \begin{cases} \epsilon \ln \frac{\omega_i^{(J_i)}}{q - z_i^{(J_i)}} & \text{if } j = J_i,\\ \widehat{\zeta}^{(J_i)} & \text{if } j \neq J_i. \end{cases}$$

PROPOSITION 3.8. *There exist $\delta_j > 0$, $0 \le j \le 4$, such that if Γ_i, $\mathscr{A}_i$, Σ_i are defined in the preceding paragraphs, then the transition map $\Pi_{\epsilon\Gamma_i}^{\Sigma_i}$ from Γ_i to Σ_i following the flow of* (1.1) *is well-defined for all small $\epsilon > 0$, and*

$$\left\| \kappa_{\epsilon i}^{(31)} \circ \Pi_{\epsilon\Gamma_i}^{\Sigma_i} - \Pi_{0\widehat{\mathscr{A}}_i}^{\widehat{\Sigma}_i} \circ \kappa_{0i}^{(31)} \circ \Pi_{0\Gamma_i}^{\mathscr{A}_i} \right\|_{C^1(\Gamma_i)} = O(\epsilon) \tag{3.39}$$

as $\epsilon \to 0$.

Proof. Analogous to the proof of Proposition 3.4, the assertions can be derived from Propositions 3.5, 3.6, and 3.7. We skip it here. ∎

Proof of Theorem 2.7. By a reversal of the time variable and applying Proposition 3.8, we have

$$\left\| \kappa_{\epsilon i}^{(31)} \circ \Pi_{\epsilon\Gamma_{i+1}}^{\Sigma_i} - \Pi_{0\widehat{\mathscr{B}}_i}^{\widehat{\Sigma}_i} \circ \kappa_{0i}^{(31)} \circ \Pi_{0\Gamma_{i+1}}^{\mathscr{B}_i} \right\|_{C^1(\Gamma_{i+1})} = O(\epsilon).$$

Taking the inverse of the mappings in the last equation, we obtain

$$\left\| \Pi_{\epsilon\Sigma_i}^{\Gamma_{i+1}} \circ \kappa_{\epsilon i}^{(13)} - \Pi_{0\mathscr{B}_i}^{\Gamma_{i+1}} \circ \kappa_{0i}^{(13)} \circ \Pi_{0\widehat{\Sigma}_i}^{\widehat{\mathscr{B}}_i} \right\|_{C^1(\widehat{\Sigma}_i)} = O(\epsilon), \tag{3.40}$$

where $\kappa_{\epsilon i}^{(13)} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{m-1}$ is the chart defined by

$$\begin{aligned}
\kappa_{\epsilon i}^{(31)}(p, \zeta) = (p, q, \widehat{\zeta}) \quad &\text{with} \;\; q = z_i + \omega_i^{(J_i)} \exp(-\zeta^{(J_i)}/\epsilon),\\
&\text{and} \;\; \widehat{\zeta}^{(j)} = \zeta^{(J_i)} \;\; \text{for} \;\; j = 1, 2, \dots, J_i - 1, J_i + 1, \dots, m.
\end{aligned}$$

By (3.39) and (3.40),

$$\begin{aligned}
\Pi_{\epsilon\Gamma_i}^{\Gamma_{i+1}} &= \left(\Pi_{\epsilon\Sigma_i}^{\Gamma_{i+1}} \circ \kappa_{i\epsilon}^{(13)} \right) \circ \left(\kappa_{i\epsilon}^{(31)} \circ \Pi_{\epsilon\Gamma_i}^{\Sigma_i} \right)\\
&= \left(\Pi_{0\mathscr{B}_i}^{\Gamma_{i+1}} \circ \kappa_{0i}^{(13)} \circ \Pi_{0\widehat{\Sigma}_i}^{\widehat{\mathscr{B}}_i} \right) \circ \left(\Pi_{0\widehat{\mathscr{A}}_i}^{\widehat{\Sigma}_i} \circ \kappa_{0i}^{(31)} \circ \Pi_{0\Gamma_i}^{\mathscr{A}_i} \right) + O(\epsilon)\\
&= \Pi_{0\mathscr{B}_i}^{\Gamma_{i+1}} \circ \kappa_{0i}^{(13)} \circ \Pi_{0\widehat{\mathscr{A}}_i}^{\widehat{\mathscr{B}}_i} \circ \kappa_{0i}^{(31)} \circ \Pi_{0\Gamma_i}^{\mathscr{A}_i} + O(\epsilon)\\
&= \Pi_{0\mathscr{B}_i}^{\Gamma_{i+1}} \circ \widehat{Q}_i \circ \Pi_{0\Gamma_i}^{\mathscr{A}_i} + O(\epsilon),
\end{aligned}$$

where $\widehat{Q}_i$ is defined in (2.14). Therefore,

$$\Pi^{\Gamma_{i+2}}_{\epsilon\Gamma_{i+1}} \circ \Pi^{\Gamma_{i+1}}_{\epsilon\Gamma_i} = \left(\Pi^{\Gamma_{i+2}}_{0\mathscr{B}_{i+1}} \circ \widehat{Q}_{i+1} \circ \Pi^{\mathscr{A}_{i+1}}_{0\Gamma_{i+1}}\right) \circ \left(\Pi^{\Gamma_{i+1}}_{0\mathscr{B}_i} \circ \widehat{Q}_i \circ \Pi^{\mathscr{A}_i}_{0\Gamma_i}\right) + O(\epsilon). \tag{3.41}$$

We denote

$$P_\epsilon = \Pi^{\Gamma_1}_{\epsilon\Gamma_N} \circ \cdots \circ \Pi^{\Gamma_3}_{\epsilon\Gamma_2} \circ \Pi^{\Gamma_2}_{\epsilon\Gamma_1}.$$

By (3.41) and the relation that $\Pi^{\mathscr{A}_i}_{0\mathscr{B}_{i-1}} = \pi_i \times \mathrm{id}$, we have

$$P_\epsilon = \Pi^{\Gamma_1}_{0\mathscr{B}_N} \circ \widehat{Q}_N \circ (\pi_N \times \mathrm{id}) \circ \cdots \circ \widehat{Q}_2 \circ (\pi_2 \times \mathrm{id}) \circ \widehat{Q}_1 \circ \Pi^{\mathscr{A}_1}_{0\Gamma_1} + O(\epsilon).$$

Writing $\Pi^{\mathscr{A}_1}_{0\Gamma_1} = \Pi^{\mathscr{A}_1}_{0\mathscr{B}_N} \circ \Pi^{\mathscr{B}_N}_{0\Gamma_1} = (\pi_1 \times \mathrm{id}) \circ \left(\Pi^{\Gamma_1}_{\mathscr{B}_N}\right)^{-1}$, it follows that

$$P_\epsilon = \Pi^{\Gamma_1}_{0\mathscr{B}_N} \circ \widetilde{P} \circ \left(\Pi^{\Gamma_1}_{0\mathscr{B}_N}\right)^{-1} + O(\epsilon),$$

where $\widetilde{P}$ is defined by (2.21). This implies that

$$\begin{aligned}
&\det(DP_\epsilon - \mathrm{id}) \\
&\quad = \det\left(D\Pi^{\Gamma_1}_{0\mathscr{B}_N} \circ D\widetilde{P} \circ \left(D\Pi^{\Gamma_1}_{0\mathscr{B}_N}\right)^{-1} - \mathrm{id}\right) + O(\epsilon) \\
&\quad = \det\left(D\widetilde{P} - \mathrm{id}\right) + O(\epsilon).
\end{aligned}$$

Hence, the linearization of the limiting return map P_0 at $(p_{01}, q_{01}, \widehat{\zeta}_1) \in \Gamma_1$ does not have a singular value equal to 1 if $\det(D\widetilde{P}(A_1, \widehat{\zeta}_1) - \mathrm{id}) \neq 0$. Consequently, by the implicit function theorem for all small $\epsilon > 0$ there exists a locally unique fixed point $(p_{\epsilon 1}, q_{\epsilon 1}, \widehat{\zeta}_{\epsilon 1}) \in \Gamma_i$ of P_ϵ. Let $(p_{\epsilon 1}, z_{\epsilon 1}) = \kappa^{(01)}_{\epsilon 1}(p_{\epsilon 1}, q_{\epsilon 1}, \widehat{\zeta}_{\epsilon 1})$, where $\kappa^{(01)}_{\epsilon 1}$: $\mathbb{R}^n \times (z^{(J_i)}_{\min}, z^{(J_i)}_{\max}) \times \mathbb{R}^{m-1} \to \mathbb{R}^n \times \mathbb{R}^m$ is defined by

$$\kappa^{(01)}_{\epsilon 1}(p, q, \widehat{\zeta}) = (p, z) \quad \text{with} \quad z^{(j)} = \begin{cases} q & \text{if } j = J_i, \\ z_i + \omega^{(j)}_i \exp(-\widehat{\zeta}^{(j)}/\epsilon) & \text{if } j \neq J_i. \end{cases}$$

Then the trajectory passing through $(p_{\epsilon 1}, z_{\epsilon 1})$ is a periodic orbit of system (1.1). If $r(D\widetilde{P}(A_1, \zeta_1)) < 1$ (resp., $r(D\widetilde{P}(A_1, \zeta_1)) > 1$), then P_ϵ is a contraction (resp., expansion), and hence the periodic orbit is orbitally asymptotically stable (resp., unstable). ∎

4. Examples. In this section we apply the main results to study the examples (1.6), (1.7), (1.9) and the planar system (1.10) mentioned in section 1.

4.1. Trade-off between encounter and growth rates. Consider system (1.7) from Example 2, which takes the form

$$\begin{aligned}
x' &= F(x, \alpha) - G(x, y, \alpha), \\
y' &= H(x, y, \alpha) - D(y), \\
\epsilon\, \alpha' &= \alpha(1 - \alpha) E(x, y, \alpha)
\end{aligned}$$

with

$$F(x,\alpha) = x(\alpha + r - kx),$$
$$G(x,y,\alpha) = H(x,y,\alpha) = \frac{xy(a\alpha^2 + b\alpha + c)}{1+x},$$
$$D(y,\beta) = dy$$

and

$$E(x,y,\alpha) = \frac{\partial}{\partial \alpha}\left(\frac{x'}{x}\right) = 1 - \frac{y(2a\alpha + b)}{1+x}.$$

The limiting fast system is

$$\dot{x} = 0, \quad \dot{y} = 0, \quad \dot{\alpha} = \alpha(1-\alpha)E(x,y,\alpha).$$

The critical manifolds are

$$M_1 = \{(x,y,\alpha) : \alpha = 0\} \quad \text{and} \quad M_2 = \{(x,y,\alpha) : \alpha = 1\}.$$

On the critical manifolds M_i, the limiting slow system is

$$\begin{aligned} x' &= F(x,\bar{\alpha}) - G(x,y,\bar{\alpha}), \\ y' &= H(x,y,\bar{\alpha}) - D(y), \end{aligned} \tag{4.1}$$

where $\bar{\alpha} = 0, 1$. Let Φ_1 and Φ_2 be the solution operators for (4.1) with $\alpha = 0$ and $\alpha = 1$, respectively. The transition maps Q_1 and Q_2 in Theorem 2.5 are determined by

$$Q_1(A_1) = \Phi_1(A_1,\tau_1) \text{ with } \int_0^{\tau_1} \left(1 - \frac{by}{1+x}\right)\Big|_{(x,y)=\Phi_1(A_1,t)} dt = 0$$

and

$$Q_2(A_2) = \Phi_2(A_2,\tau_2) \text{ with } \int_0^{\tau_2} \left(1 - \frac{y(2a+b)}{1+x}\right)\Big|_{(x,y)=\Phi_2(A_2,t)} dt = 0.$$

Following [10], we set $a = -0.1$, $b = 3$, $c = 1$, $d = 2.8$, $k = 1$, and $r = 10$. By implementing Newton's iteration, we find points $A_1 = B_2 \approx (5.57, 11.03)$ and $B_1 = A_2 \approx (9.96, 0.36)$ satisfying

$$A_2 = B_1 = Q_1(A_1) \quad \text{and} \quad A_1 = B_2 = Q_2(A_2).$$

This means that A_i and B_i satisfy the following conditions (see Figure 3(b)):
(i) A_1 and B_1 are connected by a trajectory σ_1 of (4.1) with $\bar{\alpha} = 0$;
(ii) A_2 and B_2 are connected by a trajectory σ_2 of (4.1) with $\bar{\alpha} = 1$;
(iii) $\int_{\sigma_1} E(x,y,0)\,dt = 0$ and $\int_{\sigma_2} E(x,y,1)\,dt = 0$.

Using the formulas in Proposition A.1 and Remark A.2, we obtain

$$DQ_1(A_1) \approx \begin{pmatrix} -0.0001 & -0.0029 \\ 0.0009 & 0.0258 \end{pmatrix} \quad \text{and} \quad DQ_2(A_2) \approx \begin{pmatrix} 0.02 & 18.91 \\ -0.02 & -16.95 \end{pmatrix}.$$

Hence, the eigenvalues of $DP(A_1) = DQ_2(A_2)\,DQ_1(A_1)$ are $\lambda_1 \approx 2.86 \cdot 10^{-14}$ and $\lambda_2 = -0.42$. Note that although λ_1 is close to zero, it is nonzero since P is induced by the flow of systems of differential equations and is thus a composition of diffeomorphisms. The reason that λ_1 is close to zero is that the endpoint B_1 of the slow trajectory σ_1 of system (4.1) with $\bar{\alpha} = 0$ is close to the asymptotically stable equilibrium $(10, 0)$, which causes contraction of surface areas (see, e.g., Li and Muldowney [41]).

Since λ_1 and λ_2 are both of magnitude less than one, by Theorem 2.5 or its special case, Theorem 2.1, the configuration

$$\gamma_1 \cup \sigma_1 \cup \gamma_2 \cup \sigma_2$$

corresponds to a relaxation oscillation formed by locally orbitally asymptotically stable periodic orbits.

For system (1.7) with $\epsilon = 0.1$, taking initial data $(x, y, \alpha) = (10, 0.5, 0.5)$ we find that the trajectory converges to a periodic orbit (see Figure 3(a)) near the singular configuration.

4.2. Prey switching. Consider system (1.9) from Example 3. Following Piltz et al. [49], we assume that the response functions $f_i(p_i)$ in (1.9) are linear. After rescaling, the system is converted to

$$\begin{aligned} p_1' &= (1 - qz)p_1, \\ p_2' &= (r - (1 - q)z)p_2, \\ z' &= \big(qp_1 + (1 - q)p_2 - 1\big)z, \\ \epsilon\, q' &= q(1 - q)(p_1 - p_2). \end{aligned} \tag{4.2}$$

The critical manifolds for (4.2) are

$$M_1 = \{(p_1, p_2, z, q) : q = 0\} \quad \text{and} \quad M_2 = \{(p_1, p_2, z, q) : q = 1\}.$$

On M_1, the restriction of (1.9) is

$$\begin{aligned} p_1' &= p_1, \\ p_2' &= (r - z)p_2, \\ z' &= (p_2 - 1)z, \end{aligned} \tag{4.3}$$

which means that the predators hunt exclusively only the first prey population. On M_2, the restriction of (1.9) is

$$\begin{aligned} p_1' &= (1 - z)p_1, \\ p_2' &= rzp_2, \\ z' &= (p_1 - 1)z, \end{aligned} \tag{4.4}$$

which means that the predators hunt exclusively only the second prey population.

Let Φ_1 and Φ_2 be the transition maps for (4.3) and (4.4), respectively. The transition maps Q_1 and Q_2 in Theorem 2.5 are determined by

$$Q_1(A_1) = \Phi_1(A_1, \tau_1) \ \text{ with } \ \int_0^{\tau_1} (p_1 - p_2)\Big|_{(p_1,p_2,z)=\Phi_1(A_1,t)} dt = 0$$

and

$$Q_2(A_2) = \Phi_2(A_2, \tau_2) \ \text{ with } \ \int_0^{\tau_2} (p_1 - p_2)\Big|_{(p_1,p_2,z)=\Phi_2(A_2,t)} dt = 0.$$

With the parameters given in Piltz et al. [49], $r = 0.5$ and $m = 0.4$, we find $A_1 = B_2 \approx (0.92, 1.08, 1.50)$ and $A_2 = B_1 \approx (1.08, 0.92, 1.50)$ such that the transition maps Q_i in Theorem 2.5 satisfy $Q_1(A_1) = B_1$ and $Q_2(A_2) = B_2$ (see Figure 4(b)). Using the formulas in Proposition A.1 and Remark A.2, we obtain

$$DQ_1(A_1) \approx \begin{pmatrix} -6.78 & 5.74 & -1.00 \\ 6.77 & -4.03 & 0.70 \\ 0.34 & -0.16 & 1.04 \end{pmatrix}, \quad DQ_2(A_2) \approx \begin{pmatrix} -1.56 & 3.38 & 0.55 \\ 2.80 & -2.80 & -0.99 \\ -0.07 & 0.34 & 1.06 \end{pmatrix}.$$

Hence, the eigenvalues of $DP(A_1) = DQ_2(A_2)\, DQ_1(A_1)$ are $\lambda_1 \approx 60.55$ and $\lambda_{2,3} \approx 0.97 \pm 0.26\sqrt{-1}$. Since λ_1 is greater than 1, by Theorems 2.5 or 2.1, the configuration connecting A_i and B_i corresponds to a relaxation oscillation formed by orbitally unstable periodic orbits (see Figure 4(b)).

4.3. Coevolution. Consider system (1.6) from Example 1. The system has critical manifolds M_i, $1 \le i \le 4$, corresponding to $(\alpha, \beta) = (\alpha_i, \beta_i)$ with (α_i, β_i), $i = 1, 2, 3, 4$, being equal to $(0, 0)$, $(0, 1)$, $(1, 1)$, and $(1, 0)$, respectively (see Figure 6). The limiting slow system on each M_i is

$$\begin{aligned} x' &= F(x, \alpha_i) - G(x, y, \alpha_i, \beta_i), \\ y' &= H(x, y, \alpha_i, \beta_i) - D(y, \beta_i). \end{aligned} \tag{4.5}$$

The numbers $\omega_i = (\omega_i^{(1)}, \omega_i^{(2)})$ defined by (2.17) are $\omega_1 = (1, 1)$, $\omega_2 = (1, -1)$, $\omega_3 = (-1, -1)$, and $\omega_4 = (-1, 1)$. Equations for $\zeta = (\zeta^{(1)}, \zeta^{(2)})$ in (2.11) on M_i are

$$\frac{d}{d\tau}\zeta^{(1)} = \omega_i^{(1)} E_1(x, y, \alpha_i, \beta_i) \quad \text{and} \quad \frac{d}{d\tau}\zeta^{(2)} = \omega_i^{(2)} E_2(x, y, \alpha_i, \beta_i), \tag{4.6}$$

where

$$E_1(x, y, \alpha, \beta) = \frac{\partial}{\partial \alpha}\left(\frac{F(x, \alpha) - G(x, y, \alpha, \beta)}{x}\right)$$

and

$$E_2(x, y, \alpha, \beta) = \frac{\partial}{\partial \beta}\left(\frac{H(x, \alpha) - D(x, y, \alpha, \beta)}{y}\right).$$

Let $\widehat{\Phi}_i$, $1 \le i \le 4$, be the solution operators for system (4.5)–(4.6). Then the transition maps $\widehat{Q}_i$ in Theorem 2.6 are determined by

$$\begin{aligned}
\widehat{Q}_1(A_1, \zeta) &= \widehat{\Phi}_1((A_1, \zeta), \tau_1) \ \text{ with } \ \zeta^{(2)} + \int_0^{\tau_1} E_2(x, y, 0, 0)\Big|_{(x,y)=\Phi_1(A_1,t)} dt = 0, \\
\widehat{Q}_2(A_2, \zeta) &= \widehat{\Phi}_2((A_2, \zeta), \tau_2) \ \text{ with } \ \zeta^{(1)} + \int_0^{\tau_2} E_1(x, y, 0, 1)\Big|_{(x,y)=\Phi_2(A_2,t)} dt = 0, \\
\widehat{Q}_3(A_3, \zeta) &= \widehat{\Phi}_3((A_3, \zeta), \tau_3) \ \text{ with } \ \zeta^{(2)} - \int_0^{\tau_1} E_2(x, y, 1, 1)\Big|_{(x,y)=\Phi_1(A_1,t)} dt = 0, \\
\widehat{Q}_4(A_4, \zeta) &= \widehat{\Phi}_4((A_4, \zeta), \tau_4) \ \text{ with } \ \zeta^{(1)} - \int_0^{\tau_4} E_1(x, y, 1, 0)\Big|_{(x,y)=\Phi_4(A_4,t)} dt = 0.
\end{aligned}$$

Following Cortez and Weitz [12, Supporting Information D], we consider (1.6) with

$$\begin{aligned}
F(x,\alpha) &= x(s_0+s_1\alpha)\left(1-\frac{x}{k_0+k_1\alpha}\right),\\
G(x,y,\alpha,\beta) &= \frac{(r_0+r_1\alpha+r_2\beta+r_3\alpha\beta+r_4\beta^2)xy}{1+hx},\\
H(x,y,\alpha,\beta) &= c_0G(x,y,\alpha,\beta),\\
D(y,\beta) &= y^{1.5}(\delta_0+\delta_1\beta),
\end{aligned}$$

and parameters $s_0=2.5, s_1=3.5, k_0=1, k_1=0.1, r_0=0.65, r_1=3, r_2=2.3, r_3=-0.2, r_4=0.01, c_0=1.7, \delta_0=0.76, \delta_1=1.77$, and $h=1$. Implementing Newton's iteration for $\widehat{Q}_i(A_i,\zeta_i)=(A_{i+1},\zeta_{i+1})$, $1\le i\le 4$, we find $B_4=A_1\approx(0.33,1.99)$, $B_1=A_2\approx(0.92,0.56)$, $B_2=A_3\approx(0.60,0.55)$, and $B_3=A_4\approx(0.30,0.93)$ (see Figure 2(b)), and $\zeta_1\approx(0,0.98)$, $\zeta_2\approx(3.84,0)$, $\zeta_3\approx(0,1.12)$, and $\zeta_4\approx(0.55,0)$.

Let $\{\mathsf{e}_x,\mathsf{e}_y,\mathsf{e}_\alpha,\mathsf{e}_\beta\}$ be the standard ordered basis of the (x,y,α,β)-space. Note that the tangent space of $\mathscr{A}_1\times\Lambda_1$ at (A_1,ζ_1) is spanned by $\{\mathsf{e}_x,\mathsf{e}_y,\mathsf{e}_\beta\}$, and the tangent space of $\mathscr{B}_1\times\Lambda_2$ at (B_1,ζ_2) is spanned by $\{\mathsf{e}_x,\mathsf{e}_y,\mathsf{e}_\alpha\}$. Using formulas in Proposition A.3, we obtain

$$D\widehat{Q}_1(A_1,\zeta_1)\approx \begin{array}{c} \begin{array}{ccc} \mathsf{e}_x & \mathsf{e}_x & \mathsf{e}_\beta \end{array} \\ \begin{pmatrix} 0.013 & 0.004 & -0.007\\ 0.080 & -0.254 & 0.038\\ -3.29 & -2.42 & 0.67 \end{pmatrix} \end{array} \begin{array}{c} \\ \mathsf{e}_x\\ \mathsf{e}_y\\ \mathsf{e}_\alpha \end{array}.$$

Similarly,

$$D\widehat{Q}_2(A_2,\zeta_2)\approx \begin{array}{c} \begin{array}{ccc} \mathsf{e}_x & \mathsf{e}_x & \mathsf{e}_\alpha \end{array} \\ \begin{pmatrix} -0.00040 & -0.0058 & 0.00024\\ -0.00003 & 0.00024 & 0.00030\\ 0.37 & -1.44 & -0.26 \end{pmatrix} \end{array} \begin{array}{c} \\ \mathsf{e}_x\\ \mathsf{e}_y\\ \mathsf{e}_\beta \end{array},$$

and the approximations of $D\widehat{Q}_3(A_3,\zeta_3)$ and $D\widehat{Q}_4(A_4,\zeta_4)$ are, respectively,

$$\begin{array}{c} \begin{array}{ccc} \mathsf{e}_x & \mathsf{e}_x & \mathsf{e}_\beta \end{array} \\ \begin{pmatrix} 0.29 & -0.04 & -0.22\\ 0.26 & -0.67 & 0.49\\ 2.49 & 0.13 & -0.86 \end{pmatrix} \end{array} \begin{array}{c} \\ \mathsf{e}_x\\ \mathsf{e}_y\\ \mathsf{e}_\alpha \end{array} \quad\text{and}\quad \begin{array}{c} \begin{array}{ccc} \mathsf{e}_x & \mathsf{e}_x & \mathsf{e}_\alpha \end{array} \\ \begin{pmatrix} -0.10 & -0.09 & 0.03\\ 0.42 & 0.38 & -0.13\\ -0.36 & -0.33 & 0.11 \end{pmatrix} \end{array} \begin{array}{c} \\ \mathsf{e}_x\\ \mathsf{e}_y\\ \mathsf{e}_\beta \end{array}.$$

Hence, the eigenvalues of

$$D\widehat{P}(A_1,\zeta_1)=D\widehat{Q}_4(A_4,\zeta_4)\;D\widehat{Q}_3(A_3,\zeta_3)\;D\widehat{Q}_2(A_2,\zeta_2)\;D\widehat{Q}_1(A_1,\zeta_1)$$

are $\lambda_1\approx 0.39$, $\lambda_2\approx -6.14\cdot 10^{-5}$, and $\lambda_3\approx -5.11\cdot 10^{-11}$, which are all of magnitude less than one. Therefore, by Theorem 2.6 or its special case, Theorem 2.2, this singular configuration corresponds to a relaxation oscillation formed by locally orbitally asymptotically stable periodic orbits.

4.4. A planar system. Consider system (1.10) from Example 4. The limiting fast system is

$$\dot{a}=b\,H(a,b,0),\qquad \dot{b}=b\,G(a,b,0). \tag{4.7}$$

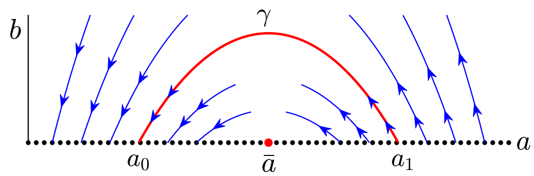


FIG. 11. *For system* (4.7) *with* $\epsilon = 0$*, the* a*-axis is a line of equilibria and* γ *is a heteroclinic orbit connecting* $(a_0, 0)$ *and* $(a_1, 0)$*.*

On the critical manifold $M = \{(a, b) : b = 0\}$, the limiting slow system is

$$a' = F(a, 0, 0).$$

We assume the following (see Figure 11):

(i) There is a trajectory γ of (4.7) satisfying

$$\lim_{t\to-\infty} \gamma(t) = (a_0, 0), \quad \lim_{t\to\infty} \gamma(t) = (a_1, 0).$$

(ii) $F(a, 0, 0) > 0$ for all $a \in [a_0, a_1]$.

(iii) $G(a_0, 0, 0) < 0$ and $G(a_1, 0, 0) > 0$.

(iv) $\displaystyle\int_{a_0}^{a_1} \frac{G(a,0,0)}{F(a,0,0)}\, da = 0$ and $\displaystyle\int_{a_0}^{s} \frac{G(a,0,0)}{F(a,0,0)}\, da < 0 \;\; \forall s \in (a_0, a_1)$.

We provide an alternative proof of the following theorem from Hsu and Wolkowicz [28].

THEOREM 4.1. *Consider system* (1.10)*. Assume* (i)–(iv) *and let*

$$\lambda = \ln\left|\frac{F(a_1,0,0)}{F(a_0,0,0)}\right| + \int_\gamma \frac{\partial_a H}{H}\, da + \int_\gamma \frac{\partial_b G}{G}\, db.$$

If $\lambda \neq 0$*, then* γ *admits a relaxation oscillation which is formed by locally unique periodic orbits for small* $\epsilon > 0$*. Moreover, the periodic orbit is orbitally asymptotically stable if* $\lambda < 0$ *and unstable if* $\lambda > 0$*.*

Remark 4.2. Assumptions (i) and (iv) are weaker than the conditions assumed in [28]. In that paper, the assumption corresponding to (i) is that there exists a smooth family of heteroclinic orbits; the assumptions corresponding to the inequalities in (iii) and (iv) are $G(a, 0, 0) < 0$ for $a < \bar{a}$ and $G(a, 0, 0) > 0$ for $a > \bar{a}$. However, the analysis in that paper is also valid under these weaker assumptions.

Proof. Define a function Q implicitly by $Q(a_0) = a_1$ and

$$\int_{a_0}^{Q(a)} \frac{G(r,0,0)}{F(r,0,0)}\, dr = 0. \tag{4.8}$$

By (A.6) in Proposition A.1,

$$\frac{dQ(a_0)}{da} = \frac{F(a_1,0,0)}{G(a_1,0,0)} \frac{G(a_0,0,0)}{F(a_0,0,0)}. \tag{4.9}$$

(Alternatively, (4.9) can be derived directly by differentiating (4.8).)

Let π be the transition map of (4.7) from a neighborhood of $(a_1, 0)$ to a neighborhood $(a_0, 0)$ in the a-axis. By (A.18) in Proposition A.4,

$$\frac{d}{da}\pi(a_1) = \frac{G(a_1,0,0)}{G(a_0,0,0)} \exp\left(\int_\gamma \partial_a H + \partial_b G\, dt\right). \tag{4.10}$$

By (4.10) and (4.9), we obtain

$$\begin{aligned}\frac{d}{da}(\pi \circ Q) &= \frac{d\pi(a_1)}{da}\,\frac{dQ(a_0)}{da}\\ &= \left(\frac{F(a_1,0,0)}{G(a_1,0,0)}\,\frac{G(a_0,0,0)}{F(a_0,0,0)}\right)\frac{G(a_1,0,0)}{G(a_0,0,0)} \exp\left(\int_0^T \partial_a H + \partial_b G\, dt\right).\end{aligned}$$

Using the relations $da/dt = H$ and $da/dt = G$ in (4.7), it follows that

$$\frac{d}{da}(\pi \circ Q)(a_0) = \frac{F(a_1,0,0)}{F(a_0,0,0)} \exp\left(\int_\gamma \frac{\partial_a H}{H}\, da + \int_\gamma \frac{\partial_b G}{H}\, db.\right).$$

Hence

$$\ln\left|\frac{d}{da}(\pi \circ Q)(a_0)\right| = \ln\left|\frac{F(a_1,0,0)}{F(a_0,0,0)}\right| + \int_\gamma \frac{\partial_a H}{H}\, da + \int_\gamma \frac{\partial_b G}{H}\, db.$$

Hence $\lambda < 0$ if and only if $\left|\frac{d}{da}(\pi \circ Q)(a_0)\right| < 1$. By Theorem 2.7, the desired result follows. ∎

Appendix A. Some computable formulas. Under Assumptions 1–5, we define $f_i(p) = f(p, z_i, 0)$ and $p_i(\tau) = \Phi_i(\tau, A_i)$ for each $i = 1, 2, \ldots, N$. Let $L_i(\tau)$ be the fundamental matrix for the variational equations of (1.4) along σ_i. This means that for any $v \in \mathbb{R}^n$, $w(\tau) = L_i(\tau)v$ is the solution of

$$\frac{d}{d\tau} w = \left[Df_i(p_i(\tau))\right] w, \quad w(0) = v_0, \quad \text{for } 0 \le \tau \le \tau_i. \tag{A.1}$$

It can be shown that, for $v \in \mathbb{R}^n$ and $0 \le \tau \le \tau_i$,

$$L_i(\tau)v = D\Phi(A_i, \tau)v, \tag{A.2}$$

which implies

$$L_i(\tau)v = v + \int_0^\tau \left[Df_i(p_i(s))\right] L_i(s)v\, ds. \tag{A.3}$$

We define a linear functional μ_i on $\mathbb{R}^n$ by

$$\mu_i(v) = \int_0^{\tau_i} \left\langle L_i(\tau)v, D\frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(p_i(\tau), z_i, 0)\right\rangle d\tau \quad \text{for } v \in \mathbb{R}^n, \tag{A.4}$$

where D denotes the derivative with respect to p.

PROPOSITION A.1. *Let Q_i be defined by* (2.9). *Then*

$$DQ_i(A_i, \zeta_i)v = L_i(\tau_i)v - \frac{\mu_i(v)}{\frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(B_i, z_i, 0)} f(B_i, z_i, 0) \quad \forall v \in \mathbb{R}^n, \tag{A.5}$$

where L_i and μ_i are defined by (A.2) *and* (A.4)*, and J_i is the index in Assumption* 2. *In particular,*

$$DQ_i(A_i,\zeta_i)f(A_i,z_i,0) = \frac{\dfrac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(A_i,z_i,0)}{\dfrac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(B_i,z_i,0)} f(B_i,z_i,0). \tag{A.6}$$

Proof. By differentiating (2.8) with respect to p we obtain

$$\begin{aligned}&\langle DT_i(p),v\rangle \frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}\big(\Phi_i(A,\tau_i),z_i,0\big)\\ &\quad + \int_0^{T_i(A)} \left\langle D\frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}\big(\Phi_i(A,\tau),z_i,0\big), D\Phi_i(A,\tau)v\right\rangle\, d\tau = 0 \quad \forall v\in\mathbb{R}^n.\end{aligned}$$

Evaluating this equation at $A = A_i$ yields

$$\langle DT_i(p),v\rangle \frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(B_i,z_i,0) = -\int_0^{\tau_i} \left\langle D\frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}\big(p_i(\tau),z_i,0\big), L_i(\tau)v\right\rangle\, d\tau.$$

By (A.4) it follows that

$$\langle DT_i(p),v\rangle = \frac{-\mu_i(v)}{\dfrac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(B_i,z_i,0)}. \tag{A.7}$$

On the other hand, since Φ_i is the solution operator for (1.4), the definition of Q_i in (2.9) means that

$$Q_i(p) = p + \int_0^{T_i(p)} f_i(\Phi_i(p,\tau))\, d\tau.$$

Differentiating both sides of the equation with respect to p gives

$$\begin{aligned}DQ_i(p)v &= v + \langle DT_i(p),v\rangle\, f_i(\Phi_i(p,T_i(p)))\\ &\qquad + \int_0^{T_i(p)} Df_i(\Phi_i(p,\tau))D\Phi_i(p,\tau)v\, d\tau \quad \forall v\in\mathbb{R}^n.\end{aligned}$$

Evaluating the equation at $p = A_i$ and using (A.2) we have

$$DQ_i(A_i)v = v + \langle DT_i(A_i),v\rangle\, f_i(B_i) + \int_0^{\tau_i} Df_i(p_i(\tau))L_i(\tau)v\, d\tau.$$

By (A.3) it follows that

$$DQ_i(A_i)v = L_i(\tau_i)v + \langle DT_i(A_i),v\rangle\, f_i(B_i). \tag{A.8}$$

Substituting (A.7) into (A.8), we then obtain (A.5).

Since $f_i(p_i(\tau))$ is a solution of (A.1) with $v_0 = f_i(A_i)$,

$$L_i(\tau)f_i(A_i) = f_i(p_i(\tau)) \quad \text{for} \quad 0\le\tau\le\tau_i. \tag{A.9}$$

Using $\frac{d}{d\tau}p_i(\tau) = f_i(p_i(\tau))$ and (A.9), evaluating (A.4) at $v = f_i(p)$ gives

$$\mu(f_i(A_i)) = \frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(p_i(\tau),z_i,0)\bigg|_{\tau=0}^{\tau_i} = \frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(B_i,z_i,0) - \frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(A_i,z_i,0). \tag{A.10}$$

Substituting (A.10) into (A.5) we obtain (A.6). ∎

Remark A.2. Numerical approximations of L_i and μ_i can be computed by extending system (1.3) of p to a system of (p, w, μ) by appending equations (A.4) and

$$\frac{d}{d\tau}\mu_i = \left\langle L_i(\tau)v, D\frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(p_i(\tau), z_i, 0)\right\rangle.$$

PROPOSITION A.3. *Let $\widehat{Q}_i$ be defined by* (2.14)*. Then*

$$\begin{aligned} &D\widehat{Q}_i(A_i, \zeta_i)(v, 0) \\ &= \left(DQ_i(A_i)v,\ \frac{-\nu_i(v)}{\frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(B_i, z_i, 0)} \sum_{j\neq J_i} \frac{\partial g^{(j)}}{\partial z^{(j)}}(B_i, z_i, 0)\, \mathsf{e}_j\right) \quad \forall v \in \mathbb{R}^n, \end{aligned} \tag{A.11}$$

where $\nu_i(v)$ is defined by (A.4)*, and*

$$\begin{aligned} &D\widehat{Q}_i(A_i, \zeta_i)(0, \mathsf{e}_j) \\ &= \begin{cases} (0, \mathsf{e}_j) & \text{if } j \neq J_i \\ \frac{1}{\frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(B_i, z_i, 0)}\left(f(B_i, z_i, 0),\ \sum_{k\neq J_i} \frac{\partial g^{(k)}}{\partial z^{(k)}}(B_i, z_i, 0)\, \mathsf{e}_\mathsf{k}\right) & \text{if } j = J_i. \end{cases} \end{aligned} \tag{A.12}$$

Proof. We identify vectors $v \in \mathbb{R}^n$ with their images $(v, 0_m) \in \mathbb{R}^n \times \mathbb{R}^m$ and identify the vector e_j, $j \in \{1, 2, \ldots, m\}$, in the standard basis of $\mathbb{R}^m$, with the vector $(0_n, \mathsf{e}_j)$ in $\mathbb{R}^n \times \mathbb{R}^m$. The function $\widehat{Q}_i(p, \zeta)$ defined by (2.14) can be written as

$$\begin{aligned} &\widehat{Q}_i(p, \zeta) \\ &= \left(\Phi\big(p, \widehat{T}_i(p, \zeta)\big), \sum_{k\neq J_i}\left[\zeta^{(k)} + \int_0^{\widehat{T}(p, \zeta^{(J_i)})} \frac{\partial g^{(k)}}{\partial z^{(k)}}(\Phi(p, \tau), z_i, 0)\, d\tau\right] \mathsf{e}_k\right). \end{aligned} \tag{A.13}$$

Since $\widehat{T}_i(p, \zeta^{(J_i)}) = T_i(p)$ and $\Phi\big(p, T_i(p)\big) = Q_i(p)$ for all $p \in \mathscr{A}_i$,

$$\widehat{Q}_i(p, \zeta_i) = \left(Q_i(p), \sum_{k\neq J_i}\left[\zeta^{(k)} + \int_0^{T(p)} \frac{\partial g^{(k)}}{\partial z^{(k)}}(\Phi(p, \tau), z_i, 0)\, d\tau\right] \mathsf{e}_k\right).$$

Hence,

$$\begin{aligned} &D\widehat{Q}_i(p, \zeta_i)(v, 0) \\ &= \left(DQ_i(p)v, \langle DT(p), v\rangle \sum_{j\neq J_i} \frac{\partial g^{(j)}}{\partial z^{(j)}}\big(\Phi(p, \tau), z_i, 0\big)\, \mathsf{e}_j\right) \quad \forall v \in \mathbb{R}^n. \end{aligned}$$

Evaluating this equation at $p = A_i$, by (A.7) we then obtain (A.11).

For each $j \in \{1, 2, \ldots, m\} \setminus \{J_i\}$, differentiating (A.13) with respect to $\zeta^{(j)}$ gives $\frac{\partial}{\partial \zeta^{(j)}}\widehat{Q}_i(p, \zeta) = \mathsf{e}_j$ for all (p, ζ). On the other hand, by differentiating (A.13) with respect to $\zeta^{(J_i)}$, from the relation $\frac{\partial}{\partial \tau}\Phi(p, \tau) = f(\Phi(p, \tau))$ we obtain

$$\begin{aligned} &\frac{\partial}{\partial \zeta^{(J_i)}}\widehat{Q}_i(p, \zeta) \\ &= \frac{\partial \widehat{T}(p, \zeta^{(J_i)})}{\partial \zeta^{(J_i)}}\left(f\big(\Phi(p, \widehat{T}_i(p, \zeta^{(J_i)})), z_i, 0\big), \sum_{k\neq J_i} \frac{\partial g^{(k)}}{\partial z^{(k)}}(B_i, z_i, 0)\, \mathsf{e}_\mathsf{k}\right). \end{aligned} \tag{A.14}$$

Note that differentiating (2.13) with respect to $\zeta^{(J_i)}$ gives

$$\text{(A.15)} \qquad \frac{\partial \widehat{T}_i(A_i, \zeta^{(J_i)})}{\partial \zeta^{(J_i)}} = \frac{-1}{\frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(B_i, z_i, 0)}.$$

By (A.14) and (A.15) it follows that

$$\frac{\partial}{\partial \zeta^{(J_i)}} \widehat{Q}_i(A_i, \zeta) = \frac{-1}{\frac{\partial g^{(J_i)}}{\partial z^{(J_i)}}(B_i, z_i, 0)} \left(f(B_i, z_i, 0), \sum_{k \neq J_i} \frac{\partial g^{(k)}}{\partial z^{(k)}}(B_i, z_i, 0)\, \mathsf{e}_\mathsf{k} \right).$$

This means that (A.12) holds. ∎

Let Ψ_i be the solution operator for (2.19). Let t_i be the positive number such that

$$\Psi_i\big(t_i, (B_{i-1}, z_{i-1}^{(J_i)})\big) = (A_{i-1}, z_i^{(J_i)}).$$

Let

$$\bar{\gamma}_i(t) = \Psi_i\big(t, (B_{i-1}, z_{i-1}^{(J_i)})\big), \quad 0 \le t \le t_i.$$

Thus $\bar{\gamma}$ has the same trajectory as the curve γ given in Assumption 2.

We define $R_i(t) : \mathbb{R}^n \to \mathbb{R}^n$ and $\nu_i(t) : \mathbb{R}^n \to \mathbb{R}$, $0 \le t \le t_i$, to be the linear operators so that for any $v_0 \in \mathbb{R}^n$, $R_i(t)[v_0]$ and $\nu_i((t)[v_0]$ are the v- and w-components, respectively, of the variational equations of (2.19) along $\bar{\gamma}_i(t)$ with initial data $(v_0, 0)$. This means that for any $(v_0, w_0) \in \mathbb{R}^n \times \mathbb{R}$, $(v, w) = \big(R_i(t)[v_0], \nu_i((t)[v_0]\big)$ is the solution of

$$\text{(A.16)} \qquad \frac{d}{dt}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} D_p h_i & D_q h_i \\ D_p g_i & D_q g_i \end{pmatrix}_{\bar{\gamma}_i(t)} \begin{pmatrix} v \\ w \end{pmatrix}, \qquad \begin{pmatrix} v \\ w \end{pmatrix}(0) = \begin{pmatrix} v_0 \\ 0 \end{pmatrix},$$

where g_i and h_i are defined by (2.18).

Proposition A.4. *Let π_i be defined by* (2.20). *Then*

$$\text{(A.17)} \qquad D\pi_i(B_{i-1})[v] = R_i(t_i)[v] - \nu_i(t_i)[v]\frac{h_i(A_i, z_i)}{g_i(A_i, z_i)} \qquad \forall v \in \mathbb{R}^n.$$

Moreover, if $n = 1$, then

$$\text{(A.18)} \qquad D\pi_i(B_{i-1}) = \frac{g_i(B_{i-1}, z_{i-1})}{g_i(A_i, z_i)} \exp\left(\int_0^{t_i} (D_p h_i + D_q g_i)(\widetilde{\gamma}_i(t))\, dt \right).$$

Proof. The first part of the proof is similar to that of Proposition A.1, so we only sketch it briefly. Define $S_i : \mathscr{B}_{i-1} \to (0, \infty)$ implicitly by $S_i(p) = t_i$ and

$$\text{(A.19)} \qquad z_{i-1}^{(J_i)} + \int_0^{S_i(p)} g_i\big(\Psi_i\big(t, (p, z_{i-1}^{(J_i)})\big)\big)\, dt = z_i^{(J_i)}.$$

Then

$$\text{(A.20)} \qquad (\pi_i(p), z_i^{(J_i)}) = \Psi_i\big(S_i(p), (p, z_{i-1}^{(J_i)})\big).$$

Differentiating (A.19) gives (similar to the derivation of (A.7))

$$\langle DS_i(p), v\rangle\, g_i(A_i, z_i^{(J_i)}) = \nu_i(t_i)[v]. \tag{A.21}$$

Differentiating (A.20) gives (similar to the derivation of (A.8))

$$D\pi_i(p)[v] = R_i(t_i)[v] - \langle DS_i(p), v\rangle\, h_i(A_i, z_i^{(J_i)}). \tag{A.22}$$

By (A.21) and (A.22) we obtain (A.17).

Now we assume $n = 1$. Then (A.22) gives

$$\begin{aligned} D\pi_i(B_{i-1}) &= \frac{R_i(t_i)g_i(A_i, z_i) - \nu_i(t_i)h_i(A_i, z_i)}{g_i(A_i, z_i)} \\ &= \frac{1}{g_i(A_i, z_i)} \det \begin{pmatrix} R_i(t) & h_i(\widetilde{\gamma}_i(t)) \\ \nu_i(t) & g_i(\widetilde{\gamma}_i(t)) \end{pmatrix}_{t=t_i}. \end{aligned} \tag{A.23}$$

On the other hand, when $n = 1$, $(R_i, \nu_i)(t)$ is the solution of (A.16) with $v_0 = 1$. Note that $(h_i, g_i)(\widetilde{\gamma}_i(t))$ also satisfies the differential equations in (A.16). Hence,

$$\frac{d}{dt}\begin{pmatrix} R_i(t) & h_i(\widetilde{\gamma}_i(t)) \\ \nu_i(t) & g_i(\widetilde{\gamma}_i(t)) \end{pmatrix} = \begin{pmatrix} D_p g & D_q g \\ D_p h & D_q h \end{pmatrix}_{(p,q)=\widetilde{\gamma}_i(t)} \begin{pmatrix} R_i(t) & h_i(\widetilde{\gamma}_i(t)) \\ \nu_i(t) & g_i(\widetilde{\gamma}_i(t)) \end{pmatrix}$$

and

$$\begin{pmatrix} R_i(t) & h_i(\widetilde{\gamma}_i(t)) \\ \nu_i(t) & g_i(\widetilde{\gamma}_i(t)) \end{pmatrix}_{t=0} = \begin{pmatrix} 1 & h_i(B_{i-1}, z_{i-1}) \\ 0 & g_i(B_{i-1}, z_{i-1}) \end{pmatrix}.$$

By Abel's formula for the Wronskian, it follows that

$$\begin{aligned} &\det \begin{pmatrix} R_i(t) & h_i(\widetilde{\gamma}_i(t)) \\ \nu_i(t) & g_i(\widetilde{\gamma}_i(t)) \end{pmatrix}_{t=t_i} \\ &= \det \begin{pmatrix} R_i(t) & h_i(\widetilde{\gamma}_i(t)) \\ \nu_i(t) & g_i(\widetilde{\gamma}_i(t)) \end{pmatrix}_{t=0} \exp\left(\int_0^{t_i} \operatorname{tr} \begin{pmatrix} D_p g_i & D_q g_i \\ D_p h_i & D_q h_i \end{pmatrix}_{(p,q)=\widetilde{\gamma}_i(t)} dt \right) \\ &= \det \begin{pmatrix} 1 & h_i(B_{i-1}, z_{i-1}) \\ 0 & g_i(B_{i-1}, z_{i-1}) \end{pmatrix} \exp\left(\int_0^{t_i} (D_p g_i + D_q h_i)(\widetilde{\gamma}_i(t))\, dt \right), \\ &= g_i(B_{i-1}, z_{i-1}) \exp\left(\int_0^{t_i} (D_p g_i + D_q h_i)(\widetilde{\gamma}_i(t))\, dt \right). \end{aligned} \tag{A.24}$$

By (A.23) and (A.24), we then obtain (A.18). ∎

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