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Amphichiral knots with large 4-genus

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Abstract

For each $g > 0$ we give infinitely many knots that are strongly negative amphichiral, hence rationally slice and representing 2-torsion in the smooth concordance group, yet which do not bound any locally flatly embedded surface in the 4-ball with genus less than or equal to g. Our examples also allow us to answer a question about the 4-dimensional clasp number of knots.

1. *Introduction*

An oriented knot K in S^3 is called *strongly negative amphichiral* if there exists an orientation reversing involution $\varphi\colon S^3 \to S^3$ such that $\varphi(K) = K^r$. Many concordance invariants vanish on such knots, including the classical Tristram-Levine signature function [**13**], [**23**] and more modern invariants coming from Heegaard Floer and Khovanov homology like the τ-invariant [**20**], ν^+-invariant [**8**], Υ-invariant [**21**], s-invariant [**22**], s_n-invariants [**18**], [**24**], $s^\#$-invariant [**12**], and $\daleth$-invariant [**14**]. Notably, this list contains almost all known lower bounds on the *4-genus*, or minimal genus of a (smoothly or locally flatly) embedded orientable surface in B^4 with boundary the given knot. However, we use Gilmer's bound on the topological 4-genus [**5**] coming from Casson-Gordon signatures [**3**] to prove the following.

Theorem 1.1. *For any $g > 0$, there exists a knot K with the following properties:*
(i) *K is strongly negative amphichiral.*
(ii) *K can be transformed to a smoothly slice knot by changing some crossings $(+)$ to $(-)$.*
(iii) *K can be transformed to a smoothly slice knot by changing some crossings $(-)$ to $(+)$.*
(iv) *the topological 4-genus of K is strictly larger than g.*

In fact, something more is true, and proven in Proposition 2.7: for any $g \in \mathbb{N}$ there exists an infinite family of knots $\{K^k\}_{k\in\mathbb{N}}$, generating a subgroup of the concordance group isomorphic to $(\mathbb{Z}_2)^\infty$, such that any nontrivial sum $K = \#_{j=1}^m K^{k_j}$ satisfies the conclusions of Theorem 1.1. Moreover, each of the knots K^k is algebraically slice, so we incidentally reprove a result of Livingston [**16**] that there is a $(\mathbb{Z}_2)^\infty$-subgroup of the concordance group consisting of algebraically slice knots.

Negative amphichiral knots, if not slice, represent 2-torsion elements of the smooth concordance group; a still-open question of Gordon asks whether all 2-torsion elements have such representatives [**6**, Problem 16]. We therefore obtain the following corollary to Theorem 1.1, which appears to be previously unknown.

Corollary 1.2. *There exist 2-torsion knots with arbitrarily large 4-genera.*

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A knot K is called *rationally slice* if there exists a smooth 4-manifold W with boundary $\partial W = S^3$ and $H_*(W;\mathbb{Q}) = H_*(B^4;\mathbb{Q})$ such that K bounds a smoothly embedded null-homologous disc in W. Every strongly negative amphichiral knot is rationally slice [**10**], and so Theorem 1.1 also answers a question of [**7**] in the affirmative.

Corollary 1.3. *There exist rationally slice knots with arbitrarily large 4-genera.*

The *4-dimensional clasp number* $c_4(K)$ of a knot K is the minimal number of transverse double points across all immersions of D^2 in B^4 with $\partial D^2 = K$. Similarly, $c_4^+(K)$ (respectively $c_4^-(K)$) is defined to be the minimal number of positive (resp. negative) transverse double points across all immersions of D^2 in B^4 with $\partial D^2 = K$. It follows immediately from the definitions that $c_4^+ + c_4^- \leq c_4$; the figure-eight knot 4_1 is the prototypical example of when this inequality is strict, since $c_4^+(4_1) = c_4^-(4_1) = 0$ and yet $c_4(4_1) = 1$. We answer a question of [**9**] by giving the first examples of knots for which $c_4(K)$ is arbitrarily larger than $c_4^+(K) + c_4^-(K)$.

Corollary 1.4. *The difference between $c_4(K)$ and $c_4^+(K) + c_4^-(K)$ can be arbitrarily large.*

Proof. For $g \in \mathbb{N}$, let K_g be a knot satisfying the conclusions of Theorem 1.1. By items (ii) and (iii), we have that $c_4^+(K_g) + c_4^-(K_g) = 0 + 0 = 0$, and by item (iv) we have that

$$g < g_4(K_g) \leq g_4^s(K_g) \leq c_4(K_g),$$

noting that standard arguments show that for any knot K the smooth 4-genus $g_4^s(K)$ is bounded above by $c_4(K)$. □

Since Casson-Gordon signatures provide bounds on the topological 4-genus, it remains open whether one can find examples for the smooth analogue of Theorem 1.1. In particular, the following three questions remain open.

Question 1. For $g \in \mathbb{N}$, is there a topologically slice knot K such that $g_4^s(K) > g$ and
(i) K has order 2 in the smooth concordance group?
(ii) K is smoothly rationally slice?
(iii) $c_4^+(K) = c_4^-(K) = 0$?

Recent work of Hom-Kang-Park-Stoffregen [**7**] has shown that $\{C_{2n+1,1}(4_1)\}_{n\in\mathbb{N}}$ generates a $\mathbb{Z}^\infty$-subgroup of rationally slice knots in the smooth concordance group. By work of [**4**], the topological 4-genus of $C_{2n+1,1}(4_1)$ equals 1 for all $n \in \mathbb{N}$, but it remains open whether the smooth 4-genus of $C_{2n+1,1}(4_1)$ is large. Since $2n+1$ is relatively prime to 2, one can combine the work of this paper with the formulas for Casson-Gordon signatures of satellite knots given in [**15**] and conclude that for our choice of K_g satisfying the conclusions of Theorem 1.1, we have that $g_4(C_{2n+1,1}(K_g)) > g$ for all $n \in \mathbb{N}$. We therefore state the following as an interesting open problem in either the smooth or topological categories.

Question 2. For any $g \in \mathbb{N}$, let K_g be one of the knots given in Section 2 that satisfies the conclusions of Theorem 1.1. For some or all $n \in \mathbb{N}$, determine whether $C_{2n+1,1}(K_g)$ is infinite order in the concordance group.

We note that it remains open even whether $C_{2n,1}(K)$ must always be slice for strongly negative amphichiral K, though it is known that many such knots are not ribbon [**19**].

REMARK 1. The key feature of Casson-Gordon signatures that allows us to use Gilmer's bound to establish Theorem 1.1 when all other lower bounds on the 4-genus fail might initially seem like a flaw: no single signature gives a 4-genus bound or even a sliceness obstruction. While we avoid stating the precise definition of these invariants, we remind the reader that $\sigma(K,\chi) \in \mathbb{Q}$ depends on not just the knot K but a choice of map χ from the first homology of the double branched cover of K to a cyclic group. The fact that K is negative amphichiral implies that there is an involution ι on the set of such maps such that $\sigma(K,\iota(\chi)) = -\sigma(K,\chi)$. As long as this involution is non-trivial, the negative amphichirality of K does not force $\sigma(K,\chi)$ to vanish and there is still the potential to obtain a sliceness obstruction–and even a lower bound on the 4-genus–by considering the set of all such signatures. This could be considered as philosophically similar to the fact that Casson-Gordon signatures can obstruct knots from being concordant to their reverses [**11**], though that result requires a careful analysis of additional structure that we are able to avoid.

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2. *Proof of Main Result*

Our examples are connected sums of certain satellites of the figure-eight knot.

EXAMPLE 1. Let J be a reversible knot and define $K(J)$ to be as in Figure 1, where $\overline{J}$ denotes the mirror image of J, which since J is reversible equals the concordance inverse $-J$. We note for later that the disc-with-bands Seifert surface for $K(J)$ visible on the left of Figure 1 demonstrates that $K(J)$ shares a Seifert form with the figure-eight knot K_0.

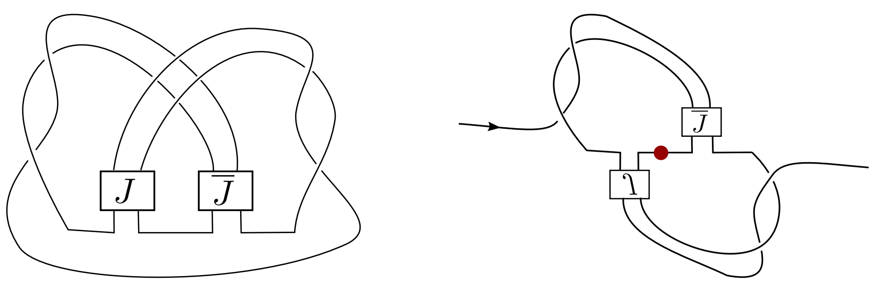

FIGURE 1. *The knot $K(J)$ from two perspectives.*

The right side of Figure 1 demonstrates that $K(J)$ is strongly negative amphichiral: rotation by 180 degrees in the plane about the marked point followed by reflection in the plane of the page takes $K(J)$ to itself, but with reversed orientation. An alternate construction of this involution comes from considering the decomposition of the exterior of $K(J)$ as the exterior of $K(U)$, the figure eight knot, with two solid tori cut out and the exteriors of J and $\bar{J}$ glued in places. One can then verify that the involution guaranteeing the strong negative amphichirality of $K(U)$ exchanges said tori, and hence yields an appropriate involution on the exterior of $K(J)$.

PROPOSITION 2.1. *If J is a reversible knot, then $K(J)$ has $c_4^+(K_J) = c_4^-(K_J) = 0$.*

Proof. Consider the knots $K_\pm$ as depicted in Figure 2, shown with genus one Seifert surfaces $F_\pm$ in disc-with-bands position. Observe that K_+ (respectively K_-) is obtained from K_J by changing a single negative (resp. positive) crossing to a positive (resp. negative) crossing. Figure 2 also depicts a curve $\gamma_\pm$ on $F_\pm$. Note that each of $\gamma_\pm$ represents a nontrivial

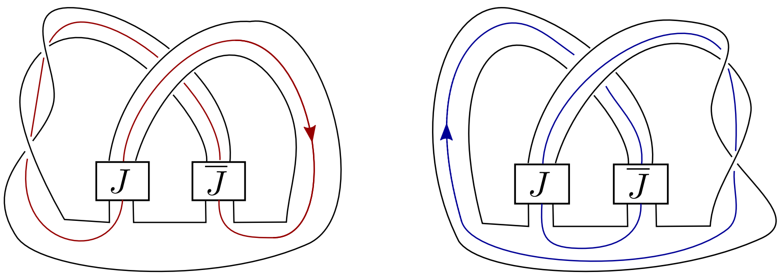

FIGURE 2. *K_+, obtained by changing a crossing from $-$ to $+$ (left) and K_-, obtained by changing a crossing from $+$ to $-$ (right).*

element of $H_1(F_\pm)$ and is 0-framed by $F_\pm$; i.e. is an *derivative curve*. Considered as a knot, γ_+ is $J\#\overline{J}$; since J is reversible this is isotopic to $J\#-J$ and hence is slice. Similarly, the knot type of γ_- is the slice knot $J\#-J$. Therefore, surgering the Seifert surface $F_\pm$ along the derivative curve $\gamma_\pm$ yields a smooth slice disc for $K_\pm$. We can convert this single crossing change from $K(J)$ to $K_\pm$ into an immersed annulus in $S^3 \times I$ from $K(J)$ to $K_\pm$. Capping each of these annuli with a smooth slice disc for $K_\pm$ yields the desired immersed discs bounded by $K(J)$, each with a single singularity of different sign. □

2.1. *Background results*

For $n \in \mathbb{N}$ and a knot K, we let $\Sigma_n(K)$ denote the nth cyclic branched cover of S^3 along K. To a knot K and a map $\chi\colon H_1(\Sigma_n(K)) \to \mathbb{Z}_q$ one can associate the Casson-Gordon signature $\sigma(K,\chi) \in \mathbb{Q}$ [3]. We avoid giving the technical definition of these invariants, noting only that they are defined in terms of the twisted intersection form of some 4-manifold and are notoriously difficult to compute precisely. We remark for those familiar with Casson-Gordon signatures that in the literature what we call $\sigma(K,\chi)$ is just $\sigma_1\tau(K,\chi)$ instead.

Our lower bound on the topological 4-genus of a knot comes from the following slightly reformulated result of Gilmer.

THEOREM 2.2. *[5, Theorem 1] Suppose that K is a knot with $g_4(K) \leq g$. Then there is a decomposition $H_1(\Sigma_2(K)) = A_1 \oplus A_2$ such that:*
(i) *A_1 has a presentation with at most $2g$ generators.*
(ii) *There is some $B \leq A_2$ with $|B|^2 = |A_2|$ such that for any prime power order $\chi\colon H_1(\Sigma(K)) \to \mathbb{Z}_q$ with χ vanishing on $A_1 \oplus B$, we have*

$$|\sigma(K,\chi) + \sigma(K)| \leq 4g.$$

We remark for later that in our applications of Theorem 2.2 we will always have that K is negative amphichiral and hence that $\sigma(K) = 0$.

In particular, we have the following corollary, which is all that we need for Theorem 1.1.

COROLLARY 2.3. *Suppose that K is a knot with $g_4(K) \leq g$ such that $H_1(\Sigma_2(K))$ is not generated as an abelian group by any $2g$ of its elements. Then there exists a prime p and a nontrivial character $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_p$ such that*

$$|\sigma(K, \chi) + \sigma(K)| \leq 4g.$$

Proof. Let A_1, A_2, and B be as in in the conclusion of Theorem 2.2. By our assumption on $H_1(\Sigma_2(K))$, we have that A_1 does not equal all of $H_1(\Sigma_2(K))$. Therefore $|A_1| < |H_1(\Sigma_2(K))|$ and so

$$|A_1 \oplus B| = |A_1| \cdot |B| = |A_1| \cdot \sqrt{|A_2|} = |A_1| \cdot \sqrt{\frac{|H_1(\Sigma_2(K))|}{|A_1|}} = \sqrt{|A_1|}\sqrt{|H_1(\Sigma_2(K))|}$$

is strictly less than the order of $H_1(\Sigma_2(K))$. That is, $A_1 \oplus B$ is a proper subgroup of $H_1(\Sigma_2(K))$. It follows that for any prime p dividing the index of $A_1 \oplus B$ in $H_1(\Sigma_2(K))$ there exists a nontrivial character $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_p$ that vanishes on $A_1 \oplus B$, and hence that by Theorem 2.2 satisfies

$$|\sigma(K, \chi) + \sigma(K)| \leq 4g$$

as desired. □

Litherland proved a much more general formula for the Casson-Gordon invariants of satellite knots, but we will only need the following special case.

THEOREM 2.4. *[**15**, Special case of Theorem 2] Suppose P is a pattern of winding number 0 described by an unknot η in the complement of $P(U)$. Let x denote the homology class of one of the lifts of η to $\Sigma_2(P(U))$. For any knot J, there is an isomorphism $\alpha\colon H_1(\Sigma_2(P(J))) \to H_1(\Sigma_2(P(U)))$ such that for any $\chi\colon H_1(\Sigma_2(P(U))) \to \mathbb{Z}_q$ we have*

$$\sigma(P(J), \chi \circ \alpha) = \sigma(P(U), \chi) + 2\sigma_J(\omega_q^{\chi(x)}),$$

where $\omega_q = e^{2\pi i/q}$ and σ_J denotes the Tristram-Levine signature function.

As well as the knot invariant $\sigma(K, \chi)$, Casson-Gordon introduced a signature invariant $\sigma(M, \phi)$ associated to a 3-manifold M and a character $\phi\colon H_1(M) \to \mathbb{Z}_q$. These are much more computable than the knot Casson-Gordon signatures and satisfy the following key property.

PROPOSITION 2.5. *[**3**, Lemma 3 and Theorem 4] Let J be a knot such that $H_1(\Sigma_2(J))$ is cyclic. Then for any prime power order character χ on $H_1(\Sigma_2(J))$, we have that*

$$|\sigma(J, \chi) - \sigma(\Sigma_2(J), \chi)| \leq 1.$$

We will need a formula due to Cimasoni-Florens for the Casson-Gordon signature of a 3-manifold in terms of the colored signature function of a surgery link. Although this result is proved in much more generality, we state it only for the case of interest: when M is obtained by surgery on a Hopf link. We thereby avoid going into the technical details of the definition of the colored signature function, noting only for the experts that the cell complex consisting of 2 discs meeting in a single arc and bounded by the Hopf link is a C-complex in the sense of [2], and the contractibility of this complex immediately implies that the colored signature function of the Hopf link is identically zero.

THEOREM 2.6. *[2, Theorem 6.7] Suppose that a 3-manifold M is obtained by surgery on a Hopf link L with linking matrix $\Lambda = \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$. Let q be prime and $\chi\colon H_1(M) \to \mathbb{Z}_q$ be a character such that the two meridians μ_1, μ_2 of L are sent to nonzero elements of $\mathbb{Z}_q$. For $i = 1, 2$ let $n_i \in \{1, \dots, q-1\}$ be the unique value satisfying $n_i \equiv \chi(\mu_i) \mod q$. Then*

$$\sigma(M, \chi) = -1 - sign(\Lambda) + \frac{2}{q^2} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}^T \cdot \begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix} \cdot \begin{bmatrix} q - n_1 \\ q - n_2 \end{bmatrix}$$

2.2. *Proof of Theorem 1.1*

In the proof of Theorem 1.1, we will need a formula for the Casson-Gordon signatures of K_J in terms of the Tristram-Levine signatures of J.

EXAMPLE 2. Let K_0 denote the figure-eight knot. Note that $K(J)$ is obtained from K_0 by two infections along curves η_1 and η_2, as depicted in Figure 3.

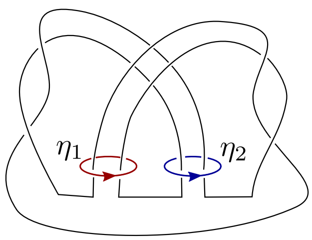

FIGURE 3. *The knot $K(J)$ is an iterated satellite of the figure-eight knot.*

By twice applying Theorem 2.4, we see that for any knot J there is an isomorphism $\alpha\colon H_1(\Sigma_2(K(J))) \to H_1(\Sigma_2(K_0))$ such that for any character $\chi\colon H_1(\Sigma_2(K_0)) \to \mathbb{Z}_q$ we have

$$\sigma(K(J), \alpha \circ \chi) = \sigma(K_0, \chi) + 2\sigma_J(\omega_q^{\chi(\widetilde{\eta_1})}) + 2\sigma_{\overline{J}}(\omega_q^{\chi(\widetilde{\eta_2})}) = \sigma(K_0, \chi) + 2\sigma_J(\omega_q^{\chi(\widetilde{\eta_1})}) - 2\sigma_J(\omega_q^{\chi(\widetilde{\eta_2})})$$

Since both η_i curves are disjoint from the usual genus one Seifert surface for K_0, we can apply Akbulut-Kirby's algorithm of [1] to obtain the following surgery diagram for $\Sigma_2(K_0)$, with lifts of η_1 and η_2 as indicated. (Note that we have only depicted one lift of each curve, since that is all we need to apply Theorem 2.6.) The first homology of $\Sigma_2(K_0)$ is generated

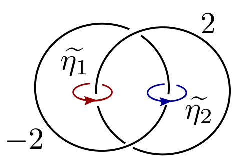

FIGURE 4. *A surgery diagram L for $\Sigma_2(K_0)$.*

by the meridians of the components of L, which are isotopic to $\widetilde{\eta_1}$ and $\widetilde{\eta_2}$. The relations are given by the rows of the linking-framing matrix, and are

$$-2[\widetilde{\eta_2}] + [\widetilde{\eta_1}] = 0 \text{ and } [\widetilde{\eta_2}] + 2[\widetilde{\eta_1}] = 0.$$

Some quick simplifications give us that $H_1(\Sigma_2(K_0)) \cong \mathbb{Z}_5$, generated by $a := [\widetilde{\eta_2}]$ and such that $[\widetilde{\eta_1}] = 2[\widetilde{\eta_2}]$. Therefore, for any character $\chi\colon H_1(\Sigma_2(K_0)) \to \mathbb{Z}_5$ we have that

$$\sigma(K_J, \chi \circ \alpha) = \sigma(K_0, \chi) + \sigma_J(\omega_5^{2\chi(a)}) - \sigma_J(\omega_5^{\chi(a)}). \tag{2.1}$$

We can also use the surgery diagram of Figure 4 to bound $|\sigma(K_0, \chi)|$. For $j \in \mathbb{Z}_5$, define $\chi_j\colon H_1(\Sigma_2(K_0)) \to \mathbb{Z}_5$ to be the map with $\chi_j(a) = j$. Observe that $\chi_1([\widetilde{\eta_1}]) = 2$ and $\chi_2([\widetilde{\eta_1}]) = 4$. Therefore, Theorem 2.6 gives us that

$$\sigma(\Sigma_2(K_0), \chi_1) = -1 - 0 + \frac{2}{25}\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = -1 + \frac{30}{25} = 1/5$$

and

$$\sigma(\Sigma_2(K_0), \chi_2) = -1 - 0 + \frac{2}{25}\begin{bmatrix} 2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = -1 + \frac{20}{25} = -1/5.$$

Moreover, basic properties of Casson-Gordon signatures (or reapplying Theorem 2.6) imply that $\sigma(\Sigma_2(K_0), \chi_3) = \sigma(\Sigma_2(K_0), \chi_2)$, $\sigma(\Sigma_2(K_0), \chi_4) = \sigma(\Sigma_2(K_0), \chi_1)$, and $\sigma(\Sigma_2(K_0), \chi_0) = 0$.

Since $H_1(\Sigma_2(K_0)) \cong \mathbb{Z}_5$ is cyclic, Proposition 2.5 implies that for any $\chi\colon H_1(\Sigma_2(K_0)) \to \mathbb{Z}_5$ we have $|\sigma(K_0, \chi) - \sigma(\Sigma_2(K_0), \chi)| \leq 1$ and hence, by the above computation, that $|\sigma(K_0, \chi)| < 2$.

We are now ready to prove the following and obtain Theorem 1.1 as a consequence.

Proposition 2.7. *Fix $g \in \mathbb{N}$. For $i \in \mathbb{N}$ define $J_i = \#^{m_i} T_{2,5}$, where $m_i = 2^{2i+1} g$. Now, for $k \in \mathbb{N}$ define $K^k := \#_{i=1}^{2g+2} K(J_{k(2g+2)+i})$. Then $S = \{K^k\}_{k \in \mathbb{N}}$ is a collection of algebraically slice knots such that any nontrivial sum $K = \#_{j=1}^{n} K^{k_j}$ satisfies the conclusions of Theorem 1.1.*

Proof. Observe that for any choice of J, the knot $K(J)$ shares a Seifert form with K_0. Therefore, each K^k shares a Seifert form with the slice knot $\#_{i=1}^{2g+2} K_0$, and hence is algebraically slice. Also, since Seifert forms determine the homology of cyclic branched covers, we record for later that for any k we have

$$H_1(\Sigma_2(K^k)) \cong H_1(\Sigma_2(\#_{i=1}^{2g+2} K_0)) \cong \mathbb{Z}_5^{2g+2}.$$

Now let $K = \#_{j=1}^{n} K^{k_j}$ be a nontrivial sum of elements of S. Since each K^{k_j} is 2-torsion, we can and do assume that $k_1 < k_2 < \cdots < k_n$. Since strong negative amphichirality, rational sliceness, and being related to a slice knot via $(+)$ to $(-)$ (or $(-)$ to $(+)$) crossing changes are all preserved under the connected sum operation, it only remains to verify item (iv).

Since $n \geq 1$,

$$H_1(\Sigma_2(K)) \cong \bigoplus_{j=1}^{n} H_1(\Sigma_2(K^{k_j})) \cong \mathbb{Z}_5^{n(2g+2)} \tag{2.2}$$

is not generated by any $2g$ of its elements. So Corollary 2.3 applies and it is enough to show that for every nontrivial character $\chi\colon H_1(\Sigma_2(K)) \to \mathbb{Z}_5$, we have $|\sigma(K, \chi)| > 4g$.

Let χ be a nontrivial character, which by the isomorphism of Equation 2.2 we can write as $\chi = ((\chi_i^j)_{i=1}^{2g+2})_{j=1}^{n}$. By the additivity of Casson-Gordon signatures with respect to connected sum, we have that

$$\sigma\left(K, \left((\chi_i^j)_{i=1}^{2g+2}\right)_{j=1}^{n}\right) = \sum_{j=1}^{n} \sigma(K^{k_j}, (\chi_i^j)_{i=1}^{2g+2}) = \sum_{j=1}^{n}\sum_{i=1}^{2g+2} \sigma(K(J_{k_j(2g+2)+i}), \chi_i^j). \tag{2.3}$$

Moreover, for each $1 \leq j \leq n$ and $1 \leq i \leq 2g+2$, Theorem 2.4 and Equation 2.1 of Example 2 tells us that there is an identification

$$\alpha_i^j \colon H_1(\Sigma_2(K(J_{k_j(2g+2)+i}))) \to H_1(\Sigma_2(K_0))$$

such that for any $\phi \colon H_1(\Sigma_2(K_0)) \to \mathbb{Z}_5$, we have

$$\sigma(K(J_{k_j(2g+2)+i}), \phi \circ \alpha_i^j) = \sigma(K_0, \phi) + 2\sigma_{J_{k_j(2g+2)+i}}(\omega_5^{2\phi(a)}) - 2\sigma_{J_{k_j(2g+2)+i}}(\omega_5^{\phi(a)}). \tag{2.4}$$

Now, for each $1 \leq j \leq n$ and $1 \leq i \leq 2g+2$, define

$$\beta_i^j := \chi_i^j \circ (\alpha_i^j)^{-1} \colon H_1(\Sigma_2(K_0)) \to \mathbb{Z}_5.$$

Equations 2.3 and 2.4 combine to tell us that

$$\sigma(K, \chi) = \sum_{j=1}^{n}\left(\sum_{i=1}^{2g+2} \sigma(K_0, \beta_i^j) + 2\sigma_{J_{k_j(2g+2)+i}}(\omega_5^{2\beta_i^j(a)}) - 2\sigma_{J_{k_j(2g+2)+i}}(\omega_5^{\beta_i^j(a)})\right) \tag{2.5}$$

Since χ is nontrivial, there exists some j such that $(\chi_i^j)_{i=1}^{2g+1}$ and hence $(\beta_i^j)_{i=1}^{2g+1}$ is not identically zero. Let j_0 be the maximal such j and let i_0 be the maximal i such that $\beta_i^{j_0}$ is nonzero. Let $\ell = k_{j_0}(2g+2) + i_0$. The following algebraic manipulations show that $\sigma(K(J_\ell), \chi_{i_0}^{j_0})$ so dominates the other terms that could contribute to $\sigma(K, \chi)$ that we have as desired that $|\sigma(K, \chi)| > 4g$.

Recalling that each $J_i = \#^{m_i} T_{2,5}$, where $m_i = 2^{2i+1}g$, we have by the additivity of Tristram-Levine signatures under connected sum that $\sigma_{J_i}(\omega_5) = \sigma_{J_i}(\omega_5^4) = -2^{2i+2}g$ and $\sigma_{J_i}(\omega_5^2) = \sigma_{J_i}(\omega_5^3) = -2^{2i+3}g$ (see KnotInfo [17] for the Tristram-Levine signature function of $T(2,5)$.) Applying Equation 2.1 from Example 2, we see that for any i and any nonzero character $\rho \colon H_1(\Sigma_2(K(J_i))) \to \mathbb{Z}_5$ we have that

$$2^{2i+3}g - 2 \leq |\sigma(K(J_i), \rho)| = |\sigma(K_0, \rho) \pm (2\sigma_{J_i}(\omega_5) - 2\sigma_{J_i}(\omega_5^2))| \leq 2^{2i+3}g + 2, \tag{2.6}$$

where we briefly suppress the identification of $H_1(\Sigma_2(K(J_i)))$ with $H_1(\Sigma_2(K_0))$.

Observe that the set of natural numbers

$$\{k_{j_0}(2g+2) + i : 1 \leq i \leq i_0 - 1\} \cup \bigcup_{j=1}^{j_0-1} \{k_j(2g+2) + i) : 1 \leq i \leq 2g+2\} \tag{2.7}$$

is a subset of $\{1, \dots, \ell - 1\}$, where $\ell = k_{j_0}(2g+2) + i_0$.

Recalling that $\sigma(K(J_{k_j(2g+2_+i)}), \chi_i^j) = 0$ whenever $j > j_0$ or $j = j_0$ and $i > i_0$, we therefore have that

$$\begin{aligned}
|\sigma(K, \chi)| &= \left|\sum_{j=1}^{n}\sum_{i=1}^{2g+2} \sigma(K(J_{k_j(2g+2)+i}), \chi_i^j)\right| \\
&= \left|\sigma(K(J_\ell), \chi_{i_0}^{j_0}) + \sum_{i=1}^{i_0-1} \sigma(K(J_{k_{j_0}(2g+2)+i}), \chi_i^{j_0}) + \sum_{j=1}^{j_0-1}\sum_{i=1}^{2g+2} \sigma(K(J_{k_j(2g+2)+i}), \chi_i^j)\right| \\
&\geq \left|\sigma(K(J_\ell), \chi_{i_0}^{j_0})\right| - \sum_{i=1}^{i_0-1} \left|\sigma(K(J_{k_{j_0}(2g+2)+i}), \chi_i^{j_0})\right| - \sum_{j=1}^{j_0-1}\sum_{i=1}^{2g+2} \left|\sigma(K(J_{k_j(2g+2)+i}), \chi_i^j)\right| \\
&\geq (2^{2\ell+3}g - 2) - \sum_{k=1}^{\ell-1}(2^{2k+3}g + 2) =: (*)
\end{aligned}$$

where in the last inequality we use our observation from Equation 2.7 together with Equation 2.6. Some algebraic simplification yields that

$$(*) = 8g\left(2^{2\ell} - \sum_{k=1}^{\ell-1} 2^{2k}\right) - 2\ell = (g/3)(2^{2\ell+3} - 32) - 2\ell.$$

Now, note that since $\ell > 2g + 2 \geq 4$ we have that $2\ell + 3 > 11$ and so certainly $2^{2\ell+3} - 32 > 2^{2\ell+2}$. Therefore

$$|\sigma(K, \chi)| \geq (*) > (g/3)2^{2\ell+2} - 2\ell > 2^{2\ell} - 2\ell.$$

Finally, we observe that for any $x > 2$ we have $2^{2x} - 2x > 2x$, since letting $f(x) = 2^{2x} - 4x$ we see that $f'(x) = \ln(4)2^{2x} - 4$ is positive for all $x \geq 1$ and $f(2) = 8$. Therefore

$$|\sigma(K, \chi)| > 2\ell > 4g + 4 > 4g,$$

as desired. □

Remark 2. The examples of Proposition 2.7 are far from the only knots satisfying the conclusions of Theorem 1.1. One could vary the base knot, for example by choosing $\{a_i\}_{i\geq 0}$ to be natural numbers such that $\{4a_i^2 + 1\}_{i\in\mathbb{N}}$ consists of pairwise relatively prime numbers. (This is easily accomplished by e.g. letting $a_0 = 1$ and $a_k = \prod_{i=1}^{k-1}(4a_i^2 + 1)$ for $k \geq 1$.) Now, let K_i be the 2-bridge knot corresponding to the rational number $\frac{4a_i^2+1}{2a_i}$, noting that indeed K_0 is the figure-eight knot. Choose $\{p_i\}_{i\geq 0}$ to be primes dividing $4a_i^2 + 1$, noting that by our choice of a_i we have that p_i divides $4a_j^2 + 1$ if and only if $j = i$. By taking connected sums of K_{a_i} analogously infected with large connected sums of T_{2,p_i} and $-T_{2,p_i}$, we can essentially repeat the arguments of Proposition 2.7 and obtain many more linearly independent knots satisfying the conclusions of Theorem 1.1.

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