BRANCHED COVERS BOUNDING RATIONAL HOMOLOGY BALLS

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Abstract. Prime power fold cyclic branched covers along smoothly slice knots all bound rational homology balls. This phenomenon, however, does not characterize slice knots. In this paper, we give a new construction of non-slice knots that have the above property. The sliceness obstruction comes from computing twisted Alexander polynomials, and we introduce new techniques to simplify their calculation.

1. Introduction

For a knot $K \subset S^3$, let $\Sigma_q(K)$ denote the q-fold cyclic branched cover of S^3 along K. Consider the set of prime powers $\mathcal{Q} = \{p^\ell \mid p \text{ prime}, \ell \in \mathbb{N}\}$. For $q \in \mathcal{Q}$, the three-manifold $\Sigma_q(K)$ is a rational homology sphere – i.e. $H_*(\Sigma_q(K);\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$. It is not hard to see that if $K \subset S^3$ is smoothly slice – i.e. bounds a smooth, properly embedded disk D in the 4-ball D^4 – then $\Sigma_q(K)$ bounds a smooth rational homology ball X^4, that is, $\Sigma_q(K) = \partial X^4$ and $H_*(X^4;\mathbb{Q}) \cong H_*(D^4;\mathbb{Q})$. Indeed, the q-fold cyclic branched cover of D^4 branched along D will be such a four-manifold. It is natural to ask if the property that all prime power fold cyclic branched covers bound rational homology balls characterizes slice knots.

To put this question in a more algebraic framework, notice that $\Sigma_q(-K) = -\Sigma_q(K)$ (where $-K$ is the reverse of the mirror image of the knot K, and $-Y$ is the three-manifold Y with reversed orientation) and $\Sigma_q(K_1 \# K_2) = \Sigma_q(K_1) \# \Sigma_q(K_2)$. Hence the map

$$K \mapsto \Sigma_q(K)$$

descends to a homomorphism $\mathcal{C} \to \Theta^3_{\mathbb{Q}}$, where $\mathcal{C}$ denotes the smooth concordance group of knots in S^3, and $\Theta^3_{\mathbb{Q}}$ is the smooth rational homology cobordism group of rational homology spheres. We then let

$$\varphi\colon \mathcal{C} \to \prod_{q\in\mathcal{Q}} \Theta^3_{\mathbb{Q}},$$

be the homomorphism given by

$$[K] \mapsto ([\Sigma_q(K)])_{q\in\mathcal{Q}},$$

and note that $[K] \in \ker\varphi$ exactly when all the prime power fold cyclic branched covers of K bound rational homology balls. The following question was posed during problem sessions at the conference on Synchronizing Smooth and Topological 4-manifolds and at the conference on 4-Manifolds and Knot Concordance, hosted by Banff International Research Station and the Max Planck Institute for Mathematics, respectively, in 2016 [1, 2].

Question 1.1. Is $\ker\varphi$ trivial?

In fact, the answer to this question can be rather swiftly shown to be 'no'. Since the 3-manifold $\Sigma_q(K)$ is independent of the orientation of K, any knot J which is not concordant to its reverse J^r yields a nontrivial element $K_J := J \# -J^r$ in $\ker\varphi$. The existence of such J was first established by Livingston; see [27, 28] for proofs. In fact, recent work of Kim and Livingston together with the above observation implies that $\ker\varphi$ contains an infinitely generated free subgroup, generated by topologically slice knots of the form $K \# -K^r$ [25].

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Considerably less seems to be known with regards to finite order elements in $\ker\varphi$. Kirk and Livingston showed that the knot 8_{17}, which is negative-amphichiral, is not concordant to its reverse [27] (see also [7]), and so the above observation implies that $8_{17}\#8_{17}^r$ represents a nontrivial element of order two in $\ker\varphi$.

In this article, we give a new construction that yields large families of knots representing elements in $\ker\varphi$. In addition, we show that four of these knots generate a $(\mathbb{Z}_2)^4$ subgroup of $\ker\varphi$. We remark that an easy extension of our arguments in the proof of Theorem 1.4 show that adding in $8_{17}\#8_{17}^r$ in fact gives a $(\mathbb{Z}_2)^5$ subgroup. It is an interesting technical challenge to show that a $(\mathbb{Z}_2)^\infty$ subgroup exists in $\ker\varphi$; we expect that our examples (see Theorem 1.3) generate such a subgroup.

While our knots are prime and hence not isotopic to any $J\#-J^r$, it is not at all obvious how to show that they are not concordant to any knot of the form $J\#-J^r$. To belabor this point, we note that the bulk of this article's work is required to merely show that these knots are not concordant to $U\#-U^r=U$. We therefore propose the following refinement of Hedden's Question 1.1 as a stimulus to future work.

Question 1.2. Is $\ker\varphi = \{[K\#-K^r] \mid K \text{ is a knot in } S^3\}$?

Our examples are constructed as follows. Let L_r be the link depicted in the left diagram of Figure 1, where the box labeled $r\in\mathbb{N}$ consists of r right-handed half-twists (and $-r$ denotes r left-handed half-twists). When r is even, L_r is a knot (a simple generalization of the figure-8 knot, which is given by L_2). As was shown in [6], these knots are rationally slice, non-slice, and strongly negative-amphichiral and moreover generate a subgroup isomorphic to $(\mathbb{Z}_2)^\infty$ in the smooth concordance group $\mathcal{C}$. If $r=2m+1$ is odd, then L_r is a 2-component link of unknots, which we redraw in the middle of Figure 1 by braiding component B_{2m+1} about component A_{2m+1}. The resulting $(2m+1)$-braid β_m is shown in the right diagram of Figure 1.

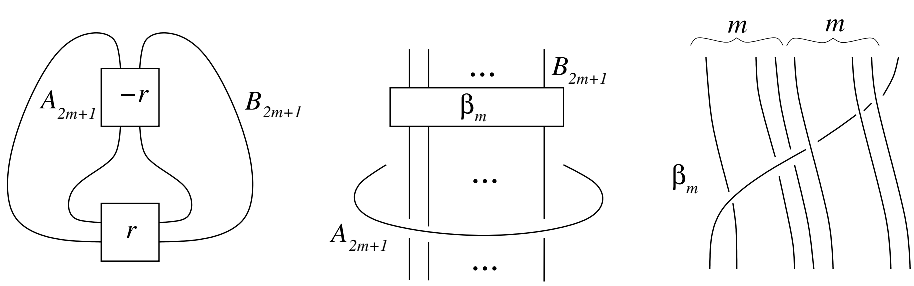

FIGURE 1. L_r (left) is a knot if r is even and is a 2-component link if $r=2m+1$ is odd. The middle diagram shows $L_{2m+1}=A_{2m+1}\cup B_{2m+1}$ redrawn as (the closure of) a $(2m+1)$-braid β_m with its braid axis. On the right we give the $(2m+1)$-braid β_m.

We define $K_{m,n}$ to be the lift of B_{2m+1} to $\Sigma_n(A_{2m+1})$, which since A_{2m+1} is an unknot is just S^3. Note that $K_{m,n}$ is a knot if $r=2m+1$ and n are relatively prime. In fact, the description of Figure 1 shows that $K_{m,n}$ is simply the braid closure of the braid β_m^n. We use the symmetry of L_{2m+1} to show that $\Sigma_q(K_{m,n})$ is diffeomorphic to $\Sigma_n(K_{m,q})$ when n and q are both relatively prime to $2m+1$. We then use the fact that $K_{m,n}$ is strongly negative-amphichiral to show that many of these knots represent elements of $\ker\varphi$.

Theorem 1.3. *If n is an odd prime power which is relatively prime to $2m+1$, then $[K_{m,n}]\in\ker\varphi$.*

For instance, if n is an odd prime power and not divisible by 3, then $K_{1,n}$ is contained in $\ker\varphi$. The knots $K_{1,n}$ previously appeared in work of Lisca [31], where it was pointed out that these knots are strongly negative-amphichiral. Therefore they are of order at most two in $\mathcal{C}$. In addition, Sartori proved in his thesis [41] that one of these knots ($K_{1,7}$ in our notation) is not slice; hence,

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by our Theorem 1.3, this knot spans $\mathbb{Z}_2 \leq \ker\varphi$. We show that some other members of this family represent non-trivial elements in $\ker\varphi$ and are moreover linearly independent. Let K_n denote $K_{1,n}$, i.e. the closure of the three-braid $(\beta_1)^n := \left(\sigma_1\sigma_2^{-1}\right)^n$.

Theorem 1.4. *The subgroup of* $\ker\varphi$ *generated by* K_7, K_{11}, K_{17}*,* K_{23}*, and* $8_{17}\#8_{17}^r$ *is isomorphic to* $(\mathbb{Z}_2)^5$.

In general, using twisted Alexander polynomials to show that a fixed knot K is not slice is not so much technically difficult as computationally intense. However, obstructing sliceness for an infinite family of knots via twisted Alexander polynomials is generally much harder, and occurs only infrequently in the literature. While we see no reason that $\{K_{6n+5}\}_{n\in\mathbb{N}}$ should not generate $(\mathbb{Z}_2)^\infty \leq \ker\varphi$, proving this via twisted Alexander polynomials would require significantly more arduous computations and arguments.

Delaying all precise definitions to Section 3, we say merely that in this context twisted Alexander polynomials are associated to a choice of $q \in \mathcal{Q}$ and a map $\chi\colon H_1(\Sigma_q(K);\mathbb{Z}) \to \mathbb{Z}_d$ for some d. In order to use twisted Alexander polynomials to obstruct a knot K from being slice, one must show that for every subgroup M of $H_1(\Sigma_q(K);\mathbb{Z})$ satisfying certain algebraic properties there exists a map χ vanishing on M such that the resulting twisted Alexander polynomial does not factor in a certain way.

By better understanding the structure of $H_1(\Sigma_q(K);\mathbb{Z})$ one can sometimes significantly reduce the number of computations that are necessary. For example, Sartori's result of [41] that K_7 is not slice requires the computation (and subsequent obstruction of factorization as a norm) of 170 different twisted Alexander polynomials, corresponding to order 13 characters vanishing on the 130 different square root order subgroups of $H_1(\Sigma_7(K_7);\mathbb{Z})$. By careful consideration of the linking form on $H_1(\Sigma_3(K_n);\mathbb{Z})$ and how its metabolizers are permuted by the induced action of an order n symmetry of K_n, we are able to prove that K_n is not slice by computing only two twisted Alexander polynomials, at least for $n = 11, 17, 23$. In fact, while we do not include these computations here, we leave as a challenge for the interested reader to reprove Sartori's result by following roughly the same argument below, but computing precisely 3 carefully chosen twisted Alexander polynomials corresponding to $\chi\colon H_1(\Sigma_3(K_7);\mathbb{Z}) \to \mathbb{Z}_7$.

In addition, we overcome the following technical difficulty, which may be of independent interest. In many settings, the easiest way to compute the homology of a knot's cyclic branched cover, with its linking form and module structure, is in terms of some nice Seifert surface. However, the standard efficient algorithms for computing the twisted Alexander polynomial corresponding to $\chi\colon H_1(\Sigma_q(K);\mathbb{Z}) \to \mathbb{Z}_d$ require one to compute a map $\phi_\chi\colon \pi_1(X_K) \to GL(q, \mathbb{Q}(\xi_d)[t^{\pm 1}])$ on the Wirtinger generators for $\pi_1(X_K)$. Relating these two perspectives is not entirely trivial, and we refer the reader to Section 3 for a discussion of this process.

Remark 1.5. One can ask an analogous question in the topological category: Is there a knot that does not bound any topologically locally flat disk in the 4-ball but all its prime power fold cyclic branched covers bound topological rational homology balls? It turns out that such examples can be constructed by using the *classical* Alexander polynomial. Let $\{n_i\}$ be the set of all natural numbers divisible by at least 3 distinct primes and K_i be a knot with Alexander polynomial the n_i^{th} cyclotomic polynomial. By Livingston [33], for each i, all the prime power fold cyclic branched covers along K_i are integral homology spheres. Hence, by Freedman [11, 12], they all bound contractible topological four-manifolds. On the other hand, since the cyclotomic polynomials are irreducible, K_i and K_j are concordant if and only if $i = j$. Hence the knots $\{K_i\}$ represent distinct elements in $\ker\varphi^{top}$, the topological analogue of $\ker\varphi$.

The results discussed in this introduction show that slice knots are not characterized by the property that each of their prime power fold cyclic branched covers bound rational homology balls. However, there is a stronger condition that one might posit as a characterization of sliceness. When

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a knot is slice, not only do its covers bound rational homology balls, but the deck transformations of the covers extend over these balls. Additionally, the lifts of the slice knot to knots in the covers bound slicing disks in these balls. This leads us to the following question.

Question 1.6.

(1) Does there exist a non-slice knot K such that $\Sigma_q(K)$ bounds a rational homology ball for each prime power q such that the deck transformations of $\Sigma_q(K)$ extend over the rational homology ball?
(2) Does there exist a non-slice knot K such that $\Sigma_q(K)$ bounds a rational homology ball for each prime power q such that the lift of K to $\Sigma_q(K)$ bounds a disk in the rational homology ball?

We remark that each of the knots $K_{m,n}$ studied in this article, as well as any knot of the form $K\# -K^r$ where K is negative-amphichiral, can be shown to have the desired properties of Question 1.6 (1) when q is odd or equal to 2, and the desired properties of Question 1.6 (2) when q is odd.

Lastly, we make a remark on some other sliceness obstructions for K_n, where n is an odd prime power not divisible by 3. Note that K_n is strongly positive-amphichiral hence it is algebraically slice [34]. Further, K_n is also strongly negative-amphichiral, which implies that it is rationally slice. Hence the τ-invariant [38], ε-invariant [18], Υ-invariant [39], Υ^2-invariant [24], ν^+-invariant [19], φ_j-invariants [9], and s-invariant [40] all vanish for K_n. Moreover, since $[K_n] \in \ker\varphi$, the sliceness obstructions from the Heegaard Floer correction term and Donaldson's diagonalization theorem (e.g. [16, 20, 30, 35]) applied to the cyclic branched covers of K_n all vanish. As mentioned above, the fact that the involution induced by the deck transformation on $\Sigma_2(K_n)$ extends to a rational homology ball (in fact it is a $\mathbb{Z}_2$ homology ball) implies that sliceness obstructions such as [3, 8] vanish.

The paper is organized as follows: in Section 2 we prove Theorem 1.3, and in Section 3 we use twisted Alexander polynomials to show Theorem 1.4.

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2. Branched covers bounding rational homology balls

In this section, we will prove Theorem 1.3 after establishing the following two propositions. We work in the smooth category.

Proposition 2.1. *Suppose that n and q are both relatively prime to $2m+1$. Then $\Sigma_q(K_{m,n})$ and $\Sigma_n(K_{m,q})$ are diffeomorphic three-manifolds.*

Proof. We can realize $\Sigma_q(K_{m,n})$ by first taking the n-fold cyclic branched cover of S^3 branched along A_{2m+1} and then the q-fold cyclic branched cover branched along the pull-back of B_{2m+1} of Figure 1. Since the roles of A_{2m+1} and B_{2m+1} are symmetric (as shown by the left diagram of Figure 1), this three-manifold is the same as the q-fold cyclic branched cover branched along A_{2m+1}, followed by the n-fold cyclic branched cover branched along the pull-back of B_{2m+1}, which is exactly $\Sigma_n(K_{m,q})$, concluding the argument. □

Proposition 2.2. *Suppose that n is relatively prime to $2m+1$. Then $K_{m,n}$ bounds a disk in a rational homology ball $X_{m,n}$ with only 2-torsion in $H_1(X_{m,n};\mathbb{Z})$.*

Recall that a knot is called *strongly negative-amphichiral* if there is an orientation-reversing involution $\tau\colon S^3 \to S^3$ such that $\tau(K) = K$ and the fixed point set of τ is a copy of $S^0 \subset K$. Proposition 2.2 follows from the following lemma, which is a special case of [21], together with a simple observation regarding the knots $K_{m,n}$. (This version of the result below was also proved in [23, Lemma 3.1].)

Lemma 2.3 ([21, Section 2])**.** *If K is a strongly negative-amphichiral knot, then K is slice in a rational homology ball X with only 2-torsion in $H_1(X;\mathbb{Z})$.*

Proof. Let τ be the orientation-reversing involution on S^3 with $\tau(K) = K$ where the fixed point set is two points. Let M_K be the three-manifold obtained by performing 0-surgery on K. Then the involution τ extends from the exterior of K to a fixed-point free orientation-reversing involution $\hat\tau$ on M_K.

The rational homology ball X of the lemma is now constructed as follows: Consider the trace W of the 0-surgery M_K, i.e. W is the four-manifold we get from $S^3 \times [0,1]$ by attaching a 0-framed 2-handle along $K \subset S^3 \times \{1\}$. Consider the quotient of W by $\hat\tau$ on its boundary component diffeomorphic to M_K. The resulting compact four-manifold X has S^3 as its boundary, and $K \subset S^3 \times \{0\}$ is obviously slice in X: the slice disk is simply the core of the 2-handle (trivially extended through $S^3 \times [0,1]$).

In order to complete the proof of the lemma, it would be enough to show that $H_*(X;\mathbb{Q}) = H_*(D^4;\mathbb{Q})$ and $H_1(X;\mathbb{Z}) \cong \mathbb{Z}_2$. For this computation, we consider an alternative description of X as follows. Factoring M_K by the free involution $\hat\tau$ we get a three-manifold M, together with a principal $\mathbb{Z}_2$-bundle $\pi\colon M_K \to M$ and an associated interval-bundle $Z \to M$. Note that $\partial Z = M_K$ and that Z retracts to M. Then X is the union of the surgery trace W with Z, glued along M_K, i.e. the four-manifold obtained by attaching a 0-framed 2-handle along the meridian of $\partial Z = M_K$. The inclusion map i induces the following exact sequence

$$H_1(\partial Z;\mathbb{Z}) \xrightarrow{i_*} H_1(Z;\mathbb{Z}) \to \mathbb{Z}_2 \to 0.$$

This implies that $H_1(X;\mathbb{Z}) \cong \mathbb{Z}_2$ since a 2-handle is attached along the generator of $H_1(\partial Z;\mathbb{Z})$ to obtain X. □

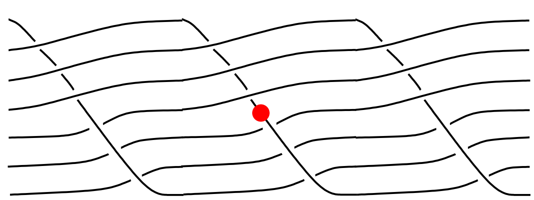

FIGURE 2. Reflection to the (red) dot in the middle provides an involution $\tau\colon S^3 \to S^3$ verifying that the knot is strongly negative-amphichiral.

Recall that $K_{m,n}$ is the braid closure of the $(2m+1)$-braid $(\beta_m)^n$ (see Figure 1 for β_m) — in the following we will view $K_{m,n}$ as this braid closure; see Figure 2 for the case of the 7-braid $(\beta_3)^3$.

Proof of Proposition 2.2. Figure 2 shows that $K_{m,n}$ is strongly negative-amphichiral: Indeed, if the (red) dot in the middle of Figure 2 is in the origin of $\mathbb{R}^3$ and the middle strand of the knot passes through $\infty = S^3 \setminus \mathbb{R}^3$, then the knot can be isotoped slightly so that the obvious extension to S^3 of the map $v \mapsto -v$ for $v \in \mathbb{R}^3$ provides the required τ. Then Lemma 2.3 completes the proof of the proposition. □

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We recall a well known lemma of Casson and Gordon and for completeness sketch its proof.

Lemma 2.4 ([5, Lemma 4.2])**.** *Suppose that $q = p^\ell$ is an odd prime power, and K is a knot that is slice in a rational homology ball X with only 2-torsion in $H_1(X;\mathbb{Z})$. Then $\Sigma_q(K)$ bounds a rational homology ball.*

Proof. Let D be the disk that K bounds in X and $\Sigma_q(D)$ be the q-fold cyclic branched cover of X branched along D. Consider the infinite cyclic cover $\widetilde{X}$ of $X \smallsetminus D$ and the following long exact sequence [36]

$$\cdots \to \widetilde{H}_i(\widetilde{X};\mathbb{Z}_p) \xrightarrow{t_*^q - \mathrm{Id}} \widetilde{H}_i(\widetilde{X};\mathbb{Z}_p) \to \widetilde{H}_i(\Sigma_q(D);\mathbb{Z}_p) \to \widetilde{H}_{i-1}(\widetilde{X};\mathbb{Z}_p) \to \cdots$$

Here t_* is the automorphism induced by the canonical covering translation. Since X is a rational homology ball with only 2-torsion in the first homology, $t_* - \mathrm{Id}$ is an isomorphism. Moreover, with $\mathbb{Z}_p$ coefficients we have $t_*^q - \mathrm{Id} = (t_* - \mathrm{Id})^q$. Hence the result follows. □

Proof of Theorem 1.3. If q is an odd prime power, then Proposition 2.2 and Lemma 2.4 together immediately imply that $\Sigma_q(K_{m,n})$ bounds a rational homology ball.

Suppose now that $q = 2^\ell$. By Proposition 2.1, we have that $\Sigma_q(K_{m,n})$ is diffeomorphic to $\Sigma_n(K_{m,q})$. Moreover n was chosen to be an odd prime power, while $q = 2^\ell$ is relatively prime to $2m+1$. Hence the statement follows from the first case of this proof. □

3. Sliceness obstructions from twisted Alexander polynomials

The goal of this section is to prove Theorem 1.4. Recall that $K_n := K_{1,n}$ is the closure of the three-braid $(\beta_1)^n := \left(\sigma_1\sigma_2^{-1}\right)^n$ (see e.g. the left diagram of Figure 3 for $n = 7$). We first prove the following theorem.

Theorem 3.1. *The knots K_{11}, K_{17}, and K_{23} are not slice; hence are of order two in $\mathcal{C}$.*

The sliceness obstruction we intend to use in the proof of Theorem 3.1 rests on a result of Kirk and Livingston [26] involving twisted Alexander polynomials. Throughout the rest of the section, $e^{2\pi i/d}$ is denoted by ξ_d, and the three-manifold obtained by performing 0-surgery on K is denoted by M_K. We generally follow the exposition of [17], and refer the reader to that work for more details.

Definition 3.2. Given a representation $\alpha\colon \pi_1(M_K) \to GL(q, \mathbb{Q}(\xi_d)[t^{\pm 1}])$, the *twisted Alexander module* $\mathcal{A}^\alpha(K)$ is the $\mathbb{Q}(\xi_d)[t^{\pm1}]$-module $H_1(M_K;\mathbb{Q}(\xi_d)[t^{\pm1}]^q)$.

Definition 3.3. The *twisted Alexander polynomial* $\Delta_K^\alpha(t)$ is a generator of the order ideal of $\mathcal{A}^\alpha(K)$; this polynomial is well-defined up to multiplication by units in $\mathbb{Q}(\xi_d)[t^{\pm1}]$.

Twisted Alexander polynomials generalize the classical Alexander polynomial. If we fix the representation $\alpha_0\colon \pi_1(M_K) \to GL(1, \mathbb{Q}[t^{\pm1}])$ (i.e. $q = d = 1$), then $\mathcal{A}^{\alpha_0}(K)$ is the classical (rational) Alexander module $\mathcal{A}(K)$ of K and $\Delta_K(t) := \Delta_K^{\alpha_0}(t)$ is the classical Alexander polynomial. (Note that α_0 is not actually uniquely determined, as it may map a meridian of K to t^n for any $n \in \mathbb{Z}$. Rather, one should consider α_0 to be any representation that comes from abelianization of $\pi_1(M_K)$.)

We will restrict to a special class of representations as follows. First, choose $q \in \mathbb{N}$ and a character $\chi\colon H_1(\Sigma_q(K);\mathbb{Z}) \to \mathbb{Z}_d$. Note that $H_1(\Sigma_q(K);\mathbb{Z}) \cong \mathcal{A}(K)/\langle t^q - 1\rangle$ and that a choice of a meridian for K determines a map from $\pi_1(M_K)$ to $\mathbb{Z} \ltimes \mathcal{A}(K)/\langle t^q - 1\rangle$, as discussed in more detail in Subsection 3.2. The character χ therefore induces $\alpha_\chi\colon \pi_1(M_K) \to GL(q, \mathbb{Q}(\xi_d)[t^{\pm1}])$, and we write $\Delta_K^\chi(t) := \Delta_K^{\alpha_\chi}(t)$.

This is a very quick explanation of twisted Alexander polynomials from the Casson-Gordon perspective, and Friedl and Vidussi [15] have a survey of twisted Alexander polynomials which we recommend for more detailed exposition.

The obstruction we will use in the proof of Theorem 3.1 is a generalization of the Fox-Milnor condition [10], which states that the Alexander polynomial of a slice knot factors as $f(t)f(t^{-1})$ for some $f(t) \in \mathbb{Z}[t^{\pm 1}]$. First, recall the following definitions.

Definition 3.4. We call a Laurent polynomial $d(t) \in \mathbb{Q}(\xi_d)[t^{\pm 1}]$ a *norm* if there exist $c \in \mathbb{Q}(\xi_d)$, $k \in \mathbb{Z}$, and $f(t) \in \mathbb{Q}(\xi_d)[t^{\pm 1}]$ such that

$$d(t) = ct^k f(t)\overline{f(t)},$$

where $\overline{\cdot}$ is induced by the $\mathbb{Q}$-linear map on $\mathbb{Q}(\xi_d)[t^{\pm 1}]$ sending t^i to t^{-i} and ξ_d to ξ_d^{-1}.

Definition 3.5. Let q be an odd prime power and $\chi\colon H_1(\Sigma_q(K);\mathbb{Z}) \to \mathbb{Z}_d$ be a character. The *reduced twisted Alexander polynomial* $\widetilde{\Delta}^{\chi}_K(t)$ is defined to be $\Delta^{\alpha_\chi}_K(t)(t-1)^{-s}$, where $s = 0$ if χ is trivial and $s = 1$ otherwise.

Theorem 3.6 (Theorem 6.2, [26])**.** *Suppose that K is a slice knot and q is an odd prime power. Then there exists a covering transformation invariant metabolizer $P \leq H_1(\Sigma_q(K);\mathbb{Z})$ such that if $\chi\colon H_1(\Sigma_q(K);\mathbb{Z}) \to \mathbb{Z}_d$ is a character of odd prime power order such that $\chi|_P = 0$, then the reduced twisted Alexander polynomial $\widetilde{\Delta}^{\chi}_K(t) \in \mathbb{Q}(\xi_d)[t^{\pm 1}]$ is a norm.* □

Let $K \in \{K_{11}, K_{17}, K_{23}\}$. We first determine the metabolizers of $H_1(\Sigma_3(K);\mathbb{Z})$ and construct prime order characters vanishing on each metabolizer in Subsection 3.1. We then show that the corresponding reduced twisted Alexander polynomials of K do not factor as a norm in Subsection 3.4.

3.1. **The metabolizers of $H_1(\Sigma_3(K_n);\mathbb{Z})$.** We assume that n is odd and not divisible by 3, so in particular K_n is a knot. Our understanding of $H_1(\Sigma_3(K_n);\mathbb{Z})$ and its metabolizers will come from a computation of the Alexander module and the Blanchfield pairing of K_n. Throughout this section, we also keep track of the order n symmetry of K_n, which will be useful later on to reduce the number of twisted Alexander polynomials we must compute.

Observe that $K := K_n$ has a genus $n-1$ Seifert surface F, illustrated in Figure 3 for $n = 7$, which is invariant under the periodic order n symmetry $r\colon S^3 \to S^3$ given diagrammatically by rotating counterclockwise by $2\pi/n$. We pick a collection of simple closed curves $\alpha_1, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_{n-1}$

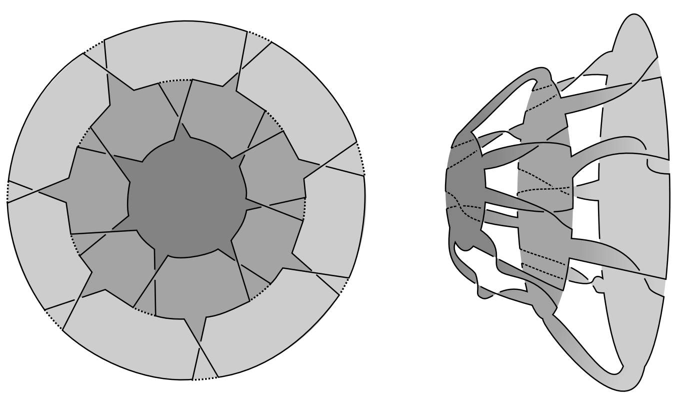

FIGURE 3. A Seifert surface F for K from two different perspectives.

on F that form a basis for $H_1(F;\mathbb{Z})$ as illustrated on the left of Figure 4. Note that $r(\alpha_i) = \alpha_{i-1}$ and $r(\beta_i) = \beta_{i-1}$ for all $i = 1, \ldots n-1$. By considering the right side of Figure 4 we see that $\sum_{i=0}^{n-1}[\alpha_i] = 0$ in $H_1(F;\mathbb{Z})$, and so the induced action of r on $[\alpha_1], [\beta_1] \in H_1(F;\mathbb{Z})$ is given by

$$r_*([\alpha_1]) = [\alpha_0] = \sum_{i=1}^{n-1} -[\alpha_i]$$

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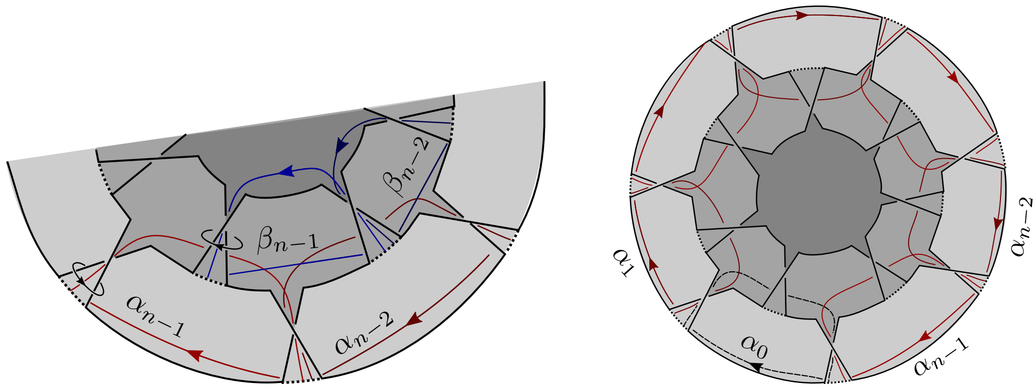

FIGURE 4. A basis of curves for $H_1(F;\mathbb{Z})$ (left) and an additional curve α_0 which does not represent an element of our basis for $H_1(F,\mathbb{Z})$ (right, depicted for $n=7$).

and, via an analogous argument,

$$r_*([\beta_1]) = \sum_{i=1}^{n-1} -[\beta_i].$$

It is straightforward to compute the Seifert matrix A for the Seifert pairing on F with respect to our fixed basis, and we obtain $A = \begin{bmatrix} -B^T & 0 \\ B & B \end{bmatrix}$, where B is the $(n-1)\times(n-1)$ matrix with entries given by $B_{i,j} = \begin{cases} 1 & i=j \\ -1 & i=j-1 \\ 0 & \text{else} \end{cases}$. Recall that Blanchfield [4] showed that the Alexander module $\mathcal{A}(K)$ supports a non-singular pairing

$$\mathrm{Bl}\colon\ \mathcal{A}(K)\times\mathcal{A}(K)\to\mathbb{Q}(t)/\,\mathbb{Z}[t^{\pm1}]$$

called the *Blanchfield pairing*. The pairing can be computed using a Seifert matrix of K as follows, for more details see [14, 22, 29].

Theorem 3.7 ([14, Theorem 1.3 and 1.4])**.** *Let F be a Seifert surface for a knot K with a collection of simple closed curves $\delta_1,\dots,\delta_{2g}$ on F that form a basis for $H_1(F;\mathbb{Z})$ and corresponding Seifert matrix A. Let $\widehat{\delta}_1,\dots,\widehat{\delta}_{2g}$ be a collection of simple closed curves in $S^3\smallsetminus\nu(F)$ representing a basis for $H_1(S^3\smallsetminus\nu(F);\mathbb{Z})$ satisfying $\mathrm{lk}(\delta_i,\widehat{\delta}_j)=\delta_{i,j}$ (i.e. the Alexander dual basis), where $\nu(F)$ denotes an open tubular neighborhood $F\times I$. Consider the standard decomposition of the infinite cyclic cover of the knot exterior as*

$$X_K^\infty = \bigcup_{i=-\infty}^{+\infty}(S^3\smallsetminus\nu(F))_i,$$

and let the homology class of the unique lift of $\widehat{\delta}_i$ to $(S^3\smallsetminus\nu(F))_0$ be denoted by d_i. Then the map

$$p\colon\ \left(\mathbb{Z}[t^{\pm1}]\right)^{2g}\to\mathcal{A}(K)$$
$$(x_1,\dots,x_{2g})\mapsto\sum_{i=1}^{2g}x_id_i.$$

is surjective and has kernel given by $(tA-A^T)\,\mathbb{Z}[t^{\pm1}]^{2g}$. Moreover, the Blanchfield pairing is given as follows: for $x,y\in\mathbb{Z}[t^{\pm1}]^{2g}$ we have

$$\mathrm{Bl}(p(x),p(y)) = (t-1)x^T(A-tA^T)^{-1}\overline{y}\in\mathbb{Q}(t)/\,\mathbb{Z}[t^{\pm1}],$$

where $\bar{\cdot}$ *is induced by the* $\mathbb{Z}$*-linear map on* $\mathbb{Z}[t^{\pm 1}]$ *sending* t^i *to* t^{-i}*.* □

Following the language above, let $\hat{\alpha}_1, \dots, \hat{\alpha}_{n-1}, \hat{\beta}_1, \dots, \hat{\beta}_{n-1}$ be the Alexander dual basis of $\alpha_1, \dots, \alpha_{n-1}, \beta_1, \dots, \beta_{n-1}$ and a_i, b_i be the homology classes of the unique lifts of $\hat{\alpha}_i, \hat{\beta}_i$, respectively. Note that $\hat{\alpha}_{n-1}$ and $\hat{\beta}_{n-1}$ are illustrated in Figure 4 as small closed curves linking F. By inspecting the matrix $tA - A^T$, illustrated below for $n = 7$,

$$\left[\begin{array}{cccccc|cccccc}
1-t & t & 0 & 0 & 0 & 0 & -1 & \mathbf{1} & 0 & 0 & 0 & 0 \\
-1 & 1-t & t & 0 & 0 & 0 & 0 & -1 & \mathbf{1} & 0 & 0 & 0 \\
0 & -1 & 1-t & t & 0 & 0 & 0 & 0 & -1 & \mathbf{1} & 0 & 0 \\
0 & 0 & -1 & 1-t & t & 0 & 0 & 0 & 0 & -1 & \mathbf{1} & 0 \\
0 & 0 & 0 & -1 & 1-t & t & 0 & 0 & 0 & 0 & -1 & \mathbf{1} \\
0 & 0 & 0 & 0 & -1 & 1-t & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
t & 0 & 0 & 0 & 0 & 0 & t-1 & 1 & 0 & 0 & 0 & 0 \\
-t & t & 0 & 0 & 0 & 0 & -t & t-1 & 1 & 0 & 0 & 0 \\
0 & -t & t & 0 & 0 & 0 & 0 & -t & t-1 & 1 & 0 & 0 \\
0 & 0 & -t & t & 0 & 0 & 0 & 0 & -t & t-1 & 1 & 0 \\
0 & 0 & 0 & -t & t & 0 & 0 & 0 & 0 & -t & t-1 & 1 \\
0 & 0 & 0 & 0 & -t & t & 0 & 0 & 0 & 0 & -t & t-1
\end{array}\right]$$

we see that we can use the bolded pivot entries to perform column operations over $\mathbb{Z}[t^{\pm 1}]$ to transform $tA - A^T$ to a matrix as below:

$$\left[\begin{array}{cccccc|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
* & * & * & * & * & * & * & * & * & * & * & * \\
\hline
* & -\mathbf{t} & 0 & 0 & 0 & 0 & t & 1 & 0 & 0 & 0 & 0 \\
* & * & -\mathbf{t} & 0 & 0 & 0 & 0 & t & 1 & 0 & 0 & 0 \\
* & * & * & -\mathbf{t} & 0 & 0 & 0 & 0 & t & 1 & 0 & 0 \\
* & * & * & * & -\mathbf{t} & 0 & 0 & 0 & 0 & t & 1 & 0 \\
* & * & * & * & * & -\mathbf{t} & 0 & 0 & 0 & 0 & t & 1 \\
* & * & * & * & * & * & -1 & -1 & -1 & -1 & -1 & t-1
\end{array}\right].$$

We now use the new bolded entries as pivots to perform column operations to obtain a matrix whose i^{th} row has a single non-zero entry that occurs in column $i+1$, for all $i = 1, \dots, n-2, n, \dots, 2n-3$. This matrix is of the following form:

$$\left[\begin{array}{cccccc|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
*_{n-1,1} & * & * & * & * & * & *_{n-1,n} & * & * & * & * & * \\
\hline
0 & -t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -t & 0 & 0 & 0 & 0 & 0 & 0 \\
*_{2n-2,1} & * & * & * & * & * & *_{2n-2,n} & * & * & * & * & *
\end{array}\right].$$

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Notice that only the $*$-entries with indices have an impact on $\mathcal{A}(K)$. In particular, $\mathcal{A}(K)$ is generated by a_{n-1} and b_{n-1}, in the language of the notation introduced just after Theorem 3.7.

For $n = 11, 17, 23$ one continues to perform column moves until the above matrix is simplified to the following form:

$$\left[\begin{array}{cccccc|cccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_n(t) & * & * & * & * & * & 0 & * & * & * & * & * \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & * & * & * & * & * & p_n(t) & * & * & * & * & *
\end{array}\right],$$

where

$$p_n(t) = \prod_{k=0}^{(n-1)/2} \left(t^2 + (\xi_n^k - 1 + \xi_n^{-k})t + 1\right).$$

This and all further computations in Subsection 3.1 were done in a Jupyter notebook and is available on the third author's website. In particular, this implies that $\Delta_{K_n}(t) = p_n(t)^2$, which one can verify for general $n \in \mathbb{N}$ by using the formula for the Alexander polynomial of a periodic knot in terms of the multivariable Alexander polynomial of the quotient link [37].

Using the above matrix, we obtain for our values of interest that

$$\mathcal{A}(K) \cong \mathbb{Z}[t^{\pm 1}]/\langle p_n(t)\rangle \oplus \mathbb{Z}[t^{\pm 1}]/\langle p_n(t)\rangle,$$

where the two summands are respectively generated by $a := a_{n-1}$ and $b := b_{n-1}$.

Remark 3.8 (The action of r_* on $\mathcal{A}(K)$)**.** We can also compute the action induced by the order n symmetry r on $\mathcal{A}(K)$. In particular, we can observe that $r(\widehat{\alpha}_{n-1})$ is a curve whose only non-trivial linkage is -1 with α_{n-1} and $+1$ with α_{n-2}. Similar observations can be made for $r(\widehat{\beta}_{n-1})$, and so it follows that the induced action of r on $[\widehat{\alpha}_{n-1}], [\widehat{\beta}_{n-1}] \in H_1(S^3 \setminus \nu(F); \mathbb{Z})$ is given by

$$r_*([\widehat{\alpha}_{n-1}]) = -[\widehat{\alpha}_{n-1}] + [\widehat{\alpha}_{n-2}] \text{ and } r_*([\widehat{\beta}_{n-1}]) = -[\widehat{\beta}_{n-1}] + [\widehat{\beta}_{n-2}].$$

Therefore, the action of r_* on the generators of $\mathcal{A}(K)$ is given by

$$r_*(a_{n-1}) = -a_{n-1} + a_{n-2} \text{ and } r_*(b_{n-1}) = -b_{n-1} + b_{n-2}.$$

Moreover, by considering the $(n-1)^{th}$ and $(2n-2)^{th}$ columns of $tA - A^T$, we obtain the relations

$$ta_{n-2} + (1-t)a_{n-1} + tb_{n-1} = 0,$$
$$a_{n-2} - a_{n-1} + b_{n-2} + (t-1)b_{n-1} = 0.$$

Simple algebraic manipulations give us that

$$r_*(a_{n-1}) = -a_{n-1} + a_{n-2} = -t^{-1}a_{n-1} - b_{n-1}, \quad (1)$$
$$r_*(b_{n-1}) = -b_{n-1} + b_{n-2} = t^{-1}a_{n-1} + (1-t)b_{n-1}. \quad (2)$$

Moreover, we obtain that if $v = f_1(t)a_{n-1} + g_1(t)b_{n-1}$ and $w = f_2(t)a_{n-1} + g_2(t)b_{n-1}$ then

$$\mathrm{Bl}(v, w) = \begin{bmatrix} f_1(t) \\ g_1(t) \end{bmatrix}^T \cdot \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \cdot \begin{bmatrix} f_2(t^{-1}) \\ g_2(t^{-1}) \end{bmatrix}$$

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where $c_{ij} = (t-1)(A - tA^T)^{-1}_{(i(n-1),j(n-1))}$. We remark that the interested reader can use this formula to algebraically verify the geometrically immediate fact that $\mathrm{Bl}(r_*(v), r_*(w)) = \mathrm{Bl}(v,w)$ for all $v, w \in \mathcal{A}(K)$.

In applying Theorem 3.6 we will take $q = 3$, that is, we will consider the 3-fold cyclic branched cover $\Sigma_3(K)$ of S^3 branched along K, and will derive the sliceness obstruction from that cover. From now on, we take n to be 11, 17, or 23. We expect that the subsequent computations of this section will hold for general $n \equiv 5 \pmod 6$, but we have not verified these results for $n > 23$.

We wish to transfer our information about $(\mathcal{A}(K), \mathrm{Bl})$ to tell us about $(H_1(\Sigma_3(K);\mathbb{Z}), \lambda)$. First, we have that

$$\begin{aligned} H_1(\Sigma_3(K);\mathbb{Z}) &\cong \mathcal{A}(K)/\langle t^3 - 1\rangle \\ &\cong \mathcal{A}(K)/\langle t^2 + t + 1\rangle \\ &\cong \mathbb{Z}[t^{\pm 1}]/\langle p_n(t), t^2+t+1\rangle \oplus \mathbb{Z}[t^{\pm 1}]/\langle p_n(t), t^2+t+1\rangle \end{aligned}$$

where the two summands are generated by a and b, the images of a_{n-1} and b_{n-1} in $H_1(\Sigma_3(K);\mathbb{Z})$ or, equivalently, the homology classes of the lifts of the curves $\widehat{\alpha}_{n-1}$ and $\widehat{\beta}_{n-1}$ to the preferred copy of $S^3 \smallsetminus \nu(F)$ in $\Sigma_3(K)$.

A straightforward computation using our explicit formula for $p_n(t)$ when $n = 11, 17, 23$ shows that

$$\mathbb{Z}[t^{\pm 1}]/\langle p_n(t), t^2+t+1\rangle \cong \mathbb{Z}_n[t^{\pm 1}]/\langle t^2+t+1\rangle$$

and hence that

$$H_1(\Sigma_3(K);\mathbb{Z}) \cong \mathbb{Z}_n[t^{\pm 1}]/\langle t^2+t+1\rangle \oplus \mathbb{Z}_n[t^{\pm 1}]/\langle t^2+t+1\rangle.$$

In particular, as a group $H_1(\Sigma_3(K);\mathbb{Z}) \cong (\mathbb{Z}_n)^4$, with natural generators a, ta, b, and tb.

The following result, which is slightly reformulated from [13], lets us compute the torsion linking form λ with respect to our preferred basis.

Proposition 3.9 ([13, Chapter 2.6])**.** *Suppose that q is a prime power and let $x, y \in H_1(\Sigma_q(K);\mathbb{Z})$. Choose $\widetilde{x}, \widetilde{y} \in \mathcal{A}(K)$ which lift x and y, and write*

$$\mathrm{Bl}(\widetilde{y}, \widetilde{x}) = \frac{p(t)}{\Delta_K(t)} \in \mathbb{Q}(t)/\,\mathbb{Z}[t^{\pm 1}].$$

Since $t^q - 1$ and $\Delta_K(t)$ are relatively prime, one can find $r(t) \in \mathbb{Z}[t^{\pm 1}]$ and $c \in \mathbb{Z}$ such that $\Delta_K(t) r(t) \equiv c \pmod{t^q - 1}$. Writing $p(t)r(t) \equiv \sum_{i=1}^{q} \alpha_i t^i \pmod{t^q-1}$, for $i = 0, \dots, q-1$ we obtain

$$\lambda_q(x, t^i y) = \frac{\alpha_{q-i}}{c} \in \mathbb{Q}/\,\mathbb{Z}\,. \qquad \square$$

When we apply this process to our formula for Bl, we obtain that with respect to the $\mathbb{Z}_n$-basis $\{a, ta, b, tb\}$ our linking form is given by the matrix

$$L = \frac{1}{n}\begin{bmatrix} -1 & -k & -k & k \\ -k & -1 & 0 & -k \\ -k & 0 & 1 & k \\ k & -k & k & 1 \end{bmatrix},$$

where $n = 2k+1$. We remind the reader that, while we expect this formula to hold for general $n \equiv 5 \pmod 6$, we only establish it for $n = 11, 17, 23$.

Remark 3.10 (The invariant metabolizers of $H_1(\Sigma_3(K_n);\mathbb{Z})$)**.** We now wish to understand the invariant metabolizers of $H_1(\Sigma_3(K_n);\mathbb{Z})$ in order to effectively apply Theorem 3.6.

First, we make the key observation that the induced action of r on $H_1(\Sigma_3(K_n); Z)$ preserves the linking form. This can be seen from a geometric viewpoint, since the action of r on the exterior X_{K_n}

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lifts to the infinite cyclic cover $X^\infty_{K_n}$ to induce a Blanchfield pairing preserving action on $\mathcal{A}(K_n)$. It can also be verified algebraically, using our formulas (1) and (2), which hold equally well for the induced action of r on $H_1(\Sigma_3(K_n);\mathbb{Z})$ once we apply the relation $t^3=1$, and our explicit formula for the linking form. Since r_* also commutes with the action of t, we have that r_* acts on the set $\mathcal{P}$ of all covering transformation invariant metabolizers.

Recalling that $n\in\{11,17,23\}$, we note that since $n\equiv 5 \pmod 6$ the polynomial t^2+t+1 is irreducible in $\mathbb{Z}_n[t^{\pm1}]$. Therefore, since n is prime, we see that $\mathbb{Z}_n[t^{\pm1}]/\langle t^2+t+1\rangle$ has no non-trivial proper submodules. Therefore, if $(k_0+k_1)a+(j_0+j_1)b\in M$ for some integers k_0,k_1,j_0,j_1 and some proper submodule M, then either $k_0=k_1=0$ and $b\in M$ or $a+(j_0'+j_1')b\in M$ for some integers j_0',j_1'.

It follows that there are exactly n^2+1 order n^2 submodules of $H_1(\Sigma_3(K);\mathbb{Z})$: first, for any $n_0,n_1\in\mathbb{Z}_n$ we have

$$P_{n_0,n_1}:=\operatorname{span}_{\mathbb{Z}_n[t^{\pm1}]}\{a+(n_0+n_1t)b\}=\operatorname{span}_{\mathbb{Z}_n}\{a+n_0b+n_1tb,\, ta-n_1b+(n_0-n_1)tb\}$$

and secondly we have

$$P':=\operatorname{span}_{\mathbb{Z}_n[t^{\pm1}]}\{b\}=\operatorname{span}_{\mathbb{Z}_n}\{b,tb\}.$$ [1]

Using the matrix L, we see that $\lambda(b,b)=\frac{1}{n}\neq 0\in\mathbb{Q}/\mathbb{Z}$, and so P' is not a metabolizer. Moreover, observe that the condition

$$\lambda(a+(n_0+n_1t)b,a+(n_0+n_1t)b)=\begin{bmatrix}1\\0\\n_0\\n_1\end{bmatrix}^T\cdot L\cdot\begin{bmatrix}1\\0\\n_0\\n_1\end{bmatrix}=0\in\mathbb{Q}/\mathbb{Z}$$

gives us a 2-variable (n_0 and n_1) quadratic polynomial over $\mathbb{Z}_n$, and hence has at most $2n$ solutions.

We have therefore shown that

$$|\mathcal{P}|\leq 2n.$$

Moreover, since the map r_* acts on $\mathcal{P}$, n is prime, and $(r_*)^n=\mathrm{Id}$, we know that the orbit of a metabolizer must be either of order n or order 1. A short algebraic argument shows that $r_*(P_{n_0,n_1})=P_{n_0,n_1}$ if and only if $n_0=n_1=1$: the 'if' direction follows immediately from Equation (1) and (2), and for the 'only if' direction, we compute

$$r_*(a+n_0b+n_1tb)=(1-n_0+n_1)a+(1-n_0)ta+(-1+n_0+n_1)b+(-n_0+2n_1)tb$$

and observe that if this element belongs to P_{n_0,n_1} then by looking at the a and ta coefficients we see that it must equal

$$(1-n_0+n_1)(a+n_0b+n_1tb)+(1-n_0)(ta-n_1b+(n_0-n_1)tb).$$

Contemplation of the coefficients of b and tb in these two expressions shows that they can only be equal if $n_0=n_1=1$. One can quickly verify that $P_{1,1}$ is in fact a metabolizer, and so we see that the action of r_* on $\mathcal{P}$ has exactly one orbit of order 1. The remaining metabolizers (of which there are at most $2n-1$ must be partitioned into orbits of order n, and so there are at most 2 orbits. It is not hard to explicitly verify that $P_{-1,-1}$ is also a metabolizer and so there are exactly two orbits.

We summarize this work in the following proposition.

Proposition 3.11. *Let $n=11,17,$ or 23. Then the action of r_* on $\mathcal{P}$, the set of invariant metabolizers for $H_1(\Sigma_3(K_n);\mathbb{Z})$, has exactly two orbits. The first orbit has order 1 and consists of*

$$P_+=P_{1,1}=\operatorname{span}_{\mathbb{Z}_n}\{a+b+tb,\, ta-b\}.$$

[1]Although at first glance the reader may be surprised by the seeming asymmetry between a and b, we note that we could have equivalently enumerated our order n^2 submodules by $Q_{m_0,m_1}:=\operatorname{span}_{\mathbb{Z}_n[t^{\pm1}]}\{b+(m_0+m_1t)a\}$ and $Q'=\operatorname{span}_{\mathbb{Z}[t^{\pm1}]}\{a\}$.

The second orbit has order n and consists of $\{r_*^i(P_-)\}_{i=0}^{n-1}$, where

$$P_- := P_{-1,-1} = \text{span}_{\mathbb{Z}_n}\{a - b - tb, ta + b\}. \qquad \square$$

It is extremely easy to construct characters

$$\chi\colon H_1(\Sigma_3(K_n); \mathbb{Z}) \to \mathbb{Z}_n$$

vanishing on either P_+ or P_-: choose $\chi(b)$ and $\chi(tb)$ freely and $\chi(a)$ and $\chi(ta)$ are determined.

Definition 3.12. Define $\chi_\pm\colon H_1(\Sigma_3(K_n); \mathbb{Z}) \to \mathbb{Z}_n$ as follows:

$$\chi_\pm(a) = \pm 1, \chi_\pm(ta) = 0, \chi_\pm(b) = 0, \text{ and } \chi_\pm(tb) = -1. \tag{3}$$

Note that χ_+ vanishes on P_+ and χ_- vanishes on P_-. To avoid confusion, we point out here that the 'd' of Definitions 3.2 and 3.3 and Theorem 3.6 happens to be n for us.

3.2. **Computation of twisted Alexander polynomials.** We are now ready to begin computing certain twisted Alexander polynomials. It is helpful to have the following naming conventions that are standard in this subfield. Given a knot K in S^3 bounding a Seifert surface F, we write:

$\nu(K)$ to denote an open tubular neighborhood of K,
$\nu(F)$ to denote an open tubular neighborhood of F,
X_K to denote $S^3 \smallsetminus \nu(K)$,
X_K^n to denote the n-fold cyclic cover of X_K, and
X_F to denote $S^3 \smallsetminus \nu(F)$.

Given a character $\chi\colon H_1(\Sigma_3(K); \mathbb{Z}) \to \mathbb{Z}_n$, we apply [17] to obtain a representation

$$\phi_\chi\colon \pi_1(X_K) \to GL(3, \mathbb{Q}(\xi_n)[t^{\pm 1}])$$

as follows. Fix a basepoint x_0 chosen for convenience in X_F and let $\widetilde{x}_0$ denote the lift of x_0 to the 0^{th} copy of $S^3 \smallsetminus \nu(F)$ in $X_K^3 \subset \Sigma_3(K)$. Let $\epsilon\colon \pi_1(X_K) \to \mathbb{Z}$ be the canonical abelianization map, and let μ_0 be a preferred meridian of K based at x_0. Given a simple closed curve γ in $S^3 \smallsetminus K$ based at x_0 and with $\text{lk}(K, \gamma) = 0$, we can obtain a well-defined lift $\widetilde{\gamma}$ of γ to $\Sigma_3(K)$, giving a map

$$l\colon \ker(\epsilon) \to H_1(\Sigma_3(K); \mathbb{Z}).$$

We note for the sake of clarity that the map l does not in general coincide with our previous method of converting elements of $H_1(S^3 \smallsetminus \nu(F); \mathbb{Z})$ to elements of $H_1(\Sigma_3(K); \mathbb{Z})$, unless γ is actually disjoint from F. In particular, $l(\mu_0 g \mu_0^{-1}) = t \cdot l(g)$ despite the fact that $\mu_0 g \mu_0^{-1}$ and g certainly represent the same class in $H_1(S^3 \smallsetminus \nu(F); \mathbb{Z})$.

Our choice of μ_0 allows us to define a map

$$\begin{aligned} \phi\colon \pi_1(X_K) &\to \mathbb{Z} \ltimes H_1(\Sigma_3(K); \mathbb{Z}) \\ g &\mapsto (t^{\epsilon(g)}, l(\mu_0^{-\epsilon(g)} g)), \end{aligned}$$

where the product structure on $\mathbb{Z} \ltimes H_1(\Sigma_3(K); \mathbb{Z})$ is given by

$$(t^{m_1}, x_1) \cdot (t^{m_2}, x_2) = (t^{m_1+m_2}, t^{-m_2} \cdot x_1 + x_2).$$

We then define $\phi_\chi = f_\chi \circ \phi$, where

$$\begin{aligned} f_\chi\colon \mathbb{Z} \ltimes H_1(\Sigma_3(K); \mathbb{Z}) &\to GL(3, \mathbb{Q}(\xi_n)[t^{\pm 1}]) \\ (t^m, x) &\mapsto \begin{bmatrix} 0 & 0 & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^m \begin{bmatrix} \xi_n^{\chi(x)} & 0 & 0 \\ 0 & \xi_n^{\chi(t\cdot x)} & 0 \\ 0 & 0 & \xi_n^{\chi(t^2\cdot x)} \end{bmatrix}. \end{aligned} \tag{4}$$

The following well-known result (see e.g. [17, 26]) reduces computation of twisted Alexander polynomials to Fox calculus and matrix algebra.

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Proposition 3.13 ([17, Section 9]). *Let $\pi_1(X_K) = \langle g_1, \ldots, g_{c(K)} : r_1, \ldots, r_{c(K)}\rangle$ be a Wirtinger presentation . Assume that $\phi_\chi\colon \pi_1(X_K) \to GL(q, \mathbb{F}[t^{\pm 1}])$ is induced by a non-trivial character χ. Then there is a natural extension $\Phi\colon \mathbb{Z}[\pi_1(X_K)] \to M_q(\mathbb{F}[t^{\pm 1}])$ where $M_q(\mathbb{F}[t^{\pm 1}])$ is the set of q by q matrices with entries from $\mathbb{F}[t^{\pm 1}]$, and the reduced twisted Alexander polynomial of (K, χ) is*

$$\widetilde{\Delta}^\chi_K(t) = \frac{\det\left(\left[\Phi\left(\frac{\partial r_i}{\partial g_j}\right)\right]_{i,j=2}^{c(K)}\right)}{(t-1)\det(\phi_\chi(g_1) - \mathrm{Id}).} \qquad \square$$

We see that in order to compute a twisted Alexander polynomial one must determine $\phi_\chi(g_i)$ for all the generators g_i in a Wirtinger presentation of $\pi_1(X_K)$. If one is not particularly attached to a given description of $H_1(\Sigma_n(K);\mathbb{Z})$, [17] gives a straightforward way to do this. However, we have very specific $\chi_\pm$, defined in terms of a particular generating set for $H_1(\Sigma_3(K);\mathbb{Z})$, and so we must be a little more careful than is usually necessary.

Our basepoint x for $S^3 \smallsetminus \nu(K)$ lies far below the diagram, which we think of as lying almost in the plane of the page. All of our curves are based at x_0, though as usual we sometimes draw meridians to components of the knots as unbased curves, with the understanding that they are based via the 'go straight down to the basepoint' path.

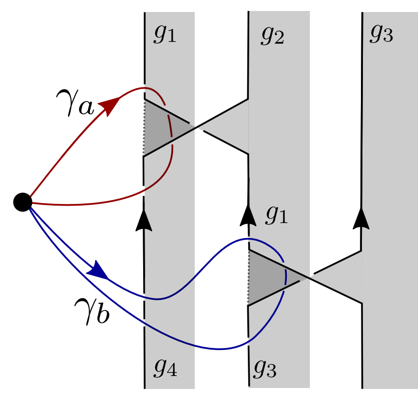

FIGURE 5. Wirtinger generators g_i.

Remark 3.14 (Computing the image of Wirtinger generators under $\phi_{\chi_\pm}$). Let $\{g_i\}_{i=1}^{2n}$ be the Wirtinger generators for $\pi_1(X_K, x_0)$, some of which are illustrated in Figure 5, and μ_0 be the preferred meridian that represents g_1. In order to compute our desired twisted Alexander polynomials, we need to know $\phi_\chi(g_i)$ for all $i = 1, \ldots, 2n$. Since K is the closure of a 3-braid, once we specify the image of the three top strand generators g_1, g_2, g_3 under ϕ_χ, the rest of the computation is simple. In fact, since $g_2 = g_1^{-1} g_4 g_1$, it suffices to determine the image of g_1, g_3, and g_4.

By considering Equation (4), we see that $\phi_\chi(g_i)$ is determined by the tuple

$$(*)_i := \left(\chi(l(g_1^{-1}g_i)), \chi(t \cdot l(g_1^{-1}g_i)), \chi(t^2 \cdot l(g_1^{-1}g_i))\right).$$

We now describe $(*)_1, (*)_3$, and $(*)_4$, and use the above discussion to compute $\phi_\chi(g_i)$ for each Wirtinger generator g_i. We obtain immediately that

$$(*)_1 = (\chi(l(g_1^{-1}g_1)), \chi(t \cdot l(g_1^{-1}g_1)), \chi(t^2 \cdot l(g_1^{-1}g_1))) = (0,0,0) \in \mathbb{Z}_n^3.$$

Given a simple closed curve γ based at x_0 and disjoint from F, recall that we obtain a curve $\widetilde{\gamma}$ in $\Sigma_3(K)$ by lifting γ to our preferred copy of $S^3 \smallsetminus \nu(F)$. As before, we let a denote the homology class of the lift of $\widehat{\alpha}_{n-1}$ and b denote the homology class of the lift of $\widehat{\beta}_{n-1}$ in $H_1(\Sigma_3(K);\mathbb{Z})$. Let γ_a

be a simple closed curve that represents $g_1 g_4^{-1}$ and γ_{-a} be its reverse, chosen to be disjoint from F as in Figure 5. Then we have that $-a = [\widetilde{\gamma}_a] \in H_1(\Sigma_3(K);\mathbb{Z})$ and

$$a = [\widetilde{\gamma}_{-a}] = l(g_4 g_1^{-1}) = l(g_1(g_1^{-1}g_4)g_1^{-1}) = t \cdot l(g_1^{-1}g_4) \in H_1(\Sigma_3(K);\mathbb{Z}).$$

Therefore

$$\begin{aligned}(*)_4 = (\chi(l(g_1^{-1}g_4)), \chi(t\cdot l(g_1^{-1}g_4)), \chi(t^2 \cdot l(g_1^{-1}g_4))) &= (\chi(t^{-1}\cdot a), \chi(a), \chi(t\cdot a)) \\ &= (-\chi(a) - \chi(t\cdot a), \chi(a), \chi(t \cdot a)) \in \mathbb{Z}_n^3.\end{aligned}$$

Similarly, let γ_b be a simple closed curve that represents $g_4 g_3 g_1^{-1} g_4^{-1}$ and is disjoint from F, as in Figure 5. So we have that

$$b = [\widetilde{\gamma}_b] = l(g_4 g_3 g_1^{-1} g_4^{-1}) = t \cdot l(g_3 g_1^{-1}) = t\cdot l(g_1(g_1^{-1}g_3)g_1^{-1}) = t^2 \cdot l(g_1^{-1}g_3) \in H_1(\Sigma_3(K);\mathbb{Z}).$$

Hence

$$\begin{aligned}(*)_3 = (\chi(l(g_1^{-1}g_3)), \chi(t\cdot l(g_1^{-1}g_3)), \chi(t^2 \cdot l(g_1^{-1}g_3))) &= (\chi(t\cdot b), \chi(t^2\cdot b), \chi(b)) \\ &= (\chi(t\cdot b), -\chi(b) - \chi(t\cdot b), \chi(b)) \in \mathbb{Z}_n^3.\end{aligned}$$

We can now straightforwardly compute $\phi_\chi(g_i)$ for the rest of the Wirtinger generators g_i.

We have now done all the work necessary to compute the reduced twisted Alexander polynomials $\widetilde{\Delta}_{K_n}^{\chi_\pm}(t) \in \mathbb{Q}(\xi_n)[t^{\pm 1}]$ what remains is simply to write down a Wirtinger presentation, construct a matrix as in Proposition 3.13, and take its determinant. However, we do not do this directly, instead choosing a slightly more efficient approach that allows us to work over finite fields.

3.3. **Working over finite fields.** To apply Theorem 3.6, we must obstruct polynomials from factoring in a certain way over $\mathbb{Q}(\xi_d)[t^{\pm 1}]$. It is easier to obstruct the existence of factorizations in $\mathbb{Z}_p[t^{\pm 1}]$, where computer programs are for finiteness reasons capable of proving that no factorization of a given kind exists, and the following propositions allow us to make this transition.

Proposition 3.15 ([17, Lemma 8.6])**.** *Let d, s be primes and suppose $s = kd+1$. Choose $\theta \in \mathbb{Z}_s$ so that $\theta \in \mathbb{Z}_s$ is a primitive d^{th} root of unity modulo s. The choice of s and θ defines a map $\pi_\theta \colon \mathbb{Z}[\xi_d][t^{\pm 1}] \to \mathbb{Z}_s[t^{\pm 1}]$ where 1 is mapped to 1 and ξ_d is mapped to θ.*

Let $d(t) \in \mathbb{Z}[\xi_d][t^{\pm 1}]$ be a polynomial of degree $2N$ such that $\pi_\theta(d(t)) \in \mathbb{Z}_s[t^{\pm 1}]$ also has degree $2N$. If $d(t) \in \mathbb{Q}(\xi_d)[t^{\pm 1}]$ is a norm then $\pi_\theta(d(t)) \in \mathbb{Z}_s[t^{\pm 1}]$ factors as the product of two polynomials of degree N. □

Proposition 3.16. *Given a knot K, a preferred meridian μ_0, and a map $\chi\colon H_1(\Sigma_q(K);\mathbb{Z}) \to \mathbb{Z}_d$ where d is a prime, we obtain as above a reduced twisted Alexander polynomial $\widetilde{\Delta}_K^\chi(t)$. By rescaling, assume that $\widetilde{\Delta}_K^\chi(t)$ is an element of $\mathbb{Z}[\xi_d][t^{\pm 1}]$.*

Let $s = kd+1$, $\theta \in \mathbb{Z}_s$, and $\pi_\theta\colon \mathbb{Z}[\xi_d][t^{\pm 1}] \to \mathbb{Z}_s[t^{\pm 1}]$ be as in Proposition 3.15. Suppose that $\pi_\theta\left(\widetilde{\Delta}_K^\chi(t)\right)$ is a degree $2\lfloor \frac{c(K)-3}{2}\rfloor$ polynomial which cannot be written as a product of two degree $\lfloor\frac{c(K)-3}{2}\rfloor$ polynomials in $\mathbb{Z}_s[t^{\pm 1}]$. Then $\widetilde{\Delta}_K^\chi(t) \in \mathbb{Q}(\xi_d)[t^{\pm 1}]$ is not a norm.

Here, degree is taken to be the degree of a Laurent polynomial – i.e. $\deg_{\max} - \deg_{\min}$. Proposition 3.16 is useful for efficient computations, since in our setting $\det(\phi_\chi(g_1) - \mathrm{Id}) = t-1$ and one can compute

$$\pi_\theta\left(\widetilde{\Delta}_K^\chi(t)\right) = \frac{\det\left(\left[\pi_\theta\left(\Phi\left(\frac{\partial r_i}{\partial g_j}\right)\right)\right]_{i,j=2}^{c(K)}\right)}{(t-1)^2} = \frac{\det\left(\left[\Phi\left(\pi_\theta\left(\frac{\partial r_i}{\partial g_j}\right)\right)\right]_{i,j=2}^{c(K)}\right)}{(t-1)^2}$$

in particular computing determinants of matrices with entries in $\mathbb{Z}_s[t^{\pm 1}]$ rather than in $\mathbb{Q}(\xi_d)[t^{\pm 1}]$.

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Proof of Proposition 3.16. By Proposition 3.15, to establish our desired result under the above hypotheses it suffices to show that the degree of $\widetilde{\Delta}^{\chi}_K(t)$ is equal to $2\lfloor\frac{c(K)-3}{2}\rfloor$, i.e. that the reduced twisted Alexander polynomial does not drop degree under π_θ. By considering Proposition 3.13 and recalling that we chose g_1 so that the determinant of $\phi_\chi(g_1)-\mathrm{Id}$ is $t-1$, we see that the degree of $\widetilde{\Delta}^{\chi}_K(t)$ is no more than $c(K)-3$ as follows.

The degree of $\widetilde{\Delta}^{\chi}_K(t)$ is 2 less than the degree of $\det\left[\Phi\left(\frac{\partial r_i}{\partial g_j}\right)\right]_{i,j=2}^{c(K)}$. The Wirtinger presentation of $\pi_1(X_K)$ has $c(K)$ generators and $c(K)$ relations of the form $r_i = g_{a_i}g_{b_i}g_{c_i}^{-1}g_{b_i}^{-1}$ for some a_i, b_i, c_i. Moreover, since $g_{a_i}g_{b_i}g_{c_i}^{-1}g_{b_i}^{-1}=1$ one can verify that for any $g_j\in\pi_1(X_K)$

$$\frac{\partial\left(g_{a_i}g_{b_i}g_{c_i}^{-1}g_{b_i}^{-1}\right)}{\partial g_j} = \frac{\partial\left((g_{a_i}g_{b_i})(g_{b_i}g_{c_i})^{-1}\right)}{\partial g_j} = \frac{\partial\left(g_{a_i}g_{b_i}\right)}{\partial g_j} - \frac{\partial(g_{b_i}g_{c_i})}{\partial g_j} = \frac{\partial g_{a_i}}{\partial g_j} + (g_{a_i}-1)\frac{\partial g_b}{\partial g_j} - g_{b_i}\frac{\partial g_{c_i}}{\partial g_j}.$$

Therefore for any i, j we have that

$$\Phi\left(\frac{\partial r_i}{\partial g_j}\right) = \begin{cases} \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} & \text{if } j=a_i,\\ \begin{bmatrix} 0 & 0 & t\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix} \xi_d^{*} & 0 & 0\\ 0 & \xi_d^{**} & 0\\ 0 & 0 & \xi_d^{***}\end{bmatrix} - \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} & \text{if } j=b_i,\\ \begin{bmatrix} 0 & 0 & t\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix} \xi_d^{*} & 0 & 0\\ 0 & \xi_d^{**} & 0\\ 0 & 0 & \xi_d^{***}\end{bmatrix} & \text{if } j=c_i,\end{cases}$$

and is the 3×3 zero matrix if $j\notin\{a_i,b_i,c_i\}$. In particular, $\Phi\left(\frac{\partial r_i}{\partial g_j}\right)$ has at most one entry which is of the form αt for $\alpha\in\mathbb{Q}(\xi_d)$ and all its other entries are elements of $\mathbb{Q}(\xi_d)$. It follows that the degree of

$$\det\left[\Phi\left(\frac{\partial r_i}{\partial g_j}\right)\right]_{i,j=2}^{c(K)}$$

is no more than $c(K)-1$ and so the degree of $\widetilde{\Delta}^{\chi}_K(t)$ is no more than $c(K)-3$.

Since polynomials of the form $f(t)\overline{f(t)}$ certainly have even degrees, either $\widetilde{\Delta}^{\chi}_K(t)$ is not a norm, or we have

$$2\left\lfloor\frac{c(K)-3}{2}\right\rfloor = \deg\pi_\theta\left(\widetilde{\Delta}^{\chi}_K(t)\right) \leq \deg\widetilde{\Delta}^{\chi}_K(t) \leq 2\left\lfloor\frac{c(K)-3}{2}\right\rfloor,$$

and hence we have equality throughout. □

Table 1 gives the degrees of the irreducible factors of $\pi_\theta\left(\widetilde{\Delta}^{\chi_\pm}_{K_n}(t)\right)$ over $\mathbb{Z}_s[t^{\pm1}]$, which is all we will need to prove our main theorem. We refer the interested reader to the Appendix A for the actual factorizations.

3.4. **Proof of the main theorems.** We are now ready to embark upon proving the main theorems of this paper.

Proof of Theorem 3.1. Let $n\in\{11,17,23\}$ and let $K=K_n$. Let $r\colon X_K\to X_K$ denote the order n symmetry of the knot exterior given in Figure 3 by rotation by $2\pi/n$. As discussed above, r extends to an order n symmetry of $\Sigma_3(K)$ and induces a covering transformation invariant, linking form preserving isomorphism $r_*\colon H_1(\Sigma_3(K);\mathbb{Z})\to H_1(\Sigma_3(K);\mathbb{Z})$. Let P be a covering transformation

n	$\pm$	$s = kn+1$	$\theta \in \mathbb{Z}_s$	degree sequence of $\pi_\theta\left(\widetilde{\Delta}^{\chi_\pm}_{K_n}(t)\right)$
11	+	23	2	(2,2,3,3,8)
	−		2	(4,14)
17	+	103	8	(2,3,9,16)
	−		9	(2, 28)
23	+	47	4	(1, 1,11,29)
	−		2	(1, 1, 2, 12, 12, 14)

Table 1. The degree sequences of $\pi_\theta\left(\widetilde{\Delta}^{\chi_\pm}_{K_n}(t)\right)$.

invariant metabolizer of $H_1(\Sigma_3(K);\mathbb{Z})$. By Proposition 3.11, we see that either $P = P_+$ or there exists some $k = 0,\dots,n-1$ such that $P = r^k_*(P_-)$.

In the former case, let χ_+ be the character defined in Equation (3) and note that χ_+ vanishes on $P = P_+$. Moreover, the computations in Table 1, the observation that $2\lfloor\frac{c(K_n)-3}{2}\rfloor = 2\lfloor\frac{2n-3}{2}\rfloor = 2(n-2)$, and Proposition 3.16 together imply that $\widetilde{\Delta}^{\chi^+}_K(t)$ does not factor as a norm over $\mathbb{Q}(\xi_n)[t^{\pm 1}]$.

In the latter case, let $\chi_-\colon H_1(\Sigma_3(K);\mathbb{Z}) \to \mathbb{Z}_n$ be the character defined in Equation (3) that vanishes on P_-. Since $r^k_*(P_-) = P$, we have that $\chi := \chi_- \circ r^{-k}_*$ vanishes on P. Moreover, since r is a diffeomorphism of the 0-surgery, we have that $\widetilde{\Delta}^{\chi}_K(t) = \widetilde{\Delta}^{\chi_-}_K(t)$. So again the computations in Table 1 and Proposition 3.16 imply that $\widetilde{\Delta}^{\chi}_K(t)$ does not factor as a norm over $\mathbb{Q}(\xi_n)[t^{\pm 1}]$.

Therefore, for each invariant metabolizer of $H_1(\Sigma_3(K);\mathbb{Z})$ we have constructed a character of prime power order vanishing on that metabolizer so that the corresponding reduced twisted Alexander polynomial of K is not a norm. By Theorem 3.6, we conclude that K is not slice. □

Recall that $J = 8_{17}\#8^r_{17}$. Kirk and Livingston [27] proved J is not slice by showing that for each invariant metabolizer of $P \le H_1(\Sigma_3(J);\mathbb{Z}) \cong (\mathbb{Z}_{13})^4$ there exists a character $\chi\colon H_1(\Sigma_3(J);\mathbb{Z}) \to \mathbb{Z}_{13}$ such that $\chi|_P = 0$ and $\widetilde{\Delta}^{\chi}_J(t) \in \mathbb{Q}(\xi_{13})[t^{\pm 1}]$ is not a norm. Now we are ready to prove our main theorem.

Proof of Theorem 1.4. For the duration of this proof we refer to J as K_J for ease of notation in the formulae below.

Suppose that

$$K = a_7K_7\#a_{11}K_{11}\#a_JK_J\#a_{17}K_{17}\#a_{23}K_{23}$$

is slice for $a_7, a_{11}, a_J, a_{17}, a_{23} \in \{0,1\}$. If $a_{11} = a_J = a_{17} = a_{23} = 0$, then Sartori's work [41] implies that $a_7 = 0$, since K_7 is not slice. So we can assume that there exists $i_0 \in \{11, J, 17, 23\}$ such that $a_{i_0} \neq 0$.

Let

$$I := \{i \in \{7, 11, J, 17, 23\} \mid a_i \neq 0\}$$

and P be an invariant metabolizer for $H_1(\Sigma_3(K);\mathbb{Z})$. With the understanding of $\mathbb{Z}_J = \mathbb{Z}_{13}$, we have

$$H_1(\Sigma_3(K);\mathbb{Z}) \cong \bigoplus_{i\in I} H_1(\Sigma_3(K_i);\mathbb{Z}) \cong \bigoplus_{i\in I} (\mathbb{Z}_i)^4,$$

and since $7, 11, 13, 17$, and 23 are relatively prime, $P' := P \cap H_1(\Sigma_3(K_{i_0});\mathbb{Z})$ is an invariant metabolizer for $H_1(\Sigma_3(K_{i_0});\mathbb{Z})$.

Moreover, if $\chi'\colon H_1(\Sigma_3(K_{i_0});\mathbb{Z}) \to \mathbb{Z}_{i_0}$ is a character vanishing on P', then we can construct a character χ vanishing on P by decomposing

$$H_1(\Sigma_3(K);\mathbb{Z}) \cong \bigoplus_{i\in I} H_1(\Sigma_3(K_i);\mathbb{Z})$$

and letting

$$\chi|_{H_1(\Sigma_3(K_i);\mathbb{Z})} = \begin{cases} \chi' & i = i_0 \\ 0 & i \neq i_0. \end{cases}$$

Moreover, for such a character we let χ_i denote $\chi|_{H_1(\Sigma_3(K_i);\mathbb{Z})}$ and observe that by [32, Corollary 1] we have

$$\widetilde{\Delta}^{\chi}_{K}(t) = \prod_i \widetilde{\Delta}^{\chi|_i}_{K_i}(t) = \widetilde{\Delta}^{\chi'}_{K_{i_0}}(t).$$

It therefore suffices to show that for any invariant metabolizer of $H_1(\Sigma_3(K_{i_0});\mathbb{Z})$ there exists a character χ' to $\mathbb{Z}_{i_0}$ vanishing on that metabolizer such that the resulting reduced twisted Alexander polynomial $\widetilde{\Delta}^{\chi'}_{K_{i_0}}(t)$ does not factor as a norm over $\mathbb{Q}(\xi_{i_0})[t^{\pm 1}]$.

This is exactly what we did in the proof of Theorem 3.1 for $i_0 = 11, 17, 23$ and what Kirk and Livingston did in [27] for the case of $i_0 = J$ thereby completing the proof. □

Appendix A. The images of the reduced twisted Alexander polynomials in $\mathbb{Z}_s[t^{\pm 1}]$.

For the convenience of the reader, we give the irreducible factors of the images of the reduced twisted Alexander polynomials $\pi_\theta\left(\widetilde{\Delta}^{\chi^\pm}_{K}(t)\right) \in \mathbb{Z}_s[t^{\pm 1}]$. These computations, necessary in order to obtain the degree sequences of Table 1, were done in Maple worksheets that are available on the third author's website.

$(\pm, s, \theta)$	Irreducible factors of $\pi_\theta\left(\widetilde{\Delta}^{\chi^\pm}_{K_{11}}(t)\right) \in \mathbb{Z}_s[t^{\pm 1}]$
$(+, \mathbf{23}, \mathbf{2})$	
degree 2	$t^2 + 13t + 1,\ t^2 + 3t + 11$
degree 3	$t^3 + 14t^2 + 3,\ t^3 + 22t^2 + 22t + 22$
degree 8	$t^8 + 22t^7 + 4t^6 + 14t^5 + 3t^4 + 3t^3 + 16t^2 + t + 20$
$(-, \mathbf{23}, \mathbf{2})$	
degree 4	$t^4 + 17t^3 + 4t^2 + 17t + 1$
degree 14	$t^{14} + 7t^{13} + 5t^{12} + 7t^{11} + 7t^{10} + 22t^9 + 22t^8 + 7t^7$ $+22t^6 + 22t^5 + 7t^4 + 7t^3 + 5t^2 + 7t + 1$

$(\pm, s, \theta)$	Irreducible factors of $\pi_\theta\left(\widetilde{\Delta}^{\chi^\pm}_{K_{17}}(t)\right) \in \mathbb{Z}_s[t^{\pm 1}]$
$(+, \mathbf{103}, \mathbf{8})$	
degree 2	$t^2 + 98t + 5$
degree 3	$t^3 + 12t^2 + 36t + 93$
degree 9	$t^9 + 33t^8 + 94t^7 + 32t^6 + 61t^5 + 20t^4 + 63t^3 + 48t^2 + 19t + 94$
degree 16	$t^{16} + 74t^{15} + 26t^{14} + 92t^{13} + 31t^{12} + 85t^{11} + 86t^{10} + 34t^9 + 35t^8$ $+67t^7 + 99t^6 + 64t^5 + 67t^4 + 11t^3 + 95t^2 + 8t + 19$
$(-, \mathbf{103}, \mathbf{9})$	
degree 2	$t^2 + 13t + 1$
degree 28	$t^{28} + 61t^{27} + 97t^{26} + 22t^{25} + 25t^{24} + 27t^{23} + 73t^{22} + 47t^{21} + 79t^{20} + 31t^{19}$ $+99t^{18} + 36t^{17} + 54t^{16} + 40t^{15} + 40t^{14} + 40t^{13} + 54t^{12} + 36t^{11} + 99t^{10}$ $+31t^9 + 79t^8 + 47t^7 + 73t^6 + 27t^5 + 25t^4 + 22t^3 + 97t^2 + 61t + 1$

$(\pm, s, \theta)$	Irreducible factors of $\pi_\theta\left(\widetilde{\Delta}^{\chi^\pm}_{K_{23}}(t)\right) \in \mathbb{Z}_s[t^{\pm 1}]$
$(+, \mathbf{47}, \mathbf{4})$	
degree 1	$t + 21,\ t + 29$
degree 11	$t^{11} + 37t^{10} + 43t^9 + 5t^8 + t^7 + 42t^6 + 34t^5 + 43t^4 + 5t^3 + 34t^2 + 44t + 9$
degree 29	$t^{29} + 25t^{28} + 9t^{27} + 19t^{26} + 38t^{25} + 46t^{24} + 27t^{23} + 40t^{22} + 41t^{21} + 18t^{20}$ $+17t^{19} + t^{18} + 34t^{17} + 6t^{16} + 21t^{15} + 25t^{14} + 18t^{13} + 25t^{12} + 34t^{11} + 9t^{10}$ $+12t^9 + 41t^8 + 46t^7 + 10t^6 + 40t^5 + 21t^4 + 10t^3 + t^2 + 40t + 13$
$(-, \mathbf{47}, \mathbf{2})$	
degree 1	$t + 46, t + 46$
degree 2	$t^2 + t + 1$
degree 12	$t^{12} + 3t^{11} + 27t^{10} + 19t^9 + 38t^8 + 25t^7 + 25t^6 + 40t^5 + 16t^4 + 25t^3$ $+44t^2 + 28t + 23,\ t^{12} + 38t^{11} + 6t^{10} + 44t^9 + 15t^8 + 14t^7 + 44t^6$ $+44t^5 + 18t^4 + 9t^3 + 40t^2 + 41t + 45$
degree 14	$t^{14} + 2t^{13} + 2t^{12} + 43t^{11} + 42t^{10} + 36t^9 + 30t^8 + 33t^7 + 30t^6$ $+36t^5 + 42t^4 + 43t^3 + 2t^2 + 2t + 1$

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