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Maximum principle preserving finite difference scheme for 1-D nonlocal-to-local diffusion problems

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ABSTRACT

In a recent paper (Du et al., 2018), a quasi-nonlocal coupling method was introduced to seamlessly bridge a nonlocal diffusion model with the classical local diffusion counterpart in a one-dimensional space. The proposed coupling framework removes interfacial inconsistency, preserves the balance of fluxes, and satisfies the maximum principle of diffusion problem. However, the numerical scheme proposed in that paper does not maintain all of these properties on a discrete level. In this paper we resolve this issue by proposing a new finite difference scheme that ensures the balance of fluxes and the discrete maximum principle. We rigorously prove these results and provide the stability and convergence analyses accordingly. In addition, we provide the Courant–Friedrichs–Lewy (CFL) condition for the new scheme and test a series of benchmark examples which confirm the theoretical findings.

1. Introduction

Since the last decade, nonlocal integro-differential type models have been employed to describe physical systems, due to their natural ability to model physical phenomena at small scales and their reduced regularity requirements which lead to greater flexibility [1–18]. These nonlocal models are defined through a length scale parameter δ, referred to as a horizon, which measures the extent of nonlocal interaction. An important feature of nonlocal models is that they restore the corresponding classical partial differential equation (PDE) models as the horizon $\delta \to 0$ [6,7].

Nonlocal models that are compatible with the local PDEs are often much computationally expensive and require additional attention to the boundary treatments since a layer of volumetric boundary conditions is needed within the physical system. Meanwhile, nonlocal models need less regularity requirements which helps the descriptions near defects and singularities. Consequently, tremendous efforts have been devoted to combining nonlocal and local methods to keep accuracy around the irregularity while retain efficiency away from the singularity (see the review paper [19] for the state-of-art).

In [20], a quasi-nonlocal (QNL) coupling method was proposed to combine the nonlocal and local diffusion operators in a seamless way using the variational approach. The coupled operator is proved to preserve many mathematical and physical properties on the continuous level, including the symmetry of operator, the balance of linear momentum, and the maximum principle. However, it is not clear how to retain these desired properties with proper numerical discretization. In this paper, we propose a new finite difference method which inherits all properties from the continuous case.

We recall that the linear local diffusion model in one-dimensional space is

$$u_t(x,t) = u_{xx}(x,t) + f(x,t). \tag{1.1}$$

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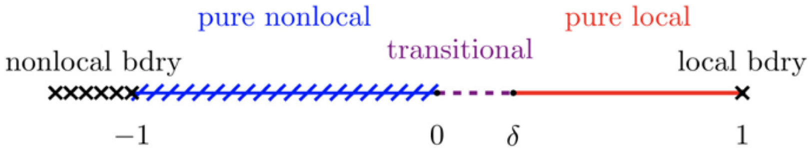


Fig. 1. Graphical illustration of 1-D Domain.

The corresponding counterpart in the nonlocal setting is the linear nonlocal diffusion model which reads

$$u_t(x,t) = \int_{-\delta}^{\delta} \gamma_\delta(s)\Big(u(x+s,t) - u(x,t)\Big)ds, \tag{1.2}$$

where $\gamma_\delta(s)$ denotes the isotropic nonlocal diffusion kernel satisfying the following convenient assumption with $\gamma_\delta(\cdot)$ being a rescaled kernel,

$$\begin{cases} \gamma_\delta(|s|) = \dfrac{1}{\delta^3}\gamma\left(\dfrac{|s|}{\delta}\right), \quad \gamma \text{ is nonnegative and nonincreasing on } (0,1), \\ \text{with } \mathrm{supp}(\gamma) \subset [0,1] \text{ and } \displaystyle\int_{-\delta}^{\delta} |s|^2\gamma(|s|)ds = 2\,. \end{cases} \tag{1.3}$$

We will display more details about the coupling and numerical schemes in the following sections.

More precisely, we organize the paper as follows: In Section 2, we recall the energy-based QNL coupling from [20] to build the coupling operator $\mathcal{L}_\delta^{qnl}$ bringing the nonlocal and local diffusion problems and introduce space–time discretizations as well as the new finite difference method (FDM). In Section 3, we estimate the consistency errors of the proposed scheme using Taylor expansions. In Section 4, we prove the discrete maximum principle and hence the stability of proposed scheme. In Section 5, we combine the consistency and stability results to conclude the convergence estimates. In Section 6, we mathematically study the Courant–Friedrichs–Lewy (CFL) condition for the space–time discretization. In Section 7, we test several benchmark examples to confirm our theoretic findings.

2. QNL coupling and finite difference scheme

Now, we consider the domain to be $\Omega_\delta = [-1-\delta, 1]$, with the coupling interface of nonlocal and local models at $x^* = 0$; $(-1, 0)$ denotes the nonlocal region with nonlocal boundary layer at $[-1-\delta, -1]$ and $(0, 1)$ denotes the local region with local boundary point at $\{1\}$, as illustrated in Fig. 1.

In [20], the QNL operator $\mathcal{L}_\delta^{qnl}u(x,t)$ is introduced to smoothly bridge the local and nonlocal regions over the transitional region $[0, \delta]$. The corresponding coupled diffusion problem is proved to be a well-posed initial value problem and is given by

$$\begin{cases} u_t(x,t) = \mathcal{L}_\delta^{qnl}u(x,t) + f(x,t), & \text{for } \ T > t > 0 \ \text{ and } \ x \in (-1,1), \\ u(x,0) = u_0(x), & \text{for } \ x \in (-1,1), \\ u(x,t) = 0, & \text{for } \ x \in [-1-\delta, -1], \ \text{ or } \ x = 1. \end{cases} \tag{2.1}$$

$\mathcal{L}_\delta^{qnl}$ employed in Eq. (2.1) is the quasi-nonlocal coupling operator which describes the diffusion within the nonlocal, transitional, and local regions, respectively. The expression of $\mathcal{L}_\delta^{qnl}$ is given below

$$\mathcal{L}_\delta^{qnl}u(x,t) = \begin{cases} \displaystyle\int_{-\delta}^{\delta}\Big(u(x+s,t) - u(x,t)\Big)\gamma_\delta(s)ds, & \text{if } \ x \in (-1,0], \\ \displaystyle\int_x^{\delta}\gamma_\delta(s)\Big(u(x-s,t) - u(x,t)\Big)ds + \left(\int_x^{\delta} s\gamma_\delta(s)ds\right)u_x(x,t) & \\ \quad \displaystyle+\left(\int_0^x s^2\gamma_\delta(s) + x\int_x^{\delta} s\gamma_\delta(s)ds\right)u_{xx}(x), & \text{if } \ x \in (0,\delta], \\ u_{xx}(x,t), & \text{if } \ x \in (\delta, 1). \end{cases} \tag{2.2}$$

Next, we discuss the numerical settings for the spatial and temporal discretization. We use u_i^n to denote the numerical approximation of the exact solution $u(x_i, t^n)$ with spatial and temporal step sizes being with $\Delta x := \frac{1}{N}$ and $\Delta t := \frac{T}{N_T}$, respectively. Hence, the spatial grid is x_i and temporal grid is $t_n = n\Delta t$. For simplicity, we drop x and t but only use i and n accordingly. The relation between Δx and Δt will be determined later by the CFL condition. Meanwhile, we assume that the horizon δ is a multiple of Δx with $\delta = r\Delta x$ and $r \in \mathbb{N}$.

Recall that the entire computational domain is $\Omega_\delta := [-1-\delta,\ 1]$, so the interior domain is $\Omega = [-1, 1]$ with interface at $x^* = 0$; the volumetric boundary layer for the nonlocal region is $\Omega_n = [-1-\delta, -1)$; and the local boundary point is $\Omega_c = \{1\}$. Next we denote the set of spatial grids by I and $I = I_\Omega \cup I_{\Omega_n} \cup I_{\Omega_c}$, where $I_\Omega = \{1, 2, \ldots, 2N-1\}$ denotes the interior grids, $I_{\Omega_n} = \{-(r-1), \ldots, 0\}$ denotes the nonlocal volumetric boundary grids, and $I_{\Omega_c} = \{2N\}$ denotes the local boundary point. Following the scope of asymptotically compatible scheme [21,22], we define the spatial discretization of the QNL coupling operator $\mathcal{L}^{qnl}_{\delta,\Delta x}$ as follows

$$
\mathcal{L}^{qnl}_{\delta,\Delta x} u_i^n := \begin{cases}
\displaystyle\sum_{j=1}^{r} \frac{u_{i+j}^n - 2u_i^n + u_{i-j}^n}{(j\Delta x)^2} \int_{(j-1)\Delta x}^{j\Delta x} s^2 \gamma_\delta(s) ds, & \text{if} \quad x_i \le 0, \\[2ex]
\displaystyle\sum_{j=\frac{x_i}{\Delta x}+1}^{r} \frac{u_{i+j-1}^n - 2u_i^n + u_{i-j+1}^n}{2(j-1)\Delta x} \int_{(j-1)\Delta x}^{j\Delta x} s\gamma_\delta(s) ds \\
\displaystyle\quad - \sum_{j=\frac{x_i}{\Delta x}+1}^{r} \frac{u_{i+j-1}^n - u_{i-j+1}^n}{2(j-1)\Delta x} \int_{(j-1)\Delta x}^{j\Delta x} s\gamma_\delta(s) ds \\
\displaystyle\quad + \left(\int_{x_i}^{\delta} s\gamma_\delta(s) ds \right) \frac{u_{i+1}^n - u_i^n}{\Delta x} \\
\displaystyle\quad + \left(\int_0^{x_i} s^2 \gamma_\delta(s) ds + x_i \int_{x_i}^{\delta} s\gamma_\delta(s) ds \right) \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}, & \text{if} \quad x_i \in (0, \delta], \\[2ex]
\displaystyle\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}, & \text{if} \quad x_i \in (\delta, 1).
\end{cases} \tag{2.3}
$$

For the temporal discretization, we employ the simplest explicit Euler scheme due to the limitation of first order accuracy in the spatial discretization, which will be proved later. Hence the full FDM discretization of (2.1) is

$$
\frac{u_i^{n+1} - u_i^n}{\Delta t} = \mathcal{L}^{qnl}_{\delta,\Delta x} u_i^n + f_i^n, \quad i \in I_\Omega, \tag{2.4}
$$

where $f_i^n = f(x_i, t^n)$.

Fig. 2 displays a sampling set of spatial stencils using $N = 5$ on domain $[-1-\delta, 1]$. The step size is $\Delta x = \frac{1}{5}$ and the horizon $\delta = r\Delta x$ with $r = 3$.

Remark 2.1. In [20], the time-integral is still approximated by the explicit Euler method, and the $\tilde{\mathcal{L}}^{qnl}_{\delta,\Delta x}$ is approximated by the following finite difference scheme given interface at $x^* = 0$:

$$
\tilde{\mathcal{L}}^{qnl}_{\delta,\Delta x} u_i^n \approx \begin{cases}
\displaystyle 2\sum_{j=1}^{r} \frac{u_{i+j}^n - 2u_i^n + u_{i-j}^n}{(j\Delta x)^2} \int_{(j-1)\Delta x}^{j\Delta x} s^2 \gamma_\delta(s) ds, & \text{if} \quad x_i \le 0. \\[2ex]
\displaystyle\sum_{j=\frac{x_i}{\Delta x}}^{r} \frac{u_{i+j}^n - 2u_i^n + u_{i-j}^n}{(j\Delta x)^2} \int_{(j-1)\Delta x}^{j\Delta x} s^2 \gamma_\delta(s) ds \\
\displaystyle - \sum_{j=\frac{x_i}{\Delta x}}^{r} \frac{u_{i+j}^n - u_{i-j}^n}{j\Delta x} \int_{(j-1)\Delta x}^{j\Delta x} s\gamma_\delta(s) ds \\
\displaystyle + 2\left(\int_{x_i}^{\delta} s\gamma_\delta(s) ds \right) \frac{u_{i+1}^n - u_i^n}{\Delta x} \\
\displaystyle + \left(2\int_0^{x_i} s^2 \gamma_\delta(s) ds + 2x_i \int_{x_i}^{\delta} s\gamma_\delta(s) ds \right) \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}, & \text{if} \quad x_i \in (0, \delta], \\[2ex]
\displaystyle\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}, & \text{if} \quad x_i \in (\delta, 1).
\end{cases} \tag{2.5}
$$

Compare (2.3) with (2.5), we notice that the difference is replacing j in the original scheme by $(j-1)$ in the new scheme across the transitional region. This is the main difference in the approximation that allows Eq. (2.3) to satisfy the discrete maximum principle whereas equation (2.5) does not. We will rigorously prove this in Section 4.

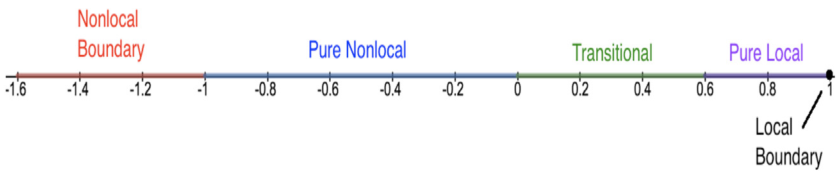


Fig. 2. Illustration of the finite difference stencil.

Remark 2.2. For numerical schemes that preserve the maximum principles in high dimensional space, recently, there are other types of coupling methods developed for two dimensional problems, such as [23,24]. These coupling schemes are based on domain-decomposition methods via Neumann or Robin type boundary conditions, and are rigorously proved to keep the maximum principles.

Regarding the conservation of flux, notice that the operator $\mathcal{L}^{qnl}_{\delta,\Delta x}$ of new scheme (2.3) is symmetric, hence, it possesses this property. In general, one have to keep interaction symmetries across the transitional region of the coupling region. However, the nonlocal neighborhood, $B_\delta(x)$, becomes a disk (in two dimensions) or a ball (in three dimensions), making the intersections with the interface highly more complex. As a result, it is not easy to preserve the flux in high dimensions.

3. Consistency

In this section, we estimate the consistency error of the scheme (2.4) with $\mathcal{L}^{qnl}_{\delta,\Delta x}$ defined in (2.3).

Theorem 3.1. *Let the horizon $\delta = r\Delta x$ with $r \in \mathbb{N}$ and being fixed, and suppose $u(x,t)$ is the strong solution to (2.1), and u_i^n is the discrete solution to the scheme (2.4) with $i \in I_\Omega$ and $t^n = n\Delta t$. Also assume that the exact solution u is sufficiently smooth, specifically $u(x,t) \in C^4([-1-\delta,1]\times[0,T])$. Suppose at any given time level $t^n = n\Delta t$ we have $u(x_i,t^n) = u_i^n,\ \forall i \in I_\Omega = \{1,\dots,2N-1\}$, then for the next time level $n+1$ the consistency error of the scheme satisfies*

$$|u_i^{n+1} - u(x_i,t^{n+1})| \le C_\delta \Delta t\,((\Delta x) + (\Delta t)),\ \ \forall i = 1,\dots,2N-1, \tag{3.1}$$

where C_δ is a constant independent of Δx and Δt.

Proof. We evolve $u(x_i,t^n)$ and u_i^n by one time step Δt according to three differential regions.

Local: If $x_i > \delta$ or simply $i \in \{N+r+1,\dots,2N-1\}$, then the continuous and discrete equations follow the expressions in the local region. So at (x_i,t^n), we have the continuous equation:

$$u_t(x_i,t_n) = u_{xx}(x_i,t_n) + f(x_i,t_n), \tag{3.2}$$

and the discrete equation:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + f_i^n \tag{3.3}$$

with $f_i^n = f(x_i,t^n)$.

Notice from consistency assumption that $u_i^n = u(x_i,t^n)$, so can rewrite the discrete equation as

$$\frac{u_i^{n+1}-u(x_i,t^n)}{\Delta t} = \frac{u(x_{i+1},t^n) - 2u(x_i,t^n) + u(x_{i-1},t^n)}{(\Delta x)^2} + f(x_i,t^n). \tag{3.4}$$

We apply the Taylor expansion at the spatial grid (x_i) up to fourth order derivative and get an estimate of u_i^{n+1}, which is

$$\begin{aligned} u_i^{n+1} =& u(x_i,t^n) + \Delta t\left(\frac{u(x_{i+1},t^n) - 2u(x_i,t^n) + u(x_{i-1},t^n)}{(\Delta x)^2} + f(x_i,t^n)\right) \\ =& u(x_i,t^n) + \Delta t\left(\frac{(\Delta x)^2 u_{xx}(x_i,t^n) + O(\Delta x^4)}{(\Delta x)^2} + f(x_i,t^n)\right) \\ =& u(x_i,t^n) + \Delta t\Big(u_{xx}(x_i,t^n) + f(x_i,t^n)\Big) + O\big(\Delta t(\Delta x)^2\big). \end{aligned} \tag{3.5}$$

Now, let us estimate the continuous solution $u(x_i,t^{n+1})$. This time, we apply Taylor expansion at the time grid (t^n) and get

$$\begin{aligned} u(x_i,t^{n+1}) =& u(x_i,t^n) + \Delta t u_t(x_i,t^n) + O(\Delta t^2) \\ =& u(x_i,t^n) + \Delta t\Big[\big(u_{xx}(x_i,t^n) + f(x_i,t^n)\big)\Big] + O(\Delta t^2), \end{aligned} \tag{3.6}$$

where we substitute $u_t(x_i, t^n)$ by the continuous equation on the local region.

By subtracting (3.5) from (3.6) we can get

$$u_i^{n+1} - u(x_i, t^{n+1}) = O\big(\Delta t(\Delta x)^2\big) + O\big((\Delta t)^2\big). \tag{3.7}$$

Nonlocal: Next we consider the fully nonlocal region where $x_i \leq 0$ or simply $i \in \{1, \dots, N\}$. We first have the continuous equation:

$$\begin{aligned} u_t(x_i, t^n) &= \int_{-\delta}^{\delta} \gamma_\delta(s)\Big(u(x_i + s, t^n) - u(x_i, t^n)\Big)ds + f(x_i, t^n) \\ &= \int_{-\delta}^{0} \gamma_\delta(s)\Big(u(x_i + s, t^n) - u(x_i, t^n)\Big)ds + \int_{0}^{\delta} \gamma_\delta(s)\Big(u(x_i + s, t^n) - u(x_i, t^n)\Big)ds + f(x_i, t^n) \\ &= \int_{0}^{\delta} \gamma_\delta(-s)\Big(u(x_i - s, t^n) - u(x_i, t^n)\Big)ds + \int_{0}^{\delta} \gamma_\delta(s)\Big(u(x_i + s, t^n) - u(x_i, t^n)\Big)ds + f(x_i, t^n). \end{aligned} \tag{3.8}$$

Because of the isotropic property of the nonlocal kernel $\gamma_\delta(s)$ summarized in (1.3), we have

$$u_t(x_i, t^n) = \int_{0}^{\delta} \gamma_\delta(s)\Big(u(x_i + s, t^n) - 2u(x_i, t^n) + u(x_i - s, t^n)\Big)ds + f(x_i, t^n). \tag{3.9}$$

Clearly, we can divide the integral into the sum of subintegrals on the union of subintervals, so we have,

$$u_t(x_i, t^n) = \sum_{j=1}^{r} \int_{(j-1)\Delta x}^{j\Delta x} \gamma_\delta(s)\Big(u(x_i + s, t^n) - 2u(x_i, t^n) + u(x_i - s, t^n)\Big)ds + f(x_i, t^n). \tag{3.10}$$

Meanwhile, we have the discrete equation to advance u_i^n to u_i^{n+1}:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \sum_{j=1}^{r} \frac{u_{i+j}^n - 2u_i^n + u_{i-j}^n}{(j\Delta x)^2} \int_{(j-1)\Delta x}^{j\Delta x} s^2 \gamma_\delta(s)ds + f_i^n. \tag{3.11}$$

Which gives,

$$u_i^{n+1} = u_i^n + \Delta t\left(\sum_{j=1}^{r} \frac{u_{i+j}^n - 2u_i^n + u_{i-j}^n}{(j\Delta x)^2} \int_{(j-1)\Delta x}^{j\Delta x} s^2 \gamma_\delta(s)ds + f_i^n\right). \tag{3.12}$$

Now we want to estimate the continuous solution $u(x_i, t^{n+1})$. We know that

$$u(x_i, t^{n+1}) = u(x_i, t^n) + \Delta t u_t(x_i, t^n) + O(\Delta t^2), \tag{3.13}$$

Hence, plugging the continuous description of nonlocal diffusion (3.10), we get

$$\begin{aligned} u(x_i, t^{n+1}) &= u(x_i, t^n) + \Delta t u_t(x_i, t^n) + O(\Delta t^2) \\ &= u(x_i, t^n) + \Delta t\Bigg[\sum_{j=1}^{r} \int_{(j-1)\Delta x}^{j\Delta x} \gamma_\delta(s)s^2\left(\frac{u(x_i + s, t^n) - 2u(x_i, t^n) + u(x_i - s, t^n)}{s^2}\right)ds \\ &\qquad + f(x_i, t^n)\Bigg] + O(\Delta t^2). \end{aligned} \tag{3.14}$$

For each integral term from $[(j-1)\Delta x, j\Delta x]$ within the summation , we then focus on the fractional term and apply Taylor expand to $u(x_i + s, t^n)$ and $u(x_i - s, t^n)$ for s at $(j\Delta x)$ up to fourth order derivative. This gives an estimate of

$$\begin{aligned} u(x_i, t^{n+1}) &= u(x_i, t^n) \\ &\quad + \Delta t\Bigg[\sum_{j=1}^{r} \int_{(j-1)\Delta x}^{j\Delta x} \gamma_\delta(s)s^2 \frac{1}{(j\Delta x)^2}\Big(\big(u(x_{i+j}, t^n) - 2u(x_i, t^n) + u(x_{i-j}, t^n)\big) + O(s^4)\Big)ds \\ &\qquad + f(x_i, t^n)\Bigg] + O(\Delta t^2) \\ &= u_i^n + \Delta t\Bigg[\sum_{j=1}^{r} \int_{(j-1)\Delta x}^{j\Delta x} \gamma_\delta(s)s^2 \frac{1}{(j\Delta x)^2}\Big(\big(u_{i+j}^n - 2u_i^n + u_{i-j}^n\big)\Big)ds + O(\Delta x^2) \\ &\qquad + f(x_i, t^n)\Bigg] + O(\Delta t^2). \end{aligned} \tag{3.15}$$

Then by subtracting (3.12) from (3.15), we can get

$$u_i^{n+1} - u(x_i, t^{n+1}) = O(\Delta t) \cdot O(\Delta x)^2 + O(\Delta t^2). \tag{3.16}$$

Transitional: Finally we consider when $x_i \in (0,\ \delta]$ or equivalently $i \in \{N+1, \ldots, N+r\}$, and again we will look at the continuous equation for the time derivative $u_t(x_i, t^n)$ first.

$$\begin{aligned} u_t(x_i, t^n) = &\Bigg[\int_{x_i}^{\delta} \gamma_\delta(s)\Big(u(x_i - s, t^n) - u(x_i, t^n)\Big)ds + \Big(\int_{x_i}^{\delta} s\gamma_\delta(s)ds\Big)u_x(x_i, t^n) \\ &+ \Big(\int_0^{x_i} s^2\gamma_\delta(s)ds + x_i\int_{x_i}^{\delta} s\gamma_\delta(s)ds\Big)u_{xx}(x_i, t^n)\Bigg] + f(x_i, t^n), \end{aligned} \tag{3.17}$$

and splitting and symmetrizing the first integral gives

$$\begin{aligned} u_t(x_i, t^n) = &\int_{x_i}^{\delta} \frac{\gamma_\delta(s)}{2}\Big(u(x_i - s, t^n) - 2u(x_i, t^n) + u(x_i + s, t^n)\Big)ds \\ &+ \int_{x_i}^{\delta} \frac{\gamma_\delta(s)}{2}\Big(u(x_i - s, t^n) - u(x_i + s, t^n)\Big)ds + \Big(\int_{x_i}^{\delta} s\gamma_\delta(s)ds\Big)u_x(x_i, t^n) \\ &+ \Big(\int_0^{x_i} s^2\gamma_\delta(s)ds + x_i\int_{x_i}^{\delta} s\gamma_\delta(s)ds\Big)u_{xx}(x_i, t^n) + f(x_i, t^n), \end{aligned} \tag{3.18}$$

and dividing these two integrals into the sum of subintegrals on the union of subintervals, and modify each integrand in the scope of asymptotically compatible scheme [22], we get

$$\begin{aligned} u_t(x_i, t^n) = &\sum_{j=\frac{x_i}{\Delta x}+1}^{r} \int_{(j-1)\Delta x}^{j\Delta x} \frac{\gamma_\delta(s)s}{2}\Big(\frac{u(x_i - s, t^n) - 2u(x_i, t^n) + u(x_i + s, t^n)}{s}\Big)ds \\ &+ \sum_{j=\frac{x_i}{\Delta x}+1}^{r} \int_{(j-1)\Delta x}^{j\Delta x} \frac{\gamma_\delta(s)s}{2}\Big(\frac{u(x_i - s, t^n) - u(x_i + s, t^n)}{s}\Big)ds + \Big(\int_{x_i}^{\delta} s\gamma_\delta(s)ds\Big)u_x(x_i, t^n) \\ &+ \Big(\int_0^{x_i} s^2\gamma_\delta(s)ds + x_i\int_{x_i}^{\delta} s\gamma_\delta(s)ds\Big)u_{xx}(x_i, t^n) + f(x_i, t^n). \end{aligned} \tag{3.19}$$

Now working with the discrete equation for u_i^{n+1}

$$\begin{aligned} \frac{u_i^{n+1} - u_i^n}{\Delta t} = &\sum_{j=\frac{x_i}{\Delta x}+1}^{r} \frac{u_{i+j-1}^n - 2u_i^n + u_{i-j+1}^n}{2(j-1)\Delta x} \int_{(j-1)\Delta x}^{j\Delta x} s\gamma_\delta(s)ds \\ &- \sum_{j=\frac{x_i}{\Delta x}+1}^{r} \frac{u_{i+j-1}^n - u_{i-j+1}^n}{2(j-1)\Delta x} \int_{(j-1)\Delta x}^{j\Delta x} s\gamma_\delta(s)ds + \Big(\int_{x_i}^{\delta} s\gamma_\delta(s)ds\Big)\frac{u_{i+1}^n - u_i^n}{\Delta x} \\ &+ \Big(\int_0^{x_i} s^2\gamma_\delta(s)ds + x_i\int_{x_i}^{\delta} s\gamma_\delta(s)ds\Big)\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + f_i^n. \end{aligned} \tag{3.20}$$

Which gives,

$$\begin{aligned} u_i^{n+1} = u_i^n + \Delta t\Bigg[&\sum_{j=\frac{x_i}{\Delta x}+1}^{r} \frac{u_{i+j-1}^n - 2u_i^n + u_{i-j+1}^n}{2(j-1)\Delta x} \int_{(j-1)\Delta x}^{j\Delta x} s\gamma_\delta(s)ds \\ &- \sum_{j=\frac{x_i}{\Delta x}+1}^{r} \frac{u_{i+j-1}^n - u_{i-j+1}^n}{2(j-1)\Delta x} \int_{(j-1)\Delta x}^{j\Delta x} s\gamma_\delta(s)ds + \Big(\int_{x_i}^{\delta} s\gamma_\delta(s)ds\Big)\frac{u_{i+1}^n - u_i^n}{\Delta x} \\ &+ \Big(\int_0^{x_i} s^2\gamma_\delta(s)ds + x_i\int_{x_i}^{\delta} s\gamma_\delta(s)ds\Big)\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + f_i^n\Bigg]. \end{aligned} \tag{3.21}$$

Again we want to estimate difference between $u(x_i, t^{n+1})$ and u_i^{n+1}.

For each integral term $[(j-1)\Delta x, j\Delta x]$ within the summation of (3.19), we then Taylor expand $u(x_i+s, t^n)$ and $u(x_i-s, t^n)$ for s at $(j-1)\Delta x$, which is similar to the processing we did for the nonlocal region.

$$\begin{aligned} u(x_i,t^{n+1}) &= u(x_i, t^n) \\ &+ \Delta t \Bigg[\sum_{j=\frac{x_i}{\Delta x}+1}^{r} \int_{(j-1)\Delta x}^{j\Delta x} \frac{\gamma_\delta(s)s}{2(j-1)\Delta x}\Big(u(x_{i+j-1}, t^n) - 2u(x_i, t^n) + u(x_{i-j+1}, t^n) + O(s^2)\Big)ds \\ &+ \sum_{j=\frac{x_i}{\Delta x}+1}^{r} \int_{(j-1)\Delta x}^{j\Delta x} \frac{\gamma_\delta(s)s}{2(j-1)\Delta x}\Big(u(x_{i+j-1}, t^n) - u(x_{i-j+1}, t^n) + O(s)\Big)ds \\ &+ \left(\int_{x_i}^{\delta} s\gamma_\delta(s)ds\right)\left(\frac{u(x_{i+1}, t^n) - (x_i, t^n)}{\Delta x} + O(\Delta x)\right) \\ &+ \left(\int_0^{x_i} s^2\gamma_\delta(s)ds + x_i\int_{x_i}^{\delta} s\gamma_\delta(s)ds\right)\left(\frac{u(x_{i+1}, t^n) - 2(x_i, t^n) + (x_{i-1}, t^n)}{\Delta x^2} + O(\Delta x^2)\right). \end{aligned} \tag{3.22}$$

$$+f(x_i, t^n)\Bigg] + O(\Delta t^2). \tag{3.23}$$

By subtracting (3.21) from (3.22) we can get

$$u_i^{n+1} - u(x_i, t^{n+1}) = O(\Delta t)O(\Delta x) + O(\Delta t^2). \tag{3.24}$$

Therefore, $\|u(x_i, t^{n+1}) - u_i^{n+1}\|_{L^\infty} = O(\Delta t)O(\Delta x) + O(\Delta t^2)$ with highest restrictions from the transitional region. Since the order of accuracy is greater than zero, the finite difference scheme is consistent. □

4. Stability

Global stability of the scheme is attained by the discrete maximum principle. To prove the discrete maximum principle for the quasi-nonlocal coupling equation with an underlying finite difference discretization the spatial operator $(-\mathcal{L}^{qnl}_{\delta,\Delta x})$ must be positive-definite, and the time discretization, that is a single explicit Euler, must be a convex scheme. Recall the interior domain $\Omega = [-1, 1]$ with interface at $x^* = 0$. The volumetric boundary layer for the nonlocal region is $\Omega_n = (-1-\delta, -1]$, and the local boundary point is $\Omega_c = \{1\}$. The corresponding sets of spatial grids are $I_\Omega = \{1, 2, \ldots, 2N-1\}$ for Ω, $I_{\Omega_n} = \{-(r-1), \ldots, 0\}$ for Ω_n, and $I_{\Omega_c} = \{2N\}$ for Ω_c. Let $I = I_\Omega \cup I_{\Omega_n} \cup I_{\Omega_c}$ denote the union of total stencils within the entire domain (Interior and Boundary), and $I_B = I_{\Omega_n} \cup I_{\Omega_c}$ denote the stencils within the boundary regions $\Omega_n \cup \Omega_c$ (Boundary).

Next we will firstly prove the positive-definiteness of $(-\mathcal{L}^{qnl}_{\delta,\Delta x})$ in Theorem 4.1, which is the discrete maximum principle for the static case; and then extend the result to the dynamic case in Theorem 4.2 where time derivative is involved.

Theorem 4.1 (*Discrete Maximum Principle for the Static Case*)**.** *The discrete operator $\mathcal{L}^{qnl}_{\delta,\Delta x}$ satisfies the maximum principle. For $u(x_i) \in \ell^1(I)$ with $\left(-\mathcal{L}^{qnl}_{\delta,\Delta x}\right)\left(u(x_j)\right) \le 0$ and $j \in I_\Omega$, and for any $i \in I = I_\Omega \cup I_B$, we have*

$$\max_{i\in I} u(x_i) \le \max_{i\in I_B} u(x_i). \tag{4.1}$$

Furthermore, equality holds, and $u(x_i)$ is a constant function on stencils I.

Proof. Suppose the discrete function u achieves its strictly maximum values at an interior grid $j^* \in I_\Omega$.

Case I Nonlocal: Consider $j^* \in \{1, 2, \ldots, N\}$. Then since $u(x_{j^*})$ is a strict maximum

$$\mathcal{L}^{qnl}_{\delta,\Delta x}u_h(x_j^*) = \sum_{k=1}^{r} \frac{u(x_{j^*+k}) - 2u(x_{j^*}) + u(x_{j^*-k})}{(k\Delta x)^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds < 0 \tag{4.2}$$

which contradicts $-\mathcal{L}^{qnl}_{\delta,\Delta x}u(x_j^*) \le 0$ unless u is constant.

Case II Transitional: Consider $j^* \in \{N+1, N+2, \ldots, N+r\}$. We observe that

$$\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds > (k-1)\Delta x\int_{(k-1)\Delta x}^{k\Delta x} s\gamma_\delta(s)ds. \tag{4.3}$$

Using $u(x_{j^*})$

$$\mathcal{L}^{qnl}_{\delta,\Delta x}u_h(x_j^*) = \sum_{k=\frac{x_{j^*}}{\Delta x}+1}^{r} \frac{u(x_{j^*+k-1}) - 2u(x_{j^*}) + u(x_{j^*-k+1})}{2(k-1)^2(\Delta x)^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds$$

$$
\begin{aligned}
&- \sum_{k=\frac{x_{j^*}}{\Delta x}+1}^{r} \frac{u(x_{j^*+k-1}) - u(x_{j^*-k+1})}{2(k-1)\Delta x} \int_{(k-1)\Delta x}^{k\Delta x} s\gamma_\delta(s)ds \\
&+ \left(\int_{x_{j^*}}^{\delta} s\gamma_\delta(s)ds\right) \frac{u(x_{j^*+1}) - u(x_{j^*})}{\Delta x} \\
&+ \left(\int_0^{x_j^*} s^2\gamma_\delta(s)ds + x_{j^*}\int_{x_{j^*}}^{\delta} s\gamma_\delta(s)ds\right) \frac{u(x_{j^*+1}) - 2u(x_{j^*}) + u(x_{j^*-1})}{(\Delta x)^2}.
\end{aligned} \tag{4.4}
$$

Also since $u(x_{j^*})$ is a strict maximum we know

$$
\frac{u(x_{j^*+k-1}) - 2u(x_{j^*}) + u(x_{j^*-k+1})}{2(k-1)^2(\Delta x)^2} < 0, \tag{4.5}
$$

combined with (4.3), this gives us

$$
\begin{aligned}
\mathcal{L}^{qnl}_{\delta,\Delta x} u(x_j^*) \le & \sum_{k=\frac{x_{j^*}}{\Delta x}+1}^{r} \frac{u(x_{j^*+k-1}) - 2u(x_{j^*}) + u(x_{j^*-k+1})}{2(k-1)^2(\Delta x)^2} \cdot (k-1)\Delta x \int_{(k-1)\Delta x}^{k\Delta x} s\gamma_\delta(s)ds \\
&- \sum_{k=\frac{x_{j^*}}{\Delta x}+1}^{r} \frac{u(x_{j^*+k-1}) - u(x_{j^*-k+1})}{2(k-1)\Delta x} \int_{(k-1)\Delta x}^{k\Delta x} s\gamma_\delta(s)ds \\
&+ \left(\int_{x_{j^*}}^{\delta} s\gamma_\delta(s)ds\right) \frac{u(x_{j^*+1}) - u(x_{j^*})}{\Delta x} \\
&+ \left(\int_0^{x_j^*} s^2\gamma_\delta(s)ds + x_{j^*}\int_{x_{j^*}}^{\delta} s\gamma_\delta(s)ds\right) \frac{u(x_{j^*+1}) - 2u(x_{j^*}) + u(x_{j^*-1})}{(\Delta x)^2}.
\end{aligned} \tag{4.6}
$$

By simplifying we conclude

$$
\begin{aligned}
\mathcal{L}^{qnl}_{\delta,\Delta x} u_h(x_j^*) \le & \sum_{k=\frac{x_{j^*}}{\Delta x}+1}^{r} \frac{-2u(x_{j^*}) + 2u(x_{j^*-k+1})}{2(k-1)\Delta x} \int_{(k-1)\Delta x}^{k\Delta x} s\gamma_\delta(s)ds \\
&+ \left(\int_{x_{j^*}}^{\delta} s\gamma_\delta(s)ds\right) \frac{u(x_{j^*+1}) - u(x_{j^*})}{\Delta x} \\
&+ \left(\int_0^{x_j^*} s^2\gamma_\delta(s)ds + x_{j^*}\int_{x_{j^*}}^{\delta} s\gamma_\delta(s)ds\right) \frac{u(x_{j^*+1}) - 2u(x_{j^*}) + u(x_{j^*-1})}{(\Delta x)^2} < 0.
\end{aligned} \tag{4.7}
$$

which contradicts $-\mathcal{L}^{qnl}_{\delta,\Delta x} u(x_j) \le 0$.

Case III Local: Consider $j^* \in \{N+r+1, \dots, 2N-1\}$. Then since $u(x_{j^*})$ is a strict maximum

$$
\mathcal{L}^{qnl}_{\delta,\Delta x} u(x_j^*) = \frac{u(x_{j^*+1}) - 2u(x_{j^*}) + u(x_{j^*-1})}{(\Delta x)^2} < 0 \tag{4.8}
$$

which contradicts $-\mathcal{L}^{qnl}_{\delta,\Delta x} u(x_j) \le 0$. □

Next, we will consider the time-dependent case.

Theorem 4.2 (*Discrete Maximum Principle for the Dynamic Case*)**.** *Suppose for $i \in I = I_\Omega \cup I_B$ and $n = 0, 1, \dots, N_T - 1$ with $T = N_T \cdot \Delta t$ that $\{u_i^n\}$ solves the following discrete QNL diffusion equation.*

$$
\begin{cases}
\frac{u_i^{n+1} - u_i^n}{\Delta t} = \mathcal{L}^{qnl}_{\delta,\Delta x} u_i^n + f_i^n, & \text{for } i \in I_\Omega, \text{ and } N_T > n \ge 0, \\
u_i^0 = g_i^0, & \text{for } i \in I \text{ (Initial Condition)}, \\
u_i^n = q_i^n, & \text{for } i \in I_B,\ n \ge 0 \text{ (Boundary Condition)},
\end{cases} \tag{4.9}
$$

then u_i^n satisfies the discrete maximum principle

$$
u_i^n \le \max\{g_i^0|_{i\in I}, \quad q_i^n|_{i\in I_B, n\ge 0}\} \tag{4.10}
$$

given that $f_i^n \le 0$ for all $i \in I_\Omega$, all $n \ge 0$, and $\frac{\Delta t}{\Delta x^2} \le \frac{1}{4}$.

Proof. We denote $M = \max\{g_i^0|_{i\in I}, \quad q_i^n|_{i\in I_B, n\geq 0}\}$. Clearly, at $n = 0$ we have $u_i^0 \leq M$ for all $i \in I = I_\Omega \cup I_B$. We assume that this holds for $n = m$ with $0 \leq m \leq N_T - 2$. Now we would like to advance it to the next time level $n = m + 1$.

Case I Nonlocal: Consider $i \in \{1, 2, \ldots, N\}$ which is the nonlocal region. Then

$$\begin{aligned} u_i^{m+1} &= u_i^m + \Delta t\left(\mathcal{L}_{\delta,\Delta x}^{qnl} u_i^m + f_i^m\right) \\ &\leq u_i^m + \Delta t \mathcal{L}_{\delta,\Delta x}^{qnl} u_i^m \\ &= \left(1 - \frac{2\Delta t}{\Delta x^2}\sum_{k=1}^{r}\frac{1}{k^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds\right)u_i^m + \frac{\Delta t}{\Delta x^2}\sum_{k=1}^{r}\frac{u_{i+k}^m + u_{i-k}^m}{k^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds. \end{aligned}$$

Notice that

$$\sum_{k=1}^{r}\frac{1}{k^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds \leq \sum_{k=1}^{r}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds = \int_0^\delta s^2\gamma_\delta(s)ds = 1 \tag{4.11}$$

and $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{4}$, so

$$\left(1 - \frac{2\Delta t}{\Delta x^2}\sum_{k=1}^{r}\frac{1}{k^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds\right) \geq 0. \tag{4.12}$$

Hence,

$$\begin{aligned} u_i^{m+1} &\leq \left(1 - \frac{2\Delta t}{\Delta x^2}\sum_{k=1}^{r}\frac{1}{k^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds\right)u_i^m + \frac{\Delta t}{\Delta x^2}\sum_{k=1}^{r}\frac{u_{i+k}^m + u_{i-k}^m}{k^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds \\ &\leq \left(1 - \frac{2\Delta t}{\Delta x^2}\sum_{k=1}^{r}\frac{1}{k^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds\right)M + \frac{\Delta t}{\Delta x^2}\sum_{k=1}^{r}\frac{M + M}{k^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds \\ &= M. \end{aligned} \tag{4.13}$$

Case II Transitional: Consider $i \in \{N + 1, \ldots, N + r\}$ which is the transitional region. Then

$$\begin{aligned} u_i^{m+1} &\leq u_i^m + \Delta t \mathcal{L}_{\delta,\Delta x}^{qnl} u_i^m \\ &= u_i^m + \Delta t\Bigg[\sum_{k=\frac{x_i}{\Delta x}+1}^{r}\frac{u_{i+k-1}^m - 2u_i^m + u_{i-k+1}^m}{2(k-1)^2\Delta x^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds \\ &\quad - \sum_{k=\frac{x_i}{\Delta x}+1}^{r}\frac{u_{i+k-1}^m - u_{i-k+1}^m}{2(k-1)\Delta x}\int_{(k-1)\Delta x}^{k\Delta x} s\gamma_\delta(s)ds + \left(\int_{x_i}^{\delta} s\gamma_\delta(s)ds\right)\frac{u_{i+1}^m - u_i^m}{\Delta x} \\ &\quad + \left(\int_0^{x_i} s^2\gamma_\delta(s)ds + x_i\int_{x_i}^{\delta} s\gamma_\delta(s)ds\right)\frac{u_{i+1}^m - 2u_i^m + u_{i-1}^m}{\Delta x^2}\Bigg] \\ &= A\cdot u_i^m + \sum_{k=\frac{x_i}{\Delta x}+1}^{r}\left(B_k\cdot u_{i+k-1}^m + C_k\cdot u_{i-k+1}^m + D\cdot u_{i+1}^m + E\cdot u_{i-1}^m\right) \end{aligned} \tag{4.14}$$

where those notations are defined as

$$\begin{aligned} A &= 1 + \frac{\Delta t}{\Delta x^2}\left(\sum_{k=\frac{x_i}{\Delta x}+1}^{r}\frac{-1}{(k-1)^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds\right) + \frac{\Delta t}{\Delta x}\left(-\int_{x_i}^{\delta} s\gamma_\delta(s)ds\right) \\ &\quad - \frac{2\Delta t}{\Delta x^2}\left(\int_0^{x_i} s^2\gamma_\delta(s)ds + x_i\int_{x_i}^{\delta} s\gamma_\delta(s)ds\right), \\ B_k &= \frac{\Delta t}{2\Delta x^2(k-1)^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds - \frac{\Delta t}{2\Delta x(k-1)}\int_{(k-1)\Delta x}^{k\Delta x} s\gamma_\delta(s)ds, \\ C_k &= \frac{\Delta t}{2\Delta x^2(k-1)^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds + \frac{\Delta t}{2\Delta x(k-1)}\int_{(k-1)\Delta x}^{k\Delta x} s\gamma_\delta(s)ds, \\ D &= \frac{\Delta t}{\Delta x}\int_{x_i}^{\delta} s\gamma_\delta(s)ds + \frac{\Delta t}{\Delta x^2}\left(\int_0^{x_i} s^2\gamma_\delta(s)ds + x_i\int_{x_i}^{\delta} s\gamma_\delta(s)ds\right), \text{ and} \end{aligned}$$

$$E = \frac{\Delta t}{\Delta x^2}\left(\int_0^{x_i} s^2\gamma_\delta(s)ds + x_i\int_{x_i}^{\delta} s\gamma_\delta(s)ds\right). \tag{4.15}$$

Clearly, $A+\sum_{k=\frac{x_i}{\Delta x}+1}^{r}(B_k+C_k)+D+E=1$, and $B_k, C_k, D, E \geq 0$ when Δx is sufficiently small and because that $-\frac{\Delta t}{2\Delta x(k-1)}\int_{(k-1)\Delta x}^{k\Delta x} s\gamma_\delta(s)ds > -\frac{\Delta t}{2(\Delta x)^2(k-1)^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds$.

Now we want to prove that $A \geq 0$. It is equivalent to prove

$$1-A = \frac{\Delta t}{\Delta x^2}\Bigg[\sum_{k=\frac{x_i}{\Delta x}+1}^{r}\frac{1}{(k-1)^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds + 2\left(\int_0^{x_i} s^2\gamma_\delta(s)ds + x_i\int_{x_i}^{\delta} s\gamma_\delta(s)ds\right)$$
$$+\ \Delta x\int_{x_i}^{\delta} s\gamma_\delta(s)ds\Bigg] \leq 1. \tag{4.16}$$

Notice that

$$1-A = \frac{\Delta t}{\Delta x^2}\Bigg[\sum_{k=\frac{x_i}{\Delta x}+1}^{r}\left(\frac{1}{(k-1)^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds + 2x_i\int_{(k-1)\Delta x}^{k\Delta x}\left(\frac{1}{s}\right)s^2\gamma_\delta(s)ds\right.$$
$$\left. +\ \Delta x\int_{(k-1)\Delta x}^{k\Delta x}\left(\frac{1}{s}\right)s^2\gamma_\delta(s)ds\right) + 2\int_0^{x_i} s^2\gamma_\delta(s)ds\Bigg]$$
$$\leq \frac{\Delta t}{\Delta x^2}\Bigg[\sum_{k=\frac{x_i}{\Delta x}+1}^{r}\left(\frac{1}{(k-1)^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds + \frac{2x_i}{(k-1)\Delta x}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds\right.$$
$$\left. +\ \frac{\Delta x}{(k-1)\Delta x}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds\right) + 2\int_0^{x_i} s^2\gamma_\delta(s)ds\Bigg]$$
$$\leq \frac{\Delta t}{\Delta x^2}\Bigg[\sum_{k=\frac{x_i}{\Delta x}+1}^{r} 4\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds + 4\int_0^{x_i} s^2\gamma_\delta(s)ds\Bigg]$$
$$= 4\frac{\Delta t}{\Delta x^2}\Bigg[\sum_{k=\frac{x_i}{\Delta x}+1}^{r}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds + \int_0^{x_i} s^2\gamma_\delta(s)ds\Bigg] = \frac{4\Delta t}{\Delta x^2}\int_0^{\delta} s^2\gamma_\delta(s)ds$$
$$= 4\frac{\Delta t}{\Delta x^2} \leq 1.$$

Since $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{4}$, so $1-A \leq 1$. Therefore,

$$A \geq 0 \text{ for } B_k \geq \frac{\Delta t}{2\Delta x^2(k-1)^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds - \frac{\Delta t}{2\Delta x^2(k-1)^2}\int_{(k-1)\Delta x}^{k\Delta x} s^2\gamma_\delta(s)ds = 0.$$

Summarizing the coefficients of Eq. (4.14) gives

- $A, B_k, C_k, D, E \geq 0$
- $A+\sum_{k=\frac{x_i}{\Delta x}+1}^{r}(B_k+C_k)+D+E=1$.

Hence $u_i^{m+1} \leq \left(A+\sum_{k=\frac{x_i}{\Delta x}+1}^{r}(B_k+C_k)+D+E\right)M = M$.

Case III Local: Consider $i \in \{N+r+1,\ldots,2N-1\}$ which is the local region. Then

$$u_i^{m+1} = u_i^m + \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^m - 2u_i^m + u_{i-1}^m\right) + \Delta t f_i^m \leq \left(1-\frac{2\Delta t}{\Delta x^2}\right)u_i^m + \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^m + u_{i-1}^m\right)$$

with $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{4}$ which gives all positive coefficients, so $u_i^{m+1} \leq M$.

Combining case I, II, III we can conclude that given $u_i^m \leq M$ for all $i \in I_\Omega$, and $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{4}$ we have $u_i^{m+1} \leq M$ for all $i \in I_\Omega$. According to the induction we prove the theorem. □

Corollary 4.3. *Suppose for $i \in I = I_\Omega \cup I_B$, $n = 0, 1, \ldots, N_T - 1$, and $T = N_T \cdot \Delta t$ that $\{u_i^n\}$ solves the following discrete QNL diffusion equation (4.9) then we have the following upper bound for u_i^n given that $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{4}$,*

$$u_i^n \leq T\cdot\|f\|_{\ell^\infty(I)} + max\{\|g_i^0\|_{\ell^\infty(I)}, \|q_i^n\|_{\ell^\infty(I_B)}\}. \tag{4.17}$$

Proof. We introduce a comparison function

$$w_i^n = u_i^n + (T - n \cdot \Delta t)\|f\|_{\ell^\infty(I)} \geq u_i^n \tag{4.18}$$

for $i \in I$, and $n \geq 0$. Then we have

$$\frac{w_i^{n+1} - w_i^n}{\Delta t} = \frac{u_i^{n+1} - u_i^n}{\Delta t} - \|f\|_{\ell^\infty(I)} = \mathcal{L}_{\delta,\Delta x}^{qnl} u_i^n + \left(f_i^n - \|f\|_{\ell^\infty(I)}\right)$$

where $\left(f_i^n - \|f\|_{\ell^\infty(I)}\right) \leq 0$. Therefore by Theorem 4.2, w_i^n satisfies the discrete maximum principle $w_i^n \leq \max\{w_i^0|_{i\in I}, w_i^n|_{i\in I_B}\}$ for all $i \in I_\Omega$ and $n \geq 0$, given that $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{4}$.

Notice that

$$w_i^0 = u_i^0 + T \cdot \|f\|_{\ell^\infty(I)} \leq \max\{\|g_i^0\|_{\ell^\infty(I)}, \|q_i^n\|_{\ell^\infty(I_B)}\} + T \cdot \|f\|_{\ell^\infty(I)} \tag{4.19}$$

and also that

$$w_i^n|_{i\in I_B} = u_i^n|_{i\in I_B} + \left(T - n \cdot \Delta t\right)\|f\|_{\ell^\infty(I)} \leq \max\{\|g_i^0\|_{\ell^\infty(I)}, \|q_i^n\|_{\ell^\infty(I_B)}\} + T \cdot \|f\|_{\ell^\infty(I)}. \tag{4.20}$$

combined with the fact that $u_i^n|_{i\in I} \leq w_i^n|_{i\in I}$ proves the corollary. □

Remark 4.1. Although in the proof of stability analysis, we require that $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{4}$ to proceed the analysis; meanwhile, we notice in the simulation that with $\frac{\Delta t}{\Delta x^2}$ close to $\frac{1}{2}$, we still have stable numerical results.

5. Convergence

In this section, we prove the convergence results of the proposed FDM scheme.

Theorem 5.1 (*Global Error Estimate of the Discrete Solution*)**.** *Suppose $u(x, t)$ is the strong solution to (2.1) and u_i^n is the discrete solution to the scheme (2.4) with $i \in I$, $n = 0, 1, \ldots, N_T - 1$, and $N_T \Delta t = T$, respectively. Then we have*

$$|u(x_i, t^n) - u_i^n| \leq T \cdot C_\delta(\Delta x + \Delta t) \tag{5.1}$$

given that $\frac{\Delta t}{\Delta x^2} \leq \frac{1}{4}$.

Proof. We define $e_i^n = u(x_i, t^n) - u_i^n$, $i = 1, 2, \ldots, 2N - 1$, $n = 0, 1, \ldots, N_T$ to be the error between the exact and discrete solutions. Then from the consistency analysis, and since $f_i^n = f(x_i, t^n)$ we have that

$$\begin{cases} \frac{e_i^{n+1} - e_i^n}{\Delta t} - \mathcal{L}_{\delta,\Delta x}^{qnl} e_i^n = \varepsilon_{c,i}, & \textit{for } i \in I_\Omega, \textit{ and } n \geq 0 \\ e_i^0 = 0, i \in I & \text{(Initial Error)} \\ e_i^n = 0, \quad i \in I_B & \text{(Boundary Error)} \end{cases} \tag{5.2}$$

where $|\varepsilon_{c,i}| < C_\delta(\Delta x + \Delta t)$ according to the consistency analysis. Hence we consider the following auxiliary function

$$w_i^n = e_i^n - (n\Delta t) \cdot C_\delta(\Delta x + \Delta t). \tag{5.3}$$

Observe that

$$\begin{aligned} \frac{w_i^{n+1} - w_i^n}{\Delta t} &- \mathcal{L}_{\delta,\Delta x}^{qnl} w_i^n \\ &= \frac{[e_i^{n+1} - C_\delta(\Delta x + \Delta t)((n+1)\Delta t)] - [e_i^n - C_\delta(\Delta x + \Delta t)(n\Delta t)]}{\Delta t} - \mathcal{L}_{\delta,\Delta x}^{qnl} e_i^n \\ &= \frac{e_i^{n+1} - e_i^n}{\Delta t} - C_\delta(\Delta x + \Delta t) - \mathcal{L}_{\delta,\Delta x}^{qnl} e_i^n \\ &= \varepsilon_{c,i} - C_\delta(\Delta x + \Delta t) \leq 0. \end{aligned} \tag{5.4}$$

Then w_i^n satisfies

$$\begin{cases} \frac{w_i^{n+1} - w_i^n}{\Delta t} - \mathcal{L}_{\delta,\Delta x}^{qnl} w_i^n \leq 0, & i \in I_\Omega, \\ w_i^0 = 0, \quad i \in I, & \text{(Initial)}, \\ w_i^n = -(n\Delta t) \cdot C_\delta(\Delta x + \Delta t), \quad i \in I_B & \text{(Boundary)}, \end{cases} \tag{5.5}$$

because of the discrete maximum principle proved in Theorem 4.2, so

$$w_i^n \leq \max\{w_i^0 | i \in I, \quad w_i^n|_{i\in I_B}\} = 0, \quad \forall i \in I_\Omega. \tag{5.6}$$

Therefore, $e_i^n \le (n\Delta t)\cdot C_\delta(\Delta x+\Delta t)$. Similarly when $w_i^n = e_i^n + (n\Delta t)\cdot C_\delta(\Delta x+\Delta t)$ we have $e_i^n \ge -(n\Delta t)\cdot C_\delta(\Delta x+\Delta t)$. Hence, $|e_i^n| \le (n\Delta t)\cdot C_\delta(\Delta x+\Delta t)$ which gives $|u(x_i,t^n)-u_i^n| \le T\cdot C_\delta(\Delta x+\Delta t)$. □

6. Study of the Courant–Friedrichs–Lewy (CFL) condition

In this section, we study the CFL condition of the new finite difference scheme by employing the Von Neumann stability analysis. We denote $\frac{\Delta t}{\Delta x}$ by λ_1 and $\frac{\Delta t}{(\Delta x)^2}$ by λ_2 and insert $u_i^n = (g(\theta))^n e^{\sqrt{-1}\theta x_i}$ into the scheme (2.3) where θ is a given wave number. We get the following three different cases:

- **Case I Nonlocal**: for $x_i \le 0$, the growth factor is

$$g(\theta) = 1 + \lambda_2 \sum_{j=1}^{r} \frac{2\big(\cos(\theta j\Delta x)-1\big)}{j^2} \int_{(j-1)\Delta x}^{j\Delta x} s^2\gamma_\delta(s)ds. \tag{6.1}$$

- **Case II Transitional**: for $0 < x_i \le \delta$, the growth factor is

$$\begin{aligned} g(\theta) =& 1 + \lambda_1 \sum_{j=\frac{x_i}{\Delta x}+1}^{r} \frac{\big(\cos(\theta(j-1)\Delta x)-1\big)}{(j-1)} \int_{(j-1)\Delta x}^{j\Delta x} s\gamma_\delta(s)ds \\ &- \lambda_1 \sum_{j=\frac{x_i}{\Delta x}+1}^{r} \frac{\sqrt{-1}\sin(\theta(j-1)\Delta x)}{(j-1)} \int_{(j-1)\Delta x}^{j\Delta x} s\gamma_\delta(s)ds \\ &+ \lambda_1 \left(\int_{x_i}^{\delta} s\gamma_\delta(s)ds\right)\Big(\cos(\theta\Delta x)+\sqrt{-1}\sin(\theta\Delta x)-1\Big) \\ &+ \lambda_2 \left(\int_0^{x_i} s^2\gamma_\delta(s)ds + x_i\int_{x_i}^{\delta} s\gamma_\delta(s)ds\right)\big(2\cos(\theta\Delta x)-2\big). \end{aligned} \tag{6.2}$$

- **Case III Local**: for $x_i > \delta$, the growth factor is

$$g(\theta) = 1 + \lambda_2\big(2\cos(\theta\Delta x)-2\big). \tag{6.3}$$

Proof. Performing Von Neumann analysis for stability we substitute $u_i^n = (g(\theta))^n e^{\sqrt{-1}\theta x_i}$

Case I:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} = \sum_{j=1}^{r} \frac{u_{i+j}^n - 2u_i^n + u_{i-j}^n}{(j\Delta x)^2} \int_{(j-1)\Delta x}^{j\Delta x} s^2\gamma_\delta(s)ds \tag{6.4}$$

Substituting $u_i^n = (g(\theta))^n e^{\sqrt{-1}\theta x_i}$ gives

$$g(\theta)^n e^{\sqrt{-1}\theta x_i}(g(\theta)-1) = \lambda_2 \sum_{j=1}^{r} \frac{g(\theta)^n e^{\sqrt{-1}\theta x_i}\big(e^{\sqrt{-1}\theta\Delta x} - 2 + e^{-\sqrt{-1}\theta\Delta x}\big)}{j^2} \int_{(j-1)\Delta x}^{j\Delta x} s^2\gamma_\delta(s)ds. \tag{6.5}$$

Therefore, we can conclude the growth factor for the nonlocal region is

$$g(\theta) = 1 + \lambda_2 \sum_{j=1}^{r} \frac{2\big(\cos(\theta j\Delta x)-1\big)}{j^2} \int_{(j-1)\Delta x}^{j\Delta x} s^2\gamma_\delta(s)ds. \tag{6.6}$$

Case II:

$$\begin{aligned} \frac{u_i^{n+1}-u_i^n}{\Delta t} =& \sum_{j=\frac{x_i}{\Delta x}+1}^{r} \frac{u_{i+j-1}^n - 2u_i^n + u_{i-j+1}^n}{2(j-1)\Delta x} \int_{(j-1)\Delta x}^{j\Delta x} s\gamma_\delta(s)ds \\ &- \sum_{j=\frac{x_i}{\Delta x}+1}^{r} \frac{u_{i+j-1}^n - u_{i-j+1}^n}{2(j-1)\Delta x} \int_{(j-1)\Delta x}^{j\Delta x} s\gamma_\delta(s)ds \\ &+ \left(\int_{x_i}^{\delta} s\gamma_\delta(s)ds\right)\frac{u_{i+1}^n - u_i^n}{\Delta x} \\ &+ \left(\int_0^{x_i} s^2\gamma_\delta(s)ds + x_i\int_{x_i}^{\delta} s\gamma_\delta(s)ds\right)\frac{u_{i+1}^n - 2u_i^n + u_{i-1}}{(\Delta x)^2}. \end{aligned} \tag{6.7}$$

Similarly to the nonlocal region substituting $u_i^n = (g(\theta))^n e^{\sqrt{-1}\theta x_i}$ gives

$$\begin{aligned} & g(\theta)^n e^{\sqrt{-1}\theta x_i}(g(\theta)-1) = \\ & \quad \lambda_1 \sum_{j=\frac{x_i}{\Delta x}+1}^{r} \frac{1}{2(j-1)} \left(g(\theta)^n e^{\sqrt{-1}\theta x_i} \left(e^{\sqrt{-1}\theta(j-1)\Delta x} - 2 + e^{-\sqrt{-1}\theta(j-1)\Delta x} \right) \right) \int_{(j-1)\Delta x}^{j\Delta x} s\gamma_\delta(s)ds \\ & \quad - \lambda_1 \sum_{j=\frac{x_i}{\Delta x}+1}^{r} \frac{1}{2(j-1)} \left(g(\theta)^n e^{\sqrt{-1}\theta x_i} \left(e^{\sqrt{-1}\theta(j-1)\Delta x} - e^{-\sqrt{-1}\theta(j-1)\Delta x} \right) \right) \int_{(j-1)\Delta x}^{j\Delta x} s\gamma_\delta(s)ds \\ & \quad + \lambda_1 \left(\int_{x_i}^{\delta} s\gamma_\delta(s)ds \right) \left(g(\theta)^n e^{\sqrt{-1}\theta x_i} \left(e^{\sqrt{-1}k\Delta x} - 1 \right) \right) \\ & \quad + \lambda_2 \left(\int_0^{x_i} s^2\gamma_\delta(s)ds + x_i \int_{x_i}^{\delta} s\gamma_\delta(s)ds \right) \left(g(\theta)^n e^{\sqrt{-1}\theta x_i} \left(e^{\sqrt{-1}\theta\Delta x} - 2 + e^{-\sqrt{-1}\theta\Delta x} \right) \right). \end{aligned} \tag{6.8}$$

Therefore, we can conclude the growth factor for the transitional region is

$$\begin{aligned} g(\theta) = & 1 + \lambda_1 \sum_{j=\frac{x_i}{\Delta x}+1}^{r} \frac{\left(\cos(\theta(j-1)\Delta x) - 1\right)}{(j-1)} \int_{(j-1)\Delta x}^{j\Delta x} s\gamma_\delta(s)ds \\ & - \lambda_1 \sum_{j=\frac{x_i}{\Delta x}+1}^{r} \frac{\sqrt{-1}\sin(\theta(j-1)\Delta x)}{(j-1)} \int_{(j-1)\Delta x}^{j\Delta x} s\gamma_\delta(s)ds \\ & + \lambda_1 \left(\int_{x_i}^{\delta} s\gamma_\delta(s)ds \right) \left(\cos(\theta\Delta x) + \sqrt{-1}\sin(k\Delta x) - 1 \right) \\ & + \lambda_2 \left(\int_0^{x_i} s^2\gamma_\delta(s)ds + x_i \int_{x_i}^{\delta} s\gamma_\delta(s)ds \right) \left(2\cos(\theta\Delta x) - 2 \right). \end{aligned} \tag{6.9}$$

Case III:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \tag{6.10}$$

Finally, substituting $u_i^n = (g(\theta))^n e^{\sqrt{-1}\theta x_i}$ gives

$$g(\theta)^n e^{\sqrt{-1}\theta x_i}(g(\theta)-1) = \lambda_2 \left(g(\theta)^n e^{\sqrt{-1}\theta x_i} \left(e^{\sqrt{-1}\theta\Delta x} - 2 + e^{-\sqrt{-1}k\Delta x} \right) \right). \tag{6.11}$$

Therefore, we can conclude the growth factor for the local region is

$$g(\theta) = 1 + \lambda_2 \left(2\cos(\theta\Delta x) - 2 \right). \tag{6.12}$$

Clearly, we have $\lambda_2 = \Delta x \lambda_1$, so once we get the CFL constraint on λ_1, the CFL condition for λ_2 will be satisfied when Δx is sufficiently small. Because it is very difficult to analytically find this upper bound we implement the growth factor $g(\theta)$ numerically to identify restrictions on λ_1 and λ_2 to ensure $|g(\theta)| \le 1$. □

For linear local diffusion models with the explicit Euler and middle point finite difference discretization, the CFL is restricted by CFL $= \frac{\Delta t}{\Delta x^2} \le 0.5$. This provides the largest step size in time to reduce computational cost while preserves stability. By numerically analyzing the growth factor in Fig. 3, we found that the nonlocal and local regions match the typical restrictions for stability, but the transitional region is slightly less than 0.5. This factor needs to be considered for stability restrictions to the CFL on the whole coupling system. On the other hand, compared with the original FDM scheme proposed in [20], the new FDM discretization can afford larger CFL condition, which suggests that the new scheme is more stable.

7. Numerical examples

In this section, we test several numerical examples to confirm the stability and convergence results.
We fix the nonlocal diffusion kernel to be a constant kernel

$$\gamma_\delta(s) = \frac{3}{\delta^3}\chi_{[-\delta,\,\delta]}(s).$$

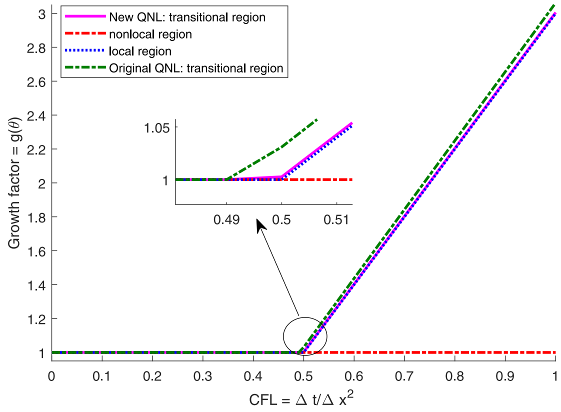
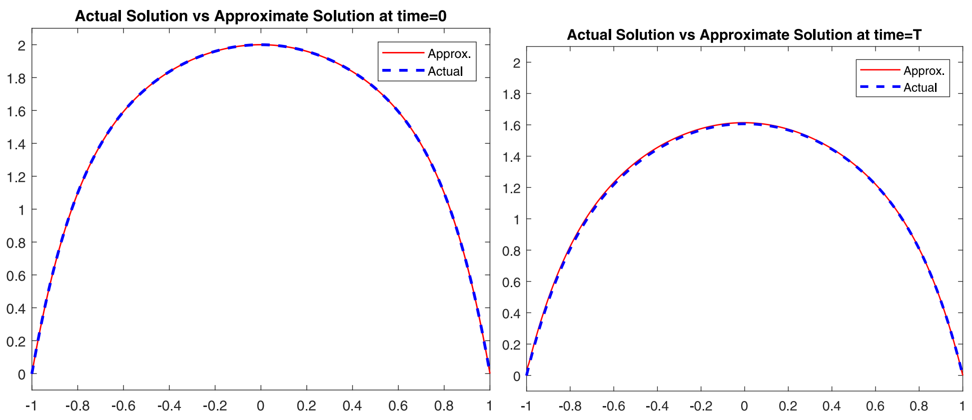


Fig. 3. Maximum Growth Rate of (6.1)–(6.3) for the new finite difference method versus that of (2.5) for the original finite difference method.

Table 1
$L^\infty_{\Omega\times[0,T]}$ differences between the local continuous solution u^ℓ and discrete solution $u^{qnl}_{\delta,\Delta x}$. We fix $\delta = 3\Delta x$, and the kernel is $\gamma_\delta(s) = \frac{3}{\delta^3}\chi_{[-\delta,\delta]}(s)$. The termination time $T = 1$ and $\Delta t = 0.2\Delta x$.

Δx	$\|\|u^\ell(x_i,t^n) - u^{qnl}_{\delta,\Delta x}(x_i,t^n)\|\|_{L^\infty_{\Omega\times[0,T]}}$	Order
$\frac{1}{50}$	0.1422	–
$\frac{1}{100}$	7.168e−2	0.988
$\frac{1}{200}$	3.614e−2	0.988
$\frac{1}{400}$	1.820e−2	0.990
$\frac{1}{800}$	9.151e−3	0.992
$\frac{1}{1600}$	4.594e−3	0.994

1. For the first example, we consider the asymptotic compatibility (AC) of the discretized operator $\mathcal{L}^{qnl}_{\delta,\Delta x}$ to the local diffusion problem as the horizon δ and spatial discretization Δx go to zero at the same time.
   We consider the external force f as
$$f(x,t) = 30x^4e^{-t} + e^{-t}(x^6 - 1) + 2. \tag{7.1}$$
   Then, the exact solution to the local diffusion $u^\ell{}_t = u^\ell_{xx} + f$ with $u^\ell(-1,t) = u^\ell(1,t) = 0$ and $u^\ell(x,0) = (1-x^2) - (x^6 - 1)$ is
$$u^\ell(x,t) = (1-x^2) - e^{-t}(x^6 - 1). \tag{7.2}$$
   To test the AC convergence, we fix $\delta = r\Delta$ with $r = 3$ and set the CFL to be $CFL = 0.45$, that is $\Delta t = 0.2\Delta x$, and the termination time is chosen to be $T = 1$.
   First order convergence with respect to Δx is observed. The convergence order and $L^\infty_{\Omega\times[0,T]}$ differences between $u^\ell(x,t)$ and discrete solution of $u^{qnl}_{\delta,\Delta x}$ are listed in Table 1. Also the visual comparison of the two solutions at $t = 0$ and $t = T$ are displayed in Fig. 4 with a nice agreement.
2. In the following example, we compare the original scheme $\tilde{\mathcal{L}}^{qnl}_\delta$ (2.5) proposed in [20] with the new proposed scheme $\mathcal{L}^{qnl}_{\delta,\Delta x}$ in (2.3).
   We are going to compare the AC convergence between (2.3) and (2.5). The exact local continuous solution is chosen to be
$$u^\ell(x,t) = e^{-t}(1-x)^2(1+x)^2x^2 \tag{7.3}$$
   and the corresponding external force is
$$\begin{aligned} f(x,t) =& u^\ell_t - u^\ell_{xx} \\ =& -e^{-t}\left((x-x^3)^2 + (2 - 24x^2 + 30x^4)\right). \end{aligned} \tag{7.4}$$
   Again the kernel used is $\gamma_\delta(s) = \frac{3}{\delta^3}$ with $\delta = 3\Delta x$. We denote the solution obtained by $\mathcal{L}^{qnl}_{\delta,\Delta x}$ by $u^{qnl}_{\delta,\Delta x}$ and the solution obtained by $\tilde{\mathcal{L}}^{qnl}_{\delta,\Delta x}$ by $\tilde{u}^{qnl}_{\delta,\Delta x}$.



(a) solutions at $t = 0$ (b) solutions at $t = 1$

Fig. 4. Plots of solutions to the approximate and actual solutions. The kernel function was chosen as $\gamma_\delta(s) = \frac{3}{\delta^3}\chi_{[-\delta,\delta]}(s)$. The coupling inference is at $x^* = 0$, and the mesh size is $\Delta x = \frac{1}{400}$ with a horizon as $\delta = \frac{3}{400}$, the temporal step size is $\Delta t = 0.45\Delta x$.

Table 2
$L^\infty_{\Omega\times[0,T]}$ differences between the local continuous solution u^ℓ and two discrete solutions $u^{qnl}_{\delta,\Delta x}$, $\tilde{u}^{qnl}_{\delta,\Delta x}$ using the FDM schemes (2.3) and (2.5), respectively. We fix $\delta = 3\Delta x$, and the kernel is $\gamma_\delta(s) = \frac{3}{\delta^3}$. The termination time is $T = 1$ and $\Delta t = 0.2\Delta x$.

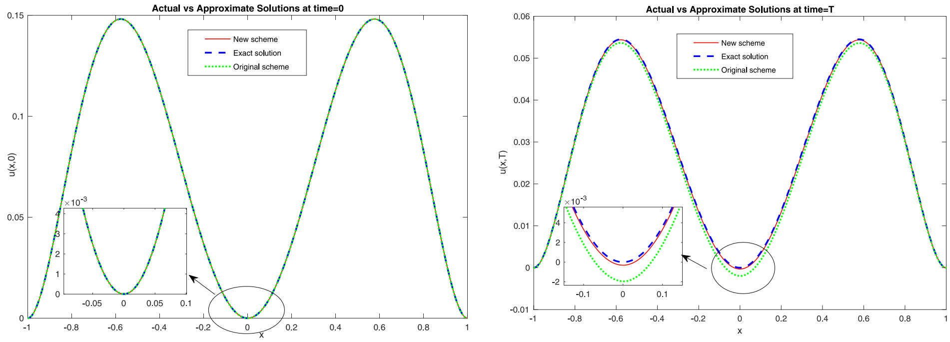
Δx	$\|\|u^\ell(x_i,t^n) - \tilde{u}^{qnl}_{\delta,\Delta x}(x_i,t^n)\|\|_{L^\infty}$	Order	$\|\|u^\ell(x_i,t^n) - u^{qnl}_{\delta,\Delta x}(x_i,t^n)\|\|_{L^\infty}$	Order
$\frac{1}{50}$	9.255e−3	–	7.200e−3	–
$\frac{1}{100}$	4.692e−3	0.980	1.698e−3	2.08
$\frac{1}{200}$	2.356e−3	0.994	4.121e−4	1.09
$\frac{1}{400}$	1.179e−3	0.998	1.931e−4	1.09
$\frac{1}{800}$	5.900e−4	0.999	9.628e−5	1.00
$\frac{1}{1600}$	2.951e−4	1.00	4.806e−5	1.00

First order AC convergence with respect to Δx are observed in Table 2 for both schemes (2.3) and (2.5), respectively. The approximation using scheme (2.3) at larger step size has second order convergence rate, and at smaller step size tends to be of first order.

Next, we compare the three solutions obtained from (1) new scheme; (2) exact local continuous solution and (3) the original scheme visually in Fig. 5. Notice that the exact local continuous solution $u^\ell(x,t)$ should remain non-negative throughout the entire computational domain $\Omega \times [0,T]$, however, both $u^{qnl}_{\delta,\Delta x}$ and $\tilde{u}^{qnl}_{\delta,\Delta x}$ become slightly negative around the interface $x^* = 0$. This does not contract the discrete maximum principle of $\mathcal{L}^{qnl}_{\delta,\Delta x}$ as the external force $f(x,t)$ defined in (7.4) does not retain negative on $[-1,1]$ as required in the assumption of Theorem 4.2. On the other hand, because $\mathcal{L}^{qnl}_{\delta,\Delta x}$ satisfies the discrete maximum principle, consequently, $u^{qnl}_{\delta,\Delta x}$ provides less artificial negativity than $\tilde{u}^{qnl}_{\delta,\Delta x}$ around the interface of coupling.

8. Conclusion

We propose a new scheme to discretize the quasi-nonlocal (QNL) coupling operator introduced in [20] for the nonlocal-to-local diffusion problem. This new finite difference approximation preserves the properties of continuous equation on a discrete level. Consistency, stability, the maximum principle and the global convergence analysis of the scheme are proved rigorously. We analytically find the CFL conditions through the Von Neumann stability analysis and numerically calculate the CFL values for a given spatial discretization. The numerical calculations of the CFL provide us addition alert around the interface when considering the temporal step size for an explicit time integrator, as the CFL restrictions on the transitional region was discovered to be slightly less than $\frac{1}{2}$ with explicit Euler method employed in a diffusion problem. Multiple numerical examples are then provided and summarized to verify the theoretical findings. A comparison with the original scheme used in [20] is also provided which confirmed the improvements of the new scheme.



(a) solutions at $t = 0$ (b) solutions at $t = 1$

Fig. 5. Numerical comparison between the new scheme (2.3) and original scheme (2.5) used to approximate (7.3) with external force given by (7.4). The spatial step size is $\Delta x = \frac{1}{200}$ and $\Delta t = 0.25\Delta x$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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