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Spherical Spin Glass Model with External Field

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Abstract

We analyze the free energy and the overlaps in the 2-spin spherical Sherrington Kirkpatrick spin glass model with an external field for the purpose of understanding the transition between this model and the one without an external field. We compute the limiting values and fluctuations of the free energy as well as three types of overlaps in the setting where the strength of the external field goes to zero as the dimension of the spin variable grows. In particular, we consider overlaps with the external field, the ground state, and a replica. Our methods involve a contour integral representation of the partition function along with random matrix techniques. We also provide computations for the matching between different scaling regimes. Finally, we discuss the implications of our results for susceptibility and for the geometry of the Gibbs measure. Some of the findings of this paper are confirmed rigorously by Landon and Sosoe in their recent paper which came out independently and simultaneously.

Keywords Spin glass · Spherical Sherrington–Kirkpatrick · SSK · External field · Overlap · Susceptibility · Geometry of Gibbs measure

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1 Introduction

1.1 The Model, Definitions, and Notation

Spin glasses are disordered magnetic alloys [8,45] that provide a physical context for the development of various mathematical models with wide ranging applications, not only in physics, but also in computer science and other areas [32]. One of the most studied spin glass models is the Sherrington–Kirkpatrick (SK) model [21,35,37,40], in which the spin variable $\boldsymbol{\sigma}$ is a random vector from the N-dimensional hypercube $\{-1, +1\}^N$. In this paper, we focus on the continuous analog of SK—the spherical Sherrington–Kirkpatrick (SSK) model. The SSK model shares many properties with the SK model but is usually easier to analyze and thus allows us to obtain results that remain out of reach for the SK model.

Particular quantities of interest in the study of the SSK model are the free energy and overlaps in the presence of an external field. In the absence of a field, the model exhibits, at large N, a transition to a spin glass phase at low temperature. In the presence of a field, this phase transition disappears. However, there are interesting regimes when the field is scaled as a power of the dimension N. Those transitional regimes (with respect to the external field) will be the focus of this paper. We compute the free energy as well as three types of overlaps, up to fluctuations, when h, the strength of the external field, converges to zero as the dimension, N, of the system grows.

For the SSK model, the spin variable $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$ is in S_{N-1}, the sphere of radius $\sqrt{N}$ in $\mathbb{R}^N$:

$$S_{N-1} = \{\boldsymbol{\sigma} \in \mathbb{R}^N : \|\boldsymbol{\sigma}\| = \sqrt{N}\}.$$

The 2-spin spherical Sherrington–Kirkpatrick (SSK) model with external field is defined by the Hamiltonian

$$\mathcal{H}(\boldsymbol{\sigma}) = -\frac{1}{2}\sum_{i,j=1}^{N} M_{ij}\, \text{œ}_i\, \text{œ}_j - h\sum_{i=1}^{N} g_i\, \text{œ}_i = -\frac{1}{2}\boldsymbol{\sigma}\cdot M\boldsymbol{\sigma} - h\,\mathbf{g}\cdot\boldsymbol{\sigma} \tag{1.1}$$

for $\boldsymbol{\sigma} \in S_{N-1}$, where M and $\mathbf{g}$ are respectively a random matrix and a random vector, specified below. The associated Gibbs measure is

$$p(\boldsymbol{\sigma}) = \frac{1}{\mathcal{Z}_N} e^{-\beta\mathcal{H}(\boldsymbol{\sigma})} \quad \text{for } \boldsymbol{\sigma} \in S_{N-1} \tag{1.2}$$

where

$$\beta = 1/T \tag{1.3}$$

denotes the inverse temperature. The partition function and the free energy per spin component are

$$\mathcal{Z}_N = \int_{S_{N-1}} e^{-\beta\mathcal{H}(\boldsymbol{\sigma})}\mathrm{d}\omega_N(\boldsymbol{\sigma}) \quad \text{and} \quad \mathcal{F}_N = \mathcal{F}_N(T,h) = \frac{1}{N\beta}\log \mathcal{Z}_N, \tag{1.4}$$

where ω_N is the normalized uniform measure on S_{N-1}. Since the disorder variables M and $\mathbf{g}$ are random, the Gibbs measure is a random measure, which we also call a thermal measure, and the free energy $\mathcal{F}_N$ is a random variable. We are interested in the fluctuations of the free energy when $h \to 0$ as $N \to \infty$.

We also consider the behavior of the spin variables taken from the Gibbs measure. We focus on the following three particular overlaps.

- (overlap with the external field) Define

$$\mathfrak{M} = \frac{\mathbf{g} \cdot \boldsymbol{\sigma}}{N}. \tag{1.5}$$

- (overlap with the ground state) Let $\mathbf{u}_1$ be a unit eigenvector corresponding to the largest eigenvalue of the disorder matrix M. The vectors $\pm\mathbf{u}_1$ are the ground state in the absence of an external field, and we simply call them the ground states. Define

$$\mathfrak{G} = \frac{|\mathbf{u}_1 \cdot \boldsymbol{\sigma}|}{\sqrt{N}} \quad \text{and} \quad \mathfrak{O} = \mathfrak{G}^2 \tag{1.6}$$

- (overlap with a replica) Let $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$ be two independent spin variables from the Gibbs measure for the same sample (i.e. disorder variables M_{ij} and g_i); $\boldsymbol{\sigma}^{(2)}$ is a replica of $\boldsymbol{\sigma}^{(1)}$. Define

$$\mathfrak{R} = \frac{\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}}{N}. \tag{1.7}$$

The factors N and $\sqrt{N}$ are included since $||\mathbf{u}_1|| = 1$, $\|\boldsymbol{\sigma}\| = \sqrt{N}$, and the expected value of $\|\mathbf{g}\|^2 = g_1^2 + \cdots + g_N^2$ is N (see below).

The overlaps depend on the spin variable and also the disorder sample. Hence, there are two different expectations to consider. We consider the thermal (Gibbs) fluctuations of the overlaps for a given disorder sample. For some quantities, we also consider the sample-to-sample fluctuations of the thermal average. We denote the thermal (Gibbs) average for a given disorder sample by the bracket $\langle \cdot \rangle$. On the other hand, the sample-to-sample average of an observable O is denoted by $\bar{O}$ or $\mathbb{E}_s[O]$. For example, the thermal averages

$$\mathcal{M} = \langle \mathfrak{M} \rangle \quad \text{and} \quad \mathcal{X} = \frac{1}{h} \langle \mathfrak{M} \rangle$$

are called magnetization and susceptibility, respectively. Many of the results of this paper are about the thermal fluctuations of overlaps for a given disorder sample, i.e. for a given quenched disorder.

1.2 Assumptions on Disorder Samples

The disorder parameters in the Hamiltonian (1.1) are chosen as follows. We define

$$M = (M_{ij})_{1 \le i,j \le N}$$

to be a disorder matrix given by a random symmetric matrix from the Gaussian orthogonal ensemble (GOE), which is a matrix ensemble whose probability is rotationally invariant. For $i \le j$, the variables M_{ij} are independent centered Gaussian random variables with variance $\frac{1}{N}(1 + \delta_{ij})$. By the symmetry matrix condition, $M_{ij} = M_{ji}$ for $i > j$. We denote by

$$\lambda_1 \ge \cdots \ge \lambda_N \quad \text{and} \quad \mathbf{u}_1, \ldots, \mathbf{u}_N \tag{1.8}$$

the eigenvalues of the disorder matrix M and corresponding unit eigenvectors. The GOE assumption implies that the eigenvalues and eigenvectors are independent of each other. The external field is given by the vector

$$\mathbf{g} = (g_1, g_2, \ldots, g_N)^T, \tag{1.9}$$

which we assume to be a standard Gaussian vector. The strength of the external field is denoted by a non-negative scalar h. We also define

$$n_i = \mathbf{u}_i \cdot \mathbf{g}, \tag{1.10}$$

the overlap of the eigenvector and the external field. The external field and eigenvectors appear in the results and analysis of this paper only as this combination. The variables λ_i and n_i are collectively called disorder variables. We call the joint realization of λ_i and n_i a disorder sample throughout the paper.

Note that $(n_1, \dots, n_N)$ is a standard Gaussian vector, whose entries are independent of the eigenvalues $\lambda_1, \dots, \lambda_N$. The analysis of this paper also applies, after some changes of formulas, to the case when $\mathbf{g} = (1, \dots, 1)^T$. However, we restrict to the Gaussian external field since the Gaussian assumption makes calculations simpler.

1.3 Summary of Prior Research

The purpose of this paper is to study the case $h \to 0$ systematically including up to the fluctuation term for the free energy and the three overlaps. Here we provide a survey of some of the existing research as it connects to our study.

The free energy for the Hamiltonian (1.1) above when $h = 0$ converges to a deterministic value which was computed by Kosterlitz et al. [24]. The Hamiltonian (1.1) is the 2-spin case of the more general p-spherical spin glass model which includes interactions between multiple spin coordinates. The limit of the free energy for the general spherical spin glass models which also includes the external field is given by the Crisanti-Sommers formula [13]. This formula is the spherical version of the Parisi formula [36] for the spins in hypercubes. The Parisi formula and Crisanti-Sommers formula are proved rigorously by Talagrand in [41,42]. The result of Kosterlitz, Thouless and Jones shows that when $h = 0$, there are two phases: the spin glass phase when $T < 1$ and the paramagnetic phase when $T > 1$. On the other hand, they argued that when $h > 0$, assuming that the external field is uniform, there is no phase transition.

The subleading (in N) term of the free energy depends on the disorder and hence it describes the fluctuations of the free energy. For $h = 0$ and $T > 1$, the fluctuation term is of order N^{-1} and has the Gaussian distribution. This is proved for both the hypercube case [2,12,18] and the spherical case [4]. For $h = 0$ and $T < 1$, for the Hamiltonian above, the fluctuation term is of order $N^{-2/3}$ and has the GOE Tracy-Widom distribution [4]. Chen et al. performed a similar calculation for the case with Ising spins where $h > 0$ is of order 1 and g is the vector of all 1s. In this case, they find [9] that the fluctuation term is of order $N^{-1/2}$ and has the Gaussian distribution for all temperature. They claim that similar results hold for the spherical case and our results confirm this claim using a different method. We note that their result also holds for mixed p-spin with even degree terms. Chen and Sen [10] computed the ground state energy for spherical mixed p-spin models (of which SSK is a specific case) and found that the fluctuations of the ground state energy are Gaussian in the presence of an external field.

In [19], the large deviations of the free energy distribution was obtained at $T = 0$ from a non-rigorous saddle point calculation of the moments of $\mathcal{Z}_N$ in the large N limit (see also [15] for a rigorous version). From this calculation a transitional regime $h \sim N^{-1/6}$ for the fluctuations of the free energy was conjectured. A proof of the existence of this regime was obtained in [23]. In the current paper, we obtain explicitly the fluctuations of the free energy in the regime $h \sim N^{-1/6}$ for any $T < 1$ and in the regime $h \sim N^{-1/4}$ for $T > 1$. As we show, our results match in the tail of the distribution with those of [19]. Note also the recent physics work [20] where a different spherical model of random optimization was considered, which exhibits a similar phenomenology.

The overlap with the external field has been studied in the context of magnetism and susceptibility. Kosterlitz et al. [24] computed the susceptibility as h tends to zero and observed a transition at the temperature $T = 1$. Cugliandolo et al. [14] computed two different versions of this limit of the susceptibility, in the first case taking $\lim_{h\to 0}\lim_{N\to\infty}$ and in the second case taking $\lim_{N\to\infty}\lim_{h\to 0}$. In the first of these cases, they get the same result as [24] with a transition at $T = 1$, but in the second case they do not observe a transition. Furthermore, they find that the two types of limits agree for $T > 1$ but not for $T < 1$. They also extend that results to a more general class of models (beyond Gaussian) and to non-linear susceptibility. We focus on the linear susceptibility and differential susceptibility in the Gaussian case, and obtain a more detailed picture. By considering the three regimes $h = O(1)$, $h \sim N^{-1/6}$, and $h \sim N^{-1/2}$, we see that the first limit considered by Cugliandolo et al agrees with our result for the $h \to 0$ limit of the $h = O(1)$ case. The second limit that they consider is analogous to our result for the $H \to 0$ limit of the $h \sim N^{-1/2}$ case where we define $H = hN^{1/2}$. However, we find in this case that the susceptibility depends on the sample and is a function of $\mathbf{g}\cdot\mathbf{u}_1$, the inner product of the external field and the ground state. This dependence was not apparent in [14], since their set-up fixes $\mathbf{g}\cdot\mathbf{u}_1 = 1$. When $\mathbf{g}\cdot\mathbf{u}_1 = 1$ we find, as they do, that there is no transition in the susceptibility between high and low temperature. However, a transition does exist for all other values of $\mathbf{g}\cdot\mathbf{u}_1$.

The overlap with the ground state is relevant to understanding the geometry of the Gibbs measure. Subag [39] examines the geometry of the Gibbs measure for general p-spin spherical models and finds that the Gibbs measure concentrates in spherical bands around the critical points of the Hamiltonian. These bands are of the form $\mathrm{Band}(\boldsymbol{\sigma}_0, q, q') = \{\boldsymbol{\sigma} \in S_{N-1} : q \le R(\boldsymbol{\sigma}, \boldsymbol{\sigma}_o) \le q'\}$ where $\boldsymbol{\sigma}_0$ is a critical point of $\mathcal{H}$ and $R(\boldsymbol{\sigma}, \boldsymbol{\sigma}_0)$ is the overlap of $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_0$. We focus specifically on the overlap with the ground state (where $\boldsymbol{\sigma}_0$ is the critical point corresponding to the largest eigenvalue). In the $h = 0$ regime, as expected, we see the Gibbs measure concentrates in a band and we examine how this geometry changes for the case of positive constant h as well as the cases of $h \sim N^{-1/6}$ and $h \sim N^{-1/3}$.

The overlap with a replica has been studied extensively, both for the Ising spin models and the spherical spin models with general p-spin interaction. For $p = 2$ the non-rigorous replica method used in [13,19,24] obtains a replica symmetric saddle point leading to a prediction for the overlap q as a function of h. In particular, at $h = 0$, the prediction is that $q = 1 - T$ for $T < 1$ and $q = 0$ for $T > 1$. These calculations were confirmed rigorously in [34]. Recently, Landon et al. extended the results further to examine the fluctuations of the overlap at high temperature [33] and low temperature [25]. They find, in particular, that the overlap has Gaussian fluctuations in the high temperature regime, whereas, in the low temperature regime, the fluctuations are of order $N^{-1/3}$ and converge to a random variable that has an explicit formula in terms of the GOE Airy point process (see Sect. 4.4 for a description of this). In this paper, we obtain similar results for $h \sim N^{-1/6}$ and $h \sim N^{-1/2}$.

1.4 Method of Analysis

Our computations are based on contour integral representations which we present in Sect. 3. The free energy and the moment generating functions of two of the overlaps can be expressed in terms of a single integral, whereas the moment generating function in the case of the overlap with a replica can be written as a double integral. The integrand for each of these integral representations contains disorder variables and hence we have random integral formulas. The single integral formula for the free energy was first observed by Kosterlitz et al. [24] and the authors use the method of the steepest descent to evaluate the limiting free energy. For the case of $h = 0$, this calculation was extended in [4] to find the fluctuation terms using the recent advancements in random matrix theory, in particular the rigidity results on the eigenvalues [16] and the linear statistics [3,22,30]. Similar ideas were also used in [5–7], including the case for the overlap with a replica in [25]. This paper extends the integral formula approach to the case when $h = O(1)$ and $h \to 0$ in the transitional regimes. When there is an external field, the analysis becomes more involved. In this case, the dot products of the eigenvectors and the external field play an important role in the analysis.

The steepest descent analysis of this paper can be made mathematically rigorous after some efforts using probability theory and random matrix theory. However, this paper will focus on computations and interpretations assuming that various estimates in the steepest descent analysis can be obtained. We use the label "Result" for findings in which we do not provide rigorous proofs and the label "Theorem" for findings that we cite from prior papers that include rigorous proof. We use the label "Lemma" for short findings that we prove in full detail.

In a recent preprint [26] which was obtained independently and simultaneously with this paper, Landon and Sosoe consider a similar SSK model in which the external field is a fixed vector and the disorder matrix has zero diagonal entries. Their work is mathematically rigorous and contains proofs of some of the results obtained in this paper, namely for the free energy and some aspects of the overlaps with the external field and with a replica in Sects. 5.2, 6.1, 7.1, 8.3, 8.5, 10.1, and 10.2. After the completion of this paper, one of us, Collins-Woodfin, also proved the results in Sect. 8.6 on the overlap with the microscopic external field rigorously in [11].

1.5 Organization of the Paper

The results of the calculations are scattered throughout the paper. In Sect. 2, we present some of the highlights of the results of this paper. The single and double integral representation of the free energy and the generating functions of the overlaps are given in Sect. 3. The next three sections of the paper address the free energy. Section 5 summarizes known results for the $h = 0$ case and explains our findings for the $h > 0$ case. Section 6 addresses the $h \to 0$ case for $T > 1$ and Sect. 7 addresses $h \to 0$ for $T < 1$. Sections 8, 9, and 10 provide our results for each of the three types of overlaps. Section 8 also provides our results for magnetization and susceptibility. Section 11 describes the geometry of the spin vector configuration under the Gibbs measure. We include as appendices the proof of the contour integral formulas and also a perturbation lemma.

2 Highlights of the Results

2.1 Results for the Free Energy

We examine the behavior of the free energy, including its leading order and the sample-to sample fluctuation term, as $N \to \infty$ when $h = O(1)$ and when $h \to 0$. We find that, in each case,

$$\mathcal{F}_N(T, h) \overset{\mathcal{D}}{\simeq} F(T, h) + \text{ sample fluctuations} \tag{2.1}$$

Table 1 This table summarizes our findings for the leading term and fluctuations of $\mathcal{F}_N(T,h)$ in the various cases we considered

Case	Limiting free energy $F(T,h)$	Sample fluctuations	Result
$h=0,\ T>1$	$\frac{1}{4T}$	N^{-1} Gaussian distribution	5.1
$h=0,\ T<1$	$1-\frac{3T}{4}+\frac{T\log T}{2}$	$N^{-\frac{2}{3}}$ TW_{GOE} distribution	5.2
$h=O(1)$	$\frac{\gamma_0}{2}-\frac{Ts_0(\gamma_0)}{2}-\frac{T-T\log T}{2}+\frac{h^2s_1(\gamma_0)}{2}$	$N^{-\frac{1}{2}}$ Gaussian distribution	5.5
$h\sim N^{-\frac{1}{4}},\ T>1$	$\frac{1}{4T}+\frac{h^2}{2T}$	N^{-1} Gaussian distribution	6.2
$h\sim N^{-\frac{1}{6}},\ T<1$	$1-\frac{3T}{4}+\frac{T\log T}{2}+\frac{h^2}{2}$	$N^{-\frac{2}{3}}$ function of the GOE Airy point process and Gaussian r.v.'s	7.2

The $h=0$ cases were already known [4] but are included here for completeness. In the limiting free energy for the $h=O(1)$ case, the quantity γ_0 is deterministic and depends only on T and h. The functions s_0 and s_1 are defined in Sect. 4. For more details on the notation, derivation, and precise formulas for the fluctuation terms, see the corresponding result

where $\overset{\mathcal{D}}{\simeq}$ denotes an asymptotic expansion in distribution with respect to the disorder variables. The limiting free energy $F(T,h)$ includes all deterministic (depending only on h and T) terms whose order exceeds that of the sample fluctuations. The "sample fluctuations" refers to the largest order term that depends on the disorder sample. Our findings in each case are summarized in Table 1. Upon computing the leading term and sample fluctuations for $\mathcal{F}_N(T,h)$ with $h=O(1)$, we made two key observations. Firstly, the free energy for $h=O(1)$ does not exhibit a transition as we see in the $h=0$ case; this observation is consistent with the result of [9] for Ising spins. Secondly, while the limiting free energy is continuous in T and h, the sample fluctuations in the $h=O(1)$ case do not agree with those in the $h=0$ case (neither for $T>1$ nor for $T<1$). This suggests the existence of transitional regimes. We found that, for $T>1$, the transition occurs at $h\sim N^{-1/4}$ while, for $T<1$, the transition occurs at $h\sim N^{-1/6}$. We computed the asymptotic expansion of $\mathcal{F}_N(T,h)$ in these transitional regimes.

When comparing the fluctuations in each regime, we observe that the order of the fluctuations are largest in the $h=O(1)$ case, where they have order $N^{-1/2}$ and Gaussian distribution. This holds for all temperatures. When $T>1$ but $h=0$ or $h\to 0$, the fluctuations remain Gaussian, but their order shrinks to N^{-1}. When $T<1$ and $h=0$ or $h\to 0$, the fluctuations have order $N^{-2/3}$. In the case of $h=0$ they have GOE Tracy-Widom distribution while, in the case of $h\sim N^{-1/6}$, their distribution is a function of the GOE Airy point process and of a sequence of i.i.d. standard Gaussian random variables. See Table 1 for the equations corresponding to each of these results.

2.2 Results for the Overlaps

In the next three (Tables 2, 3, and 4) we state our findings for the overlap with the external field, with the ground state and with a replica. In each case the thermal average and thermal fluctuations are presented in interesting regimes of h and T. The thermal average and fluctuations in most cases depend on the disorder sample. Our findings also have implications for magnetization and susceptibility, which will be described in more detail in Sect. 8.

Table 2 This table summarizes our finding for $\mathfrak{M}$, the overlap with the external field

Case	Thermal average $\langle\mathfrak{M}\rangle$	Thermal fluctuations of $\mathfrak{M}$	Result
$h = O(1)$ for all T	$hs_1(\gamma_0) + O(N^{-\frac{1}{2}})$	$N^{-\frac{1}{2}}$ Gaussian	8.2
(and $h = 0,\ T > 1$)			8.3
$h \sim N^{-\frac{1}{6}},\ T < 1$	$h + O(N^{-\frac{1}{2}})$	$N^{-\frac{1}{2}}$ Gaussian	8.5
$h \sim N^{-\frac{1}{2}},\ T < 1$	$h + \frac{\lvert n_1\rvert\sqrt{1-T}}{\sqrt{N}} \tanh\left(\frac{\lvert n_1\rvert h\sqrt{N(1-T)}}{T}\right)$	$N^{-\frac{1}{2}}$ [Gaussian + Bernoulli]	8.7
(and $h = 0,\ T < 1$)			8.3

Here, $\gamma_0 = \gamma_0(h, T)$ in the first row is deterministic. The variable n_1 in the third row is $n_1 = \mathbf{u}_1 \cdot \mathbf{g}$. For the top two rows, the leading term in $\langle\mathfrak{M}\rangle$ and the thermal fluctuations of $\mathfrak{M}$ do not depend on the disorder sample. However, the $O(N^{-\frac{1}{2}})$ subleading terms in $\langle\mathfrak{M}\rangle$ for the top two cases and both the leading term in $\langle\mathfrak{M}\rangle$ and the thermal fluctuations of $\mathfrak{M}$ of the last row do depend on the disorder sample

Table 3 This table summarizes our finding for $\mathfrak{G}^2 = \mathfrak{Q}$, the squared overlap with the ground state

Case	Thermal average $\langle\mathfrak{G}^2\rangle$	Thermal fluctuations of $\mathfrak{G}^2$	Result
$h = O(1)$ for all T	$\frac{1}{N}\left(\frac{h^2 n_1^2}{(\gamma_0-2)^2} + \frac{T}{\gamma_0-2}\right)$	N^{-1} χ-squared (non-centered)	9.2
(and $h = 0,\ T > 1$)			9.7
$h \sim N^{-\frac{1}{6}},\ T < 1$	$1 - T - \sum_{i=2}^{N} \frac{n_i^2 h^2 N^{1/3}}{(t+a_1-a_i)^2}$	$N^{-\frac{1}{6}}$ Gaussian	9.4
$h \sim N^{-\frac{1}{3}},\ T < 1$	$1 - T + O(N^{-\frac{1}{3}})$	$N^{-\frac{1}{3}}$ r.v. that depends on disorder	9.6
(and $h = 0,\ T < 1$)			9.7

Here $n_i = \mathbf{u}_i \cdot \mathbf{g}$ and $a_i = N^{2/3}(\lambda_i - 2)$. The quantity γ_0 in the top row is the same term from Table 2. In the second row, the variable t and the total sum, which is $O(1)$, depends on the disorder sample. All leading and subleadings terms of $\langle\mathfrak{G}^2\rangle$, and the thermal fluctuations of $\mathfrak{G}^2$, except the leading term, $1 - T$, of $\langle\mathfrak{G}^2\rangle$ in the last row, depend on the disorder sample

Table 4 This table summarizes our finding for $\mathfrak{R}$, the overlap between two independent spins

Case	Thermal average $\langle\mathfrak{R}\rangle$	Thermal fluctuations of $\mathfrak{R}$	Result
$h = O(1)$ for all T	$h^2 s_2(\gamma_0) + O(N^{-\frac{1}{2}})$	$N^{-\frac{1}{2}}$ Gaussian	10.3
(and $h = 0,\ T > 1$)			10.8
$h \sim N^{-\frac{1}{6}},\ T < 1$	$1 - T + O(N^{-\frac{1}{3}})$	$N^{-\frac{1}{3}}$ r.v. that depends on disorder	10.5
$h \sim N^{-\frac{1}{2}},\ T < 1$	$(1 - T)\tanh^2\left(\frac{\lvert n_1\rvert h\sqrt{N(1-T)}}{T}\right)$	$O(1)$ Bernoulli	10.7
(and $h = 0,\ T < 1$)			10.8

The quantity γ_0 is the same term from Table 2 and $n_1 = \mathbf{u}_1 \cdot \mathbf{g}$. The subleading terms of $\langle\mathfrak{R}\rangle$ in the top two rows and the leading term of $\langle\mathfrak{R}\rangle$ in the third row depend on the disorder sample. The thermal fluctuations of $\mathfrak{R}$ also depend on the disorder sample for the bottom two rows

2.3 Geometry of the Gibbs Measure

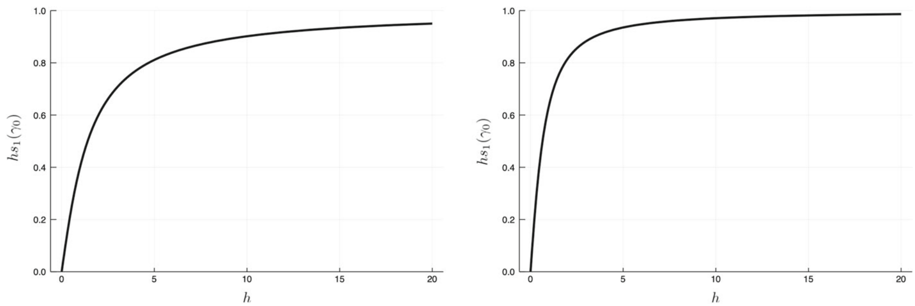
The results for the overlaps give us information on the geometry of the spin configuration under the Gibbs measure, some of which we summarize here. Recall that the spin configuration is parameterized by the vector $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$ which belongs to the $N-1$ dimensional sphere of radius $\sqrt{N}$ and we consider the limit of large N. At high temperature, $T > 1$, the spin vector $\boldsymbol{\sigma}$ is nearly orthogonal to the ground state $\pm\mathbf{u}_1$ when $h = 0$. For $h = O(1)$, the spin vector concentrates on the intersection of the sphere and the single cone around the vector $\mathbf{g}$. The leading term of the cosine of the angle between the spin and the external field $\mathbf{g}$ depends on the temperature and the field but not on the disorder sample, and, as one can expect, is an increasing function of the field. See Fig. 1a. This implies that as the field becomes stronger, the cone becomes narrower. There are no transitions between $h = 0$ and $h = O(1)$.

Now consider the low temperature regime $0 < T < 1$. When $h = 0$, the spins are concentrated on the intersection of the sphere with the double cone around the ground state $\pm\mathbf{u}_1$ such that the leading term of the cosine of the angle is $\sqrt{1-T}$. This angle was found in [13,19,24] and in particular, [34] showed that spins are distributed uniformly on the intersection of this double cone with the sphere. Consider increasing the external field strength h. When $h = O(1)$, the spin vector concentrates on the intersection of the sphere and the single cone around the vector $\mathbf{g}$ just like the high temperature case. See Fig. 1b, which is qualitatively same as Figure (a). However now between $h = 0$ and $h = O(1)$, there are two interesting transitional regimes, $h \sim N^{-1/2}$, which we call the microscopic regime, and $h \sim N^{-1/6}$, the mesoscopic regime.

In the microscopic regime, $h \sim N^{-1/2}$, at low temperature $0 < T < 1$, the results of this paper lead us to the Conjecture 11.1, which implies that the double cone becomes polarized into a single cone. The spin vector prefers the cone which is closer to $\mathbf{g}$ to the other cone by the

$$e^{\frac{2h\sqrt{N}|n_1|\sqrt{1-T}}{T}} \text{ to 1 probability ratio.}$$

The spin vector is more or less uniformly distributed on the cones. In this regime, the response of the spin to the field is the sum of (i) a linear response in the direction transverse to $\pm\mathbf{u}_1$ (i.e. along the cones) and, (ii) the response of an effective 2-level system, which may be modeled as a single one-component effective Ising spin $\frac{\boldsymbol{\sigma}}{\sqrt{N}} = \pm S\mathbf{u}_1$ of size $S = \frac{|n_1|\sqrt{1-T}}{\sqrt{N}}$ with energy



(a) $T = 2$ **(b)** $T = 0.5$

Fig. 1 These are plots of the leading term of the angle between the spin and $\mathbf{g}$. The formula is given by $\mathfrak{M}^0$ in Sect. 8.3.2. The function depends only on T and h. (a) $T = 2$, (b) $T = 0.5$

Table 5 This table summarized the findings of the decomposition of the spin variable $\hat{\boldsymbol{\sigma}} \simeq a\mathbf{u}_1 + b\hat{\mathbf{g}} + \mathbf{v}$ in different regimes for $0 < T < 1$

Case	$a = \hat{\boldsymbol{\sigma}} \cdot \mathbf{u}_1$	$b \simeq \hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{g}}$	$\|\mathbf{v}\|$	$\hat{\mathbf{v}}^{(1)} \cdot \hat{\mathbf{v}}^{(2)}$
$h \to \infty$	0	1	0	0
$h = O(1)$	$\frac{\sqrt{\mathfrak{D}^0}}{\sqrt{N}}$	$hs_1(\gamma_0)$	$\sqrt{1 - h^2 s_1(\gamma_0)^2}$	$\frac{h^2 s_1(\gamma_0)^4}{(1-s_1(\gamma_0)^2)(1-h^2 s_1(\gamma_0)^2)}$
$h \to 0, hN^{\frac{1}{6}} \to \infty$	$\frac{4(1-T)^2 \|n_1\|}{h^3\sqrt{N}}$	h	1	$1 - T$
$h \sim N^{-\frac{1}{6}}$	$\mathcal{A}(T, hN^{1/6})$	h	$\sqrt{1 - \mathcal{A}^2}$	$\frac{1-T-\mathcal{A}^2}{1-\mathcal{A}^2}$
$hN^{\frac{1}{6}} \to 0, hN^{\frac{1}{2}} \to \infty$	$\sqrt{1 - T}$	h	$\sqrt{T}$	$o(1)$
$h \sim N^{-\frac{1}{2}}$ (and $h = 0$)	$\sqrt{1 - T}\, \mathfrak{B}(\alpha)$	$h + \frac{\sqrt{T}\mathfrak{N}}{\sqrt{N}}$	$\sqrt{T}$	$o(1)$

We indicate the leading order terms, except that we have $o(1)$ at two places. The $o(1)$ term in the fifth row is complicated to state and the $o(1)$ term in the last row is not determined from our analysis. The unit transversal vector is $\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$

scale $E = NhS = \sqrt{N}h|n_1|\sqrt{1-T}$ (leading to a mean magnetization $S\tanh(E/T)$). Note that both S and E are sample dependent, but depend only on $|n_1|$, the overlap of the ground state and the field.

For $h \sim N^{-1/6}$, progressively all eigenvectors and eigenvalues become important. In this regime, the spins are concentrated on the intersection of the sphere and a single cone around the ground state, but the cone depends on the disorder sample. The cosine of the angle between the spin and $\mathbf{u}_1$ changes from $\sqrt{1-T}$ to a function which depends on all eigenvalues λ_i and the overlaps $n_i = \mathbf{u}_i \cdot \mathbf{g}$ of the eigenvectors and the external field. Furthermore, the spins are no longer uniformly distributed on the cone. They are pulled into the direction of $\mathbf{g}$. This regime can be called "mesoscopic" as sample to sample fluctuations are strong and non trivial. Note that in the present model the magnetic response to the field, although non-trivial and sample dependent, does not exhibit jumps (so-called static avalanches or shocks) at very low temperature, as were observed and studied in other mean-field models such as the SK model; see [27–29,44,46].

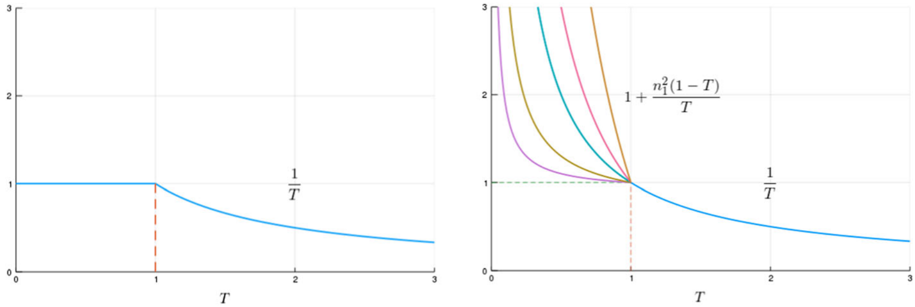
For more details on the geometry of the Gibbs measure see Sect. 11, in particular, the Table 5 and the summary in Sect. 11.3.

2.4 Magnetization and Susceptibility

We also evaluate the magnetization, susceptibility, and differential susceptibility. One of the results is that at low temperature $0 < T < 1$, for $h \sim N^{-1/2}$, the linear susceptibility defined by $\mathcal{X} = \frac{1}{h}\langle \mathfrak{M} \rangle$ satisfies

$$\mathcal{X} \simeq 1 + \frac{|n_1|\sqrt{1-T}}{hN^{1/2}} \tanh\left(\frac{hN^{1/2}|n_1|\sqrt{1-T}}{T}\right) \tag{2.2}$$

for asymptotically almost every disorder sample. This formula is a consequence of the spin geometry which has an interpretation as an effective 2 level system, as discussed above.



(a) $\lim_{h\to 0} \lim_{\substack{N\to\infty \\ h>0}} \mathcal{X}$ **(b)** $\lim_{H\to 0} \lim_{\substack{N\to\infty \\ h=HN^{-1/2}}} \mathcal{X}$ for sevaral values of n_1^2

Fig. 2 Graph of the zero external field limit of the susceptibility as a function of T

The above formula implies the zero external field limit of the susceptibility:

$$\lim_{H\to 0} \lim_{\substack{N\to\infty \\ h=HN^{-1/2}}} \mathcal{X} = 1 + \frac{n_1^2(1-T)}{T} \quad \text{and} \quad \lim_{H\to 0} \lim_{\substack{N\to\infty \\ h=HN^{-1/2}}} \bar{\mathcal{X}} = \frac{1}{T}. \tag{2.3}$$

The limit of $\mathcal{X}$ depends on the disorder variable n_1^2. See Fig. 2b. This result shows that the Curie law holds for the sample-to-sample average, but not for a given disorder sample. If we take a different limit, namely if we let $N \to \infty$ with $h > 0$ first and then let $h \to 0$, then the limit of the susceptibility is deterministic and given by $\min\{T^{-1}, 1\}$. See Fig. 2a. This formula was previously obtained in [24], and also in [14]. See Sects. 8.7 and 8.8 for details.

3 Contour Integral Representations

The partition function is an N-fold integral over a sphere. Using the Laplace transform and Gaussian integrations, Kosterlitz, Thouless and Jones showed in [24] that this integral can be expressed as a single contour integral which involve the disorder sample. We state this result and also include its derivation in Sect. 3.1. By the same method, the moment generating functions of the overlaps can also be written as a ratio of single or double contour integrals. These results are presented in Sect. 3.2.

3.1 Free Energy

The following result holds for any disorder sample.

Lemma 3.1 ([24]) *Let M be an arbitrary N by N symmetric matrix and let $\mathbf{g}$ be an N dimensional vector. Let $\lambda_1 \geq \cdots \geq \lambda_N$ be the eigenvalues of the matrix M and let $\boldsymbol{u}_i$ be a corresponding unit eigenvector. Then, the partition function $\mathcal{Z}_N$ defined in (1.4) can be written as*

$$\mathcal{Z}_N = C_N \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{\frac{N}{2}\mathcal{G}(z)} \mathrm{d}z \quad \textit{where} \quad C_N = \frac{\Gamma(N/2)}{2\pi \mathrm{i}(N\beta/2)^{N/2-1}} \tag{3.1}$$

and

$$\mathcal{G}(z)=\beta z-\frac{1}{N}\sum_{i=1}^{N}\log(z-\lambda_i)+\frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2}{z-\lambda_i}\quad \textit{with}\quad n_i=\boldsymbol{u}_i\cdot\boldsymbol{g}. \tag{3.2}$$

Here, the integration is over the vertical line $\gamma+\mathrm{i}\mathbb{R}$ *where* γ *is an arbitrary constant satisfying* $\gamma>\lambda_1$.

Proof Let $\Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\cdots,\lambda_N)$. Let $O=(\mathbf{u}_1,\cdots,\mathbf{u}_N)$ be an orthogonal matrix so that $M=O\Lambda O^T$. Let S^{N-1} be the sphere of radius 1 in $\mathbb{R}^N$ and let $\mathrm{d}\Omega_{N-1}$ be the surface area element on S^{N-1}. Then, using the changes of variables $\frac{1}{\sqrt{N}}O^T\boldsymbol{\sigma}=x$,

$$\begin{aligned}\mathcal{Z}_N&=\frac{1}{|S^{N-1}|}I\left(\frac{\beta N}{2},h\sqrt{2\beta}\right)\quad\text{where}\quad I(t,s)\\&=\int_{S^{N-1}}e^{t\sum_{i=1}^N\lambda_i x_i^2+s\sqrt{t}\sum_{i=1}^N n_i x_i}\mathrm{d}\Omega_{N-1}(x).\end{aligned}$$

where $n_i=(O^T\mathbf{g})_i=\mathbf{u}_i\cdot\mathbf{g}$. We take the Laplace transform of $J(t)=t^{N/2-1}I(t,s)$. Making a simple change of variables $t=r^2$ and using Gaussian integrals, the Laplace transform is equal to

$$L(z)=\int_0^\infty e^{-zt}J(t)\mathrm{d}t=2\int_{\mathbb{R}^N}e^{-\sum_{i=1}^N(z-\lambda_i)y_i^2+s\sum_{i=1}^N n_i y_i}\mathrm{d}^N y=2\prod_{i=1}^N e^{\frac{s^2n_i^2}{4(z-\lambda_i)}}\sqrt{\frac{\pi}{z-\lambda_i}}$$

for z satisfying $z>\lambda_1$. We obtain a single integral formula of the partition function by taking the inverse Laplace transform. □

Note that the sign ambiguity of $\mathbf{u}_i$ does not affect the result since the formula depends only on n_i^2.

3.2 Overlaps

In this section, we give the moment generating function of each of the overlaps, expressed as a ratio of contour integrals. The proofs are similar to the computations for the free energy case and we give the proof in Appendix A.

Definition 3.2 The following three functions are related to the function $\mathcal{G}$ and will be used to compute the three overlaps respectively. We denote by $\eta\in\mathbb{R}$ the parameter that will be used for the moment generating function of each overlap.

- For the overlap with the external field, we use the function

$$\mathcal{G}_{\mathfrak{M}}(z):=\beta z-\frac{1}{N}\sum_{i=1}^{N}\log(z-\lambda_i)+\frac{(h+\frac{\eta}{N})^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2}{z-\lambda_i}. \tag{3.3}$$

  Note that this is $\mathcal{G}(z)$ with h replaced by $h+\eta N^{-1}$.
- For the (square of the) overlap with the ground state, we use the function

$$\begin{aligned}\mathcal{G}_{\mathfrak{O}}(z):=&\beta z-\frac{1}{N}\log\left(z-\left(\lambda_1+\frac{2\eta}{N}\right)\right)-\frac{1}{N}\sum_{i=2}^{N}\log(z-\lambda_i)\\&+\frac{h^2\beta}{N}\frac{n_1^2}{z-(\lambda_1+\frac{2\eta}{N})}+\frac{h^2\beta}{N}\sum_{i=2}^{N}\frac{n_i^2}{z-\lambda_i}.\end{aligned} \tag{3.4}$$

Note that this is $\mathcal{G}(z)$ with λ_1 replaced by $\lambda_1 + \frac{\eta}{\beta N}$.

- For the overlap with a replica, we use the function

$$\mathcal{G}_{\mathfrak{R}}(z, w; a) := \beta(z + w) - \frac{1}{N}\sum_{i=1}^{N} \log\left((z - \lambda_i)(w - \lambda_i) - a^2\right) + \frac{h^2\beta}{N}\sum_{i=1}^{N} \frac{n_i^2(z + w - 2\lambda_i + 2a)}{(z - \lambda_i)(w - \lambda_i) - a^2}. \tag{3.5}$$

Lemma 3.3 *For real parameter η, the moment generation functions of the three overlaps are as follows:*

$$\langle e^{\beta\eta\mathfrak{M}}\rangle = \frac{\int e^{\frac{N}{2}\mathcal{G}_{\mathfrak{M}}(z)}\mathrm{d}z}{\int e^{\frac{N}{2}\mathcal{G}(z)}\mathrm{d}z}, \qquad \langle e^{\beta\eta\mathfrak{O}}\rangle = \frac{\int e^{\frac{N}{2}\mathcal{G}_{\mathfrak{O}}(z)}\mathrm{d}z}{\int e^{\frac{N}{2}\mathcal{G}(z)}\mathrm{d}z},$$
$$\langle e^{\eta\mathfrak{R}}\rangle s = \frac{\iint e^{\frac{N}{2}\mathcal{G}_{\mathfrak{R}}(z,w;\frac{\eta}{\beta N})}\mathrm{d}z\mathrm{d}w}{\iint e^{\frac{N}{2}\mathcal{G}_{\mathfrak{R}}(z,w;0)}\mathrm{d}z\mathrm{d}w}. \tag{3.6}$$

The contours are vertical lines in the complex plane such that all singularities lie on the left of the contour. See Appendix A for the derivation.

4 Results from Random Matrices

Since the disorder matrix M is a GOE matrix, the eigenvectors are uniformly distributed on the sphere. On the other hand, the eigenvalues statistics are well studied in random matrix theory. We summarize several definitions and properties of the eigenvalues and other related quantities that we use in this paper.

4.1 Probability Notations

There are two types of randomness, one from the disorder sample M and $\mathbf{g}$, and the other from the Gibbs (thermal) measure. We often need to distinguish them. We add the subscript s to denote sample probability or sample expectation such as $\mathbb{P}_s$ and $\mathbb{E}_s$. In addition, we use the following notations.

Definition 4.1 When describing the limiting distributions in our results, we consider two classes of random variables, which we refer to as sample random variables and thermal random variables. To distinguish between these two classes, we denote them with the calligraphic font and the gothic font respectively. For example a standard Gaussian sample random variable and a standard Gaussian thermal variable will be denoted below by

$$\mathcal{N} \quad \text{and} \quad \mathfrak{N} \tag{4.1}$$

respectively.

Definition 4.2 Asymptotic notations:

- If $\{E_N\}_{N=1}^{\infty}$ is a sequence of events, we say that E_N holds asymptotically almost surely (or everywhere) if $\mathbb{P}_s(E_N) \to 1$ as $N \to \infty$. This probability is with respect to the choice of disorder sample.

- For two N-dependent random variables $A := A_N$ and $B := B_N$, the notation

$$A = \mathcal{O}(B) \tag{4.2}$$

means that, for any $\varepsilon > 0$, the inequality $A \leq BN^{\varepsilon}$ holds asymptotically almost surely.
- The notation $\simeq$ means an asymptotic expansion up to the terms indicated on the right-hand side and the notation $\asymp$ denotes two sides are of the same order. When we say $A \asymp \mathcal{O}(B)$ we mean that, for any $\varepsilon > 0$, the inequality $BN^{-\varepsilon} < A < BN^{\varepsilon}$ holds asymptotically almost surely.

Definition 4.3 Convergence notations:

- The convergence in distribution of a sequence of random variables X_N to a random variable X with respect to the disorder variables is denoted by $X_N \Rightarrow X$.
- We use the notations $\stackrel{\mathcal{D}}{=}$ and $\stackrel{\mathcal{D}}{\simeq}$ to denote an equality and an asymptotic expansion in distribution with respect to the disorder sample, respectively.
- We use similar notations with a different font, $\stackrel{\mathfrak{D}}{=}$ and $\stackrel{\mathfrak{D}}{\simeq}$, to denote an equality and an asymptotic expansion in distribution with respect to the Gibbs (thermal) measure, respectively.

It is worth noting that many of our results actually hold with high probability (i.e., there exist some $D > 0$, $N_0 > 0$ such that, for all $N \geq N_0$, $\mathbb{P}(E_N) > 1 - N^{-D}$). While high probability is much stronger than asymptotically almost sure probability, it is much more delicate to prove and we do not discuss those proofs in the current paper.

4.2 Semicircle Law

The empirical distribution of eigenvalues of M converges to the semicircle law [31]: for every continuous bounded function $f(x)$,

$$\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i) \to \int f(x)\mathrm{d}\sigma_{scl}(x) \quad \text{where} \quad \mathrm{d}\sigma_{scl}(x) = \frac{\sqrt{4-x^2}}{2\pi}\mathbb{1}_{x\in[-2,2]}\mathrm{d}x \tag{4.3}$$

with probability 1 as $N \to \infty$.

Definition 4.4 We define the following functions for later use:

$$s_0(z) := \int \log(z-x)\mathrm{d}\sigma_{scl}(x) \quad \text{and} \quad s_k(z) := \int \frac{\mathrm{d}\sigma_{scl}(x)}{(z-x)^k} \quad \text{for } k = 1, 2, \cdots, \tag{4.4}$$

Properties: These functions can be evaluated explicitly as

$$\begin{aligned} s_0(z) =& \frac{1}{4}z(z-\sqrt{z^2-4}) + \log(z+\sqrt{z^2-4}) - \log 2 - \frac{1}{2}, \\ s_1(z) =& \frac{z-\sqrt{z^2-4}}{2}, \quad s_2(z) = \frac{z-\sqrt{z^2-4}}{2\sqrt{z^2-4}}, \quad s_3(z) = \frac{1}{(z^2-4)^{3/2}}, \\ s_4(z) =& \frac{z}{(z^2-4)^{5/2}} \end{aligned} \tag{4.5}$$

for z not in the real interval $[-2, 2]$. As $z \to 2$, we have

$$\begin{aligned} &s_1(z) \simeq 1 - \sqrt{z-2}, \quad s_2(z) \simeq \frac{1}{2\sqrt{z-2}} - \frac{1}{2}, \quad s_3(z) \simeq \frac{1}{8(z-2)^{3/2}}, \\ &s_4(z) \simeq \frac{1}{16(z-2)^{5/2}}. \end{aligned} \tag{4.6}$$

4.3 Rigidity

Definition 4.5 For $i = 1, 2, \cdots, N$, let $\widehat{\lambda}_i$ be the classical location defined by the quantile conditions

$$\int_{\widehat{\lambda}_i}^{2} \mathrm{d}\sigma_{scl}(x) = \frac{i}{N}. \tag{4.7}$$

We set $\widehat{\lambda}_0 = 2$. We also set $\widehat{a}_i = (\widehat{\lambda}_i - 2)N^{2/3}$.

Rigidity property: The rigidity result [16,17] states that

$$|\lambda_i - \widehat{\lambda}_i| \leq (\min\{i, N+1-i\})^{-1/3}\mathcal{O}\left(N^{-2/3}\right) \tag{4.8}$$

uniformly for $i = 1, 2, \cdots, N$.

The rigidity property allows us to apply the method of steepest descent to evaluate the integrals involving the eigenvalues since the eigenvalues are close enough to the classical location, and the fluctuations are small enough.

4.4 Edge Behavior

Definition 4.6

- Define the rescaled eigenvalues

$$a_i := N^{2/3}(\lambda_i - 2). \tag{4.9}$$

- Define $\{\alpha_i\}_{i=1}^{\infty}$ to be the GOE Airy point process process to which the rescaled eigenvalues converge in distribution as $N \to \infty$ [38,43]:

$$\{a_i\} \Rightarrow \{\alpha_i\}. \tag{4.10}$$

Properties: The rightmost point α_1 of the GOE Airy point process has the GOE Tracy-Widom distribution

$$a_1 \Rightarrow \alpha_1 \stackrel{\mathcal{D}}{=} \mathrm{TW}_{GOE}\,. \tag{4.11}$$

The GOE Airy point process satisfies the asymptotic property that

$$\alpha_i \simeq -\left(\frac{3\pi i}{2}\right)^{2/3} \quad \text{as } i \to \infty. \tag{4.12}$$

This asymptotic is due to the fact that the semicircle law is asymptotic to $\frac{\sqrt{2-x}}{\pi}\mathrm{d}x$ as $x \to 2$. The above formula and the rigidity imply that, with high probability,

$$a_i \asymp -i^{2/3} \quad \text{as } i, N \to \infty \text{ satisfying } i \leq N \tag{4.13}$$

4.5 Central Limit Theorem of Linear Statistics

For a function f which is analytic in an open neighborhood of $[-2, 2]$ in the complex plane, consider the sum of $f(\lambda_i)$. The semicircle law (4.3) gives its leading behavior. If we subtract the leading term, the difference

$$\sum_{i=1}^{N} f(\lambda_i) - N\int f(x)\mathrm{d}\sigma_{scl}(x) \tag{4.14}$$

converges to a Gaussian distribution with explicit mean and variance; see, for example, [3,22,30]. Note that unlike the classical central limit theorem, we do not divide by $\sqrt{N}$.

Definition 4.7 Define

$$\mathcal{L}_N(z) := \sum_{i=1}^{N} \log(z-\lambda_i) - N s_0(z). \tag{4.15}$$

for $z > 2$ where $s_0(z)$ is given by (4.5).

Properties:
The above-mentioned central limit theorem implies in this case that

$$\mathcal{L}_N(z) \Rightarrow \mathcal{N}(M(z), V(z)) \tag{4.16}$$

where (see Lemma A.1 in [4])

$$M(z) = \frac{1}{2}\log\left(\frac{2\sqrt{z^2-4}}{z+\sqrt{z^2-4}}\right), \qquad V(z) = 2\log\left(\frac{z+\sqrt{z^2-4}}{2\sqrt{z^2-4}}\right). \tag{4.17}$$

For later uses, we record that for $0 < \beta < 1$,

$$M(\beta+\beta^{-1}) = \frac{1}{2}\log(1-\beta^2), \qquad V(\beta+\beta^{-1}) = -2\log(1-\beta^2). \tag{4.18}$$

4.6 Special Sums

In this section we collect several important results about convergence of various types of sums that we will use in this paper. Many of the results are motivated by the need to work with sums of the form

$$\frac{1}{N}\sum_{i=2}^{N} \frac{1}{(\lambda_1-\lambda_i)^k}, \qquad k = 1, 2, \cdots, \tag{4.19}$$

or its variations. The above quantity looks superficially close to the linear statistics (4.14) with $f(x) = \frac{1}{(\lambda_1-x)^k}$ with one term removed but the function $f(x)$ is singular at $x = \lambda_1$. We note that if we replace $f(x)$ by $\frac{1}{(2-x)^k}$ and use the semicircle law, we obtain $s_k(2)$ which diverges for $k \geq 2$. Hence, the result of the previous subsection does not apply. On the hand, for $k = 1$, $s_1(2) = 1$. This fact indicates that the above sum still converges when $k = 1$.

We present several definitions, followed by their related convergence results and some brief explanation of why these results hold. Recall the definition $a_i = (\lambda_i - 2)N^{2/3}$ and $n_i = \mathbf{u}_i \cdot \mathbf{g}$.

Definition 4.8 We define the following random sums, which depend on the disorder sample:

- Define

$$\Xi_N := N^{1/3}\left(\frac{1}{N}\sum_{i=2}^{N}\frac{1}{\lambda_1-\lambda_i} - 1\right) = \sum_{i=2}^{N}\frac{1}{a_1-a_i} - N^{1/3}. \tag{4.20}$$

- Define, for $w > 0$,

$$\mathcal{E}_N(w) := N^{1/3}\left[\frac{1}{N}\sum_{i=1}^{N}\frac{n_i^2}{wN^{-2/3}+\lambda_1-\lambda_i} - 1\right] = \sum_{i=1}^{N}\frac{n_i^2}{w+a_1-a_i} - N^{1/3}. \tag{4.21}$$

- Define, for $z > 2$,

$$\mathcal{S}_N(z;k) := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{n_i^2 - 1}{(z - \widehat{\lambda}_i)^k} \qquad \textit{for } k \geq 1. \tag{4.22}$$

Definition 4.9 We define the following limits, which depend on the GOE Airy point process $\{\alpha_i\}$:

- Define

$$\Xi := \lim_{n\to\infty} \left(\sum_{i=2}^{n} \frac{1}{\alpha_1 - \alpha_i} - \frac{1}{\pi} \int_0^{\left(\frac{3\pi n}{2}\right)^{2/3}} \frac{\mathrm{d}x}{\sqrt{x}} \right). \tag{4.23}$$

  Landon and Sosoe showed that the limit exists almost surely [25].
- Define $\mathcal{E}(s)$ as follows, where ν_i are i.i.d. Gaussian random variables with mean 0 and variance 1 independent of the GOE Airy point process α_i:

$$\mathcal{E}(s) := \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{\nu_i^2}{s + \alpha_1 - \alpha_i} - \frac{1}{\pi} \int_0^{\left(\frac{3\pi n}{2}\right)^{2/3}} \frac{\mathrm{d}x}{\sqrt{x}} \right). \tag{4.24}$$

  This limit exists almost surely by a similar argument as in [25] showing that Ξ exists.

Result 4.10 *Using the notations above, we have the following convergence results.*

- *Landon and Sosoe proved in [25] that*

$$\Xi_N \Rightarrow \Xi. \tag{4.25}$$

  They use this result to describe the fluctuations of the overlap with a replica when $h = 0$ and $T < 1$.
- *We also need another version of the result (4.25) where the constant numerators are replaced n_i^2:*

$$N^{1/3}\left(\frac{1}{N} \sum_{i=2}^{N} \frac{n_i^2}{\lambda_1 - \lambda_i} - 1 \right) \Rightarrow \lim_{n\to\infty} \left(\sum_{i=2}^{n} \frac{\nu_i^2}{\alpha_1 - \alpha_i} - \frac{1}{\pi} \int_0^{\left(\frac{3\pi n}{2}\right)^{2/3}} \frac{\mathrm{d}x}{\sqrt{x}} \right) \tag{4.26}$$

  where ν_i are i.i.d standard Gaussians, independent of the GOE Airy point process α_i. This follows from (4.25) and the fact that

$$\frac{1}{N^{2/3}} \sum_{i=2}^{N} \frac{n_i^2 - 1}{\lambda_1 - \lambda_i} \Rightarrow \sum_{i=2}^{\infty} \frac{\nu_i^2 - 1}{\alpha_1 - \alpha_i} \tag{4.27}$$

  which is a convergent series due to Kolmogorov's three series theorem and (4.12).
- *By the same argument as for 4.26,*

$$\mathcal{E}_N(w) \Rightarrow \mathcal{E}(w) \quad \textit{for } w > 0. \tag{4.28}$$

- *By the Lyapunov central limit theorem and the definition of $\widehat{\lambda}_i$, we have*

$$\mathcal{S}_N(z;k) \Rightarrow \mathcal{N}(0, 2s_{2k}(z)) \tag{4.29}$$

  as $N \to \infty$ for $z > 2$. (Note that the variance of $n_i^2 - 1$ is 2.)

Result 4.11 *In addition to the convergence results listed above, we also need estimates that hold for asymptotically almost every disorder sample.*

- *A consequence of* (4.25) *is that*

$$\frac{1}{N}\sum_{i=2}^{N}\frac{1}{\lambda_1-\lambda_i}=\frac{1}{N^{1/3}}\sum_{i=2}^{N}\frac{1}{a_1-a_i}=1+\mathcal{O}\left(N^{-1/3}\right). \tag{4.30}$$

for asymptotically almost every disorder sample.

- *We have*

$$\sum_{i=2}^{N}\frac{1}{(a_1-a_i)^k}=\mathcal{O}(1) \quad \text{and} \quad \sum_{i=2}^{N}\frac{n_i^2}{(a_1-a_i)^k}=\mathcal{O}(1), \qquad k\geq 2, \tag{4.31}$$

for asymptotically almost every disorder sample. This follows from the fact that the $a_i \asymp -i^{2/3}$ *and the difference* $(a_1-a_2)^{-1}$ *is of order 1 with vanishing probability.*

- *We also need the result*

$$\frac{1}{N}\sum_{i=1}^{N}\frac{n_i^2}{(z-\lambda_i)^k}=s_k(z)+\frac{\mathcal{S}_N(z;k)}{\sqrt{N}}+\mathcal{O}\left(N^{-1}\right), \qquad z>2, \quad k>1 \tag{4.32}$$

for asymptotically almost every disorder sample.
To justify (4.32)*, we observe that*

$$\frac{1}{N}\sum_{i=1}^{N}\frac{n_i^2}{(z-\lambda_i)^k}=\frac{1}{N}\sum_{i=1}^{N}\frac{1}{(z-\lambda_i)^k}+\frac{1}{N}\sum_{i=1}^{N}\frac{n_i^2-1}{(z-\lambda_i)^k}. \tag{4.33}$$

We then use the central limit theorem (4.14) *for linear statistics for the first sum and replace* λ_i *by* $\widehat{\lambda}_i$ *in the second sum using the rigidity* (4.8).

5 Fluctuations of the Free Energy

From the integral formula (3.1), using

$$C_N=\frac{\sqrt{N}\beta}{2\mathrm{i}\sqrt{\pi}(\beta e)^{N/2}}(1+O(N^{-1})), \tag{5.1}$$

the free energy can be written as

$$\mathcal{F}_N=\frac{1}{2\beta}(\mathcal{G}(\gamma)-1-\log\beta)+\frac{1}{N\beta}\log\left(\frac{\sqrt{N}\beta}{2\mathrm{i}\sqrt{\pi}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z\right)+O(N^{-2}) \tag{5.2}$$

where $O(N^{-2})$ is a constant that does not depend on the disorder sample M and $\mathbf{g}$. We evaluate the integral asymptotically using the method of steepest descent. The formula for $\mathcal{G}(z)$ is given in (3.2) and

$$\mathcal{G}'(z)=\beta-\frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-\lambda_i}-\frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2}{(z-\lambda_i)^2} \qquad \text{where} \quad n_i=\mathbf{u}_i\cdot\mathbf{g}. \tag{5.3}$$

For real z, $\mathcal{G}'(z)$ is an increasing function taking values from $-\infty$ to β as z moves from λ_1 to ∞. Hence, there is a unique real critical point γ satisfying

$$\mathcal{G}'(\gamma)=0, \qquad \gamma>\lambda_1.$$

We set γ for the contour of (5.2) to be this critical point.

In this section, we use the formula (5.2) to evaluate the fluctuations of the free energy when the external field strength h is fixed. For the case $h = 0$, this computation was done in [24] for the leading deterministic term and in [4] for the subleading term. For fixed $h > 0$, the fluctuations for the SK model were computed in [9] using a method different from the one of this paper. We first review the computation of [4] for $h = 0$ and then give a new computation for fixed $h > 0$ using the above integral formula.

The following formula will be used in one of the subsections: Since $\mathcal{G}(z) - \mathcal{G}(\gamma) = \mathcal{G}(z) - \mathcal{G}(\gamma) - \mathcal{G}'(\gamma)(z-\gamma)$, we can write

$$N(\mathcal{G}(z) - \mathcal{G}(\gamma)) = -\sum_{i=1}^{N}\left[\log(1+\frac{z-\gamma}{\gamma-\lambda_i}) - \frac{z-\gamma}{\gamma-\lambda_i}\right] + h^2\beta\sum_{i=1}^{N}\frac{n_i^2(z-\gamma)^2}{(z-\lambda_i)(\gamma-\lambda_i)^2}. \tag{5.4}$$

5.1 No External Field: $h = 0$

5.1.1 High Temperature Regime: $T > 1$

When $h = 0$, we write, using the notation (4.15),

$$\mathcal{G}(z) = \beta z - \frac{1}{N}\sum_{i=1}^{N}\log(z-\lambda_i) = \beta z - s_0(z) - \frac{\mathcal{L}_N(z)}{N}, \qquad s_0(z) = \int \log(z-x)\mathrm{d}\sigma_{scl}(x). \tag{5.5}$$

From (4.16), $\mathcal{L}_N(z) = \mathcal{O}(1)$ for fixed $z > 2$. Thus, $\mathcal{G}_0(z) := \beta z - s_0(z)$ is an approximation of the function $\mathcal{G}(z)$ and we first find the critical point γ_0 of $\mathcal{G}_0(z)$ satisfying $\gamma_0 > 2$, where we recall that the largest eigenvalue $\lambda_2 \to 2$. Since $\mathcal{G}_0''(z) > 0$, we find that $\min_{z\geq 2}\mathcal{G}_0'(z) = \mathcal{G}_0'(2) = \beta - 1$ from the formula (4.5) of $s_0'(z) = s_1(z)$. Thus, the critical point of $\mathcal{G}_0(z)$ exists only when $\frac{1}{\beta} = T > 1$. From the formula, we find that for $T > 1$, it is given by

$$\gamma_0 := \beta + \beta^{-1} = T + T^{-1}. \tag{5.6}$$

In this case, a simple perturbation argument (see Appendix B) implies that $\gamma = \gamma_0 + \mathcal{O}\left(N^{-1}\right)$ and

$$\mathcal{G}(\gamma) = \mathcal{G}(\gamma_0) - \frac{\mathcal{L}_N(\gamma_0)}{N} + \mathcal{O}\left(N^{-2}\right) = \frac{\beta^2}{2} + 1 + \log\beta - \frac{\mathcal{L}_N(\gamma_0)}{N} + \mathcal{O}\left(N^{-2}\right). \tag{5.7}$$

Even though the integral in (5.2) involves the disorder sample, the rigidity of the eigenvalues from Sect. 4.3 implies that, with high probability, the eigenvalues are close to the non-random classical locations (i.e. the quantiles of the semicircle law). Thus, we can still apply the method of steepest descent when the disorder sample is in an event of the high probability. Using

$$\mathcal{G}''(\gamma) \simeq \mathcal{G}_0''(\gamma_0) = s_2(\gamma_0) = \frac{\beta^2}{1-\beta^2}$$

and $\mathcal{G}^{(k)}(\gamma) = \mathcal{O}(1)$ for all $k \geq 2$, the Gaussian approximation of the integral is valid and we find that

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z \simeq \frac{\mathrm{i}\sqrt{4\pi}}{\sqrt{Ns_2(\gamma_0)}} = \frac{\mathrm{i}\sqrt{4\pi(1-\beta^2)}}{\sqrt{N\beta^2}}. \tag{5.8}$$

Inserting everything into (5.2) and using the fact that $\mathcal{L}_N(\gamma_0)$ converges to a Gaussian distribution with mean and variance given by (4.18), we obtain the following result. This result was proved rigorously in [4].

Theorem 5.1 ([4]) *For $h = 0$ and $T > 1$,*

$$\mathcal{F}_N(T, 0) = \frac{1}{4T} + \frac{T}{2N}\left[\log(1 - T^{-2}) - \mathcal{L}_N(\gamma_0)\right] + \mathcal{O}\left(N^{-3/2}\right) \tag{5.9}$$

as $N \to \infty$ with high probability, where $\gamma_0 = T + T^{-1}$ and $\mathcal{L}_N(z)$ is defined in (4.15). As a consequence,

$$\mathcal{F}_N(T, 0) \overset{\mathcal{D}}{\simeq} \frac{1}{4T} + \frac{T}{2N}\mathcal{N}(-\alpha, 4\alpha) \qquad \alpha := -\frac{1}{2}\log(1 - T^{-2}), \tag{5.10}$$

where $\mathcal{N}(a, b)$ is a (sample) Gaussian distribution of mean a and variance b.

5.1.2 Low Temperature Regime: $T < 1$

In contrast to the previous section, the function $\mathcal{G}_0(z) = \beta z - s_0(z)$ is no longer a good approximation of $\mathcal{G}(z)$ for $0 < T < 1$ when $h = 0$. Indeed, the function $\mathcal{G}_0(z)$ does not have a critical point satisfying $z > 2$. Hence, we need to find the critical point γ of $\mathcal{G}(z)$ directly. Since the critical point of $\mathcal{G}_0(z)$ when $T = 1$ is given by $\gamma_0 = 2$, it is reasonable to assume that when $0 < T < 1$, γ is close to the large eigenvalue λ_1. It turns out that $\gamma = \lambda_1 + \mathcal{O}\left(N^{-1}\right)$. We set $\gamma = \lambda_1 + sN^{-1}$ with $s = \mathcal{O}(1)$ and determine s. Separating out the term with $i = 1$ in the equation (5.3) and using (4.30),

$$\mathcal{G}'(\gamma) = \beta - \frac{1}{N(\gamma - \lambda_1)} - \frac{1}{N}\sum_{i=2}^{N}\frac{1}{\gamma - \lambda_i} = \beta - \frac{1}{s} - 1 + \mathcal{O}\left(N^{-1/3}\right) = 0. \tag{5.11}$$

Thus $s = \frac{1}{\beta - 1} + \mathcal{O}\left(N^{-1/3}\right)$, which is consistent with our assumption that $s = \mathcal{O}(1)$. To evaluate

$$\mathcal{G}(\gamma) = \beta\gamma - \frac{1}{N}\sum_{i=1}^{N}\log(\gamma - \lambda_i),$$

we use (4.15)–(4.17). We need to evaluate $\sum_{i=1}^{N}\log(z - \lambda_i)$ for $z = 2 + O(N^{-2/3})$. Observe that

$$M(z) = O(\log(z - 2)) \quad \text{and} \quad V(z) = O(\log(z - 2)) \quad \text{as } z \to 2.$$

Hence, a formal application of (4.16) to this case using $s_0(z) = \frac{1}{2} + (z - 2) + O((z - 2)^{3/2})$ implies that for $z \to 2$ such that $|z - 2| \geq N^{-d}$ for some $d > 0$,

$$\frac{1}{N}\sum_{i=1}^{N}\log(z - \lambda_i) = s_0(z) + \mathcal{O}\left(N^{-1}\right) = \frac{1}{2} + (z - 2) + \mathcal{O}\left(N^{-1}\right) + O((z - 2)^{3/2}). \tag{5.12}$$

This heuristic computation indicates that

$$\mathcal{G}(\gamma) = \beta\gamma - \frac{1}{N}\sum_{i=1}^{N}\log(\gamma - \lambda_i) = 2\beta - \frac{1}{2} + (\beta - 1)(\lambda_1 - 2) + \mathcal{O}\left(N^{-1}\right). \tag{5.13}$$

We now consider the integral in (5.2). For $k \geq 2$, we have, using the notation (4.9) for the scaled eigenvalues $a_i = N^{2/3}(\lambda_i - 2)$ and the estimate (4.31),

$$\begin{aligned}\frac{\mathcal{G}^{(k)}(\gamma)}{(-1)^k(k-1)!} &= \frac{1}{N}\sum_{i=1}^{N}\frac{1}{(\gamma-\lambda_i)^k} = \frac{N^{k-1}}{s^k} + N^{\frac{2}{3}k-1}\sum_{i=2}^{N}\frac{1}{(a_1+sN^{-1/3}-a_i)^k} \\ &= \mathcal{O}\left(N^{k-1}\right)\end{aligned} \tag{5.14}$$

with high probability. The estimate $\mathcal{G}''(\gamma) = \mathcal{O}(N)$ indicates that the main contribution to the integral comes from a neighborhood of radius N^{-1} of the critical point. However, all terms of the Taylor series

$$N\left(\mathcal{G}(\gamma+uN^{-1})-\mathcal{G}(\gamma)\right) = \sum_{k=2}^{N} N^{1-k}\frac{\mathcal{G}^{(k)}(\gamma)}{k!}u^k$$

are of the same order $\mathcal{O}(1)$ for finite u. Hence, we cannot replace the integral with a Gaussian integral. Instead, we proceed as follows. Using the formula (5.4), separating out the $i = 1$ term from the sum, using a Taylor approximation for the remaining sum, and using (4.31),

$$\begin{aligned}N\left(\mathcal{G}(\gamma+uN^{-1})-\mathcal{G}(\gamma)\right) &= -\log\left(1+\frac{u}{s}\right)+\frac{u}{s}+O\left(\sum_{i=2}^{N}\frac{u^2N^{-2/3}}{(a_1-sN^{-1/3}-a_i)^2}\right) \\ &= -\log\left(1+\frac{u}{s}\right)+\frac{u}{s}+\mathcal{O}\left(N^{-2/3}\right)\end{aligned} \tag{5.15}$$

with high probability for finite u. From this,

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z \simeq \frac{1}{N}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{e^{\frac{u}{s}}}{1+\frac{u}{s}}\mathrm{d}u \asymp \mathcal{O}\left(N^{-1}\right). \tag{5.16}$$

We do not need the exact value of the integral, but only the estimate that its log is $\mathcal{O}(\log N)$.

We thus obtain the following result, which was proved rigorously in [4].

Theorem 5.2 ([4]) *For $h = 0$ and $0 \leq T < 1$,*

$$\mathcal{F}_N(T,0) = 1-\frac{3T}{4}+\frac{T\log T}{2}+\frac{1-T}{2N^{2/3}}a_1+\mathcal{O}\left(N^{-1}\right) \tag{5.17}$$

as $N \to \infty$ with high probability. As a consequence,

$$\mathcal{F}_N(T,0) \overset{\mathcal{D}}{\simeq} 1-\frac{3T}{4}+\frac{T\log T}{2}+\frac{1-T}{2N^{2/3}}\mathrm{TW}_{GOE}\,. \tag{5.18}$$

Remark 5.3 The zero temperature case $T = 0$ of the theorem is the standard random matrix theory result that the largest eigenvalue of a GOE matrix converges to the Tracy-Widom distribution. We see that a formal $T \to 0$ limit of the result implies this statement. Similarly, all results of this paper, other than those that have $T > 1$ restrictions, have a convergent formal limit if we take $T \to 0$. Hence, even though we need a separate argument since there is no integral representation, we expect that all results are valid for the $T = 0$ case as well.

5.2 Positive External Field: $h = O(1)$

Fix $h > 0$. We use (4.16) and (4.32) to write

$$\mathcal{G}(z) = \beta z - s_0(z) + h^2\beta\left[s_1(z) + \frac{1}{\sqrt{N}}\mathcal{S}_N(z;1)\right] + \mathcal{O}\left(N^{-1}\right)$$

for $z > \lambda_1$. The random variable $\mathcal{S}_N(z;k)$ is defined in (4.22) and it converges in distribution to $\mathcal{N}(0, 2s_{2k}(z))$; see (4.29). This time, $\mathcal{G}(z)$ is approximated by the function $\mathcal{G}_0(z) = \beta z - s_0(z) + h^2\beta s_1(z)$. Its derivative $\mathcal{G}_0'(z) = \beta - s_1(z) - h^2\beta s_2(z)$ is an increasing function for $z > 2$ and $\mathcal{G}'(z) \to -\infty$ as $z \downarrow 2$ while $\mathcal{G}'(z) \to +\infty$ as $z \to +\infty$. Hence, unlike in the case of $h = 0$, there is a point $\gamma_0 > 2$ satisfying $\mathcal{G}_0'(\gamma_0) = 0$ for all $T > 0$. It satisfies the equation

$$\mathcal{G}_0'(\gamma_0) = \beta - s_1(\gamma_0) - h^2\beta s_2(\gamma_0) = 0. \tag{5.19}$$

A perturbation argument (see Appendix B) implies that the critical point γ of $\mathcal{G}(z)$ has the form

$$\gamma = \gamma_0 + \gamma_1 N^{-1/2} + \mathcal{O}\left(N^{-1}\right). \tag{5.20}$$

We do not need a formula for γ_1 in this section, but we record it here since we use it in later sections;

$$\gamma_1 = \frac{h^2\beta\mathcal{S}_N(\gamma_0;2)}{s_2(\gamma_0) + 2h^2\beta s_3(\gamma_0)} \tag{5.21}$$

where we used the fact that $\frac{\mathrm{d}}{\mathrm{d}z}\mathcal{S}_N(z;1) = -\mathcal{S}_N(z;2)$. The perturbation argument also implies that

$$\mathcal{G}(\gamma) = \beta\gamma_0 - s_0(\gamma_0) + h^2\beta s_1(\gamma_0) + \frac{h^2\beta}{\sqrt{N}}\mathcal{S}_N(\gamma_0;1) + \mathcal{O}\left(N^{-1}\right). \tag{5.22}$$

The integral term in (5.2) can be evaluated using the steepest descent method as in the case of $h = 0$ and $T > 1$ since $\mathcal{G}^{(k)}(\gamma) = \mathcal{O}(1)$ for all $k \geq 2$. From the Gaussian integral approximation,

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z \simeq \frac{\mathrm{i}\sqrt{4\pi}}{\sqrt{N\mathcal{G}''(\gamma)}} \asymp \mathcal{O}\left(N^{-1/2}\right). \tag{5.23}$$

Remark 5.4 We do not focus in this paper on justifying the use of steepest descent in this context, but instead provide the computations based on this method. One can rigorously check that the steepest descent method works here, but it is also worth noting that all the contour integral computations needed in this paper can be achieved without the use of steepest descent. In fact, for the contour integrals in Sects. 5.2, 6.1, 7.1, 8.3, 8.5, 9.1, 9.2, 9.3, 10.1, and 10.2 require no contour deformation at all. Using the straight line contour and crude bounds on the order of the integrand, one can compute, up to leading order, the value of the integral in a neighborhood of γ and then show that the tails are of smaller order. These computations are fairly lengthy and will be omitted from this paper. The integrals in Sects. 8.6 and 10.3 can be treated by a similar method, but require a slight deformation of the original contour. For ease of computation, we instead employ the steepest descent method here, but without providing rigorous justification.

Combining the preceding information in this section, we obtain the following result.

Result 5.5 *For fixed $h > 0$ and $T > 0$,*

$$\mathcal{F}_N(T,h) = F(T,h) + \frac{h^2 \mathcal{S}_N(\gamma_0; 1)}{2\sqrt{N}} + \mathcal{O}\left(N^{-1}\right) \tag{5.24}$$

as $N \to \infty$ with high probability where $\mathcal{S}_N(z;k)$ is defined in (4.22) and

$$F(T,h) := \frac{\gamma_0}{2} - \frac{T s_0(\gamma_0)}{2} - \frac{T - T\log T}{2} + \frac{h^2 s_1(\gamma_0)}{2} \tag{5.25}$$

with γ_0 being the solution of the equation

$$1 - T s_1(\gamma_0) - h^2 s_2(\gamma_0) = 0, \qquad \gamma_0 > 2. \tag{5.26}$$

Since $\mathcal{S}_N(\gamma_0; 1)$ converges in distribution to $\mathcal{N}(0, 2s_2(\gamma_0))$ from (4.29), we conclude the following result.

Result 5.6 *For fixed $h > 0$ and $T > 0$, as $N \to \infty$,*

$$\mathcal{F}_N(T,h) \overset{\mathcal{D}}{\simeq} F(T,h) + \frac{1}{\sqrt{N}} \mathcal{N}\left(0, \frac{h^4 s_2(\gamma_0)}{2}\right). \tag{5.27}$$

This result shows that the order of the fluctuations of the free energy is $N^{-1/2}$ for all $T > 0$, which is different from both N^{-1} for $h = 0$, $T > 1$ and $N^{-2/3}$ for $h = 0$, $0 < T < 1$.

5.3 Comparison with the Result of Chen, Dey, and Panchenko

Chen, Dey, and Penchenko computed the fluctuations of the free energy of the SK model with $h > 0$ in [9] when $\mathbf{g} = \mathbf{1}$. We compare our result with theirs. The adaptation of the approach of [9] to the SSK model with $\mathbf{g} = \mathbf{1}$ implies that $\sqrt{N}\,(\mathcal{F}_N(T,h) - \mathbb{E}[F(T,h)])$ converges in distribution as $N \to \infty$ to the centered Gaussian distribution with variance

$$\frac{h^4(1-q_0)^4}{2T^2(T^2 - (1-q_0))} \tag{5.28}$$

where

$$q_0 + h^2 = \frac{T^2 q_0}{(1-q_0)^2}. \tag{5.29}$$

The quantity q_0 has the interpretation as the overlap of two independent spins from the Gibbs measure involving the same disorder sample, i.e. the overlap of a spin with a replica. The formula (5.29) was predicted using the replica saddle point method in [13] (equation (4.5)) and [19] (equation (29) with $n = 0$).

Our result (5.27) above is for the SSK model when $\mathbf{g}$ is a Gaussian vector, but it extends to the case $\mathbf{g} = \mathbf{1}$. The only difference is that the variance of the limiting Gaussian distribution (5.27) changes to

$$\frac{h^4}{2}(s_2(\gamma_0) - (s_1(\gamma_0))^2). \tag{5.30}$$

Using the fact that $s_2(z) = \frac{s_1(z)^2}{1-s_1(z)^2}$ for $z > 2$, it is easy to check that (5.28) and (5.30) are same with q_0 and γ_0 related by the equation

$$q_0 = 1 - T s_1(\gamma_0). \tag{5.31}$$

5.4 Matching Between $h > 0$ and $h = 0$

We have considered three different regimes: (a) $h = 0$ and $T < 1$, (b) $h = 0$ and $T > 1$, and (c) $h = O(1)$. The order of the fluctuations of the free energy in these regimes are N^{-1}, $N^{-2/3}$, and $N^{-1/2}$, respectively. In these cases, the fluctuations are governed by the disorder variables given by (a) all eigenvalues $\lambda_1, \cdots, \lambda_N$, (b) the top eigenvalue λ_1, and (c) the combinations $n_i = \mathbf{u}_i \cdot \mathbf{g}$ of the eigenvectors and the external field. These differences indicate that there should be transitional regimes as $h \to 0$. We now study the limit $h \to 0$ of the result obtained for the case $h > 0$ and determine the transitional scaling of h heuristically by matching the order of the fluctuations. We need to consider the high temperature case and the low temperature case separately.

5.4.1 Asymptotic Property of γ_0

Throughout this paper, we will make use of following property of the leading term γ_0 of the critical point of $\mathcal{G}(z)$ when $h = O(1)$.

Lemma 5.7 *Let $\gamma_0 > 2$ be the solution of the Eq. (5.26), $1 - Ts_1(\gamma_0) - h^2 s_2(\gamma_0) = 0$. Then, as $h \to 0$,*

$$\gamma_0 = \begin{cases} T + T^{-1} + \frac{h^2}{T} + O(h^4) & \text{for } T > 1, \\ 2 + \frac{h^4}{4(1-T)^2} - \frac{h^6}{4(1-T)^4} + O(h^8) & \text{for } 0 < T < 1. \end{cases} \tag{5.32}$$

On the other hand, as $h \to \infty$,

$$\gamma_0 = h + \frac{T}{2} + O(h^{-1}) \qquad \text{for all } T > 0. \tag{5.33}$$

Proof Consider the limit of γ_0 as $h \to 0$. For $T > 1$, the equation for γ_0 becomes $1 - Ts_1(\gamma_0) = 0$ when $h = 0$, and its solution is $T + T^{-1}$. A simple perturbation argument applied to the equation for small h implies the result. For $0 < T < 1$, we use the asymptotics

$$s_2(z) = \frac{1}{2\sqrt{z-2}} + O(1) \quad \text{and} \quad s_1(z) = 1 + O(\sqrt{z-2}) \quad \text{as } z \to 2,$$

which follow from the formulas in (4.5). Then, the equation for γ_0 becomes

$$1 - T - \frac{h^2}{2\sqrt{\gamma_0 - 2}} + O(h^2) + O(\sqrt{\gamma_0 - 2}) = 0 \tag{5.34}$$

as $h \to 0$ and $\gamma_0 \to 2$. From this equation we find the result as $h \to 0$. The limit as $h \to \infty$ follows from $s_k(z) = z^{-k} + O(z^{-k-1})$ as $z \to \infty$. □

5.4.2 High Temperature Case, $T > 1$

From (5.32), we find that for $T > 1$, as $h \to 0$,

$$s_0(\gamma_0) = \frac{1}{2T^2} + \log T + \frac{h^2}{T^2} + \mathcal{O}\left(h^4\right), \quad s_1(\gamma_0) = \frac{1}{T} - \frac{h^2}{T(T^2-1)} + \mathcal{O}\left(h^4\right),$$
$$s_2(\gamma_0) = \frac{1}{T^2-1} - \frac{2T^2h^2}{(T^2-1)^3} + \mathcal{O}\left(h^4\right).$$

Inserting the formulas into (5.25),

$$F(T,h) = \frac{1}{2T} + \frac{h^2}{2T} - \frac{h^4}{4T(T^2-1)} + O(h^6). \tag{5.35}$$

Therefore, we find that if we first take $N \to \infty$ with fixed $h > 0$ and then let $h \to 0$, then

$$\mathcal{F}_N(T,h) \stackrel{\mathcal{D}}{\simeq} \left[\frac{1}{2T} + \frac{h^2}{2T} - \frac{h^4}{4T(T^2-1)}\right] + \frac{h^2}{\sqrt{2N(T^2-1)}}\mathcal{N}(0,1) \tag{5.36}$$

where the terms of orders h^6 and $h^4 N^{-1/2}$ have been dropped. The fluctuations are of order $\frac{h^2}{\sqrt{N}}$. On the other hand, when $h = 0$, the fluctuations are of order N^{-1} (see (5.10)). These two terms are of same order when $h \sim N^{-1/4}$.

5.4.3 Low Temperature Case, $T < 1$

Using the $T < 1$ case of (5.32), the leading term (5.25) becomes

$$F(T,h) = 1 - \frac{3T}{4} + \frac{T\log T}{2} + \frac{h^2}{2} - \frac{h^4}{8(1-T)} + O(h^6) \tag{5.37}$$

and the variance of the Gaussian distribution in (5.27) becomes $\frac{h^4 s_2(\gamma_0)}{2} = \frac{h^2(T-1)}{2} + O(h^4)$. Thus, from (5.27), for $T < 1$, we find that if we take $N \to \infty$ first and then take $h \to 0$, then

$$\mathcal{F}_N(T,h) \stackrel{\mathcal{D}}{\simeq} \left[1 - \frac{3T}{4} + \frac{T\log T}{2} + \frac{h^2}{2} - \frac{h^4}{8(1-T)}\right] + \frac{1}{\sqrt{N}}\mathcal{N}\left(0, \frac{h^2(1-T)}{2}\right) \tag{5.38}$$

where the terms of orders h^6 and $h^3 N^{-1/2}$ have been dropped. This implies that the fluctuations of the free energy are of order $\frac{h}{\sqrt{N}}$. On the other hand, when $h = 0$, the fluctuations are of order $N^{-2/3}$ (see (5.18)). These two terms are of same order when $h \sim N^{-1/6}$.

5.4.4 Summary

In summary, a heuristic matching computation suggests that the transitional scaling is

$$\begin{aligned} h &= O(N^{-1/4}) \quad \text{for } T > 1, \\ h &= O(N^{-1/6}) \quad \text{for } T < 1. \end{aligned} \tag{5.39}$$

In next two sections, we compute the fluctuations of the free energy in the above transitional regimes.

6 Free Energy for $T > 1$ and $h \sim N^{-1/4}$

6.1 Analysis

Assume that $T > 1$ and set

$$h = HN^{-1/4} \tag{6.1}$$

for fixed $H > 0$. In this case, using the notations (4.15) and (4.22),

$$\mathcal{G}(z) = \beta z - s_0(z) - \frac{\mathcal{L}_N(z)}{N} + \frac{H^2\beta}{\sqrt{N}}\left[s_1(z) + \frac{\mathcal{S}_N(z;1)}{\sqrt{N}}\right] + \mathcal{O}\left(N^{-3/2}\right) \tag{6.2}$$

where we recall that $\mathcal{L}_N(z)$ and $\mathcal{S}_N(z;1)$ are $\mathcal{O}(1)$ for $z > 2$. We approximate the function by $\mathcal{G}_0(z) = \beta z - s_0(z)$ and, as we discussed in Sect. 5.1.1, this function has the critical point $\gamma_0 = \beta + \beta^{-1}$ for $T > 1$. Applying a perturbation argument (see Appendix B) and using the formulas of $s_0(z)$ and $s_1(z)$, the critical point of $\mathcal{G}(z)$ is given by

$$\gamma = \gamma_0 + \mathcal{O}\left(N^{-1/2}\right) \qquad \text{with } \gamma_0 = \beta + \beta^{-1}. \tag{6.3}$$

Furthermore,

$$\begin{aligned}\mathcal{G}(\gamma) = &\frac{\beta^2}{2} + 1 + \log\beta + \frac{H^2\beta^2}{\sqrt{N}}\\ &+ \frac{1}{N}\left[-\frac{H^4\beta^4}{2(1-\beta^2)} + H^2\beta\mathcal{S}_N(\gamma_0;1) - \mathcal{L}_N(\gamma_0)\right] + \mathcal{O}\left(N^{-3/2}\right).\end{aligned} \tag{6.4}$$

Since

$$\mathcal{G}''(\gamma) = \frac{1}{N}\sum_{i=1}^N \frac{1}{(\gamma-\lambda_i)^2} + \frac{2H^2\beta}{N^{3/2}}\sum_{i=1}^N \frac{n_i^2}{(\gamma-\lambda_i)^3} \simeq s_2(\gamma) + \frac{2H^2\beta}{N^{1/2}}s_3(\gamma) \simeq s_2(\gamma_0) \tag{6.5}$$

and $\mathcal{G}^{(k)}(\gamma) = O(1)$ for all $k \geq 2$, the method of steepest descent implies that

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z \simeq \frac{\mathrm{i}}{N^{1/2}}\sqrt{\frac{4\pi}{s_2(\gamma_0)}} \asymp \mathcal{O}\left(N^{-1/2}\right). \tag{6.6}$$

Result 6.1 *For $h = HN^{-1/4}$ with fixed $H > 0$ and $T > 1$,*

$$\begin{aligned}\mathcal{F}_N(T,h) = &\frac{1}{4T} + \frac{H^2}{2T\sqrt{N}} + \frac{T}{2N}\\ &\times\left[\log(1-T^{-2}) - \frac{H^4}{2T^2(T^2-1)} + \frac{H^2}{T}\mathcal{S}_N(\gamma_0;1) - \mathcal{L}_N(\gamma_0)\right]\end{aligned} \tag{6.7}$$

plus $\mathcal{O}\left(N^{-3/2}\right)$, as $N \to \infty$ with high probability where $\mathcal{L}_N(z)$ and $\mathcal{S}_N(z;1)$ are defined in (4.15) and (4.22), respectively, and $\gamma_0 = \gamma_0(h=0) = T + T^{-1}$.

The sample random variables $\mathcal{S}_N(\gamma_0;1)$ and $\mathcal{L}_N(\gamma_0)$ both converge to Gaussian distributions. Since $\mathcal{S}_N(\gamma_0;1)$ depends only on n_i's and $\mathcal{L}_N(\gamma_0)$ depends only on λ_i's, these two random variables are independent. Therefore, we obtain the following result.

Result 6.2 *For $h = HN^{-1/4}$ and $T > 1$, as $N \to \infty$,*

$$\mathcal{F}_N(T,h) \overset{\mathcal{D}}{\simeq} \left[\frac{1}{4T} + \frac{H^2}{2T\sqrt{N}}\right] + \frac{T}{2N}\mathcal{N}(-\alpha, 4\alpha), \qquad \alpha := \frac{H^4}{2T^2(T^2-1)} - \frac{1}{2}\log(1-T^{-2}). \tag{6.8}$$

6.2 Matching with $h = 0$ and $h = O(1)$ Cases

If we set $H = 0$ in (6.7), we recover the result (5.10) for the case of $h = 0$. We now consider the limit $H \to \infty$. If we formally set $H = hN^{1/4}$ in (6.7) with h small but fixed and N large, then we have

$$\mathcal{F}_N(T, h) \simeq \frac{1}{4T} + \frac{h^2}{2T} - \frac{h^4}{4T(T^2-1)} + \frac{h^2}{2\sqrt{N}}\mathcal{S}_N(\gamma_0; 1) \tag{6.9}$$

for asymptotically almost every disorder sample. This is the same as (5.24) when $h \to 0$ since $F(T, h)$ satisfies (5.35) as $h \to 0$. Therefore, (6.7) matches well with both regimes.

7 Free Energy for $T < 1$ and $h \sim N^{-1/6}$

7.1 Analysis

Assume that $0 < T < 1$ and we set

$$h = HN^{-1/6} \tag{7.1}$$

for fixed $H > 0$. We find the critical point $\gamma > \lambda_1$. Previously we had $\gamma = \lambda_1 + \mathcal{O}\left(N^{-1}\right)$ when $h = 0$ and $\gamma = \lambda_1 + \mathcal{O}\left(1\right)$ when $h > 0$. For $h \sim N^{-1/6}$, we make the ansatz

$$\gamma = \lambda_1 + sN^{-2/3} \tag{7.2}$$

and find $s > 0$ assuming that $s = \mathcal{O}\left(1\right)$. From the equation $\mathcal{G}'(\gamma) = 0$, see (5.3), the equation of s is

$$\beta - \frac{1}{N^{1/3}}\sum_{i=1}^{N}\frac{1}{s + a_1 - a_i} - h^2\beta N^{1/3}\sum_{i=1}^{N}\frac{n_i^2}{(s + a_1 - a_i)^2} = 0 \tag{7.3}$$

where we recall $a_i = N^{2/3}(\lambda_i - 2)$. Here, we did not change h to $HN^{-1/6}$ since we will cite this equation in several places in the paper. From (4.31), the second sum converges with high probability. The first sum is $1 + \mathcal{O}\left(N^{-1/3}\right)$ from (4.30). Thus, with $h = HN^{-1/6}$ the equation becomes, under the assumption that $s = \mathcal{O}\left(1\right)$,

$$\beta - 1 - H^2\beta\sum_{i=1}^{N}\frac{n_i^2}{(a_1 + s - a_i)^2} + \mathcal{O}\left(N^{-1/3}\right) = 0. \tag{7.4}$$

Let t be the solution of the equation

$$\beta - 1 - H^2\beta\sum_{i=1}^{N}\frac{n_i^2}{(t + a_1 - a_i)^2} = 0, \qquad t > 0. \tag{7.5}$$

Using the rigidity, we can show that $t \asymp \mathcal{O}\left(1\right)$ with high probability. From this, comparing the equations for s and t, we find that

$$s = t + \mathcal{O}\left(N^{-1/3}\right). \tag{7.6}$$

which is consistent with the ansatz. The last equation can also be verified by checking the inequalities

$$\mathcal{G}'(\lambda_1 + tN^{-2/3}(1 - N^{-\varepsilon})) < 0 < \mathcal{G}'(\lambda_1 + tN^{-2/3}(1 + N^{-\varepsilon}))$$

for any $0 < \epsilon < 1/3$.

We now evaluate $G(\gamma)$ which is given by

$$\mathcal{G}(\gamma) = \beta\gamma - \frac{1}{N}\sum_{i=1}^{N}\log(\gamma - \lambda_i) + \frac{H^2\beta}{N^{4/3}}\sum_{i=1}^{N}\frac{n_i^2}{\gamma - \lambda_i}. \tag{7.7}$$

Insert $\gamma = \lambda_1 + sN^{-2/3} = 2 + (a_1 + s)N^{2/3}$. By (5.12), the sum involving the log function becomes

$$\frac{1}{N}\sum_{i=1}^{N}\log(\gamma - \lambda_i) = \frac{1}{2} + N^{-2/3}(a_1 + s) + \mathcal{O}\left(N^{-1}\right).$$

The other sum is equal to

$$\frac{H^2\beta}{N^{2/3}}\sum_{i=1}^{N}\frac{n_i^2}{a_1 + s - a_i} = \frac{H^2\beta}{N^{2/3}}\left(N^{1/3} + \mathcal{E}_N(s)\right)$$

using the random variable $\mathcal{E}_N(w)$ defined by (4.21), which is $\mathcal{O}(1)$ outside of a set whose probability shrinks to zero. Thus,

$$\mathcal{G}(\gamma) = 2\beta - \frac{1}{2} + \frac{H^2\beta}{N^{1/3}} + \frac{1}{N^{2/3}}\left[(\beta - 1)(a_1 + s) + H^2\beta\mathcal{E}_N(s)\right] + \mathcal{O}\left(N^{-1}\right). \tag{7.8}$$

To evaluate the integral in (5.2), we observe that for $k \geq 2$,

$$\begin{aligned}\frac{\mathcal{G}^{(k)}(\gamma)}{(-1)^k(k-1)!} &= N^{\frac{2k}{3}-1}\sum_{i=1}^{N}\frac{1}{(s + a_1 - a_i)^k} + kN^{\frac{2}{3}k-\frac{2}{3}}H^2\beta\sum_{i=1}^{N}\frac{n_i^2}{(s + a_1 - a_i)^{k+1}} \\ &= \mathcal{O}\left(N^{\frac{2}{3}k-\frac{2}{3}}\right).\end{aligned}$$

For $k = 2$, the leading term is

$$\mathcal{G}''(\gamma) = 2N^{2/3}H^2\beta\sum_{i=1}^{N}\frac{n_i^2}{(s + a_1 - a_i)^3} + \mathcal{O}\left(N^{1/3}\right). \tag{7.9}$$

Since $\mathcal{G}''(\gamma) \sim N^{2/3}$, the main contribution to the integral comes from a neighborhood of radius $N^{-5/6}$ near the critical point. By the Taylor series, for $u = O(1)$,

$$\begin{aligned}N\left(\mathcal{G}(\gamma + uN^{-5/6}) - \mathcal{G}(\gamma)\right) &= \sum_{k=2}^{\infty}\frac{N^{1-\frac{5}{6}k}}{k!}\mathcal{G}^{(k)}(\gamma)u^k \\ &= H^2\beta\left(\sum_{i=1}^{N}\frac{n_i^2}{(s + a_1 - a_i)^3}\right)u^2 + \mathcal{O}\left(N^{-5/6}\right)\end{aligned} \tag{7.10}$$

where all terms but $k = 2$ are $\mathcal{O}\left(N^{-5/6}\right)$. Thus, from the Gaussian integral approximation,

$$\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-G(\gamma))}\mathrm{d}z \simeq \frac{1}{N^{5/6}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} e^{H^2\beta\left(\sum_{i=1}^{N}\frac{n_i^2}{(s+a_1-a_i)^3}\right)u^2}\mathrm{d}u \asymp \mathcal{O}\left(N^{-5/6}\right). \tag{7.11}$$

Combining all together in (5.2) and replacing s by t, we obtain the following

Result 7.1 *For $h = HN^{-1/6}$ and $0 < T < 1$,*

$$\mathcal{F}_N = F_0(T,h) + \frac{\widetilde{\mathcal{F}}(T,H)}{N^{2/3}} + \mathcal{O}\left(N^{-1}\right), \qquad F_0(T,h) := 1 - \frac{3T}{4} + \frac{T\log T}{2} + \frac{h^2}{2}, \quad (7.12)$$

as $N \to \infty$ for asymptotically almost every disorder sample. Here,

$$\widetilde{\mathcal{F}}(T,H) = \frac{1}{2}(1-T)(t+a_1) + \frac{1}{2}H^2\mathcal{E}_N(t) \quad (7.13)$$

where $\mathcal{E}_N(z)$ is defined in (4.21) and t is the solution of the Eq. (7.5),

$$1 - T = H^2 \sum_{i=1}^{N} \frac{n_i^2}{(t + a_1 - a_i)^2}, \qquad t > 0. \quad (7.14)$$

The function $F_0(T,h)$ is equal to $F(T,h)$ of (5.25) if we set $\gamma_0 = 2$. The order of fluctuations is $N^{-2/3}$ as in the $h = 0$ case. But the fluctuations depend on all eigenvalues and $n_1, \cdots, n_N$. In contrast, when $h = 0$ they depend only on the largest eigenvalue. Using (4.28) for $\mathcal{E}_N(t)$, we obtain the next distributional result.

Result 7.2 *For $h = HN^{-1/6}$ and $0 < T < 1$,*

$$\mathcal{F}_N \overset{\mathcal{D}}{\simeq} F_0(T,h) + \frac{(1-T)(\varsigma + \alpha_1) + H^2\mathcal{E}(\varsigma)}{2N^{2/3}} \quad (7.15)$$

as $N \to \infty$, where

$$\mathcal{E}(w) = \lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{\nu_i^2}{w + \alpha_1 - \alpha_i} - \frac{1}{\pi}\int_0^{\left(\frac{3\pi n}{2}\right)^{2/3}} \frac{\mathrm{d}x}{\sqrt{x}}\right) \quad (7.16)$$

and ς is the solution of the equation

$$1 - T = H^2 \sum_{i=1}^{\infty} \frac{\nu_i^2}{(\varsigma + \alpha_1 - \alpha_i)^2}, \qquad \varsigma > 0, \quad (7.17)$$

where α_i is the GOE Airy point process and ν_i are independent standard normal sample random variables.

7.2 Asymptotic Behavior of the Scaled Limiting Critical Point *t*

The solution t of the Eq. (7.5),

$$1 - T - H^2 \sum_{i=1}^{N} \frac{n_i^2}{(t + a_1 - a_i)^2} = 0, \qquad t > 0, \quad (7.18)$$

is the scaled limiting critical point that is used in the result (7.12) above. We now describe the behavior of t as $H \to 0$ and $H \to \infty$. The following result is useful in the next two subsections and in two later sections.

Result 7.3 *The solution t of the Eq. (7.18) satisfies:*

$$t = \frac{|n_1|}{\sqrt{1-T}}H + O(H^2) \qquad \text{as } H \to 0 \quad (7.19)$$

and

$$\sqrt{t} \simeq \frac{H^2}{2(1-T)}\left[1+\frac{H^2\mathcal{S}_N\big(2+\frac{H^4N^{-2/3}}{4(1-T)^2};2\big)}{(1-T)N^{5/6}}\right] \quad \text{as } H\to\infty. \tag{7.20}$$

The second term inside the bracket of the equation (7.20) is $\mathcal{O}\left(H^{-3}\right)$.

For the $H \to 0$ limit, we see from the equation (7.18) that $t \to 0$ as $H \to 0$. If we set $t = yH$, then separating the term $i = 1$, the equation becomes $1 - T = \frac{n_1^2}{y^2} + O(H^2)$. Solving it, we obtain (7.19).

We now consider the large H behavior of t. We write the Eq. (7.18) as

$$\frac{1-T}{H^2} = \sum_{i=1}^{N}\frac{n_i^2}{(t+a_1-a_i)^2} = \frac{1}{N^{4/3}}\sum_{i=1}^{N}\frac{n_i^2}{(z-\lambda_i)^2}, \qquad z = 2+(t+a_1)N^{-2/3}. \tag{7.21}$$

Note that $t \to \infty$ as $H \to \infty$. We evaluate the leading term of the right-hand of the above equation when $z \to 2$ such that $z - 2 \gg N^{-2/3}$. The Eq. (4.32) when $k = 2$ is

$$\frac{1}{N}\sum_{i=1}^{N}\frac{n_i^2}{(z-\lambda_i)^2} = s_2(z) + \frac{\mathcal{S}_N(z;2)}{\sqrt{N}} + \mathcal{O}\left(N^{-1}\right)$$

for $z - 2 = O(1)$. We expect that this formula is still applicable to $z = 2 + (t + a_1)N^{-2/3}$ since $t \to \infty$. Since $z \to 2$, we have $s_2(z) \simeq \frac{1}{2\sqrt{z-2}}$ from (4.6). The Eq. (7.21) becomes

$$\frac{1-T}{H^2} \simeq \frac{1}{2N^{1/3}\sqrt{z-2}} + \frac{\mathcal{S}_N(z;2)}{N^{5/6}}. \tag{7.22}$$

The sample expectation of $\mathcal{S}_N(z; 2)$ with respect to n_is is 0 and the variance is

$$\mathbb{E}_s[\mathcal{S}_N(z;2)^2] = \frac{2}{N}\sum_{i=1}^{N}\frac{1}{(z-\widehat{\lambda}_i)^4} \simeq 2s_4(z) \simeq \frac{1}{8(z-2)^{5/2}}$$

from (4.6). Thus, we expect that $\mathcal{S}_N(z; 2) = \mathcal{O}\left((z-2)^{-5/4}\right)$ as $z \to 0$ and (7.22) becomes

$$\frac{1-T}{H^2} \simeq \frac{1}{2\sqrt{t}} + \frac{\mathcal{S}_N(2+tN^{-2/3};2)}{N^{5/6}} \simeq \frac{1}{2\sqrt{t}} + \mathcal{O}\left(t^{-5/4}\right).$$

Solving it gives $t \simeq \frac{H^4}{4(1-T)^2}$, the leading term of (7.20), as $H \to \infty$. Inserting it bask to the same equation, we obtain the next term and obtain (7.20). The last computation also shows that the second term in the bracket of (7.20) is $\mathcal{O}\left(H^2 t^{-5/4}\right) = \mathcal{O}\left(H^{-3}\right)$.

7.3 Matching with $h = 0$

We show that a formal limit (7.12) as $H \to 0$ agrees with (5.17) which is the result for $h = 0$. The leading term satisfies

$$F_0(T,h) = 1 - \frac{3T}{4} + \frac{T\log T}{2} + O(H^2N^{-1/3}). \tag{7.23}$$

For the subleading term (7.13), we use (7.19) for t and find that

$$\mathcal{E}_N(t) = \frac{n_1^2}{t} + \sum_{i=2}^{N}\frac{n_i^2}{t+a_1-a_i} - N^{1/3} = \frac{|n_1|\sqrt{1-T}}{H} + \mathcal{O}(1) \tag{7.24}$$

where the $\mathcal{O}(1)$ term follows from (4.30). Therefore, if we set $h = HN^{-1/6}$ and take the limits $N \to \infty$ first and $H \to 0$ second, then

$$\mathcal{F}_N(T, h) = 1 - \frac{3T}{4} + \frac{T \log T}{2} + \frac{1-T}{2N^{2/3}} a_1 + O\left(H^2 N^{-1/3}\right) + \mathcal{O}\left(HN^{2/3}\right) \tag{7.25}$$

for asymptotically almost every disorder sample. This agrees with result (5.17) obtained when $h = 0$.

We remark that the two subleading terms in (7.25) are comparable in size when $H = O(N^{-1/3})$, or equivalently when $h = O(N^{-1/2})$. This regime is not important for the computation of the free energy, but it will become important when we discuss the overlap of the spin variable with the external field in Sect. 8.6.

7.4 Matching with $h > 0$

We show that the formal limit of (7.12) as $H \to \infty$ is consistent with the result (5.25) for $h > 0$.

7.4.1 Large w Limit of $\mathcal{E}_N(w)$

We first consider the behavior of $\mathcal{E}_N(w)$, defined in (4.21), as $w \to \infty$ and then we insert $w = t$ which tends to ∞ from (7.20). This result is also used in other sections later.

Result 7.4 *As $w \to \infty$,*

$$\mathcal{E}_N(w) \simeq -\sqrt{w} + \frac{\mathcal{S}_N(W; 1)}{N^{1/6}} + \mathcal{O}\left(w^{-1/2}\right), \qquad W := 2 + wN^{-2/3}. \tag{7.26}$$

where $\mathcal{S}_N(z; k)$ is defined in (4.22).

Let $\widehat{a}_i := N^{2/3}(\widehat{\lambda}_i - 2)$ be the scaled classical location of the eigenvalues. Write

$$\mathcal{E}_N(w) = \sum_{i=1}^{N} \frac{n_i^2}{w - \widehat{a}_i} - N^{1/3} + \sum_{i=1}^{N} \frac{n_i^2 (a_i - \widehat{a}_i - a_1)}{(w + a_1 - a_i)(w - \widehat{a}_i)}. \tag{7.27}$$

Since $a_i \asymp -i^{2/3}$, we find that for any $\epsilon > 0$,

$$\sum_{i=1}^{N} \frac{1}{(w - a_i)^2} \leq \frac{1}{w^{1/2-\epsilon}} \sum_{i=1}^{N} \frac{1}{(w - a_i)^{3/2+\epsilon}} = \mathcal{O}\left(w^{-1/2}\right)$$

as $w \to \infty$. Thus, considering in a similar way, the last sum in (7.27) is $\mathcal{O}\left(w^{-1/2}\right)$ since $a_1 = \mathcal{O}(1)$, $a_i - \widehat{a}_i = \mathcal{O}(1)$, and $w \to \infty$. Setting $W = 2 + wN^{-2/3}$, (7.27) can be written as

$$\mathcal{E}_N(w) = N^{1/3} \left[\frac{1}{N} \sum_{i=1}^{N} \frac{1}{W - \widehat{\lambda}_i} - 1 \right] + \frac{\mathcal{S}_N(W; 1)}{N^{1/6}} + \mathcal{O}\left(w^{-1/2}\right).$$

From a formal application of the semicircle law,

$$\frac{1}{N} \sum_{i=1}^{N} \frac{1}{W - \widehat{\lambda}_i} \simeq s_1(W) = 1 - \sqrt{W - 2} + O(W - 2) = 1 - \frac{\sqrt{w}}{N^{1/3}} + O(wN^{-2/3}).$$

Thus, we obtain (7.26).

The Eqs. (7.20) and (7.26) imply the next result.

Result 7.5 *Let t be the solution of* (7.14). *Then, as* $H \to \infty$,

$$\mathcal{E}_N(t) \simeq -\frac{H^2}{2(1-T)} - \frac{H^4 \mathcal{S}_N(\Gamma_0; 2)}{2(1-T)^2 N^{5/6}} + \frac{\mathcal{S}_N(\Gamma_0; 1)}{N^{1/6}}, \qquad \Gamma_0 = 2 + \frac{H^4 N^{-2/3}}{4(1-T)^2}. \quad (7.28)$$

7.4.2 Large *H* Limit

From (7.28), we see that the $N^{-2/3}$ term in (7.12) satisfies

$$\frac{\widetilde{\mathcal{F}}(T, H)}{N^{2/3}} \simeq \frac{(1-T)a_1}{2N^{2/3}} - \frac{H^4}{8(1-T)N^{2/3}} + \frac{H^2 \mathcal{S}_N(\Gamma_0; 1)}{2N^{5/6}} \simeq -\frac{h^4}{8(1-T)} + \frac{h^2 \mathcal{S}_N(\Gamma_0; 1)}{2\sqrt{N}} \quad (7.29)$$

writing in terms of $h = HN^{-1/6}$. Thus, we find that if we take $h = HN^{-1/6}$ and $N \to \infty$ and then take $H \to \infty$, then

$$\mathcal{F}_N \simeq \left[1 - \frac{3T}{4} + \frac{T \log T}{2} + \frac{h^2}{2} - \frac{h^4}{8(1-T)}\right] + \frac{h^2 \mathcal{S}_N(\Gamma_0; 1)}{2\sqrt{N}}, \qquad \Gamma_0 = 2 + \frac{H^4 N^{-2/3}}{4(1-T)^2} \quad (7.30)$$

for asymptotically almost every disorder sample. The point Γ_0 is approximately equal to γ_0. The terms is the bracket are the same as the limit of $F(T, h)$ as $h \to 0$ in (5.37). The $O(N^{-1/2})$ term in (5.37) agrees with the last term of (7.30) since $\gamma_0 \simeq 2 + \frac{h^4}{4(1-T)^2} = \Gamma_0$ from (5.32). Hence, we find that the above formula is the same as the formal $h \to 0$ limit of the result (5.25), which was obtained by taking $N \to \infty$ first with $h = O(1)$ fixed. Hence, the result matches with the $h = O(1)$ regime.

The last term of (7.30) depends on the disorder sample. We consider its sample distribution and show that the sample distributions of the $h = HN^{-1/6}$ regime and $h > 0$ regime match for $0 < T < 1$. Using (4.29), we replace $\mathcal{S}_N(\Gamma_0; 1)$ by $\mathcal{N}(0; 2s_2(\Gamma_0))$. Using $s_2(z) \simeq \frac{1}{2\sqrt{z-2}}$ as $z \to 2$, we find that

$$\frac{h^2 \mathcal{S}_N(\Gamma_0; 1)}{2\sqrt{N}} \overset{\mathcal{D}}{\simeq} \frac{h\sqrt{1-T}}{\sqrt{2N}} \mathcal{N}(0, 1). \quad (7.31)$$

The right-hand side is same as the fluctuation term in (5.38), which shows the matching. This computation shows the matching of $h = HN^{-1/6}$ regime and $h > 0$ regime for $0 < T < 1$ in terms of the sample distribution as well.

7.5 Comparison with the Large Deviation Result of [19]

We now compare our results with the large deviation result of [19]. To this aim we first extend their calculation from $T = 0$ to any $0 < T < 1$, which is straightforward. Denoting by $\mathbb{E}_s$ the sample expectation, we find that

$$\mathbb{E}_s[\mathcal{Z}_N^n] = \mathbb{E}_s[e^{\beta N n \mathcal{F}_N}] \simeq e^{\beta N n F^0} e^{N 2^6 h^6 G(\frac{\beta n}{8h^2})} \quad (7.32)$$

where F^0 is the same as the terms in the bracket in (7.30), the sample-independent terms, and

$$G(x) = \frac{(1-T)^3}{3} x^3 + \frac{1-T}{4} x^2. \quad (7.33)$$

This formula is valid for fixed $T < 1$, n, and h to the leading order as $N \to \infty$ and in a second stage as $n, h \to 0$ so that $\frac{n}{h^2}$ is fixed. The full result for fixed n and h is in (94) and (95) of [19] and the above formula follows from it after changing $T \to 2T$, $\sigma \to 2h$, and $J_0 = 2$. Note that the term $e^{N2^6h^6G(\frac{n}{8Th^2})}$ is $O(1)$ when $h = O(N^{-1/6})$ and $n = O(h^2) = O(N^{-1/3})$. We have

$$N2^6h^6G\left(\frac{n}{8Th^2}\right) = \frac{N(1-T)^3n^3}{24T^3} + \frac{Nh^2(1-T)n^2}{4T^2}. \tag{7.34}$$

We compare the above formula with the one obtained using the result (7.12). From (7.12), we find that

$$\mathbb{E}_s[\mathcal{Z}_N^n] = \mathbb{E}_s[e^{\frac{Nn}{T}\mathcal{F}_N}] \simeq e^{\frac{Nn}{T}F_0(T,h)}\,\mathbb{E}_s[e^{\frac{N^{1/3}n}{T}\widetilde{\mathcal{F}}(T,H)}]. \tag{7.35}$$

Now we let $H \to \infty$. This term was computed in (7.29) in which we neglected the contribution from a_1. Including this term, using (7.31), and also noting that $\mathcal{S}_N(z;1)$ and a_1 are independent, we obtain

$$\mathbb{E}_s[e^{\frac{N^{1/3}n}{T}\widetilde{\mathcal{F}}(T,H)}] \simeq e^{-\frac{N^{1/3}nH^4}{8T(1-T)}}e^{\frac{N^{2/3}n^2H^4}{8T^2\sqrt{t}}}\,\mathbb{E}_s\left[e^{\frac{N^{1/3}n(1-T)}{2T}a_1}\right]. \tag{7.36}$$

We can replace $\sqrt{t} \simeq \frac{H^2}{2(1-T)}$ from (7.20) in the middle term. For the remaining expectation, using the right tail of the GOE Tracy Widom distribution $F_1(s) = \mathbb{P}(\alpha_1 < s) \sim \exp(-\frac{2}{3}s^{3/2})$,

$$\mathbb{E}[e^{\frac{N^{1/3}n(1-T)}{2T}a_1}] \simeq \int e^{\frac{N^{1/3}n(1-T)}{2T}a_1-\frac{2}{3}\alpha_1^{3/2}}\mathrm{d}\alpha_1 \simeq \exp\left(\frac{1}{3}\left(\frac{N^{1/3}n(1-T)}{2T}\right)^3\right). \tag{7.37}$$

Combining the calculations together, we find that

$$\mathbb{E}[\mathcal{Z}_N^n] \simeq e^{\frac{Nn}{T}F_0(T,h)}e^{-\frac{N^{1/3}nH^4}{8T(1-T)}}e^{\frac{N^{2/3}n^2H^2(1-T)}{4T^2}}e^{\frac{Nn^3(1-T)^3}{24T^3}}. \tag{7.38}$$

The exponents of the last two factors, upon writing $H = hN^{1/6}$ agree with (7.34). Since $F^0 = F_0(T,h) - \frac{h^4}{8(1-T)}$, we find that (7.38) is the same as (7.32). This shows that the tail of the typical fluctuations obtained here matches the large deviation tails at the exponential order.

8 Overlap with the External Field

The overlap of a spin with the external field is

$$\mathfrak{M} = \frac{\mathbf{g}\cdot\boldsymbol{\sigma}}{N}.$$

We study the thermal fluctuation of the overlap for a given disorder sample in several regimes: $h = O(1)$, $h \sim N^{-1/6}$ and $h \sim N^{-1/2}$. We also consider the magnetization, susceptibility, and differential susceptibility,

$$\mathcal{M} = \langle\mathfrak{M}\rangle, \qquad \mathcal{X} = \frac{\mathcal{M}}{h}, \qquad \mathcal{X}_d = \frac{\mathrm{d}\mathcal{M}}{\mathrm{d}h}.$$

8.1 Thermal Average from Free Energy

Before we discuss the thermal fluctuations of $\mathfrak{M}$, we first derive the thermal average, the magnetization, from the results for the free energy in two regimes, $h = O(1)$ and $h \sim N^{-1/6}$, using

$$\mathcal{M} = \langle\mathfrak{M}\rangle = \frac{\mathrm{d}\mathcal{F}_N}{\mathrm{d}h}. \tag{8.1}$$

Case $h = O(1)$:

For $h > 0$ and $T > 0$, the result (5.24) for the free energy implies that

$$\langle \mathfrak{M} \rangle = \frac{\mathrm{d}\mathcal{F}_N}{\mathrm{d}h} \simeq \frac{\mathrm{d}F(T,h)}{\mathrm{d}h} + \frac{1}{2\sqrt{N}} \frac{\mathrm{d}}{\mathrm{d}h}(h^2 \mathcal{S}_N(\gamma_0; 1)) \tag{8.2}$$

for asymptotically almost every disorder sample. Using $s_0'(z) = s_1(z)$ and $s_1'(z) = -s_2(z)$,

$$\frac{\mathrm{d}F(T,h)}{\mathrm{d}h} = h s_1(\gamma_0) + \frac{1}{2}(1 - T s_1(\gamma_0) - h^2 s_2(\gamma_0)) \frac{d\gamma_0}{\mathrm{d}h} \tag{8.3}$$

However, the equation for γ_0 implies that the second term is zero. On the other hand, since $\mathcal{S}_N'(z; 1) = -\mathcal{S}_N(z; 2)$,

$$\frac{\mathrm{d}}{\mathrm{d}h}(h^2 \mathcal{S}_N(\gamma_0; 1)) = 2h\mathcal{S}_N(\gamma_0; 1) - h^2 \mathcal{S}_N(\gamma_0; 2) \frac{d\gamma_0}{\mathrm{d}h}. \tag{8.4}$$

Using the equation for γ_0 and $s_2'(z) = -2s_3(z)$, we find that

$$\frac{d\gamma_0}{\mathrm{d}h} = \frac{2h s_2(\gamma_0)}{T s_2(\gamma_0) + 2h^2 s_3(\gamma_0)}. \tag{8.5}$$

Therefore, we conclude that, for fixed $h > 0$ and $T > 0$,

$$\langle \mathfrak{M} \rangle \simeq h s_1(\gamma_0) + \frac{1}{\sqrt{N}} \left[h\mathcal{S}_N(\gamma_0; 1) - \frac{h^3 s_2(\gamma_0)\mathcal{S}_N(\gamma_0; 2)}{T s_2(\gamma_0) + 2h^2 s_3(\gamma_0)} \right] \tag{8.6}$$

for asymptotically almost every disorder sample.

Case $h \sim N^{-1/6}$ and $T < 1$:

If we use the result (7.12) for the free energy when $h = HN^{-1/6}$ and $0 < T < 1$, we find that

$$\langle \mathfrak{M} \rangle = N^{1/6} \frac{\mathrm{d}\mathcal{F}_N}{\mathrm{d}H} \simeq h + \frac{H\mathcal{E}_N(t)}{\sqrt{N}} + \frac{\left(1 - T + H^2 \mathcal{E}_N'(t)\right)}{2\sqrt{N}} \frac{\mathrm{d}t}{\mathrm{d}H} \tag{8.7}$$

for asymptotically almost every disorder sample. The formula for $\mathcal{E}_N$ is given in (4.21) and

$$\mathcal{E}_N'(w) = -\sum_{i=1}^{N} \frac{n_i^2}{(w + a_1 - a_i)^2}. \tag{8.8}$$

Since t satisfies the Eq. (7.14), we see that the term $1 - T + H^2 \mathcal{E}_N'(t) = 0$. Hence, for $h = HN^{-1/6}$ and $0 < T < 1$,

$$\langle \mathfrak{M} \rangle \simeq h + \frac{H\mathcal{E}_N(t)}{\sqrt{N}} \tag{8.9}$$

for asymptotically almost every disorder sample.

In both of these regimes, it turns out that the thermal average is indeed the leading term. However, this calculation does not give us the thermal fluctuation term. To obtain that, we use the integral representation of the overlap in the following subsections. For the overlap and magnetization, it turns out that there is another interesting regime, $h \sim N^{-1/2}$, for $0 < T < 1$. This is the regime that occurs when the two terms in (8.9) have the same order; it was shown in (7.24) that $H\mathcal{E}_N(t) \simeq \mathcal{O}(1)$ as $H \to 0$. See the following subsections for the details.

8.2 Setup

We obtain the thermal probability of the overlap by considering the moment generating function $\langle e^{\beta\eta\mathfrak{M}}\rangle$ with respect to the Gibbs measure (1.2). Here, η is the variable for the generating function and we added β for the convenience of subsequent formulas. It turns out that the thermal fluctuations of $\mathfrak{M}$ are of order $N^{-1/2}$ in all regimes. Hence, we scale $\eta = \xi\sqrt{N}$ and use ξ as the scaled variable for the moment generating function. From Lemma 3.3, we have the following formula:

$$\langle e^{\beta\xi\sqrt{N}\mathfrak{M}}\rangle = e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})-\mathcal{G}(\gamma))} \frac{\int_{\gamma_{\mathfrak{M}}-\mathrm{i}\infty}^{\gamma_{\mathfrak{M}}+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z)-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))}\mathrm{d}z}{\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z} \tag{8.10}$$

where

$$\mathcal{G}_{\mathfrak{M}}(z) = \beta z - \frac{1}{N}\sum_{i=1}^{N}\log(z-\lambda_i) + \frac{(h+\frac{\xi}{\sqrt{N}})^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2}{z-\lambda_i}. \tag{8.11}$$

Here, we take $\gamma_{\mathfrak{M}} > \lambda_1$ to be the critical point of $\mathcal{G}_{\mathfrak{M}}(z)$ satisfying

$$\mathcal{G}'_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) = 0 \tag{8.12}$$

and we take $\gamma > \lambda_1$ to be the critical point of $\mathcal{G}(z)$. The only difference between $\mathcal{G}_{\mathfrak{M}}$ and $\mathcal{G}$, which we studied extensively in the previous sections, is that h is changed to $h + \xi N^{-1/2}$.

We record two formulas that we use below. From the explicit formulas for $\mathcal{G}_{\mathfrak{M}}$ and $\mathcal{G}$, the equation $\mathcal{G}'_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}'(\gamma) = 0$ implies that

$$\begin{aligned}(\gamma_{\mathfrak{M}}-\gamma)&\left[\frac{1}{N}\sum_{i=1}^{N}\frac{1}{(\gamma_{\mathfrak{M}}-\lambda_i)(\gamma-\lambda_i)} + \frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2(\gamma+\gamma_{\mathfrak{M}}-2\lambda_i)}{(\gamma_{\mathfrak{M}}-\lambda_i)^2(\gamma-\lambda_i)^2}\right]\\ &= \left(\frac{2\xi h}{N^{3/2}}+\frac{\xi^2}{N^2}\right)\beta\sum_{i=1}^{N}\frac{n_i^2}{(\gamma_{\mathfrak{M}}-\lambda_i)^2}.\end{aligned} \tag{8.13}$$

The other formula that we will need is

$$\begin{aligned}&N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})-\mathcal{G}(\gamma))\\ &= -\sum_{i=1}^{N}\left[\log\left(1+\frac{\gamma_{\mathfrak{M}}-\gamma}{\gamma-\lambda_i}\right) - \frac{\gamma_{\mathfrak{M}}-\gamma}{\gamma-\lambda_i}\right] + h^2\beta\sum_{i=1}^{N}\frac{n_i^2(\gamma_{\mathfrak{M}}-\gamma)^2}{(\gamma_{\mathfrak{M}}-\lambda_i)(\gamma-\lambda_i)^2}\\ &\quad + \left(\frac{2\xi h}{\sqrt{N}}+\frac{\xi^2}{N}\right)\beta\sum_{i=1}^{N}\frac{n_i^2}{\gamma_{\mathfrak{M}}-\lambda_i} =: A_1 + A_2 + A_3,\end{aligned} \tag{8.14}$$

which can be seen using $\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma) = \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma) - \mathcal{G}'(\gamma)(\gamma_{\mathfrak{M}}-\gamma)$.

8.3 Positive External Field: $h = O(1)$

8.3.1 Analysis

Fix $h > 0$. The critical point γ of $\mathcal{G}(z)$ is evaluated in Sect. 5.2. It is shown in (5.20) that

$$\gamma = \gamma_0 + \gamma_1 N^{-1/2} + \mathcal{O}\left(N^{-1}\right)$$

where γ_0 and γ_1 are deterministic functions of h and T. From the formulas for $\mathcal{G}$ and $\mathcal{G}_{\mathfrak{M}}$, we see that $\mathcal{G}'_{\mathfrak{M}}(z) = \mathcal{G}'(z) + \mathcal{O}\left(N^{-1/2}\right)$ for $z > \lambda_1 + O(1)$ (cf. (4.32)). This implies that $\gamma_{\mathfrak{M}} - \gamma = \mathcal{O}\left(N^{-1/2}\right)$. We need to evaluate the difference precisely. From (8.13), we find, using the semicircle law, that

$$(\gamma_{\mathfrak{M}} - \gamma)\left(s_2(\gamma) + 2h^2\beta s_3(\gamma) + \mathcal{O}\left(N^{-1/2}\right)\right) = \frac{2\xi h\beta}{\sqrt{N}} s_2(\gamma) + \mathcal{O}\left(N^{-1}\right).$$

Thus,

$$\gamma_{\mathfrak{M}} = \gamma + \Delta N^{-1/2}, \qquad \Delta = \frac{2h\beta\xi s_2(\gamma_0)}{s_2(\gamma_0) + 2h^2\beta s_3(\gamma_0)} + \mathcal{O}\left(N^{-1/2}\right). \tag{8.15}$$

We now evaluate $N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma))$ for (8.10) via the Eq. (8.14). Using the Taylor expansion of the logarithm function,

$$A_1 = \frac{\Delta^2}{2N}\sum_{i=1}^{N}\frac{1}{(\gamma-\lambda_i)^2} + \mathcal{O}\left(\frac{1}{N^{3/2}}\sum_{i=1}^{N}\frac{1}{(\gamma-\lambda_i)^3}\right) = \frac{\Delta^2 s_2(\gamma)}{2} + \mathcal{O}\left(N^{-1/2}\right). \tag{8.16}$$

Similarly,

$$A_2 = \frac{h^2\beta\Delta^2}{N}\sum_{i=1}^{N}\frac{n_i^2}{(\gamma-\lambda_i)^3} + \mathcal{O}\left(N^{-1/2}\right) = h^2\beta\Delta^2 s_3(\gamma) + \mathcal{O}\left(N^{-1/2}\right). \tag{8.17}$$

In these two equations, we replaced $\gamma_{\mathfrak{M}}$ by γ . For A_3, using (8.15) and the notation (4.22), we have

$$\begin{aligned} A_3 &= 2\xi h\beta(s_1(\gamma_{\mathfrak{M}})\sqrt{N} + \mathcal{S}_N(\gamma_{\mathfrak{M}};1)) + \xi^2\beta s_1(\gamma_{\mathfrak{M}}) + \mathcal{O}\left(N^{-1/2}\right) \\ &= 2\xi h\beta s_1(\gamma)\sqrt{N} + \left[2\xi h(\mathcal{S}_N(\gamma;1) - s_2(\gamma)\Delta) + \xi^2 s_1(\gamma)\right]\beta + \mathcal{O}\left(N^{-1/2}\right). \end{aligned} \tag{8.18}$$

Combining the three terms and inserting the formulas of γ and Δ,

$$\begin{aligned} N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma)) =& 2\xi h\beta\left[\sqrt{N}s_1(\gamma_0) - s_2(\gamma_0)\gamma_1 + \mathcal{S}_N(\gamma_0;1)\right] \\ &+ \xi^2\left[\beta s_1(\gamma_0) - \frac{2h^2\beta^2 s_2(\gamma_0)^2}{s_2(\gamma_0) + 2h^2\beta s_3(\gamma_0)}\right] + \mathcal{O}\left(N^{-1/2}\right). \end{aligned} \tag{8.19}$$

Now we consider the integrals in (8.10). Since $\mathcal{G}^{(k)}(\gamma) = \mathcal{O}(1)$ for all $k \geq 2$, the method of steepest descent applies. It is also straightforward to check that

$$\mathcal{G}''_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) = \mathcal{G}''(\gamma_{\mathfrak{M}}) + \mathcal{O}\left(N^{-1/2}\right) = \mathcal{G}''(\gamma) + \mathcal{O}\left(N^{-1/2}\right).$$

Hence,

$$\frac{\int_{\gamma_{\mathfrak{M}}-\mathrm{i}\infty}^{\gamma_{\mathfrak{M}}+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))}\mathrm{d}z}{\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z} \simeq \sqrt{\frac{\mathcal{G}''(\gamma)}{\mathcal{G}''_{\mathfrak{M}}(\gamma_{\mathfrak{M}})}} \simeq 1.$$

Inserting the above computations into (8.10), moving the term involving $\sqrt{N}$ to the left, replacing $\beta\xi$ by ξ, using $\beta = 1/T$, and inserting the formula (5.21) for γ_1, we obtain the following.

Result 8.1 *For $h = O(1)$ and $T > 0$,*

$$\langle e^{\xi\sqrt{N}(\mathfrak{M} - hs_1(\gamma_0))}\rangle \simeq e^{\xi h\left[\mathcal{S}_N(\gamma_0;1) - \frac{h^3 s_2(\gamma_0)\mathcal{S}_N(\gamma_0;2)}{Ts_2(\gamma_0)+2h^2 s_3(\gamma_0)}\right] + \frac{\xi^2}{2}\left[Ts_1(\gamma_0) - \frac{2Th^2 s_2(\gamma_0)^2}{2Ts_2(\gamma_0)+h^2 s_3(\gamma_0)}\right]} \tag{8.20}$$

as $N \to \infty$ for asymptotically almost every disorder sample, where $\gamma_0 > 2$ is the solution of the Eq. (5.19) and $\mathcal{S}_N(z; k)$ is defined in (4.22).

Since the right-hand side is an exponential of a quadratic function of ξ, we obtain the following distributional result.

Result 8.2 *For $h = O(1)$ and $T > 0$,*

$$\mathfrak{M} \overset{\mathfrak{D}}{\simeq} hs_1(\gamma_0) + \frac{1}{\sqrt{N}} \left[h\mathcal{S}_N(\gamma_0; 1) - \frac{h^3 s_2(\gamma_0)\mathcal{S}_N(\gamma_0; 2)}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)} + \sigma_{\mathfrak{M}} \mathfrak{N} \right] \tag{8.21}$$

as $N \to \infty$ for asymptotically almost every disorder sample. The thermal random variable $\mathfrak{N}$ is a standard normal random variable and the term $\sigma_{\mathfrak{M}} > 0$ is given by the formula

$$\sigma_{\mathfrak{M}}^2 = Ts_1(\gamma_0) - \frac{2Th^2 s_2(\gamma_0)^2}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)}. \tag{8.22}$$

The thermal average is given by the first three terms and they agree with the formula (8.6) obtained from the free energy.

8.3.2 Discussion on the Leading Term

The leading term

$$\mathfrak{M}^0(h, T) := hs_1(\gamma_0(h)) \tag{8.23}$$

in (8.21) is deterministic. See Fig. 3a for a graph as a functions of h. The function $\mathfrak{M}^0$ satisfies the following properties:

- For every $T > 0$, $\mathfrak{M}^0(h, T)$ is an increasing function of h.
- As $h \to \infty$,

$$\mathfrak{M}^0(h, T) = 1 - \frac{T}{2h} + O(h^{-2}) \quad \text{for all } T > 0. \tag{8.24}$$

- As $h \to 0$,

$$\mathfrak{M}^0(h, T) \simeq \begin{cases} \frac{h}{T} - \frac{h^3}{T(T^2-1)} & \text{for } T > 1, \\ h - \frac{h^3}{2(1-T)} & \text{for } 0 < T < 1. \end{cases} \tag{8.25}$$

The first property is consistent with the intuition that the overlap of the spin with the external field becomes larger as the external field becomes stronger. The proof follows from

$$\frac{\mathrm{d}}{\mathrm{d}h}\mathfrak{M}^0 = s_1(\gamma_0) - hs_2(\gamma_0)\gamma_0' = \frac{Ts_1(\gamma_0)s_2(\gamma_0) + 2h^2(s_1(\gamma_0)s_3(\gamma_0) - s_2(\gamma_0)^2)}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)} \tag{8.26}$$

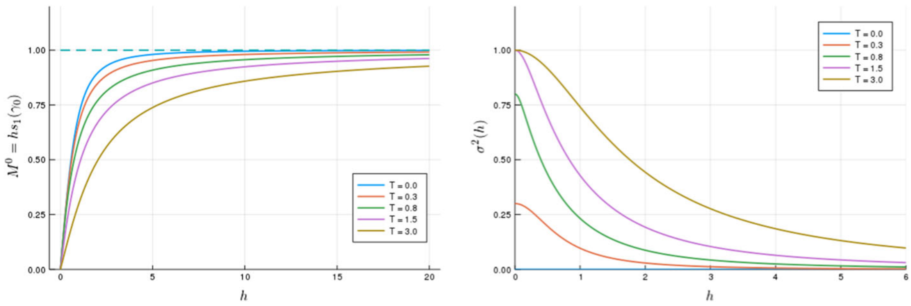
and from checking that $s_1(z)s_3(z) - s_2(z)^2 > 0$ for all $z > 2$ using (4.5). The large h and small h limits follow from Lemma 5.7.

8.3.3 Discussion on the Variance

The variance of the overlap satisfies

$$\langle \mathfrak{M}^2 \rangle - \langle \mathfrak{M} \rangle^2 \simeq \frac{\sigma_{\mathfrak{M}}^2}{N}. \tag{8.27}$$

The term $\sigma_{\mathfrak{M}}^2(h, T) = \sigma_{\mathfrak{M}}^2$ is given in (8.22) and does not depend on the disorder sample. See Fig. 3 for the graph. Here are some properties of $\sigma_{\mathfrak{M}}^2$.



(a) $\mathfrak{M}^0$ **(b)** $\sigma^2_{\mathfrak{M}}$

Fig. 3 Graph of $\mathfrak{M}^0$ and $\sigma^2_{\mathfrak{M}}$ as function of h for various values of T

- For every T, $\sigma^2_{\mathfrak{M}}(h, T)$ is a decreasing function.
- As $h \to \infty$,

$$\sigma^2_{\mathfrak{M}} = \frac{T}{h} + O(h^{-2}) \quad \text{for all } T > 0. \tag{8.28}$$

- As $h \to 0$,

$$\sigma^2_{\mathfrak{M}} \to \begin{cases} 1 & \text{for } T > 1, \\ T & \text{for } 0 < T < 1. \end{cases} \tag{8.29}$$

The first property follows from

$$\frac{\mathrm{d}}{\mathrm{d}h}\sigma^2_{\mathfrak{M}} = -\frac{T^2 s_2(\gamma_0)\left[\left(T s_2(\gamma_0)^2 - 12h^4 s_3(\gamma_0)^2 + 12h^4 s_2(\gamma_0)s_4(\gamma_0)\right)\gamma_0'(h) + 4hTs_2(\gamma_0)^2\right]}{(T s_2(\gamma_0) + 2h^2 s_3(\gamma_0))^2} \tag{8.30}$$

by checking that $s_2(z)s_4(z) - s_3(z)^2 > 0$ for all $z > 2$. The large and small h limits follow from Lemma 5.7.

8.3.4 Limit as $h \to \infty$

Consider the formal limit of the result (8.21) as $h \to \infty$. Using (5.33), we have

$$h^k \mathcal{S}_N(\gamma_0; k) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{n_i^2 - 1}{(\frac{\gamma_0}{h} - \frac{\widehat{\lambda}_i}{h})^k} \simeq \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (n_i^2 - 1) \tag{8.31}$$

and $s_k(\gamma_0) \simeq h^{-k}$ as $h \to \infty$. Therefore, using (8.24) and (8.28), we find that if we take $N \to \infty$ with $h > 0$ and then take $h \to \infty$, we get

$$\mathfrak{M} \overset{\mathfrak{D}}{\simeq} 1 - \frac{T}{2h} + \frac{1}{\sqrt{N}} \left[\frac{\sum_{i=1}^{N}(n_i^2 - 1)}{2\sqrt{N}} + \frac{\sqrt{T}}{\sqrt{h}} \mathfrak{N} \right]. \tag{8.32}$$

The leading term $\mathfrak{M} \simeq 1$ is trivial since the spin is likely to be pulled to the direction of the external field if h is large.

8.3.5 Limit as $h \to 0$ When $T > 1$

Since $\gamma_0 \to T + T^{-1}$ as $h \to 0$ for $T > 1$ from (5.32), the terms $\mathcal{S}_N(\gamma_0; 1)$ and $\mathcal{S}_N(\gamma_0; 2)$ remain $O(1)$. Hence the deterministic terms in the square brackets in (8.21) converge to zero as $h \to 0$. We thus find, using (8.25) and (8.29) that, if we take the limit $N \to \infty$ with $h > 0$ and then take $h \to 0$, the result for $T > 1$ becomes

$$\mathfrak{M} \overset{\mathfrak{D}}{\simeq} \frac{h}{T} - \frac{h^3}{T(T^2-1)} + \frac{1}{\sqrt{N}}\left[\mathfrak{N} + h\mathcal{S}_N(T + \frac{1}{T}; 1)\right]. \tag{8.33}$$

8.3.6 Limit as $h \to 0$ When $T < 1$

The small h limit (5.32) of γ_0 and the limit of $s_k(z)$ as $z \to 2$ obtained in (4.6) imply that, as $h \to 0$,

$$s_2(\gamma_0) \simeq \frac{(1-T)}{h^2} + \frac{T}{2(1-T)}, \qquad s_3(\gamma_0) \simeq \frac{(1-T)^3}{h^6}, \qquad s_4(\gamma_0) \simeq \frac{2(1-T)^5}{h^{10}} \tag{8.34}$$

when $0 < T < 1$. From these, we see that

$$h\mathcal{S}_N(\gamma_0; 1) - \frac{h^3 s_2(\gamma_0)\mathcal{S}_N(\gamma_0; 2)}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)} \simeq h\mathcal{S}_N(\gamma_0; 1) - \frac{h^5\mathcal{S}_N(\gamma_0; 2)}{2(1-T)^2}. \tag{8.35}$$

Thus, by (8.25) and (8.29), if take the limit $N \to \infty$ with $h > 0$ and then take $h \to 0$, then

$$\mathfrak{M} \overset{\mathfrak{D}}{\simeq} h - \frac{h^3}{2(1-T)} + \frac{1}{\sqrt{N}}\left[h\mathcal{S}_N(\gamma_0; 1) - \frac{h^5\mathcal{S}_N(\gamma_0; 2)}{2(1-T)^2} + \sqrt{T}\mathfrak{N}\right], \quad \gamma_0 \simeq 2 + \frac{h^4}{4(1-T)^2}. \tag{8.36}$$

Finally, we consider the terms $h\mathcal{S}_N(\gamma_0; 1)$ and $h^5\mathcal{S}_N(\gamma_0; 2)$. The sample-to-sample variance of $\mathcal{S}_N(\gamma_0; k)$ is

$$\frac{2}{N}\sum_{i=1}^{N} \frac{1}{(\gamma_0 - \widehat{\lambda}_i)^{2k}} \simeq 2s_{2k}(\gamma_0), \tag{8.37}$$

which is expected to hold for $\gamma_0 - 2 \gg N^{-2/3}$, i.e., $h \gg N^{-1/6}$, Thus the sample-to-sample variance is $\mathcal{O}\left(h^{-2}\right)$ for $k = 1$ and $\mathcal{O}\left(h^{-10}\right)$ for $k = 2$ from (8.34). Hence, we expect that $h\mathcal{S}_N(\gamma_0; 1)$ and $h^5\mathcal{S}_N(\gamma_0; 2)$ are $\mathcal{O}(1)$ for $h \gg N^{-1/6}$.

8.4 No External Field: $h = 0$

When $h = 0$ and $T > 1$, it is well-known in spin glass theory [33,34] that the overlap of two independently chosen spins are asymptotically orthogonal, indicating that the spin variable becomes uniformly distributed on the sphere $\|\boldsymbol{\sigma}\| = \sqrt{N}$ as $N \to \infty$. For $h = 0$ the Gibbs measure is independent of $\mathbf{g}$. Hence, the overlap $\mathfrak{M} = \frac{1}{N}\mathbf{g} \cdot \boldsymbol{\sigma}$ of the spin with the random Gaussian vector $\boldsymbol{\sigma}$ is the cosine of the angle of two independent vectors which are chosen more or less uniformly at random from the sphere. Thus, we expect that $\mathfrak{M}$ is approximately $\frac{1}{\sqrt{N}}$ times a standard normal distribution. The formal limit of (8.33) as $h \to 0$ coincides with this result. Indeed when $T > 1$, the analysis for $h > 0$ with $h = O(1)$ extends to $h \geq 0$ and (8.21) holds.

When $h = 0$ and $T < 1$, it was argued in [24] that $\frac{\langle|\mathbf{u}_1 \cdot \boldsymbol{\sigma}|\rangle}{\sqrt{N}}$ converges to $\sqrt{1-T}$. (In [24], the authors claim that $\frac{\langle \mathbf{u}_1 \cdot \boldsymbol{\sigma}\rangle}{\sqrt{N}} \to \sqrt{1-T}$, but this seems to be a typographical error since $\langle \mathbf{u}_1 \cdot \boldsymbol{\sigma}\rangle = 0$ due to the symmetry of the Gibbs measure under the transformation $\boldsymbol{\sigma} \mapsto -\boldsymbol{\sigma}$.) It was also proven in [34] that the absolute value of the overlap of two independently chosen spins converges to $1-T$. Hence, a spin variable may be written as $\frac{\boldsymbol{\sigma}}{\sqrt{N}} = \pm\sqrt{1-T}\mathbf{u}_1 + \sqrt{T}\mathbf{v}$, where the unit vector $\mathbf{v}$ is taken uniformly at random from the hyperplane perpendicular to $\mathbf{u}_1$ and the signs $\pm$ are each taken with probability $1/2$: See more discussions on such decomposition of the spin variable in Sect. 11. Thus, using the notation $n_1 = \mathbf{u}_1 \cdot \mathbf{g}$, we expect that $\mathfrak{M} \simeq \frac{\pm n_1 \sqrt{1-T}+\sqrt{T}\mathfrak{N}}{\sqrt{N}}$. Recall that $\mathbf{u}_1$ has the sign ambiguity and hence n_1 is defined up to its sign. Thus, we find the following result for $h = 0$.

Result 8.3 *For $h = 0$,*

$$\mathfrak{M} \simeq \begin{cases} \frac{1}{\sqrt{N}}\mathfrak{N} & \text{for } T >, \\ \frac{|n_1|\sqrt{1-T}\mathfrak{B}+\sqrt{T}\mathfrak{N}}{\sqrt{N}} & \text{for } 0 < T < 1 \end{cases} \tag{8.38}$$

as $N \to \infty$, for asymptotically almost every disorder sample, where $\mathfrak{N}$ is a standard normal random variable, and $\mathfrak{B}$ is independent of $\mathfrak{N}$ and has the distribution $\mathbb{P}(\mathfrak{B} = 1) = \mathbb{P}(\mathfrak{B} = -1) = \frac{1}{2}$.

The right-hand side of (8.36) involves the thermal random variable $\mathfrak{N}$ but does not involve the other thermal random variable $\mathfrak{B}$ in (8.38). Hence, the formal limit of (8.36) as $h \to 0$ is not equal to (8.38) when $T < 1$. This implies that there should be a transitional regime. It turns out that there are two transitional regimes, $h \sim N^{-1/6}$ and $h \sim N^{-1/2}$. The first regime can be expected, since $\gamma_0 = 2 + O(h^4)$ as $h \to 0$, and the subleading term $O(h^4)$ is of same order as the fluctuations of the top eigenvalue λ_1 when $h \sim N^{-1/6}$. This is the same transitional regime that was observed for the free energy. The second regime $h \sim N^{-1/2}$ arises because the ratio of the integrals in (8.10), which was approximately equal to 1 when $h > 0$ (and when $h \sim N^{-1/6}$ as well), is no longer close to 1 when $h \sim N^{-1/2}$. This will be responsible for the appearance of $\mathfrak{B}$. We discuss these two transitional regimes in the next subsections. We will see in Sect. 8.6 that the result for $h = HN^{-1/2}$ actually holds even when $H = 0$, implying that (8.38) indeed holds.

8.5 Mesoscopic External Field: $h \sim N^{-1/6}$ and $T < 1$

8.5.1 Analysis

We scale h as

$$h = HN^{-1/6}$$

for fixed $H > 0$. This scale is the same as the one considered in Sect. 7.1. We showed in that section that the critical point of $\mathcal{G}(z)$ is $\gamma = \lambda_1 + sN^{-2/3}$ where $s > 0$ satisfies the Eq. (7.4). To find the critical point of $\mathcal{G}_{\mathfrak{M}}(z)$, we make the ansatz that $\gamma_{\mathfrak{M}} \simeq \gamma$, Then, the Eq. (8.13) becomes

$$(\gamma_{\mathfrak{M}} - \gamma)\left[N^{1/3} \sum_{i=1}^{N} \frac{1}{(s + a_1 - a_i)^2} + H^2 \beta N^{2/3} \sum_{i=1}^{N} \frac{2n_i^2}{(s + a_1 - a_i)^3} \right]$$
$$\simeq \left(\frac{2\xi H}{N^{5/3}} + \frac{\xi^2}{N^2} \right) \beta N^{4/3} \sum_{i=1}^{N} \frac{n_i^2}{(s + a_1 - a_i)^2},$$

implying that

$$\gamma_{\mathfrak{M}} - \gamma = \mathcal{O}\left(N^{-1}\right), \tag{8.39}$$

which is consistent with the ansatz. We do not need to determine the $\mathcal{O}\left(N^{-1}\right)$ term in this subsection.

We now evaluate $N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma))$ using (8.14). From the Taylor series of the log function,

$$A_1 \simeq \sum_{i=1}^{N} \frac{(\gamma_{\mathfrak{M}} - \gamma)^2}{(\gamma - \lambda_i)^2} = \sum_{i=1}^{N} \frac{(\gamma_{\mathfrak{M}} - \gamma)^2 N^{4/3}}{(s + a_1 - a_i)^2} = \mathcal{O}\left(N^{-2/3}\right).$$

Inserting $h = HN^{-1/6}$,

$$A_2 \simeq \frac{H^2\beta}{N^{1/3}} \left[N^2 \sum_{i=1}^{N} \frac{n_i^2}{(s + a_1 - a_i)^2} \right] (\gamma_{\mathfrak{M}} - \gamma)^2 = \mathcal{O}\left(N^{-1/3}\right).$$

The third term is

$$A_3 = \left(2\xi H + \frac{\xi^2}{N^{1/3}}\right) \beta \left[\sum_{i=1}^{N} \frac{n_i^2}{s + a_1 - a_i} + \mathcal{O}\left(N^{-1/3}\right) \right].$$

Using the random variable $\mathcal{E}_N(s)$ defined in (4.21), which is $\mathcal{O}(1)$, and combining all three terms,

$$N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma)) = 2\xi H\beta N^{1/3} + 2\beta\xi H\mathcal{E}_N(s) + \beta\xi^2 + \mathcal{O}\left(N^{-1/3}\right). \tag{8.40}$$

Finally, consider the integrals in (8.10). The denominator is computed in Section 7.1. The numerator can be computed in the same manner. Indeed, we can check, as with the denominator, that $\mathcal{G}_{\mathfrak{M}}^{(k)}(\gamma_{\mathfrak{M}}) = \mathcal{O}\left(N^{\frac{2}{3}k - \frac{2}{3}}\right)$ for all $k \geq 2$ and

$$\mathcal{G}_{\mathfrak{M}}''(\gamma_{\mathfrak{M}}) = 2N^{2/3}H^2\beta \sum_{i=1}^{N} \frac{n_i^2}{(s + a_1 - a_i)^3} + \mathcal{O}\left(N^{1/2}\right), \tag{8.41}$$

which is the same as the denominator. Hence, the Gaussian integral approximations of the integrals imply that

$$\frac{\int_{\gamma_{\mathfrak{M}} - \mathrm{i}\infty}^{\gamma_{\mathfrak{M}} + \mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z) - \mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))} \mathrm{d}z}{\int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z) - \mathcal{G}(\gamma))} \mathrm{d}z} \simeq \sqrt{\frac{\mathcal{G}''(\gamma)}{\mathcal{G}_{\mathfrak{M}}''(\gamma_{\mathfrak{M}})}} \simeq 1. \tag{8.42}$$

Combining the above computations into (8.10), replacing s by t (the solution to (7.5)), replacing $\beta\xi$ by ξ, and using $1/\beta = T$, we obtain the following result.

Result 8.4 *For $h = HN^{-1/6}$ and $0 < T < 1$,*

$$\langle e^{\xi\sqrt{N}(\mathfrak{M} - h)} \rangle \simeq e^{\xi H\mathcal{E}_N(t) + \frac{T\xi^2}{2}}, \qquad \mathcal{E}_N(t) := \sum_{i=1}^{N} \frac{n_i^2}{t + a_1 - a_i} - N^{1/3}, \tag{8.43}$$

as $N \to \infty$ for asymptotically almost every disorder sample, where $t > 0$ is the solution of the equation (7.14).

Since the exponent of the right-hand side of (8.43) is a quadratic function of ξ, we obtain

Result 8.5 *For $h = HN^{-1/6}$ and $0 < T < 1$,*

$$\mathfrak{M} \overset{\mathfrak{D}}{\simeq} h + \frac{1}{\sqrt{N}} \left[H\mathcal{E}_N(t) + \sqrt{T}\mathfrak{N} \right] \tag{8.44}$$

as $N \to \infty$ for asymptotically almost every disorder sample, where the thermal random variable $\mathfrak{N}$ has the standard Gaussian distribution.

The thermal average is obtained from the first two terms. The average is the same as (8.9) that we obtained from the free energy.

8.5.2 Matching with $h = O(1)$

We take the formal limit $H \to \infty$ of (8.44). The limit of $\mathcal{E}_N(t)$ as $H \to \infty$ is obtained in (7.28). From this, we find that, if we take $h = HN^{-1/6}$ and let $N \to \infty$ first and then take $H \to \infty$, then

$$\mathfrak{M} \overset{\mathcal{D}}{\simeq} h - \frac{h^3}{2(1-T)} + \frac{1}{\sqrt{N}}\left[h\mathcal{S}_N(\gamma_0; 1) - \frac{h^5 \mathcal{S}_N(\gamma_0; 2)}{2(1-T)^2} + \sqrt{T}\mathfrak{N} \right] \tag{8.45}$$

as $H \to \infty$ where $\gamma_0 \simeq 2 + \frac{h^4}{4(1-T)^2}$. This result agrees with (8.36), which is obtained by taking $h > 0$ fixed and letting $N \to \infty$ first and then taking $h \to 0$.

8.5.3 Formal Limit as $H \to 0$

We take the formal limit $H \to 0$ of (8.44). We obtained the limit of $\mathcal{E}_N(t)$ as $H \to 0$ in (7.24). Hence, we find that, if we take $N \to \infty$ with $h = HN^{-1/6}$ first and then take $H \to 0$, then

$$\mathfrak{M} \overset{\mathcal{D}}{\simeq} h + \frac{1}{\sqrt{N}}\left[|n_1|\sqrt{1-T} + \sqrt{T}\mathfrak{N} \right]. \tag{8.46}$$

This formula evaluated at $H = 0$ is different from (8.38). In particular, the Bernoulli random variable $\mathfrak{B}(1/2)$ is missing. In the next subsection, we consider a new regime $h = O(N^{-1/2})$ in which the two terms in (8.46) are of the same order. We will show that this new regime interpolates between $h = O(N^{-1/6})$ and $h = 0$.

8.6 Microscopic External Field: $h \sim N^{-1/2}$ and $T < 1$

8.6.1 Analysis

We set, for fixed $H > 0$,

$$h = HN^{-1/2}. \tag{8.47}$$

This is a new regime which did not appear in previous sections. The appearance of this scaling regime was first noticed in [19] for the zero temperature case.

Critical Points

We first compute the critical point γ of $\mathcal{G}(z)$. In previous sections, we had $\gamma = \lambda_1 + \mathcal{O}\left(N^{-2/3}\right)$ for $h \sim N^{-1/6}$ and $\gamma = \lambda_1 + \mathcal{O}\left(N^{-1}\right)$ for $h = 0$. For $h \sim N^{-1/2}$, it turns out that $\gamma = \lambda_1 + \mathcal{O}\left(N^{-1}\right)$. We make the ansatz that

$$\gamma = \lambda_1 + pN^{-1} \tag{8.48}$$

with $p = \mathcal{O}(1)$. Then, the critical point equation becomes

$$\beta - \frac{1}{N}\sum_{i=1}^{N} \frac{1}{\lambda_1 + pN^{-1} - \lambda_i} - \frac{H^2\beta}{N^2}\sum_{i=1}^{N} \frac{n_i^2}{(\lambda_1 + pN^{-1} - \lambda_i)^2} = 0. \tag{8.49}$$

Separating out $i = 1$ in both sums and using (4.30) and (4.31) for the remaining sums, the equation becomes

$$\beta - 1 - \frac{1}{p} - \frac{H^2\beta n_1^2}{p^2} + \mathcal{O}\left(N^{-1/3}\right) = 0. \tag{8.50}$$

The solution is

$$p = \frac{1 + \sqrt{1 + 4(\beta - 1)H^2\beta n_1^2}}{2(\beta - 1)} + \mathcal{O}\left(N^{-1/3}\right). \tag{8.51}$$

Hence, $p = \mathcal{O}(1)$, which is consistent with the ansatz.

Now consider the critical point of $\mathcal{G}_{\mathfrak{M}}(z)$. Due to the scale $h = HN^{-1/2}$, the function $\mathcal{G}_{\mathfrak{M}}(z)$ is the same as $\mathcal{G}(z)$ with H replaced by $H + \xi$. Thus, we find that

$$\gamma_{\mathfrak{M}} = \lambda_1 + p_{\mathfrak{M}} N^{-1} \tag{8.52}$$

where $p_{\mathfrak{M}} > 0$ solves the equation

$$\beta - 1 - \frac{1}{p_{\mathfrak{M}}} - \frac{(H + \xi)^2\beta n_1^2}{p_{\mathfrak{M}}^2} + \mathcal{O}\left(N^{-1/3}\right) = 0. \tag{8.53}$$

Exponential Terms

We evaluate $N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma))$ using (8.14). For A_1, the sum with $i \geq 2$, using a Taylor approximation, is $\mathcal{O}\left(N^{-2/3}\right)$. Hence,

$$A_1 = -\log\left(\frac{p_{\mathfrak{M}}}{p}\right) + \frac{p_{\mathfrak{M}}}{p} - 1 + \mathcal{O}\left(N^{-2/3}\right).$$

The sum with $i \geq 2$ for A_2 is $\mathcal{O}\left(N^{-1}\right)$ and we obtain

$$A_2 = \frac{H^2\beta n_1^2(p_{\mathfrak{M}} - p)^2}{p_{\mathfrak{M}} p^2} + \mathcal{O}\left(N^{-1}\right).$$

Finally, again separating the term with $i = 1$ and using (4.26) for the rest of the sum,

$$A_3 = (2\xi H + \xi^2)\beta\left(\frac{n_1^2}{p_{\mathfrak{M}}} + 1\right) + \mathcal{O}\left(N^{-1/3}\right). \tag{8.54}$$

Therefore,

$$\begin{aligned} &N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma)) \\ &= -\log(\frac{p_{\mathfrak{M}}}{p}) + \frac{p_{\mathfrak{M}}}{p} - 1 + \frac{H^2\beta n_1^2(p_{\mathfrak{M}} - p)^2}{p_{\mathfrak{M}} p^2} \\ &\quad + (2\xi H + \xi^2)\beta\left(\frac{n_1^2}{p_{\mathfrak{M}}} + 1\right) + \mathcal{O}\left(N^{-1/3}\right). \end{aligned} \tag{8.55}$$

Using the Eqs. (8.50) and (8.53) satisfied by p and $p_{\mathfrak{M}}$, the equation (8.55) can be written as

$$N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) - \mathcal{G}(\gamma)) = -\log(\frac{p_{\mathfrak{M}}}{p}) + 2(\beta - 1)(p_{\mathfrak{M}} - p) + (2H\xi + \xi^2)\beta + \mathcal{O}\left(N^{-1/3}\right). \tag{8.56}$$

Integrals

We now consider the integrals in the formula (8.10). The ratio of the integrals in this regime turns out to give a non-trivial contribution. We first show that we cannot use a Taylor series approximation. Consider the numerator; the denominator is the same as the numerator with $\xi = 0$. For $k \geq 2$, we use the formula for $\mathcal{G}_{\mathfrak{M}}(z)$ to get

$$\begin{aligned}\frac{\mathcal{G}_{\mathfrak{M}}^{(k)}(\gamma_{\mathfrak{M}})}{(-1)^k(k-1)!} =& \frac{1}{N}\sum_{i=1}^{N}\frac{N^{\frac{2}{3}k}}{(a_1+p_{\mathfrak{M}}N^{-1/3}-a_i)^k} \\ &+ \frac{k(H+\xi)^2\beta}{N^2}\sum_{i=1}^{N}\frac{n_i^2 N^{\frac{2}{3}(k+1)}}{(a_1+p_{\mathfrak{M}}N^{-1/3}-a_i)^{k+1}} \\ =& N^{\frac{2}{3}k-1}\left(\frac{N^{\frac{1}{3}k}}{p_{\mathfrak{M}}^k}+\mathcal{O}(1)\right) \\ &+ k(H+\xi)^2\beta N^{\frac{2}{3}k-\frac{4}{3}}\left(\frac{N^{\frac{1}{3}(k+1)}}{p_{\mathfrak{M}}^{k+1}}+\mathcal{O}(1)\right) = \mathcal{O}\left(N^{k-1}\right).\end{aligned}$$

Since $\mathcal{G}_{\mathfrak{M}}^{(2)} = \mathcal{O}(N)$, the main contribution to the integral comes from a neighborhood of radius N^{-1} around the critical point. If we use the new variable $z = \gamma_{\mathfrak{M}} + uN^{-1}$ and the Taylor series

$$N\left(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}+uN^{-1})-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})\right) = \sum_{k=2}^{\infty}\frac{N^{-k+1}}{k!}\mathcal{G}_{\mathfrak{M}}^{(k)}(\gamma_{\mathfrak{M}})u^k,$$

we find that all terms in the series are $\mathcal{O}(1)$ for finite u. Since all terms in the Taylor series contribute to the integral, this method will not work and we instead proceed as follows. Using $\mathcal{G}'_{\mathfrak{M}}(\gamma_{\mathfrak{M}}) = 0$, we have

$$\begin{aligned}&N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}+w)-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})) \\ &= N(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}+w)-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})-\mathcal{G}'_{\mathfrak{M}}(\gamma_{\mathfrak{M}})w) \\ &= -\sum_{i=1}^{N}\left[\log\left(1+\frac{w}{\gamma_{\mathfrak{M}}-\lambda_i}\right)-\frac{w}{\gamma_{\mathfrak{M}}-\lambda_i}\right] \\ &\quad+\left(h+\frac{\xi}{\sqrt{N}}\right)^2\beta\sum_{i=1}^{N}\frac{n_i^2w^2}{(\gamma_{\mathfrak{M}}+w-\lambda_i)(\gamma_{\mathfrak{M}}-\lambda_i)^2}.\end{aligned}$$

Separating out $i = 1$, using a Taylor approximation of the log function, and using (4.31),

$$\begin{aligned}&N\left(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}+uN^{-1})-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})\right) \\ &= -\log\left(1+\frac{u}{p_{\mathfrak{M}}}\right)+\frac{u}{p_{\mathfrak{M}}}+\frac{(H+\xi)^2\beta n_1^2u^2}{(p_{\mathfrak{M}}+u)p_{\mathfrak{M}}^2}+\mathcal{O}\left(u^2N^{-2/3}\right).\end{aligned}$$

We temporarily write the middle two terms with $x := (H+\xi)^2\beta n_1^2$ and get

$$\frac{u}{p_{\mathfrak{M}}}+\frac{xu^2}{(p_{\mathfrak{M}}+u)p_{\mathfrak{M}}^2} = u\left(\frac{1}{p_{\mathfrak{M}}}+\frac{x}{p_{\mathfrak{M}}^2}\right)-\frac{x}{p_{\mathfrak{M}}}+\frac{x}{p_{\mathfrak{M}}+x}.$$

Using (8.53) twice, the above formula can be written as

$$\begin{aligned} &N\left(\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}+uN^{-1})-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}})\right) \\ &\simeq -\log\left(1+\frac{u}{p_{\mathfrak{M}}}\right)+(\beta-1)(u-p_{\mathfrak{M}})+1+\frac{(H+\xi)^2\beta n_1^2}{(p_{\mathfrak{M}}+u)}. \end{aligned} \tag{8.57}$$

Thus,

$$\begin{aligned} &\int_{\gamma_{\mathfrak{M}}-\mathrm{i}\infty}^{\gamma_{\mathfrak{M}}+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z)-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))}\mathrm{d}z \\ &\simeq \frac{1}{N}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\sqrt{\frac{p_{\mathfrak{M}}}{p_{\mathfrak{M}}+u}}\,e^{\frac{(\beta-1)(u-p_{\mathfrak{M}})}{2}+\frac{1}{2}+\frac{(H+\xi)^2\beta n_1^2}{2(p_{\mathfrak{M}}+u)}}\mathrm{d}u \\ &= \frac{p_{\mathfrak{M}}^{1/2}e^{-(\beta-1)p_{\mathfrak{M}}+\frac{1}{2}}}{N}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{e^{\frac{(\beta-1)(p_{\mathfrak{M}}+u)}{2}+\frac{(H+\xi)^2\beta n_1^2}{2(p_{\mathfrak{M}}+u)}}}{\sqrt{p_{\mathfrak{M}}+u}}\mathrm{d}u. \end{aligned} \tag{8.58}$$

The last integral is an integral formula of a modified Bessel function which can be evaluated explicitly (see e.g. [1]):

$$\int_{0_+ + \mathrm{i}\mathbb{R}}\frac{e^{aw+\frac{b}{w}}}{\sqrt{w}}\mathrm{d}w = 2\pi\mathrm{i}\left(\frac{b}{a}\right)^{1/4} I_{-\frac{1}{2}}(2\sqrt{ab}) = \frac{2\mathrm{i}\sqrt{\pi}}{\sqrt{a}}\cosh(2\sqrt{ab}). \tag{8.59}$$

Hence, we obtain

$$\begin{aligned} &\int_{\gamma_{\mathfrak{M}}-\mathrm{i}\infty}^{\gamma_{\mathfrak{M}}+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z)-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))}\mathrm{d}z \\ &\simeq \frac{2\mathrm{i}\sqrt{2\pi\, p_{\mathfrak{M}}}e^{-(\beta-1)p_{\mathfrak{M}}+\frac{1}{2}}}{N\sqrt{\beta-1}}\cosh\left((H+\xi)|n_1|\sqrt{\beta(\beta-1)}\right). \end{aligned} \tag{8.60}$$

Note that the integral depends on ξ, unlike in the cases $h>0$ and $h\sim N^{-1/6}$. The denominator is the case when $\xi=0$. Thus,

$$\frac{\int e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{M}}(z)-\mathcal{G}_{\mathfrak{M}}(\gamma_{\mathfrak{M}}))}\mathrm{d}z}{\int e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z} \simeq \sqrt{\frac{p_{\mathfrak{M}}}{p}}\,e^{-(\beta-1)(p_{\mathfrak{M}}-p)}\frac{\cosh\left((H+\xi)|n_1|\sqrt{\beta(\beta-1)}\right)}{\cosh\left(H|n_1|\sqrt{\beta(\beta-1)}\right)}. \tag{8.61}$$

Combining all terms together, replacing $\beta\xi$ by ξ and using $T=1/\beta$, we obtain the following.

Result 8.6 *For $h=HN^{-1/2}$ and $0<T<1$,*

$$\langle e^{\xi\sqrt{N}\mathfrak{M}}\rangle \simeq e^{H\xi+\frac{T\xi^2}{2}}\frac{\cosh\left((H+T\xi)|n_1|\frac{\sqrt{1-T}}{T}\right)}{\cosh\left(H|n_1|\frac{\sqrt{1-T}}{T}\right)} \tag{8.62}$$

as $N\to\infty$ for asymptotically almost every disorder sample.

The right-hand side is the product of two terms, implying that $\sqrt{N}\mathfrak{M}$ is a sum two independent random variables. The exponential term on right-hand side is the moment generating function of a Gaussian distribution, while the ratio of the cosh functions is the moment generating function of a shifted Bernoulli distribution. Indeed, if $\mathbb{P}(X=1)=p$ and $\mathbb{P}(X=-1)=1-p$ with $p=\frac{e^a}{e^a+e^{-a}}$, then

$$\mathbb{E}[e^{\xi X}] = pe^{\xi}+(1-p)e^{-\xi} = \frac{\cosh(a+\xi)}{\cosh(a)}.$$

Hence, we deduce the following result.

Result 8.7 *For $h = HN^{-1/2}$ and $0 < T < 1$,*

$$\mathfrak{M} \overset{\mathfrak{D}}{\simeq} h + \frac{|n_1|\sqrt{1-T}\mathfrak{B}(\alpha) + \sqrt{T}\mathfrak{N}}{\sqrt{N}} \tag{8.63}$$

as $N \to \infty$ for asymptotically almost every disorder sample. Here, $\mathfrak{B}(c)$ is a shifted Bernoulli thermal random variable with the probability mass function $P(\mathfrak{B}(c) = 1) = c$ and $P(\mathfrak{B}(c) = -1) = 1 - c$ and α in (8.63) is given by

$$\alpha := \frac{e^{\frac{H|n_1|\sqrt{1-T}}{T}}}{e^{\frac{H|n_1|\sqrt{1-T}}{T}} + e^{-\frac{H|n_1|\sqrt{1-T}}{T}}}. \tag{8.64}$$

The thermal random variable $\mathfrak{N}$ has the standard Gaussian distribution and it is independent of $\mathfrak{B}(\alpha)$.

8.6.2 Matching with $h \sim N^{-1/6}$ and $h = 0$

As $H \to \infty$, the random variable $\mathfrak{B}(\alpha) \to 1$. The formal limit of (8.63) as $H \to \infty$ is

$$\mathfrak{M} \overset{\mathfrak{D}}{\simeq} h + \frac{1}{\sqrt{N}}\left[|n_1|\sqrt{1-T} + \sqrt{T}\mathfrak{N}\right], \tag{8.65}$$

which is the same as (8.46) from the $h = HN^{-1/6}$ regime. On the other hand, if we take $H \to 0$, then $\mathfrak{B}(\alpha) \overset{D}{\to} \mathfrak{B}(1/2)$. Hence, the formal limit of (8.63) as $H \to 0$ is the same as the $h = 0$ case (8.38). Therefore, the result (8.63) matches with both the $h \sim N^{-1/6}$ and $h = 0$ results.

8.7 Susceptibility

In this subsection, we discuss properties of the susceptibility, defined as the magnetization per external field strength. In the next subsection we discuss differential susceptibility

$$\mathcal{X} = \frac{\mathcal{M}}{h} = \frac{\langle \mathfrak{M} \rangle}{h} = \frac{1}{h}\frac{\mathrm{d}\mathcal{F}_N}{\mathrm{d}h}. \tag{8.66}$$

We denote by $\bar{\mathcal{X}}$ or $\mathbb{E}_s[\mathcal{X}]$ the sample average of $\mathcal{X}$. We denote by Var_s the sample variance. As described in Section 4, we use the font $\overset{\mathcal{D}}{\simeq}$ to denote an asymptotic expansion in distribution with respect to the disorder sample.

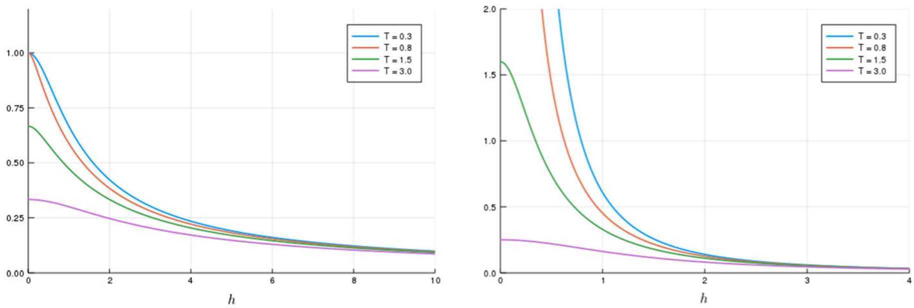
8.7.1 Macroscopic Field $h = O(1)$

From Sect. 8.6, for fixed $h > 0$ and $T > 0$,

$$\mathcal{X} \simeq \mathcal{X}^0 + \frac{\mathcal{X}^1}{\sqrt{N}} \tag{8.67}$$

for asymptotically almost every disorder sample, where

$$\mathcal{X}^0 = s_1(\gamma_0) \quad \text{and} \quad \mathcal{X}^1 = \mathcal{S}_N(\gamma_0; 1) - \frac{h^2 s_2(\gamma_0)\mathcal{S}_N(\gamma_0; 2)}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)} \tag{8.68}$$



(a) Graph of $\mathcal{X}^0(h)$ **(b)** Graph of $\sigma_s(h)$

Fig. 4 Graph of $\mathcal{X}^0(h)$ and $\sigma_s^2(h)$ as function of h for various values of T

and γ_0 is the solution of the Eq. (5.19) and $\mathcal{S}_N(z;k)$ is defined in (4.29).

The leading term $\mathcal{X}^0$ is deterministic and satisfies:

- $\mathcal{X}^0$ is a decreasing function of h,
- As $h \to \infty$,

$$\mathcal{X}^0(h,T) = \frac{1}{h} - \frac{T}{2h^2} + O(h^{-3}) \quad \text{for all } T > 0 \tag{8.69}$$

- As $h \to 0$,

$$\mathcal{X}^0(h,T) \simeq \begin{cases} \frac{1}{T} - \frac{h^2}{T(T^2-1)} & \text{for } T > 1 \\ 1 - \frac{h^2}{2(1-T)} & \text{for } 0 < T < 1. \end{cases} \tag{8.70}$$

See Fig. 4a for the graph of $\mathcal{X}^0$ as a function of h.

The subleading term $\mathcal{X}^1$ depends on the disorder sample. We consider its sample-to-sample fluctuations. From (4.29), $\mathcal{S}_N(\gamma_0;1)$ and $\mathcal{S}_N(\gamma_0;2)$ converge to the centered bivariate Gaussian distribution with

$$\mathrm{Var}_s[\mathcal{S}_N(\gamma_0;1)] \to 2s_2(\gamma_0), \qquad \mathrm{Var}_s[\mathcal{S}_N(\gamma_0;2)] \to 2s_4(\gamma_0), \tag{8.71}$$

and

$$\mathrm{Cov}_s(\mathcal{S}_N(\gamma_0;1), \mathcal{S}_N(\gamma_0;2)) = \mathbb{E}_s\left[\frac{1}{N}\sum_{i=1}^{N}\frac{(n_i^2-1)^2}{(\gamma_0-\widehat{\lambda}_i)^3}\right] \to 2s_3(\gamma_0). \tag{8.72}$$

as $N \to \infty$. Hence, as $N \to \infty$,

$$\mathcal{X}^1 \overset{\mathcal{D}}{\simeq} \mathcal{N}(0,\sigma_s^2) \tag{8.73}$$

where the sample variance is

$$\sigma_s^2 = \frac{2s_2(\gamma_0)^2\left(T^2 s_2(\gamma_0) + 2Th^2 s_3(\gamma_0) + h^4 s_4(\gamma_0)\right)}{\left(Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)\right)^2}. \tag{8.74}$$

See Fig. 4b for the graph of σ_s^2. The graph shows that σ_s^2 is a monotonically decreasing function of h. It is easy to check that:

- As $h \to \infty$,

$$\sigma_s^2 \simeq \frac{1}{2h^2} \qquad \text{for all } T > 0. \tag{8.75}$$

- As $h \to 0$,

$$\sigma_s^2 \simeq \begin{cases} \frac{2}{T^2-1} & \text{for } T > 1, \\ \frac{1-T}{h^2} & \text{for } T < 1. \end{cases} \tag{8.76}$$

The above formula suggests that there is an interesting transition as T approaches the critical temperature $T = 1$ in the case where $h \to 0$. The behavior near the $(T, h) = (1, 0)$ is worth studying, but we leave this subject for the future.

8.7.2 Mesoscopic External Field: $h \sim N^{-1/6}$ and $T < 1$

From Sect. 8.6, for $h = HN^{-1/6}$ with fixed $H > 0$ and $0 < T < 1$,

$$\mathcal{X} \simeq 1 + \frac{\mathcal{E}_N(t)}{N^{1/3}} \tag{8.77}$$

for asymptotically almost every disorder sample, where $\mathcal{E}_N(t)$ is given in (8.43).

The behavior of $\mathcal{E}_N(t)$ as $H \to \infty$ and $H \to 0$ is discussed in Sects. 8.5.2 and 8.5.3. The sample-to-sample fluctuation of $\mathcal{E}_N(t)$ is shown in Sect. 7.1 and we see that $\mathcal{E}_N(t) \overset{\mathcal{D}}{\simeq} \mathcal{E}(\varsigma)$ where

$$\mathcal{E}(\varsigma) = \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{\nu_i^2}{\varsigma + \alpha_1 - \alpha_i} - \frac{1}{\pi} \int_0^{\left(\frac{3\pi n}{2}\right)^{2/3}} \frac{\mathrm{d}x}{\sqrt{x}} \right) \tag{8.78}$$

and $\varsigma > 0$ solves $1 - T = H^2 \sum_{i=1}^{\infty} \frac{\nu_i^2}{(\varsigma+\alpha_1-\alpha_i)^2}$. Here, α_i is the GOE Airy point process and ν_i are i.i.d standard normal random variables independent of α_i.

8.7.3 Microscopic External Field: $h \sim N^{-1/2}$ and $T < 1$

The thermal average of (8.63) implies that for $h = HN^{-1/2}$ with fixed $H > 0$ and $0 < T < 1$,

$$\mathcal{X} \simeq 1 + \frac{|n_1|\sqrt{1-T}}{H} \tanh\left(\frac{H|n_1|\sqrt{1-T}}{T} \right) =: \mathcal{X}_{\text{micro}} \tag{8.79}$$

for asymptotically almost every disorder sample. The function $\mathcal{X}_{\text{micro}}$ is a decreasing function in both H and T (see Figs. 5 and 6). From the formula for $\mathcal{X}_{\text{micro}}$, we conclude that

$$\mathcal{X}_{\text{micro}} \simeq 1 + \frac{|n_1|\sqrt{1-T}}{H} \qquad \text{as } H \to \infty \tag{8.80}$$

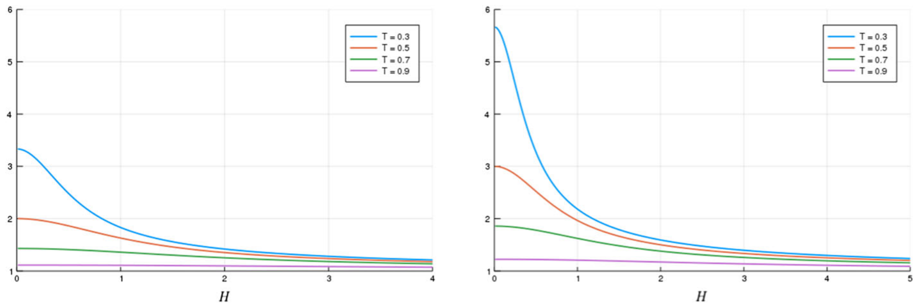
and

$$\mathcal{X}_{\text{micro}} \simeq 1 + \frac{n_1^2(1-T)}{T} - \frac{H^2 n_1^4 (1-T)^2}{3T^3} \qquad \text{as } H \to 0. \tag{8.81}$$

8.7.4 The Zero External Field Limit of the Susceptibility

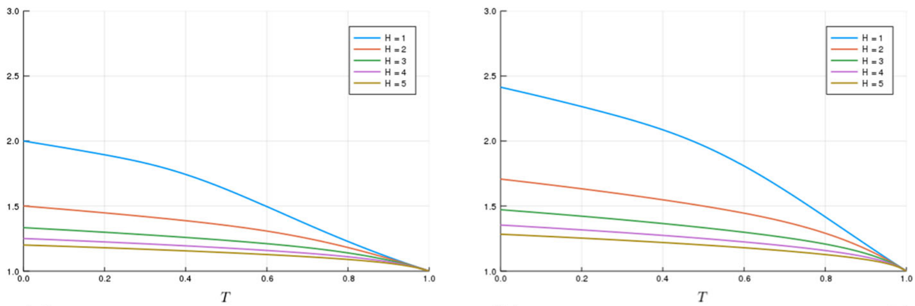
We consider two different limits of the susceptibility depending on how $h \to 0$ and $N \to \infty$ are taken. The first limit is obtained from (8.70):

$$\lim_{h\to 0} \lim_{\substack{N\to\infty \\ h>0}} \mathcal{X} = \begin{cases} \frac{1}{T} & \text{for } T > 1 \\ 1 & \text{for } T < 1. \end{cases} \tag{8.82}$$



(a) $\mathcal{X}_{\text{micro}}$ as a function of H when $|n_1| = 1$ **(b)** $\mathcal{X}_{\text{micro}}$ as a function of H when $|n_1| = \sqrt{2}$

Fig. 5 Graph of $\mathcal{X}_{\text{micro}}(H, T)$ as a functions of H for various values of $0 < T < 1$



(a) $\mathcal{X}_{\text{micro}}$ as a function of T when $|n_1| = 1$ **(b)** $\mathcal{X}_{\text{micro}}$ as a function of T when $|n_1| = \sqrt{2}$

Fig. 6 Graph of $\mathcal{X}_{\text{micro}}(H, T)$ as a functions of T for various values of H

See Fig. 2a. This result (8.82) was previously obtained in [24], and also in [14]. The limit does not depend on the disorder sample.

The second limit is obtained from (8.81) for $0 < T < 1$:

$$\lim_{H\to 0} \lim_{\substack{N\to\infty \\ h=HN^{-1/2}}} \mathcal{X} = 1 + \frac{n_1^2(1-T)}{T} \qquad \text{for } 0 < T < 1. \tag{8.83}$$

See Fig. 2b. This limit depends on the disorder sample, but only on one variable, n_1^2. Observe that this limit blows up at $T = 0$ while the limit (8.82) is finite at $T = 0$. The sample-to-sample average of (8.83) satisfies

$$\lim_{H\to 0} \lim_{\substack{N\to\infty \\ h=HN^{-1/2}}} \bar{\mathcal{X}} = \frac{1}{T} \qquad \text{for } 0 < T < 1. \tag{8.84}$$

8.8 Differential Susceptibility

We also consider the differential susceptibility given by

$$\mathcal{X}_d = \frac{\mathrm{d}}{\mathrm{d}h}\langle \mathfrak{M} \rangle = \frac{\mathrm{d}^2 \mathcal{F}_N}{\mathrm{d}h^2} = \frac{N}{T}\left(\langle \mathfrak{M}^2\rangle - \langle \mathfrak{M}\rangle^2\right). \tag{8.85}$$

The results (8.21), (8.44), and (8.63) imply the following formulas. All formulas hold for asymptotically almost every disorder sample.

(a) For fixed $h > 0$ and $T > 0$,

$$\mathcal{X}_d \simeq s_1(\gamma_0) - \frac{2h^2 s_2(\gamma_0)^2}{T s_2(\gamma_0) + 2h^2 s_3(\gamma_0)} =: \mathcal{X}_d^0. \tag{8.86}$$

(b) For $h = HN^{-1/6}$ with fixed $H > 0$ and $0 < T < 1$,

$$\mathcal{X}_d \simeq 1. \tag{8.87}$$

(c) For $h = HN^{-1/2}$ with fixed $H > 0$ and $0 < T < 1$,

$$\mathcal{X}_d \simeq 1 + \frac{n_1^2(1-T)}{T \cosh^2\left(\frac{H|n_1|\sqrt{1-T}}{T}\right)} =: \mathcal{X}_{d,\text{micro}}. \tag{8.88}$$

The limits for the macroscopic and mescopic regimes do not depend on the disorder samples, but the limit for the microscopic regime depends on the disorder variable n_1^2. The macroscopic limit satisfies the following property as $h \to 0$:

$$\mathcal{X}_d^0 \simeq \begin{cases} \frac{1}{T} - \frac{3h^2}{T(T^2-1)} + O(h^4) & T > 1, \\ 1 - \frac{3h^2}{2(1-T)^2} + O(h^4) & 0 < T < 1. \end{cases} \tag{8.89}$$

On the other hand the microscopic limit satisfies, for $0 < T < 1$,

$$\mathcal{X}_{d,\text{micro}} \simeq \begin{cases} 1 + O(e^{-\frac{2H|n_1|\sqrt{1-T}}{T}}) & \text{as } H \to \infty. \\ 1 + \frac{n_1^2(1-T)}{T} - \frac{H^2 n_1^4 (1-T)^2}{T^3} & \text{as } H \to 0. \end{cases} \tag{8.90}$$

The zero external field limit is the same as the susceptibility of the last section even though the subleading terms differ by a factor of 3. In both cases the limit is

$$\lim_{H \to 0} \mathcal{X}_{d,\text{micro}} = \lim_{H \to 0} \mathcal{X}_{\text{micro}} = 1 + \frac{n_1^2(1-T)}{T} \tag{8.91}$$

and this value depends on the disorder variable n_1^2. Note that the sample-to-sample average of n_1^2 is 1. This result shows that both susceptibilities satisfy Curie's law in the sample-to-sample average sense, but not in the quenched disorder sense.

We note that if we take $T \to 0$ with $H > 0$ fixed in (8.79) and (8.88), then

$$\mathcal{X}_{\text{micro}} \simeq 1 + \frac{|n_1|}{H} \quad \text{and} \quad \mathcal{X}_{d,\text{micro}} \simeq 1 \qquad \text{at } T = 0. \tag{8.92}$$

This shows that $\mathcal{X}_{d,\text{micro}}(T = 0)$ does not diverge as $H \to 0$ but $\mathcal{X}_{\text{micro}}(T = 0)$ does.

9 Overlap with the ground state

Recall that $\pm\mathbf{u}_1$ denote the unit eigenvectors corresponding to the largest eigenvalue of M. The overlap of the spin with the ground state and the squared overlap are defined as

$$\mathfrak{G} = \frac{|\mathbf{u}_1 \cdot \boldsymbol{\sigma}|}{\sqrt{N}}, \qquad \mathfrak{O} = \mathfrak{G}^2 = \frac{1}{N}(\mathbf{u}_1 \cdot \boldsymbol{\sigma})^2, \tag{9.1}$$

respectively. The overlap $\mathfrak{G} = 1$ when $T = h = 0$ since the Hamiltonian is maximized when $\boldsymbol{\sigma}$ is parallel to $\pm\mathbf{u}_1$. The overlap measures how close the spin is to the ground state. Since it is more convenient to analyze, we consider $\mathfrak{O}$ in this section.

As with the overlap with the external field, there are no transitions when $T > 1$ as $h \to 0$. However, when $T < 1$, there are two interesting transitional regimes given by $h \sim N^{-1/6}$ and $h \sim N^{-1/3}$. The second regime did not appear for the overlap with the external field. On the other hand, the regime $h \sim N^{-1/2}$, which we studied for the free energy and the overlap with the external field, does not reveal any new features of $\mathfrak{O}$. Instead, $\mathfrak{O}$ has the same properties for $h \sim N^{-1/2}$ as it does for $h = 0$.

The moment generating function of $\mathfrak{O}$ has the integral formula given in Lemma 3.3,

$$\langle e^{\beta\eta\mathfrak{O}}\rangle = e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{O}}(\gamma_{\mathfrak{O}})-\mathcal{G}(\gamma))} \frac{\int_{\gamma_{\mathfrak{O}}-\mathrm{i}\infty}^{\gamma_{\mathfrak{O}}+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{O}}(z)-\mathcal{G}_{\mathfrak{O}}(\gamma_{\mathfrak{O}}))}\mathrm{d}z}{\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z} \tag{9.2}$$

where

$$\begin{aligned} \mathcal{G}_{\mathfrak{O}}(z) &= \beta z - \frac{1}{N}\log(z-\lambda_1-b) - \frac{1}{N}\sum_{i=2}^{N}\log(z-\lambda_i) + \frac{h^2\beta n_1^2}{N(z-\lambda_1-b)} \\ &\quad + \frac{h^2\beta}{N}\sum_{i=2}^{N}\frac{n_i^2}{z-\lambda_i}. \end{aligned} \tag{9.3}$$

We take take $\gamma_{\mathfrak{O}}$ and γ to be the critical points of $\mathcal{G}_{\mathfrak{O}}$ and $\mathcal{G}$ respectively, and we use the notation

$$b := \frac{2\eta}{N}. \tag{9.4}$$

The difference between $\mathcal{G}_{\mathfrak{O}}$ and $\mathcal{G}$ is that, in the case of $\mathcal{G}_{\mathfrak{O}}$, λ_1 is changed to $\lambda_1 + b$.

The following two formulas will be used in the analysis below. First, we have

$$N(\mathcal{G}_{\mathfrak{O}}(\gamma_{\mathfrak{O}})-\mathcal{G}(\gamma)) = N(\mathcal{G}_{\mathfrak{O}}(\gamma_{\mathfrak{O}})-\mathcal{G}(\gamma)-\mathcal{G}'(\gamma)(\gamma_{\mathfrak{O}}-\gamma)) = D_1+D_2+D_3+D_4 \tag{9.5}$$

where

$$\begin{aligned} D_1 &= -\log\left(1+\frac{\gamma_{\mathfrak{O}}-\gamma-b}{\gamma-\lambda_1}\right)+\frac{\gamma_{\mathfrak{O}}-\gamma}{\gamma-\lambda_1}, \\ D_2 &= -\sum_{i=2}^{N}\left[\log\left(1+\frac{\gamma_{\mathfrak{O}}-\gamma}{\gamma-\lambda_i}\right)-\frac{\gamma_{\mathfrak{O}}-\gamma}{\gamma-\lambda_i}\right], \\ D_3 &= h^2\beta n_1^2\left[\frac{1}{\gamma_{\mathfrak{O}}-\lambda_1-b}-\frac{1}{\gamma-\lambda_1}+\frac{\gamma_{\mathfrak{O}}-\gamma}{(\gamma-\lambda_1)^2}\right], \\ D_4 &= h^2\beta(\gamma_{\mathfrak{O}}-\gamma)^2\sum_{i=2}^{N}\frac{n_i^2}{(\gamma_{\mathfrak{O}}-\lambda_i)(\gamma-\lambda_i)^2}. \end{aligned} \tag{9.6}$$

Second, we can show from the equation $\mathcal{G}'_{\mathfrak{O}}(\gamma_{\mathfrak{O}}) - \mathcal{G}'(\gamma) = 0$ that

$$\begin{aligned} &(\gamma_{\mathfrak{O}}-\gamma)\Bigg[\frac{1}{N(\gamma_{\mathfrak{O}}-\lambda_1-b)(\gamma-\lambda_1)}+\frac{1}{N}\sum_{i=2}^{N}\frac{1}{(\gamma_{\mathfrak{O}}-\lambda_i)(\gamma-\lambda_i)} \\ &\quad + \frac{h^2\beta n_1^2}{N}\frac{\gamma+\gamma_{\mathfrak{O}}-2\lambda_1-b}{(\gamma_{\mathfrak{O}}-\lambda_1-b)^2(\gamma-\lambda_1)^2} \end{aligned}$$

$$+\frac{h^2\beta}{N}\sum_{i=2}^{N}\frac{n_i^2(\gamma+\gamma_{\mathfrak{D}}-2\lambda_i)}{(\gamma_{\mathfrak{D}}-\lambda_i)^2(\gamma-\lambda_i)^2}\Bigg]=b\Bigg[\frac{1}{N(\gamma_{\mathfrak{D}}-\lambda_1-b)(\gamma-\lambda_1)}$$
$$+\frac{h^2\beta n_1^2}{N}\frac{\gamma+\gamma_{\mathfrak{D}}-2\lambda_1-b}{(\gamma_{\mathfrak{D}}-\lambda_1-b)^2(\gamma-\lambda_1)^2}\Bigg]. \tag{9.7}$$

9.1 Macroscopic External Field: $h = O(1)$

9.1.1 Analysis

Fix $h > 0$. The fluctuations of $\mathfrak{D}$ turn out to be of order N^{-1}. Thus we set

$$\eta = \xi N \quad \text{so that} \quad b = 2\xi. \tag{9.8}$$

The critical point of $\mathcal{G}(z)$ is obtained in Sect. 5.2 and is given by $\gamma = \gamma_0 + \mathcal{O}\left(N^{-1/2}\right)$ where γ_0 solves the Eq. (5.19). We do not need an explicit formula for the term $\mathcal{O}\left(N^{-1/2}\right)$ in this section. Since $\mathcal{G}'_{\mathfrak{D}}(z) = \mathcal{G}'(z) + \mathcal{O}\left(N^{-1}\right)$ for $z > 2$, a perturbation argument implies that the critical point of $\mathcal{G}_{\mathfrak{D}}(z)$ is given by

$$\gamma_{\mathfrak{D}} = \gamma + \mathcal{O}\left(N^{-1}\right). \tag{9.9}$$

We use (9.5) to compute $N(\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}}) - \mathcal{G}(\gamma))$. From the semi-circle law, we have $D_2 = \mathcal{O}\left((\gamma_{\mathfrak{D}}-\gamma)^2 N\right) = \mathcal{O}\left(N^{-2}\right)$ and $D_4 = \mathcal{O}\left(N^{-1}\right)$. On the other hand, D_1 and D_3 are easy to compute and we find that

$$N(\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}})-\mathcal{G}(\gamma)) = -\log\left(1-\frac{2\xi}{\gamma_0-2}\right)+\frac{2h^2\beta n_1^2\xi}{(\gamma_0-2)^2(1-\frac{2\xi}{\gamma_0-2})}+\mathcal{O}\left(N^{-1/2}\right). \tag{9.10}$$

Since $\mathcal{G}^{(k)}_{\mathfrak{D}}(\gamma_{\mathfrak{D}}) = \mathcal{O}(1)$ for all $k \geq 2$, the ratio of the integrals (9.2) can be evaluated using the method of steepest descent. For $k = 2$,

$$\mathcal{G}''_{\mathfrak{D}}(\gamma_{\mathfrak{D}}) \simeq s_2(\gamma_0) + h^2\beta s_3(\gamma_0),$$

which does not depend on ξ. Since $\mathcal{G}(\gamma)$ is the special case of $\mathcal{G}_{\mathfrak{D}}(\gamma)$ when $\xi = 0$, we conclude that

$$\frac{\int_{\gamma_{\mathfrak{D}}-\mathrm{i}\infty}^{\gamma_{\mathfrak{D}}+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{D}}(z)-\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}}))}\mathrm{d}z}{\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z} \simeq \sqrt{\frac{\mathcal{G}''(\gamma)}{\mathcal{G}''_{\mathfrak{D}}(\gamma_{\mathfrak{D}})}} \simeq 1.$$

Inserting these results into (9.2), replacing ξ with $(\gamma_0-2)\xi$, and using $\beta = 1/T$, we obtain the following.

Result 9.1 *For $h = O(1)$ and $T > 0$,*

$$\langle e^{\frac{\gamma_0-2}{T}\xi N\mathfrak{D}}\rangle \simeq (1-2\xi)^{-1/2}\, e^{\frac{h^2 n_1^2\xi}{T(\gamma_0-2)(1-2\xi)}} \tag{9.11}$$

as $N \to \infty$ for asymptotically almost every disorder, where $\gamma_0 > 2$ is the solution of the Eq. (5.19).

Note that if X is a non-central Gaussian random variable $\mu + \mathfrak{N}$, i.e. if X^2 is a non-centered chi-squared distribution with 1 degree of freedom, then

$$E[e^{\xi X^2}] = (1-2\xi)^{-1/2}e^{\frac{\mu^2\xi}{1-2\xi}}. \tag{9.12}$$

Therefore, we obtain the next result from the one above.

Result 9.2 *For $h = O(1)$ and $T > 0$,*

$$\mathfrak{O} \overset{\mathfrak{D}}{\simeq} \frac{\mathfrak{O}^0}{N} \quad \text{where} \quad \mathfrak{O}^0 = \frac{T}{\gamma_0 - 2}\left|\frac{h|n_1|}{\sqrt{T(\gamma_0 - 2)}} + \mathfrak{N}\right|^2 \tag{9.13}$$

as $N \to \infty$ for asymptotically almost every disorder, where the thermal random variable $\mathfrak{N}$ has the standard Gaussian distribution.

9.1.2 Limits as $h \to \infty$ and $h \to 0$

Consider the formal limit of (9.13) as $h \to \infty$. From (5.33), we find that if we take $h > 0$ and let $N \to \infty$ first and then $h \to \infty$, we get

$$\mathfrak{O} \overset{\mathcal{D}}{\simeq} \frac{1}{N}\left[n_1^2 + \frac{2|n_1|\sqrt{T}}{\sqrt{h}}\mathfrak{N}\right] \tag{9.14}$$

for all $T > 0$. On the other hand, the Eq. (5.32) implies that if we take $h > 0$ and let $N \to \infty$ first and then $h \to 0$, we obtain

$$\mathfrak{O} \overset{\mathfrak{D}}{\simeq} \frac{T^2}{N(T-1)^2}\left[\mathfrak{N}^2 + \frac{2h|n_1|}{T-1}\mathfrak{N}\right] \qquad \text{for } T > 1 \tag{9.15}$$

and

$$\mathfrak{O} \overset{\mathfrak{D}}{\simeq} \frac{16}{N}\left[\frac{(1-T)^4 n_1^2}{h^6} + \frac{\sqrt{T}(1-T)^3|n_1|}{h^5}\mathfrak{N}\right] \qquad \text{for } 0 < T < 1. \tag{9.16}$$

For $0 < T < 1$, the above result indicates that the overlap is of order 1 when $h \sim N^{-1/6}$. We study this regime in the next subsection.

9.2 Mesoscopic External Field: $h \sim N^{-1/6}$ and $T < 1$

9.2.1 Analysis

We set

$$h = HN^{-1/6} \tag{9.17}$$

for fixed $H > 0$. If we insert $h = HN^{-1/6}$ in to the formula, the Eq. (9.16) indicates that the fluctuations are of order $N^{-1/6}$. Thus, we set

$$\eta = \xi N^{1/6} \quad \text{so that} \quad b = 2\xi N^{-5/6} \tag{9.18}$$

in (9.2) and (9.4).

The critical point γ of $\mathcal{G}(z)$ is obtained in Sect. 7.1 and it is given by $\gamma = \lambda_1 + sN^{-2/3}$ where $s > 0$ is the solution of the Eq. (7.4). We now consider the critical point of $\mathcal{G}_{\mathfrak{O}}(z)$. From the formula, we see that $\mathcal{G}'_{\mathfrak{O}}(z)$ is an increasing function of z for $z > \lambda_1 + b$. Using $b > 0$ and the explicit formula of the functions, we can easily check that $\mathcal{G}'_{\mathfrak{O}}(\gamma) < \mathcal{G}'(\gamma) = 0$ and $\mathcal{G}'_{\mathfrak{O}}(\gamma + b) > \mathcal{G}'(\gamma) = 0$. Hence, we find that $\gamma < \gamma_{\mathfrak{O}} < \gamma + b$, and thus, $\gamma_{\mathfrak{O}} - \gamma = \mathcal{O}(N^{-5/6})$. We now set

$$\gamma_{\mathfrak{O}} = \gamma + \Delta N^{-5/6} \tag{9.19}$$

and determine Δ using (9.7). The right-hand side of the Eq. (9.7) is equal to

$$\frac{2\xi}{N^{5/6}}\left[\frac{N^{1/3}}{s^2}+\frac{2H^2\beta n_1^2 N^{2/3}}{s^3}\right]=\frac{4\xi H^2\beta n_1^2}{N^{1/6}s^3}\left(1+\mathcal{O}\left(N^{-1/3}\right)\right).$$

For the left-hand side of the equation, the first two terms are of smaller order than the last two terms. Using $\gamma_{\mathfrak{D}}=\gamma+\mathcal{O}\left(N^{-5/6}\right)$ and $b=\mathcal{O}\left(N^{-5/6}\right)$ for the other two sums, the left-hand side is equal to

$$\frac{\Delta}{N^{5/6}}\left[\frac{2H^2\beta n_1^2 N^{2/3}}{s^3}+2H^2\beta N^{2/3}\sum_{i=2}^{N}\frac{n_i^2}{(s+a_1-a_i)^3}+\mathcal{O}\left(N^{1/3}\right)\right].$$

Therefore,

$$\Delta=\frac{2\xi n_1^2 s^{-3}}{\sum_{i=1}^{N} n_i^2(s+a_1-a_i)^{-3}}+\mathcal{O}\left(N^{-1/6}\right). \tag{9.20}$$

We now evaluate $N(\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}})-\mathcal{G}(\gamma))$ using (9.5). It is easy to check that $D_1=\mathcal{O}\left(N^{-1/6}\right)$ and $D_2=\mathcal{O}\left(N^{-1/3}\right)$. Evaluating the first two leading terms,

$$D_3=H^2\beta n_1^2\left[\frac{2\xi}{s^2}N^{1/6}+\frac{(\Delta-2\xi)^2}{s^3}+\mathcal{O}\left(N^{-1/6}\right)\right].$$

Finally,

$$D_4=H^2\beta\Delta^2\sum_{i=2}^{N}\frac{n_i^2}{(s+a_1-a_i)^3}+\mathcal{O}\left(N^{-1/6}\right).$$

Putting these together and also using the explicit formula of Δ, we obtain

$$\begin{aligned} &N(\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}})-\mathcal{G}(\gamma)) \\ &=\frac{2H^2\beta n_1^2}{s^2}\xi N^{1/6}+\frac{4H^2\beta n_1^2\left[\sum_{i=2}^{N} n_i^2(s+a_1-a_i)^{-3}\right]}{s^3\left[\sum_{i=1}^{N} n_i^2(s+a_1-a_i)^{-3}\right]}\xi^2+\mathcal{O}\left(N^{-1/6}\right). \end{aligned} \tag{9.21}$$

It remains to consider the integrals in (9.2). The scale $h=HN^{-1/6}$ is the same as the one in Sect. 7.1. Since $\gamma_{\mathfrak{D}}=\lambda_1+sN^{-2/3}+\Delta N^{-5/6}=\lambda_1+sN^{-2/3}+\mathcal{O}\left(N^{-5/6}\right)$ and $b=O(N^{-5/6})$, the calculation from Sect. 7.1 applies with only small changes. We find from the explicit formulas that $\mathcal{G}_{\mathfrak{D}}^{(k)}(\gamma_{\mathfrak{D}})=\mathcal{O}\left(N^{\frac{2}{3}k-\frac{2}{3}}\right)$ for all $k\geq 2$ and

$$\mathcal{G}_{\mathfrak{D}}''(\gamma_{\mathfrak{D}})=H^2\beta t^2\sum_{i=1}^{N}\frac{n_i^2}{(s+a_1-a_i)^3}+\mathcal{O}\left(N^{-1/6}\right).$$

Thus, as in Sect. 7.1, the main contribution to the integral comes from a neighborhood of radius $N^{-5/6}$ around the critical point, and the numerator can be evaluated using a Gaussian integral. Since the leading term of $\mathcal{G}_{\mathfrak{D}}''(\gamma_{\mathfrak{D}})$ does not depend on ξ and the denominator is the case of the numerator with $\xi=0$, we find that

$$\frac{\int_{\gamma_{\mathfrak{D}}-\mathrm{i}\infty}^{\gamma_{\mathfrak{D}}+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{D}}(z)-\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}}))}\mathrm{d}z}{\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty} e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z}\simeq 1. \tag{9.22}$$

From the above computations, we obtain an asymptotic formula for $\langle e^{\beta\xi N^{1/6}\mathfrak{O}}\rangle$. Moving a term of order $N^{1/6}$ to the left, changing $\beta\xi$ to ξ, replacing β by $1/T$, and replacing s by t, which solves the Eq. (7.14), we arrive at the following result.

Result 9.3 *For $h = HN^{-1/6}$ and $0 < T < 1$,*

$$\langle e^{\xi N^{1/6}(\mathfrak{O}-\frac{H^2n_1^2}{t^2})}\rangle \simeq e^{\frac{2H^2Tn_1^2\left[\sum_{i=2}^N n_i^2(t+a_1-a_i)^{-3}\right]}{t^3\left[\sum_{i=1}^N n_i^2(t+a_1-a_i)^{-3}\right]}\xi^2} \tag{9.23}$$

as $N \to \infty$ for asymptotically almost every disorder sample, where $t > 0$ is the solution of the Eq. (7.14).

The right-hand side depends on the disorder sample heavily, as the formula involves all of the a_i and n_i. The above result implies the following.

Result 9.4 *For $h = HN^{-1/6}$ and $0 < T < 1$,*

$$\mathfrak{O} \overset{\mathfrak{D}}{\simeq} \frac{H^2n_1^2}{t^2} + \frac{\sigma_{\mathfrak{O}}\mathfrak{N}}{N^{1/6}} = \left[1 - T - H^2\sum_{i=2}^N \frac{n_i^2}{(t+a_1-a_i)^2}\right] + \frac{\sigma_{\mathfrak{O}}\mathfrak{N}}{N^{1/6}} \tag{9.24}$$

as $N \to \infty$ for asymptotically almost every disorder sample, where the thermal random variable $\mathfrak{N}$ has the standard normal distribution and $\sigma_{\mathfrak{O}} > 0$ satisfies

$$\sigma_{\mathfrak{O}}^2 = \frac{4H^2Tn_1^2\left[\sum_{i=2}^N n_i^2(t+a_1-a_i)^{-3}\right]}{t^3\left[\sum_{i=1}^N n_i^2(t+a_1-a_i)^{-3}\right]}. \tag{9.25}$$

The equality of the leading terms in the two formulas of (9.24) follows from the Eq. (7.14) that t satisfies.

9.2.2 Matching with $h = O(1)$

We consider the $H \to \infty$ limit. From (7.20), we have $t \simeq \frac{H^4}{4(1-T)^2}$. Hence, the term (9.24) satisfies

$$\sigma_{\mathfrak{O}}^2 \simeq \frac{4H^2Tn_1^2}{t^3} \simeq \frac{4^4Tn_1^2(1-T)^6}{H^{10}}.$$

Therefore, the first formula of (9.24) implies that if we take $h = HN^{-1/6}$ and let $N \to \infty$ first and then $H \to \infty$, we get

$$\mathfrak{O} \overset{\mathfrak{D}}{\simeq} \frac{16}{N}\left[\frac{(1-T)^4n_1^2}{h^6} + \frac{\sqrt{T}(1-T)^3|n_1|}{h^5}\mathfrak{N}\right]. \tag{9.26}$$

This formula matches the formal limit given in (9.16). Thus this regime matches with the $h = O(1)$ regime.

9.2.3 Formal Limit as $H \to 0$

Using (7.19) for t, the denominator of (9.25) becomes $n_1^2 + \mathcal{O}\left(H^3\right)$ as $H \to 0$. Thus, if we take $h = HN^{-1/6}$ and let $N \to \infty$ first and then take $H \to 0$, we get

$$\mathfrak{O} \overset{\mathfrak{D}}{\simeq} 1 - T - H^2\sum_{i=2}^N \frac{n_i^2}{(a_1-a_i)^2} + \frac{2H\sqrt{T}}{N^{1/6}}\left[\sum_{i=2}^N \frac{n_i^2}{(a_1-a_i)^3}\right]^{1/2}\mathfrak{N}. \tag{9.27}$$

The last two terms of (9.27) are of orders $H^2 = h^2 N^{1/3}$ and $HN^{-1/6} = h$, respectively. These two terms have the same order if $h \sim N^{-1/3}$. We study this regime in the next subsection. Note that, in this regime, the two terms are of order $N^{-1/3}$.

9.3 Microscopic External Field: $h \sim N^{-1/3}$ and $T < 1$

9.3.1 Analysis

Set

$$h = HN^{-1/3} \tag{9.28}$$

for fixed $H > 0$. In the last part of the previous sub-subsection, a formal calculation indicated that the order of fluctuation in this regime is $N^{-1/3}$. We set

$$\eta = \xi N^{1/3} \quad \text{so that} \quad b = 2\xi N^{-2/3}. \tag{9.29}$$

The regime $h \sim N^{-1/3}$ did not appear in previous sections. Hence, we first find the critical point γ of $\mathcal{G}(z)$. Previously we saw that $\gamma = \lambda_1 + \mathcal{O}\left(N^{-2/3}\right)$ when $h \sim N^{-1/6}$ and $\gamma = \lambda_1 + \mathcal{O}\left(N^{-1}\right)$ when $h \sim N^{-1/2}$. We expect that, in this regime, γ is between the above two cases, so we set $\gamma = \lambda_1 + w$ for some w and we assume $N^{-1} \ll w \ll N^{-2/3}$. The equation for the critical point is, using (4.30),

$$\begin{aligned}\mathcal{G}'(\gamma) &= \beta - \frac{1}{N}\sum_{i=1}^{N}\frac{1}{\gamma-\lambda_i} - \frac{H^2\beta}{N^{5/3}}\sum_{i=1}^{N}\frac{n_i^2}{(\gamma-\lambda_i)^2} \\ &= \beta - \frac{1}{Nw} - 1 + \mathcal{O}\left(N^{-1/3}\right) - \frac{H^2\beta n_1^2}{N^{5/3}w^2} = 0.\end{aligned} \tag{9.30}$$

Under the assumption for w, we see that $\frac{1}{NW} \ll \frac{1}{N^{5/3}w^2}$, and hence $w = \mathcal{O}\left(N^{-5/6}\right)$. Explicitly solving the equation $\beta - 1 - \frac{H^2\beta n_1^2}{N^{5/3}w^2} = 0$, we find that

$$\gamma = \lambda_1 + rN^{-5/6} \quad \text{where} \quad r = \sqrt{\frac{H^2\beta n_1^2}{\beta-1}} + \mathcal{O}\left(N^{-1/6}\right). \tag{9.31}$$

For later use, we record that, upon inserting $\gamma = \lambda_1 + rN^{-5/6}$ into the Eq. (9.30), r satisfies the following more detailed equation, using the notation Ξ_N defined in (4.25):

$$\beta - \frac{1}{rN^{1/6}} - 1 - \frac{\Xi_N}{N^{1/3}} + \mathcal{O}\left(N^{-1/2}\right) - \frac{H^2\beta n_1^2}{r^2} - \frac{H^2\beta}{N^{1/3}}\sum_{i=2}^{N}\frac{n_i^2}{(a_1-a_i)^2} = 0. \tag{9.32}$$

The critical point $\gamma_{\mathfrak{D}}$ of $\mathcal{G}_{\mathfrak{D}}(z)$ is easy to obtain since $b = \frac{2\xi}{N^{2/3}}$ has the same order as the fluctuations of the eigenvalues λ_i. The critical point equation is the same as in the case of $\mathcal{G}(z)$ except that λ_1 is changed to $\lambda_1 + b$. Thus we have

$$\gamma_{\mathfrak{D}} = \lambda_1 + b + r_{\mathfrak{D}}N^{-5/6} \quad \text{where} \quad r_{\mathfrak{D}} = r + \mathcal{O}\left(N^{-1/6}\right). \tag{9.33}$$

For our computation, it turns out that we need an improved estimate for $r_{\mathfrak{D}} - r$. The equation $\mathcal{G}'_{\mathfrak{D}}(\gamma_{\mathfrak{D}}) = 0$ is, in terms of $r_{\mathfrak{D}}$,

$$\beta - \frac{1}{r_{\mathfrak{D}}N^{1/6}} - 1 - \frac{H^2\beta n_1^2}{r_{\mathfrak{D}}^2} + \mathcal{O}\left(N^{-1/3}\right) = 0.$$

This equation is the same as the Eq. (9.32) up to order $N^{-1/6}$. Therefore, we obtain an improved estimate $r_{\mathfrak{D}} = r + \mathcal{O}\left(N^{-1/3}\right)$. As a consequence,

$$\gamma_{\mathfrak{D}} - \gamma = b + \mathcal{O}\left(N^{-7/6}\right) = 2\xi N^{-2/3} + \mathcal{O}\left(N^{-7/6}\right). \tag{9.34}$$

We now evaluate $N(\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}}) - \mathcal{G}(\gamma))$ using (9.5). We have

$$D_1 = \frac{2\xi N^{1/6}}{r} + \mathcal{O}\left(N^{-1/3}\right), \qquad D_2 = -\sum_{i=2}^{N}\left[\log\left(1+\frac{2\xi}{a_1-a_i}\right) - \frac{2\xi}{a_1-a_i}\right] + \mathcal{O}\left(N^{-1/6}\right),$$

$$D_3 = \frac{2\xi H^2\beta n_1^2}{r^2}N^{1/3} + \mathcal{O}\left(N^{-1/6}\right), \qquad D_4 = 4\xi^2H^2\beta\sum_{i=2}^{N}\frac{n_i^2}{(a_1+2\xi-a_i)(a_1-a_i)^2} + \mathcal{O}\left(N^{-1/6}\right).$$

Note that r appears only in D_1 and D_3. Using the Eq. (9.32), the sum $D_1 + D_3$ can be expressed without using r:

$$D_1 + D_3 = 2\xi N^{1/3}\left[\beta - 1 - \frac{\Xi_N}{N^{1/3}} - \frac{H^2\beta}{N^{1/3}}\sum_{i=2}^{N}\frac{n_i^2}{(a_1-a_i)^2}\right] + \mathcal{O}\left(N^{-1/6}\right). \tag{9.35}$$

On the other hand, using the notation Ξ_N in (4.25) again, we can write

$$D_2 = -\left[\sum_{i=2}^{N}\log\left(1+\frac{2\xi}{a_1-a_i}\right) - 2\xi N^{1/3}\right] + 2\xi\,\Xi_N + \mathcal{O}\left(N^{-1/6}\right). \tag{9.36}$$

Adding D_1, D_2, D_3, and D_4, and combining two sums that are multiplied by $H^2\beta$, we find that

$$\begin{aligned} N(\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}}) - \mathcal{G}(\gamma)) = {}& 2\xi(\beta-1)N^{1/3} + \left[2\xi N^{1/3} - \sum_{i=2}^{N}\log\left(1+\frac{2\xi}{a_1-a_i}\right)\right] \\ & - 2\xi H^2\beta\sum_{i=2}^{N}\frac{n_i^2}{(a_1+2\xi-a_i)(a_1-a_i)} + \mathcal{O}\left(N^{-1/6}\right) \end{aligned} \tag{9.37}$$

We note that the term in brackets is $\mathcal{O}(1)$ due to (4.25).

Finally, we consider the integrals in (9.2), beginning with the numerator. Using $\gamma_{\mathfrak{D}} = \lambda_1 + b + rN^{-5/6} + \mathcal{O}\left(N^{7/6}\right)$ and the explicit formula for $\mathcal{G}_{\mathfrak{D}}(z)$, we find that

$$\mathcal{G}_{\mathfrak{D}}^{(k)}(\gamma_{\mathfrak{D}}) = \mathcal{O}\left(N^{\frac{5}{6}k-\frac{5}{6}}\right)$$

for $k \geq 2$. Since $\mathcal{G}_{\mathfrak{D}}''(\gamma_{\mathfrak{D}}) = \mathcal{O}\left(N^{\frac{5}{6}}\right)$, the main contribution to the integral comes from a neighborhood of radius $N^{-\frac{11}{12}}$ about the critical point. For $k = 2$, we find explicitly that

$$\mathcal{G}_{\mathfrak{D}}''(\gamma_{\mathfrak{D}}) = \frac{2H^2\beta n_1^2}{r^3}N^{-5/6} + \mathcal{O}\left(N^{-1}\right).$$

Hence,

$$N(\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}}+wN^{-\frac{11}{12}})-\mathcal{G}_{\mathfrak{D}}(\gamma_{\mathfrak{D}}))=\sum_{k=2}^{\infty}\frac{N^{1-\frac{11}{12}k}\mathcal{G}_{\mathfrak{D}}^{(k)}(\gamma_{\mathfrak{D}})w^k}{k!}=\frac{H^2\beta n_1^2}{r^3}w^2+\mathcal{O}\left(N^{-\frac{1}{12}}\right) \tag{9.38}$$

for finite w, and the integral can be evaluated as a Gaussian integral. Since the leading term of (9.38) does not depend on ξ, we find that the ratio of the integrals in (9.2) is asymptotically equal to 1.

Combining the computations above, we obtain an asymptotic formula for $\left\langle e^{\beta\xi N^{1/3}\mathfrak{D}}\right\rangle$. Moving a term and using $\beta=1/T$, we arrive at the following result.

Result 9.5 *For $h=HN^{-1/3}$ and $0<T<1$,*

$$\left\langle e^{\frac{\xi}{T}N^{1/3}(\mathfrak{D}-(1-T))}\right\rangle\simeq e^{\xi N^{1/3}}\prod_{i=2}^{N}\frac{e^{-\frac{\xi H^2 n_i^2}{T(a_1+2\xi-a_i)(a_1-a_i)}}}{\sqrt{1+\frac{2\xi}{a_1-a_i}}} \tag{9.39}$$

as $N\to\infty$ for asymptotically almost every disorder sample.

We remark that the right-hand side is $\mathcal{O}(1)$ since

$$\xi N^{1/3}-\frac{1}{2}\sum_{i=2}^{N}\log\left(1+\frac{2\xi}{a_1-a_i}\right)=\mathcal{O}(1)\,.$$

The formula (9.39) is a product of the moment generating functions of non-centered chi-squared distributions (see (9.12)). Hence, we obtain the following.

Result 9.6 *For $h=HN^{-1/3}$ and $0<T<1$,*

$$\mathfrak{D}\overset{\mathfrak{D}}{\simeq}1-T+\frac{T}{N^{1/3}}\mathfrak{W}_N,\qquad \mathfrak{W}_N=N^{1/3}-\sum_{i=2}^{N}\frac{\left|\frac{H|n_i|}{\sqrt{T(a_1-a_i)}}+\mathfrak{n}_i\right|^2}{a_1-a_i} \tag{9.40}$$

as $N\to\infty$ for asymptotically almost every disorder sample, where the thermal random variables $\mathfrak{n}_i$ are independent standard normal random variables.

Here, we emphasize that n_i are sample random variables (given by the dot product of each eigenvector of M with the external field) while $\mathfrak{n}_i$ are thermal random variables. Note that $\mathfrak{W}_N=\mathcal{O}(1)$ since $N^{1/3}-\sum_{i=2}^{N}\frac{\mathfrak{n}_i^2}{a_1-a_i}=\mathcal{O}(1)$.

9.3.2 Matching with the Mesocopic Field, $h\sim N^{-1/6}$

We take the formal limit $H\to\infty$ of (9.40) and compare with (9.27). Then, using $N^{1/3}-\sum_{i=2}^{N}\frac{1}{a_1-a_i}=\mathcal{O}(1)$ from (4.30),

$$\mathfrak{W}_N=-\frac{H^2}{T}\sum_{i=2}^{n}\frac{n_i^2}{(a_1-a_i)^2}-\frac{2H}{\sqrt{T}}\sum_{i=2}^{N}\frac{|n_i|\mathfrak{n}_i}{(a_1-a_i)^{3/2}}+\mathcal{O}(1)\,. \tag{9.41}$$

The second sum is a sum of independent (thermal) Gaussian random variables,and hence it has a Gaussian distribution. Therefore, if take $h=HN^{-1/3}$ and let $N\to\infty$ first and then take $H \to \infty$, we get

$$\mathfrak{O} \stackrel{\mathfrak{D}}{\simeq} 1 - T - \frac{H^2}{N^{1/3}} \sum_{i=2}^{N} \frac{n_i^2}{(a_1 - a_i)^2} + \frac{2H\sqrt{T}}{N^{1/3}} \left[\sum_{i=2}^{N} \frac{n_i^2}{(a_1 - a_i)^3} \right]^{1/2} \mathfrak{N}. \tag{9.42}$$

In order to compare this with the result (9.27), we use the notation $h = H_{\text{micro}} N^{-1/3} = H_{\text{meso}} N^{-1/6}$. The Eqs. (9.42) and (9.27) are same once we set $H = H_{\text{micro}}$ and $H = H_{\text{meso}}$, respectively.

9.4 No External Field: $h = 0$

For $0 < T < 1$, the calculations of the previous subsection for $h = HN^{-1/3}$ go through; we obtain the result by setting $H = 0$ in (9.40). For $T > 1$, the computations in Sect. 9.1 for $h = O(1)$ also apply to $h = 0$; see (9.15).

Result 9.7 *For $h = 0$,*

$$\mathfrak{O} \stackrel{\mathfrak{D}}{\simeq} \begin{cases} \frac{T^2}{N(T-1)^2} \mathfrak{N}^2 & \text{for } T > 1, \\ 1 - T + \frac{T}{N^{1/3}} \left(N^{1/3} - \sum_{i=2}^{N} \frac{\mathfrak{n}_i^2}{a_1 - a_i} \right) & \text{for } 0 < T < 1. \end{cases} \tag{9.43}$$

where the thermal random variable $\mathfrak{N}$ has the standard normal distribution, and $\mathfrak{n}_i$ are independent standard normal thermal random variables.

9.5 The Thermal Average

We use the notation

$$\Omega = \langle \mathfrak{O} \rangle = \langle \mathfrak{G}^2 \rangle \tag{9.44}$$

to denote the thermal average of the squared overlap of a spin with the ground state. Previous subsections imply the following results.

(i) For $h \geq 0$ and $T > 1$, or for $h = O(1)$ with $h > 0$ and $0 < T < 1$,

$$\Omega \simeq \frac{\Omega^0}{N}, \qquad \Omega^0 = \frac{T}{\gamma_0 - 2} \left[\frac{h^2 n_1^2}{T(\gamma_0 - 2)} + 1 \right]. \tag{9.45}$$

From the asymptotic formulas (5.33) and (5.32) of γ_0,

$$\Omega^0 \simeq n_1^2 + \frac{T - (T-4)n_1^2}{h} \qquad \text{as } h \to \infty \text{ for all } T > 0 \tag{9.46}$$

and

$$\Omega^0 \simeq \begin{cases} \frac{T^2}{(T-1)^2} + \frac{h^2(n_1^2 - 1)T^2}{(T-1)^4} & \text{as } h \to 0 \text{ for } T > 1 \\ \frac{16 n_1^2 (1-T)^4}{h^6} + \frac{4T(1-T)^2 + 32 n_1^2 (1-T)^4}{h^4} & \text{as } h \to 0 \text{ for } 0 < T < 1. \end{cases} \tag{9.47}$$

See Fig. 7 for graphs of Ω^0.

(ii) For $h = HN^{-1/6}$ with $0 < T < 1$,

$$\Omega \simeq \frac{H^2 n_1^2}{t^2} = 1 - T - H^2 \sum_{i=2}^{N} \frac{n_i^2}{(t + a_1 - a_i)^2}. \tag{9.48}$$

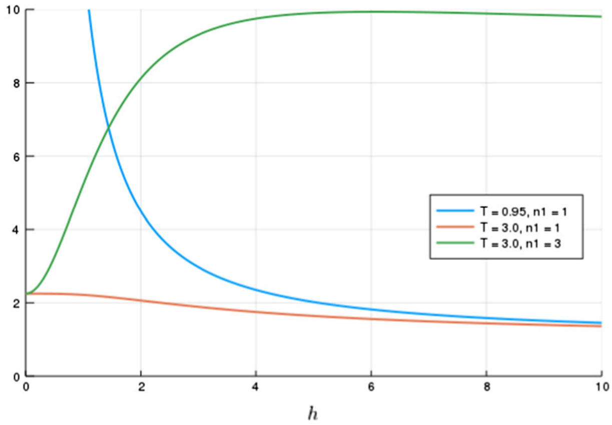


Fig. 7 Graphs of Ω^0 for $h = O(1)$ as function of h for different combinations of T and n_1

(iii) For $h = HN^{-1/3}$ with $T < 1$ (including the case when $H = 0$),

$$\Omega \simeq 1 - T + \frac{1}{N^{1/3}}\left[T\left(N^{1/3} - \sum_{i=2}^{N}\frac{1}{a_1 - a_i}\right) - H^2\sum_{i=2}^{N}\frac{n_i^2}{(a_1 - a_i)^2}\right]. \tag{9.49}$$

If we collect only the order 1 terms, then as $N \to \infty$ with $T < 1$,

$$\Omega \to \begin{cases} 0 & \text{for } h > 0 \\ 1 - T - H^2\sum_{i=2}^{N}\frac{n_i^2}{(t+a_1-a_i)^2} & \text{for } h = HN^{-1/6} \\ 1 - T & \text{for } h = HN^{-1/3} \text{ (including } H = 0). \end{cases} \tag{9.50}$$

The sample-to-sample standard deviation of the thermal average of squared overlap satisfies for $0 < T < 1$,

$$\sqrt{\overline{\Omega^2} - (\overline{\Omega})^2} = \begin{cases} O(N^{-1}) & \text{for } h = O(1) \\ O(1) & \text{for } h \sim N^{-1/6} \\ O(N^{-1/3}) & \text{for } h \sim N^{-1/3} \text{ (including } h = 0). \end{cases} \tag{9.51}$$

The order is largest when $h \sim N^{-1/6}$.

9.6 Order of Thermal Fluctuations

For $0 < T < 1$, the standard deviation of the thermal fluctuations satisfies

$$\sqrt{\langle\mathfrak{O}^2\rangle - \langle\mathfrak{O}\rangle^2} = \begin{cases} \mathcal{O}\left(N^{-1}\right) & \text{for } h = O(1) \\ \mathcal{O}\left(N^{-1/6}\right) & \text{for } h \sim N^{-1/6} \\ \mathcal{O}\left(N^{-1/3}\right) & \text{for } h \sim N^{-1/3} \text{ (including } h = 0). \end{cases} \tag{9.52}$$

for asymptotically almost every disorder sample. The thermal fluctuations are largest when $h \sim N^{-1/6}$.

10 Overlap with a Replica

Let

$$\mathfrak{R} = \mathfrak{R}^{1,2} = \frac{\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}}{N} \tag{10.1}$$

be the overlap of a spin $\boldsymbol{\sigma}^{(1)}$ and its replica $\boldsymbol{\sigma}^{(2)}$, chosen independently from S_{N-1} using the Gibbs measure with the same disorder sample. From Lemma 3.3, we have

$$\langle e^{\eta\mathfrak{R}}\rangle = e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a)-2\mathcal{G}(\gamma))} \frac{\iint e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a))}\mathrm{d}z\mathrm{d}w}{\left(\int e^{\frac{N}{2}(\mathcal{G}(z)-\mathcal{G}(\gamma))}\mathrm{d}z\right)^2} \tag{10.2}$$

where

$$\begin{aligned}\mathcal{G}_{\mathfrak{R}}(z,w;a) = \beta(z+w) &- \frac{1}{N}\sum_{i=1}^{N}\log\left((z-\lambda_i)(w-\lambda_i)-a^2\right)\\ &+ \frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2(z+w-2\lambda_i+2a)}{(z-\lambda_i)(w-\lambda_i)-a^2}\end{aligned} \tag{10.3}$$

and we set

$$a = \frac{\eta}{\beta N}. \tag{10.4}$$

We take γ to be the critical point of $\mathcal{G}(z)$ and we chose $\gamma_{\mathfrak{R}} > \lambda_1 + |a|$ such that $(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}})$ is a critical point of $\mathcal{G}_{\mathfrak{R}}(z, w; a)$. We calculate $\gamma_{\mathfrak{R}}$ below.

The partial derivative of $\mathcal{G}_{\mathfrak{R}}$ with respect to z is

$$\frac{\partial \mathcal{G}_{\mathfrak{R}}}{\partial z} = \beta - \frac{1}{N}\sum_{i=1}^{N}\frac{w-\lambda_i}{(z-\lambda_i)(w-\lambda_i)-a^2} - \frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2(w-\lambda_i+a)^2}{((z-\lambda_i)(w-\lambda_i)-a^2)^2} \tag{10.5}$$

and $\frac{\partial \mathcal{G}_{\mathfrak{R}}}{\partial w}$ is similar. Since $\frac{\partial \mathcal{G}_{\mathfrak{R}}}{\partial z}$ is an increasing function for real z (and similarly with $\frac{\partial \mathcal{G}_{\mathfrak{R}}}{\partial w}$), there exists a critical point of the form $(z, w) = (\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}})$ where $\gamma_{\mathfrak{R}}$ solves the equation

$$\beta - \frac{1}{N}\sum_{i=1}^{N}\frac{\gamma_{\mathfrak{R}}-\lambda_i}{(\gamma_{\mathfrak{R}}-\lambda_i-a)(\gamma_{\mathfrak{R}}-\lambda_i+a)} - \frac{h^2\beta}{N}\sum_{i=1}^{N}\frac{n_i^2}{(\gamma_{\mathfrak{R}}-\lambda_i-a)^2} = 0, \quad \gamma_{\mathfrak{R}} > \lambda_1+|a|. \tag{10.6}$$

There may be other critical points, but $(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}})$ is the one that we use for our steepest descent analysis. For simplicity, we refer to this critical point as $\gamma_{\mathfrak{R}}$ rather than $(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}})$. For $a = 0$, $\mathcal{G}_{\mathfrak{R}}(z, w; 0) = \mathcal{G}(z) + \mathcal{G}(w)$, and in this case, the critical point is $(z, w) = (\gamma, \gamma)$.

We use the following two formulas in this section. The first formula is

$$\begin{aligned}&N\left(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a)-2\mathcal{G}(\gamma)\right)\\ &\quad= N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a)-2\mathcal{G}(\gamma)-2\mathcal{G}'(\gamma)(\gamma_{\mathfrak{R}}-\gamma)) = B_1+B_2\end{aligned} \tag{10.7}$$

where

$$B_1 = -\sum_{i=1}^{N}\left[\log\left(1+\frac{2(\gamma_{\mathfrak{R}}-\gamma)}{\gamma-\lambda_i}+\frac{(\gamma_{\mathfrak{R}}-\gamma)^2-a^2}{(\gamma-\lambda_i)^2}\right)-\frac{2(\gamma_{\mathfrak{R}}-\gamma)}{\gamma-\lambda_i}\right]$$

and

$$B_2 = 2h^2\beta \sum_{i=1}^{N} n_i^2 \left[\frac{1}{\gamma_{\mathfrak{R}} - \lambda_i - a} - \frac{1}{\gamma - \lambda_i} + \frac{\gamma_{\mathfrak{R}} - \gamma}{(\gamma - \lambda_i)^2} \right].$$

The second formula is

$$\begin{aligned} &(\gamma_{\mathfrak{R}} - \gamma - a) \left[\sum_{i=1}^{N} \frac{\gamma_{\mathfrak{R}} - \lambda_i}{(\gamma_{\mathfrak{R}} - \lambda_i - a)(\gamma_{\mathfrak{R}} - \lambda_i + a)(\gamma - \lambda_i)} \right. \\ &\quad \left. + h^2\beta \sum_{i=1}^{N} \frac{n_i^2(\gamma + \gamma_{\mathfrak{R}} - 2\lambda_i - a)}{(\gamma_{\mathfrak{R}} - \lambda_i - a)^2(\gamma - \lambda_i)^2} \right] \\ &= -a \sum_{i=1}^{N} \frac{1}{(\gamma_{\mathfrak{R}} - \lambda_i + a)(\gamma - \lambda_i)}, \end{aligned} \tag{10.8}$$

which follows from subtracting the critical point equations for $\gamma_{\mathfrak{R}}$ and γ.

We also make use of the following lemma.

Lemma 10.1 *The point $\gamma_{\mathfrak{R}}$ satisfies $\gamma < \gamma_{\mathfrak{R}} < \gamma + a$.*

Proof Let

$$g(z) = \beta - \frac{1}{N} \sum_{i=1}^{N} \frac{z - \lambda_i}{(z - \lambda_i - a)(z - \lambda_i + a)} - \frac{h^2\beta}{N} \sum_{i=1}^{N} \frac{n_i^2}{(z - \lambda_i - a)^2}.$$

Since $g(\gamma_{\mathfrak{R}}) = 0$, it is enough to show that $g(\gamma) < 0$ and $g(\gamma + a) > 0$. Using $a > 0$, we see that

$$g(\gamma) < \beta - \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\gamma - \lambda_i} - \frac{h^2\beta}{N} \sum_{i=1}^{N} \frac{n_i^2}{(\gamma - \lambda_i)^2} = \mathcal{G}'(\gamma) = 0.$$

On the other hand,

$$\begin{aligned} g(\gamma + a) &= \beta - \frac{1}{N} \sum_{i=1}^{N} \frac{\gamma - \lambda_i + a}{(\gamma - \lambda_i)(\gamma - \lambda_i + 2a)} - \frac{h^2\beta}{N} \sum_{i=1}^{N} \frac{n_i^2}{(\gamma - \lambda_i)^2} \\ &> \beta - \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\gamma - \lambda_i} - \frac{h^2\beta}{N} \sum_{i=1}^{N} \frac{n_i^2}{(\gamma - \lambda_i)^2} = \mathcal{G}'(\gamma) = 0. \end{aligned}$$

□

10.1 Macroscopic External Field: $h = O(1)$

10.1.1 Analysis

Fix $h > 0$. It turns out that the fluctuations are of order $N^{-1/2}$. Hence, we set

$$\eta = \beta\xi\sqrt{N} \quad \text{so that} \quad a = \xi N^{-1/2}. \tag{10.9}$$

The critical point of $\mathcal{G}(z)$ is given in (5.20) by $\gamma = \gamma_0 + \gamma_1 N^{-1/2} + \mathcal{O}\left(N^{-1}\right)$. Consider the critical point $\gamma_{\mathfrak{R}}$. By Lemma 10.1, $\gamma_{\mathfrak{R}} = \gamma + O(N^{-1/2})$. We now use the Eq. (10.8). Using the semi-circle law approximation, we find that

$$\gamma_{\mathfrak{R}} - \gamma - a = -\frac{a\left(s_2(\gamma_0) + O(N^{-\frac{1}{2}})\right)}{s_2(\gamma_0) + 2h^2\beta s_3(\gamma_0) + O(N^{-\frac{1}{2}})}. \tag{10.10}$$

Thus,

$$\gamma_{\mathfrak{R}} = \gamma + \frac{\xi A}{\sqrt{N}} + O(N^{-1}) \quad \text{where} \quad A = \frac{2h^2\beta s_3(\gamma_0)}{s_2(\gamma_0) + 2h^2\beta s_3(\gamma_0)}. \tag{10.11}$$

We evaluate $N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a) - 2\mathcal{G}(\gamma))$ using (10.7). From a Taylor approximation,

$$B_1 = \sum_{i=1}^{N} \frac{(\gamma_{\mathfrak{R}} - \gamma)^2 + a^2}{(\gamma - \lambda_i)^2} + \mathcal{O}\left(N^{-1/2}\right) = \left((\gamma_{\mathfrak{R}} - \gamma)^2 + a^2\right) N s_2(\gamma) + \mathcal{O}\left(N^{-1/2}\right). \tag{10.12}$$

On the other hand, using the geometric series for $\frac{1}{\gamma_{\mathfrak{R}} - \lambda_i - a} = \frac{1}{(\gamma - \lambda_i) + (\gamma_{\mathfrak{R}} - \gamma - a)}$ and using (4.32),

$$\begin{aligned} B_2 &= \sum_{i=1}^{N} n_i^2 \left[\frac{a}{(\gamma - \lambda_i)^2} + \frac{(\gamma_{\mathfrak{R}} - \gamma - a)^2}{(\gamma - \lambda_i)^3} + O\left(\frac{(\gamma_{\mathfrak{R}} - \gamma - a)^3}{(\gamma - \lambda_i)^4}\right)\right] \\ &= a\left(s_2(\gamma) + N^{-1/2}\mathcal{S}_N(\gamma; 2)\right) + (\gamma_{\mathfrak{R}} - \gamma - a)^2 s_3(\gamma) + \mathcal{O}\left(N^{-1/2}\right) \end{aligned} \tag{10.13}$$

where $\mathcal{S}_N(z; k)$ is defined in (4.22). The leading term is $as_2(\gamma)$ which is $O(N^{1/2})$ and the rest is $\mathcal{O}(1)$. Inserting $\gamma = \gamma_0 + \gamma_1 N^{-1/2} + \mathcal{O}\left(N^{-1}\right)$ and using $s_2'(z) = -2s_3(z)$, we find that

$$\begin{aligned} &N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a) - 2\mathcal{G}(\gamma)) = \xi^2(1 + A^2)s_2(\gamma_0) \\ &\quad + 2h^2\beta\left(\xi^2(A-1)^2 s_3(\gamma_0) + \xi\mathcal{S}_N(\gamma_0; 2) + \xi\sqrt{N}s_2(\gamma_0) - 2\xi s_3(\gamma_0)\gamma_1\right) + \mathcal{O}\left(N^{-\frac{1}{2}}\right). \end{aligned} \tag{10.14}$$

We now consider the integrals in (10.2). Since all partial derivatives of $\mathcal{G}_{\mathfrak{R}}(z, w)$ evaluated at the critical point $(z, w) = (\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}})$ are $\mathcal{O}(1)$, the two dimensional method of steepest descent applies. Since the second derivatives evaluated at the critical point do not depend on ξ, we find that the ratio of the integrals in (10.2) is asymptotically equal to 1.

Combining the computations above, we find that

$$\begin{aligned} \log\langle e^{\beta\xi\sqrt{N}\mathfrak{R}}\rangle &\simeq \frac{1}{2}\xi^2(1 + A^2)s_2(\gamma_0) \\ &\quad + h^2\beta\left(\xi^2(A-1)^2 s_3(\gamma_0) + \xi\mathcal{S}_N(\gamma_0; 2) + \xi\sqrt{N}s_2(\gamma_0) - 2\xi s_3(\gamma_0)\gamma_1\right) \end{aligned} \tag{10.15}$$

where A is given by (10.11). Using the formula (5.21) of γ_1, we obtain

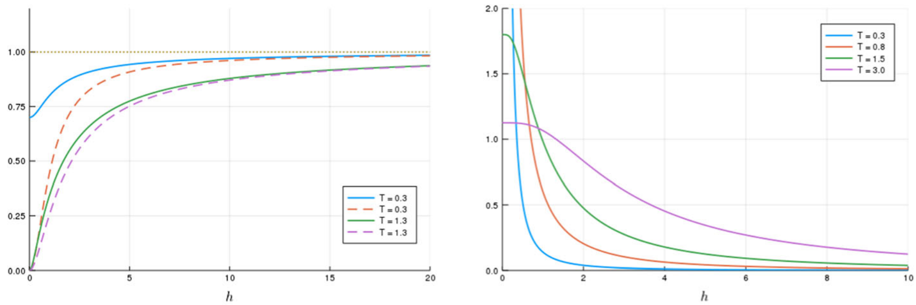
$$\mathcal{S}_N(\gamma_0; 2) - 2s_3(\gamma_0)\gamma_1 = \frac{Ts_2(\gamma_0)}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)}\mathcal{S}_N(\gamma_0; 2). \tag{10.16}$$

Hence, we conclude the following.

Result 10.2 *For $h > 0$ and $T > 0$,*

$$\log\langle e^{\xi\sqrt{N}(\mathfrak{R} - h^2 s_2(\gamma_0))}\rangle \simeq \frac{h^2 T s_2(\gamma_0)\mathcal{S}_N(\gamma_0; 2)}{Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0)}\xi + \frac{T^2 s_2(\gamma_0)(Ts_2(\gamma_0) + 4h^2 s_3(\gamma_0))}{2(Ts_2(\gamma_0) + 2h^2 s_3(\gamma_0))}\xi^2 \tag{10.17}$$

as $N \to \infty$ for asymptotically almost every disorder sample, where $\gamma_0 > 2$ is the solution of the equation $1 - Ts_1(\gamma_0) - h^2 s_2(\gamma_0) = 0$, and $\mathcal{S}_N(z; k)$ is defined in (4.22).



(a) Graph of $\mathfrak{R}^0$ (solid line) and $(\mathfrak{M}^0)^2$ (dashed line) as a function of h for $T = 0.3$ and $T = 1.3$

(b) Graph of $\sigma^2_{\mathfrak{R}}$ as a function of h.

Fig. 8 Graphs of $\mathfrak{R}^0$ and $\sigma^2_{\mathfrak{R}}$

As a consequence, we obtain the following.

Result 10.3 *For $h > 0$ and $T > 0$,*

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} h^2 s_2(\gamma_0) + \frac{1}{\sqrt{N}} \left[\frac{h^2 T s_2(\gamma_0) \mathcal{S}_N(\gamma_0; 2)}{T s_2(\gamma_0) + 2h^2 s_3(\gamma_0)} + \sigma_{\mathfrak{R}} \mathfrak{N} \right] \tag{10.18}$$

as $N \to \infty$ for asymptotically almost every disorder sample, where the thermal random variable $\mathfrak{N}$ has the standard normal distribution and $\sigma_{\mathfrak{R}} > 0$ satisfies

$$\sigma^2_{\mathfrak{R}} = \frac{T^2 s_2(\gamma_0)(T s_2(\gamma_0) + 4h^2 s_3(\gamma_0))}{T s_2(\gamma_0) + 2h^2 s_3(\gamma_0)}. \tag{10.19}$$

10.1.2 Discussion of the Leading Term

The leading term

$$\mathfrak{R}^0 = \mathfrak{R}^0(T, h) = h^2 s_2(\gamma_0) = 1 - T s_1(\gamma_0) \tag{10.20}$$

in (10.18) depends on neither the choice of spin configuration nor the disorder sample. See Fig. 8a for the graph of $\mathfrak{R}^0$ as a function of h.

The value (10.20) for $\mathfrak{R}^0$ reproduces the prediction q_0 for the overlap obtained in [13, 19] from the replica saddle methods which predicts that q_0 is determined by (5.29), The equivalence is checked using that $s_2(z) = s_1(z)^2/(1 - s_1(z)^2)$ and $q_0 = 1 - T s_1(\gamma_0)$.

It is easy to check the following properties using a computation similar to the one in Sect. 8.3:

- For every $T > 0$, $\mathfrak{R}^0$ is an increasing function of $h > 0$.
- As $h \to \infty$,

$$\mathfrak{R}^0 = 1 - \frac{T}{h} + O(h^{-2}) \qquad \text{for all } T > 0. \tag{10.21}$$

- As $h \to 0$,

$$\mathfrak{R}^0 = \begin{cases} \frac{h^2}{T^2-1} - \frac{2T^2h^4}{(T^2-1)^2} + O(h^6) & \text{for } T > 1, \\ 1 - T + \frac{Th^2}{2(1-T)} + O(h^4) & \text{for } 0 < T < 1. \end{cases} \tag{10.22}$$

10.1.3 Discussion of the Thermal Variance

The thermal variance of $\mathfrak{R}$ satisfies

$$\langle \mathfrak{R}^2 \rangle - \langle \mathfrak{R} \rangle^2 \simeq \frac{\sigma_{\mathfrak{R}}^2}{N} \tag{10.23}$$

for $\sigma_{\mathfrak{R}}^2$ given in (10.19) and it does not depend on the disorder sample. See Fig. 8b for the graph. It is a decreasing function of h, and satisfies

$$\sigma_{\mathfrak{R}}^2 = \frac{2T^2}{h^2} - \frac{5T^3}{2h^3} + O(h^{-4}) \quad \text{as } h \to \infty \text{ for all } T > 0 \tag{10.24}$$

and

$$\sigma_{\mathfrak{R}}^2(h, T) = \begin{cases} \frac{T^2}{T^2-1} + O(h^4) & \text{as } h \to 0 \text{ for } T > 1, \\ \frac{2T^2(1-T)}{h^2} + O(1) & \text{as } h \to 0 \text{ for } 0 < T < 1. \end{cases} \tag{10.25}$$

10.1.4 Limit as $h \to \infty$

As $h \to \infty$, using (5.33) and $s_k(z) = z^{-k} + O(z^{-k-2})$ as $z \to \infty$, we find that

$$\frac{h^2 T s_2(\gamma_0) \mathcal{S}_N(\gamma_0; 2)}{T s_2(\gamma_0) + 2h^2 s_3(\gamma_0)} \simeq \frac{T \sum_{i=1}^N (n_i^2 - 1)}{2h\sqrt{N}}. \tag{10.26}$$

Thus, we see that, for every $T > 0$, if we take $N \to \infty$ with $h > 0$ and then take $h \to \infty$,

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} 1 - \frac{T}{h} + \frac{T}{h\sqrt{N}} \left[\frac{\sum_{i=1}^N (n_i^2 - 1)}{2\sqrt{N}} + \sqrt{2}\mathfrak{N} \right]. \tag{10.27}$$

10.1.5 Limit as $h \to 0$ When $T > 1$

Using (5.32), if we take $N \to \infty$ with $h > 0$ and then take $h \to 0$, we see that, for $T > 1$,

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} \frac{h^2}{T^2 - 1} - \frac{2T^2 h^4}{(T^2 - 1)^2} + \frac{1}{\sqrt{N}} \left[\frac{T}{\sqrt{T^2 - 1}} \mathfrak{N} + h^2 \mathcal{S}_N(T + \frac{1}{T}; 2) \right]. \tag{10.28}$$

10.1.6 Limit as $h \to 0$ When $T < 1$

Similarly, from (5.32), if we take $N \to \infty$ with $h > 0$ and then take $h \to 0$, we see that, for $0 < T < 1$,

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} (1 - T) + \frac{Th^2}{2(1 - T)} + \frac{T}{h\sqrt{N}} \left[\frac{h^5 \mathcal{S}_N(\gamma_0; 2)}{2(1 - T)^2} + \sqrt{2(1 - T)}\mathfrak{N} \right]. \tag{10.29}$$

From the discussions around the Eq. (8.37), we expect that $h^5 \mathcal{S}_N(\gamma_0; 2) = \mathcal{O}(1)$ as $h \to 0$ if $h \gg N^{-1/6}$. This indicates that there may be a transition when $h \sim N^{-1/6}$. We study this regime in the next subsection. On the other hand, the thermal fluctuation term becomes of order 1 if $h^{-1} N^{-1/2} = O(1)$. This indicates a new regime $h \sim N^{-1/2}$, which we study in a later section.

10.2 Mescoscopic External Field: $h \sim N^{-1/6}$ and $T < 1$

10.2.1 Analysis

Set

$$h = HN^{-1/6} \tag{10.30}$$

for fixed $H > 0$. It turns out that the order of the fluctuations of $\mathfrak{R}$ is $N^{-1/3}$. Hence, we set

$$\eta = \beta\xi N^{1/3} \quad \text{so that} \quad a = \xi N^{-2/3}. \tag{10.31}$$

The critical point of $\mathcal{G}(z)$ is given by $\gamma = \lambda_1 + sN^{-2/3}$ where $s > 0$ solves the Eq. (7.3). Inserting $h = HN^{-1/6}$, the equation takes the form

$$\beta - \frac{1}{N^{1/3}} \sum_{i=1}^{N} \frac{1}{s + a_1 - a_i} - H^2\beta \sum_{i=1}^{N} \frac{n_i^2}{(s + a_1 - a_i)^2} = 0. \tag{10.32}$$

The solution satisfies $s = t + \mathcal{O}\left(N^{-1/3}\right)$ where t solves the Eq. (7.14).

For the critical point of $\mathcal{G}_{\mathfrak{R}}$, Lemma 10.1) shows that $\gamma < \gamma_{\mathfrak{R}} < \gamma + a$. Hence, $\gamma_{\mathfrak{R}} - \gamma - a = O(N^{-2/3})$. However, we can get a sharper bound on this difference. The right-hand side of (10.8) is $\mathcal{O}\left(aN^{4/3}\right)$ and the bracket term of the left-hand side of the same equation is $\mathcal{O}\left(N^{5/3}\right)$, with the leading contribution coming from the second sum. Hence, we find that

$$\gamma_{\mathfrak{R}} = \gamma + a - \epsilon, \qquad \epsilon = \mathcal{O}\left(N^{-1}\right). \tag{10.33}$$

We now evaluate (10.7). The first sum B_1 is

$$\begin{aligned} &-\sum_{i=1}^{N}\left[\log\left(1 + \frac{2(a-\epsilon)}{\gamma - \lambda_i} - \frac{(2a-\epsilon)\epsilon}{(\gamma-\lambda_i)^2}\right) - \frac{2(a-\epsilon)}{\gamma-\lambda_i}\right] \\ &\simeq -\sum_{i=1}^{N}\left[\log\left(1 + \frac{2\xi}{s + a_1 - a_i}\right) - \frac{2\xi}{s + a_1 - a_i}\right] \end{aligned}$$

and this sum is $\mathcal{O}(1)$. For the second sum, we get

$$B_2 = 2\xi N^{1/3} H^2 \beta \sum_{i=1}^{N} \frac{n_i^2}{(s + a_1 - a_i)^2} + \mathcal{O}\left(N^{-1/3}\right).$$

Therefore, $N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a) - 2\mathcal{G}(\gamma))$ is equal to

$$\begin{aligned} &-\sum_{i=1}^{N}\left[\log\left(1 + \frac{2\xi}{s + a_1 - a_i}\right) - \frac{2\xi}{s + a_1 - a_i}\right] \\ &+ 2\xi N^{1/3} H^2 \beta \sum_{i=1}^{N} \frac{n_i^2}{(s + a_1 - a_i)^2} + \mathcal{O}(N^{-1/3}). \end{aligned} \tag{10.34}$$

Using the Eq. (10.32) for s, we can write

$$N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a) - 2\mathcal{G}(\gamma)) = 2\xi\beta N^{1/3} - \sum_{i=1}^{N} \log\left(1 + \frac{2\xi}{s + a_1 - a_i}\right) + \mathcal{O}(N^{-1/3}). \tag{10.35}$$

Finally, we compute the integrals in (10.2). A calculation similar to the one from Sect. 7.1 shows that the kth partial derivatives of $\mathcal{G}_{\mathfrak{R}}$ evaluated at $(z, w) = (\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}})$ are $\mathcal{O}\left(N^{\frac{2}{3}k-\frac{2}{3}}\right)$. Since the second derivatives are $\mathcal{O}\left(N^{\frac{2}{3}}\right)$, the main contribution to the integral comes from a neighborhood of radius $N^{-5/6}$ around the critical point. Moreover, from explicit computations, we find that

$$\frac{\partial^2 \mathcal{G}_{\mathfrak{R}}}{\partial z^2}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}) = \frac{\partial^2 \mathcal{G}_{\mathfrak{R}}}{\partial w^2}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}) \simeq x N^{2/3}, \qquad \frac{\partial^2 \mathcal{G}_{\mathfrak{R}}}{\partial z \partial w}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}) \simeq y N^{2/3}$$

where

$$x = 2H^2\beta \sum_{i=1}^{N} \frac{n_i^2(s + a_1 - a_i + \xi)}{(s + a_1 - a_i)^3(s + a_1 - a_i + 2\xi)},$$

$$y = 2H^2\beta \sum_{i=1}^{N} \frac{n_i^2 \xi}{(s + a_1 - a_i)^3(s + a_1 - a_i + 2\xi)}.$$

Using the method of steepest descent with the change of variables $z = \gamma_{\mathfrak{R}} + uN^{-5/6}$ and $w = \gamma_{\mathfrak{R}} + vN^{-5/6}$, the integral becomes

$$\int_{\gamma_{\mathfrak{R}}+\mathrm{i}\mathbb{R}} \int_{\gamma_{\mathfrak{R}}+\mathrm{i}\mathbb{R}} e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a))} \mathrm{d}z\mathrm{d}w \simeq \frac{1}{N^{5/3}} \int_{\mathrm{i}\mathbb{R}} \int_{\mathrm{i}\mathbb{R}} e^{\frac{1}{4}(xu^2+xv^2+2yuv)} \mathrm{d}u\mathrm{d}v. \tag{10.36}$$

Evaluating the Gaussian integral, inserting the formulas of x and y, and noting that the denominator is the same as the numerator when $\xi = 0$, the ratio of the integrals becomes

$$\frac{\int\int e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a))} \mathrm{d}z\mathrm{d}w}{\left(\int e^{\frac{N}{2}(\mathcal{G}_{\mathfrak{R}}(z)-\mathcal{G}(\gamma))} \mathrm{d}z\right)^2} \simeq \sqrt{\frac{\sum_{i=1}^{N} \frac{n_i^2}{(s+a_1-a_i)^3}}{\sum_{i=1}^{N} \frac{n_i^2}{(s+a_1-a_i)^2(s+a_1-a_i+2\xi)}}}. \tag{10.37}$$

Combining the above calculations and replacing s by t, we obtain the following result after moving a term of order $N^{1/3}$.

Result 10.4 *For $h = HN^{-1/6}$ and $0 < T < 1$,*

$$\langle e^{\frac{1}{T}\xi N^{1/3}(\mathfrak{R}-(1-T))} \rangle \simeq e^{\xi N^{1/3} - \frac{1}{2}\sum_{i=1}^{N} \log\left(1+\frac{2\xi}{t+a_1-a_i}\right)} \sqrt{\frac{\sum_{i=1}^{N} \frac{n_i^2}{(t+a_1-a_i)^3}}{\sum_{i=1}^{N} \frac{n_i^2}{(t+a_1-a_i)^2(t+a_1-a_i+2\xi)}}} \tag{10.38}$$

as $N \to \infty$ for asymptotically almost every disorder sample, where $t > 0$ is the solution of the Eq. (7.14).

The term in the exponent on the right-hand side is $\mathcal{O}(1)$.

Result 10.5 *For $h = HN^{-1/6}$ and $0 < T < 1$,*

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} 1 - T + \frac{T}{N^{1/3}} \Upsilon_N(t) \tag{10.39}$$

as $N \to \infty$ for asymptotically almost every disorder sample, where $t > 0$ is the solution of the Eq. (7.14) and $\Upsilon_N(t)$ is a random variable defined by the generating function given by the right-hand side of (10.38).

10.2.2 Matching with $h = O(1)$

We take the formal limit of the result (10.39) as $H \to \infty$. From (7.20), $t \to \infty$. The big square root term of the generating function on the right-hand side of (10.38) is approximately is approximately 1. On the other hand,

$$\xi N^{1/3} - \frac{1}{2}\sum_{i=1}^{N} \log\left(1 + \frac{2\xi}{t + a_1 - a_i}\right) \simeq \xi\left(N^{1/3} - \sum_{i=1}^{N} \frac{1}{t + a_1 - a_i}\right) + \xi^2 \sum_{i=1}^{N} \frac{1}{(t + a_1 - a_i)^2}$$

Setting $x = \lambda_1 + tN^{-2/3}$, we have, using a formal application of the semi-circle law,

$$N^{1/3} - \sum_{i=1}^{N} \frac{1}{t + a_1 - a_i} = N^{1/3}\left(1 - \frac{1}{N}\sum_{i=1}^{N} \frac{1}{x - \lambda_i}\right) \simeq N^{1/3}\,(1 - s_1(x))\,.$$

Using (4.6), the above equation becomes

$$N^{1/3} - \sum_{i=1}^{N} \frac{1}{t + a_1 - a_i} \simeq N^{1/3}\sqrt{x - 2} \simeq \sqrt{t}.$$

For the other term,

$$\sum_{i=1}^{N} \frac{1}{(t + a_1 - a_i)^2} = \frac{1}{N^{4/3}} \sum_{i=1}^{N} \frac{1}{(x - \lambda_i)^2} \simeq \frac{1}{N^{1/3}} s_2(x) \simeq \frac{1}{N^{1/3} 2\sqrt{x - 2}} \simeq \frac{1}{2\sqrt{t}}.$$

Hence, the generating function on the right-hand side of (10.38) is approximately $e^{\sqrt{t}\xi + \frac{\xi^2}{2\sqrt{t}}}$. Therefore,

$$\Upsilon_N(t) \overset{\mathfrak{D}}{\simeq} \sqrt{t} + t^{-1/4}\mathfrak{N}$$

for a thermal standard normal random variable $\mathfrak{N}$. Inserting the large H formula (7.20) for t and replacing $H = hN^{1/6}$, we find that if we take $h = HN^{-1/6}$ and let $N \to \infty$ first and then take $H \to \infty$, we get

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} 1 - T + \frac{Th^2}{2(1 - T)} + \frac{T}{hN^{1/2}}\left[\frac{h^5 \mathcal{S}_N(\gamma_0; 2)}{2(1 - T)^2} + \sqrt{2(1 - T)}\mathfrak{N}\right]. \tag{10.40}$$

This is the same as (10.29) which is obtained by first taking $N \to \infty$ with $h > 0$ fixed and then taking $h \to 0$. Therefore, the result matches with the $h = O(1)$ case.

10.2.3 Limit as $H \to 0$

From (7.19), $t = O(H) \to 0$ as $H \to 0$. The generating function on the right-hand side of (10.38) converges to

$$e^{\xi N^{1/3} - \frac{1}{2}\sum_{i=2}^{N} \log\left(1 + \frac{2\xi}{t + a_1 - a_i}\right)}$$

where the term $i = 1$ cancels out with the limit of the big square root term. Using the moment generating function (9.12) for the chi-squared distribution, we find that if we take

$h = HN^{-1/6}$ and $N \to \infty$ and then take $H \to 0$, then

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} 1 - T + \frac{T}{N^{1/3}} \left(N^{1/3} - \sum_{i=2}^{N} \frac{\mathfrak{n}_i^2}{a_1 - a_i} \right). \tag{10.41}$$

for independent thermal standard Gaussian random variables $\mathfrak{n}_i$.

10.3 Microscopic External Field: $h \sim HN^{-1/2}$ and $T < 1$

10.3.1 Analysis

Set

$$h = HN^{-1/2} \tag{10.42}$$

for fixed $H > 0$. It turns out that the fluctuations are of order $\mathcal{O}(1)$. In other words, the leading term of $\mathfrak{R}$ converges to a random variable. We set

$$\eta = \beta\xi \quad \text{so that} \quad a = \xi N^{-1}. \tag{10.43}$$

The critical point of $\mathcal{G}(z)$ is $\gamma = \lambda_1 + pN^{-1}$ from (8.48). Consider the critical point of $\mathcal{G}_{\mathfrak{R}}$. Lemma 10.1 implies that $\gamma_{\mathfrak{R}} = \lambda_1 + \mathcal{O}\left(N^{-1}\right)$. We set

$$\gamma_{\mathfrak{R}} = \lambda_1 + q_{\mathfrak{R}} N^{-1}, \qquad q_{\mathfrak{R}} > |\xi|, \tag{10.44}$$

for some $q_{\mathfrak{R}}$. Separating $i = 1$ in the equation (10.6), we find that $q_{\mathfrak{R}}$ is the solution of the equation

$$\beta - 1 - \frac{q_{\mathfrak{R}}}{q_{\mathfrak{R}}^2 - \xi^2} - \frac{H^2 \beta n_1^2}{(q_{\mathfrak{R}} - \xi)^2} + \mathcal{O}\left(N^{-1/3}\right) = 0. \tag{10.45}$$

When $\beta = T^{-1} > 1$, the equation $\beta - 1 - \frac{x}{x^2 - \xi^2} - \frac{H^2 \beta n_1^2}{(x-\xi)^2} = 0$ has a unique solution and $q_{\mathfrak{R}}$ is approximated by this solution with error $\mathcal{O}\left(N^{-1/3}\right)$.

Using (10.7) and separating out the $i = 1$ term, we find that $N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a) - 2\mathcal{G}(\gamma))$ is equal to

$$-\log\left(\frac{q_{\mathfrak{R}}^2 - \xi^2}{p^2}\right) + \frac{2(q_{\mathfrak{R}} - p)}{p} + 2H^2 \beta n_1^2 \left[\frac{1}{q_{\mathfrak{R}} - \xi} - \frac{1}{p} + \frac{q_{\mathfrak{R}} - p}{p^2}\right] + \mathcal{O}\left(N^{-1/3}\right). \tag{10.46}$$

Using the equation for p, this can be written as

$$\begin{aligned} &N(\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}}, \gamma_{\mathfrak{R}}; a) - 2\mathcal{G}(\gamma)) \\ &\quad = -\log\left(\frac{q_{\mathfrak{R}}^2 - \xi^2}{p^2}\right) + 2(\beta - 1)(q_{\mathfrak{R}} - p) + 2H^2 \beta n_1^2 \left[\frac{1}{q_{\mathfrak{R}} - \xi} - \frac{1}{p}\right] + \mathcal{O}\left(N^{-1/3}\right). \end{aligned} \tag{10.47}$$

We now consider the integrals in (10.2). As in Sect. 8.6 of the overlap with the external field when $h \sim N^{-1/2}$, the main contribution to the integral comes from a neighborhood of radius N^{-1} around the critical point in both variables. Changing variables to $z = \gamma_{\mathfrak{R}} + uN^{-1}$ and $w = \gamma_{\mathfrak{R}} + vN^{-1}$, we find that all terms of the Taylor series are of the same order, so we see, as in Sect. 8.6, that the integral is not approximated by a Gaussian integral. Therefore, we proceed by writing

$$\begin{aligned}
&N(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a))\\
&\quad= N(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a)-(\mathcal{G}_{\mathfrak{R}})_z(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a)(z-\gamma_{\mathfrak{R}})\\
&\qquad-(\mathcal{G}_{\mathfrak{R}})_w(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a)(w-\gamma_{\mathfrak{R}}))\\
&\quad= -\sum_{i=1}^{N}\left[\log\left(\frac{(z-\lambda_i)(w-\lambda_i)-a^2}{(\gamma_{\mathfrak{R}}-\lambda_i)^2-a^2}\right)-\frac{(\gamma_{\mathfrak{R}}-\lambda_i)(z+w-2\gamma_{\mathfrak{R}})}{(\gamma_{\mathfrak{R}}-\lambda_i)^2-a^2}\right]\\
&\qquad+h^2\beta\sum_{i=1}^{N}n_i^2\left[\frac{z+w-2\lambda_i+2a}{(z-\lambda_i)(w-\lambda_i)-a^2}-\frac{2}{\gamma_{\mathfrak{R}}-\lambda_i-a}+\frac{z+w-2\gamma_{\mathfrak{R}}}{(\gamma_{\mathfrak{R}}-\lambda_i-a)^2}\right].
\end{aligned}$$

Inserting the change of variables and separating $i=1$ out,

$$\begin{aligned}
&N(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a))\\
&\quad\simeq -\log\left(\frac{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}{q_{\mathfrak{R}}^2-\xi^2}\right)+\frac{q_{\mathfrak{R}}(u+v)}{q_{\mathfrak{R}}^2-\xi^2}\\
&\qquad+H^2\beta n_1^2\left[\frac{u+v+2q_{\mathfrak{R}}+2\xi}{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}-\frac{2}{q_{\mathfrak{R}}-\xi}+\frac{u+v}{(q_{\mathfrak{R}}-\xi)^2}\right]
\end{aligned}$$

for finite u and v. Using the Eq. (10.45), this can be written as

$$\begin{aligned}
&N(\mathcal{G}_{\mathfrak{R}}(z,w;a)-\mathcal{G}_{\mathfrak{R}}(\gamma_{\mathfrak{R}},\gamma_{\mathfrak{R}};a))\\
&\quad\simeq -\log\left(\frac{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}{q_{\mathfrak{R}}^2-\xi^2}\right)+(\beta-1)(u+v)\\
&\qquad+H^2\beta n_1^2\left[\frac{u+v+2q_{\mathfrak{R}}+2\xi}{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}-\frac{2}{q_{\mathfrak{R}}-\xi}\right].
\end{aligned}$$

Thus, the numerator integral in (10.2) is asymptotically equal to

$$\frac{\sqrt{q_{\mathfrak{R}}^2-\xi^2}}{N^2}\int\int\frac{e^{\frac{1}{2}(\beta-1)(u+v)+\frac{H^2\beta n_1^2}{2}\left[\frac{u+v+2q_{\mathfrak{R}}+2\xi}{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}-\frac{2}{q_{\mathfrak{R}}-\xi}\right]}}{\sqrt{(u+q_{\mathfrak{R}})(v+q_{\mathfrak{R}})-\xi^2}}\mathrm{d}u\mathrm{d}v$$

where the contours are from $-\mathrm{i}\infty$ to $\mathrm{i}\infty$ such that all singularities lie on the left of the contours. The denominator integral is the same with $\xi=0$.

Combining the above calculations into (10.2) and making simple translations for the integral, we find that

$$\langle e^{\beta\xi\mathfrak{R}}\rangle\simeq\frac{\int\int\frac{1}{\sqrt{uv-\xi^2}}e^{\frac{1}{2}(\beta-1)(u+v)+\frac{H^2\beta n_1^2(u+v+2\xi)}{2(uv-\xi^2)}}\mathrm{d}u\mathrm{d}v}{\left(\int\frac{1}{\sqrt{u}}e^{\frac{1}{2}(\beta-1)u+\frac{H^2\beta n_1^2}{2u}}\mathrm{d}u\right)^2}\tag{10.48}$$

where the contours are vertical lines such that the points ξ or 0 lie on the left of the contours. We now evaluate the integrals using (recall (8.59))

$$\int\frac{e^{au+\frac{b}{u}}}{\sqrt{u}}\mathrm{d}u=\frac{2\mathrm{i}\sqrt{\pi}}{\sqrt{a}}\cosh(2\sqrt{ab}).\tag{10.49}$$

Consider the double integral in the numerator. For each v, we change the variable u to z by setting $uv - \xi^2 = z$. We can define the branch cut appropriately such that the contour for z does not cross the branch cut. The numerator becomes

$$\iint \frac{1}{v\sqrt{z}} e^{\frac{\beta-1}{2}(\frac{z+\xi^2}{v}+v)+\frac{H^2\beta n_1^2}{2z}(\frac{z+\xi^2}{v}+v+2\xi)} \mathrm{d}z\mathrm{d}v.$$

The z-integral can be evaluated using (10.49). Writing the resulting cosh term as the sum of two exponentials, we can evaluate the w-integral again using (10.49). The above double integral becomes

$$-\frac{2\pi}{\beta-1}\Big[e^{\sqrt{(\beta-1)\beta}H|n_1|}\cosh\left(\sqrt{(\beta-1)\beta}H|n_1| + (\beta-1)\xi\right) \\ + e^{-\sqrt{(\beta-1)\beta}H|n_1|}\cosh\left(\sqrt{(\beta-1)\beta}H|n_1| - (\beta-1)\xi\right)\Big].$$

Writing cosh as the sum of two exponentials again, the expression above becomes a linear combination of $e^{(\beta-1)\xi}$ and $e^{-(\beta-1)\xi}$. The denominator in (10.48) is the same as the numerator when $\xi = 0$. Thus, using $\beta = 1/T$ and re-scaling ξ, we obtain the following

Result 10.6 *For* $h = HN^{-1/2}$ *and* $0 < T < 1$,

$$\langle e^{\xi \frac{\mathfrak{R}}{1-T}} \rangle \simeq \frac{\cosh\left(\frac{2\sqrt{1-T}H|n_1|}{T}\right)e^{\xi} + e^{-\xi}}{\cosh\left(\frac{2\sqrt{1-T}H|n_1|}{T}\right)+1} \tag{10.50}$$

as $N \to \infty$ *for asymptotically almost every disorder sample.*

Recognizing that the right-hand side is the moment generating function of a shifted Bernoulli random variable, we obtain the following result.

Result 10.7 *For* $h = HN^{-1/2}$ *and* $0 < T < 1$,

$$\frac{\mathfrak{R}}{1-T} \overset{\mathfrak{D}}{\simeq} \mathfrak{B}(\theta), \qquad \theta := \frac{\cosh\left(\frac{2\sqrt{1-T}H|n_1|}{T}\right)}{\cosh\left(\frac{2\sqrt{1-T}H|n_1|}{T}\right)+1} \tag{10.51}$$

as $N \to \infty$ *for asymptotically almost every disorder sample, where the thermal random variable* $\mathfrak{B}(c)$ *is the (shifted) Bernoulli distribution taking values* 1 *and* -1 *with probability* c *and* $1-c$*, respectively.*

10.3.2 Limits as $H \to \infty$

If we formally take the limit as $H \to \infty$ of the result (10.51), then

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} 1 - T. \tag{10.52}$$

This is the same as the leading term of (10.41) which is obtained by taking $h = HN^{-1/6}$ and letting $N \to \infty$ first and then taking $H \to 0$.

10.4 No External Field: $h = 0$

For $0 < T < 1$, the analysis in Sect. 10.3 for $h = HN^{-1/2}$ extends to $H = 0$ case as well. For $T > 1$, the analysis in Sect. 10.1 applies to all $h \geq 0$. We note that, for $h = 0$ and $T > 1$, $\gamma_0 = T + T^{-1}$ and $s_2(\gamma_0) = \frac{1}{T^2-1}$. We have the following result.

Result 10.8 *For $h = 0$,*

$$\mathfrak{R} \overset{\mathfrak{D}}{\simeq} \begin{cases} \frac{T}{\sqrt{N(T^2-1)}}\mathfrak{N} & \text{for } T > 1, \\ (1-T)\,\mathfrak{B}(1/2) & \text{for } 0 < T < 1. \end{cases} \tag{10.53}$$

11 Geometry of the Spin Configuration

The results on three types of overlaps tell us how the spin variables are distributed on the sphere. We discuss the geometry of the spin configuration vector $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_N)$ from the Gibbs measure in this section. Recall that $\mathbf{u}_1$ is a unit vector which is parallel to the eigenvector corresponding to the largest eigenvalue of the disorder matrix. In this section, we choose $\mathbf{u}_1$, among two opposite directions, as the one satisfying $\mathbf{u}_1 \cdot \mathbf{g} \geq 0$. Recall the notation $n_1 = \mathbf{g} \cdot \mathbf{u}_1$ and that the external field $\mathbf{g}$ is a standard Gaussian vector. Note that $n_1 = |n_1|$ because of the choice of $\mathbf{u}_1$. The normalized spin vector can be decomposed as

$$\hat{\boldsymbol{\sigma}} := \frac{\boldsymbol{\sigma}}{\sqrt{N}} = a\mathbf{u}_1 + b\frac{\mathbf{g} - n_1\mathbf{u}_1}{\|\mathbf{g} - n_1\mathbf{u}_1\|} + \mathbf{v}, \qquad \mathbf{v} \cdot \mathbf{u}_1 = \mathbf{v} \cdot \mathbf{g} = 0, \tag{11.1}$$

where a and b are components of the normalized spin vector in the $\mathbf{u}_1$ and $\mathbf{g}_1 - n_1\mathbf{u}_1$ directions, respectively. The vector $\mathbf{v}$ is perpendicular to both $\mathbf{u}_1$ and $\mathbf{g}$, and it satisfies

$$\|\mathbf{v}\|^2 = 1 - a^2 - b^2. \tag{11.2}$$

Note that $\|\mathbf{g} - n_1\mathbf{u}_1\|^2 = \|\mathbf{g}\|^2 - n_1^2 \simeq N + \mathcal{O}\left(N^{1/2}\right)$ and $n_1 = \mathcal{O}(1)$. Thus, if we ignore subleading terms from each component, the above decomposition becomes

$$\hat{\boldsymbol{\sigma}} \simeq a\mathbf{u}_1 + b\frac{\mathbf{g}}{\sqrt{N}} + \mathbf{v} = a\mathbf{u}_1 + b\hat{\mathbf{g}} + \mathbf{v}, \qquad \hat{\mathbf{g}} := \frac{\mathbf{g}}{\sqrt{N}}. \tag{11.3}$$

The components a and b are related to the overlaps by the formulas

$$\mathfrak{O} = (\hat{\boldsymbol{\sigma}} \cdot \mathbf{u}_1)^2 = a^2, \qquad \mathfrak{M} = \hat{\boldsymbol{\sigma}} \cdot \hat{\mathbf{g}} = \frac{an_1}{\sqrt{N}} + b\frac{\|\mathbf{g} - n_1\mathbf{u}_1\|}{\sqrt{N}} \simeq \frac{an_1}{\sqrt{N}} + b \tag{11.4}$$

up to $\mathcal{O}\left(N^{-1}\right)$ terms. Furthermore, $\mathbf{v}$ satisfies the equation

$$\mathfrak{R} = \hat{\boldsymbol{\sigma}}^{(1)} \cdot \hat{\boldsymbol{\sigma}}^{(2)} = a_1a_2 + b_1b_2 + \mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)}. \tag{11.5}$$

11.1 The Signed Overlap with a Replica for Microscopic Field, $h \sim N^{-1/2}$ and $T < 1$

Consider the decomposition for $h = HN^{-1/2}$ and $0 < T < 1$. The overlap with the ground state is given in Result 9.6 for $h \sim N^{-1/3}$ and Result 9.7 for $h = 0$. Since the leading terms of the both results are same, given by $1 - T$, the leading term holds also for $h \sim N^{-1/2}$. Thus, we find that $a^2 \simeq 1 - T$ in this regime, and hence $|a| \simeq \sqrt{1-T}$. On the other hand, Result 8.7 on $\mathfrak{M}$ implies that

$$\frac{an_1}{\sqrt{N}} + b \overset{\mathfrak{D}}{\simeq} h + \frac{|n_1|\sqrt{1-T}\mathfrak{B}(\alpha)}{\sqrt{N}} + \frac{\sqrt{T}\mathfrak{N}}{\sqrt{N}}. \tag{11.6}$$

Noting $h \sim N^{-1/2}$, we find that $b = \mathcal{O}\left(N^{-1/2}\right)$. From the formulas of a and b, we also find that $\|\mathbf{v}\|^2 = 1 - a^2 - b^2 \simeq T$. Finally, Result 10.7 implies that

$$a_1 a_2 + b_1 b_2 + \mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)} \overset{\mathfrak{D}}{\simeq} (1-T)\mathfrak{B}(\theta). \tag{11.7}$$

Here, θ is given in (10.51) and α in (11.6) is given by (8.64). They satisfy the relation $\theta = \alpha^2 + (1-\alpha)^2$. Now, we make the following ansatz on a. For $h = 0$ and $0 < T < 1$, the spin configurations are equally likely to be on either of the double cones around $\mathbf{u}_1$ with the cosine of the angle given by $\sqrt{1-T}$. This means that $a \overset{\mathfrak{D}}{\simeq} \sqrt{1-T}\,\mathfrak{B}(1/2)$ for $h = 0$ and $0 < T < 1$. For $h \sim N^{-1/2}$, we make the ansatz that

$$a = \hat{\boldsymbol{\sigma}} \cdot \mathbf{u}_1 \overset{\mathfrak{D}}{\simeq} \sqrt{1-T}\,\mathfrak{B}(\varphi) \tag{11.8}$$

for some φ which we determine now. Note that if X_1 and X_2 are independent (thermal) random variables distributed as $\mathfrak{B}(\varphi)$, then their product $X_1 X_2$ is $\mathfrak{B}(\varphi^2 + (1-\varphi)^2)$-distributed. Thus, the Eqs. (11.6) and (11.7) become

$$\frac{|n_1|\sqrt{1-T}\,\mathfrak{B}(\varphi)}{\sqrt{N}} + b \overset{\mathfrak{D}}{\simeq} h + \frac{|n_1|\sqrt{1-T}\mathfrak{B}(\alpha)}{\sqrt{N}} + \frac{\sqrt{T}\mathfrak{N}}{\sqrt{N}}$$

and

$$(1-T)\,\mathfrak{B}(\varphi^2 + (1-\varphi)^2) + \mathcal{O}\left(N^{-1}\right) + \mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)} \overset{\mathfrak{D}}{\simeq} (1-T)\mathfrak{B}(\theta).$$

Since $\theta = \alpha^2 + (1-\alpha)^2$, it is reasonable to assume that the solutions are $\varphi = \alpha$, and

$$a \overset{\mathfrak{D}}{\simeq} \sqrt{1-T}\,\mathfrak{B}(\alpha), \qquad b \overset{\mathfrak{D}}{\simeq} h + \frac{\sqrt{T}\mathfrak{N}}{\sqrt{N}}.$$

This calculation leads us to the following conjecture on the signed overlap of the spin variable with a replica.

Conjecture 11.1 *For a given disorder sample, let $\boldsymbol{u}_1$ be the unit vector corresponding to the ground state such that $\boldsymbol{u}_1 \cdot \boldsymbol{g} \geq 0$. Then, for $h = HN^{-1/2}$ and $0 < T < 1$, the signed overlap with the ground state satisfies*

$$\frac{\boldsymbol{\sigma} \cdot \boldsymbol{u}_1}{\sqrt{N}} \overset{\mathfrak{D}}{\simeq} \sqrt{1-T}\,\mathfrak{B}(\alpha), \qquad \alpha = \frac{e^{\frac{H|n_1|\sqrt{1-T}}{T}}}{e^{\frac{H|n_1|\sqrt{1-T}}{T}} + e^{-\frac{H|n_1|\sqrt{1-T}}{T}}}, \tag{11.9}$$

as $N \to \infty$ for asymptotically almost every disorder sample.

The above conjecture implies that for $h = HN^{-1/2}$ the spin configuration vector concentrates on the intersection of the sphere and the double cone around $\mathbf{u}_1$ where the cosine of the angle is $\sqrt{1-T}$, just like the $h = 0$ case. However, while for $H = 0$ the spin vector is equally likely to be on either of the cones, for $H > 0$ the spin prefers the cone that is closer to $\mathbf{g}$ than the other cone. As $H \to \infty$, the polarization parameter $\alpha \to 1$ and hence for $h \gg N^{-1/2}$, the spin vector is concentrated on one of the cones.

11.2 Spin Decompositions in Various Regimes

The results of the overlaps give us information about the decomposition of the spin for other regimes of h as well. From the first equation of (11.4), we find a^2, and hence $|a|$. The discussion of the previous subsection implies that for $h \gg N^{-1/2}$, the spin vector concentrates on one of the cones. Thus, we expect that $a = |a|$ for such h. Using this formula of a, we then obtain b from the second equation of (11.4), from which we also find $\|\mathbf{v}\|^2 = 1 - a^2 - b^2$. Finally, the equation (11.5) implies $\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)}$, and hence, the overlap $\hat{\mathbf{v}}^{(1)} \cdot \hat{\mathbf{v}}^{(2)}$ of the unit transversal vector $\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ with its replica. We summarize the findings in Table 5. The result for the last row follows from the last subsection.

The result for the regime $h \sim N^{-1/6}$ (fourth row) follows from Results 9.4, 8.5, and 10.5. The term $\mathcal{A} = \mathcal{A}(T, hN^{1/6})$ is given by the leading term in Result 9.4,

$$\mathcal{A} = \sqrt{1 - T - h^2 N^{1/3} \sum_{i=2}^{N} \frac{n_i^2}{(t + a_1 - a_i)^2}} = \frac{hN^{1/6}|n_1|}{t}, \tag{11.10}$$

where $t > 0$ is the number that makes the two formulas of $\mathcal{A}$ equal. For every disorder sample, $\mathcal{A}$ is a decreasing function of $H = hN^{1/6}$, changing from $\sqrt{1-T}$ for $H = 0$ to 0 as $H \to \infty$.

The result for the regime $h = O(1)$ (second row) follows from Result 9.2, 8.2, and 10.3. The variable $\gamma_0 = \gamma_0(T, h) > 2$ is the solution of the equation (5.26). It satisfies $\gamma_0 \simeq h + \frac{T}{2}$ as $h \to \infty$ and $\gamma_0 \simeq 2 + \frac{h^4}{4(1-T)^2}$ as $h \to 0$: See Lemma 5.7. The function $s_1(z)$ is the Stieltjes transform of the semicircle law. It satisfies $s_1(z) = z^{-1} + O(z^{-3})$ as $z \to \infty$ and $s_1(z) \simeq 1 - \sqrt{z-2}$ as $z \to 2$: see (4.6). See Sect. 8.3.2 for properties of $\mathfrak{M}^0 = hs_1(\gamma_0)$. The term $\frac{\sqrt{\mathfrak{O}^0}}{\sqrt{N}}$ is from Result 9.2 and is given by

$$\frac{\sqrt{\mathfrak{O}^0}}{\sqrt{N}} = \frac{1}{\sqrt{N}} \left| \frac{h|n_1|}{\gamma_0 - 2} + \frac{\sqrt{T}\mathfrak{N}}{\sqrt{\gamma_0 - 2}} \right|. \tag{11.11}$$

For the last column, Result 10.3 and the formula $b \simeq hs_1(\gamma_0)$ imply that $\mathbf{v}^{(1)} \cdot \mathbf{v}^{(2)} \simeq h^2 s_2(\gamma_0) - h^2 s_1(\gamma_0)^2$. We use the identity $s_2(z) = s_1(z)^2/(1 - s_1(z)^2)$ for $z > 0$ to simplify the formula.

The third row follows either from the fourth row or from the second row. Starting from the fourth row, we use (9.26), which shows that

$$\mathcal{A}^2 \simeq \frac{16(1-T)^4 n_1^2}{h^6 N} \tag{11.12}$$

as $hN^{1/6} \to \infty$. We can also see this formula from (11.10) because $t \simeq \frac{h^4 N^{2/3}}{4(1-T)^2}$ (see (7.20)). Note that $\mathcal{A} = o(1)$ in this regime. On the other hand, if we start from the second row, we use (9.16) to find the same formula for a. Other columns can be found from $s_1(\gamma_0) \simeq 1 - \frac{h^2}{2(1-T)}$ as $h \to 0$. Note that the two components a and b are comparable in size for $h \sim N^{-1/8}$.

The quantity a in the fifth row follows either from the fourth row or from the last row. The formula (9.27) shows that $\mathcal{A}^2 \simeq 1 - T$ as $hN^{1/6} \to 0$. We also see this formula from (11.10) by dropping the $o(1)$ term. If we start from the last row, the polarization parameter α satisfies $\alpha \to 1$ as $hN^{1/2} \to \infty$, and hence $a \simeq \sqrt{1-T}$, giving the same formula for a. The other columns follow from this result. One can show using Result 9.6 and (9.41) that the subleading term in a (not shown in Table 5) is comparable to the leading term of b, which is h, when $h \sim N^{-1/3}$.

11.3 Summary

Three quantities contain thermal random variables: a for the regimes $h = O(1)$ and $h \sim N^{-1/2}$, and b for the regime $h \sim N^{-1/2}$. Among those, a for the regime $h \sim N^{-1/2}$ is $\mathcal{O}(1)$ but the other two quantities are of smaller order $\mathcal{O}\left(N^{-1/2}\right)$.

The table shows that $a = O(1)$ for $h \le O(N^{-1/6})$ and $b = O(1)$ for $h \ge O(1)$. As h increases, the $\mathbf{u}_1$ component of a typical spin vector decreases while the $\hat{\mathbf{g}}$ component increases. The above result shows that the crossover occurs in the regime $N^{-1/6} \ll h \ll O(1)$ in which both components are $o(1)$.

The last column of the table is the overlap of the unit transversal vector $\hat{\mathbf{v}}$ with its replica. This overlap is $o(1)$ for $h \ll N^{-1/6}$. If the error were $O(N^{-1/2})$, it would give a strong indication that the thermal distribution of $\hat{\mathbf{v}}$ is uniform on the transverse space (i.e. the set of unit vectors that are perpendicular to $\mathbf{u}_1$ and $\mathbf{g}$). The above result does not show the error, but we expect that the distribution on the transverse space is close to being uniform. On the other hand, for $h \ge O(N^{-1/6})$, the overlap of the unit transversal vector is non-zero and $\mathcal{O}(1)$. This implies that $\hat{\mathbf{v}}$ is not uniformly distributed on the transverse space.

Overall, for $0 < T < 1$, as we increase the external field, we expect the following geometry of the spin vector that is randomly chosen using the Gibbs (thermal) measure for a quenched disorder, i.e. for asymptotically almost every disorder sample.

- For $h \ll N^{-1/6}$, the spin vector is on a double cone around $\mathbf{u}_1$ (possibly preferring one cone to the other), and the thermal distribution on the transverse space is close to being uniform.
- For $h \sim N^{-1/6}$, the spin vector is polarized to a single cone around $\mathbf{u}_1$, but the cone itself depends non-trivially on the disorder sample. The thermal distribution on the transverse space is not uniform and depends on the disorder sample.
- For $N^{-1/6} \ll h \ll O(1)$, the spin vector entirely lies on the transverse space with only $o(1)$ components on the ground state and external field directions. Although the thermal distribution is not uniform, it does not depend on the disorder sample.
- For $h = O(1)$, the spin vector is on a cone around $\mathbf{g}$ and the thermal distribution on the transverse space is not uniform. The cone and the distribution on the transverse space do not depend on the disorder sample.
- For $h \to \infty$, the spin vector is parallel to $\mathbf{g}$.

The result of this paper does not describe the distribution of $\hat{\mathbf{v}}$ on the transverse space in detail. This can be achieved by studying the overlaps $\sigma \cdot \mathbf{u}_i$ with other eigenvectors. This analysis can be done using the method of this paper and we leave this work as a future project.

The items in the table can be written in an uniform formula across all regimes as the following decomposition formula of the spin configuration vector:

$$\hat{\boldsymbol{\sigma}} \overset{\mathfrak{D}}{\simeq} \mathcal{A}\mathfrak{B}(\alpha)\mathbf{u}_1 + hs_1(\gamma_0)\hat{\mathbf{g}} + \sqrt{1 - \mathcal{A}^2 - h^2 s_1(\gamma_0)^2}\hat{\mathbf{v}} + \mathcal{O}\left(N^{-1/2}\right) \tag{11.13}$$

where $\hat{\mathbf{v}}$ is a unit vector in the transverse space, i.e. $\hat{\mathbf{v}} \cdot \mathbf{u}_1 = \hat{\mathbf{v}} \cdot \hat{\mathbf{g}} = 0$ and $\|\hat{\mathbf{v}}\| = 1$. All items in the middle three columns of the table other than two items, a for the regime $h = O(1)$ and b for the regime $h \sim N^{-1/2}$, are of order greater than $\mathcal{O}\left(N^{-1/2}\right)$. Hence, the above formula is meaningful for all items except those two.

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Appendix A Proof of Lemma 3.3

We prove Lemma 3.3. First,

$$\langle e^{\beta\eta\mathfrak{M}}\rangle = \frac{1}{\mathcal{Z}_N(h)}\int_{S_{N-1}} e^{\beta\frac{\eta}{N}\mathbf{g}\cdot\boldsymbol{\sigma}} e^{\beta\left(\frac{1}{2}\boldsymbol{\sigma}\cdot M\boldsymbol{\sigma}+h\mathbf{g}\cdot\boldsymbol{\sigma}\right)}\mathrm{d}\omega_N(\boldsymbol{\sigma}) = \frac{\mathcal{Z}_N(h+\eta N^{-1})}{\mathcal{Z}_N(h)}.$$

Secondly, by definition,

$$\langle e^{\beta\eta\mathfrak{O}}\rangle = \frac{1}{\mathcal{Z}_N}\int_{S_{N-1}} e^{\beta\frac{\eta}{N}(\mathbf{u}_1\cdot\sigma)^2} e^{\beta(\boldsymbol{\sigma}\cdot M\boldsymbol{\sigma}+h\mathbf{g}\cdot\boldsymbol{\sigma})}\mathrm{d}\omega_N(\boldsymbol{\sigma}). \tag{A.1}$$

Since

$$\frac{1}{2}\boldsymbol{\sigma}\cdot M\boldsymbol{\sigma} + \frac{\eta}{N}(\mathbf{u}_1\cdot\sigma)^2 = \frac{1}{2}\sum_{i=1}^{N}\lambda_i(\mathbf{u}_i\cdot\sigma)^2 + \frac{\eta}{N}(\mathbf{u}_1\cdot\sigma)^2,$$

the integral in (A.1) is the same as that of $\mathcal{Z}_N$ with $\lambda_1 \mapsto \lambda_1 + \frac{2\eta}{N}$. Finally, using the eigenvalue-eigenvector decomposition $M = O\Lambda O^T$ and changing variables $\frac{1}{\sqrt{N}}O^T\sigma = x$ and $\frac{1}{\sqrt{N}}O^T\tau = y$, we find that

$$\langle e^{\eta\mathfrak{R}}\rangle = \frac{J(\frac{\beta N}{2}, \frac{\beta N}{2}; \frac{\eta}{N\beta}, \frac{\sqrt{\beta}h}{\sqrt{2}}, \frac{\sqrt{\beta}h}{\sqrt{2}})}{J(\frac{\beta N}{2}, \frac{\beta N}{2}; 0, \frac{\sqrt{\beta}h}{\sqrt{2}}, \frac{\sqrt{\beta}h}{\sqrt{2}})}. \tag{A.2}$$

where we use the notation

$$\begin{aligned} &J(u, v; a, b, c) \\ &= (uv)^{\frac{N}{2}-1}\int\int e^{2a\sqrt{uv}\sum_{i=1}^{N}x_iy_i + u\sum_{i=1}^{N}\lambda_i x_i^2 + 2b\sqrt{u}\sum_{i=1}^{N}n_ix_i + v\sum_{i=1}^{N}\lambda_i y_i^2 + 2c\sqrt{v}\sum_{i=1}^{N}n_iy_i}\mathrm{d}\Omega_{N-1}^{\otimes 2}(x, y). \end{aligned}$$

We evaluate the Laplace transform of $J(u, v, a, b, c)$. Changing of variable as $u = r^2, v = s^2$ and $rx \mapsto x, sy \mapsto y$, the Laplace transform

$$Q(z, w) = \int_0^\infty\int_0^\infty e^{-zu-wv}J(u, v)\mathrm{d}u\mathrm{d}v$$

becomes a 2-dimensional Gaussian integral which evaluates to

$$Q(z, w) = 4\prod_{i=1}^{N}\frac{\pi}{\sqrt{(z-\lambda_i)(w-\lambda_i)-a^2}}e^{\frac{n_i^2((w-\lambda_i)b^2+2abc+(z-\lambda_i)c^2}{(z-\lambda_i)(w-\lambda_i)-a^2}}.$$

The inverse Laplace transform gives a double integral formula for $J(u, v)$.

Appendix B A perturbation argument

The following perturbation lemma is used to obtain (5.7), (5.22) and (6.4).

Lemma B.1 *Let I be a closed interval of $\mathbb{R}$. Let $G(z; N)$ be a sequence of random C^4-functions for $z \in I$. Let $\epsilon = \epsilon(N) := N^{-\delta}$ for some $\delta > 0$ and assume that*

$$G(z; N) = G_0(z; N) + G_1(z; N)\epsilon + G_2(z; N)\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) \tag{B.1}$$

and

$$G'(z; N) = G_0'(z; N) + G_1'(z; N)\epsilon + G_2'(z; N)\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) \tag{B.2}$$

for random C^4-functions $G_k(z; N)$. Suppose that

$$G_k^{(\ell)}(z; N) = \mathcal{O}(1) \tag{B.3}$$

uniformly for $z \in I$ for all $k = 0, 1, 2$, $0 \le \ell \le 4$ and also assume that there is a $\gamma_0 \in I$ satisfying

$$G_0'(\gamma_0; N) = 0, \qquad |G_0''(\gamma_0; N)| \ge C > 0 \tag{B.4}$$

for a positive constant C. Then there is a critical point $\gamma = \gamma(N)$ of $G(z; N)$ admitting the asymptotic expansion

$$\gamma = \gamma_0 + \gamma_1\epsilon + \gamma_2\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) \tag{B.5}$$

where

$$\gamma_1 = -\frac{G_1'(\gamma_0; N)}{G_0''(\gamma_0; N)}, \qquad \gamma_2 = -\frac{G_2'(\gamma_0; N) + G_1''(\gamma_0; N)\gamma_1 + \frac{1}{2}G_0'''(\gamma_0; N)\gamma_1^2}{G_0''(\gamma_0; N)}. \tag{B.6}$$

Furthermore,

$$G(\gamma; N) = G_0(\gamma_0; N) + G_1(\gamma_0; N)\epsilon + \left(\frac{1}{2}G_1'(\gamma_0; N)\gamma_1 + G_2(\gamma_0; N)\right)\epsilon^2 + \mathcal{O}\left(\epsilon^3\right). \tag{B.7}$$

Proof This lemma is standard when $G(z; N)$ is deterministic. The proof for the random $G(z; N)$ does not change. For simplicity, we suppress the dependence on N in the notations; for example we write $G_0(z)$ instead of $G_0(z; N)$. In order to prove (B.5), it is enough to show that for any $0 < t < \delta$, $G'(\gamma_+)G'(\gamma_-) < 0$ with $\gamma_\pm = \gamma_0 + \gamma_1\epsilon + \gamma_2\epsilon^2 \pm \epsilon^3 N^t$. From the Taylor expansion,

$$\begin{aligned} G'(\gamma_\pm) &= G_0'(\gamma_0) + (G_0''(\gamma)\gamma_1 + G_1'(\gamma_0))\epsilon \\ &\quad + \left(G_0''(\gamma_0)\gamma_2 + G_2'(\gamma_0) + G_1''(\gamma_0)\gamma_1 + \frac{1}{2}G_0'''(\gamma_0)\gamma_1^2\right)\epsilon^2 \\ &\quad \pm G_0''(\gamma_0)\epsilon^3 N^t + \mathcal{O}\left(\epsilon^3\right). \end{aligned} \tag{B.8}$$

The definitions of γ_0, γ_1, and γ_2 imply that

$$G'(\gamma_\pm) = \pm G_0''(\gamma_0)\epsilon^3 N^t + \mathcal{O}\left(\epsilon^3\right) \tag{B.9}$$

Thus, $G'(\gamma_+)G'(\gamma_-) < 0$ for all large enough N and we obtain (B.5). The Eq. (B.7) follows from

$$\begin{aligned} G(\gamma) &= G_0(\gamma) + G_1(\gamma)\epsilon + G_2(\gamma)\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) = G_0(\gamma_0) + (G_0'(\gamma_0)\gamma_1 + G_1(\gamma_0))\epsilon \\ &\quad + \left(G_0'(\gamma_0)\gamma_2 + \frac{1}{2}G_0''(\gamma_0)\gamma_1^2 + G_1'(\gamma_0)\gamma_1 + G_2(\gamma_0)\right)\epsilon^2 + \mathcal{O}\left(\epsilon^3\right), \end{aligned} \tag{B.10}$$

together with $G_0'(\gamma_0) = 0$ and (B.6). □

Remark B.2 Here, we consider the asymptotic expansion of $G(z)$ up to the third order term. One can also consider the case where the expansion is up to the second order, then (B.7) is still valid up to the second order.

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