

# BRANES AND MODULI SPACES OF HIGGS BUNDLES ON SMOOTH PROJECTIVE VARIETIES

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**ABSTRACT.** Given a smooth complex projective variety  $M$  and a smooth closed curve  $X \subset M$  such that the homomorphism of fundamental groups  $\pi_1(X) \longrightarrow \pi_1(M)$  is surjective, we study the restriction map of Higgs bundles, namely from the Higgs bundles on  $M$  to those on  $X$ . In particular, we investigate the interplay between this restriction map and various types of branes contained in the moduli spaces of Higgs bundles on  $M$  and  $X$ . We also consider the set-up where a finite group is acting on  $M$  via holomorphic automorphisms or anti-holomorphic involutions, and the curve  $X$  is preserved by this action. Branes are studied in this context.

## 1. INTRODUCTION

Lagrangian and holomorphic spaces have been of much interest within the study of Higgs bundles both on Riemann surfaces and on higher dimensional varieties. We shall consider here an irreducible smooth complex projective variety  $M$  of dimension  $d$ , and consider the following moduli spaces:

- the Betti moduli space  $\mathcal{B}_M(r)$  of equivalence classes of representations of  $\pi_1(M, x_0)$  in  $\mathrm{GL}(r, \mathbb{C})$ ;
- the moduli space  $\mathcal{H}_M(r)$  of semistable Higgs bundles on  $M$ , of rank  $r$  and vanishing Chern classes; and
- the Deligne–Hitchin moduli space  $\mathcal{M}_M^{DH}(r)$  which is the twistor space of the hyperKähler space  $\mathcal{H}_M(r)$ .

We shall dedicate this paper to the study of Lagrangian and holomorphic spaces appearing within the above moduli spaces arising through the inclusion of curves and hypersurfaces in  $M$ , as well as the action of anti-holomorphic involutions of  $M$  and also the finite group actions on  $M$ . In the case of Riemann surfaces, real structures have long been studied in relation to Hitchin systems. In particular, [1, 2] initiated the study of branes arising through anti-holomorphic involutions from the perspective of [13]. Lagrangians within moduli spaces of Higgs bundles arising through finite group actions were considered also in the setting of Riemann surfaces in [9]. In the present manuscript, we shall follow the lines of thought of the above papers.

After introducing Higgs bundles from a differential geometric perspective, the Betti moduli space of representations, and a brief description of nonabelian Hodge theory in

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Section 2, we study the implications of considering a smooth closed curve  $X$  on  $M$  such that the homomorphism of fundamental groups  $\pi_1(X, x_0) \rightarrow \pi_1(M, x_0)$  induced by the inclusion map  $X \hookrightarrow M$  is surjective (this is done in Section 3). This inclusion map induces maps

$$\Phi : \mathcal{B}_M(r) \rightarrow \mathcal{B}_X(r)$$

on the Betti moduli spaces and

$$\Psi : \mathcal{H}_M(r) \rightarrow \mathcal{H}_X(r)$$

on the Higgs moduli spaces, as seen in (3.3) and (3.4) respectively. The given condition that the homomorphism  $\pi_1(X, x_0) \rightarrow \pi_1(M, x_0)$  induced by the inclusion map  $X \hookrightarrow M$  is surjective ensures that the above maps  $\Phi$  and  $\Psi$  are actually embeddings. By studying these maps, through Theorem 3.2 and Corollary 3.4 we are able to show the following.

**Theorem A.** *The above map  $\Psi$  makes  $\mathcal{H}_M(r)$  a hyperKähler subspace of  $\mathcal{H}_X(r)$ , and thus the subspace  $\mathcal{H}_M(r)$  is a  $(B, B, B)$ -brane in  $\mathcal{H}_X(r)$ .*

The case of Deligne–Hitchin moduli spaces is considered in Section 4, where we look into the relationship of these spaces with twistor spaces of certain hyperKähler manifolds. Let  $\mathcal{M}_M^{DH}(r)$  and  $\mathcal{M}_X^{DH}(r)$  be the Deligne–Hitchin moduli spaces for  $M$  and  $X$  respectively. In this setting, through an induced map

$$\Upsilon : \mathcal{M}_M^{DH}(r) \rightarrow \mathcal{M}_X^{DH}(r)$$

constructed using restriction to  $X$  (see (4.2)) we prove the following in Theorem 4.2:

**Theorem B.** *The above map  $\Upsilon$  makes  $\mathcal{M}_M^{DH}(r)$  a twistor subspace of  $\mathcal{M}_X^{DH}(r)$ .*

We begin the study of actions by considering real structures  $\sigma : M \rightarrow M$  in Section 5. By observing that one may choose the real structure and the subvariety  $X$  such that they are compatible (Lemma 5.1), we show in Theorem 5.2 the following.

**Theorem C.** *Choose the closed curve  $X$  such that it is preserved by the real structure  $\sigma$  on  $M$ . Take  $S$  to be  $X$  or  $M$ . Then, the  $C^\infty$  involution  $\mathcal{I}_S$  induced on  $\mathcal{B}_S(r)$  by  $\sigma$  (the restriction of  $\sigma$  when  $S = X$ ) is anti-holomorphic with respect to the complex structure  $I$  (that gives the complex structure of the moduli space of Higgs bundles  $\mathcal{H}_S(r)$ ) and holomorphic with respect to the complex structure  $J$  (that gives the complex structure of  $\mathcal{B}_S(r)$ ), and thus the fixed point set  $\mathcal{H}_S(r)^{\mathcal{I}_S}$  is a  $(A, B, A)$ -brane in the corresponding moduli space  $\mathcal{H}_S(r)$ . Furthermore,*

$$\mathcal{I}_X \circ \Psi = \Psi \circ \mathcal{I}_M$$

for  $\Psi$  in Theorem A.

The case of finite group actions on  $M$  is studied in Section 6. This line of research begun in [9] for rank two Higgs bundles on Riemann surfaces where it was shown that equivariant Higgs bundles under a finite group give natural  $(B, B, B)$ -branes in the moduli space of Higgs bundles. Here, we shall address the higher rank cases and consider finite group actions on the moduli space of fixed determinant Higgs bundles for which the trace of the Higgs field being zero ( $\mathrm{SL}(r, \mathbb{C})$ -Higgs bundles). One of our main results, appearing in Theorem 6.2 and Proposition 6.4 is the following:

**Theorem D.** *Let  $M$  be a compact Riemann surface of genus  $g$  on which a finite group  $\Gamma$  is acting via holomorphic automorphisms. Consider the moduli space  $\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})} \subset \mathcal{H}_M(r)$  of semistable Higgs bundles on  $M$  whose structure group is  $\mathrm{SL}(r, \mathbb{C})$ . Then, for  $r > 2$  the following hold:*

- *For even genus  $g$ , the fixed point locus in  $\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}$  giving a  $(B, B, B)$ -brane will never be a mid dimensional space.*
- *For odd genus  $g$ , the fixed point free action of  $\Gamma := \mathbb{Z}/2\mathbb{Z}$  on  $M$  defines a mid dimensional space as its fixed point locus on  $\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}$ .*

The subspaces constructed in Theorem D can be further studied, and as shown in Proposition 6.5, all these mid-dimensional spaces constructed in Theorem 6.2 and Proposition 6.4 are in fact  $(B, B, B)$ -branes.

## 2. NONABELIAN HODGE THEORY

In what follows we shall introduce Higgs bundles from a differential geometric perspective, and also the Betti moduli space of representations, and then give a brief description of nonabelian Hodge theory.

**2.1. Higgs bundles.** Let  $M$  be an irreducible smooth complex projective variety of dimension  $d$ . The holomorphic cotangent and tangent bundles of  $M$  will be denoted by  $\Omega_M^1$  and  $TM$  respectively. The  $i$ -fold exterior product  $\bigwedge^i \Omega_M^1$  will be denoted by  $\Omega_M^i$ . Fixing a very ample line bundle  $L$  on  $M$ , the degree of a torsionfree coherent sheaf  $F$  on  $X$  is defined to be

$$\mathrm{degree}(F) := (c_1(\det F) \cup c_1(L)^{d-1}) \cap [M] \in \mathbb{Z}$$

(see [14, Ch. V, § 6] for the determinant bundle  $\det F$ ). The number

$$\mu(F) := \frac{\mathrm{degree}(F)}{\mathrm{rank}(F)} \in \mathbb{Q}$$

is called the *slope* of  $F$ . From [10, 11, 15, 16] one has the following definition:

**Definition 2.1.** A *Higgs field* on a vector bundle  $E$  over  $M$  is a holomorphic section

$$\theta \in H^0(M, \mathrm{End}(E) \otimes \Omega_M^1)$$

such that the section  $\theta \wedge \theta$  of  $\mathrm{End}(E) \otimes \Omega_M^2$  vanishes identically. A *Higgs bundle* on  $M$  is a pair  $(E, \theta)$ , where  $E$  is a holomorphic vector bundle and  $\theta$  is a Higgs field on  $E$ .

**Definition 2.2.** A Higgs bundle  $(E, \theta)$  is called *stable* (respectively, *semistable*) if

$$\mu(F) < \mu(E) \quad (\text{respectively, } \mu(F) \leq \mu(E))$$

for all coherent subsheaves  $F \subset E$  with  $0 < \mathrm{rank}(F) < \mathrm{rank}(E)$  and  $\theta(F) \subset F \otimes \Omega_M^1$  [10, 15, 16]. A Higgs bundle  $(E, \theta)$  is called *polystable* if

- $(E, \theta)$  is semistable, and
- $(E, \theta) = \bigoplus_{i=1}^{\ell} (E_i, \theta_i)$ , where each  $(E_i, \theta_i)$  is a stable Higgs bundle.

Let  $(E, \theta)$  be a Higgs bundle on  $M$  such that

$$\text{degree}(E) = 0 = (ch_2(E) \cup c_1(L)^{d-2}) \cap [M]; \quad (2.1)$$

we recall that  $ch_2(E) = \frac{1}{2}c_1(E)^2 - c_2(E)$ . Given a Hermitian structure  $h$  on  $E$ , the Chern connection on  $E$  corresponding to  $h$  will be denoted by  $\nabla^h$ , and the curvature of the connection  $\nabla^h$  will be denoted by  $\mathcal{K}(\nabla^h)$ . Let

$$\theta^* \in C^\infty(M; \text{End}(E) \otimes \Omega_M^{0,1})$$

be the adjoint of  $\theta$  with respect to the Hermitian structure  $h$ . Then, from [10, 15, 16], the following is the Hermitian–Yang–Mills equation:

$$\mathcal{K}(\nabla^h) + [\theta, \theta^*] = 0. \quad (2.2)$$

A theorem of Simpson says that  $E$  admits a Hermitian metric satisfying the Hermitian–Yang–Mills equation for  $(E, \theta)$  if and only if  $(E, \theta)$  is a polystable Higgs bundle [15, Theorem 1], [15, Proposition 3.4], [16, Theorem 1]. When  $E$  is a rank two vector bundle on a smooth complex projective curve, the result was proven earlier by Hitchin in [10].

We note that the Hermitian–Yang–Mills equation in (2.2) implies that

$$\nabla^h + \theta + \theta^* \quad (2.3)$$

is a flat connection on  $E$ . Let

$$\nabla^h = (\nabla^h)^{1,0} + (\nabla^h)^{0,1}$$

be the decomposition of the Chern connection  $\nabla^h$  into  $(1, 0)$  and  $(0, 1)$  components.

**Remark 2.3.** The holomorphic structure on  $E$  is given by the Dolbeault operator  $(\nabla^h)^{0,1}$ . Note that the Dolbeault operator for the holomorphic structure on the  $C^\infty$  bundle  $E$  given by the flat connection in (2.3) is  $(\nabla^h)^{0,1} + \theta^*$ . Therefore, the holomorphic vector bundle given by the flat connection corresponding to a Higgs bundle does not coincide, in general, with the holomorphic vector bundle underlying the Higgs bundle.

**2.2. Flat connections.** Given a base point  $x_0 \in M$ , a representation

$$\rho : \pi_1(M, x_0) \longrightarrow \text{GL}(r, \mathbb{C})$$

is called *irreducible* if the standard action of  $\rho(\pi_1(M, x_0)) \subset \text{GL}(r, \mathbb{C})$  on  $\mathbb{C}^r$  does not preserve any nonzero proper subspace of  $\mathbb{C}^r$ . The homomorphism  $\rho$  is called *completely reducible* if it is a direct sum of irreducible representations. Two homomorphisms

$$\rho_1, \rho_2 : \pi_1(M, x_0) \longrightarrow \text{GL}(r, \mathbb{C})$$

are called *equivalent* if there is an element  $g \in \text{GL}(r, \mathbb{C})$  such that

$$\rho_1(\gamma) = g^{-1}\rho_2(\gamma)g$$

for all  $\gamma \in \pi_1(M, x_0)$ . Clearly, this equivalence relation preserves both irreducibility and complete reducibility. The space of equivalence classes of completely reducible homomorphisms from  $\pi_1(M, x_0)$  to  $\text{GL}(r, \mathbb{C})$  has the structure of an affine scheme defined over  $\mathbb{C}$ , which can be seen as follows. Note that  $\pi_1(M, x_0)$  is a finitely presented group and  $\text{GL}(r, \mathbb{C})$  is a complex affine algebraic group. Therefore, the space of all homomorphisms

$\text{Hom}(\pi_1(M, x_0), \text{GL}(r, \mathbb{C}))$  is a complex affine scheme. The adjoint action of  $\text{GL}(r, \mathbb{C})$  on itself produces an action of  $\text{GL}(r, \mathbb{C})$  on  $\text{Hom}(\pi_1(M, x_0), \text{GL}(r, \mathbb{C}))$ . The geometric invariant theoretic quotient

$$\text{Hom}(\pi_1(M, x_0), \text{GL}(r, \mathbb{C})) // \text{GL}(r, \mathbb{C})$$

is the moduli space of equivalence classes of completely reducible homomorphisms from  $\pi_1(M, x_0)$  to  $\text{GL}(r, \mathbb{C})$  [17, 18].

We shall let  $\mathcal{B}_M(r)$  denote this moduli space of equivalence classes of completely reducible homomorphisms from  $\pi_1(M, x_0)$  to  $\text{GL}(r, \mathbb{C})$ , which is known as the Betti moduli space.

A homomorphism  $\rho : \pi_1(M, x_0) \rightarrow \text{GL}(r, \mathbb{C})$  produces an algebraic vector bundle  $E$  on  $M$  of rank  $r$  equipped with a flat (i.e., integrable) algebraic connection, together with an isomorphism of the fiber  $E_{x_0}$  with  $\mathbb{C}^r$ . Equivalence classes of such homomorphisms correspond to algebraic vector bundles of rank  $r$  equipped with a flat algebraic connection; this is an example of the Riemann–Hilbert correspondence.

**Definition 2.4.** A connection  $\nabla$  on a vector bundle  $E$  is called *irreducible* if there is no subbundle  $0 \neq F \subsetneq E$  which is preserved by  $\nabla$ . A connection  $\nabla$  on a vector bundle  $E$  is called *completely reducible* if

$$(E, \nabla) = \bigoplus_{i=1}^N (E_i, \nabla^i),$$

where each  $\nabla^i$  is an irreducible connection on  $E_i$ .

We note that irreducible (respectively, completely reducible) flat algebraic connections of rank  $r$  on  $M$  correspond to irreducible (respectively, completely reducible) equivalence classes of homomorphisms from  $\pi_1(M, x_0)$  to  $\text{GL}(r, \mathbb{C})$ .

**2.3. Harmonic structures.** Let  $(E, \nabla)$  be a flat vector bundle of rank  $r$  on  $M$ , and let

$$\rho : \pi_1(M, x_0) \rightarrow \text{GL}(E_{x_0})$$

be the monodromy homomorphism for  $\nabla$ .

Given the universal cover  $\varpi : (\widetilde{M}, \widetilde{x}_0) \rightarrow (M, x_0)$ , using the pulled back connection  $\varpi^*\nabla$ , the pulled back bundle  $\varpi^*E$  is identified with the trivial bundle  $\widetilde{M} \times E_{x_0} \rightarrow \widetilde{M}$ . Therefore, a Hermitian structure  $h$  on  $E$  gives a  $C^\infty$  map

$$F^h : \widetilde{M} \rightarrow \text{GL}(E_{x_0})/\text{U}(E_{x_0}), \quad (2.4)$$

where  $\text{U}(E_{x_0})$  consists of all automorphisms of  $E_{x_0}$  that preserve the Hermitian structure  $h(x_0)$  on it. The group  $\pi_1(M, x_0)$  acts on  $\text{GL}(E_{x_0})/\text{U}(E_{x_0})$  through the left translation action of  $\rho(\pi_1(M, x_0))$ . We note that the map  $F^h$  in (2.4) is  $\pi_1(M, x_0)$ -equivariant. The quotient space  $\text{GL}(E_{x_0})/\text{U}(E_{x_0})$  is equipped with a Riemannian metric; it is given by the restriction of the Killing form on the orthogonal complement of  $\text{Lie}(\text{U}(E_{x_0})) \subset \text{Lie}(\text{GL}(E_{x_0}))$ .

The Hermitian structure  $h$  is called *harmonic* if the map  $F^h$  in (2.4) is harmonic with respect to a Kähler structure on  $M$ . It should be emphasized that the harmonicity condition for  $F^h$  does not depend on the choice of the Kähler form on  $M$ .

A theorem of Corlette says that  $(E, \nabla)$  admits a harmonic Hermitian structure if and only if  $\nabla$  is completely reducible [6] (when  $E$  is of rank two on a smooth complex projective curve, this was proved in [7] by Donaldson).

Let  $(E, \theta)$  be a polystable Higgs bundle on  $M$  satisfying (2.1), and consider a Hermitian structure  $h$  on  $E$  satisfying the Hermitian–Yang–Mills equation in (2.2). Then, the flat connection  $\nabla^h + \theta + \theta^*$  on  $E$  in (2.3) is completely reducible, and  $h$  is a harmonic Hermitian structure for the flat connection  $\nabla^h + \theta + \theta^*$ . Conversely, if  $h$  is a harmonic Hermitian structure for a flat connection  $(E, \nabla)$ , then  $h$  and  $\nabla$  together define a holomorphic structure on  $E$  and a Higgs field  $\theta$  on  $E$  for this holomorphic structure such that  $h$  satisfies the Hermitian–Yang–Mills equation for the Higgs bundle  $(E, \theta)$ .

As shown in [16, p. 20, Corollary 1.3], the above constructions produce an equivalence of categories between the following two categories:

- (1) Objects are completely reducible flat algebraic connections on  $M$ , and morphisms are connection preserving homomorphisms.
- (2) Objects are polystable Higgs bundles  $(E, \theta)$  on  $M$  satisfying (2.1); the morphisms are homomorphisms of Higgs bundles.

We also note that if  $(E, \theta)$  is a polystable Higgs bundle on  $M$  satisfying (2.1), then all the rational Chern classes of  $E$  of positive degree vanish [15, p. 878–879, Proposition 3.4]. (See also [20] for an exposition for dimension one.)

### 3. RESTRICTION TO CURVES

We shall dedicate this section to understanding the restriction of the ideas of Section 1 to hypersurfaces, which will become useful when studying Lagrangians within the moduli space of Higgs bundles.

**3.1. Higgs bundles and hypersurfaces.** The homotopical version of the Lefschetz hyperplane theorem says that for a smooth very ample hypersurface  $H \subset M$ , the homomorphism of fundamental groups  $\pi_1(H, x_0) \longrightarrow \pi_1(M, x_0)$  induced by the inclusion map  $H \hookrightarrow M$ , where  $x_0 \in H$ , is surjective, and it is an isomorphism if  $d = \dim M \geq 3$  (see [8, p. 48, (8.1.1)]). Consequently, using induction on the number of hypersurfaces we conclude that for a smooth closed curve  $X$  on  $M$  which is the intersection of  $d - 1$  smooth very ample hypersurfaces on  $M$ , the homomorphism of fundamental groups  $\pi_1(X, x_0) \longrightarrow \pi_1(M, x_0)$  induced by the inclusion map  $X \hookrightarrow M$  is surjective.

Fix a smooth closed curve  $X$  on  $M$  such that the homomorphism of fundamental groups  $\pi_1(X, x_0) \longrightarrow \pi_1(M, x_0)$  induced by the inclusion map  $X \hookrightarrow M$  is surjective, and let

$$\phi : X \hookrightarrow M \tag{3.1}$$

be the inclusion map.

As done previously, consider a polystable Higgs bundle  $(E, \theta)$  on  $M$  satisfying (2.1), and let  $h$  be a Hermitian structure on  $E$  that satisfies the Hermitian–Yang–Mills equation in (2.2) for the Higgs bundle  $(E, \theta)$ .

**Lemma 3.1.** *The Hermitian structure  $\phi^*h$  on  $\phi^*E$  satisfies Hermitian–Yang–Mills equation for the induced Higgs bundle  $(\phi^*E, \phi^*\theta)$  on  $X$ .*

*Proof.* The pulled back section  $\phi^*\theta$  defines a Higgs field on  $\phi^*E$  using the natural homomorphism

$$(d\phi)^* : \phi^*\Omega_M^1 \longrightarrow \Omega_X^1,$$

where  $d\phi : TX \longrightarrow \phi^*TM$  is the differential of the map  $\phi$  in (3.1). The vector bundle  $\phi^*E$  has the pulled back Hermitian structure  $\phi^*h$ . Note that the Chern connection  $\nabla^{\phi^*h}$  on  $\phi^*E$  for the Hermitian structure  $\phi^*h$  coincides with  $\phi^*\nabla^h$ , where  $\nabla^h$  is the Chern connection on  $E$  for the Hermitian structure  $h$ . Moreover, we also have that  $\phi^*(\theta^*) = (\phi^*\theta)^*$ . Using these, and the fact that  $h$  satisfies the Hermitian–Yang–Mills equation for the Higgs bundle  $(E, \theta)$ , the lemma follows.  $\square$

From Lemma 3.1 one has that

- the Higgs bundle  $(\phi^*E, \phi^*\theta)$  is polystable, and
- $\nabla^{\phi^*h} + \phi^*\theta + \phi^*\theta^*$  is a flat connection on  $\phi^*E$ .

In particular, we have  $\text{degree}(\phi^*E) = 0$ . Let  $(W, \nabla)$  be the flat connection on  $M$  corresponding to the Higgs bundle  $(E, \theta)$ ; recall that  $E$  and  $W$  are the same  $C^\infty$  vector bundle with possibly different holomorphic structures. From Lemma 3.1 it follows that the flat connection corresponding to  $(\phi^*E, \phi^*\theta)$  coincides with the pulled back flat connection  $(\phi^*W, \phi^*\nabla)$ .

**3.2. Harmonic structures and hypersurfaces.** Consider now an algebraic vector bundle  $V$  of rank  $r$  on  $M$  equipped with a flat algebraic completely reducible connection  $\nabla$ . For  $x_0 \in X$  and the map  $\phi$  in (3.1), the homomorphism

$$\phi_* : \pi_1(X, x_0) \longrightarrow \pi_1(M, \phi(x_0)), \quad (3.2)$$

is surjective, and thus it follows immediately that  $\phi^*\nabla$  is a flat algebraic completely reducible connection on  $\phi^*V$ .

Let  $h_V$  be a harmonic Hermitian structure on  $V$  for  $\nabla$ , and let  $(W, \theta)$  be the Higgs bundle corresponding to  $(V, \nabla)$ . Recall that  $V$  and  $W$  are the same  $C^\infty$  vector bundle with possibly different holomorphic structures. It is straightforward to check that the Hermitian structure  $\phi^*h$  on  $\phi^*V$  is harmonic for the connection  $\phi^*\nabla$ . Indeed, this follows from the facts that  $M$  is Kähler and the embedding  $\phi$  in (3.1) is holomorphic. Consequently, the Higgs bundle for  $(\phi^*V, \phi^*\nabla)$  coincides with  $(\phi^*W, \phi^*\theta)$ .

As before, let  $\mathcal{B}_X(r)$  (respectively,  $\mathcal{B}_M(r)$ ) be the Betti moduli space of equivalence classes of completely reducible representations of  $\pi_1(X, x_0)$  (respectively,  $\pi_1(M, \phi(x_0))$ ) in  $\text{GL}(r, \mathbb{C})$ . Let

$$\Phi : \mathcal{B}_M(r) \longrightarrow \mathcal{B}_X(r) \quad (3.3)$$

be the map induced by  $\phi_*$  in (3.2), which clearly is an algebraic map. We note that the map  $\Phi$  is injective, because the homomorphism  $\phi_*$  is surjective.

Let  $\mathcal{H}_X(r)$  be the moduli space of semistable Higgs bundles on  $X$  of rank  $r$  and degree zero; this moduli space was constructed in [17], [18] (see [5] for its properties). Let  $\mathcal{H}_M(r)$  be the moduli space of semistable Higgs bundles  $(E, \theta)$  on  $M$  of rank  $r$  such that all the rational Chern classes of  $E$  of positive degree vanish. It may be recalled that any semistable Higgs bundle  $(E, \theta)$  satisfying (2.1) has the property that all the rational Chern classes of  $E$  of positive degree vanish [16, p. 39, Theorem 2]. In view of Lemma 3.1, we have a map

$$\Psi : \mathcal{H}_M(r) \longrightarrow \mathcal{H}_X(r) \quad (3.4)$$

defined by  $(E, \theta) \longmapsto (\phi^*E, \phi^*\theta)$ . It is clearly an algebraic map. Indeed, the restriction map from an algebraic family of Higgs bundles on a variety to a family of Higgs bundles on a subvariety is evidently algebraic. As observed above, using the  $C^\infty$  identification between  $\mathcal{B}_X(r)$  and  $\mathcal{H}_X(r)$ , and also between  $\mathcal{B}_M(r)$  and  $\mathcal{H}_M(r)$ , the map  $\Phi$  in (3.3) coincides with the map  $\Psi$  in (3.4).

As noted before, the map  $\Phi$  in (3.3) is injective, because  $\phi_*$  is surjective. Therefore, from the above observation that  $\Phi$  coincides with  $\Psi$  using the  $C^\infty$  identification between  $\mathcal{B}_X(r)$  and  $\mathcal{H}_X(r)$ , and also between  $\mathcal{B}_M(r)$  and  $\mathcal{H}_M(r)$ , we conclude that  $\Psi$  is an embedding. As mentioned above, the map  $\Phi$  is algebraic.

**3.3. HyperKähler structure.** Recall from [10, 11] that the moduli space  $\mathcal{H}_X(r)$  has a natural hyperKähler structure, and thus we may fix three complex structures  $I, J$  and  $K$  on  $\mathcal{H}_X(r)$  satisfying the quaternionic equations. We shall let the complex structures  $I$  and  $J$  be the complex structures of  $\mathcal{H}_X(r)$  and  $\mathcal{B}_X(r)$  respectively; so  $K$  is determined by the equation

$$IJ = K.$$

Therefore, from the holomorphicity of the maps  $\Phi$  and  $\Psi$  in (3.3) and (3.4) we obtain the following:

**Theorem 3.2.** *The map  $\Psi$  in (3.4) makes  $\mathcal{H}_M(r)$  a hyperKähler subspace of  $\mathcal{H}_X(r)$ .*

**Remark 3.3.** Consider the holomorphic symplectic form  $\sigma_I$  on  $\mathcal{H}_X(r)$ . This symplectic manifold  $(\mathcal{H}_X(r), \sigma_I)$  has a natural algebraically completely integrable structure given by the Hitchin map. Although  $\Psi^*\sigma_I$  is a holomorphic symplectic structure on  $\mathcal{H}_M(r)$ , the map  $\Psi$  is not Poisson because it is an embedding and  $\dim \mathcal{H}_X(r) > \dim \mathcal{H}_M(r)$  in general.

Given the hyperKähler spaces  $\mathcal{H}_M(r)$  and  $\mathcal{H}_X(r)$ , and the three fixed complex structures  $(I, J, K)$  as before, we shall denote by  $\omega_I, \omega_J$ , and  $\omega_K$  the corresponding Kähler forms defined by

$$\omega_I(X, Y) = g(IX, Y), \quad \omega_J(X, Y) = g(JX, Y), \quad \omega_K(X, Y) = g(KX, Y),$$

where  $g$  is the Riemannian metric for the hyperKähler structure. Then, the induced complex symplectic forms  $\Omega_I, \Omega_J$ , and  $\Omega_K$  are given by

$$\Omega_I = \omega_J + \sqrt{-1}\omega_K, \quad \Omega_J = \omega_K + \sqrt{-1}\omega_I, \quad \Omega_K = \omega_I + \sqrt{-1}\omega_J. \quad (3.5)$$



Therefore, one may consider subspaces of  $\mathcal{H}_M(r)$  and  $\mathcal{H}_X(r)$  which are Lagrangian with respect to one of the symplectic structures in (3.5), or holomorphic with respect to one of the fixed complex structures  $I, J$  or  $K$ . Following [13], we shall say that a Lagrangian subspace with respect to a symplectic structure is an  $A$ -brane, and a holomorphic subspace with respect to a complex structure is a  $B$ -brane. Hence, with respect to  $(I, J, K)$  and  $(\omega_I, \omega_J, \omega_K)$  one may have branes given by triples of letters. In particular, from Theorem 3.2 we have the following:

**Corollary 3.4.** *The subspace  $\mathcal{H}_M(r)$  is a  $(B, B, B)$ -brane in  $\mathcal{H}_X(r)$ .*

#### 4. DELIGNE–HITCHIN MODULI SPACE AND TWISTOR SPACE

In what follows we shall first recall the notion of  $\lambda$ -connections and the Deligne–Hitchin moduli space (for details and proofs see Simpson [19]). Then, we shall look into the relationship of these spaces with the twistor spaces of certain hyperKähler manifolds.

**4.1. Deligne–Hitchin moduli spaces.** Let  $M$  be an irreducible smooth complex projective variety of dimension  $d$ . We fix a  $C^\infty$  complex vector bundle  $V \rightarrow M$  such that all its rational Chern classes of positive degree vanish. A  $\lambda$ -connection on  $V \rightarrow M$  is a triple  $(\lambda, \bar{\partial}, D)$  consisting of a complex number  $\lambda$ , a holomorphic structure  $\bar{\partial}$  on  $M$ , i.e.,  $\bar{\partial}^2 = 0$ , and a holomorphic first order differential operator  $D$  satisfying the following two conditions:

- $D^2 = 0$ , and
- $D(fs) = \lambda(\partial f)s + fDs$  (similar to the Leibniz rule), where  $f$  is any locally defined holomorphic function and  $s$  is any locally defined holomorphic section of  $V$ .

**Remark 4.1.** A  $\lambda$ -connection for  $\lambda = 0$  is a Higgs bundle  $(\bar{\partial}, D)$ , and for  $\lambda = 1$  it is a flat holomorphic connection  $D + \bar{\partial}$ .

Consider the moduli space  $\mathcal{M}_M^{Hod}(r)$  of completely reducible  $\lambda$ -connections (to clarify,  $\lambda$  is not fixed), and let

$$f_M : \mathcal{M}_M^{Hod}(r) \rightarrow \mathbb{C}, \quad (\lambda, \bar{\partial}, D) \mapsto \lambda$$

be the fibration. The analogous construction on the complex conjugate space  $\overline{M}$  (the  $C^\infty$  manifold underlying  $\overline{M}$  is  $M$  itself while the complex structure on it is  $-J_M$ , where  $J_M$  is the complex structure on  $M$ ) gives a fibration

$$f_{\overline{M}} : \mathcal{M}_{\overline{M}}^{Hod}(r) \rightarrow \mathbb{C}.$$

Moreover, there is the following gluing constructed by Deligne:

$$\begin{aligned} \varphi : \mathcal{M}_M^{Hod}(r)_{|\mathbb{C}^*} &\rightarrow \mathcal{M}_{\overline{M}}^{Hod}(r)_{|\mathbb{C}^*} \\ (\lambda, \bar{\partial}, D)_M &\mapsto \left(\frac{1}{\lambda}, \frac{1}{\lambda}D, \frac{1}{\lambda}\bar{\partial}\right)_{\overline{M}}; \end{aligned}$$

this covers the map  $\mathbb{C}^* \rightarrow \mathbb{C}^*$  defined by  $\lambda \mapsto \frac{1}{\lambda}$ .

The Deligne–Hitchin moduli space

$$\mathcal{M}_M^{DH}(r) = \mathcal{M}_M^{Hod}(r) \cup_{\varphi} \mathcal{M}_{\overline{M}}^{Hod}(r)$$

is obtained by gluing the two Hodge moduli spaces via  $\varphi$ . It admits a fibration to  $\mathbb{CP}^1$  which is  $f_M$  on  $\mathcal{M}_M^{Hod}(r)$  and  $1/f_{\overline{M}}$  on  $\mathcal{M}_{\overline{M}}^{Hod}(r)$ . It should be mentioned that there does not exist a natural algebraic structure on Deligne–Hitchin moduli spaces.

**4.2. Twistor space.** The twistor space of a hyperKähler manifold  $(N, g, I, J, K)$  encodes the hyperKähler geometry in terms of complex analytic data. As a complex manifold it is given by  $\mathcal{P} = N \times \mathbb{CP}^1$  with (integrable) complex structure at  $(p, \lambda) = (p, a + \sqrt{-1}b)$  given by

$$I_{(p, a + \sqrt{-1}b)} = (1 - aK_p + bJ_p)^{-1}I_p(1 - aK_p + bJ_p).$$

There is the holomorphic fibration  $\mathcal{P} \rightarrow \mathbb{CP}^1$ , and the manifold  $N$  can be recovered as the space of *twistor lines*, i.e., as a component of holomorphic sections of  $\mathcal{P} \rightarrow \mathbb{CP}^1$  which are real with respect to the real structures

$$\begin{aligned} \rho : \mathcal{P} &\rightarrow \mathcal{P}, \quad (p, \lambda) \mapsto (p, -\overline{\lambda}^{-1}) \\ \mathbb{CP}^1 &\rightarrow \mathbb{CP}^1, \quad \lambda \mapsto -\overline{\lambda}^{-1} \end{aligned}$$

of  $\mathcal{P}$  and  $\mathbb{CP}^1$ .

The various complex structures on  $N$  are obtained by evaluating at specific  $\lambda \in \mathbb{CP}^1$ . Moreover, the Riemannian metric can be computed from a holomorphic twisted symplectic form on  $\mathcal{P}$ ; see [12].

As shown in [19, Theorem 4.2], the Deligne–Hitchin moduli space is the twistor space of the hyperKähler space  $\mathcal{H}_M(r)$ . The real structure  $\rho$  on  $\mathcal{M}_M^{DH}(r)$  is given by

$$\rho((\lambda, \overline{\partial}, D)_M) = (-\overline{\lambda}^{-1}, \overline{\partial}^*, -D^*)_{\overline{M}},$$

where  $\overline{\partial}^*$  and  $D^*$  are the adjoint operators with respect to a Hermitian metric  $h$  on the underlying vector bundle  $V$  (for the operators  $\overline{\partial}$  and  $D$ ).

Consider a polystable Higgs bundle  $(E, \theta)$  on  $M$  satisfying (2.1), and let  $h$  be a Hermitian structure on  $E$  that satisfies the Hermitian–Yang–Mills equation in (2.2) for the Higgs bundle  $(E, \theta)$ . Let  $(\nabla^h)^{0,1}$  be the holomorphic structure on  $E$  determined by  $E$ . Then, the twistor line through the point  $p = ((\nabla^h)^{0,1}, \theta) \in \mathcal{H}_M(r)$ , over the open subset  $\mathbb{C} \subset \mathbb{CP}^1$ , is given by

$$\lambda \mapsto (\lambda, (\nabla^h)^{0,1} + \lambda\theta^*, \theta + \lambda(\nabla^h)^{1,0});$$

see [19]. Note that this family interpolates between the Higgs pair  $(E, \theta)$  at  $\lambda = 0$  and the corresponding flat connection (2.3) at  $\lambda = 1$ .

**4.3. Twistor subspaces.** Let

$$\pi : \mathcal{P} \rightarrow \mathbb{CP}^1 \tag{4.1}$$

be a twistor space of a hyperKähler space  $M$ . We call a complex subspace  $\mathcal{N} \subset \mathcal{P}$  a twistor subspace if the following condition holds: whenever  $s(\lambda_0) \in \mathcal{N}$  for a twistor line  $s$  and some  $\lambda_0 \in \mathbb{CP}^1$ , then  $s(\lambda) \in \mathcal{N}$  for all  $\lambda \in \mathbb{CP}^1$ .

If  $\mathcal{N} \subset \mathcal{P}$  is a twistor subspace then the restriction  $\pi|_{\mathcal{N}}$  is a fibration and

$$\rho(\mathcal{N}) = \mathcal{N}.$$

Moreover, the twisted holomorphic symplectic structure on  $\mathcal{P}$  in (4.1) remains non-degenerate when restricted to the vertical tangent bundle of  $\mathcal{N}$ . It follows that there exists a hyperKähler subspace  $N \subset M$  whose twistor space is given by  $\mathcal{N}$ . Conversely, every hyperKähler subspace  $N \subset M$  gives rise to a twistor subspace of  $\mathcal{P}$  in (4.1).

The construction of the Deligne–Hitchin moduli space can be applied to the projective curve  $\phi : X \hookrightarrow M$  with induced surjective map

$$\phi_* : \pi_1(X, x_0) \longrightarrow \pi_1(M, x_0)$$

as in Section 3. In that set-up, let

$$\Upsilon : \mathcal{M}_M^{DH}(r) \longrightarrow \mathcal{M}_X^{DH}(r) \quad (4.2)$$

be the map defined by  $(\lambda, E, D) \longmapsto (\lambda, \phi^*E, \phi^*D)$ . Theorem 3.2 implies the following:

**Theorem 4.2.** *The map  $\Upsilon$  in (4.2) makes  $\mathcal{M}_M^{DH}(r)$  a twistor subspace of  $\mathcal{M}_X^{DH}(r)$ .*

## 5. REAL STRUCTURES

We shall dedicate this section to the study of real structures and their fixed point sets, and in particular, their appearance in the context of Higgs bundles.

**5.1. Real structures and Higgs bundles.** As before,  $M$  is an irreducible smooth complex projective variety of dimension  $d$ . Let

$$\sigma : M \longrightarrow M \quad (5.1)$$

be an anti-holomorphic involution of  $M$ .

**Lemma 5.1.** *It is possible to choose  $\phi$  in (3.1) such that*

- (1)  $\sigma(\phi(X)) = \phi(X)$ , and
- (2) *the homomorphism  $\phi_* : \pi_1(X, x_0) \longrightarrow \pi_1(M, x_0)$  is surjective.*

*Proof.* Let  $\mathcal{L}$  be a very ample line bundle on  $M$ . Then the holomorphic line bundle  $\sigma^*\overline{\mathcal{L}}$  on  $M$  is also ample. The very ample line bundle

$$\mathbb{L} := \mathcal{L} \otimes \sigma^*\overline{\mathcal{L}}$$

is equipped with a lift of the involution  $\sigma$ . This involution of  $\mathbb{L}$ , which we shall denote by  $\tilde{\sigma}$ , sends any  $v_1 \otimes v_2 \in \mathbb{L}_x = \mathcal{L}_x \otimes \overline{\mathcal{L}}_{\sigma(x)}$  to  $v_2 \otimes v_1 \in \mathbb{L}_{\sigma(x)} = \mathcal{L}_{\sigma(x)} \otimes \overline{\mathcal{L}}_x$ . Moreover,  $\tilde{\sigma}$  produces a conjugate linear involution

$$\eta : H^0(M, \mathbb{L}) \longrightarrow H^0(M, \mathbb{L})$$

that sends any  $s \in H^0(M, \mathbb{L})$  to  $\tilde{\sigma}(s)$ . The fixed point set

$$H^0(M, \mathbb{L})^\eta \subset H^0(M, \mathbb{L})$$

is a totally real subspace, meaning  $\dim_{\mathbb{R}} H^0(M, \mathbb{L})^\eta = \dim_{\mathbb{C}} H^0(M, \mathbb{L})$ , and

$$H^0(M, \mathbb{L}) = H^0(M, \mathbb{L})^\eta \oplus \overline{H^0(M, \mathbb{L})^\eta}.$$

Now the intersection of divisors of general  $d-1$  elements of  $H^0(M, \mathbb{L})^\eta$  is a smooth closed curve  $Y$  in  $M$  such that  $\sigma(Y) = Y$ , and the homomorphism  $\pi_1(Y) \longrightarrow \pi_1(M)$  induced by the inclusion map of  $Y$  in  $M$  is surjective, as required.  $\square$

Fix a smooth closed curve  $X$  as in (3.1) such that

- (1)  $\sigma(\phi(X)) = \phi(X)$ , and
- (2) the homomorphism  $\phi_* : \pi_1(X, x_0) \longrightarrow \pi_1(M, x_0)$  induced by  $\phi$  in (3.1) is surjective, where  $x_0 \in X$ .

We shall denote the restriction of the map  $\sigma$  to  $X$  by  $\tau$ . The anti-holomorphic involution  $\tau$  on  $X$  produces a  $C^\infty$  involution

$$\mathcal{I}_X : \mathcal{B}_X(r) \longrightarrow \mathcal{B}_X(r) \quad (5.2)$$

of  $\mathcal{B}_X(r)$  in (3.3) that sends a flat connection  $(E, \nabla)$  on  $X$  to  $(\tau^*E, \tau^*\nabla)$ . The anti-holomorphic involution  $\sigma$  on  $M$  produces a  $C^\infty$  involution

$$\mathcal{I}_M : \mathcal{B}_M(r) \longrightarrow \mathcal{B}_M(r) \quad (5.3)$$

of  $\mathcal{B}_M(r)$  in (3.3) that sends a flat connection  $(E, \nabla)$  on  $X$  to  $(\sigma^*E, \sigma^*\nabla)$ .

From the above, the following is straightforward to prove.

**Theorem 5.2.** *For any  $S \in \{X, M\}$ , the involution  $\mathcal{I}_S$  in  $\{(5.2), (5.3)\}$  is anti-holomorphic with respect to the complex structure  $I$  (that gives the complex structure of the moduli space of Higgs bundles  $\mathcal{H}_S(r)$ ). The involution  $\mathcal{I}_S$  is holomorphic with respect to the complex structure  $J$  (that gives the complex structure of  $\mathcal{B}_S(r)$ ), and thus the fixed point set  $\mathcal{H}_S(r)^{\mathcal{I}_S}$  is a  $(A, B, A)$ -brane in the corresponding moduli space  $\mathcal{H}_S(r)$ . Furthermore,*

$$\mathcal{I}_X \circ \Psi = \Psi \circ \mathcal{I}_M,$$

where  $\Psi$  is the embedding in (3.4).

**Remark 5.3.** When the variety  $M$  is a Riemann surface, the involution  $\mathcal{I}_M$  gives the map  $f$  in [1] fixing an  $(A, B, A)$ -brane in the moduli space of Higgs bundles on  $M$ . Moreover, in such a situation  $\mathcal{I}_M$  can naturally be seen as part of a triple of involutions fixing Lagrangian subspaces, as shown in [2].

**5.2. Real structures and  $\lambda$ -connections.** Let  $X \hookrightarrow M$  be an embedded real curve as above, and let  $\tau$  and  $\sigma$  be the corresponding real structures. Note that for another irreducible smooth complex projective variety  $N$  and an algebraic map  $f : N \longrightarrow M$ , we obtain an induced map

$$f^{M,N} : \mathcal{M}_M^{DH}(r) \longrightarrow \mathcal{M}_N^{DH}(r)$$

generalizing  $\Upsilon$  in (4.2). Thus, the commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{\sim} & \overline{X} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\sim} & \overline{M} \end{array}$$

yields for the twistor fibrations

$$\begin{array}{ccc} \mathcal{M}_M^{DH}(r) & \xrightarrow{\sim} & \mathcal{M}_{\overline{M}}^{DH}(r) \\ \downarrow & & \downarrow \\ \mathcal{M}_X^{DH}(r) & \xrightarrow{\sim} & \mathcal{M}_{\overline{X}}^{DH}(r). \end{array} \quad (5.4)$$

In particular, at  $\lambda = 1$  we obtain

$$\begin{array}{ccc} \mathcal{B}_M(r) & \xrightarrow{I_M = f^{M, \overline{M}}|_{\lambda=1}} & \mathcal{B}_{\overline{M}}(r) \\ \downarrow \Phi & & \downarrow \Phi \\ \mathcal{B}_X(r) & \xrightarrow{I_X = f^{X, \overline{X}}|_{\lambda=1}} & \mathcal{B}_{\overline{X}}(r) \end{array}$$

giving the  $J$ -part of Theorem 5.2. Recall that the fibers of a twistor space at  $\lambda = 0$  and  $\lambda = \infty$  are complex conjugate spaces. Hence, evaluating (5.4) at  $\lambda = 0$ , we obtain the commutative diagram

$$\begin{array}{ccccc} \mathcal{H}_M(r) & \xrightarrow{f^{M, \overline{M}}|_{\lambda=0}} & \mathcal{H}_{\overline{M}}(r) & \xrightarrow{\sim} & \overline{\mathcal{H}_M(r)} \\ \downarrow \Psi & & \downarrow \Psi & & \downarrow \\ \mathcal{H}_X(r) & \xrightarrow{f^{X, \overline{X}}|_{\lambda=1}} & \mathcal{H}_{\overline{X}}(r) & \xrightarrow{\sim} & \overline{\mathcal{H}_X(r)} \end{array}$$

which is in full accordance with the  $I$ -part of Theorem 5.2.

## 6. HOLOMORPHIC ACTION OF A FINITE GROUP

As in previous sections, let  $M$  be an irreducible smooth complex projective variety of complex dimension  $d$ , and consider a finite group  $\Gamma$  acting faithfully on  $M$  via holomorphic automorphisms of  $M$ . For any  $\gamma \in \Gamma$ , the automorphism of  $M$  given by the action of  $\gamma$  will also be denoted by  $\gamma$ . For a very ample line bundle  $L$  on  $M$ , the holomorphic line bundle

$$\mathcal{L} := \bigotimes_{\gamma \in \Gamma} \gamma^* L$$

is both very ample and  $\Gamma$ -equivariant.

**6.1. Actions on higher dimensional varieties.** Consider the action of  $\Gamma$  on  $H^0(M, \mathcal{L})$  given by the action of  $\Gamma$  on  $\mathcal{L}$ . For general  $d-1$  elements  $s_1, \dots, s_{d-1}$  of  $H^0(M, \mathcal{L})^\Gamma$ , the intersection

$$X = \prod_{i=1}^{d-1} \text{divisor}(s_i) \subset M$$

is a smooth projective curve satisfying the following two conditions:

- the homomorphism  $\pi_1(X, x_0) \longrightarrow \pi_1(M, x_0)$  induced by the inclusion map

$$X \hookrightarrow M$$

is surjective, where  $x_0 \in X$ , and

- the action of  $\Gamma$  on  $M$  preserves the curve  $X$ .

Fix a smooth closed curve  $X$  as in (3.1) such that

- (1) the action of  $\Gamma$  on  $M$  preserves  $\phi(X)$ , and
- (2) the homomorphism  $\phi_* : \pi_1(X, x_0) \longrightarrow \pi_1(M, x_0)$  induced by  $\phi$  in (3.1) is surjective, where  $x_0 \in X$ .

**Lemma 6.1.** *There is an embedding of the  $\Gamma$ -fixed locus  $\mathcal{H}_M(r)^\Gamma$  in  $\mathcal{H}_X(r)^\Gamma$  induced by the map  $\Psi$  in (3.4).*

*Proof.* The action of  $\Gamma$  on  $M$  (respectively,  $X$ ) produces an action of  $\Gamma$  on the moduli space  $\mathcal{H}_M(r)$  (respectively,  $\mathcal{H}_X(r)$ ) of Higgs bundles on  $M$  (respectively,  $X$ ) in (3.4); the action of any  $\gamma \in \Gamma$  sends any  $(E, \theta)$  to  $(\gamma^*E, \gamma^*\theta)$ . The  $\Gamma$ -fixed locus in  $\mathcal{H}_M(r)$  (respectively,  $\mathcal{H}_X(r)$ ) will be denoted by  $\mathcal{H}_M(r)^\Gamma$  (respectively,  $\mathcal{H}_X(r)^\Gamma$ ). The map  $\Psi$  in (3.4) is evidently  $\Gamma$ -equivariant. Consequently,  $\Gamma$  produces a map

$$\Psi^\Gamma : \mathcal{H}_M(r)^\Gamma \longrightarrow \mathcal{H}_X(r)^\Gamma. \quad (6.1)$$

The map  $\Psi^\Gamma$  in (6.1) is an embedding because  $\Psi$  is so.  $\square$

**6.2. Actions on Riemann surfaces.** We shall now restrict our attention to the case where  $M$  is a compact Riemann surface, and thus consider the moduli space  $\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}$  of semistable Higgs bundles as introduced in [10] whose structure group is  $\mathrm{SL}(r, \mathbb{C})$ : that is, we consider  $\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})} \subset \mathcal{H}_M(r)$ . In this setting, we can see the following.

**Theorem 6.2.** *Assume that a finite group  $\Gamma \subset \mathrm{Aut}(M)$  is acting on  $M$  such that the action is not free (so there are points with nontrivial isotropy). Then, for  $r > 2$  the fixed point locus in  $\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}$  for the action of  $\Gamma$  will never be a mid-dimensional space, and thus will never be a Lagrangian subspace of  $\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}$ .*

*Proof.* Assume that  $r > 2$ . Then any connected component of the fixed point locus

$$\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}^\Gamma \subset \mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}$$

for the action of  $\Gamma$  is a component of a moduli space of parabolic Higgs bundles of rank  $r$  on the quotient surface  $M/\Gamma$  of fixed determinant for which the trace of the Higgs field is zero [3, 4].

Consider the quotient map

$$q : M \longrightarrow M/\Gamma =: Y.$$

The genus of  $Y$  will be denoted by  $g_Y$ . Let  $x_1, \dots, x_a \in Y$  be the points over which the map  $q$  is ramified. For any  $1 \leq i \leq a$ , let  $r_i$  be the order of ramification of  $q$  at any point of  $q^{-1}(x_i)$  (clearly all points over  $x_i$  have same order of ramification). By Hurwitz formula,

$$2(g - 1) = 2N \cdot (g_Y - 1) + \sum_{i=1}^a \frac{Nr_i}{r_i + 1}, \quad (6.2)$$

where  $N$  is the order of the group  $\Gamma$ .

We shall compute an upper bound for the dimension of the moduli space of  $\mathrm{SL}(r, \mathbb{C})$  parabolic Higgs bundles with parabolic structure at the ramification points for  $M$ . The maximal possible dimension of quasi-parabolic filtrations at a point on a given bundle is  $r(r-1)/2$ , and the number of ramification points is  $\sum_{i=1}^a \frac{N}{r_i+1}$ . So an upper bound for the dimension of the moduli space of parabolic Higgs bundles of rank  $r$  is

$$B := 2 \left( (r^2 - 1)(g_Y - 1) + \frac{r(r-1)}{2} \sum_{i=1}^a \frac{N}{r_i+1} \right); \quad (6.3)$$

note that the dimension of the moduli space of vector bundles of rank  $r$  of fixed determinant on the curve  $Y$  is  $(r^2 - 1)(g_Y - 1)$ . Now using (6.2) it follows immediately that  $B$  in (6.3) satisfies the inequality

$$2B < 2(r^2 - 1)(g - 1) = \dim \mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}.$$

The theorem follows from this inequality.  $\square$

**Remark 6.3.** It can be shown that the action of the group  $\Gamma =: (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$  considered in [9] does not give mid-dimensional subspaces of the moduli spaces of higher rank Higgs bundles.

**Proposition 6.4.** *Assume that  $M$  admits a holomorphic involution which is fixed point free. Then the corresponding fixed point locus*

$$\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}^{\mathbb{Z}/2\mathbb{Z}} \subset \mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}$$

*is mid-dimensional for  $r > 1$ .*

*Proof.* Since  $\Gamma = \mathbb{Z}/2\mathbb{Z}$  acts freely on  $M$ , the genus  $g_Y$  of the quotient Riemann surface  $Y := M/\Gamma$  is

$$g_Y = \frac{g+1}{2}.$$

We note that the given condition that the involution of  $M$  is fixed point free implies that the genus  $g$  of  $M$  is odd. Let  $\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}^Y$  denote the moduli space of semistable  $\mathrm{SL}(r, \mathbb{C})$ -Higgs bundles on  $Y$ . Then we have

$$\dim \mathcal{M}_{\mathrm{SL}(r, \mathbb{C})} = 2(r^2 - 1)(g - 1) = 4(r^2 - 1)(g_Y - 1) = 2 \dim \mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}^Y.$$

The proposition follows from this.  $\square$

From the above results, one can further understand the structure of the fixed point locus as a brane of Higgs bundles. Indeed, in this setting one has the following.

**Proposition 6.5.** *All the fixed point loci  $\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}^\Gamma$  constructed in Theorem 6.2 and Proposition 6.4 are in fact  $(B, B, B)$ -branes.*

*Proof.* The fixed point locus

$$\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}^\Gamma \subset \mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}$$

is clearly holomorphic with respect to the natural holomorphic structure on the moduli space  $\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}$  of Higgs bundles. On the other hand, the action of  $\Gamma$  on  $M$  produces a homomorphism

$$f_\Gamma : \Gamma \longrightarrow \mathrm{Out}(\pi_1(M))$$

to the group of outer automorphisms of the fundamental group of  $M$ . Using this homomorphism  $f_\Gamma$ , the group  $\Gamma$  acts on the character variety

$$\mathrm{Hom}(\pi_1(M), \mathrm{SL}(r, \mathbb{C})) // \mathrm{SL}(r, \mathbb{C}).$$

This action of  $\Gamma$  on  $\mathrm{Hom}(\pi_1(M), \mathrm{SL}(r, \mathbb{C})) // \mathrm{SL}(r, \mathbb{C})$  is evidently given by algebraic automorphisms.

We note that if  $\mathrm{Hom}(\pi_1(M), \mathrm{SL}(r, \mathbb{C})) // \mathrm{SL}(r, \mathbb{C})$  is identified with the moduli space of flat  $\mathrm{SL}(r, \mathbb{C})$  connections on  $M$ , then the above action of  $\Gamma$  on

$$\mathrm{Hom}(\pi_1(M), \mathrm{SL}(r, \mathbb{C})) // \mathrm{SL}(r, \mathbb{C})$$

corresponds to the following action of  $\Gamma$  on the moduli space of flat  $\mathrm{SL}(r, \mathbb{C})$  connections: the action of any  $\gamma \in \Gamma$  sends a flat connection  $(F, \nabla)$  to the flat connection  $(\gamma^*F, \gamma^*\nabla)$ .

The  $C^\infty$  diffeomorphism

$$\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})} \xrightarrow{\sim} \mathrm{Hom}(\pi_1(M), \mathrm{SL}(r, \mathbb{C})) // \mathrm{SL}(r, \mathbb{C})$$

given by the nonabelian Hodge theory is  $\Gamma$ -equivariant.

From these we conclude that the action of  $\Gamma$  on  $\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}$  preserves all the complex structures in the family of complex structures on

$$\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})} = \mathrm{Hom}(\pi_1(M), \mathrm{SL}(r, \mathbb{C})) // \mathrm{SL}(r, \mathbb{C})$$

defining the hyperKähler structure. Consequently, the fixed point locus  $\mathcal{M}_{\mathrm{SL}(r, \mathbb{C})}^\Gamma$  is a  $(B, B, B)$ -brane.  $\square$

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