ON THE GENERALIZED SO$(2n,\mathbb{C})$–OPERS

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Abstract. Since their introduction by Beilinson-Drinfeld [BD1, BD2], opers have seen several generalizations. In [BSY] a higher rank analog was studied, named generalized B-opers, where the successive quotients of the oper filtration are allowed to have higher rank and the underlying holomorphic vector bundle is endowed with a bilinear form which is compatible with both the filtration and the oper connection. Since the definition didn't encompass the even orthogonal groups, we dedicate this paper to study generalized B-opers whose structure group is SO$(2n,\mathbb{C})$, and show their close relationship with geometric structures on a Riemann surface.

1. Introduction

Motivated by the works of Drinfeld and Sokolov [DS1, DS2], Beilinson and Drinfeld introduced opers, in [BD1, BD2], for a semisimple complex Lie group G. A G–oper on a compact Riemann surface X is

- a holomorphic principal G–bundle P on X equipped with a holomorphic connection ∇, and
- a holomorphic reduction of structure group of P to a Borel subgroup of G,

such that the reduction satisfies the Griffiths transversality condition with respect to the connection ∇ and the second fundamental form of ∇ for the reduction satisfies certain nondegeneracy conditions.

In recent years, different extensions of the above objects have been introduced and studied – examples are $\mathfrak{g}$-opers (a $\mathfrak{g}$-oper is a Aut$(\mathfrak{g})$-oper [BD2]) and Miura opers [Fr], as well as (G,P)-opers [CS]. Very recently, the authors introduced the notion of a *generalized B-oper* in [BSY]. The definition was inspired by [Bi2], where a particular class of opers was studied for which rank$(E) = nr$ and the rank of each successive quotient E_i/E_{i-1} is r (the above two conditions remain unchanged). In [BSY], the authors incorporated a non-degenerate bilinear form B and required the (not necessarily full) filtration and the connection appearing in a G–oper to be compatible with it. However, the work done in [BSY] did not apply to opers with structure group SO$(2n,\mathbb{C})$. The case of SO$(2n,\mathbb{C})$ is subtler than SO$(2n+1,\mathbb{C})$ and Sp$(2n,\mathbb{C})$. We dedicate the present paper to the study the SO$(2n,\mathbb{C})$ case.

We begin our work by considering filtered SO$(2n,\mathbb{C})$–bundles with connections in Section 2, leading to the introduction and study of a *generalized* SO$(2n,\mathbb{C})$–*quasioper*: a

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quadruple $(E,\, B_0,\, \mathcal{F}_\bullet,\, D)$, where $(E,\, B_0,\, \mathcal{F}_\bullet)$ is a filtered $\mathrm{SO}(2n,\mathbb{C})$–bundle over a compact Riemann surface X, and D is a holomorphic connection on $(E,\, B_0,\, \mathcal{F}_\bullet)$ (see Definition 2.5). Let $2m+1$ be the length of the filtration. Then $r = n/(m+1)$ is an integer.

The quasiopers have naturally induced isomorphic dual quasiopers, as shown in Proposition 2.6. The properties of generalized $\mathrm{SO}(2n,\mathbb{C})$–quasiopers are studied in Section 3, in the spirit of [BSY] and in relation to the jet bundles.

The main goal of the paper is to introduce $\mathrm{SO}(2n,\mathbb{C})$–*opers*, and to show that generalized $\mathrm{SO}(2n,\mathbb{C})$–opers are closely related to projective structures on the base Riemann surface X, and this is done in Section 4. After constructing and studying $\mathrm{SO}(2n,\mathbb{C})$–opers through $\mathrm{SO}(2n,\mathbb{C})$–quasiopers, we consider their relation to geometric structures on X.

Let X be a compact connected Riemann surfaces of genus at least two. Fix positive integers n and m such that $r := n/(m+1)$ is an integer. Let

$$\mathbb{O}_X(n,m)$$

denote the space of all isomorphism classes of generalized $\mathrm{SO}(2n,\mathbb{C})$–opers on X of filtration length $2m+1$ (see Definition 2.2 and Definition 4.1). Let $\mathcal{C}_X$ be the space of all isomorphism classes of holomorphic principal $\mathrm{SO}(r,\mathbb{C})$–bundles on X equipped with a holomorphic connection, and let $\mathfrak{P}(X)$ be the space of all projective structures on the Riemann surface X.

We prove the following (see Theorem 4.3):

Theorem 1.1. *If the integer r is odd, then there is a canonical bijection between $\mathbb{O}_X(n,m)$ and the Cartesian product*

$$\mathcal{C}_X\times\mathfrak{P}(X)\times\left(H^0(X,\, K_X^{\otimes(m+1)})\oplus\left(\bigoplus_{i=2}^{m}H^0(X,\, K_X^{\otimes 2i})\right)\right)\times J(X)_2\ ,$$

for $J(X)_2$ the group of holomorphic line bundles on X of order two, and K_X the holomorphic cotangent bundle of X.

If r is even, then there is a canonical bijection between $\mathbb{O}_X(n,m)$ and the Cartesian product

$$\mathcal{C}_X\times\mathfrak{P}(X)\times\left(H^0(X,\, K_X^{\otimes(m+1)})\oplus\left(\bigoplus_{i=2}^{m}H^0(X,\, K_X^{\otimes 2i})\right)\right).$$

We note that in the cases of $\mathrm{SO}(2n+1,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C})$, the decomposition is same for even and odd r, unlike in Theorem 1.1.

2. Filtered $\mathrm{SO}(2n,\mathbb{C})$-bundles with connections

Let X be a compact connected Riemann surface of genus g, with $g\geq 2$. The holomorphic cotangent bundle and the holomorphic tangent bundle of X will be denoted by

K_X and TX respectively. Let E be a holomorphic vector bundle on X of rank $2n$, where $n \geq 2$, such that

$$\det E = \bigwedge^{2n} E = \mathcal{O}_X$$

An SO($2n,\mathbb{C}$) structure on E is a holomorphic symmetric bilinear form

$$B_0 \in H^0(X, \operatorname{Sym}^2(E^*)) \tag{2.1}$$

on E which is fiberwise nondegenerate. In other words, $B_0(x)$ is a nondegenerate symmetric bilinear form on E_x for every $x \in X$. A pair of the form (E, B_0), where B_0 is an SO($2n,\mathbb{C}$) structure on a holomorphic vector bundle on X E, would be called an SO($2n,\mathbb{C}$)–bundle on X.

We note that for an SO($2n,\mathbb{C}$)–bundle (E, B_0), the determinant line bundle $\bigwedge^{2n} E$ is holomorphically identified with $\mathcal{O}_X$ uniquely up to a sign. More precisely, for any $x \in X$, consider all isomorphisms of $(E_x, B_0(x))$ with $\mathbb{C}^{2n}$ equipped with the standard symmetric bilinear form. Then the space of corresponding isomorphisms of $\bigwedge^{2n} E_x$ with $\bigwedge^{2n} \mathbb{C}^{2n}$ has exactly two elements, and these two elements just differ by a sign.

2.1. **Filtered SO($2n,\mathbb{C}$)-bundles.** An SO($2n,\mathbb{C}$) structure B_0 on E produces a holomorphic isomorphism

$$B : E \longrightarrow E^* \tag{2.2}$$

that sends any $v \in E_x$, $x \in X$, to the element of E_x^* defined by $w \longmapsto B_0(x)(w, v)$. The annihilator of a holomorphic subbundle $F \subset E$, for the bilinear form B_0, will be denoted by $F^\perp$. So, for any $x \in X$, the subspace $F_x^\perp \subset E_x$ consists of all $v \in E_x$ such that $B_0(x)(w, v) = 0$ for all $w \in F_x$. The bilinear form B_0 produces C^∞ homomorphisms

$$E \otimes (E \otimes \Omega 0,1_X) \longrightarrow \Omega 0,1_X \quad \text{and} \quad (E \otimes \Omega 0,1_X) \otimes E \longrightarrow \Omega 0,1_X$$

simply by tensoring with the identity map of $\Omega 0,1_X$. Since the bilinear form B_0 is holomorphic, we have

$$\overline{\partial} B_0(s, t) = B_0(\overline{\partial}_E s, t) + B_0(s, \overline{\partial}_E t)\,, \tag{2.3}$$

where s and t are locally defined C^∞ sections of E and $\overline{\partial}_E : C^\infty(X, E) \longrightarrow C^\infty(X, E \otimes \Omega 0,1_X)$ is the Dolbeault operator defining the holomorphic structure on E. If t is a locally defined holomorphic section of F and s is a locally defined C^∞ section of $F^\perp$, then from (2.3) we have

$$B_0(\overline{\partial}_E s, t) = 0\,,$$

because $\overline{\partial}_E t = 0\, B_0(s, t)$. This implies that $F^\perp$ is actually a holomorphic subbundle of E; its rank is $2n - \operatorname{rank}(F)$.

Definition 2.1. A *filtration* of an SO($2n,\mathbb{C}$)–bundle (E, B_0) is a filtration of holomorphic subbundles of E

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_i \subset \cdots \subset F_{2m} \subset F_{2m+1} = E \tag{2.4}$$

satisfying the following two conditions:

(1) $\text{rank}(F_{m+1}/F_m) = 2 \cdot \text{rank}(F_1)$, and $\text{rank}(F_i/F_{i-1}) = \text{rank}(F_1)$, for all $i \in \{1, \cdots, 2m+1\} \setminus \{m+1\}$, and

(2) $F_i^{\perp} = F_{2m+1-i}$ for all $0 \leq i \leq m$.

Note that the first condition implies that

$$(m+1) \cdot \text{rank}(F_1) = n .$$

The second condition implies that the restriction $B_0|_{F_{m+1}}$ of the form B_0 to the subbundle $F_{m+1} \subset E$ has has the following two properties:

- the subbundle $F_m \subset F_{m+1}$ is annihilated by $B_0|_{F_{m+1}}$, meaning $F_m \subset F_m^{\perp}$, and
- the restriction $B_0\big|_{F_{m+1}}$ descends to the quotient bundle F_{m+1}/F_m as a fiberwise nondegenerate symmetric bilinear form.

For notational convenience, the filtration $\{F_i\}_{i=0}^{2m+1}$ in (2.4) will henceforth be denoted by $\mathcal{F}_\bullet$.

Definition 2.2. An $\mathrm{SO}(2n,\mathbb{C})$–bundle equipped with a filtration will be called a *filtered* $\mathrm{SO}(2n,\mathbb{C})$–bundle. The odd integer $2m+1$ in (2.4) will be called the *length* of the filtration.

We shall always assume that $m \geq 2$. This is because

$$\mathrm{SO}(4,\mathbb{C}) = (\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}))/(\mathbb{Z}/2\mathbb{Z}) .$$

2.2. **SO$(2n,\mathbb{C})$-quasiopers: Filtered SO$(2n,\mathbb{C})$-bundles with connections.** Recall that a *holomorphic connection* on a holomorphic vector bundle E on X is a first order holomorphic differential operator

$$D : E \longrightarrow E \otimes K_X$$

satisfying the Leibniz identity, this is,

$$D(fs) = fD(s) + s \otimes df$$

for any locally defined holomorphic function f on X and any locally defined holomorphic section s of E [At]. In particular, a holomorphic connection is automatically flat because $\Omega_X^{2,0} = 0$. The bilinear form B_0 in (2.1) produces holomorphic homomorphisms

$$E \otimes (E \otimes K_X) \longrightarrow K_X \quad \text{and} \quad (E \otimes K_X) \otimes E \longrightarrow K_X$$

simply by tensoring with the identity map of K_X. A holomorphic connection on an $\mathrm{SO}(2n,\mathbb{C})$–bundle $(E,\, B_0)$ is a holomorphic connection D on the holomorphic vector bundle E satisfying the identity

$$\partial B_0(s,\, t) = B_0(D(s),\, t) + B_0(s,\, D(t))$$

for all locally defined holomorphic sections s and t of E.

We note that for a holomorphic connection D on an $\mathrm{SO}(2n,\mathbb{C})$–bundle $(E,\, B_0)$, the connection on the determinant line bundle $\bigwedge^{2n} E = \mathcal{O}_X$ induced by D coincides with the trivial connection on the trivial holomorphic line bundle given by the de Rham differential

d (it is the unique rank one holomorphic connection on X with trivial monodromy). Indeed, this follows immediately from the fact that the isomorphism B in (2.2) takes the connection D on E to the dual connection on E^* induced by D.

Let D a holomorphic connection on E, and let

$$F_1 \subset F_2 \subset E \quad \text{and} \quad F_3 \subset F_4 \subset E$$

be holomorphic subbundles such that

$$D(F_1) \subset F_3 \otimes K_X \quad \text{and} \quad D(F_2) \subset F_4 \otimes K_X .$$

Definition 2.3. The second fundamental form of (F_1, F_2, F_3, F_4) for the connection D is the map

$$\begin{aligned} S(D; F_1, F_2, F_3, F_4) \,:\, F_2/F_1 &\longrightarrow (F_4/F_3) \otimes K_X \\ s &\longmapsto D(\widetilde{s}) \end{aligned} \tag{2.5}$$

that sends any locally defined holomorphic section s of F_2/F_1 to the image of $D(\widetilde{s})$ in $(F_4/F_3) \otimes K_X$, where $\widetilde{s}$ is any locally defined holomorphic section of the subbundle F_2 that projects to s under the quotient map $F_2 \longrightarrow F_2/F_1$.

It is straightforward to check that the image of $D(\widetilde{s})$ in $(F_4/F_3) \otimes K_X$ does not depend on the choice of the above lift $\widetilde{s}$ of s (see [BSY, Lemma 2.10]).

Definition 2.4. Let $(E,\, B_0,\, \mathcal{F}_\bullet)$ be a filtered SO$(2n, \mathbb{C})$–bundle. A *holomorphic connection on* $(E,\, B_0,\, \mathcal{F}_\bullet)$ is a holomorphic connection D on $(E,\, B_0)$ satisfying the following three conditions:

(1) $D(F_i) \subset F_{i+1} \otimes K_X$ for all $1 \le i \le 2m$ (see (2.4)),

(2) the second fundamental form

$$S(D, i) \,:\, F_i/F_{i-1} \longrightarrow (F_{i+1}/F_i) \otimes K_X$$

is an isomorphism for all $i \in \{1, \cdots, 2m+1\} \setminus \{m,\, m+1\}$, and

(3) the composition of homomorphisms

$$(S(D, m+1) \otimes \mathrm{Id}_{K_X}) \circ S(D, m) \,:\, F_m/F_{m-1} \longrightarrow (F_{m+2}/F_{m+1}) \otimes K_X^{\otimes 2}$$

is an isomorphism.

Definition 2.5. A *generalized* SO$(2n, \mathbb{C})$–*quasioper* on X is a quadruple $(E,\, B_0,\, \mathcal{F}_\bullet,\, D)$, where $(E,\, B_0,\, \mathcal{F}_\bullet)$ is a filtered SO$(2n, \mathbb{C})$–bundle, and D is a holomorphic connection on the filtered SO$(2n, \mathbb{C})$–bundle $(E,\, B_0,\, \mathcal{F}_\bullet)$.

Two generalized SO$(2n, \mathbb{C})$–quasiopers $(E,\, B_0,\, \mathcal{F}_\bullet,\, D)$ and $(E',\, B'_0,\, \mathcal{F}'_\bullet,\, D')$ are called *isomorphic* if there is a holomorphic isomorphism of vector bundles

$$\Phi \,:\, E \longrightarrow E'$$

such that

- Φ takes the bilinear form B_0 on E to the bilinear form B'_0 on E',
- Φ takes the filtration $\mathcal{F}_\bullet$ of E to the filtration $\mathcal{F}'_\bullet$ of E', and

- Φ takes the connection ∇ on E to the connection ∇' on E'.

Proposition 2.6. *Given a generalized* $\mathrm{SO}(2n,\mathbb{C})$*–quasioper* $(E,\, B_0,\, \mathcal{F}_\bullet,\, D)$*, there is a naturally associated isomorphic dual quasioper.*

Proof. Let $(E,\, B_0,\, \mathcal{F}_\bullet,\, D)$ be a generalized $\mathrm{SO}(2n,\mathbb{C})$–quasioper on X, where $\mathcal{F}_\bullet$, as in (2.4), is a filtration of E. Consider the dual vector bundle E^*. It is equipped with a holomorphic connection D^* induced by the connection D on E.

Since the symmetric bilinear form B_0 on E is nondegenerate, it produces a holomorphic symmetric nondegenerate bilinear form B_0^* on E^*. For any F_i in (2.4), define

$$G_{2m+1-i} \,\subset\, E^* \tag{2.6}$$

to be the kernel of the natural projection $E^* \,\longrightarrow\, (F_i)^*$. Then

$$(E^*,\, B_0^*, \{G_j\}_{j=0}^{2m+1},\, D^*) \tag{2.7}$$

is also a generalized $\mathrm{SO}(2n,\mathbb{C})$–quasioper.

It is straightforward to check that the holomorphic isomorphism B in (2.2) takes the generalized $\mathrm{SO}(2n,\mathbb{C})$–quasioper $(E,\, B_0,\, \mathcal{F}_\bullet,\, D)$ to the generalized $\mathrm{SO}(2n,\mathbb{C})$–quasioper $(E^*,\, B_0^*, \{G_j\}_{j=0}^{2m+1},\, D^*)$ constructed in (2.7). □

3. Properties of a generalized $\mathrm{SO}(2n,\mathbb{C})$–quasioper

Let W be holomorphic vector bundle over X equipped with a holomorphic connection D_W, and let $V \,\subset\, W$ be any holomorphic subbundle.

Lemma 3.1. *There is a unique minimal holomorphic subbundle* $\widehat{D}_W(V)$ *of* W *containing* V *such that the connection* D_W *takes* V *into* $\widehat{D}_W(V)\otimes K_X$*.*

Proof. From Definition 2.3, consider the second fundamental form of V for the connection D_W

$$S(D_W;V) \,:\, V \,\longrightarrow\, (W/V)\otimes K_X$$

by letting $F_1 \,=\, 0, F_2 \,=\, F_3 \,=\, V$ and $F_4 \,=\, W$ in Eq. (2.5). Let

$$\mathcal{T} \,\subset\, ((W/V)\otimes K_X)/(S(D_W;V)(V))$$

be the torsion part of the coherent analytic sheaf $((W/V)\otimes K_X)/(S(D_W;V)(V))$. The inverse image of $\mathcal{T}$ under the quotient map

$$(W/V)\otimes K_X \,\longrightarrow\, ((W/V)\otimes K_X)/(S(D;V)(V))$$

will be denoted by $\mathcal{F}$. So $\mathcal{F}\otimes TX$ is a holomorphic subbundle of

$$(W/V)\otimes K_X\otimes TX \,=\, W/V\,.$$

The inverse image of the subbundle $\mathcal{F}\otimes TX \,\subset\, W/V$ under the quotient map $W \,\longrightarrow\, W/V$ will be denoted by $\widehat{D}_W(V)$.

Note that $\widehat{D}_W(V)$ is a holomorphic subbundle of W, because $\mathcal{F}\otimes TX$ is a holomorphic subbundle of W/V. Also, V is a holomorphic subbundle of $\widehat{D}_W(V)$. From the construction of $\widehat{D}_W(V)$ it is evident that we have

$$D_W(V) \subset \widehat{D}_W(V)\otimes K_X .$$

Also it is clear that $\widehat{D}_W(V)$ is the smallest among all subbundles U of W such that $D_W(V) \subset U\otimes K_X$. □

Note that V is preserved by the connection D_W if and only if $\widehat{D}_W(V) = V$, where $\widehat{D}_W(V)$ is constructed in Lemma 3.1.

The holomorphic subbundle $\widehat{D}_W(\widehat{D}_W(V)) \subset W$ will be denoted by $\widehat{D}^2_W(V)$. Moreover, for ease of notation, inductively define the subbundle

$$\widehat{D}^{k+1}_W(V) := \widehat{D}_W(\widehat{D}^k_W(V)) \subset W , \tag{3.1}$$

for $k \geq 2$. So $\{\widehat{D}^j_W(V)\}_{j\geq 1}$ is an increasing sequence of holomorphic subbundles of W.

Through Lemma 3.1, we can construct a holomorphic subbundle of a generalized SO($2n,\mathbb{C}$)–quasioper. Indeed, let

$$(E,\, B_0,\, \mathcal{F}_\bullet,\, D)$$

be a generalized SO($2n,\mathbb{C}$)–quasioper on X, where $\mathcal{F}_\bullet$, as in (2.4), is a filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_i \subset \cdots \subset F_{2m} \subset F_{2m+1} = E \tag{3.2}$$

of length $2m+1$. For the holomorphic subbundle $F_1 \subset E$ in (3.2), define the holomorphic subbundle

$$\mathbb{F} := \widehat{D}^{2m}_E(F_1) \subset E \tag{3.3}$$

(see (3.1)). We note that the subbundle $\mathbb{F}$ in general is not preserved by the connection D on E.

Now consider the generalized SO($2n,\mathbb{C}$)–quasioper $(E^*,\, B^*_0,\, \{G_j\}_{j=0}^{2m+1},\, D^*)$ in (2.7) associated to $(E,\, B_0,\, \mathcal{F}_\bullet,\, D)$ via Proposition 2.6. As in (3.3), define the holomorphic subbundle

$$\mathbb{G} := (\widehat{D}^*)^{2m}_{E^*}(G_1) \subset E^* , \tag{3.4}$$

where G_1 is constructed in (2.6). The dual of the natural quotient map $E^* \longrightarrow E^*/\mathbb{G}$ is a fiberwise injective holomorphic homomorphism $(E^*/\mathbb{G})^* \longrightarrow E^{**}$. Therefore, we have the holomorphic subbundle

$$\mathcal{S} := (E^*/\mathbb{G})^* \subset E^{**} = E \tag{3.5}$$

given by the image of the above fiberwise injective homomorphism.

Lemma 3.2. *For the holomorphic subbundles $\mathbb{F}$ and $\mathcal{S}$ of E, in (3.3) and (3.5) respectively, the natural homomorphism*

$$\mathbb{F}\oplus\mathcal{S} \longrightarrow E$$

is an isomorphism. Moreover, the resulting holomorphic decomposition

$$E = \mathbb{F}\oplus\mathcal{S}$$

of E is orthogonal with respect to the bilinear form B_0 on E.

Proof. From the properties of the filtration $\mathcal{F}_\bullet$ and the connection D it follows that the natural homomorphism

$$\mathbb{F}\oplus\mathcal{S} \longrightarrow E$$

is surjective. Note that from the properties of the filtration $\mathcal{F}_\bullet$ and D we also have $\text{rank}(\mathbb{F}) = 2n - \frac{n}{m+1} = \text{rank}(E) - \text{rank}(F_1)$ and $\text{rank}(\mathcal{S}) = \frac{n}{m+1} = \text{rank}(F_1)$. Furthermore, we have $B_0(\mathbb{F},\, \mathcal{S}) = 0$. These together imply that $\mathbb{F}\bigcap\mathcal{S} = 0$ and $\mathbb{F}^\perp = \mathcal{S}$. □

In what follows we shall describe an alternative construction of the subbundle $\mathcal{S}$ in (3.5) by considering the jet bundle approach given in [BSY]. Let

$$Q := E/F_{2m} \tag{3.6}$$

be the quotient in (3.2), and let

$$q : E \longrightarrow E/F_{2m} = Q \tag{3.7}$$

be the quotient map.

For any nonnegative integer i, let $J^i(Q)$ be the i-th order jet bundle of Q in (3.6) (see [BSY, Section 3.1], [Bi2], [Bi1] for jet bundles). As shown in [BSY, Eq. (3.3)], [BSY, Eq. (3.5)], the connection D on E produces an $\mathcal{O}_X$–linear homomorphism

$$f_i : E \longrightarrow J^i(Q)\,. \tag{3.8}$$

We briefly recall the construction of f_i as this homomorphism plays a crucial role.

Take any point $x \in X$, and let $x \in U_x \subset X$ be a simply connected analytic open neighborhood of the point x. For any $v \in E_x$, let $\widetilde{v}$ be the unique flat section of $E\big|_{U_x}$, for the flat connection D on E, such that $\widetilde{v}(x) = v$. Consider the holomorphic section $q(\widetilde{v})$ of the vector bundle $Q|_{U_x}$ in (3.6), where q is the projection in (3.7). Restricting this section $q(\widetilde{v})$ to the i–th order infinitesimal neighborhood of x, we get an element of $J^i(Q)_x$; this element of $J^i(Q)_x$ given by $q(\widetilde{v})$ will be denoted by $q(\widetilde{v})_i$. The map f_i in (3.8) sends any $v \in E_x$, $x \in X$, to $q(\widetilde{v})_i \in J^i(Q)_x$ constructed above using v and the connection D.

From the three conditions in Definition 2.4 it follows that the homomorphism

$$f_{2m} : E \longrightarrow J^{2m}(Q)$$

in (3.8) is surjective. Moreover, the subbundle $\mathcal{S}$ in (3.5) coincides with the kernel of the above homomorphism f_{2m}. Consequently, we have a short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathcal{S} = \text{kernel}(f_{2m}) \longrightarrow E \xrightarrow{f_{2m}} J^{2m}(Q) \longrightarrow 0 \tag{3.9}$$

on X. Therefore, Lemma 3.2 has the following corollary.

Corollary 3.3. *The composition of homomorphisms*

$$\mathbb{F} \hookrightarrow E \xrightarrow{f_{2m}} J^{2m}(Q)\,,$$

where f_{2m} is the homomorphism in (3.8)*, and $\mathbb{F}$ is the subbundle of E in Lemma 3.2, is an isomorphism.*

Let

$$f'_{2m} \,:\, \mathbb{F} \,\stackrel{\sim}{\longrightarrow}\, J^{2m}(Q) \tag{3.10}$$

be the composition of homomorphisms in Corollary 3.3; recall from Corollary 3.3 that f'_{2m} is an isomorphism.

Using the decomposition $\mathbb{F}\oplus\mathcal{S} \,=\, E$ in Lemma 3.2, consider the composition of homomorphisms

$$\mathbb{F} \,\hookrightarrow\, E \,\stackrel{D}{\longrightarrow}\, E\otimes K_X \,\stackrel{q_{\mathbb{F}}\otimes \mathrm{Id}}{\longrightarrow}\, \mathbb{F}\otimes K_X\,, \tag{3.11}$$

where $q_{\mathbb{F}} \,:\, E \,=\, \mathbb{F}\oplus\mathcal{S} \,\longrightarrow\, \mathbb{F}$ is the natural projection to factor $\mathbb{F}$. This composition of homomorphisms is a holomorphic connection on $\mathbb{F}$, because it satisfies the Leibniz identity. The holomorphic connection on $\mathbb{F}$ constructed in (3.11) will be denoted by $D^{\mathbb{F}}$.

Similarly, the composition of homomorphisms

$$\mathcal{S} \,\hookrightarrow\, E \,\stackrel{D}{\longrightarrow}\, E\otimes K_X \,\stackrel{q_{\mathcal{S}}\otimes \mathrm{Id}}{\longrightarrow}\, \mathcal{S}\otimes K_X\,, \tag{3.12}$$

where $q_{\mathcal{S}} \,:\, E \,=\, \mathbb{F}\oplus\mathcal{S} \,\longrightarrow\, \mathcal{S}$ is the natural projection to factor $\mathcal{S}$ in Lemma 3.2, is a holomorphic connection on the holomorphic vector bundle $\mathcal{S}$. The holomorphic connection on $\mathcal{S}$ constructed in (3.12) will be denoted by $D^{\mathcal{S}}$.

The holomorphic connections $D^{\mathbb{F}}$ and $D^{\mathcal{S}}$, on $\mathbb{F}$ and $\mathcal{S}$ respectively, together define a holomorphic connection $D^{\mathbb{F}}\oplus D^{\mathcal{S}}$ on $\mathbb{F}\oplus\mathcal{S}$. It should be emphasized that the isomorphism $\mathbb{F}\oplus\mathcal{S} \,=\, E$ in Lemma 3.2 does not, in general, take the holomorphic connection $D^{\mathbb{F}}\oplus D^{\mathcal{S}}$ on $\mathbb{F}\oplus\mathcal{S}$ to the connection D on E. Indeed, for the connection D on E the subbundles $\mathbb{F}$ and $\mathcal{S}$ of E may have nontrivial second fundamental form. On the other hand, for the direct sum of connections $D^{\mathbb{F}}\oplus D^{\mathcal{S}}$ the second fundamental form of both $\mathbb{F}$ and $\mathcal{S}$ vanish identically.

From Lemma 3.2 it follows immediately that the restrictions of the bilinear form B_0 on E to both $\mathbb{F}$ and $\mathcal{S}$ are nondegenerate. The holomorphic symmetric nondegenerate bilinear form on $\mathbb{F}$ (respectively, $\mathcal{S}$) obtained by restricting B_0 to $\mathbb{F}$ (respectively, $\mathcal{S}$) will be denoted by $B_{\mathbb{F}}$ (respectively, $B_{\mathcal{S}}$); in particular, we have

$$B_{\mathbb{F}} \,\in\, H^0(X,\, \mathrm{Sym}^2(\mathbb{F}^*)) \quad \text{and} \quad B_{\mathbb{S}} \,\in\, H^0(X,\, \mathrm{Sym}^2(\mathbb{S}^*))\,.$$

As the decomposition $\mathbb{F}\oplus\mathcal{S} \,=\, E$ in Lemma 3.2 is orthogonal, we have

$$B_0 \,=\, B_{\mathbb{F}}\oplus B_{\mathcal{S}}\,. \tag{3.13}$$

Since the connection D preserves the bilinear form B_0 on E, and (3.13) holds, from the constructions of the connections $D^{\mathbb{F}}$ in (3.11) and the connection $D^{\mathcal{S}}$ in (3.12) we have the following:

Corollary 3.4. *The connection $D^{\mathbb{F}}$ on $\mathbb{F}$ in* (3.11) *preserves the bilinear form $B_{\mathbb{F}}$ on $\mathbb{F}$.*

The connection $D^{\mathcal{S}}$ on $\mathcal{S}$ in (3.12) *preserves the bilinear form $B_{\mathcal{S}}$ on $\mathcal{S}$.*

Proposition 3.5. *The connection $D^{\mathbb{F}}$ on $\mathbb{F}$ produces a holomorphic connection D_Q on $J^{2m}(Q)$.*

Proof. This can be deduced from the fact that the homomorphism f'_{2m} in (3.10) is an isomorphism. So D_Q is the holomorphic connection on $J^{2m}(Q)$ that corresponds to the connection $D^{\mathbb{F}}$ on $\mathbb{F}$ by this isomorphism f'_{2m}.

We shall give a direct construction of this connection D_Q on $J^{2m}(Q)$. Let

$$0 \longrightarrow Q\otimes K_X^{\otimes(2m+1)} \stackrel{\iota}{\longrightarrow} J^{2m+1}(Q) \stackrel{q}{\longrightarrow} J^{2m}(Q) \longrightarrow 0$$

be the natural short exact sequence of jet bundles. It fits in the following commutative diagram of homomorphisms:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Q\otimes K_X^{\otimes(2m+1)} & \stackrel{\iota}{\longrightarrow} & J^{2m+1}(Q) & \stackrel{q}{\longrightarrow} & J^{2m}(Q) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow\lambda & & \| & & \\
0 & \longrightarrow & J^{2m}(Q)\otimes K_X & \stackrel{\iota'}{\longrightarrow} & J^1(J^{2m}(Q)) & \stackrel{q'}{\longrightarrow} & J^{2m}(Q) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & J^{2m-1}(Q)\otimes K_X & \stackrel{=}{\longrightarrow} & J^{2m-1}(Q)\otimes K_X & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{3.14}$$

(see [Bi2, p. 4, (2.4)] and [Bi2, p. 10, (3.4)]).

Consider the homomorphism

$$(f_{2m+1}|_{\mathbb{F}})\circ (f'_{2m})^{-1}\,:\, J^{2m}(Q)\,\longrightarrow\, J^{2m+1}(Q)\,,$$

where $f_{2m+1}|_{\mathbb{F}}$ is the restriction of the homomorphism in (3.8), and f'_{2m} is the isomorphism in (3.10). It is straightforward to check that

$$q\circ ((f_{2m+1}|_{\mathbb{F}})\circ (f'_{2m})^{-1})\,=\,\mathrm{Id}_{J^{2m}(Q)}\,,$$

where q is the projection in (3.14). Therefore, from the commutativity of the diagram in (3.14) we conclude that

$$q'\circ\lambda\circ ((f_{2m+1}|_{\mathbb{F}})\circ (f'_{2m})^{-1})\,=\,\mathrm{Id}_{J^{2m}(Q)}\,,$$

where q' and λ are the homomorphisms in (3.14). Consequently, the homomorphism

$$\lambda\circ ((f_{2m+1}|_{\mathbb{F}})\circ (f'_{2m})^{-1})\,:\, J^{2m}(Q)\,\longrightarrow\, J^1(J^{2m}(Q))$$

produces a holomorphic splitting of the short exact sequence

$$0\longrightarrow J^{2m}(Q)\otimes K_X \stackrel{\iota'}{\longrightarrow} J^1(J^{2m}(Q)) \stackrel{q'}{\longrightarrow} J^{2m}(Q)\longrightarrow 0 \tag{3.15}$$

in (3.14). But (3.15) is the twisted dual of the Atiyah exact sequence for $J^{2m}(Q)$. More precisely, let

$$0\,\longrightarrow\, J^{2m}(Q)^*\otimes J^{2m}(Q)\,\longrightarrow\, J^1(J^{2m}(Q))^*\otimes J^{2m}(Q)$$
$$\stackrel{\eta}{\longrightarrow}\,(J^{2m}(Q)\otimes K_X)^*\otimes J^{2m}(Q)\,=\,\mathrm{End}(J^{2m}(Q))\otimes TX\,\longrightarrow\, 0$$

be the dual of (3.15) tensored with $\text{Id}_{J^{2m}(Q)}$. Then $\eta^{-1}(\text{Id}_{J^{2m}(Q)} \otimes TX) \subset J^1(J^{2m}(Q))^* \otimes J^{2m}(Q)$ is the Atiyah bundle $\text{At}(J^{2m}(Q))$ of $J^{2m}(Q)$; furthermore, the short exact sequence

$$0 \longrightarrow \text{End}(J^{2m}(Q)) = J^{2m}(Q)^* \otimes J^{2m}(Q) \longrightarrow \text{At}(J^{2m}(Q)) \xrightarrow{\eta} TX \longrightarrow 0$$

obtained from the above short exact sequence is in fact the Atiyah exact sequence for $J^{2m}(Q)$. Consequently, a holomorphic splitting of (3.15) is a holomorphic connection on the holomorphic vector bundle $J^{2m}(Q)$ [At]. Therefore, the above homomorphism $\lambda \circ ((f_{2m+1}|_{\mathbb{F}}) \circ (f'_{2m})^{-1})$ is a holomorphic connection on $J^{2m}(Q)$.

The holomorphic connection on $J^{2m}(Q)$ defined by $\lambda \circ ((f_{2m+1}|_{\mathbb{F}}) \circ (f'_{2m})^{-1})$ coincides with the holomorphic connection D_Q on $J^{2m}(Q)$ produced by the connection $D^{\mathbb{F}}$ on $\mathbb{F}$ (see (3.11)) using the isomorphism f'_{2m} in (3.10). $\square$

Let $\mathcal{L}$ be holomorphic line bundle on X of order two. So the holomorphic line bundle $\mathcal{L} \otimes \mathcal{L}$ is holomorphically isomorphic to $\mathcal{O}_X$. Fix a holomorphic isomorphism

$$\rho : \mathcal{L} \otimes \mathcal{L} \longrightarrow \mathcal{O}_X . \tag{3.16}$$

There is a unique holomorphic connection

$$D^{\mathcal{L}} \tag{3.17}$$

on $\mathcal{L}$ such that the isomorphism ρ in (3.16) takes the holomorphic connection $D^{\mathcal{L}} \otimes \text{Id} + \text{Id} \otimes D^{\mathcal{L}}$ on $\mathcal{L} \otimes \mathcal{L}$ to the trivial connection on $\mathcal{O}_X$ given by the de Rham differential d. It should be clarified that this connection $D^{\mathcal{L}}$ does not depend on the choice of the isomorphism ρ.

Let $(E,\, B_0,\, \mathcal{F}_\bullet,\, D)$ be a generalized SO$(2n, \mathbb{C})$–quasioper on X. Consider the holomorphic vector bundle $E^1 := E \otimes \mathcal{L}$. Note that

$$\bigwedge^{2n} E^1 = (\bigwedge^{2n} E) \otimes (\bigwedge^{2n} \mathcal{L}) = \bigwedge^{2n} E = \mathcal{O}_X .$$

Since $E^1 \otimes E^1 = (E \otimes E) \otimes (\mathcal{L} \otimes \mathcal{L})$, we conclude that

$$B_0^1 = B_0 \otimes \rho$$

is a fiberwise nondegenerate symmetric holomorphic bilinear form on E^1, where ρ is the isomorphism in (3.16). The filtration $\mathcal{F}_\bullet$ of holomorphic subbundles of E produces a filtration $\mathcal{F}^1_\bullet$ of holomorphic subbundles of E^1. The i–th term F^1_i of $\mathcal{F}^1_\bullet$ is simply $F_i \otimes \mathcal{L}$ (see (3.2)). Let

$$D^1 := D \otimes \text{Id}_{\mathcal{L}} + \text{Id}_E \otimes D^{\mathcal{L}} \tag{3.18}$$

be the holomorphic connection on $E \otimes \mathcal{L} = E^1$, where $D^{\mathcal{L}}$ is the holomorphic connection in (3.17).

The following lemma is straightforward to prove.

Lemma 3.6. *The quadruple*

$$(E^1,\, B_0^1,\, \mathcal{F}^1_\bullet,\, D^1)$$

constructed above is a generalized SO$(2n, \mathbb{C})$*–quasioper on* X*.*

The holomorphic vector bundle $\mathbb{F} = \widehat{D}_E^{2m}(F_1)$ in (3.3) has the following filtration of holomorphic subbundles

$$0 \subset F_1 \subset \widehat{D}_E(F_1) \subset \widehat{D}_E^2(F_1) \subset \cdots \subset \widehat{D}_E^{2m-1}(F_1) \subset \widehat{D}_E^{2m}(F_1) = \mathbb{F}. \tag{3.19}$$

From Definition 2.4 it follows that the filtration of $\mathbb{F}$ in (3.19) coincides with the filtration of $\mathbb{F}$ obtained by intersecting the filtration of E in (3.2) with the subbundle $\mathbb{F}$ of E. Moreover, the isomorphism f'_{2m} in (3.10) takes the filtration of $\mathbb{F}$ in (3.19) to the filtration of $J^{2m}(Q)$ given by the short exact sequence of jet bundles

$$0 \longrightarrow Q \otimes K_X^{\otimes i} \longrightarrow J^i(Q) \longrightarrow J^{i-1}(Q) \longrightarrow 0 \tag{3.20}$$

for $i \geq 1$. More precisely, for any $1 \leq j \leq 2m-1$, the subbundle $\widehat{D}_E^j(F_1)$ in (3.19) corresponds to the kernel of the projection $J^{2m}(Q) \longrightarrow J^{2m-j-1}(Q)$ by the isomorphism f'_{2m} in (3.10).

Let

$$0 \longrightarrow Q \otimes K_X^{\otimes 2m} \longrightarrow \mathbb{F} = J^{2m}(Q) \longrightarrow J^{2m-1}(Q) \longrightarrow 0 \tag{3.21}$$

be the short exact sequence of jet bundles where, $\mathbb{F}$ is identified with $J^{2m}(Q)$ using the isomorphism f'_{2m} in (3.10).

As explained before, the connection D on E need not preserve the subbundle $\mathcal{S}$ in (3.5). Consider the decomposition $E = \mathbb{F} \oplus \mathcal{S}$ in Lemma 3.2. Assume that

$$\widehat{D}_E(\mathcal{S}) = \mathcal{S} \oplus (Q \otimes K_X^{\otimes 2m}) \subset \mathcal{S} \oplus \mathbb{F} = E, \tag{3.22}$$

where $Q \otimes K_X^{\otimes 2m}$ is the subbundle of $\mathbb{F}$ in (3.21), and $\widehat{D}_E(\mathcal{S}) \subset E$ is the holomorphic subbundle given by Lemma 3.1. Then the second fundamental form $S(D; \mathcal{S})$ of $\mathcal{S}$ for the connection D is a holomorphic section

$$S(D; \mathcal{S}) \in H^0(X, \mathrm{Hom}(\mathcal{S}, Q \otimes K_X^{\otimes(2m+1)})) \tag{3.23}$$
$$\subset H^0(X, \mathrm{Hom}(\mathcal{S}, \mathbb{F})) = H^0(X, \mathrm{Hom}(\mathcal{S}, E/\mathcal{S}));$$

note that Lemma 3.2 identifies $E/\mathcal{S}$ with $\mathbb{F}$.

4. Generalized SO$(2n, \mathbb{C})$-opers and projective structures

Through the construction of generalized SO$(2n, \mathbb{C})$-quasiopers in Definition 2.5 and that of generalized B-opers in [BSY, Definition 2.11] we define a generalized SO$(2n, \mathbb{C})$-oper.

Definition 4.1. A *generalized* SO$(2n, \mathbb{C})$–*oper* on X is a generalized SO$(2n, \mathbb{C})$–quasioper $(E, B_0, \mathcal{F}_\bullet, D)$ on X (see Definition 2.5) satisfying the following three conditions:

(1) $\mathcal{S} = Q \otimes K_X^{\otimes m}$, where $\mathcal{S}$ and Q are defined in (3.5) and (3.6) respectively,
(2) $\widehat{D}(\mathcal{S}) = \mathcal{S} \oplus (Q \otimes K_X^{\otimes 2m})$ (see (3.22) for this condition), and
(3) there is a holomorphic section

$$\phi \in H^0(X, K_X^{\otimes(m+1)})$$

such that the second fundamental form $S(D; \mathcal{S})$ in (3.23) is:

$$S(D; \mathcal{S}) = \mathrm{Id}_Q \otimes \phi.$$

Note that using the isomorphism $\mathcal{S} = Q \otimes K_X^{\otimes m}$ in (1), the second fundamental form $S(D; \mathcal{S})$ in (3.23) is a holomorphic section of

$$\text{Hom}(Q \otimes K_X^{\otimes m},\, Q \otimes K_X^{\otimes (2m+1)}) \,=\, \text{End}(Q) \otimes K_X^{\otimes (m+1)}\,;$$

the condition says that this section $S(D; \mathcal{S})$ coincides with $\text{Id}_Q \otimes \phi$.

Two generalized $\text{SO}(2n, \mathbb{C})$–opers are called *isomorphic* if the underlying generalized $\text{SO}(2n, \mathbb{C})$–quasiopers are isomorphic (see Definition 2.5).

The following lemma is straightforward to prove.

Lemma 4.2. *Take a holomorphic line bundle $\mathcal{L}$ on X of order two, and fix a holomorphic isomorphism ρ as in* (3.16)*. Let $(E,\, B_0,\, \mathcal{F}_\bullet,\, D)$ be a generalized* $\text{SO}(2n, \mathbb{C})$*–oper on X. Then the generalized* $\text{SO}(2n, \mathbb{C})$*–quasioper $(E^1,\, B_0^1,\, \mathcal{F}_\bullet^1,\, D^1)$ in Lemma 3.6 is also a generalized* $\text{SO}(2n, \mathbb{C})$*–oper.*

Fix integers n and m as in Definition 2.2; note that $r := n/(m+1)$ is an integer, in fact it is the rank of F_1 in (2.4). Let

$$\mathbb{O}_X(n, m)$$

denote the space of all isomorphism classes of generalized $\text{SO}(2n, \mathbb{C})$–opers on X of filtration length $2m + 1$ (see Definition 2.2 and Definition 4.1).

Let

$$J(X)_2 \,\subset\, \text{Pic}^0(X)$$

be the group of holomorphic line bundles on X of order two; it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2g}$, where $g \,=\, \text{genus}(X)$.

Let

$$\mathcal{C}_X \tag{4.1}$$

be the space of all isomorphism classes of holomorphic $\text{SO}(r, \mathbb{C})$–bundles on X equipped with a holomorphic connection. So $\mathcal{C}_X$ in (4.1) parametrizes isomorphism classes of pairs $(V,\, B_V)$, where V is a holomorphic vector bundle on X of rank r with $\bigwedge^r V \,=\, \mathcal{O}_X$, and $B_V \,\in\, H^0(X,\, \text{Sym}^2(V^*))$ is a fiberwise nondegenerate symmetric bilinear form on V. We recall that a holomorphic connection on $(V,\, B_V)$ is a holomorphic connection D_V on V such that

$$\partial B_V(s,\, t) \,=\, B_V(D_V(s),\, t) + B_V(s,\, D_V(t))$$

for all locally defined holomorphic sections s and t of V. Let

$$\mathfrak{P}(X)$$

be the space of all projective structures on X; see [Gu], [Bi1] for projective structures on X. Then, one has the following correspondence between generalized $\text{SO}(2n, \mathbb{C})$–opers and geometric structures.

Theorem 4.3. *First assume that the integer $r = n/(m+1)$ is odd. There is a canonical bijection between $\mathbb{O}_X(n,m)$ and the Cartesian product*

$$\mathcal{C}_X \times \mathfrak{P}(X) \times \left(H^0(X,\, K_X^{\otimes(m+1)}) \oplus \left(\bigoplus_{i=2}^{m} H^0(X,\, K_X^{\otimes 2i})\right)\right) \times J(X)_2\ .$$

If r is even, then there is a canonical bijection between $\mathbb{O}_X(n,m)$ and the Cartesian product

$$\mathcal{C}_X \times \mathfrak{P}(X) \times \left(H^0(X,\, K_X^{\otimes(m+1)}) \oplus \left(\bigoplus_{i=2}^{m} H^0(X,\, K_X^{\otimes 2i})\right)\right)\ .$$

Proof. Assume that $r = n/(m+1)$ is an odd integer. Take an element

$$(\alpha,\, \beta,\, \gamma,\, \delta,\, \mathcal{L}) \tag{4.2}$$

in

$$\mathcal{C}_X \times \mathfrak{P}(X) \times \left(H^0(X,\, K_X^{\otimes(m+1)}) \oplus \left(\bigoplus_{i=2}^{m} H^0(X,\, K_X^{\otimes 2i})\right)\right) \times J(X)_2\,,$$

such that

- $\alpha = (V,\, B_V,\, D_V) \in \mathcal{C}_X$, where $(V,\, B_V)$ is a holomorphic $\mathrm{SO}(r,\mathbb{C})$–bundle on X equipped with a holomorphic connection D_V,
- β is a projective structure on X,
- γ is a holomorphic section
$$\gamma \,\in\, H^0(X,\, K_X^{\otimes(m+1)})\,, \tag{4.3}$$
- $\delta \in \bigoplus_{i=2}^{m} H^0(X,\, K_X^{\otimes 2i})$, and
- $\mathcal{L}$ is a holomorphic line bundle on X of order two.

Using [BSY, Theorem 4.6], the triple $(\alpha,\, \beta,\, \delta)$ produces the following:

- a nondegenerate holomorphic symmetric bilinear form B_1 on $J^{2m}(V\otimes K_X^{-\otimes m})$, and
- a holomorphic connection $\mathbf{D}_1$ on $J^{2m}(V\otimes K_X^{-\otimes m})$ that preserves the bilinear form B_1.

Furthermore, the triple $(J^{2m}(V\otimes K_X^{-\otimes m}),\, B_1,\, \mathbf{D}_1)$, together with the filtration of $J^{2m}(V\otimes K_X^{-\otimes m})$ given by

$$0 \,=\, A_0 \,\subset\, A_1 \,\subset\, A_2 \,\subset\, \cdots \,\subset\, A_{2m} \,\subset\, A_{2m+1} \,=\, J^{2m}(V\otimes K_X^{-\otimes m})\,, \tag{4.4}$$

where A_i is the kernel of the natural projection $J^{2m}(V\otimes K_X^{-\otimes m}) \longrightarrow J^{2m-i}(V\otimes K_X^{-\otimes m})$, define a generalized B–oper (see [BSY, Definition 2.11] and [BSY, Theorem 4.6]).

Now consider the holomorphic vector bundle

$$E \,:=\, J^{2m}(V\otimes K_X^{-\otimes m})\oplus V$$

on X. Note that it is equipped with nondegenerate holomorphic symmetric bilinear form $B_1\oplus B_V$. The holomorphic connection $\mathbf{D}_1$ on $J^{2m}(V\otimes K_X^{-\otimes m})$ and the holomorphic connection D_V on V together produce the holomorphic connection $\mathbf{D}_1\oplus D_V$ on $J^{2m}(V\otimes$

$K_X^{-\otimes m}) \oplus V \,=\, E$. This connection $\mathbf{D}_1 \oplus D_V$ on E evidently preserves the bilinear form $B_1 \oplus B_V$ on E.

Using the holomorphic connection $\mathbf{D}_1 \oplus D_V$ on E and the section γ in (4.3), we shall construct another holomorphic connection on E. Since $E \,=\, J^{2m}(V\otimes K_X^{-\otimes m}) \oplus V$, using the filtration in (4.4), we have

$$\mathrm{Hom}(V,\, V\otimes K_X^{\otimes m}) \,=\, \mathrm{Hom}(V,\, A_1) \,\subset\, \mathrm{Hom}(V,\, J^{2m}(V\otimes K_X^{-\otimes m}))\,; \tag{4.5}$$

in (4.4), note that $A_1 \,=\, V\otimes K_X^{\otimes m}$. Similarly, we have

$$\mathrm{Hom}(V\otimes K_X^{-\otimes m},\, V) \,=\, \mathrm{Hom}(A_{2m+1}/A_{2m},\, V) \,\subset\, \mathrm{Hom}(J^{2m}(V\otimes K_X^{-\otimes m}),\, V)\,; \tag{4.6}$$

in (4.4), note that

$$A_{2m+1}/A_{2m} \,=\, V\otimes K_X^{-\otimes m}\,,$$

so the quotient map $A_{2m+1} \longrightarrow A_{2m+1}/A_{2m}$ produces the inclusion map

$$\mathrm{Hom}(A_{2m+1}/A_{2m},\, V) \,\hookrightarrow\, \mathrm{Hom}(J^{2m}(V\otimes K_X^{-\otimes m}),\, V)\,.$$

On the other hand,

$$\mathrm{Hom}(V,\, J^{2m}(V\otimes K_X^{-\otimes m})) \oplus \mathrm{Hom}(J^{2m}(V\otimes K_X^{-\otimes m}),\, V)$$
$$\subset\, \mathrm{End}(J^{2m}(V\otimes K_X^{-\otimes m}) \oplus V) \,=\, \mathrm{End}(E)\,.$$

Hence from (4.5), (4.6) we conclude that

$$(\mathrm{End}(V)\otimes K_X^{\otimes m}) \oplus (\mathrm{End}(V)\otimes K_X^{\otimes m}) \,=\, \mathrm{Hom}(V,\, V\otimes K_X^{\otimes m}) \oplus \mathrm{Hom}(V\otimes K_X^{-\otimes m},\, V) \tag{4.7}$$
$$\subset\, \mathrm{Hom}(V,\, J^{2m}(V\otimes K_X^{-\otimes m})) \oplus \mathrm{Hom}(J^{2m}(V\otimes K_X^{-\otimes m}),\, V) \,\subset\, \mathrm{End}(E)\,.$$

From (4.7) it follows immediately that

$$(\mathrm{Id}_V\otimes\gamma,\, -\mathrm{Id}_V\otimes\gamma) \,\in\, H^0(X,\, \mathrm{End}(E)\otimes K_X)\,, \tag{4.8}$$

where γ is the section in (4.3).

Any two holomorphic connections on the holomorphic vector bundle E differ by a holomorphic section of $\mathrm{End}(E)\otimes K_X$. Since $\mathbf{D}_1 \oplus D_V$ is a holomorphic connection on E, from (4.8) we conclude that

$$\mathbf{D}_E \,:=\, (\mathbf{D}_1 \oplus D_V) + (\mathrm{Id}_V\otimes\gamma,\, -\mathrm{Id}_V\otimes\gamma) \tag{4.9}$$

is a holomorphic connections on the holomorphic vector bundle E. Since the connection $\mathbf{D}_1 \oplus D_V$ on E preserves the bilinear form $B_1 \oplus B_V$ on E, from the construction of $(\mathrm{Id}_V\otimes\gamma,\, -\mathrm{Id}_V\otimes\gamma) \,\in\, H^0(X, \mathrm{End}(E)\otimes K_X)$ in (4.8) it follows that the connection $\mathbf{D}_E$ on E in (4.9) also preserves the bilinear form $B_1 \oplus B_V$ on E.

Using the filtration of $J^{2m}(V\otimes K_X^{-\otimes m})$ in (4.4) we shall construct a filtration of holomorphic subbundles on E. Let

$$0 \,=\, A'_0 \,\subset\, A'_1 \,\subset\, A'_2 \,\subset\, \cdots \,\subset\, A'_{2m} \,\subset\, A'_{2m+1} \,=\, E \,=\, J^{2m}(V\otimes K_X^{-\otimes m}) \oplus V \tag{4.10}$$

be the filtration where, $A'_i \,=\, A_i \oplus 0$ for all $0 \,\leq\, i \,\leq\, m$ and $A'_i \,=\, A_i \oplus V$ for all $m+1 \,\leq\, i \,\leq\, 2m+1$.

From the above, we have that the holomorphic vector bundle E, the bilinear form $B_1 \oplus B_V$, the filtration $\{A'_i\}_{i=0}^{2m+1}$ in (4.10), and the holomorphic connection $\mathbf{D}_E$ in (4.9) together define a generalized $\mathrm{SO}(2n, \mathbb{C})$–oper.

In view of Lemma 4.2, the above generalized $\mathrm{SO}(2n, \mathbb{C})$–oper

$$(E,\, B_1 \oplus B_V,\, \{A'_i\}_{i=0}^{2m+1},\, \mathbf{D}_E)$$

and the line bundle $\mathcal{L}$ in (4.2) together produce a generalized $\mathrm{SO}(2n, \mathbb{C})$–oper. It is evident that this generalized $\mathrm{SO}(2n, \mathbb{C})$–oper is an element of $\mathbb{O}_X(n, m)$.

Now assume that the integer r is even. Let V be a holomorphic vector bundle on X of rank r, and let $B_V \in H^0(X,\, \mathrm{Sym}^2(V^*))$ is a fiberwise nondegenerate symmetric bilinear form on V. Then we have $\bigwedge^r V \,=\, \mathcal{O}_X$, because r is even. Therefore, if $(V,\, B_V)$ is an holomorphic $\mathrm{SO}(r, \mathbb{C})$–bundle, then for any $\mathcal{L} \,\in\, J(X)_2$, that pair $(V \otimes \mathcal{L},\, B_V \otimes \rho)$, where $\rho \,:\, \mathcal{L}^{\otimes 2} \,\longrightarrow\, \mathcal{O}_X$ is an isomorphism (as in (3.16)), is again a holomorphic $\mathrm{SO}(r, \mathbb{C})$–bundle. Hence in the case of even r, when we consider $\mathcal{C}_X$, tensoring with line bundles of order two are already taken into account, so we no longer need to take line bundles of order two separately (which was needed in the previous case of r being odd).

Therefore, the above constructions identify $\mathbb{O}_X(n, m)$ with

$$\mathcal{C}_X \times \mathfrak{P}(X) \times \left(H^0(X,\, K_X^{\otimes(m+1)}) \oplus \left(\bigoplus_{i=2}^{m} H^0(X,\, K_X^{\otimes 2i})\right)\right).$$

We shall now describe the reverse construction. Again first assume that the integer r is odd.

Let

$$(E,\, B_0,\, \mathcal{F}_\bullet,\, D) \,\in\, \mathbb{O}_X(n, m)$$

be a generalized $\mathrm{SO}(2n, \mathbb{C})$–oper on X. Consider the decomposition

$$\mathbb{F} \oplus \mathcal{S} \,=\, E$$

in Lemma 3.2. As noted in (3.13), we have that

$$B_0 \,=\, B_{\mathbb{F}} \oplus B_{\mathcal{S}}\,.$$

Moreover, from Corollary 3.4, the connection $D^{\mathbb{F}}$ (respectively, $D^{\mathcal{S}}$) on $\mathbb{F}$ (respectively, $\mathcal{S}$) preserves the bilinear form $B_{\mathbb{F}}$ (respectively, $B_{\mathcal{S}}$). The vector bundle $\mathbb{F}$ has a filtration of holomorphic subbundles (see (3.19)), which we shall denote by $\widetilde{\mathcal{F}}_\bullet$. Recall that the isomorphism f'_{2m} in (3.10) takes the filtration $\widetilde{\mathcal{F}}_\bullet$ to the filtration of $J^{2m}(Q)$ given by the exact sequences in (3.20).

Note that $(\mathbb{F},\, B_{\mathbb{F}},\, \widetilde{\mathcal{F}}_\bullet,\, D^{\mathbb{F}})$ satisfies all the conditions needed to define a generalized B–oper (see [BSY, Definition 2.11]) except possibly the only condition

$$\det \mathbb{F} \,=\, \mathcal{O}_X\,.$$

In any case,

$$\mathcal{L} \,:=\, \det \mathbb{F} \,\in\, J(X)_2\,. \tag{4.11}$$

For the vector bundle $\mathbb{F}' \,:=\, \mathbb{F} \otimes \mathcal{L}$, we have $\det \mathbb{F}' \,=\, \mathcal{O}_X$.

Let $\rho\,:\,\mathcal{L}^{\otimes 2}\,\longrightarrow\,\mathcal{O}_X$ be an isomorphism (as in (3.16)). Define the nondegenerate symmetric bilinear form

$$B'_{\mathbb{F}}\,:=\,B_{\mathbb{F}}\otimes\rho$$

on $\mathbb{F}'\,=\,\mathbb{F}\otimes\mathcal{L}$. Tensoring the above filtration $\widetilde{\mathcal{F}}_\bullet$ of $\mathbb{F}$ by $\mathcal{L}$ we get a filtration of holomorphic subbundles of $\mathbb{F}'$; this filtration of $\mathbb{F}'$ will be denoted by $\widetilde{\mathcal{F}}'_\bullet$. The holomorphic connection $D^{\mathbb{F}}$ on $\mathcal{F}$ and the canonical connection $D^{\mathcal{L}}$ on $\mathcal{L}$ in (3.17) together define a holomorphic connection

$$D'_{\mathbb{F}}\,:=\,D^{\mathbb{F}}\otimes \mathrm{Id}_{\mathcal{L}}+\mathrm{Id}_{\mathbb{F}}\otimes D^{\mathcal{L}}$$

on $\mathbb{F}'$ (as done in (3.18)).

Now $(\mathbb{F}',\, B'_{\mathbb{F}},\, \widetilde{\mathcal{F}}'_\bullet,\, D'_{\mathbb{F}})$ is a generalized B–oper [BSY]. Therefore, from [BSY, Theorem 4.6] we obtain a triple

$$(\alpha,\,\beta,\,\delta)\,\in\,\mathcal{C}_X\times\mathfrak{P}(X)\times\left(\bigoplus_{i=2}^{m}H^0(X,\,K_X^{\otimes 2i})\right) \tag{4.12}$$

associated to $(\mathbb{F}',\, B'_{\mathbb{F}},\, \widetilde{\mathcal{F}}'_\bullet,\, D'_{\mathbb{F}})$.

Next consider the second fundamental form for the subbundle $\mathcal{S}\,\subset\,E\,=\,\mathbb{F}\oplus\mathcal{S}$ for the connection D on E. Let

$$S(D;\mathcal{S})\,\in\,H^0(X,\,\mathrm{Hom}(\mathcal{S},\,\mathbb{F})\otimes K_X)$$

be the second fundamental form for the subbundle $\mathcal{S}$ for the connection D on E. From Definition 4.1 and (3.23) we know that

$$S(D;\mathcal{S})\,\in\,H^0(X,\,\mathrm{Hom}(\mathcal{S},\,F_1)\otimes K_X)\,=\,H^0(X,\,\mathrm{Hom}(\mathcal{S},\,Q\otimes K_X^{\otimes(2m+1)}))$$

$$\subset\,H^0(X,\,\mathrm{Hom}(\mathcal{S},\,\mathbb{F})\otimes K_X)\,;$$

we note that $F_1\,=\,Q\otimes K_X^{\otimes 2m}$; this follows from the fact that isomorphism f'_{2m} in (3.10) takes the filtration $\mathcal{F}'_\bullet$ to the filtration of $J^{2m}(Q)$ given by the exact sequences in (3.20). Since $\mathcal{S}\,=\,Q\otimes K_X^{\otimes m}$ (see Definition 4.1), we have

$$S(D;\mathcal{S})\,\in\,H^0(X,\,\mathrm{End}(\mathcal{S})\otimes K_X^{\otimes(m+1)})\,.$$

We recall from Definition 4.1 that $S(D;\mathcal{S})\,=\,\mathrm{Id}_{\mathcal{S}}\otimes\phi$, where $\phi\,\in\,H^0(X,\,K_X^{\otimes(m+1)})$. Then, we have

$$(\alpha,\,\beta,\,\phi,\,\delta,\,\mathcal{L})\,\in\,\mathcal{C}_X\times\mathfrak{P}(X)\times\left(H^0(X,\,K_X^{\otimes(m+1)})\oplus\left(\bigoplus_{i=2}^{m}H^0(X,\,K_X^{\otimes 2i})\right)\right)\times J(X)_2\,,$$

where $(\alpha,\,\beta,\,\delta)$ is constructed in (4.12) and $\mathcal{L}$ is the line bundle in (4.11). It is straightforward to check that the two constructions are inverses of each other.

When the integer r is even, the above reverse construction is simpler because in that case $(\mathbb{F},\, B_{\mathbb{F}},\, \widetilde{\mathcal{F}}_\bullet,\, D^{\mathbb{F}})$ is already a generalized B–oper, so the construction of $(\mathbb{F}',\, B'_{\mathbb{F}},\, \widetilde{\mathcal{F}}'_\bullet,\, D'_{\mathbb{F}})$ from it is not needed. $\square$

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