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OPTIMAL INVESTMENT AND DIVIDEND STRATEGY UNDER RENEWAL RISK MODEL*

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Abstract. In this paper we continue investigating the optimal dividend and investment problems under the Sparre Andersen model. More precisely, we try to give a more complete description of the optimal strategy when the claim frequency is a renewal process and therefore semi-Markovian, for which it is well-known that the barrier strategy is no longer optimal (cf. [H. Albrecher and J. Hartinger, *Hermis J. Comp. Math. Appl.*, 7 (2006), pp. 109–122]). Building on our previous work [L. Bai, J. Ma, and X. Xing, *Ann. Appl. Probab.*, 27 (2017), pp. 3588–3632], where we established the dynamic programming principle via a *backward Markovization* procedure and proved that the value function is the unique *constrained* viscosity solution of the Hamilton–Jacobi–Bellman (HJB) equation, which is a nonlocal, nonlinear, and degenerate parabolic partial integro-differential equation on an unbounded domain, in this paper we show that the optimal strategy is still of a *band* type but in a more complicated dynamic fashion. The main technical obstacles in constructing and validating the optimal strategy include the regularity of the value function, due to the fundamental degeneracy of the HJB equation caused by the Markovization procedure, and the well-posedness of the closed-loop stochastic system, given the "band" nature of the optimal strategy. Some of the technical results in this paper are purely analytical and therefore interesting in their own right.

Key words. optimal dividend control, Sparre Andersen model, Hamilton–Jacobi–Bellman equation, viscosity solution, Krylov estimate

AMS subject classifications. 60H07, 15, 30, 35R60, 34F05

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1. Introduction.

In this paper we continue our investigation on the optimal dividend and investment problems under a Sparre Andersen insurance model. More precisely, we assume that the claim number process is a *renewal* process instead of a standard Poisson process; therefore, it is also referred to as a *renewal risk model*. Finding the optimal strategy for such a problem has been considered as an intriguing but challenging open problem for quite some time (cf., e.g., [2] and references cited therein) mainly due to the semi-Markov nature of the renewal process, as well as the nonoptimality of the well-known barrier strategy (see [1]). More specifically, for a general insurance model involving investments, even under the simplest Cramér–Lundberg form, direct calculation of optimal strategy becomes almost impossible, and the solution procedure often depends on some more general stochastic control technique. In particular, the approach of dynamic programming and consequently the study of the associated Hamilton–Jacobi–Bellman (HJB) equation and its viscosity solution become a natural way to attack the problem (cf., e.g., [4, 5]). However, as was pointed out in [2], the non-Markovian nature of Sparre Andersen model drastically complicated this approach, as it took away the basis of dynamic programming. On the other hand, since the commonly believed barrier type of dividend strategy was shown

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to be nonoptimal in [1], the structure of the optimal dividend-investment strategy under a renewal risk model has naturally become an intriguing issue to explore.

To better understand the main difficulties in this effort let us first briefly recall the "toy" model studied in our recent paper [7], where we assumed that the surplus process satisfies the following dynamics of Sparre Andersen type, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$: for $t \in [s, T]$,

$$(1.1) \quad dX_t^\pi = pdt + rX_t^\pi dt + (\mu - r)\gamma_t X_t^\pi dt + \sigma\gamma_t X_t^\pi dB_t - dQ_t - dL_t, \quad X_s^\pi = x,$$

where $T > 0$ is a given time horizon, $s \in [0, T]$ is the initial time and x is the initial surplus, p the premium rate, r the interest rate, and (μ, σ) the appreciation rate and the volatility of the stock, respectively, all assumed to be positive constants; B is a $(\mathbb{P}, \mathbb{F})$-Brownian motion representing the market noise, $Q_t = \sum_{i=1}^{N_t} U_i$ is the (renewal) claim process, and $\pi = (\gamma_t, L_t)$, $t \ge 0$, is the investment-dividend pair in which $\gamma = \{\gamma_t\}_{t\ge 0}$ represents the proportion of the surplus invested in the stock at each time t (hence $\gamma_t \in [0, 1]$) and $L = \{L_t\}_{t\ge 0}$ is the cumulative dividends process (hence increasing). Denoting $\mathscr{U}_{ad}$ to be all such investment-dividend strategies and the solution to (1.1) by $X_t = X_t^\pi = X_t^{\pi,x}$, define $\tau_s^\pi = \tau_s^{\pi,x} := \inf\{t \ge s; X_t^{\pi,x} < 0\}$ to be the ruin time of the insurance company. The goal is to maximize the following expected cumulated dividends: for $(s, x) \in [0, T] \times \mathbb{R}_+$,

$$(1.2) \quad J(s, x; \pi) := \mathbb{E}\left\{\int_s^{\tau_s^{\pi,x}\wedge T} e^{-c(t-s)} dL_t\right\} := \mathbb{E}\left\{\int_s^{\tau_s^{\pi}\wedge T} e^{-c(t-s)} dL_t \Big| X_s^\pi = x\right\},$$

where $c > 0$ is the *discounting factor*. We should note that even as the simplest model, the solution to the problem (1.1)–(1.2) is surprisingly challenging. The first obstacle is the *non-Markovian* nature of the renewal claim counting process N; thus the usual dynamic programming approach does not apply directly. To overcome this difficulty we invoke a standard "backward Markovization" procedure by adding an extra state process $W = \{W_t\}_{t\ge 0}$, a random clock measuring the time elapsed since the last claim (see section 2 for details), so that the model becomes Markovian again.

In [7] we verified the *dynamic programming principle* and proved that the value function of problem (1.1)–(1.2) is the unique *constrained viscosity solution* of the corresponding HJB equation which is a *fully nonlinear, nonlocal*, and *degenerate* parabolic partial integro-differential equation (PIDE) over an unbounded domain. However, in [7] we did not address the existence and the structure of the optimal control, which is particularly interesting given the counterexample of Albrecher and Hartinger [1].

The main purpose of this paper is to give a more complete answer to the open problem suggested in [2], that is, the structure of the optimal strategy of the problem (1.1)–(1.2), by using the solution (whence the value function) of the HJB equation. In fact, by simply calculating the maximizer of the Hamiltonian from the HJB equation (see (2.9) below), one could suspect the following candidate of optimal strategy:

$$(1.3) \quad \begin{cases} \gamma_t^* = \Big[-\frac{(\mu-r)V_x(t,X_t^*,W_t)}{\sigma^2 X_t^* V_{xx}(t,X_t^*,W_t)}\Big] \vee 0 \wedge 1; \\ a_t^* = \dot{L}_t^* = M\mathbf{1}_{\{V_x(t,X_t^*,W_t)<1\}} + p\mathbf{1}_{\{V_x(t,X_t^*,W_t)=1\}}, \end{cases}$$

where V is the viscosity solution and $M \ge p > 0$ is the given upper bound of the dividend rate, that is, assuming $0 \le a_t = \dot{L}_t \le M$. From (1.3) we immediately see that the optimal strategy should still have a "barrier type" but with a dynamic nature. However, there are two major technical issues. First, the validity of (1.3)

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depends on the *regularity* of the viscosity solution (i.e., the derivatives V_x and V_{xx}), which seems to be a tall order in this case due to the *nonlocal* and *degenerate* nature of the HJB equation. Second, the optimal dividend rate displays a "band" nature with the state-dependent switching times, which raises some serious questions about the well-posedness of the resulting closed-loop system. A natural way to get around these difficulties is to add some additional Brownian motions to the system so that the corresponding HJB equation becomes nondegenerate and hence possesses classical solutions and an argument of "vanishing viscosity" might lead to at least some ε-optimal strategy. Unfortunately, such a method does not work easily in this model, since the random clock W, the key for the Markovization, cannot be perturbed by a Brownian motion. Therefore the degeneracy of the HJB in the variable W is not removable by this approach. To overcome this dilemma we shall take a less standard route. That is, we shall perturb the HJB equation directly and consider an auxiliary nondegenerate PIDE and prove that its solution can be used to construct the ε-optimal strategy. The difficulty, however, is that such a PIDE does not correspond to any control problem, so the analysis will have to be purely analytic without using any control theoretic arguments. Our discussion benefitted greatly from a recent work on nonlocal HJB equations (cf. [9]), except that in our case the domain is unbounded.

Finally, we should note that, while this paper still treats only the simplest "toy model," as we shall see, the technicality involved is already overwhelming. In order not to be distracted by the main message of this paper, that is, to understand the structure of the optimal strategy and the procedure of obtaining the ε-optimal strategies, we will not pursue the generality of the model. We should also note that while the optimal strategy still has a "barrier" nature, the switching times will depend on not only the surplus level but also the random clock, and it is possible to have multiple barrier levels. We shall therefore call it a *generalized band strategy.*

The rest of the paper is organized as follows. In section 2 we briefly recall the original problem and introduce all the concepts and notations. In section 3 we prove the existence and uniqueness of the viscosity solution of our key auxiliary PIDE, keeping in mind that such a PIDE does not corresponding to an actual control problem(!). In section 4 we prove the desired convergence of the solutions of the approximating PIDEs to the value function. In section 5 we construct a prospective ε-optimal strategy in terms of the solutions to the approximating PIDEs. In section 6 we prove the well-posedness of the closed-loop system corresponding this strategy, and in section 7 we verify that the constructed strategy does produce the desired ε optimality.

2. Preliminaries.

Throughout this paper we consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which is defined standard Brownian motion $B = \{B_t : t \geq 0\}$ and a *renewal* counting process $N = \{N_t\}_{t\geq 0}$, independent of B. More precisely, denoting $\{\sigma_n\}_{n=1}^{\infty}$ to be the jump times ($\sigma_0 := 0$) of N and $T_i = \sigma_i - \sigma_{i-1}$, $i = 1, 2, \ldots$, to be its waiting times, we assume that T_i's are independent and identically distributed (i.i.d.) with a common distribution $F : \mathbb{R}_+ \mapsto \mathbb{R}_+$. We shall assume that there exists an *intensity function* $\lambda : [0, \infty) \mapsto [0, \infty)$ such that $\bar{F}(t) := \mathbb{P}\{T_1 > t\} = \exp\{-\int_0^t \lambda(u)du\}$, so that $\lambda(t) = f(t)/\bar{F}(t)$, $t \geq 0$, where f is the density function of T_i's. Clearly, if $\lambda(t) \equiv \lambda$ is a constant, then N becomes a standard Poisson process.

Let $T > 0$ be a given time horizon, $\mathbb{X}$ be a generic Euclidean space, and $\mathcal{G} \subseteq \mathcal{F}$ be any sub-σ-field. We denote $\mathbb{C}([0, T]; \mathbb{X})$ to be the space of continuous functions taking values in $\mathbb{X}$ with the usual sup-norm; $L^p(\mathcal{G}; \mathbb{X})$ to be the space of all $\mathbb{X}$-valued, $\mathcal{G}$-measurable random variables ξ such that $\mathbb{E}|\xi|^p < \infty$, $1 \leq p \leq \infty$; and $L^p_{\mathbb{F}}([0, T]; \mathbb{X})$ to be the space of all $\mathbb{X}$-valued, $\mathbb{F}$-progressively measurable processes ξ satisfying

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$\mathbb{E}\int_0^T |\xi_t|^p dt < \infty$, where $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ is a given filtration in $\mathcal{F}$ and $1 \leq p \leq \infty$. Here $p = \infty$ means that all elements are bounded.

Given a renewal counting process N, we shall consider the following *claim process* for our reserve mode: $Q_t = \sum_{i=1}^{N_t} U_i$, $t \geq 0$, where $\{U_i\}_{i=1}^{\infty}$ is a sequence of random variables representing the "size" of the incoming claims. We assume that $\{U_i\}$ are i.i.d. with a common distribution function G (and density g), independent of (N, B). We note that the process Q is non-Markovian in general (unless the counting process N is Poisson) but can be "Markovized" by the so-called *backward Markovization* technique (cf., e.g., [17]). More precisely, if we denote $W_t = t - \sigma_{N_t}$, $t \geq 0$, that is, the time elapsed since the last jump, then it is known (see, e.g., [17]) that the process (t, Q_t, W_t), $t \geq 0$, is a piecewise deterministic Markov process. We note that at each jump time σ_i, $|\Delta W_{\sigma_i}| = \sigma_i - \sigma_{i-1} = T_i$ and $0 \leq W_t \leq t \leq T$, $t \in [0, T]$.

Now let us denote $\{\mathcal{F}_t^\xi : t \geq 0\}$ to be the natural filtration generated by process $\xi = B, Q, W$, respectively, with the usual $\mathbb{P}$-augmentation such that it satisfies the *usual hypotheses* (cf., e.g., [16]). Throughout this paper we consider the filtration $\mathbb{F} = \mathbb{F}^{(B,Q,W)} = \{\mathcal{F}_t\}_{t\geq 0}$, where $\mathcal{F}_t := \mathcal{F}_t^B \vee \mathcal{F}_t^Q \vee \mathcal{F}_t^W$, $t \geq 0$. For any $s \in [0, T]$, let us consider the process (B, Q, W) starting from $s \in [0, T]$. First assume $W_s = w$, $\mathbb{P}$-a.s.; let us consider the *regular conditional probability distribution* $\mathbb{P}_{sw}(\cdot) := \mathbb{P}[\,\cdot\,|W_s = w]$ on $(\Omega, \mathcal{F})$, and consider the "shifted" version of processes (B, Q, W) on the space $(\Omega, \mathcal{F}, \mathbb{P}_{sw}; \mathbb{F}^s)$, where $\mathbb{F}^s = \{\mathcal{F}_t\}_{t\geq s}$. Define $B_t^s := B_t - B_s$, $t \geq s$. Clearly, since B is independent of (Q, W), B^s is an $\mathbb{F}^s$-Brownian motion under $\mathbb{P}_{sw}$, defined on $[s, T]$, with $B_s^s = 0$. Next, we restart the clock at time $s \in [0, T]$ by defining the new counting process $N_t^s := N_t - N_s$, $t \in [s, T]$. Then, under $\mathbb{P}_{sw}$, N^s is a "delayed" renewal process in the sense that while its waiting times T_i^s, $i \geq 2$, remain i.i.d. as the original T_i's, its "time-to-first jump," denoted by $T_1^{s,w} := T_{N_s+1} - w = \sigma_{N_s+1} - s$, should have the survival probability

$$\mathbb{P}_{sw}\{T_1^{s,w} > t\} = \mathbb{P}\{T_1 > t + w | T_1 > w\} = e^{-\int_w^{w+t} \lambda(u)du}. \tag{2.1}$$

In what follows we shall denote $N_t^s\big|_{W_s=w} := N_t^{s,w}$, $t \geq s$, to emphasize the dependence on w as well. Correspondingly, we shall denote $Q_t^{s,w} = \sum_{i=1}^{N_t^{s,w}} U_i$ and $W_t^{s,w} := w + W_t - W_s = w + [(t-s) - (\sigma_{N_t} - \sigma_{N_s})]$, $t \geq s$. It is readily seen that $(B_t^s, Q_t^{s,w}, W_t^{s,w})$, $t \geq s$, is an $\mathbb{F}^s$-adapted process defined on $(\Omega, \mathcal{F}, \mathbb{P}_{sw})$, and it remains Markovian.

The Markovized optimal investment-dividend problem. Taking the process W into account, we now reformulate the renewal risk model (1.1)-(1.2) so that it is Markovian. Similar to our previous work [7], we shall make use of the following *standing hypothesis*.

Hypothesis 2.1. (a) The parameters (r, μ, σ) and premium rate p are all constants.

(b) The distribution functions F and G are continuous on $[0, \infty)$ with densities f and g, respectively. Furthermore, we assume that $\lambda(t) := f(t)/\bar{F}(t) > 0$, $t \in [0, T]$.

(c) The cumulative dividend process L is absolutely continuous. That is, there exists $a \in L^2_{\mathbb{F}}([0, T]; \mathbb{R}_+)$ such that $L_t = \int_0^t a_s ds$, $t \geq 0$. We assume further that for some constant $M \geq p > 0$, it holds that $0 \leq a_t \leq M$, $dt \times d\mathbb{P}$-a.e.

Remark 2.2. (i) The main technical difficulty in this paper is the *degeneracy* of the HJB equation, caused by the Markovization procedure, even under Hypothesis 2.1(a). Our method is applicable to more general models with "nice" coefficients, but this is not the main point of the paper.

(ii) Hypothesis 2.1(c) is purposely restrictive, which removes the possible "singular" behavior of the optimal strategy. Such a restriction is due largely to our goal of

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constructing the "ε-optimal strategy" in this paper, which essentially eliminates the need to consider the singular case.

In what follows, given $[s,t]\subseteq[0,T]$, we say that a strategy $\pi=(\gamma,a)$ is *admissible* on $[s,t]$ if $\pi\in L^2_{\mathbb{F}}([s,t];\mathbb{R}^2)$ with $(\gamma_u,a_u)\in[0,1]\times[0,M]$, $u\in[s,t]$, $\mathbb{P}$-a.s. Moreover, for any $(s,w)\in[0,T]^2$, we denote the set of all admissible strategies on $[s,T]$, defined on the probability space $(\Omega,\mathcal{F},\mathbb{P}_{sw})$ by $\mathscr{U}_{ad}^{s,w}[s,T]$. In particular, we denote $\mathscr{U}_{ad}^{0,0}[0,T]$ by $\mathscr{U}_{ad}[0,T]=\mathscr{U}_{ad}$ for simplicity.

Let $\pi=(\gamma,a)\in\mathscr{U}_{ad}^{s,w}[s,T]$, we now consider the "Markovized" reserve model:

$$\text{(2.2)}\quad \begin{cases} dX_t=pdt+[r+(\mu-r)\gamma_t]X_tdt+\gamma_tX_tdB_t-dQ_t^{s,w}-a_tdt, X_s=x;\\ W_t=w+(t-s)-(\sigma_{N_t}-\sigma_{N_s}), \qquad\qquad t\in[s,T],\end{cases}$$

with the expected cumulated dividends up to ruin and the value function:

$$\text{(2.3)}\qquad J(s,x,w;\pi):=\mathbb{E}_{sxw}\left\{\int_s^{\tau_s^\pi\wedge T}e^{-c(t-s)}a_tdt\right\},$$

$$\text{(2.4)}\qquad V(s,x,w):=\sup_{\pi\in\mathscr{U}_{ad}^{sw}[s,T]}J(s,x,w;\pi).$$

In the above $\tau_s^\pi:=\inf\{t>s:X_t^\pi<0\}$ is the ruin time, (X^π,W) is the solution to (2.2), and $\mathbb{E}_{swx}\{\cdot\}:=\mathbb{E}_{sw}\{\cdot|X_s^\pi=x\}$.

The HJB equation and its viscosity solution. We now briefly recall the main result of [7]. We first note that there is a natural domain for the initial state (s,x,w), denoted by $D:=\{(s,x,w):0\le s\le T, x\ge 0, 0\le w\le s\}$. Here $w\le s$ is due to the fact that $W_t\le t$. We thus assume that the value function V is defined on D and that $V(s,x,w)=0$, for $(s,x,w)\notin D$. We also define the following two sets:

$$\text{(2.5)}\qquad \begin{aligned}\mathscr{D}&:=\text{int}D=\{(s,x,w)\in D:0<s<T,\ x>0,\ 0<w<s\};\\ \mathscr{D}^*&:=\{(s,x,w)\in D:0\le s<T,\ x\ge 0,\ 0\le w\le s\}.\end{aligned}$$

Clearly $\mathscr{D}\subset\mathscr{D}^*\subseteq\bar{\mathscr{D}}=D$, and $\mathscr{D}^*$ does not include boundary at the terminal time $s=T$. Furthermore, we denote $\mathbb{C}_0^{1,2,1}(D)$ to be the set of all functions $\varphi\in\mathbb{C}^{1,2,1}(\mathscr{D})$ such that for $\eta=\varphi,\varphi_t,\varphi_x,\varphi_{xx},\varphi_w$, it holds that $\lim_{\substack{(t,y,v)\to(s,x,w)\\(t,y,v)\in\mathscr{D}}}\eta(t,y,v)=\eta(s,x,w)$ for all $(s,x,w)\in D$ and $\varphi(s,x,w)=0$ for $(s,x,w)\notin D$. We note that while a function $\varphi\in\mathbb{C}_0^{1,2,1}(D)$ is well-defined on D, it is not necessarily continuous on the boundaries $\{(s,x,w):x=0 \text{ or } w=0 \text{ or } w=s\}$.

Now, for $\theta=(s,x,w)\in D$, $\xi=(\xi^1,\xi^2)\in\mathbb{R}^2$, $y,A,z\in\mathbb{R}$, and $(\gamma,a)\in[0,1]\times[0,M]$, we define the following Hamiltonian:

$$\text{(2.6)}\qquad \begin{aligned}H(\theta,y,\xi,A,z,\gamma,a):=&\frac{\sigma^2}{2}\gamma^2x^2A+[p+(r+(\mu-r)\gamma)x-a]\xi^1+\xi^2\\&+\lambda(w)z+(a-cy),\end{aligned}$$

and for $\varphi\in\mathbb{C}_0^{1,2,1}(D)$ we define the second-order partial integro-differential operator:

$$\text{(2.7)}\qquad \mathscr{L}[\varphi](\theta):=\sup_{\gamma\in[0,1],a\in[0,M]}H(\theta,\varphi,\nabla\varphi,\varphi_{xx},I(\varphi),\gamma,a),$$

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where $\nabla\varphi := (\varphi_x, \varphi_w)$, and $I[\varphi]$ is the integral operator defined by

$$I[\varphi] := \int_0^\infty [\varphi(s, x-u, 0) - \varphi(\theta)]dG(u) = \int_0^x \varphi(s, x-u, 0)dG(u) - \varphi(\theta). \tag{2.8}$$

Here the last equality is due to the fact that $\varphi(\theta) = \varphi(s, x, w) = 0$ for $x < 0$.

The main result of [7] is that the value function V is the unique *constrained* viscosity solution of the following HJB equation:

$$\{V_s + \mathscr{L}[V]\}(\theta) = 0; \quad \theta = (s, x, w) \in \mathscr{D}; \qquad V(T, x, w) = 0. \tag{2.9}$$

We end this section by recalling the definition of the "constrained viscosity solution" to the PIDE (2.9) (cf. [7]).

DEFINITION 2.3. *Let $\mathcal{O} \subseteq \mathscr{D}^*$ such that $\partial_T\mathcal{O} := \{(T, y, v) \in \partial\mathcal{O}\} \neq \emptyset$.*

(a) *$v \in \mathbb{C}(\mathcal{O})$ is called a viscosity subsolution of (2.9) on $\mathcal{O}$ if $v(T, y, v) \le 0$, $(T, y, v) \in \partial_T\mathcal{O}$ and if for any $(s, x, w) \in \mathcal{O}$ and $\varphi \in \mathbb{C}_0^{1,2,1}(\mathcal{O})$ such that $0 = [v - \varphi](s, x, w) = \max_{(t,y,v)\in\mathcal{O}}[v - \varphi](t, y, v)$, it holds that $\varphi_s(s, x, w) + \mathscr{L}[\varphi](s, x, w) \ge 0$.*

(b) *$v \in \mathbb{C}(\mathcal{O})$ is called a viscosity supersolution of (2.9) on $\mathcal{O}$ if $v(T, y, v) \ge 0$ for $(T, y, v) \in \partial_T\mathcal{O}$ and if for any $(s, x, w) \in \mathcal{O}$ and $\varphi \in \mathbb{C}_0^{1,2,1}(\mathcal{O})$ such that $0 = [v - \psi](s, x, w) = \min_{(t,y,v)\in\mathcal{O}}[v - \varphi](t, y, v)$, it holds that $\varphi_s(s, x, w) + \mathscr{L}[\varphi](s, x, w) \le 0$.*

(c) *$v \in \mathbb{C}(D)$ is called a "constrained viscosity solution" of (2.9) on $\mathscr{D}^*$ if it is both a viscosity subsolution on $\mathscr{D}^*$ and a viscosity supersolution on $\mathscr{D}$.*

3. An auxiliary equation. As we pointed out, in order to construct a sensible approximation of the optimal strategy based on the explicit form (1.3) using the solution to the HJB equation (2.9), the main obstacle is the degeneracy of the Hamiltonian (2.6), especially in the variable w, since the random clock $W = \{W_t\}$ *cannot* be perturbed by an extra Brownian noise for it would destroy the Markovization procedure. As a remedy we shall introduce an auxiliary nondegenerate PIDE that is of the same structure as (2.9), with which the approximating strategies will be constructed. It should be noted, however, that such a PIDE cannot be associated to any stochastic control problem. As a consequence our argument will be purely analytical and therefore interesting in its own right. In fact, to the best of our knowledge, the regularity of the constrained viscosity solution to a nonlocal HJB equation of this particular type on a unbounded domain is new.

Our plan of attack is quite similar to that of the recent work [14]. More precisely, we begin with the following extended domain of D: for each $\delta > 0$,

$$D_\delta = \{(s, x, w) : 0 < s \le T + \delta, x \ge -\delta, -\delta \le w \le s + \delta\}. \tag{3.1}$$

As before, we denote $\mathscr{D}_\delta := \text{int}D_\delta$ and consider the "truncated" complement" of D_δ:

$$\mathscr{D}_\delta^{*,c} := (\{T + \delta\} \times \mathbb{R}^2) \cup \big(\cup_{0<s<T+\delta} \mathscr{D}_{\delta,s}^c\big), \tag{3.2}$$

where for $0 < s < T + \delta$, $\mathscr{D}_{\delta,s} = \{(x, w) : x > -\delta, -\delta < w < s + \delta\}$ is the s-section of $\mathscr{D}_\delta$, and $\mathscr{D}_{\delta,s}^c$ is the complement of $\mathscr{D}_{\delta,s}$. Clearly, $D_\delta \cup \mathscr{D}_\delta^{*,c} = (0, T + \delta] \times \mathbb{R}^2$.

Next, we define a "perturbed" nondegenerate Hamiltonian. Let $\varepsilon_n > 0$, $n = 1, 2, \dots$ be a sequence such that $\varepsilon_n \downarrow 0$, as $n \to \infty$. We define for $\theta = (s, x, w) \in D_\delta$, $\xi = (\xi^1, \xi^2) \in \mathbb{R}^2$, $y, A_1, A_2, z \in \mathbb{R}$, and $(\gamma, a) \in [0, 1] \times [0, M]$,

$$H^n(\theta, y, \xi, A_1, A_2, z, \gamma, a) := H(\theta, y, \xi, A_1, z, \gamma, a) + \frac{\varepsilon_n}{2}A_1 + \frac{\varepsilon_n}{2}A_2, \tag{3.3}$$

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where H is the Hamiltonian defined by (2.6). Consider the following auxiliary PIDE:

$$\begin{cases} v_t(s,x,w) + \mathscr{L}^{n,\delta}[v](s,x,w) = 0 & \text{on } \mathscr{D}_\delta, \\ v(s,x,w) = \Psi(s,x,w), & (s,x,w) \in \mathscr{D}_\delta^{*,c}. \end{cases} \tag{3.4}$$

Here, as before, for a smooth function φ and $\nabla\varphi = (\varphi_x, \varphi_w)$,

$$\begin{cases} \mathscr{L}^{n,\delta}[\varphi](s,x,w) := \sup_{\gamma\in[0,1],a\in[0,M]} H^n(s,x,w,\varphi,\nabla\varphi,\varphi_{xx},\varphi_{ww},I^\delta[\varphi],\gamma,a), \\ I^\delta[\varphi](s,x,w) := \int_0^{x+\delta} \varphi(s,x-u,-\delta)dG(u) - \varphi(s,x,w), \end{cases} \tag{3.5}$$

and Ψ is a function to be determined later. We shall argue that there exists a unique classical solution to (3.4), denoted by $V^{n,\delta}$, such that $\lim_{n\to\infty,\delta\to 0} V^{n,\delta} = V$, the value function defined by (2.4), uniformly on compacta.

We should note that since (3.4) does not necessarily correspond to any stochastic control problem, the existence of the solution, even in the viscosity sense, is not clear. In the rest of this section we shall first show that there is indeed a viscosity solution to this equation, and in the next section we shall argue that such a solution is actually the unique classical solution. To simplify the argument we shall assume $0 < \delta < 1$ throughout our discussion.

The function Ψ. We now give a detailed description of the function Ψ, which is crucial for our construction of the viscosity solution. We first note that once such a function is chosen, we can modify the PIDE (3.4) to one with homogeneous boundary condition via the following standard transformation. Assume that Ψ is a (smooth) boundary condition. Let $\tilde{v} = v - \Psi$; then we have

$$\begin{cases} (\tilde{v}+\Psi)_t + \mathscr{L}^{n,\delta}[\tilde{v}+\Psi] = v_t + \mathscr{L}^{n,\delta}_\Psi[\tilde{v}] = 0, \\ \tilde{v}(s,x,w) = 0, \quad (s,x,w) \in \mathscr{D}_\delta^{*,c}, \end{cases} \tag{3.6}$$

where $\mathscr{L}^{n,\delta}_\Psi[\varphi] := \Psi_t + \mathscr{L}^{n,\delta}[\varphi + \Psi]$ will have the same properties as $\mathscr{L}^{n,\delta}$. Furthermore, we shall make the following hypotheses. Recall the set D_δ and the constants $M > 0$ in Hypothesis 2.1.

Hypothesis 3.1. There exists $\Psi \in \mathbb{C}^{1,3,3}(\mathbb{R}^3)$ such that

(i) there exists $K_1 > 0$ such that $0 \le \Psi(\theta) \le K_1$, $\theta = (s,x,w) \in D_1$, and $\Psi(\theta) = 0$, $\theta \in D_1^c$;

(ii) there exists $0 < K_2 < M$ such that for any $\theta \in D_1$,

$$M - K_2 \le \Psi_t + H^n(\theta,\Psi,\nabla\Psi,\Psi_{xx},\Psi_{ww},I^\delta[\Psi],0,M), \quad 0<\delta<1, \quad n\ge 1;$$

(iii) $\Psi(s,x,w)$ is strictly increasing with respect to x, and for some $0 < \delta_0 < 1$,

$$b := \inf_{(s,x,w)\in(0,T]\times[-\delta_0,0]\times[0,s]} \Psi_x(s,x,w) > 1. \tag{3.7}$$

We should note that under Hypothesis 2.1, Hypothesis 3.1(ii) holds if M is large enough, but (iii) is a special requirement that is important in our convergence analysis. In the rest of the paper we shall fix a function Ψ satisfying Hypothesis 3.1 and consider a viscosity solution within a special class of functions associated to Ψ. More precisely, we have the following definition.

Definition 3.2. *We say that a function v is of class (Ψ) if it satisfies the following conditions:*

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(1) $v(s,x,w) = \Psi(s,x,w), \quad (s,x,w) \in \mathscr{D}_\delta^{*,c}$;
(2) $v(s,x,w)$ *is increasing with respect to* x *on* D_δ;
(3) $v(s,x,w)$ *is bounded on* D_δ;
(4) $v(s,x,w) - v(s,-\delta,w) \geq x+\delta$ *as* $x \downarrow -\delta$ *for any* $0 \leq s \leq T+\delta, -\delta \leq w \leq s+\delta$.

We shall construct a viscosity solution of (3.4) that is of class (Ψ) by the well-known Perron's method. To begin with, we need the following lemma. Since its proof is merely computational, we give only a sketch of the proof.

LEMMA 3.3. *Assume Hypothesis* 2.1 *and Hypothesis* 3.1. *There exist both viscosity supersolution* $\overline{\psi}$ *and subsolution* $\underline{\psi}$ *of class* (Ψ) *to* (3.4) *on* $\mathscr{D}_\delta$. *Furthermore, it holds that* $\overline{\psi} = \underline{\psi} = \Psi$ *on* $\mathscr{D}_\delta^{*,c}$.

Proof. We shall argue only the subsolution case; the supersolution case is similar to [9]. First recall the distance function $d(x;D) := \inf_{y \in D} |x-y|$, for $x \in \mathbb{R}^m$, and $D \subset \mathbb{R}^m$; and we define $d_{\mathscr{D}_\delta}(\theta) := d(\theta; \mathscr{D}_\delta^c)$, $\theta = (s,x,w)$. Then one can check that

$$d_{\mathscr{D}_\delta}(\theta) = (x+\delta) \wedge (w+\delta) \wedge \frac{\sqrt{2}}{2}(s+\delta-w) \wedge (T+\delta-s) \wedge s, \quad \theta \in \mathscr{D}_\delta. \tag{3.8}$$

Now consider the function $\psi(\theta) := -k d_{\mathscr{D}_\delta}(\theta)$, $\theta := (s,x,w) \in (0, T+\delta) \times \mathbb{R}^2$, where $k > 0$ satisfies the constraint

$$k \leq \min \left\{ b-1,\ M-K_2,\ \inf_{\theta \in \Gamma_s^\delta} \Psi_x(\theta),\ \frac{M-K_2}{\sup\limits_{w \in [0,T+1]} \left|c + \dfrac{f(w)}{\bar{F}(w)} - r\right|(T+4\delta)} \right\}. \tag{3.9}$$

In the above b is the constant defined by (3.7), and $\Gamma_s^\delta := [0,T] \times [-\delta, T+3\delta] \times [0,s]$. It is then straightforward, albeit tedious, to check that $\psi + \Psi$ is a viscosity subsolution of class (Ψ) in the sense of Definition 3.2. We leave it to the interested reader. ◻

Next, for given Ψ, we consider the following set:

$$\mathscr{F} = \{v : v \text{ is a viscosity subsolution of class } (\Psi) \text{ to (3.4) on } \mathscr{D}_\delta \text{ s.t. } \underline{\psi} \leq v \leq \bar{\psi}\},$$

where $\underline{\psi}$ and $\bar{\psi}$ are the viscosity subsolution and supersolution, respectively, of class (Ψ) mentioned in Lemma 3.3. Define

$$u(s,x,w) := \sup_{v \in \mathscr{F}} v(s,x,w), \qquad (s,x,w) \in D_\delta, \tag{3.10}$$

and let u^* (resp., u_*) be the *upper semicontinuous envelope* (resp., *lower semicontinuous envelope*) of u, defined, respectively, by

$$\begin{cases} u^*(s,x,w) := \\ \limsup\limits_{r \downarrow 0} \left\{ u(t,y,v) : (t,y,v) \in \mathscr{D}_\delta, \sqrt{|t-s|^2 + |y-x|^2 + |v-w|^2} \leq r \right\}, \\ u_*(s,x,w) := \\ \liminf\limits_{r \downarrow 0} \left\{ u(t,y,v) : (t,y,v) \in \mathscr{D}_\delta, \sqrt{|t-s|^2 + |y-x|^2 + |v-w|^2} \leq r \right\}. \end{cases} \tag{3.11}$$

The main result of this section is the following theorem, which obviously implies the existence of the viscosity solution to (3.4).

THEOREM 3.4. *Assume that Hypotheses* 2.1 *and* 3.1 *are in force. Then* u^* *(resp.,* u_**) is a viscosity subsolution (resp., supersolution) of class* (Ψ) *to* (3.4) *on* $\mathscr{D}_\delta$.

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Proof. The fact that u^* is a subsolution is more or less straightforward; we shall omit the proof and accept it as a fact and prove only that u_* is a supersolution of class (Ψ). It is easy to verify that u_* is of class (Ψ). Suppose that u_* is not a supersolution; then there exists $\theta_0 = (s_0, x_0, w_0) \in \mathscr{D}_\delta$ and $\varphi \in \mathbb{C}_0^{1,2,2}(\mathscr{D}_\delta)$ such that $0 = [u_* - \varphi](\theta_0) < [u_* - \varphi](\theta)$ for all $\theta \in \mathscr{D}_\delta$, but

$$\partial_t\varphi(\theta_0) + \sup_{\gamma\in[0,1],a\in[0,M]} H^n(\theta_0, u_*, \nabla\varphi, \varphi_{xx}, \varphi_{ww}, I^\delta[\varphi], \gamma, a) =: \varepsilon_0 > 0.$$

By continuity, we can then find $\eta_0 > 0$ such that, for any $\theta \in B_{\eta_0}(\theta_0) \subset \mathscr{D}_\delta$,

$$\partial_t\varphi(\theta) + \sup_{\gamma\in[0,1],a\in[0,M]} H^n(\theta, u_*, \nabla\varphi, \varphi_{xx}, \varphi_{ww}, I^\delta[\varphi], \gamma, a) > \varepsilon_0/4. \tag{3.12}$$

We shall argue that (3.12) means that one can construct a subsolution $\psi^* \in \mathscr{F}$ such that $\psi^*(\theta_0) > u(\theta_0)$, which would contradict the definition of u. To this end, note that being of class (Ψ) u_* is increasing in x. Thus for $0 < \varepsilon_1 < \frac{\varepsilon_0}{2}$, we can modify φ slightly so that on $B_{\eta_0}(\theta_0)$ (or choose a smaller ball if necessary) φ is increasing in x, but it is decreasing in x for x sufficiently large such that

$$\inf_{\theta\in B^c_{\eta_0}(\theta_0)\cap D_\delta} \{u_*(\theta) - \varphi(\theta)\} \geq \varepsilon_1 > 0. \tag{3.13}$$

Note that, by definition of u, we have $\varphi \leq u_* \leq \bar\psi$ in $\mathscr{D}_\delta$. We claim that $\varphi(\theta_0) < \bar\psi(\theta_0)$. Indeed, if $\varphi(\theta_0) = u_*(\theta_0) = \bar\psi(\theta_0)$, then $\bar\psi - \varphi$ has a strict minimum at θ_0. Since $\bar\psi$ is a viscosity supersolution (3.4) on $\mathscr{D}_\delta$, we have

$$\partial_t\varphi(\theta_0) + \sup_{\gamma\in[0,1],a\in[0,M]} H^n(\theta_0, \varphi, \nabla\varphi, \partial_{xx}\varphi, \partial_{ww}\varphi, I^\delta[\varphi], \gamma, a) \leq 0,$$

contradicting (3.12). Therefore, by continuity of $\bar\psi$ and φ, we can find $0 < \eta_2 < \eta_0$ and $\varepsilon_2 > 0$ such that $\varphi(\theta) < \bar\psi(\theta) - \varepsilon_2$, $\theta \in B_{\eta_2}(\theta_0)$. Note that $u_* - \varphi$ has a strict minimum at θ_0; we have

$$\Delta_r := \inf_{\theta\in B^c_r(\theta_0)\cap D_\delta}\{u_*(\theta) - \varphi(\theta)\} = \inf_{\theta\in \overline{B^c_r}(\theta_0)\cap \overline{D}_\delta}\{u_*(\theta) - \varphi(\theta)\} > 0, \quad r > 0. \tag{3.14}$$

Let us now fix $r_0 \in (0, \eta_2)$. Recall that we have modified φ so that for some $\hat x > 0$ large enough, it is decreasing in x, for $x > \hat x$. We assume without loss of generality that $\hat x > x_0 + r_0$. Define $E_\delta(\hat x) := \{\hat\theta := (s, \hat x, w) : 0 \leq s < T + \delta, -\delta < w < s + \delta\}$. Clearly, $E_\delta(\hat x) \subset \overline{B^c}_{r_0} \cap \overline{D}_\delta$; thus by (3.14) we have $u_*(\hat\theta) - \varphi(\hat\theta) \geq \Delta_{r_0}$ for $\hat\theta \in E_\delta(\hat x)$. Now for fixed $\hat\theta_1 = (s_1, \hat x, w_1) \in E_\delta(\hat x)$, by definition of u_* we can choose $\hat v_1 \in \mathscr{F}$ such that $\hat v_1(\hat\theta_1) - \varphi(\hat\theta_1) \geq \frac{3\Delta_{r_0}}{4}$. But since $\hat v_1 \in \mathscr{F}$ (whence increasing in x) and φ is decreasing in x for $x > \hat x$, we have

$$\hat v_1(s_1, x, w_1) - \varphi(s_1, x, w_1) \geq \hat v_1(\hat\theta_1) - \varphi(\hat\theta_1) \geq \frac{3\Delta_{r_0}}{4} \qquad \text{for } x \geq \hat x. \tag{3.15}$$

On the other hand, by continuity of $(\hat v_1 - \varphi)(\cdot, \hat x, \cdot)$, there exists $\hat\eta_1 > 0$ such that

$$\inf_{(s,w)\in \bar B_{\hat\eta_1}(s_1,w_1)\cap \bar E_\delta(\hat x)} \{\hat v_1(s, \hat x, w) - \varphi(s, \hat x, w)\} \geq \frac{\Delta_{r_0}}{2}. \tag{3.16}$$

Note that $\bar E_\delta(\hat x)$ is compact; there exists a finite set $\{(s_j, w_j)\}_{j=1}^{m_0} \subset \bar E_\delta(\hat x)$, together with $\hat v_j \in \mathscr{F}$ and constants $\hat\eta_j > 0$, $j = 1, \ldots, m_0$, such that $\bar E_\delta(\hat x) \subset \cup_{j=1}^{m_r} \bar B_{\hat\eta_j}(s_j, w_j)$,

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and both (3.15) and (3.16) hold for each j. Now let us define $\ell_0(\theta) = \sup_{1\le j\le m_0} \hat{v}_j(\theta)$, $\theta \in \mathscr{D}_\delta$. Then one can check, as before, that $\ell_0 \in \mathcal{F}$ and is increasing with x on $\mathscr{D}_\delta$. Furthermore, since each $\hat{v}_j$ satisfies (3.15) and (3.16), it is readily seen that

$$\inf_{(s,x,w)\in\mathscr{D}_\delta\setminus D_{\delta,\hat{x}}} \{\ell_0(s,x,w) - \varphi(s,x,w)\} \ge \frac{\Delta_{r_0}}{2}, \tag{3.17}$$

where $D_{\delta,\hat{x}} := \{(s,x,w) : 0 < s < T+\delta, -\delta < x < \hat{x}, -\delta < w < s+\delta\}$.

Now let us consider the set $\bar{D}_{\delta,\hat{x}}\setminus B_{r_0}(\theta_0)$. By (3.14) we have $u_*(\theta) - \varphi(\theta) \ge \Delta_{r_0}$ for all $\theta \in \bar{D}_{\delta,\hat{x}} \setminus B_{r_0}(\theta_0)$. Since $\bar{D}_{\delta,\hat{x}}\setminus B_{r_0}(\theta_0)$ is compact, we can repeat the same argument as before to obtain a $\ell_1 \in \mathscr{F}$ so that

$$\inf_{(s,x,w)\in\bar{D}_{\delta,\hat{x}}\setminus B_{r_0}(\theta_0)} \{\ell_1(s,x,w) - \varphi(s,x,w)\} \ge \frac{\Delta_{r_0}}{2}. \tag{3.18}$$

Let $0 < \alpha_0 < \min\{\frac{\varepsilon_2}{\Delta_{r_0}}, \frac{1}{2}\}$, and define

$$U(\theta) := \begin{cases} \max\{\varphi(\theta) + \alpha_0\Delta_{r_0}, \ell_0(\theta), \ell_1(\theta)\} & \text{if } \theta \in B_{r_0}(\theta_0), \\ \max\{\ell_0(\theta), \ell_1(\theta)\} & \text{if } \theta \in B^c_{r_0}(\theta_0) \cap \mathscr{D}_\delta. \end{cases} \tag{3.19}$$

Then, by the choice of r_0 and α_0, we have $\underline{\psi} \le U \le \bar{\psi}$ in $\mathscr{D}_\delta$, and

$$U(\theta_0) \ge \varphi(\theta_0) + \alpha_0\Delta_{r_0} > \varphi(\theta_0) = u_*(\theta_0). \tag{3.20}$$

We claim that U is a viscosity subsolution of class (Ψ) to (3.4) in $\mathscr{D}_\delta$, which would be a contradiction to the the definition of u_* and prove the theorem.

To this end, for any $\bar{\theta} := (t,y,v) \in \mathscr{D}_\delta$, suppose that there is a function $\phi \in \mathbb{C}_0^{1,2,2}(\mathscr{D}_\delta)$ such that $0 = U(\bar{\theta}) - \phi(\bar{\theta})$ is a strict maximum over $\mathscr{D}_\delta$. Consider two possible cases *Case* 1: $U(\bar{\theta}) = \ell_0(\bar{\theta})$ or $\ell_1(\bar{\theta})$. We shall only consider the case $U(\bar{\theta}) = \ell_0(\bar{\theta})$, as the other case is similar. Since $\ell_0 \le U \le \phi$ on $\mathscr{D}_\delta$, $\ell_0 - \phi$ has a maximum at $\bar{\theta}$. Recall again that, as the "sup" of subsolutions, ℓ_0 is a viscosity subsolution of (3.4) on $\mathscr{D}_\delta$ as well; hence we have

$$\partial_t\phi(\bar{\theta}) + \sup_{\gamma\in[0,1],a\in[0,M]} H^n(\bar{\theta}, \phi, \nabla\phi, \phi_{xx}, \phi_{ww}, I^\delta[\phi], \gamma, a) \ge 0. \tag{3.21}$$

Case 2: $U(\bar{\theta}) = \varphi(\bar{\theta}) + \alpha_0\Delta_{r_0}$. In this case we must have $\bar{\theta} \in B_{r_0}(\theta_0)$ by definition of U. But since $\varphi + \alpha_0\Delta_{r_0} \le U \le \phi$ in $B_{r_0}(\theta_0)$ by our choices of r_0 and α_0, we have $\varphi + \alpha_0\Delta_{r_0} - \phi \le 0$ in $B_{r_0}(\theta_0)$. On the other hand, note that $\phi \ge U = \max\{\ell_0, \ell_1\}$ in $B^c_{r_0}(\theta_0) \cap \mathscr{D}_\delta$; we conclude that

$$\varphi + \alpha_0\Delta_{r_0} - \phi \le \varphi + \alpha_0\Delta_{r_0} - \max\{\ell_0, \ell_1\} \le -\frac{\Delta_{r_0}}{2} + \alpha_0\Delta_{r_0} \le 0$$

in $B^c_{r_0}(\theta_0) \cap \mathscr{D}_\delta$. That is, $\varphi + \alpha_0\Delta_{r_0} - \phi$ has a maximum at $\bar{\theta} \in B_{r_0}(\theta_0) \subset B_{\eta_1}(\theta_0)$. Then, by (3.12), choosing α_0 sufficiently small if necessary we have

$$\begin{aligned} &\partial_t\phi(\bar{\theta}) + \sup_{\substack{\gamma\in[0,1]\\ a\in[0,M]}} H^n(\bar{\theta}, \phi, \nabla\phi, \phi_{xx}, \phi_{ww}, I^\delta[\phi], \gamma, a) \\ \ge &\partial_t\varphi(\bar{\theta}) + \sup_{\substack{\gamma\in[0,1]\\ a\in[0,M]}} H^n(\bar{\theta}, \varphi + \alpha_0\Delta_{r_0}, \nabla\varphi, \varphi_{xx}, \varphi_{ww}, I^\delta[\varphi + \alpha_0\Delta_{r_0}], \gamma, a) \ge 0. \end{aligned} \tag{3.22}$$

Combining (3.21) and (3.22) we conclude that U is a viscosity subsolution of class (Ψ) to (3.4) in $\mathscr{D}_\delta$, and $U(\theta_0) > u(\theta_0)$, a contradiction. This proves the theorem. ∎

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Let us now denote the solution to (3.4) by $V^{n,\delta}$. We shall argue that such a viscosity solution is unique and is actually a classical solution. The proof of uniqueness will depend on the *comparison theorem* as usual, and in this case it can be argued along the same lines of that in [7], except for some slight modifications. We shall only state the result and omit the proof, so as to keep the paper in a proper length.

THEOREM 3.5 (comparison principle). *Let $\bar{u}$ be a viscosity supersolution and $\underline{u}$ be a viscosity subsolution of* (3.4) *on $\mathscr{D}_\delta$, and both are of class* (Ψ). *Then $\underline{u} \le \bar{u}$ on D_δ. Consequently, $u^* = u_* =: u$ defined by* (3.11) *is a unique continuous viscosity solution of class* (Ψ) *to* (3.4).

Remark 3.6. We recall that in [7] we proved the existence and uniqueness of the constrained viscosity solution. But the proof of the existence was essentially based on verifying that the value function is the desired viscosity solution. This fact sometimes causes logical confusion, since a "practical" version of the value function is actually the solution to the HJB equation. Thus is it often desirable, especially when an optimal strategy is based on the value function, to be able to "construct" a constrained viscosity solution to the original problem, which we now describe. First note that by uniqueness we need only show that we can construct a constrained viscosity subsolution u^*. Similar to the viscosity solution of class (Ψ), we consider the class of constrained viscosity solution v to (2.9) such that (i) $v(T,x,w) = 0$; (ii) $x \mapsto v(t,x,w)$ is increasing for $\theta = (t,x,w) \in D$; and (iii) $v(t,x,w)$ is bounded on D and $-Q_2 T \le v(\theta) \le (2+Q_1)T$, $\theta \in D$, for some $Q_1, Q_2 > 0$. We shall call such viscosity solutions of class (Q). Now let $d_{\mathscr{D}}(\theta) := \inf_{\eta\in\mathscr{D}} |\eta-\theta|$ be the distance between θ and the set $\mathscr{D}$. One can easily check that the functions $\bar{\Upsilon}(\theta) = 2d_{\mathscr{D}}(\theta) + Q_1(T-s)$ and $\underline{\Upsilon}(\theta) = d_{\mathscr{D}}(\theta) - Q_2(T-s)$, $\theta \in D$, where

$$Q_1 = \max\{2+M, 2(p+\mu T)\}; \quad Q_2 = \left[c + \sup_{0\le w\le T}\left|\frac{f(w)}{\bar{F}(w)}\right|\right] T + 1, \tag{3.23}$$

are, respectively, the viscosity supersolution on $\mathscr{D}$ and subsolutions on $\mathscr{D}^*$ to (2.9) of class (Q) with constants (Q_1, Q_2). Furthermore, $\underline{\Upsilon} \le \bar{\Upsilon}$ on D. Now let $\mathscr{M}$ be the set of all viscosity subsolution u of (2.9) on $\mathscr{D}^*$ of class (Q) such that $\underline{\Upsilon} \le u \le \bar{\Upsilon}$, and define $\mathfrak{u}(s,x,w) := \sup_{u\in\mathscr{M}} u(s,x,w)$. Then similar to Theorem 3.4 one can show that $\mathfrak{u}^*$, defined by

$$\mathfrak{u}^*(s,x,w) = \varlimsup_{r\downarrow 0}\left\{\mathfrak{u}(t,y,v); (t,y,v) \in D, \sqrt{|t-s| + |y-x|^2 + |v-w|^2} \le r\right\}, \tag{3.24}$$

is a (constrained) viscosity subsolution of (2.9) on $\mathscr{D}^*$, of class (Q). In particular, by uniqueness ([7]), $\mathfrak{u}^* = V$, the value function of the original optimal dividend problem.

4. The regularity and convergence of $\{V^{n,\delta}\}$. We now turn our attention to the family $\{V^{n,\delta}\}_{n\ge1,\delta>0}$, the solutions to the auxiliary equations (3.4). We shall argue that each $V^{n,\delta}$ has desired the regularity and that $V^{n,\delta} \to V$, the original value function in a satisfactory way, as $n \to \infty$ and $\delta \to 0$. We first look at the regularity issue. To begin with, we note that if u is a viscosity solution of (3.4) on D_δ, and we consider the change of variable, $y := \ln(1+x+\delta)$, $x \ge -\delta$, and define $v(s,y,w) := u(s, e^y - 1 - \delta, w)$, then it is easy to verify that v is viscosity solution of the PIDE:

$$v_t(\theta) + \sup_{\gamma\in[0,1], a\in[0,M]} \mathscr{G}^n\left(\theta, v, v_y, v_w, v_{yy}, v_{ww}, I^\delta[v], \gamma, a\right) = 0 \quad \text{on } B_\delta, \tag{4.1}$$

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where $\theta = (s, y, w)$, $B_\delta := \{\theta = (s, y, w) : 0 \le s < T + \delta, y > 0, -\delta < w < s + \delta\}$, and

(4.2)

$$\begin{aligned}
&\mathscr{G}^n(\theta, v, v_y, v_w, v_{yy}, v_{ww}, I^\delta[v], \gamma, a)\\
&\quad := \left[\frac{\varepsilon_n e^{-2y}}{2} + \frac{\sigma^2\gamma^2}{2}\left(\frac{e^y - \delta - 1}{e^y}\right)^2\right] v_{yy}(\theta) + \frac{\varepsilon_n}{2} v_{ww}(\theta)\\
&\qquad + \left[pe^{-y} - \frac{\varepsilon_n}{2}e^{-2y} - \frac{\sigma^2\gamma^2}{2}\left(\frac{e^y - \delta - 1}{e^y}\right)^2 + (r + (\mu - r)\gamma)\frac{e^y - \delta - 1}{e^y}\right] v_y(\theta)\\
&\qquad + a(1 - e^{-y} v_y(\theta)) + v_w(\theta) - cv(\theta) + \frac{f(w)}{\overline{F}(w)} I^\delta[v].
\end{aligned}$$

It is worth noting that the main difference between (4.1) and (3.4) is that all the coefficients of (4.1) are bounded and continuous, and for each fixed $n \ge 1$ and $\delta > 0$, the function $\mathscr{G}^n$ is uniformly *elliptic*. Therefore, a straightforward application of a combination of [8, Lemma 2.9, Corollary 2.12, and Theorem 9.1] (see also [19] and [20, Theorem 1.1]) leads to the following result.

THEOREM 4.1. *Assume Hypothesis* 3.1. *Let u be the unique viscosity solution of class $(\tilde{\Psi})$ to* (4.1) *with $\tilde{\Psi}(s, y, w) := \Psi(s, e^y - 1 - \delta, w)$, $(s, y, w) \in D_\delta$. Then, $u \in \mathbb{C}^{2+\alpha}_{loc}(D_\delta)$*[1] *in the sense that for any compact set $D' \subset D_\delta$, there exists a constant $C > 0$ such that $\|u\|_{C^{2+\alpha}(D')} \le C$, where $C > 0$ depends on the uniform constants in Hypothesis* 3.1 *and the time duration $T > 0$.*

Remark 4.2. A direct consequence of Theorem 4.1 is that the unique viscosity solution $V^{n,\delta}$ to the PIDE (3.4) in Theorem 3.5 has the same regularity for each fixed $n \ge 1$ and $\delta > 0$. This fact will be important for the construction of ε-optimal control in the sections to follow.

In the rest of the section we shall focus on an important and more involved issue: the convergence of the family $\{V^{n,\delta}\}$ as $n \to \infty$ and $\delta \to 0$. We shall first look at the limit as $n \to \infty$ (or as $\varepsilon_n \to 0$). Naturally, let us consider an intermediate PIDE:

$$V_t(\theta) + \sup_{\gamma \in [0,1], a \in [0,M]} H(\theta, V, \nabla V, V_{xx}, V_{ww}, I^\delta[V], \gamma, a) = 0, \quad \theta \in \mathscr{D}_\delta, \tag{4.3}$$

where H is defined by (2.6). Following the same argument as that in section 2, we now argue that (4.3) admits a unique viscosity solution of class (Ψ). To see this, for any $(t, y, v) \in D_\delta$, let

$$\tilde{V}_\delta(t, y, v) := \lim_{k\to\infty} \sup\{V^{n,\delta}(\theta) : n \ge k, \theta \in \bar{B}_{1/k}(t, y, v) \cap \bar{\mathscr{D}}_\delta\} \text{ and}$$

$$\tilde{V}^\delta(t, y, v) := \lim_{k\to\infty} \inf\{V^{n,\delta}(\theta) : n \ge k, \theta \in \bar{B}_{1/k}(t, y, v) \cap \bar{\mathscr{D}}_\delta\},$$

where $B_r(t, y, v)$ is the open ball with radius r centered at (t, y, v) and $V^{n,\delta}$'s are the viscosity solutions of class (Ψ) to PIDE (3.4).

LEMMA 4.3. *For any Ψ satisfying Hypothesis* 3.1, *the function $\tilde{V}_\delta$ (resp., $\tilde{V}^\delta$) is a viscosity subsolution (resp., supersolution) of class (Ψ) on $\mathscr{D}_\delta$ to* (4.3).

[1]A function $u \in \mathbb{C}^{1+\alpha}_{loc}([0, T] \times \mathbb{R})$ means $u \in L^\infty([0, T] \times \mathbb{R})$ and $Du \in \mathbb{C}^\alpha_{loc}([0, T] \times \mathbb{R})$; $u \in \mathbb{C}^{2+\alpha}_{loc}([0, T] \times \mathbb{R})$ means $Du \in \mathbb{C}^{1+\alpha}_{loc}([0, T] \times \mathbb{R})$ and u_t, $D^2u \in \mathbb{C}^\alpha_{loc}([0, T] \times \mathbb{R})$.

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Proof. We shall discuss only $\tilde{V}_\delta$ as the proof for $\tilde{V}^\delta$ is similar. First, it is easy to see that $\tilde{V}_\delta$ is of class (Ψ) since all $V^{n,\delta}$'s are uniformly bounded, uniformly in n, δ. Next, suppose that for some $\theta_0 := (t_0, y_0, v_0) \in D_\delta$, $0 = [\tilde{V}_\delta - \varphi](\theta_0)$ is a (strict) maximum of $\tilde{V}_\delta - \varphi$ over D_δ, where $\varphi \in C^{1,2,2}(\mathscr{D}_\delta)$. For any $N > y_0$ we define $D_{\delta,N} = [0, T+\delta] \times [-\delta, N] \times [-\delta, s+\delta]$ so that $\theta_0 \in D_{\delta,N}$. Since θ_0 is the strict maximum of $\tilde{V}_\delta - \varphi$, for $\varepsilon > 0$, there exists a modulus of continuity $\omega_1(\cdot)$ such that

$$\sup_{\theta \in B^c_\varepsilon(\theta_0) \cap D_{\delta,N}} (\tilde{V}_\delta(\theta) - \varphi(\theta)) \le -\omega_1(\varepsilon) < 0.$$

Now for $\bar{\theta} := (t, y, v) \in D_{\delta,N}$, by definition of $\tilde{V}_\delta$, there exists $k_0 := k_0(\bar{\theta}) = k_0(\bar{\theta}; \varepsilon)$ such that

$$\sup_{\theta \in \bar{B}_{1/k_0}(\bar{\theta}) \cap \bar{\mathscr{D}}_\delta} V^{n,\delta}(\theta) - \tilde{V}_\delta(\bar{\theta}) < \frac{\omega_1(\varepsilon)}{4}, \qquad n \ge k_0.$$

Let us denote $\omega^{\delta,N}_\varphi(\cdot)$ to be the modulus of continuity of φ on $D_{\delta,N}$. Then, for $\varepsilon > 0$, there exists $\eta_0 := \eta_0(\varepsilon) > 0$ such that $\omega^{\delta,N}_\varphi(\eta_0) < \omega_1(\varepsilon)/4$. Thus, for $\bar{\theta} \in D_{\delta,N} \backslash B_\varepsilon(\theta_0)$ and $n \ge k_0(\bar{\theta})$,

$$\begin{aligned}
&\sup_{\theta \in \bar{B}_{\frac{1}{k_0} \wedge \eta_0}(\bar{\theta}) \cap \bar{\mathscr{D}}_\delta} (V^{n,\delta}(\theta) - \varphi(\theta)) \\
&\quad = \sup_{\theta \in \bar{B}_{\frac{1}{k_0} \wedge \eta_0}(\bar{\theta}) \cap \bar{\mathscr{D}}_\delta} (V^{n,\delta}(\theta) - \tilde{V}_\delta(\bar{\theta}) + \tilde{V}_\delta(\bar{\theta}) - \varphi(\bar{\theta}) + \varphi(\bar{\theta}) - \varphi(\theta)) \\
&\quad \le \frac{\omega_1(\varepsilon)}{4} - \omega_1(\varepsilon) + \omega^{\delta,N}_\varphi(\eta_0) \le \frac{\omega_1(\varepsilon)}{4} - \omega_1(\varepsilon) + \frac{\omega_1(\varepsilon)}{4} = -\frac{\omega_1(\varepsilon)}{2}.
\end{aligned}$$

Since $B^c_\varepsilon(\theta_0) \cap D_{\delta,N}$ is compact and $\bigcup_{\bar{\theta} \in D_{\delta,N}} B_{\frac{1}{k_0(\bar{\theta})} \wedge \eta_0}(\bar{\theta}) \supset B^c_\varepsilon(\theta_0) \cap D_{\delta,N}$, there exist $N_1 > 0$ and $\theta_i \in B^c_\varepsilon(\theta_0) \cap D_{\delta,N}$, $i = 1, 2, 3 \ldots N_1$, such that $\bigcup_{i=1}^{N_1} B_{\frac{1}{k_0(\theta_i)} \wedge \eta_0}(\theta_i) \supset B^c_\varepsilon(\theta_0) \cap D_{\delta,N}$. Hence, for any $n \ge \max_{1 \le i \le N_1} k_0(\theta_i)$,

$$V^{n,\delta}(\bar{\theta}) - \varphi(\bar{\theta}) \le -\frac{\omega_1(\varepsilon)}{2}, \qquad \bar{\theta} \in B^c_\varepsilon(\theta_0) \cap D_{\delta,N}.$$

Finally, let $\{\varepsilon_\ell\}_{\ell \in \mathbb{N}}$ be a positive sequence such that $\varepsilon_\ell \downarrow 0$ as $\ell \to \infty$. For each $\ell > 0$, let $\bar{\theta}_\ell \in B^c_{\varepsilon_1}(\theta_0) \cap D_{\delta,N}$ and $n_\ell \ge \max\{\max_{1 \le i \le N_1(\varepsilon_\ell)} k_0(\theta_i(\varepsilon_\ell)), \frac{1}{\varepsilon_\ell}\}$ be such that

$$V^{n_\ell,\delta}(\bar{\theta}_\ell) - \varphi(\bar{\theta}_\ell) = \max_{\bar{\theta} \in \bar{D}_\delta} (V^{n_\ell,\delta}(\bar{\theta}) - \varphi(\bar{\theta})) > -\frac{\omega_1(\varepsilon_\ell)}{2}. \tag{4.4}$$

Next, denoting $\varphi^{n_\ell,\delta}(\theta) := \varphi(\theta) + V^{n_\ell,\delta}(\bar{\theta}_\ell) - \varphi(\bar{\theta}_\ell)$, $\theta \in D_\delta$, we see that $\varphi^{n_\ell,\delta} \in \mathbb{C}^{1,2,2}(D_\delta)$ and $0 = V^{n_\ell,\delta}(\bar{\theta}_\ell) - \varphi^{n_\ell,\delta}(\bar{\theta}_\ell) = \max_{\theta \in D_\delta} V^{n_\ell,\delta}(\theta) - \varphi^{n_\ell,\delta}(\theta)$, and therefore

$$\varphi_t(\bar{\theta}_\ell) + \sup_{\gamma \in [0,1], a \in [0,M]} H^{n_\ell}(\bar{\theta}_\ell, \varphi^{n_\ell,\delta}, \nabla\varphi, \varphi_{xx}, \varphi_{ww}, I^\delta[\varphi^{n_l,\delta}], \gamma, a) \ge 0. \tag{4.5}$$

Letting $\ell \to \infty$ in (4.4) and (4.5), we have

$$\begin{aligned}
0 &\le \lim_{n_\ell \to \infty} V^{n_\ell,\delta}(\bar{\theta}_\ell) \\
&\le \lim_{\varepsilon_\ell \to 0} \sup\{V^{n,\delta}(s, x, w) : n \ge \frac{1}{\varepsilon_\ell}, (s, x, w) \in \bar{B}_{\varepsilon_\ell}(\theta_0) \cap \bar{\mathscr{D}}_\delta\} - \varphi(\theta_0) \\
&= \lim_{k \to \infty} \sup\{V^{n,\delta}(s, x, w) : n \ge k, (s, x, w) \in \bar{B}_{\frac{1}{k}}(\theta_0) \cap \bar{\mathscr{D}}_\delta\} - \varphi(\theta_0) \\
&= \tilde{V}_\delta(\theta_0) - \varphi(\theta_0) = 0,
\end{aligned}$$

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and $\varphi_t(\theta_0) + \sup_{\gamma\in[0,1],a\in[0,M]} H(\theta_0, \varphi, \nabla\varphi, \varphi_{xx}, \varphi_{ww}, I^\delta[\varphi], \gamma, a) \geq 0$. That is, $\tilde{V}_\delta$ is a viscosity subsolution of (4.3). ∎

We should note that Lemma 4.3 and the comparison principle (Theorem 3.5) imply that $\tilde{V}_\delta \leq \tilde{V}^\delta$. On the other hand, by definitions of $\tilde{V}_\delta$ and $\tilde{V}^\delta$, we also have $\tilde{V}_\delta \geq \tilde{V}^\delta$. Thus we have $\tilde{V}_\delta = \tilde{V}^\delta$, and we shall denote it by V^δ. Clearly, $V^\delta \in \mathbb{C}(D_\delta)$.

Next, we recall the value function V defined by (2.4). We know from [7] that it is the unique constrained viscosity solution of (2.9), and from Remark 3.6 we see that it can be constructed as $\mathfrak{u}^*$ defined by (3.24). In what follows we shall assume that, modulo a further approximation, we can always find a function Ψ satisfying Hypothesis 3.1 such that $\Psi(\theta) = \mathfrak{u}^*(\theta) = V(\theta)$, $\theta \in \partial D$. We should note that if Ψ satisfies Hypothesis 3.1, then Ψ will be smooth and have $\partial_x\Psi > 1$ on the boundary ∂D. However, these two conditions are *not* necessarily satisfied by the value function V. The following lemma is thus useful for our discussion.

Lemma 4.4. *Let V be the value function defined by* (2.4). *Then there exist a sequence of functions $\{\Psi_m\}_{m\geq1}$ satisfying Hypothesis* 3.1 *and continuous viscosity solutions v^m of*

$$\text{(4.6)}\qquad \begin{cases} v_t(s,x,w) + \mathscr{L}[v](s,x,w) = 0, & (s,x,w) \in \mathscr{D}, \\ v(s,x,w) = \Psi_m(s,x,w), & (s,x,w) \in \partial D \end{cases}$$

such that

(i) $\lim_{m\to\infty} \sup_{\theta\in\partial D} |\Psi_m(\theta) - V(\theta)| = 0$ *and*

(ii) $\lim_{m\to\infty} \|v^m - V\|_{L^\infty(D)} \to 0$.

Proof. Let V be the (viscosity) solution to (2.9) and $\varphi_m : D \mapsto \mathbb{R}$ the standard mollifiers of V. Then, since V is continuous, we have $\lim_{m\to\infty} \|\varphi_m - V\|_{L^\infty(D)} = 0$. Next, we define

$$\text{(4.7)}\qquad \Psi_m(\theta) = \varphi_m(\theta) + (2 + N_m)d(\theta, \partial D_m), \qquad \theta := (s,x,w) \in D,$$

where $N_m := \sup_{(s,w)\in[0,T]\times[0,s]} |\partial_x\varphi_m(s,0,w)|$, and $\{D_m\}_{m\geq1}$ is a sequence of smooth area such that $D \subset D_m$, $d(D, D_m) < \delta_m := \frac{1}{m(2+N_m)}$, and D_m is parallel to the plane $\{(s,x,w), -\delta_m \leq s \leq T + \delta_m, x = 0, -\delta_m \leq w \leq s + \delta_m\}$. It is then easy to check that $\sup_{\theta\in\partial D} |(2 + N_m)d(\theta, \partial D_m)| \leq \frac{2}{m}$ and $\partial_x\Psi_m(s,0,w) = \partial_x f_m(s,0,w) + (2 + N_m) \geq -N_m + 2 + N_m = 2$. Consequently, one can further check that, by defining $\Psi_m \equiv 0$ on D_1^c, all Ψ_m's satisfy Hypothesis 3.1. Now let v^m be the unique viscosity solution of (2.9) on $\mathscr{D}$ with $v^m = \Psi_m$ on ∂D. Then by definition (4.7) we can easily check that $a_m := \sup_{\theta\in\partial D} |v^m(\theta) - V(\theta)| = \sup_{\theta\in\partial D} |\Psi_m(\theta) - V(\theta)| \to 0$ as $m \to \infty$ and $v^m - a_m \leq V \leq v^m + a_m$ on ∂D. Since $v^m - a_m$ and $v^m + a_m$ are the viscosity subsolution and supersolution of (2.9) on $\mathscr{D}$, respectively, by the comparison theorem we can then deduce that $\lim_{m\to\infty} \|v^m - V\|_{L^\infty(D)} \to 0$, proving the lemma. ∎

We can now prove the main result of this section.

Theorem 4.5. *Let V be the value function defined by* (2.4). *Then for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$, and $\delta > 0$, depending only on ε, such that $\|V^{n,\delta} - V\|_{L^\infty(D)} < \varepsilon$, where $V^{n,\delta} \in \mathbb{C}^{2+\alpha}(D_\delta)$ is a (viscosity) solution to* (3.4) *of class* (Ψ) *for some function Ψ satisfying Hypothesis* 3.1.

Proof. In light of Lemma 4.4, we can assume without loss of generality that we can find Ψ satisfying Hypothesis 3.1 such that $\Psi = \mathfrak{u}^* = V$ on ∂D. (Otherwise for any $\varepsilon > 0$ we can first choose Ψ_m so that it satisfies Hypothesis 3.1, and the corresponding

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viscosity solution v^m satisfies $\Psi_m = v^m$ on ∂D, and $\|v^m - V\|_{L^\infty(D)} < \varepsilon/3$, and then prove the theorem for Ψ_m and v^m.) For convenience we shall also define $\mathfrak{u}^*(\theta) = \Psi(\theta)$ for $\theta \in \mathscr{D}_\delta^{*,c}$ (see (3.2)).

Now let $V^{n,\delta}$ be the solutions of (3.4) of class (Ψ). We first show that

$$\lim_{n\to\infty} \|V^{n,\delta} - V^\delta\|_{L^\infty(D_\delta)} = 0.$$

Indeed, if not, then there exist $\varepsilon_0 > 0$, $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$, and $\{\theta_k := (t_k, x_k, w_k)\}_{k\in\mathbb{N}} \subset D_\delta$ such that $n_k \uparrow \infty$ as $k \to \infty$ and

$$|V^{n_k,\delta}(t_k, x_k, w_k) - V^\delta(t_k, x_k, w_k)| > \varepsilon_0.$$

By definition of $\bar{D}_\delta$ we see that, taking a subsequence if necessary, we can assume that there exists $\theta_0 := (t_0, x_0, w_0) \in \bar{D}_\delta$ (allowing $x_0 = +\infty$) such that $\theta_k \to \theta_0$. Now let $k \to \infty$. If $x_0 < +\infty$, then we have $\tilde{V}_\delta(\theta_0) - V^\delta(\theta_0) \geq \varepsilon_0$ or $\tilde{V}^\delta(\theta_0) - V^\delta(\theta_0) \leq -\varepsilon_0$, which contradicts the fact that $\tilde{V}_\delta = \tilde{V}^\delta = V^\delta$ in D_δ. If $x_0 = +\infty$, then we have $\tilde{V}_\delta(t_0, N, w_0) - V^\delta(t_0, N, w_0) \geq \varepsilon_0$ or $\tilde{V}^\delta(t_0, N, w_0) - V^\delta(t_0, N, w_0) \leq -\varepsilon_0$ for some $N > 0$, also a contradiction. This proves the claim.

Next, let us denote $a_\delta := \sup_{\theta\in D_\delta\setminus D} |V^\delta(\theta) - V(\theta)|$. Then, noting that $\Psi = V = \mathfrak{u}$ on ∂D_δ, for $\bar{\theta} = (t, y, v) \in \partial D_\delta$, we have

$$\begin{aligned} a_\delta = \sup_{\theta\in D_\delta\setminus D} |V^\delta(\theta) - \psi(\bar{\theta}) + \psi(\bar{\theta}) - V(\theta)| &\leq \sup_{\theta\in D_\delta\setminus D} [|V^\delta(\theta) - V^\delta(\bar{\theta})| + |\psi(\bar{\theta}) - \psi(\theta)|] \\ \leq \sup_{\theta\in D_\delta\setminus D} [\omega(|\theta - \bar{\theta}|) + |\psi(\bar{\theta}) - \psi(\theta)|] &= o_\delta(1) \qquad \text{as } \delta \to 0. \end{aligned}$$

Here $\omega(\cdot)$ is the modulus of continuity of $V^{n,\delta}$ (which can be chosen to be independent of δ(!)). Furthermore, it is easy to verify that $V^\delta - a_\delta$ and $V^\delta + a_\delta$ are viscosity subsolution and viscosity supersolution of (4.3), respectively, and $V^\delta - a_\delta \leq V \leq V^\delta + a_\delta$ on ∂D. It then follows from the comparison principle that $\|V^\delta - V\|_{L^\infty(D)} = o_\delta(1)$ as $\delta \to 0$.

Combining the above, for $\varepsilon > 0$, we can first choose $\delta = \delta(\varepsilon) > 0$ so that $\|V^\delta - V\|_{L^\infty(D)} < \varepsilon/2$ and then choose $n = n(\delta(\varepsilon)) \in \mathbb{N}$ such that $\|V^{n,\delta} - V^\delta\|_{L^\infty(D)} \leq \|V^{n,\delta} - V^\delta\|_{L^\infty(D_\delta)} < \varepsilon/2$. We note that $V^{n,\delta} \in \mathbb{C}^{2+\alpha}_{loc}(D_\delta)$, thanks to Theorem 4.1 and Remark 4.2. The proof is now complete. ☐

5. Construction of ε-optimal strategy.

We are now ready to construct the desired ε-optimal strategy. The idea is simple: for each $\varepsilon > 0$, we choose an approximating solution $V^{n,\delta}$, guaranteed by Theorem 4.5, and define a strategy in the form of (1.3). It is then reasonable to believe that such a strategy should be ε-optimal.

To be more precise, let $\{\varepsilon_k\}$ be any sequence such that $\varepsilon_k \downarrow 0$ as $k \to \infty$, and let $V^k := V^{n_k,\delta_k} \in \mathbb{C}^2_{loc}(D_{\delta_k})$ be the corresponding solutions of (3.4) as those in Theorem 4.5. That is,

$$\|V^{n_k,\delta_k} - V\|_{L^\infty(D)} < \varepsilon_k \to 0 \qquad \text{as } k \to \infty. \tag{5.1}$$

Since $V(\theta) \equiv 0$ for $\theta \in D^c$, we can and shall assume that $V^k(\theta) \equiv 0$ for $\theta \in D^c$ for all k. Furthermore, since each $V^{n,\delta}$ is of class (Ψ) for some Ψ satisfying Hypothesis 3.1, we can assume $V^{n,\delta}_{x+}(s, -\delta, w) > 1$. Therefore $V^k_{x+}(s, 0, w) > 1$ for large k.

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We now make the candidate optimal strategy (1.3) more specific. Consider the sequence of strategies $\{(\gamma^k, a^k)\}_{k\in\mathbb{N}}$:

$$\left\{\begin{array}{l} \gamma^k_t := \mathbf{1}_{\{V^k_{xx}(t,X_t,W_t)\ge 0\}} + (\Gamma^k(t,X_t,W_t)\wedge 1)\mathbf{1}_{\{V^k_{xx}(t,X_t,W_t)<0\}},\\ a^k_t := \Xi^k(t,X_t,W_t), \end{array}\right. \tag{5.2}$$

where for each $k\in\mathbb{N}$ and $(t,x,w)\in D$,

$$\begin{aligned} \Gamma^k(s,x,w) &:= -\frac{(\mu-r)V^k_x(s,x,w)}{\sigma^2 x V^k_{xx}(s,x,w)};\\ \Xi^k(t,x,w) &:= M\mathbf{1}_{\{V^k_x(s,x,w)<1\}} + p\mathbf{1}_{\{V^k_x(s,x,w)=1\}}, \end{aligned} \tag{5.3}$$

and (X^k, W) is the, say, weak solution to the closed-loop dynamics of the reserve (recall (2.2)), defined on some probability space $(\Omega,\mathcal{F},\mathbb{P},\mathbb{F})$:

$$\left\{\begin{array}{ll} dX_t = b^k(t,X_t,W_t)dt + \sigma^k(t,X_t,W_t)dB_t - dQ^{s,w}_t, & X_s = x;\\ W_t = w + (t-s) - (\sigma_{N_t} - \sigma_{N_s}), & 0\le s\le t\le T. \end{array}\right. \tag{5.4}$$

Here, denoting $\theta := (s,x,w)\in D$, we have (noting that $V^k_x(\theta)>0$ for $\theta\in D$)

$$b^k(\theta) := \left\{\begin{array}{ll} p + rx - (\mu-r)\Gamma^k(\theta)x - \Xi^k(\theta), & 0<\Gamma^k(\theta)\le 1,\\ p+\mu x - \Xi^k(\theta) & \text{otherwise}; \end{array}\right. \tag{5.5}$$

$$\sigma^k(\theta) := \left\{\begin{array}{ll} \sigma x\Gamma^k(\theta), & 0<\Gamma^k(\theta)\le 1,\\ \sigma x & \text{otherwise}. \end{array}\right. \tag{5.6}$$

We observe that the function Γ^k in (5.3) is continuous. In fact, by a further approximation (cf. [13]) if necessary, we can even assume further that Γ^k is Lipschitz continuous (with Lipschitz constant depending on k). The function Ξ^k, on the other hand, presents some "barrier" nature, and its discontinuity in the state variable x causes some main difficulties in the closed-loop analysis.

In the rest of the paper we shall verify two main results: (i) the closed-loop system (5.4) is well-posed, and (ii) (γ^k, a^k) provides an ε-optimal strategy for k large. We note that the discontinuous nature of the function Ξ^k, as well as the presence of jumps, makes finding the strong solution to SDE (5.4) a rather involved task. Our plan of attack is the following. We shall begin by looking at the *weak solution* to (5.4). Then using the fact that the SDE is one-dimensional, we shall argue that the weak solution is actually strong and is pathwise unique, up to the ruin time $\tau = \inf\{t>0, X_t<0\}$, following a scheme initiated by [10] (see also [6, 15]).

To this end, let us modify the function σ^k slightly: for $m\in\mathbb{N}$, we consider $\varphi^m(x) = \frac{1}{m}\vee x\wedge m$ and define $\sigma^{m,k}(\theta) := \sigma\varphi^m(x)\Gamma^k(\theta)$, $\theta\in D$. Since both φ^m and Γ^k are bounded and Lipschitz, so is $\sigma^{m,k}$. Furthermore, it is readily seen that for some constant $c_m>0$, one has

$$0<c_m\le\sigma^{m,k}(\theta)\le\sigma(x\wedge m),\quad \theta:=(s,x,w)\in D. \tag{5.7}$$

To continue our discussion we shall now consider the *canonical space*. Let $\Omega^1 = \mathbb{C}([0,T])$, the space of all continuous functions, null at zero, and endowed with the usual sup-norm. Let $\mathcal{F}^1_t \overset{\triangle}{=} \sigma\{\omega(\cdot\wedge t)|\,\omega\in\Omega^1\}$, $t\ge 0$, $\mathcal{F}^1\overset{\triangle}{=}\mathcal{F}^1_T$, $\mathbb{F}^1 = \{\mathcal{F}^1_t\}_{t\in[0,T]}$, and $\mathbb{P}^0$ be the Wiener measure on $(\Omega^1,\mathcal{F}^1)$ so that the *canonical process* $B_t(\omega)\overset{\triangle}{=}\omega^1(t)$,

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$(t,\omega^1)\in[0,T]\times\Omega^1$ is a $(\mathbb{P}^0,\mathbb{F}^1)$-Brownian motion. Let $\Omega^2=\mathbb{D}([0,T])$, the space of all real-valued, càdlàg (right-continuous with left limit) functions, endowed with the Skorohod topology, and similarly define $\mathbb{F}^2=\{\mathcal{F}^2_t\}_{t\in[0,T]}$ and $\mathcal{F}^2\stackrel{\triangle}{=}\mathcal{F}^2_T$. Let $\mathbb{P}^Q$ be the law of the renewal claim process Q on $\mathbb{D}([0,T])$ so that the coordinate process $Q_t(\omega^2)=\omega^2(t)$, $(t,\omega^2)\in[0,T]\times\Omega^2$. Now we consider the product space:

$$\Omega\stackrel{\triangle}{=}\Omega^1\times\Omega^2;\quad \mathcal{F}\stackrel{\triangle}{=}\mathcal{F}^1\otimes\mathcal{F}^2;\ \mathbb{P}\stackrel{\triangle}{=}\mathbb{P}^0\otimes\mathbb{P}^Q;\ \mathcal{F}_t\stackrel{\triangle}{=}\mathcal{F}^1_t\otimes\mathcal{F}^2_t,\quad t\in[0,T]. \tag{5.8}$$

We now consider the following SDE on the canonical space $(\Omega^1,\mathcal{F}^1,\mathbb{P}^0;\mathbb{F}^1)$:

$$\begin{cases} dX_t=\sigma^{m,k}(t,X_t,W_t)dB_t-dQ_t, & X_0=x;\\ W_t=t-\sigma_{N_t}, \end{cases}\qquad t\in[0,T]. \tag{5.9}$$

We have the following result.

Proposition 5.1. *Under Hypothesis* 2.1*, the SDE* (5.9) *has a strong solution.*

Proof. We write the element of Ω as $\omega=(\omega^1,\omega^2)\in\Omega$. Then, the two marginal coordinate processes are defined by $B_t(\omega)\stackrel{\triangle}{=}\omega^1(t)$, $Q_t(\omega)\stackrel{\triangle}{=}\omega^2(t)$, $(t,\omega)\times[0,T]\times\Omega$. Then under our hypotheses B and Q are independent, and the process $Q_t(\omega)=\omega^2(t)$ is piecewise constant jumping at $0<\sigma_1(\omega^2)<\cdots<\sigma_{N_T(\omega^2)}(\omega^2)<T$, where $N_t(\omega^2)$ denotes the number of jumps of Q up to time t and hence is a renewal counting process. We then define $W_t(\omega)=t-\sigma_{N_t(\omega^2)}(\omega^2)$, $t\geq 0$.

Now on the canonical process, for $\mathbb{P}^Q$-a.s. $\omega^2\in\Omega^2$ we define

$$\tilde{\sigma}^{m,k,\omega^2}(t,x):=\sigma^{m,k}(t,x-\omega^2(t),t-\sigma_{N_t(\omega^2)}(\omega^2)),\ (t,x)\in[0,T]\times\mathbb{R}, \tag{5.10}$$

and consider the SDE on the space $(\Omega^1,\mathcal{F}^1,\mathbb{P}^0;\mathbb{F}^1)$:

$$d\tilde{X}_t=\tilde{\sigma}^{\omega^2,m,k}(t,\tilde{X}_t)dB_t,\quad \tilde{X}_0=x,\quad t\in[0,T]. \tag{5.11}$$

Clearly, by definition (5.10) and the facts (5.7) and that $\sigma^{m,k}$ is Lipschitz, SDE (5.11) has a unique strong solution $\tilde{X}^{\omega^2}_t:=\tilde{X}_t(\cdot,\omega^2)$ on $(\Omega^1,\mathcal{F}^1,\mathbb{P}^0;\mathbb{F}^1)$ for $\mathbb{P}^Q$-a.s. $\omega^2\in\Omega^2$. Consequently, by (5.10), if we define $X:=\tilde{X}-Q$ and $W_t=t-\sigma_{N_t}$, then (X,W) satisfies (5.9).

The uniqueness of the solution (X,W) follows from that of $\tilde{X}$ as Q is a coordinate process, completing the proof. ∎

Now let (X,W) be a strong solution of (5.9) on $(\Omega,\mathcal{F},\mathbb{P})$, and denote it by $(X^{m,k},W)$ if the dependence on m,k is important. Clearly, for fixed $\omega^2\in\Omega^2$, $X^{m,k}_t(\omega)=\tilde{X}^{\omega^2}_t-\omega^2(t)$. It is well-known (cf., e.g., [3] and [11]) that the solution $\tilde{X}^{\omega^2}$ of (5.11) has a transition density, denoted by $p^{\omega^2}(t,y;s,x)$ to indicate its dependence on ω^2, and it satisfies

$$p^{\omega^2}(t,y;s,x)\leq M_0|t-s|^{-\frac{1}{2}}\exp\left\{\frac{-\Lambda(y-x)^2}{t-s}\right\},\quad s\leq t,\ x,y\in\mathbb{R}, \tag{5.12}$$

where constants M_0 and Λ depend only on m,k but are independent of ω^2. Consequently, for fixed $\omega^2\in\Omega^2$, $X^{m,k}(\cdot,\omega^2)$ has the density function $p^{\omega^2}(t,y+\omega^2(t);s,x)$ under $\mathbb{P}^0$. Furthermore, by renewal theory (see, e.g., [18]), the random variable σ_{N_t} has a density function

$$f_{\sigma_{N_t}}(u)=\bar{F}(t-u)m'(u)=\bar{F}(t-u)\sum_{n=1}^{\infty}f_n(u),\qquad t\geq u\geq 0, \tag{5.13}$$

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where $m(t) = \mathbb{E}[N_t] = \sum_{n=1}^{\infty} F_n(t)$, F is the law of the waiting time T_i's, F_n is the n-fold convolution of F with itself, and f_n is corresponding density function. Therefore, we can write down the joint distribution of $(X_t^{m,k}, \sigma_{N_t})$:

$$\mathbb{P}(X_t^{m,k} \in A, \sigma_{N_t} \in B) = \int_{\Omega^1}\int_{\Omega^2} \mathbf{1}_{\{X_t^{m,k}(\omega^1,\omega^2)\in A\}} \mathbf{1}_{\{\sigma_{N_t}(\omega^2)\in B\}} \mathbb{P}^0(d\omega^1)\mathbb{P}^Q(d\omega^2)$$
$$= \int_{\Omega^2} \left[\int_A p^{\omega^2}(t, y+\omega^2(t); s, x)dy\right] \mathbf{1}_{\{\sigma_{N_t}(\omega^2)\in B\}} \mathbb{P}^Q(d\omega^2). \tag{5.14}$$

In what follows we shall make use of an extra assumption on the jump times σ_{N_t}.

Hypothesis 5.2. There exists a constant $\gamma^{'} > 1$ such that

$$\int_0^T \int_0^t t^{\frac{1-\gamma'}{2}} f_{\sigma_{N_t}}^{\gamma'}(u)dudt < +\infty. \tag{5.15}$$

Remark 5.3. We remark that the Assumption 5.2 is merely technical, but it covers a large class of cases that are commonly seen in applications. In particular, we note that if we take $\frac{3-\gamma^{'}}{2} > -1$, then $\gamma^{'} < 5$. Furthermore, if T_i is of exponential distribution with λ (that is, the renewal process N becomes Poisson), then $m(t) = \mathbb{E}N(t) = \lambda t$ and $f_{\sigma_{N_t}}(u) = \lambda e^{-\lambda(t-u)}$. Then,

$$\int_0^T \int_0^t t^{\frac{1-\gamma'}{2}} f_{\sigma_{N_t}}^{\gamma'}(u)dudt = \int_0^T \int_0^t t^{\frac{1-\gamma'}{2}} (\lambda e^{-\lambda(t-u)})^{\gamma'} dudt$$
$$\leq \int_0^T \int_0^t t^{\frac{1-\gamma'}{2}} \lambda^{\gamma'} dudt = \frac{2\lambda^{\gamma'}}{5-\gamma^{'}} T^{\frac{5-\gamma'}{2}}.$$

Also, if $T_i \sim$ Erlang(k,λ), that is, $F(u,k,\lambda) = 1 - \Sigma_{i=0}^{k-1} \frac{1}{i!} e^{-\lambda x}(\lambda x)^i$, as we often see in the Sparre Andersen models, then $\sum_{n=1}^{\infty} f_n(u,k,\lambda) \leq \sum_{n=1}^{\infty} f_n(u,1,\lambda) = \lambda$, and one can check that

$$\int_0^T \int_0^t t^{\frac{1-\gamma'}{2}} f_{\sigma_{N_t}}^{\gamma'}(u)dudt \leq \frac{2\lambda^{\gamma'}}{5-\gamma^{'}} T^{\frac{5-\gamma'}{2}}.$$

In both cases Hypothesis 5.2 holds.

6. Strong well-posedness of the closed-loop system. We now ready to study the existence and (pathwise) uniqueness of the closed-loop system (5.4). Again, for each $m \in \mathbb{N}$ we consider the "truncated" version of b^k: $b^{m,k}(t,x,w) := \beta^k(t, -m \vee x \wedge m, w)$. Then $b^{m,k}$ is a bounded and measurable function. Let $(X^{m,k}, W)$ be the strong solution of (5.9) on $(\Omega^1, \mathbb{F}^1, \mathbb{P}^0)$, and for $\omega^2 \in \Omega^2$, define

$$\theta_t^{m,k}(\cdot,\omega^2) := \frac{b^{m,k}(t, X_t^{m,k}(\cdot,\omega^2), t - \sigma_{N_t(\omega^2)}(\omega^2))}{\sigma^{m,k}(t, X_t^{m,k}(\cdot,\omega^2), t - \sigma_{N_t(\omega^2)}(\omega^2))}. \tag{6.1}$$

Since $b^{m,k}$ is bounded, by (5.7) we see that, modulo a $\mathbb{P}^Q$-null set $N^2 \subset \Omega^2$, $\theta^{m,k}(\cdot,\omega^2)$ is a bounded, $\mathbb{F}^1$-adapted process, for all $\omega^2 \in \Omega^2 \setminus N^2$. We can then define the following exponential martingale on $(\Omega^1, \mathcal{F}^1, \mathbb{P}^0; \mathbb{F}^1)$:

$$L_t^{m,k}(\cdot,\omega^2) := \exp\left\{\int_0^t \theta_s^{m,k}(\cdot,\omega^2)dB_s - \frac{1}{2}\int_0^t |\theta_s^{m,k}(\cdot,\omega^2)|^2 ds\right\}, \omega^2 \notin N^2, \tag{6.2}$$

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and a new probability measure $\tilde{\mathbb{P}}^{m,k}$ on $(\Omega,\mathcal{F})$ by

$$(6.3)\ \tilde{\mathbb{P}}^{m,k}(A_1\times A_2):=\int_{A_2}\int_{A_1}L_T^{m,k}(\omega^1,\omega^2)\mathbb{P}^0(d\omega^1)\mathbb{P}^Q(d\omega^2),\ A_1\in\mathcal{F}^1, A_2\in\mathcal{F}^2.$$

Then, it is readily seen that, under $\tilde{\mathbb{P}}^{m,k}$, $\tilde{B}_t^{m,k}:=B_t-\int_0^t\theta_s^{m,k}ds$, $t\in[0,T]$, is a Brownian motion, still independent of Q, and on the space $(\Omega,\mathcal{F},\tilde{\mathbb{P}}^{m,k})$, $(X^{m,k},W)$ satisfies, for $t\in[0,T]$,

$$(6.4)\ \begin{cases} dX_t^{m,k}=b^{m,k}(t,X_t^{m,k},W_t)dt+\sigma^{m,k}(t,X_t^{m,k},W_t)d\tilde{B}_t^{m,k}-dQ_t, X_0^{m,k}=x,\\ W_t=t-\sigma_{N_t}.\end{cases}$$

In other words, $(\Omega,\mathcal{F},\tilde{\mathbb{P}}^{m,k},\tilde{B}^{m,k},X^{m,k},W)$ is a *weak solution* to a truncated version of (5.4). Our task in this subsection is to show that this weak solution can actually be strong and that it is pathwise unique. Furthermore, we shall argue that, as $m\to\infty$, the sequence $\{X^{m,k}\}$ would converge to a process X^k, which satisfies the SDE (5.4) on the interval $[0,\tau_k)$, where $\tau_k:=\inf\{t>0: X_t^k<0\}$. This is clearly sufficient for our purpose.

We should note that since the coefficient $b^{m,k}$ is discontinuous, the pathwise unique strong solution is only possible because the SDE (5.4) is one-dimensional. Our argument borrows the idea initiated in [10] (see also, e.g., [6]), using the so-called Krylov estimate (cf. [12]). To this end, let us begin with some observations. Let $(X^{m,k},W,B)$ be any weak solution of SDE (6.4) defined on some filtered probability space $(\Omega,\mathcal{F},\mathbb{P};\mathbb{F})$; we may assume that $(\Omega,\mathcal{F})$ is the canonical space defined before, except that $\mathbb{P}$ is any probability measure, and $\mathbb{F}$ is augmented by all the $\mathbb{P}$-null sets. Recalling θ and M defined by (6.1) and (6.2), respectively, define $\bar{\theta}:=-\theta$ and $\bar{L}:=L^{-1}$. Note that the process θ actually depends on ω^2; namely, we should have $\theta=\theta^{\omega^2}$ for $\omega^2\in\Omega^2$ and hence $L=L^{\omega^2}$ as well. We now define, for fixed ω^2, a new probability measure $\frac{d\mathbb{P}^{0,\omega^2}}{d\mathbb{P}}\big|_{\mathcal{F}_T^1}=\bar{L}_T^{\omega^2}$ on $(\Omega^1,\mathcal{F}^1)$, so that $B_t^0:=B_t-\int_0^t\bar{\theta}_s^{\omega^2}ds$, $t\ge 0$, is a Brownian motion on $(\Omega^1,\mathcal{F}^1,\mathbb{P}^{0,\omega^2})$. We next define a new probability measure $\bar{\mathbb{P}}$ on $(\Omega,\mathcal{F})$ such that for $A\in\mathcal{F}^1$, $B\in\mathcal{F}^2$,

$$(6.5)\qquad \bar{\mathbb{P}}(A\times B)=\int_B\int_A\mathbb{P}^{0,\omega^2}(d\omega^1)\mathbb{P}^Q(d\omega^2)=\int_B\int_A\bar{L}_T^{\omega^2}(\omega^1)\mathbb{P}(d\omega^1\otimes d\omega^2).$$

Then it is readily seen that $\bar{L}_t(\omega)=\bar{L}_t(\omega^1,\omega^2):=[L^{\omega^2}]^{-1}(\omega^1)$, $t\in[0,T]$ is a martingale under $\bar{\mathbb{P}}$, $\frac{d\bar{\mathbb{P}}}{d\mathbb{P}}\big|_{\mathcal{F}_T}=\bar{L}_T$, and $(X^{m,k},W,B^0)$ solves SDE (5.9) on the space $(\Omega,\mathcal{F},\bar{\mathbb{P}})$. We are now ready to prove the following *Krylov estimate*.

LEMMA 6.1. *Assume Hypothesis* 2.1 *and Hypothesis* 5.2. *Let $X^{m,k}$ be a weak solution of SDE* (6.4). *Then, for any bounded and measurable function $g:[0,T]\times[0,+\infty)\times[0,T]\to\mathbb{R}_+$, it holds that*

$$(6.6)\qquad \mathbb{E}\int_0^T g(t,X_t,W_t)dt\le G\left\{\int_0^T\int_{\mathbb{R}}\int_0^t g^{\beta\gamma}(t,y,t-u)dydudt\right\}^{1/\beta\gamma}.$$

Here in the above G is a constant defined by

$$(6.7)\qquad G=C(M_0,\Lambda,\gamma',\beta)\{\bar{\mathbb{E}}\bar{L}_T^{-\alpha}\}^{1/\alpha}\left[\int_0^T\int_0^t t^{\frac{1-\gamma'}{2}}f_{\sigma_{N_t}}^{\gamma'}(u)dudt\right]^{\frac{1}{\beta\gamma'}},$$

where $\bar{\mathbb{E}}=\mathbb{E}^{\bar{\mathbb{P}}}$, and $\bar{L}_T=\frac{d\bar{\mathbb{P}}}{d\mathbb{P}}$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, and γ' is given in Hypothesis 5.2.

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Proof. Throughout this proof we fix m and k and thus omit them in the notation for simplicity. For any bounded, nonnegative measurable function $g : [0,T]\times[0,+\infty)\times[0,T] \to \mathbb{R}_+$ and any $\beta > 1$ we have

$$\begin{aligned}(6.8)\qquad &\bar{\mathbb{E}}\left[\int_0^T g^\beta(t, X_t, t-\sigma_{N_t})dt\right]\\ &= \int_0^T\int_{\Omega^2}\left[\int_{\mathbb{R}} g^\beta(t,y,t-\sigma_{N_t}(\omega^2))p^{\omega^2}(t,y+\omega^2(t),0,x)dy\right]\mathbb{P}^Q(d\omega^2)dt\\ &\le \int_0^T\int_{\Omega^2}\left[\int_{\mathbb{R}} g^\beta(t,y,t-\sigma_{N_t}(\omega^2))M_0|t|^{-\frac{1}{2}}e^{\frac{-\Lambda(y+\omega^2(t)-x)^2}{t}}dy\right]\mathbb{P}^Q(d\omega^2)dt.\end{aligned}$$

Note that, by Hölder's inequality again, we have

(6.9)

$$\begin{aligned}&\int_{\mathbb{R}} g^\beta(t,y,t-\sigma_{N_t}(\omega^2))M_0|t|^{-\frac{1}{2}}\exp\left\{\frac{-\Lambda(y+\omega^2(t)-x)^2}{t}\right\}dy\\ &\le \left[\int_{\mathbb{R}} g^{\beta\gamma}(t,y,t-\sigma_{N_t}(\omega^2))dy\right]^{\frac{1}{\gamma}}\left[\int_{\mathbb{R}}\left(M_0|t|^{-\frac{1}{2}}\exp\left\{\frac{-\Lambda(y+\omega^2(t)-x)^2}{t}\right\}\right)^{\gamma'}dy\right]^{\frac{1}{\gamma'}},\end{aligned}$$

where $1/\gamma + 1/\gamma' = 1$. By the direct calculation, we have

$$(6.10)\qquad \int_{\mathbb{R}}\left(M_0|t|^{-\frac{1}{2}}\exp\left\{\frac{-\Lambda(y+\omega^2(t)-x)^2}{t}\right\}\right)^{\gamma'}dy \le C(M_0,\Lambda,\gamma')|t|^{\frac{1-\gamma'}{2}},$$

where $C(M_0,\Lambda,\gamma')$ is some constant depending only on M_0, Λ, and γ'. Keeping (6.8), (6.9), and (6.10) in mind, we have

$$\begin{aligned}&\mathbb{E}\left\{\int_0^T g(t,X_t,W_t)dt\right\} = \bar{\mathbb{E}}\left\{\bar{L}_T^{-1}\int_0^T g(t,X_t,W_t)dt\right\}\\ &\le \{\bar{\mathbb{E}}\bar{L}_T^{-\alpha}\}^{\frac{1}{\alpha}}\left\{\bar{\mathbb{E}}\left[\int_0^T g^\beta(t,X_t,t-\sigma_{N_t})dt\right]\right\}^{\frac{1}{\beta}}\\ &\le \{\bar{\mathbb{E}}\bar{L}_T^{-\alpha}\}^{\frac{1}{\alpha}}\left\{\int_0^T\int_{\Omega^2}\left[\int_{\mathbb{R}} g^{\beta\gamma}(t,y,t-\sigma_{N_t}(\omega^2))dy\right]^{\frac{1}{\gamma}}C(M_0,\Lambda,\gamma')|t|^{\frac{1-\gamma'}{2\gamma'}}\mathbb{P}^Q(d\omega^2)dt\right\}^{\frac{1}{\beta}}\\ &\le C(M_0,\Lambda,\gamma',\beta)\{\bar{\mathbb{E}}\bar{L}_T^{-\alpha}\}^{\frac{1}{\alpha}}\left\{\int_0^T\int_0^t\left[\int_{\mathbb{R}} g^{\beta\gamma}(t,y,t-u)dy\right]^{\frac{1}{\gamma}}|t|^{\frac{1-\gamma'}{2\gamma'}}f_{\sigma_{N_t}}(u)dudt\right\}^{\frac{1}{\beta}}\\ &\le G\left[\int_0^T\int_0^t\int_{\mathbb{R}} g^{\beta\gamma}(t,y,t-u)dydudt\right]^{\frac{1}{\beta\gamma}},\end{aligned}$$

where $C(M_0,\Lambda,\gamma',\beta) := C^{\frac{1}{\beta}}(M_0,\Lambda,\gamma')$ and G is defined by (6.7). This proves (6.6), whence the lemma. ∎

We are now ready to prove that, for fixed m, k, SDE (6.4) actually has a pathwise unique strong solution on the interval $[0,\tau_{m,k})$, where $\tau_{m,k} := \inf\{t > 0 : X_t^{m,k} < 0\}$. For notational simplicity, we again fix m and k and denote $b = b^{m,k}$ and $\sigma = \sigma^{m,k}$, so that (6.4) now reads

$$(6.11)\qquad \begin{cases} dX_t = b(t,X_t,W_t)dt + \sigma(t,X_t,W_t)dB_t - dQ_t, & X_0 = x;\\ W_t = t-\sigma_{N_t}, & t\in[0,T].\end{cases}$$

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Recalling from (5.3) and (5.5) that the function $b = b^{m,k}$ is discontinuous but has a linear growth,

$$|b(t,x,w)| \le C(1+|x|), \qquad (t,x,w) \in [0,T]\times\mathbb{R}\times[0,T] \tag{6.12}$$

for some constant $C > 0$ depending only on the coefficients but independent of m,k. In what follows we shall allow such generic constant to vary from line to line.

The scheme for constructing the strong solution for (6.11) goes as follows (see, e.g., [10, 15] or [6]). For any $N > 0$ define $b_N(t,x,w) = b(t, x\wedge N \vee (-N), w)$. Then (6.12) implies that b_N is a bounded measurable function. Let ρ be a smooth mollifier with compact support in $\mathbb{R}$ such that $\int_{\mathbb{R}} \rho(z)dz = 1$. For $n = 1, 2 \ldots$, define

$$b_{N,j}(t,x,w) = j\int b_N(t,z,w)\rho(j(x-z))dz;$$

then $b_{N,j}$'s are smooth functions, having the same bound N, and satisfying the linear growth condition (6.12) with the same constant $C > 0$, and $b_{N,j} \to b_N$ almost everywhere on $[0,T]\times\mathbb{R}\times[0,T]$ as $j\to\infty$.

Next, for any $K \in \mathbb{N}$ and $j \le K$ we define $\tilde{b}_{N,j,K} \triangleq \bigwedge_{k=j}^{K} b_{N,j}$ and $\tilde{b}_{N,j} \triangleq \bigwedge_{k=j}^{\infty} b_{N,j}$, where $a\wedge b = \min\{a,b\}$. Then clearly, each $\tilde{b}_{N,j,K}$ is continuous, and uniformly Lipschitz in x, uniformly in (t,w). Furthermore, for almost all x, for any (t,w), it holds that $\tilde{b}_{N,j,K} \downarrow \tilde{b}_{N,j}$ as $K\to\infty$ and $\tilde{b}_{N,j} \uparrow b_N$ as $j\to\infty$. Now let us fix N, j, and K and consider the following SDE:

$$\begin{cases} dY_t = \tilde{b}_{N,j,K}(t,Y_t,W_t)dt + \sigma(t,Y_t,W_t)dB_t, & Y_0 = x; \\ W_t = t - \sigma_{N_t}, & t \ge 0. \end{cases} \tag{6.13}$$

Clearly, (6.13) has a unique strong solution; denote it by $\tilde{Y}^{N,j,K}$. By the standard comparison theorem, we see that $\{\tilde{Y}^{N,j,K}\}$ is decreasing with K, and thus we can define $\tilde{Y}^{N,j}_t \triangleq \lim_{K\to\infty} \tilde{Y}^{N,j,K}_t$, $t\in[0,T]$, $\mathbb{P}$-a.s. Since $\tilde{b}^{N,j}$'s and σ are bounded, one can easily check that $\tilde{Y}^{N,j}_t < \infty$, $\mathbb{P}$-a.s. We shall argue that the limiting process $\tilde{Y}^{N,j}$ solves the SDE:

$$\begin{cases} dY_t = \tilde{b}_{N,j}(t,Y_t,W_t)dt + \sigma(t,Y_t,W_t)dB_t, & Y_0 = x; \\ W_t = t - \sigma_{N_t}, \end{cases} \qquad t \ge 0. \tag{6.14}$$

To see this, we first need the following crucial lemma.

Lemma 6.2. *Suppose that Hypothesis* 2.1 *and Hypothesis* 5.2 *are in force. Assume also that* $\{\hat{b}_K\}_{n=1}^{\infty}$ *are measurable functions defined on* $[0,T]\times\mathbb{R}\times[0,T]$*, bounded uniformly in* K*, and there exists a measurable function* $\hat{b}$ *such that*

$$\lim_{K\to\infty} \hat{b}_K(s,x,w) = \hat{b}(s,x,w) \quad \text{for } \textit{a.e. } (s,x,w)\in[0,T]\times\mathbb{R}\times[0,T].$$

Suppose that for each K*,* $(\hat{Y}^K, W)$ *is a strong solution of* (6.13) *with drift being replaced by* $\hat{b}_K$ *and that there exists* $\hat{Y}$ *such that for every* $t\in[0,T]$*,* $\lim_{K\to\infty}\hat{Y}^K_t = \hat{Y}_t$*,* $\mathbb{P}$*-a.s. Then, it holds that*

$$\lim_{K\to\infty} \mathbb{E}\left[\int_0^T |\hat{b}_K(t,\hat{Y}^K_t,W_t) - \hat{b}(t,\hat{Y}_t,W_t)|ds\right] = 0. \tag{6.15}$$

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Proof. The proof of lemma follows the almost identical arguments of those in [15] or [6], with the help of the Krylov estimate established in Lemma 6.1. We leave it to the interested reader. ∎

Let us fix N, j and denote $\hat{b}_K = \tilde{b}_{N,j,K}$, $\hat{Y}^K = \tilde{Y}^{N,j,K}$, $K \in \mathbb{N}$, and $\hat{b} = \tilde{b}_{N,j}$, $\hat{Y} = \tilde{Y}^{N,j}$. Then Lemma 6.1 shows that, possibly along a subsequence and may assume itself, we have

$$(6.16)\quad \lim_{K\to\infty}\int_0^t \tilde{b}_{N,j,K}(s, \tilde{Y}_s^{N,j,K}, W_s)ds = \int_0^t \tilde{b}_{N,j}(s, \tilde{Y}_s^{N,j}, W_s)ds, t \in [0,T], \mathbb{P}\text{-a.s.}$$

Furthermore, since σ is bounded and continuous, the bounded convergence theorem yields that $\lim_{K\to\infty}\mathbb{E}\big[\big|\int_0^T[\sigma(s, \tilde{Y}_s^{N,j,K}, W_s) - \sigma(s, \tilde{Y}_s^{N,j}, W_s)dB_s\big|^2\big] = 0$; thus along a subsequence we have

$$(6.17)\quad \lim_{K\to\infty}\int_0^t \sigma(s, \tilde{Y}_s^{N,j,K}, W_s)dB_s = \int_0^t \sigma(s, \tilde{Y}_s^{N,j}, W_s)dB_s, \quad t \in [0,T], \mathbb{P}\text{-a.s.}$$

Since $\tilde{Y}^{N,j,K}$ solves SDE (6.13) and $\tilde{Y}^{N,j,K} \downarrow \tilde{Y}^{N,j}$, we conclude that $\tilde{Y}^{N,j}$ solves the SDE (6.14).

Next, since $\tilde{Y}^{N,j,K} \le \tilde{Y}^{N,i,K}$, for $j \le i \le K$, we see that $\tilde{Y}^{N,j}$ increases as j increases; thus $\tilde{Y}_t^{N,j} \uparrow Y_t^N$, $t \in [0,T]$, $\mathbb{P}$-almost surely, where Y^N is some process with $Y_t^N < \infty$, $t \in [0,T]$, $\mathbb{P}$-a.s. By the same argument as before, using Lemma 6.2 with $\hat{b}_j = b_{N,j}$, $\hat{b} = b_N$, and $\hat{Y}^j = Y^{N,j}$, we can show that Y^N solves the SDE:

$$(6.18)\quad \begin{cases} dY_t = b_N(t, Y_t, W_t)dt + \sigma(t, Y_t, W_t)dB_t, \quad Y_0 = x; \\ W_t = t - \sigma_{N_t}, \end{cases} \qquad t \in [0,T].$$

Moreover, we can show, as in [6], that Y^N is pathwise unique. Let us now define $\tau_N = \inf\{t : |Y_t^N| \ge N\} \wedge T$. Then on the interval $[0, \tau_N]$, $b_N(t, Y_t^N, W) = b(t, Y_t^N, W)$; thus Y^N is a unique strong solution to the SDE

$$(6.19)\quad \begin{cases} dY_t = b(t, Y_t, W_t)dt + \sigma(t, Y_t, W_t)dB_t, \quad Y_0 = x; \\ W_t = t - \sigma_{N_t}, \end{cases} \qquad t \in [0, \tau_N].$$

Now observe that if $N_1 > N_2$, we have $\tau_{N_1} \ge \tau_{N_2}$. Thus by uniqueness we have $Y_t^{N_2} = Y_t^{N_1}$ on the interval $[0, \tau_{N_2}]$. We can now define a process Y such that $Y_t = Y_t^N$, $t \in [0, \tau_N]$. Then Y is well-defined on the interval $[0, \tau)$, where $\tau = \lim_{N\uparrow\infty}\tau_N$. Since b is of linear growth and σ is bounded, it is not hard to show that $\mathbb{E}[\sup_{t\in[0,T]}|Y_t^N|^2] < \infty$, which implies that $\mathbb{P}\{|Y_t| < \infty, t \in [0, \tau)\} = 1$ and hence $\tau = T$, $\mathbb{P}$-a.s. In other words, Y is a unique strong solution to (6.19) on $[0,T]$.

We can now prove the main result of this section.

THEOREM 6.3. *Assume that the Hypothesis* 2.1 *and Hypothesis* 5.2 *are in force. Then, for each $k > 0$, the closed-loop system* (5.4) *possesses a unique strong solution* (X^k, W) *on the random interval* $[0, \tau_k)$, *where* $\tau_k = \inf\{t > 0 : X^k < 0\} \wedge T$.

Proof. We begin by recalling the SDE (6.4). Without loss of generality we consider only the case $s = 0$; that is, we write SDE (6.4) as

$$(6.20)\quad \begin{cases} dX_t = b^{m,k}(t, X_t, W_t)dt + \sigma^{m,k}(t, X_t, W_t)dB_t - dQ_t, X_0 = x; \\ W_t = t - \sigma_{N_t}, \end{cases} t \in [0,T].$$

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We shall follow the same argument as that in Proposition 5.1 to construct the strong solution on the canonical space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^0; \mathbb{F}^1)$ defined by (5.8). For any $\omega = (\omega^1, \omega^2) \in \Omega$, we write the coordinate processes as $B_t(\omega) \stackrel{\triangle}{=} \omega^1(t)$, $Q_t(\omega) \stackrel{\triangle}{=} \omega^2(t)$, $(t, \omega) \times [0,T] \times \Omega$. Assuming that the process $Q_t(\omega) = \omega^2(t)$ jumps at $0 < \sigma_1(\omega^2) < \cdots < \sigma_{N_T(\omega^2)}(\omega^2) < T$, where $N_t(\omega^2)$ denotes the number of jumps of Q up to time t, we define $W_t(\omega) = t - \sigma_{N_t(\omega^2)}(\omega^2)$, $t \geq 0$.

Now for $\mathbb{P}^Q$-a.s. $\omega^2 \in \Omega^2$ we define $\tilde{b}^{m,k,\omega^2}$ and $\tilde{\sigma}^{m,k,\omega^2}$ by (5.10), respectively, and consider the SDE on the space $(\Omega^1, \mathcal{F}^1, \mathbb{P}^0; \mathbb{F}^1)$:

$$(6.21)\quad d\tilde{X}_t = b^{\omega^2,m,k}(t, \tilde{X}_t)ds + \tilde{\sigma}^{\omega^2,m,k}(t, \tilde{X}_t)dB_t, \quad X_0 = x; \qquad t \in [0,T],$$

Clearly, this equation is the same as (6.19), and we have shown that it has a unique strong solution on $(\Omega^1, \mathcal{F}^1, \mathbb{P}^0; \mathbb{F}^1)$; denote it by $\tilde{X}^{m,k,\omega^2}_t := \tilde{X}^{m,k}_t(\cdot, \omega^2)$ for $\mathbb{P}^Q$-a.s. $\omega^2 \in \Omega^2$. We then define $X^{m,k} := \tilde{X}^{m,k} - Q$ and $W_t = t - \sigma_{N_t}$; then $(X^{m,k}, W)$ is the unique strong solution to (6.20).

To complete the proof, let us define $\tau_{m,k} := \inf\{t > 0, X^{m,k}_t \notin [\frac{1}{m}, m]\} \wedge T$. Again, observe that $b^{m,k}(t, X^{m,k}_t, W) = b^k(t, X^{m,k}_t, W)$ and $\sigma^{m,k}(t, X^{m,k}_t, W) = \sigma^k(t, X^{m,k}_t, W)$. Thus $(X^{m,k}, W)$ is the unique strong solution of (5.4) on $[0, \tau_{m,k}]$. Furthermore, note that if $m_1 > m_2$, then $\tau_{m_1,k} \geq \tau_{m_2,k}$. Thus by uniqueness we have $X^{m_2,k}_t = X^{m_1,k}_t$ on the interval $[0, \tau_{m_2}]$. Thus the process X^k defined by $X^k_t = X^{m,k}_t$, $t \in [0, \tau_{m,k}]$, is well-defined, and with the linear growth of b^k and σ^k, we see that $\mathbb{E}[\sup_{t\in[0,T]} |X^{m,k}_t|^2] < \infty$. We can then conclude that X^k is the unique strong solution of SDE (5.4) on the interval $[0, \tau_k)$, where $\tau_k = \lim_{m\uparrow\infty} \tau_{m,k} = \inf\{t > 0 : X^k < 0\} \wedge T$. ∎

7. Verification of the ε-optimality. Having proved the well-posedness of the closed-loop system (5.4), we now verify that the strategy defined by (5.2) is indeed ε-optimal. That is, it does produce the cost functional V^{n_k,δ_k} as desired. We should note that the auxiliary PIDE (3.4) actually does not correspond to any variation of the original control problem (2.2)–(2.4); the verification is not automatic.

Recall that our ε-optimal strategy is based on the approximating solution $V^{n,\delta}$, guaranteed by Theorem 4.5. More precisely, let $V^k := V^{n_k,\delta_k} \in \mathbb{C}^2_{loc}([0,T] \times \mathbb{R})$ be the solutions of (3.4) as those in Theorem 4.5 such that (5.1) holds. Namely,

$$\|V^k - V\|_{L^\infty(D)} < \varepsilon_k \searrow 0 \qquad \text{as } k \to \infty.$$

Now let us define $\hat{V}^k(s,x,w) = V^k(s,x,w)\mathbf{1}_D(s,x,w)$. Then $\hat{V}^k \in \mathbb{C}^{1,2,1}(D)$, and it follows from (5.1) that $\|\hat{V}^k - V\|_{L^\infty(D)} \to 0$ as $k \to \infty$. Furthermore, by the construction of V^k, we see that $V^k_{x+}(s, -\delta, w) > 1$, and hence $\hat{V}^k_{x+}(s,0,w) = V^k_{x+}(s,0,w) > 1$ for k large enough. We should note that $\hat{V}^k_x(s,x,w) = V^k_x(s,x,w) > 0$ for $(s,x,w) \in D$ always holds.

We now recall the strategy $\pi^k = (\gamma^k, a^k)$ defined by (5.2) and denote X^k as the corresponding strong solution to (2.2), which exists on $[0, \tau^k)$, where $\tau^k := \inf\{t > 0 : X^k_t \notin [0, \infty)\}$. It is useful to remember that π^k is actually the maximizer of the Hamiltonian (2.6), namely, it holds that

$$(7.1)\quad \gamma^k_t = \operatorname*{argmax}_{\gamma\in[0,1]} \left[\frac{1}{2}\sigma^2\gamma^2(X^k_t)^2\hat{V}^k_{xx}(t, X^k_t, W_t) + (\mu - r)\gamma X^k_t \hat{V}^k_x(t, X^k_t, W_t)\right].$$

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In the rest of the section we shall consider, for $s \in [0,T]$, the closed-loop system (5.4) on the interval $[s,T]$ and write it as

$$\text{(7.2)}\quad \begin{cases} dX_t = b^k(t,X_t)dt + \sigma^k(t,X_t)dB_t - dQ_t^{s,w}; & X_s = x;\\ W_t = w + (t-s) - (\sigma_{N_t} - \sigma_{N_s}), \end{cases} \qquad t \in [s,T],$$

where $b^k(t,x) = (p - a_t^k) + [r + (\mu - r)\gamma_t^k]x$; $\sigma^k(t,x) = \gamma_t^k x$, and $\pi^k = (\gamma^k, a^k)$ is the aforementioned approximating strategy. We denote the solution by $X^k = X^{k,s,x}$ and $W = W^{s,w}$ when the context is clear. For given $(s,x,w) \in D$ we define $\tau_s^k := \inf\{t > s : X_t^k \notin [0,\infty)\}$ and denote $\mathbb{E}_{sxw}[\,\cdot\,] := \mathbb{E}[\,\cdot\,|X_s^k = x, W_s = w]$.

To show that the strategy $\pi^k = (\gamma^k, a^k)$ does satisfy the ε-optimality we shall argue that $J(s,x,w;\pi^k)$ satisfies, for $\theta = (s,x,w) \in D$, that $J(\theta;\pi^k) \to V(\theta)$ as $k \to \infty$. But note that $J(\theta;\pi^k) = \mathbb{E}_\theta[\int_0^{\tau_s^k \wedge T} e^{-c(t-s)} a_t^k dt]$ and $\lim_{k\to\infty} \|V^k - V\|_{L^\infty(D)} = 0$; the following theorem would suffice.

THEOREM 7.1. *Assume that Hypothesis* 2.1*and Hypothesis* 5.2 *are in force. Then, uniformly for* $(s,x,w) \in D$, *it holds that*

$$\text{(7.3)}\qquad \lim_{k\to\infty} \left| \mathbb{E}_{sxw}\left[\int_s^{\tau_s^k \wedge T} e^{-c(t-s)} a_t^k dt - \hat{V}^k(s,x,w) \right] \right| = 0.$$

Proof. The proof is straightforward. Applying Itô's formula from s to $\tau_s^k \wedge T$ to $e^{-c(t-s)}\hat{V}^k(t, X_t^k, W_t)$ and then taking expectation on both sides we can easily derive

$$\begin{aligned}
&\mathbb{E}\left[e^{-c(\tau_s^k \wedge T - s)} \hat{V}^k(\tau_s^k \wedge T, X^\kappa_{\tau_s^k \wedge T}, W_{\tau_s^k \wedge T})\right]\\
&\quad = \hat{V}^k(s,x,w) + \mathbb{E}\Bigg[\int_s^{\tau_s^k \wedge T} e^{-c(t-s)} \Big[-c\hat{V}^k + \hat{V}_t^k + \hat{V}_w^k\\
&\qquad + [(p - a_t^k) + (r + (\mu - r)\gamma_t^k) X_t^k]\hat{V}_x^k + \frac{1}{2}\sigma^2 (\gamma_t^k)^2 (X_t^k)^2 \hat{V}_{xx}^k \Big](t, X_t^k, W_t)dt \Bigg]\\
&\qquad + \mathbb{E}\left\{ \int_s^{\tau_s^k \wedge T} e^{-c(t-s)} \frac{f(W_t)}{\overline{F}(W_t)} \left[\int_0^{X_t^k} \hat{V}^k(t, X_t^k - u, 0) g(u) du - \hat{V}^k(t, X_t^k, W_t) \right] dt \right\}.
\end{aligned}$$

Since $\hat{V}^k(s,x,w)$ satisfies the HJB equation (4.3) and $\pi^k = (\gamma^k, a^k)$ is the maximizer in terms of $\hat{V}^k$, a simple calculation shows that (suppressing variables)

$$\begin{aligned}
&-c\hat{V}^k + \hat{V}_t^k + \hat{V}_w^k + [(p - a_t^k) + (r + (\mu - r)\gamma_t^k) X_t^k]\hat{V}_x^k\\
&\qquad + \frac{1}{2}\sigma^2 (\gamma_t^k)^2 (X_t^k)^2 \hat{V}_{xx}^k - \frac{f(W_t)}{\overline{F}(W_t)} \hat{V}^k\\
&\quad = -a_t^k - \frac{f(W_t)}{\overline{F}(W_t)} \int_0^{X_t^k + \delta_k} V^k(t, X_t^k - u, -\delta_k) g(u) du - \frac{\varepsilon_k}{2} \hat{V}_{xx}^k - \frac{\varepsilon_k}{2} \hat{V}_{ww}^k.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mathbb{E}\left[e^{-c(\tau_s^k \wedge T - s)} \hat{V}^k(\tau_s^k \wedge T, X^\kappa_{\tau_s^k \wedge T}, W_{\tau_s^k \wedge T})\right] - \hat{V}^k(s,x,w) + \mathbb{E}\left[\int_s^{\tau_s^k \wedge T} e^{-c(t-s)} a_t^k dt \right]\\
&\quad = \mathbb{E}\Bigg\{ \int_s^{\tau_s^k \wedge T} e^{-c(t-s)} \frac{f(W_t)}{\overline{F}(W_t)} \times \Bigg[\int_0^{X_t^k} [\hat{V}^k(t, X_t^k - u, 0) - V^k(t, X_t^k - u, -\delta_k)] g(u) du\\
&\qquad - \int_{X_t^k}^{X_t^k + \delta_k} V^k(t, X_t^k - u, -\delta_k) g(u) du \Bigg] \Bigg\} - \frac{\varepsilon_k}{2} \mathbb{E}\left[\int_s^{\tau_s^k \wedge T} e^{-c(t-s)} \hat{V}_{xx}^k(t, X_t^k, W_t) dt \right]
\end{aligned}$$

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$$- \frac{\varepsilon_k}{2}\mathbb{E}\left[\int_s^{\tau_s^k\wedge T} e^{-c(t-s)}\hat{V}^k_{ww}(t,X^k_t,W_t)dt\right]$$
$$\le C\delta_k - \frac{\varepsilon_k}{2}\mathbb{E}\left[\int_s^{\tau_s^k\wedge T} e^{-c(t-s)}\big[\hat{V}^k_{xx}(t,X^k_t,W_t)+\hat{V}^k_{ww}(t,X^k_t,W_t)\big]dt\right].$$

Finally, letting $k\to\infty$ and noting that $\delta_k,\varepsilon_k\to 0$, (7.3) follows from the fact that

$$\lim_{k\to\infty}\mathbb{E}_{sxw}\big[\hat{V}^k(\tau_s^k\wedge T, X^k_{\tau_s^k\wedge T}, W_{\tau_s^k\wedge T})\big] = \lim_{k\to\infty}\mathbb{E}_{sxw}\big[\mathbf{1}_{\{\tau_s^k\ge T\}}\hat{V}^k(T,X^k_T,W_T)\big]$$
$$= \lim_{k\to\infty}\mathbb{E}_{sxw}\big[\mathbf{1}_{\{\tau_s^k\ge T\}}V(T,X^k_T,W_T)\big] = 0.$$

This proves the theorem. □

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