Event-Triggered Prediction-Based Delay Compensation Approach

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***Abstract*—We provide a new event-triggered delay compensation approach for linear systems with arbitrarily long constant input delays. Our prediction map is expressible as a solution of a discrete time system. Our method ensures input-to-state stability. We also provide an analog under measurement delays, where the prediction map is expressible as a solution of a continuous-discrete system. Significant novel features are our combined use of matrices of absolute values and our prediction based event triggers, instead of Euclidean norms, and the fact that the predictor dynamics always has the same dimension as that of the original system. Our marine robotic example illustrates an advantage of using our new methods.**

***Index Terms*—Event-triggered, delay, prediction.**

I. INTRODUCTION

EVENT-TRIGGERED control provides the basis for considerable significant ongoing research, owing to the need to take communication constraints into account; see [1] and [5]. One important focus in event-triggered control theory is delay compensation. Chain predictors have been used to compensate for arbitrarily long input delays in event-triggered systems, using a dynamic extension whose dimension grows linearly with the length of the delay [8]. Standard event triggers use standard Euclidean norms to measure deviations of the current state from a reference state.

By contrast, recent synergies of event triggering, chain predictors, and interval observers [3] were based on replacing the usual Euclidean norm by vectors of absolute values; see [8]–[11], which illustrate the benefits of this replacement. Although (as indicated, e.g., in [2]) chain predictors eliminate the need for the distributed terms arising in standard delay compensating predictors by enlarging the dimension of the dynamic extension to compensate for longer delays, it is beneficial to bypass the need for distributed terms without having the dimension of the dynamic extension increase for longer delays. This motivates this letter, where we use a new predictive approach to compensate for arbitrarily long constant input delays, where the predictor can be expressed as a solution of a discrete time system. The predictor is a dynamic extension having the same dimension as the given system, regardless of the size of the input delay. We also provide an analog for output delays, whose predictor map is a solution of a continuous-discrete system. This contrasts with notable works such as [15], whose pole conditions are not needed here.

We use standard notation, which we simplify when no confusion would arise. The dimensions of our Euclidean spaces are arbitrary, unless we indicate otherwise. Our inequalities involving matrices of the same size are to be understood as being entrywise, i.e., if $M = [m_{ij}]$ and $N = [n_{ij}]$, then $M \leq N$ means $m_{ij} \leq n_{ij}$ for all i and j and similarly for $<$. Set $\mathbb{Z}_0 = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \mathbb{Z}_0 \backslash \{0\}$. Given $G = [g_{ij}] \in \mathbb{R}^{r\times s}$, we set $|G| = [|g_{ij}|]$, i.e., the entries of $|G|$ are the absolute values of the corresponding entries of G. We also set $G^+ = [\max\{g_{ij}, 0\}]$ and $G^- = G^+ - G$. For a matrix valued function $G(r) = [g_{ij}(r)]$ and an interval J in G's domain on which all of its entries g_{ij} are bounded real valued functions, we set $\sup_{r\in J} |G(r)| = [\sup_{r\in J} |g_{ij}(r)|]$, i.e., the supremum is entrywise. A square matrix is called Metzler provided all of its off-diagonal entries are nonnegative. We let $\|\cdot\|$ denote the standard Euclidean norm of matrices, $\|h\|_S$ denote the sup norm in this norm for functions h over a subset S of the domain of h, 0 be the matrix whose entries are all zeros, and I denote the identity matrix. We use the standard definitions of input-to-state stability (or ISS, which we also use to abbreviate input-to-state stable) from [14].

II. SYSTEM WITH DELAY IN THE INPUT

A. Studied System

We consider the system

$$\dot{x}(t) = Ax(t) + Bu(t - \tau) + \delta(t) \tag{1}$$

with x valued in $\mathbb{R}^n$, the control u valued in $\mathbb{R}^p$, $\tau > 0$ being a constant, constant matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$, and each component of δ being piecewise continuous. We use two assumptions (but see Remark 1 for sufficient conditions under which Assumption 1 holds after a change of variables).

Assumption 1: There is a matrix $K \in \mathbb{R}^{p\times n}$ such that the matrix $H = A + BK$ is Hurwitz and Metzler.

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Assumption 2: There is a known piecewise continuous function $\overline{\delta} : [0, +\infty) \to [0, +\infty)^n$ such that

$$|\delta(t)| \leq \overline{\delta}(t) \tag{2}$$

for all $t \geq 0$.

Under Assumption 1, there are a constant $p > 0$ and a vector $V \in \mathbb{R}^n$ such that $V > 0$ and such that the inequality

$$V^\top H \leq -pV^\top \tag{3}$$

holds (e.g., by [4, Lemma 2.3, p.41]). We use the function

$$\omega(s) = e^{As} + \int_0^s e^{A\ell}\mathrm{d}\ell BK. \tag{4}$$

We can then fix a small enough constant $\nu > 0$ such that $\omega(s)$ is nonsingular for all $s \in [0, \nu]$ and such that the matrix

$$\Gamma_1 = \sup_{r\in[0,\nu]} \left|\omega(r)^{-1} - I\right| \tag{5}$$

is such that the inequality

$$-pV^\top + V^\top |BK|\Gamma_1 < 0 \tag{6}$$

is satisfied. The existence of such a constant ν is a consequence of the facts that $p > 0$, $V > 0$, and $\omega(0) = I$.

Remark 1: Assumption 1 can be satisfied for systems $\dot{x} = A_0x + B_0u$ after a change of coordinates, when (A_0, B_0) is a controllable pair. This is done by first finding a matrix K_0 such that $A_0 + B_0K_0$ is Hurwitz with distinct real eigenvalues, then choosing a matrix P such that $P^{-1}(A_0 + B_0K_0)P$ is diagonal, and then choosing $A = P^{-1}A_0P$, $B = P^{-1}B_0$, and $K = K_0P$, to obtain an H that is Hurwitz and Metzler.

B. Main Result

Our event-triggered control will use the matrices

$$\lambda = \sup_{m\in[0,\nu]} \left|\omega(m)^{-1}\right| \text{ and } \Gamma_2 = \nu\lambda e^{|A|\nu}|BK|e^{|A|\tau} \tag{7}$$

with ν defined above and the simplifying notation

$$\xi_i = e^{A\tau}x(t_i - \tau) + \int_{t_i-2\tau}^{t_i-\tau} e^{A(t_i-\tau-m)}Bu(m)\mathrm{d}m. \tag{8}$$

Given any constant $T \geq \nu + \tau$ and K from the previous subsection, we propose the event triggered control

$$\begin{cases} u(t) = 0 & \text{if } \ t \in [-2\tau, \nu) \\ u(t-\tau) = K\xi_i & \text{if } \ t \in [t_i, t_{i+1}) \text{ and } i \geq 1, \end{cases} \tag{9}$$

where the sequence t_i is defined by these three conditions:

$$\begin{cases} (i)\ t_0 = 0, \\ (ii)\ t_1 = \nu + \tau, \text{ and} \\ (iii) \text{ for each } i \geq 1, \text{ the value } t_{i+1} \text{ is defined by} \\ t_{i+1} = \sup\{b \in [t_i, t_i + T) : |z_i(s) - x(t_i)| \\ \leq \Gamma_1|z_i(s)| + \Gamma_2 \int_{s-\nu-\tau}^{s} \overline{\delta}(\ell)\mathrm{d}\ell \text{ for all } s \in [t_i, b]\} \end{cases} \tag{10}$$

where z_i is the solution of the initial value problem

$$\dot{z}_i(t) = Az_i(t) + BK\xi_i, \quad z_i(t_i) = x(t_i) \tag{11}$$

for each $i \geq 1$ (so $u(t-\tau) = 0$ if $t \in [-\tau, t_1)$). We prove:

Theorem 1: The system (1) with u and the sequence $\{t_i\}$ defined in (9)-(11) is ISS with respect to δ. Also, $t_{i+1} - t_i \geq \nu$ for all $i \geq 0$.

Remark 2: The control u and triggering times t_i can be computed in the following recursive way. The formulas (9) with $i = 1$ define the control on $[0, t_2)$, where t_2 is defined as follows: Either t_2 is the supremum of times $b \geq t_1$ for which

$$|z_1(s) - x(t_1)| \leq \Gamma_1|z_1(s)| + \Gamma_2 \int_{s-\nu-\tau}^{s} \overline{\delta}(\ell)\mathrm{d}\ell \tag{12}$$

holds for all $s \in [t_1, b]$ if this supremum lies in $[t_1, t_1 + T)$, or $t_2 = t_1 + T$ otherwise. Then we use (8) to define ξ_2 to define the control values on $[t_2, t_3)$, where t_3 is found in the same way that we found t_2 except with the index i increased by 1. Continuing inductively defines all triggering times t_i and the control values for all times $t \geq 0$. Using the test inequalities

$$|z_i(s) - x(t_i)| \leq \Gamma_1|z_i(s)| + \Gamma_2 \int_{s-\nu-\tau}^{s} \overline{\delta}(\ell)\mathrm{d}\ell \tag{T_i}$$

and the constant values $B_i = \sup\{b \in [t_i, t_i + T) : (T_i) \text{ holds for all } s \in [t_i, b]\}$, our recursive algorithm is summarized in this table, which explains how to choose t_{i+1} after having chosen t_i for each $i \geq 0$ in the two cases:

Case	Choice
(T_i) holds for all $s \in [t_i, t_i + T]$	$t_{i+1} = t_i + T$
$B_i < t_i + T$	$t_{i+1} = B_i$

This implies that $t_{i+1} - t_i \in [0, T]$ for all $i \geq 1$. This differs significantly from standard triggers, which use Euclidean norms instead of the vectors of absolute values.

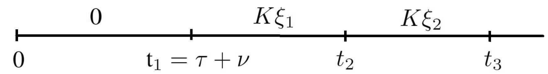

Fig. 1. Implementing Control from Theorem 1, Illustrating Control Values on Time Intervals and Feasibility of Method.

Remark 3: Our proof of Theorem 1 will show how condition (iii) is needed to allow cases where δ is not the zero function. It is used to compare solutions of the z_i dynamics (11) and the x dynamics. We use the values $z_i(s)$ in the triggering conditions instead of $x(s)$ to eliminate the need to know future $\delta(t)$ values when determining future triggering times. Moreover, by using the values $z_i(s)$ of the dynamic extensions (11) instead of $x(s)$ in the event trigger condition in (iii), we eliminate the need to continuously measure the state $x(t)$ to determine future triggering times.

Remark 4: It is tempting to surmise from (8)-(9) that the control u is not available in explicit form, because substituting (9) into (8) produces ξ_i on both sides of (8). However, this is not the case, because for each $i \geq 1$, we can write ξ_i as the state of the discrete time system

$$\begin{aligned} \xi_i &= e^{A\tau}x(t_i - \tau) \\ &+ e^{At_i}\Bigg[\sum_{j=J(i,\tau)}^{i-1} \int_{t_j}^{t_{j+1}} e^{-Am}\mathrm{d}mBK\xi_j \\ &+ \int_{t_i-\tau}^{t_{J(i,\tau)}} e^{-Am}\mathrm{d}mBK\xi_{J(i,\tau)-1}\Bigg] \end{aligned} \tag{13}$$

with the initial state $\xi_0 = e^{A\tau}x(-\tau)$, where $J(i, \tau)$ is the smallest integer j such that $t_j \geq t_i - \tau$ (so $J(i, \tau) \leq i$), and with the notational convention that the sum in (13) is not present if $J(i, \tau) = i$. The preceding allow us to write the control u from (9) in closed form. Also, by induction on i, ξ_i does not depend on event times t_j for any values $j > i$, and it also does not depend on any values $x(\ell)$ at times $\ell \geq t_i - \tau$. Since $u(t-\tau) = K\xi_i$ is the constant control value that we use in (1) for all times $t \in [t_i, t_{i+1})$ for each $i \geq 1$, this ensures the implementability of our control; see Fig. 1.

C. Proof of Theorem 1

The proof has two parts. In the first part, we prove the lower bound condition $\inf_i\{t_{i+1} - t_i\} \geq \nu$, to rule out Zeno's phenomenon (which would have allowed infinitely many triggering times on an interval of finite length). In the second part, we use interval observers to prove the ISS assertion.

1) Ruling Out Zeno's Phenomenon: Consider any $i \geq 1$. Then substituting (8) into (11) gives

$$\dot{z}_i(t) = Az_i(t) + BK\bigg[e^{A\tau}x(t_i - \tau) + \int_{t_i-2\tau}^{t_i-\tau} e^{A(t_i-\tau-m)}Bu(m)\mathrm{d}m\bigg] \quad (14)$$

for all $t \geq t_i$, and (1) and variation of parameters give

$$x(t) = e^{A\tau}x(t-\tau) + \int_{t-2\tau}^{t-\tau} e^{A(t-\ell-\tau)}Bu(\ell)\mathrm{d}\ell + \int_{t-\tau}^{t} e^{A(t-\ell)}\delta(\ell)\mathrm{d}\ell \quad (15)$$

for all $t \geq \tau$. As an immediate consequence,

$$\dot{z}_i(t) = Az_i(t) + BK\bigg[x(t_i) - \int_{t_i-\tau}^{t_i} e^{A(t_i-\ell)}\delta(\ell)\mathrm{d}\ell\bigg] \quad (16)$$

for all $t \geq t_i$. Since $x(t_i) = z_i(t_i)$, it follows that

$$\dot{z}_i(t) = Az_i(t) + BKz_i(t_i) + \delta_\sharp(t) \quad (17)$$

for all $t \geq t_i$. where

$$\delta_\sharp(t) = -BK\int_{t_i-\tau}^{t_i} e^{A(t_i-m)}\delta(m)\mathrm{d}m. \quad (18)$$

By applying variation of parameters to (17) on the interval $[t_i, t]$ for any $t \in [t_i, t_i + \nu]$ with the choice (18), we get

$$z_i(t) = \omega(t-t_i)z_i(t_i) + \int_{t_i}^{t} e^{A(t-\ell)}\delta_\sharp(\ell)\mathrm{d}\ell \quad (19)$$

where ω was defined in (4). As an immediate consequence,

$$z_i(t_i) = \omega(t-t_i)^{-1}z_i(t) - \omega(t-t_i)^{-1}\int_{t_i}^{t} e^{A(t-\ell)}\delta_\sharp(\ell)\mathrm{d}\ell \quad (20)$$

for all $t \in [t_i, t_i + \nu]$. It follows that

$$\begin{aligned}|z_i(t) - z_i(t_i)| &\leq \Big|I - \omega(t-t_i)^{-1}\Big||z_i(t)| + \bigg|\omega(t-t_i)^{-1}\int_{t_i}^{t} e^{A(t-\ell)}\delta_\sharp(\ell)\mathrm{d}\ell\bigg| \\ &\leq \Gamma_1|z_i(t)| + \bigg|\omega(t-t_i)^{-1}\int_{t_i}^{t} e^{A(t-\ell)}\delta_\sharp(\ell)\mathrm{d}\ell\bigg|\end{aligned} \quad (21)$$

for all $t \in [t_i, t_i + \nu]$. By our choice of $\delta_\sharp$, we deduce that

$$|z_i(t) - z_i(t_i)| \leq \Gamma_1|z_i(t)| + \lambda\int_{t_i}^{t}\Big|e^{A(t-\ell)}\Big|\bigg|BK\int_{t_i-\tau}^{t_i} e^{A(t_i-m)}\delta(m)\mathrm{d}m\bigg|\mathrm{d}\ell \quad (22)$$

and so also

$$|z_i(t) - z_i(t_i)| \leq \Gamma_1|z_i(t)| + \nu\lambda e^{|A|\nu}|BK|e^{|A|\tau}\int_{t-\nu-\tau}^{t}|\delta(m)|\mathrm{d}m, \quad (23)$$

where we used the fact that $|e^{Ar}| \leq e^{\|A\||r|}$ for all real values r (which follows from the Maclaurin series representation of the matrix exponential and subadditivity of the matrix norm) and the fact that $t - \nu - \tau \leq t_i - \tau \leq t_i \leq t$ for all $t \in [t_i, t_i + \nu]$. Thus, since (11) gives $x(t_i) = z_i(t_i)$, (23) gives

$$|z_i(t) - x(t_i)| \leq \Gamma_1|z_i(t)| + \Gamma_2\int_{t-\nu-\tau}^{t}|\delta(m)|\mathrm{d}m \quad (24)$$

for all $t \in [t_i, t_i + \nu]$. We conclude that $t_{i+1} - t_i \geq \nu$.

2) Stability Analysis: We perform a stability analysis of (1) with u and (t_i) defined by (9)-(10). We consider any $i \geq 1$.

From (8)-(9) and (15), it follows that

$$u(t-\tau) = K\bigg[x(t_i) - \int_{t_i-\tau}^{t_i} e^{A(t_i-\ell)}\delta(\ell)\mathrm{d}\ell\bigg] \quad (25)$$

for all $t \in [t_i, t_{i+1})$. Hence, the system (1) is

$$\dot{x}(t) = Hx(t) + BK[x(t_i) - x(t)] + \bar{\delta}_\sharp(t) \quad (26)$$

for all $t \in [t_i, t_{i+1})$, where $\bar{\delta}_\sharp = \delta_\sharp + \delta$ with $\delta_\sharp$ as defined by (18) as before. Then we define a comparison system by

$$\begin{cases}\dot{\overline{x}}(t) = H\overline{x}(t) + (BK[x(t_i) - x(t)])^+ + \big(\bar{\delta}_\sharp(t)\big)^+ \\ \dot{\underline{x}}(t) = H\underline{x}(t) - (BK[x(t_i) - x(t)])^- - \big(\bar{\delta}_\sharp(t)\big)^-\end{cases} \quad (27)$$

for all $t \in [t_i, t_{i+1})$ and $i \geq 1$.

Consider solutions of (26)-(27) such that $\underline{x}(t_1) \leq x(t_1)$ and $x(t_1) \leq \overline{x}(t_1)$. Since H is Metzler, it follows that

$$\underline{x}(t) \leq x(t) \leq \overline{x}(t) \quad (28)$$

holds for all $t \geq t_1$; this follows by noting that $X = \overline{x} - x$ and $X = x - \underline{x}$ are both solutions of a dynamics of the form $\dot{X}(t) = MX(t) + G(t)$ for a Metzler M and a nonnegative valued piecewise continuous $G(t)$ and by then applying [6, Lemma 1]. Similarly, $0 \leq \overline{x}(t)$ and $0 \leq -\underline{x}(t)$ for all $t \geq t_1$. We deduce that $\underline{x}(t) - \overline{x}(t) \leq x(t) \leq \overline{x}(t) - \underline{x}(t)$ and so also

$$|x(t)| \leq \overline{x}(t) - \underline{x}(t) \text{ for all } t \geq \nu + \tau. \quad (29)$$

We next use the variable s and the function U that are defined by $s(t) = \overline{x}(t) - \underline{x}(t)$ and $U(s) = V^\top s$. From (27), we get $\dot{s}(t) = Hs(t) + |BK[x(t_i) - x(t)]| + |\bar{\delta}_\sharp(t)|$ for all $t \in [t_i, t_{i+1})$ and $i \geq 1$. Hence, by (3), the time derivative of U satisfies

$$\dot{U}(t) \leq -pV^\top s(t) + V^\top|BK||x(t_i) - x(t)| + V^\top\big|\bar{\delta}_\sharp(t)\big| \quad (30)$$

for all $t \in [t_i, t_{i+1})$. Also, by the structures of the dynamics (1) and (11) and the fact that $z_i(t_i) = x(t_i)$, we have

$$\|z_i(t) - x(t)\| = \bigg\|\int_{t_i}^{t} e^{(t-\ell)A}\delta(\ell)\mathrm{d}\ell\bigg\| \leq e^{\|A\|T}T\|\delta\|_{[t_i,t]}$$

for all $t \in [t_i, t_{i+1})$, by applying variation of parameters to the dynamics for $x - z_i$. It follows from (18), (24), (29), (30), and the fact that $T \geq \tau$ that for all $t \geq t_1$, we have

$$\begin{aligned}\dot{U}(t) &\leq -pV^\top s(t) + \delta_B(t) + V^\top|BK|\bigg(\Gamma_1|x(t)| + \Gamma_2\int_{t-\nu-\tau}^{t}\bar{\delta}(\ell)\mathrm{d}\ell\bigg) \\ &\leq \Big(-pV^\top + V^\top|BK|\Gamma_1\Big)s(t) + V^\top|BK|\bigg(\Gamma_2\int_{t-\nu-\tau}^{t}\bar{\delta}(\ell)\mathrm{d}\ell\bigg) + \delta_B(t)\end{aligned} \quad (31)$$

where $\delta_B(t) = (\|V^\top BK\|e^{\|A\|T}T[2 + \|\Gamma_1\|] + \|V\|)\|\delta\|_{[0,t]}$, the 2 in the δ_B formula came from our using the bound $|x(t_i) - x(t)| \leq |z_i(t) - x(t_i)| + |z_i(t) - x(t)|$ to bound the second right side term in (30), and the Γ_1 in the δ_B formula came from the bound $\Gamma_1|z_i(t)| \leq \Gamma_1|x(t)| + \Gamma_1|x(t) - z_i(t)|$.

Using (6), we can find a constant $\mu > 0$ such that

$$\dot{U}(t) \leq -\mu U(s(t)) + \beta_1\|\bar{\delta}\|_{[0,t]} \text{ for all } t \geq t_1, \text{ where} \quad (32)$$

$$\beta_1 = \|V\|\Big(T\big\||BK|\Gamma_2\big\| + (2 + \|\Gamma_1\|)\|BK\|e^{\|A\|T}T + 1\Big). \quad (33)$$

It follows from integrating (32) that for all $t \geq t_1$, we have

$$\begin{aligned}U(s(t)) &\leq e^{-\mu(t-t_1)}U(s(t_1)) + \int_{t_1}^{t} e^{-\mu(t-m)}\beta_1\|\bar{\delta}\|_{[0,m]}\mathrm{d}m \\ &\leq e^{-\mu(t-t_1)}U(s(t_1)) + \frac{\beta_1}{\mu}\|\bar{\delta}\|_{[0,t]}.\end{aligned} \quad (34)$$

Since $V > 0$, the last inequality allows us to conclude that there are constants $\beta_i > 0$ for $i = 2, 3$ such that

$$\|s(t)\| \leq \beta_2 e^{-\mu(t-t_1)}\|s(t_1)\| + \beta_3\|\bar{\delta}\|_{[0,t]} \quad (35)$$

for all $t \geq t_1$. From (29), we deduce that

$$\|x(t)\| \leq \beta_2 e^{-\mu(t-t_1)}\|\overline{x}(t_1) - \underline{x}(t_1)\| + \beta_3\|\overline{\delta}\|_{[0,t]}$$

Since we can assume that $\overline{x}(t_1) \leq 2x(t_1)^+$ and $\underline{x}(t_1) \geq -2x(t_1)^-$, this gives

$$\|x(t)\| \leq 2\beta_2 e^{-\mu(t-t_1)}\|x(t_1)\| + \beta_3\|\overline{\delta}\|_{[0,t]} \tag{36}$$

for all $t \geq t_1$. Also, (1) gives $\|x(t)\| \leq \|x(0)\|e^{\|A\|(2t_1-t)} + t_1 e^{\|A\|t_1}\|\delta\|_{[0,t]}$ for all $t \in [0, t_1]$, which we can add to (36) to get the desired ISS estimate.

III. System With a Delay in the Measurements

A. Studied System

We next consider the system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + \delta(t) \\ y(t) = Cx(t-\tau) \end{cases} \tag{37}$$

where x is valued in $\mathbb{R}^n$, the known function u is valued in $\mathbb{R}^p$ and will be specified, $C \in \mathbb{R}^{q\times n}$ is known, y is valued in $\mathbb{R}^q$, $\tau > 0$ is a constant delay, and the unknown function δ has piecewise continuous components. In addition to Assumptions 1 and 2, we now assume:

Assumption 3: The pair (A, C) is observable.

Note that Assumption 3 implies that with the choice

$$C_\Delta = Ce^{-A\tau}, \tag{38}$$

the pair (A, C_Δ) is observable for all measurement delays $\tau > 0$. We let ν, p, V, Γ_1, ω and λ be as defined in Section II.

B. Preliminary Step: Finite Time Observer

The following finite time observer differs from works such as [7] and is more amenable to event-triggered control:

Lemma 1: With the preceding notation and under Assumptions 1-3, and in terms of the matrix an functions

$$\mathcal{E} = \int_{-\tau}^{0} e^{A^\top \ell} C_\Delta^\top C_\Delta e^{A\ell}\mathrm{d}\ell, \tag{39}$$

$$y_{\pounds}(t) = \mathcal{E}^{-1}\int_{t-\tau}^{t} e^{A^\top(m-t)}C_\Delta^\top\Big[y(m) + C_\Delta\int_{m-\tau}^{t} e^{A(m-\ell)}Bu(\ell)\mathrm{d}\ell\Big]\mathrm{d}m, \tag{40}$$

and

$$\delta_{\pounds}(t) = -\mathcal{E}^{-1}\int_{t-\tau}^{t} e^{A^\top(m-t)}C_\Delta^\top C_\Delta \int_{m-\tau}^{t} e^{A(m-\ell)}\delta(\ell)\mathrm{d}\ell\mathrm{d}m, \tag{41}$$

we have

$$y_{\pounds}(t) = x(t) + \delta_{\pounds}(t) \tag{42}$$

for all $t \geq 2\tau$ for all solutions of the system (37).

Proof: The matrix (39) is invertible because (A, C_Δ) is observable; see, e.g., [13, Sec. 3.5]. Next notice that

$$x(t-\tau) = e^{-A\tau}x(t) - \int_{t-\tau}^{t} e^{A(t-\ell-\tau)}Bu(\ell)\mathrm{d}\ell - \int_{t-\tau}^{t} e^{A(t-\ell-\tau)}\delta(\ell)\mathrm{d}\ell \text{ for all } t \geq \tau, \tag{43}$$

by applying variation of parameters to (37). Hence,

$$y(t) = Ce^{-A\tau}x(t) - C\int_{t-\tau}^{t} e^{A(t-\ell-\tau)}Bu(\ell)\mathrm{d}\ell - C\int_{t-\tau}^{t} e^{A(t-\ell-\tau)}\delta(\ell)\mathrm{d}\ell \tag{44}$$

for all $t \geq \tau$. We next use the function

$$y_\star(t) = y(t) + C\int_{t-\tau}^{t} e^{A(t-\ell-\tau)}Bu(\ell)\mathrm{d}\ell \tag{45}$$

for all $t \geq \tau$. The function (45) is available for measurement, because y and its integral term are known. Moreover,

$$y_\star(t) = C_\Delta x(t) + \delta_\Delta(t) \tag{46}$$

with C_Δ defined in (38) and

$$\delta_\Delta(t) = -C\int_{t-\tau}^{t} e^{A(t-\ell-\tau)}\delta(\ell)\mathrm{d}\ell \tag{47}$$

is available for all $t \geq \tau$. By applying variation of parameters to (37) on $[m, t]$ and then substituting the result into the relation $y_\star(m) - \delta_\Delta(m) = C_\Delta x(m)$ from (46), we get

$$C_\Delta e^{A(m-t)}x(t) = y_\star(m) - \delta_\Delta(m) + C_\Delta\int_{m}^{t} e^{A(m-\ell)}Bu(\ell)\mathrm{d}\ell + C_\Delta\int_{m}^{t} e^{A(m-\ell)}\delta(\ell)\mathrm{d}\ell \tag{48}$$

for all $m \in [t-\tau, t]$ and $t \geq \tau$. It follows that

$$\begin{aligned} &e^{A^\top(m-t)}C^\sharp e^{A(m-t)}x(t) \\ &= e^{A^\top(m-t)}C_\Delta^\top y_\star(m) - e^{A^\top(m-t)}C_\Delta^\top\delta_\Delta(m) \\ &\quad + e^{A^\top(m-t)}C^\sharp\int_{m}^{t} e^{A(m-\ell)}Bu(\ell)\mathrm{d}\ell \\ &\quad + e^{A^\top(m-t)}C^\sharp\int_{m}^{t} e^{A(m-\ell)}\delta(\ell)\mathrm{d}\ell, \end{aligned} \tag{49}$$

where $C^\sharp = C_\Delta^\top C_\Delta$. By integrating both sides of (49) over all $m \in [t-\tau, t]$ for a fixed value of t and then left multiplying the result by $\mathcal{E}^{-1}$, it follows that for all $t \geq 2\tau$, we get

$$\begin{aligned} x(t) &= \mathcal{E}^{-1}\int_{t-\tau}^{t} e^{A^\top(m-t)}C_\Delta^\top y_\star(m)\mathrm{d}m \\ &\quad + \mathcal{E}^{-1}\int_{t-\tau}^{t} e^{A^\top(m-t)}C^\sharp\int_{m}^{t} e^{A(m-\ell)}Bu(\ell)\mathrm{d}\ell\mathrm{d}m \\ &\quad - \mathcal{E}^{-1}\int_{t-\tau}^{t} e^{A^\top(m-t)}C_\Delta^\top\delta_\Delta(m)\mathrm{d}m \\ &\quad + \mathcal{E}^{-1}\int_{t-\tau}^{t} e^{A^\top(m-t)}C^\sharp\int_{m}^{t} e^{A(m-\ell)}\delta(\ell)\mathrm{d}\ell\mathrm{d}m. \end{aligned} \tag{50}$$

From the definition (45) of $y_\star(m)$ and (38), we deduce that

$$\begin{aligned} x(t) &= \mathcal{E}^{-1}\int_{t-\tau}^{t} e^{A^\top(m-t)}C_\Delta^\top y(m)\mathrm{d}m \\ &\quad + \mathcal{E}^{-1}\int_{t-\tau}^{t} e^{A^\top(m-t)}C_\Delta^\top C\int_{m-\tau}^{m} e^{A(m-\ell-\tau)}Bu(\ell)\mathrm{d}\ell\mathrm{d}m \\ &\quad + \mathcal{E}^{-1}\int_{t-\tau}^{t} e^{A^\top(m-t)}C^\sharp\int_{m}^{t} e^{A(m-\ell)}Bu(\ell)\mathrm{d}\ell\mathrm{d}m \\ &\quad - \mathcal{E}^{-1}\int_{t-\tau}^{t} e^{A^\top(m-t)}C_\Delta^\top\delta_\Delta(m)\mathrm{d}m \\ &\quad + \mathcal{E}^{-1}\int_{t-\tau}^{t} e^{A^\top(m-t)}C^\sharp\int_{m}^{t} e^{A(m-\ell)}\delta(\ell)\mathrm{d}\ell\mathrm{d}m \\ &= \mathcal{E}^{-1}\int_{t-\tau}^{t} e^{A^\top(m-t)}C_\Delta^\top y(m)\mathrm{d}m \\ &\quad + \mathcal{E}^{-1}\int_{t-\tau}^{t} e^{A^\top(m-t)}C^\sharp\int_{m-\tau}^{t} e^{A(m-\ell)}Bu(\ell)\mathrm{d}\ell\mathrm{d}m \\ &\quad - \mathcal{E}^{-1}\int_{t-\tau}^{t} e^{A^\top(m-t)}C_\Delta^\top\delta_\Delta(m)\mathrm{d}m \\ &\quad + \mathcal{E}^{-1}\int_{t-\tau}^{t} e^{A^\top(m-t)}C^\sharp\int_{m}^{t} e^{A(m-\ell)}\delta(\ell)\mathrm{d}\ell\mathrm{d}m \end{aligned} \tag{51}$$

for all $t \geq 2\tau$, which we can combine with the formula (47) for δ_Δ to obtain the final decomposition (42). ■

C. Main Result

In terms of the function $y_£$ from (40) and the matrix

$$\Gamma_3 = \lambda e^{|A|\nu}\nu\tau|BK||\mathcal{E}^{-1}|e^{|A^\top|\tau}|C_\Delta^\top C_\Delta|e^{2\tau|A|} \quad (52)$$

and the other notation from the previous sections, we define the control u by

$$\begin{cases} u(t) = 0 & \text{if } \ t \in [-2\tau, t_1) \\ u(t) = Ky_£(t_i) & \text{if } \ t \in [t_i, t_{i+1}) \text{ and } i \geq 1, \end{cases} \quad (53)$$

where the sequence of event trigger times t_i are defined by

$$\begin{cases} (i)\ t_0 = 0, \\ (ii)\ t_1 = \nu + 2\tau, \ \text{ and} \\ (iii) \text{ for each } i \geq 1, t_{i+1} \text{ is defined by} \\ t_{i+1} = \sup\{b \in [t_i, t_i + T) : |z_i(s) - x(t_i)| \\ \leq \Gamma_1|z_i(s)| + \Gamma_3 \int_{s-\nu-2\tau}^{s} \overline{\delta}(\ell)\mathrm{d}\ell \text{ for all } s \in [t_i, b]\} \end{cases} \quad (54)$$

where z_i is the solution of the initial value problem

$$\dot{z}_i(t) = Az_i(t) + BKy_£(t_i), \quad z_i(t_i) = x(t_i) \quad (55)$$

for each $i \geq 1$, and where we now require our constant T to be such that $T \geq \nu + 2\tau$. We prove:

Theorem 2: Let Assumptions 1-3 hold. Then, the system (37) with u and (t_i) defined in (53)-(55) is ISS with respect to δ. Also, $t_{i+1} - t_i \geq \nu$ for all $i \geq 0$.

Remark 5: The reasoning from Remarks 1-3 also applies to Theorem 2, except with Γ_2 replaced by Γ_3, and τ replaced by 2τ in the lower bound of integration in (iii) from (10). In particular, $t_{i+1} - t_i \in [0, T]$ for all $i \geq 1$.

Remark 6: By reasoning analogously to the input delay case in Remark 4, we can express the function $y_£$ as a solution of a continuous-discrete system. To see how it can be done, notice that differentiating the formula for $y_£$ from (40), and then substituting the formula $u(t) = Ky_£(t_i)$ from (53), implies that for each $t \in [t_i, t_{i+1})$ and $i \geq 1$, we have

$$\begin{aligned} \dot{y}_£(t) = {} & \mathcal{E}^{-1}C_\Delta^\top\Big[y(t) + C_\Delta e^{At}\mathcal{M}_1(t)\Big] \\ & - \mathcal{E}^{-1}A^\top\mathcal{E}y_£(t) + BKy_£(t_i) \\ & - \mathcal{E}^{-1}e^{-A^\top\tau}C_\Delta^\top\Big[y(t-\tau) + C_\Delta e^{A(t-\tau)}\mathcal{M}_2(t)\Big], \end{aligned} \quad (56)$$

where for $k = 1, 2$, we use the definitions and formulas

$$\begin{aligned} \mathcal{M}_k(t) &= \int_{t-k\tau}^{t} e^{-A\ell}Bu(\ell)\mathrm{d}\ell \\ &= \int_{t_i}^{t} e^{-A\ell}\mathrm{d}mBKy_£(t_i) + \sum_{j=J(t-k\tau)}^{i-1}\int_{t_j}^{t_{j+1}} e^{-Am}\mathrm{d}mBKy_£(t_j) \\ &\quad + \int_{t-k\tau}^{t_{J(t-k\tau)}} e^{-Am}\mathrm{d}mBKy_£(t_{J(t-k\tau)-1}) \end{aligned} \quad (57)$$

if $t_i \geq t - k\tau$ with $J(t-k\tau)$ being the smallest j such that $t_j \geq t - k\tau$ and with the notational convention that the sum in (57) is not present if $J(t-k\tau) = i$, and

$$\mathcal{M}_k(t) = \int_{t-k\tau}^{t} e^{-A\ell}\mathrm{d}mBKy_£(t_i) \quad (58)$$

if $t_i < t - k\tau$, and with the initial function

$$y_£(\ell) = \mathcal{E}^{-1}\int_{\nu+\tau}^{t_1} e^{A^\top(m-t_1)}C_\Delta^\top y(m)\mathrm{d}m \quad (59)$$

for all $\ell \in [0, t_1]$. The equations (56)-(59) show how (40) is expressible as a solution of a continuous-discrete system.

D. Proof of Theorem 2

The proof has two parts. First, we prove the lower bound condition $\inf_i\{t_{i+1} - t_i\} \geq \nu$, to rule out Zeno's phenomenon. Then, we use interval observers to prove the ISS assertion.

1) Ruling Out Zeno's Phenomenon: To prove that Zeno's phenomenon does not occur, we fix an $i \geq 1$, and we introduce the function $\delta_\star$ defined by $\delta_\star(t) = BK\delta_£(t_i)$ for all $t \geq t_i$. Then from the decomposition of $y_£$ from (42) and $z_i(t_i) = x(t_i)$, the dynamics in (55) satisfy $\dot{z}_i(t) = Az_i(t) + BKz_i(t_i) + \delta_\star(t)$ for all $t \geq t_i$. It follows that

$$z_i(t) = \omega(t - t_i)z_i(t_i) + \int_{t_i}^{t} e^{A(t-\ell)}\delta_\star(\ell)\mathrm{d}\ell \quad (60)$$

for all $t \in [t_i, t_i + \nu]$, and so also

$$\begin{aligned} z_i(t_i) = {} & \omega(t - t_i)^{-1}z_i(t) \\ & - \omega(t - t_i)^{-1}\int_{t_i}^{t} e^{A(t-\ell)}\delta_\star(\ell)\mathrm{d}\ell, \end{aligned} \quad (61)$$

by the reasoning that gave (20). We deduce that

$$\begin{aligned} |z_i(t) - z_i(t_i)| \leq {} & \left|\omega(t - t_i)^{-1} - I\right||z_i(t) \\ & + \left|\omega(t - t_i)^{-1}\right|\int_{t_i}^{t} e^{|A|(t-\ell)}|\delta_\star(\ell)|\mathrm{d}\ell. \end{aligned} \quad (62)$$

Thus, for all $t \in [t_i, t_i + \nu]$, we have

$$|z_i(t) - z_i(t_i)| \leq \Gamma_1|z_i(t)| + \lambda e^{|A|\nu}\int_{t_i}^{t}|\delta_\star(\ell)|\mathrm{d}\ell. \quad (63)$$

Setting $C^\sharp = C_\Delta^\top C_\Delta$ as before, it follows from our formula (41) for $\delta_£$ that for all $\ell \in [t_i, t]$, we have

$$|\delta_\star(\ell)| \leq |BK|\left|\mathcal{E}^{-1}\int_{t_i-\tau}^{t_i} e^{A^\top(m-t_i)}C^\sharp\int_{m-\tau}^{t_i} e^{A(m-\ell)}\delta(\ell)\mathrm{d}\ell\mathrm{d}m\right|. \quad (64)$$

Consequently, for all $t \in [t_i, t_i + \nu]$, we have

$$\begin{aligned} & \int_{t_i}^{t}|\delta_\star(\ell)|\mathrm{d}\ell \\ & \leq (t - t_i)\tau|BK||\mathcal{E}^{-1}|e^{|A^\top|\tau}|C^\sharp|e^{2\tau|A|}\int_{t_i-2\tau}^{t_i}|\delta(\ell)|\mathrm{d}\ell \\ & \leq \nu\tau|BK||\mathcal{E}^{-1}|e^{|A^\top|\tau}|C^\sharp|e^{2\tau|A|}\int_{t_i-2\tau}^{t_i}|\delta(\ell)|\mathrm{d}\ell. \end{aligned} \quad (65)$$

Using the last inequality in (65) to bound the integral in (63), it follows that for $t \in [t_i, t_i + \nu]$, we have

$$|z_i(t) - z_i(t_i)| \leq \Gamma_1|z_i(t)| + \Gamma_3\int_{t-\nu-2\tau}^{t}|\delta(m)|\mathrm{d}m. \quad (66)$$

Hence, $t_{i+1} - t_i \geq \nu$, so Zeno's phenomenon does not occur.

2) Stability: We have $\dot{x}(t) = Ax(t) + BKy_£(t_i) + \delta(t)$ for all $t \in [t_i, t_{i+1})$ and $i \geq 1$. We deduce from (42) that

$$\dot{x}(t) = Hx(t) + BK(x(t_i) - x(t)) + BK\delta_£(t_i) + \delta(t) \quad (67)$$

for all $t \in [t_i, t_{i+1})$. Then the remaining part of the proof is similar to the second part of the proof of Theorem 1.

IV. Illustration

We revisit a dynamics for the control of the depth and pitch degrees-of-freedom (or DOF) of an autonomous underwater vehicle that we studied in [11], e.g., the BlueROV2 vehicle, which is widely used in environmental surveys such as the study of corals. As in [11], we assume that the vehicle has a Doppler Velocity Logger (or DVL) for estimating its velocity. The DVL commonly experiences bottom lock, making it impractical to continuously change the control values. Hence, we show how Theorems 1-2 apply, and so cover measurement delays which were beyond the scope of [11] or other event-triggered studies of the dynamics.

As noted in [12, eq. (9.28)], after linearization and assuming that the vehicle is neutrally buoyant, the linearized dynamics for the depth plane are given by

$$\begin{aligned}&(m - X_{\dot{w}(t)})\dot{w}(t) - (mx_g + Z_{\dot{q}})\dot{q}(t)\\&\quad - Z_w w(t) - (mU + z_q)q(t) = Z_{\gamma_s} u_Z\\&(mx_g + M_{\dot{w}}(t))\dot{w}(t) + (I_{yy} - M_{\dot{q}})\dot{q}(t)\\&\quad - M_w w(t) + (mx_g U - M_q)q(t) - M_\theta \theta = M_{\gamma_s} u_M \end{aligned} \tag{68}$$

whose parameter values were experimentally computed and presented in [12]. As in [11], we assume that the surge nominal velocity is $U = 0.1$m/s. The states $x = [w,\ q]^\top$ represent the depth and pitch velocity, and the controls u_Z and u_M are the force and moment required to produce motion of the vehicle. Using the parameter values and controller from [12], the system (68) becomes $\dot{x}(t) = A_0 x(t) + B_0 u$ with

$$A_0 = \begin{bmatrix} -0.17742 & -0.3027 \\ 0.5394 & -1.4685 \end{bmatrix} \text{ and } B_0 = \begin{bmatrix} -0.2063 \\ -0.7629 \end{bmatrix}. \tag{69}$$

Choosing K_0 such that $A_0 + B_0 K_0$ has any two distinct negative eigenvalues (e.g., using the command `FeedbackGains` in the Mathematica program) and then applying a diagonalizing similarity transformation to obtain the matrices $A = P^{-1} A_0 P$, $B = P^{-1} B_0$, and $K = K_0 P$ as in Remark 1 above, we can satisfy the assumptions of Theorem 1. Also, when (A, C) is observable, the assumptions of Theorem 2 can be satisfied. Hence, Theorems 1-2 from the previous sections provide a useful alternative to results that could be obtained from [8], by only requiring a 2-dimensional piecewise continuous predictor using the values ξ_i (which are needed to compensate for the input delays) regardless of the input delay length (instead of the chains of dynamical extensions that would be required when using [8]), and also covering measurement delays which were beyond the scope of event-triggered works like [8] and [11].

In Figs. 2a and 2b, we show our MATLAB plots of the closed loop solutions for the preceding system obtained from applying Theorems 1 and 2, respectively. We chose $\tau = 0.1$ s, $\nu = 0.4077$, $K_0 = [0.941, 0.637]$, P as in the preceding paragraph, and $C = [1, 1]P$ (i.e., only the sum of the components of the states in the original variables is measured). The elements of Γ_1 were 0.001, and both elements of V were 1. We used $p = 0.5$, and a constant $\delta = [\delta_1, \delta_2]^\top$, with each δ_i for $i = 1, 2$ found with the function $\mathtt{rand}(\cdot)$ in MATLAB which draws values from the open interval $(0, 1)$. Since our plots show desired convergence, they illustrate the value of our method, in the special case of the BlueROV2 with input or measurement delay.

V. Conclusion

We proposed new event-triggered controls for linear time-invariant systems with a known arbitrarily long constant delays. Key novel features included (a) our alternative event triggers based on vectors of absolute values and (b) that our predictor maps were expressible as solutions of a discrete time system (under arbitrarily large constant input delays) or as solutions of continuous-discrete dynamic extensions (under

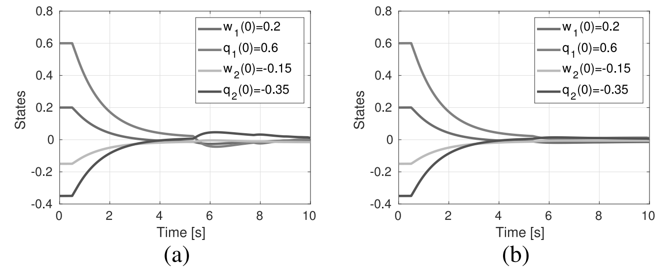

Fig. 2. Simulation for (a) input and (b) measurement delays.

measurement delays), with each of these predictor systems having its dimension equaling the dimension of the original systems that we render ISS and no distributed terms. This is a useful alternative to chain predictors whose dynamic extensions become arbitrarily large in dimension for bigger delays. Time-varying extensions are expected.

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