Vector Extensions of Halanay's Inequality

Frédéric Mazenc, Michael Malisoff, *Senior Member, IEEE*, and Miroslav Krstic, *Fellow, IEEE*

***Abstract*—We provide two extensions of Halanay's inequality, where the scalar function in the usual Halanay's inequality is replaced by a vector valued function, under a Metzler condition. We provide an easily checked necessary and sufficient condition for asymptotic convergence of the function to the zero vector in the time-invariant case. For the time-varying cases, we provide a sufficient condition for this convergence, which can be easily checked when the systems are periodic. We illustrate our results in cases that are beyond the scope of prior asymptotic stability results.**

***Index Terms*—Delay, interval observer, stability.**

I. Introduction

Halanay's inequality is an efficient stability analysis tool, especially for systems with time-varying and poorly known delays, because for such cases, no general Lyapunov–Krasovskii functional construction is available in general. This celebrated inequality has the following form:

$$\dot{v}(t) \leq -av(t) + b \sup_{\ell \in [t-\tau, t]} v(\ell) \tag{1}$$

where $a > 0$, $b \geq 0$, and $\tau \geq 0$ are constants and $v : [-\tau, +\infty) \to [0, +\infty)$ is a scalar function of class C^1. Here and in the sequel, derivatives at endpoints are one sided ones. The usual Halanay's inequality result [8] is the following: if $a > b$, then $v(t)$ exponentially converges to zero as $t \to +\infty$. In addition to our works in [16] and [17] that relax the requirement that the decay rate a is strictly larger than the gain b, several other extensions of this result are available in the literature e.g., in [6], [20], [23], and [24]. Time-varying versions have been studied in [2] and [15].

The fact that v in (1) is scalar valued is a limitation, because when one analyzes a system with delay, such a function may not be available, but functions $v_i : [-\tau, +\infty) \to [0, +\infty)$ of class C^1 and a Metzler matrix M, and a matrix P with all positive entries, such that

$$\begin{pmatrix} \dot{v}_1(t) \\ \vdots \\ \dot{v}_n(t) \end{pmatrix} \leq M \begin{pmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{pmatrix} + P \begin{pmatrix} \sup\limits_{l \in [t-\tau, t]} v_1(l) \\ \vdots \\ \sup\limits_{l \in [t-\tau, t]} v_n(l) \end{pmatrix} \tag{2}$$

for all $t \geq 0$ may be available, where the inequality in (2) is componentwise. Using (2) to obtain a scalar valued function v satisfying the requirements of Halanay's theory does not seem to be possible in general.

These remarks motivate this article, which continues our search for generalized or relaxed versions of Halanay's inequality, which we began in [16] and [17]. While Mazenc *et al.* [16] provided less restrictive versions of Halanay's inequality, where the gain in the overshoot term can exceed the decay rate, including applications to systems with scarce arbitrarily long sample intervals, whereas Mazenc *et al.* [17] covered sampled cases that were beyond the scope of earlier Halanay's inequality formulations, such as Mazenc *et al.* [16], here we pursue a very different direction, where the usual scalar decaying function in Halanay's inequality is replaced by a vector-valued function. This provides an analog to the vector Lyapunov function results that is applicable to stabilization problems that were beyond the scope of earlier Razumikhin function or diagonal-stability-based methods; see the work in [10], [22], and [26] for vector Lyapunov functions and the work in [5] for input-to-state stability (or ISS) for interconnected systems via combinations of Lyapunov functions, or under small-gain conditions that we do not require here.

We propose two extensions of Halanay's result in the case where vector Halanay's inequalities are satisfied. First, we consider a vector and time-invariant version of this inequality. In Section II, we provide an easily checked necessary and sufficient conditions for the convergence of the v_i's to zero as time converges to $+\infty$. Second, we propose sufficient conditions for this convergence, for a vector and time-varying version of Halanay's inequality in Section III. We prove the results using ideas for Metzler matrices and cooperative systems, e.g., from Berman and Plemmons [4] and Haddad *et al.* [7]. Then, in Section IV, we provide three examples that illustrate how our results add value to the literature.

We use standard notation, which is simplified when no confusion would arise, where the dimensions of our Euclidean spaces are arbitrary unless otherwise noted. The standard Euclidean norm and induced matrix norm are denoted by $|\cdot|$, and $|\cdot|_\infty$ is the usual sup norm. We define Ξ_t by $\Xi_t(s) = \Xi(t+s)$ for all Ξ, $s \leq 0$, and $t \geq 0$ for which $t+s$ is in the domain of Ξ, $\mathbb{N} = \{1, 2, \ldots\}$, and $\lfloor \cdot \rfloor$ denotes the floor function. For matrices $\mathcal{M} \in \mathbb{R}^{n \times p}$ and $\mathcal{N} \in \mathbb{R}^{n \times p}$ with entries $m_{i,j}$ and $n_{i,j}$ in row i and column j, respectively, we write $\mathcal{M} \leq \mathcal{N}$ when $m_{i,j} \leq n_{i,j}$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, p\}$, and similarly for $<$ and for vectors. A matrix is called nonnegative (resp., positive), provided all its entries are nonnegative (resp., positive). A matrix is called Metzler, provided its off diagonal entries are nonnegative, and I is the identity matrix. A continuous linear system of the form $\dot{\Xi}(t) = L(t)\Xi_t$ having a delay that is bounded above by a constant $\bar{\tau} > 0$ is called cooperative, provided for each initial function satisfying $\Xi(t) \geq 0$ for all $t \in [-\bar{\tau}, 0]$, the corresponding solution satisfies $\Xi(t) \geq 0$ for all $t \geq 0$. We also use the n-fold product notation $[0, +\infty)^n = [0, +\infty) \times \ldots [0, +\infty)$ and usual definitions and properties for the state-transition matrices (i.e., fundamental solutions) from [25, Appendix C.4].

Manuscript received May 18, 2020; revised November 11, 2020; accepted February 23, 2021. Date of publication February 26, 2021; date of current version February 28, 2022. This work was supported by the U.S. National Science Foundation under Grant 1711299 (Malisoff) and Grant 1711373 (Krstic). Recommended by Associate Editor C.-Y. Kao.
(Corresponding author: Frédéric Mazenc.)

Frédéric Mazenc is with the Inria Saclay, L2S-CNRS-CentraleSupélec, 91192 Gif-Sur-Yvette, France (e-mail: frederic.mazenc@l2s.centralesupelec.fr).

Michael Malisoff is with the Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918 USA (e-mail: malisoff@lsu.edu).

Miroslav Krstic is with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093-0411 USA (e-mail: krstic@ucsd.edu).
Digital Object Identifie 10.1109/TAC.2021.3062565

II. Time-Invariant Case

A. Statement of Result and Remarks

Let $M \in \mathbb{R}^{n\times n}$ be a Metzler and Hurwitz matrix and $P \in \mathbb{R}^{n\times n}$ be a nonnegative matrix. Let $\tau > 0$ be a constant and $V : [-\tau, +\infty) \to [0, +\infty)^n$ be C^1 and

$$\dot{V}(t) = MV(t) + P\mathcal{S}(V_t) \tag{3}$$

hold for all $t \geq 0$, where $V = (v_1 \dots v_n)^\top$ and

$$\mathcal{S}(V_t) = \sup_{l\in[t-\tau,t]} V(l) \tag{4}$$

and where

$$\sup_{l\in[t-\tau,t]} V(l) = \left[\sup_{l\in[t-\tau,t]} v_1(l) \ \dots \ \sup_{l\in[t-\tau,t]} v_n(l)\right]^\top \tag{5}$$

where v_i is the ith component of V for each i, and similarly for vector-valued functions W below. We prove the following.

Theorem 1: All C^1 solutions $V : [-\tau, +\infty) \to [0, +\infty)^n$ of (3) converge exponentially to the origin as $t \to +\infty$ if and only if $M + P$ is Hurwitz.

Remark 1: One can prove that if $M + P$ is Hurwitz, then a C^1 function $V : [-\tau, +\infty) \to [0, +\infty)^n$ such that

$$\dot{V}(t) \leq MV(t) + P\mathcal{S}(V_t) \tag{6}$$

holds for all $t \geq 0$ converges to 0 as $t \to +\infty$. This can be proved by the following variant of the usual comparison principle. Consider a function W such that

$$\dot{W}(t) = MW(t) + P\mathcal{S}(W_t) \tag{7}$$

with $\mathcal{S}(W_0) > \mathcal{S}(V_0)$, and suppose there was a $t_c > 0$ such that $W(t) > V(t)$ for all $t \in [0, t_c)$ and such that there is a $i \in \{1, \dots, n\}$ such that $v_i(t_c) = w_i(t_c)$. Then, we get the following:

$$\begin{aligned} V(t_c) &\leq e^{Mt_c}V(0) + \textstyle\int_0^{t_c} e^{M(t_c-\ell)}P\mathcal{S}(V_\ell)\mathrm{d}\ell \text{ and} \\ W(t_c) &= e^{Mt_c}W(0) + \textstyle\int_0^{t_c} e^{M(t_c-\ell)}P\mathcal{S}(W_\ell)\mathrm{d}\ell \end{aligned} \tag{8}$$

where we used the fact that the Metzler property of M implies that $e^{Ms} \geq 0$ for all $s \geq 0$ [11].

If we subtract the equality in (8) from the inequality in (8) and recall that $e^{Ms} \geq 0$ for all $s \geq 0$ (so, e^{Mt_c} is nonzero and nonnegative) and $P \geq 0$, we get the contradiction $V(t_c) < W(t_c)$. Hence, $W(t) > V(t)$ for all $t \geq 0$, and Theorem 1 ensures that $W(t)$ exponentially converges to 0 as $t \to +\infty$. Hence, since V is nonnegative valued, $V(t)$ also exponentially converges to 0 as $t \to +\infty$.

Remark 2: We can use Theorem 1 to find novel sufficient conditions for the origin to be a globally exponential stable equilibrium on $\mathbb{R}^n$ for systems of the following form:

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{p} B_i X(t - \tau_i(t)) \tag{9}$$

with multiple bounded delays τ_i for any integer $p \geq 1$, including cases where A is not required to be Metzler and B_i's might not be nonnegative (see Section IV-B).

We can rewrite (3) in the form $\dot{V}(t) = MV(t) + P[v_1(t - \tau_1(t)), \dots, v_n(t - \tau_n(t))]^\top$ with time-varying delays τ_i, which is reminiscent of but beyond the scope of [19, Th. 4.1], because Ngoc [19] was confined to constant delays. Thus, no extension of [19, Th. 4.1] to (9) is possible. When M is not Metzler or P is not nonnegative, then the nonnegative orthant may not be positively invariant for (3), and this motivates our conditions on M and P.

B. Proof of Theorem 1

1) First Part. Necessity: We prove that if $M + P$ is not Hurwitz, then the asymptotic convergence condition of the theorem does not hold. To this end, notice that if $M + P$ was not Hurwitz, then the system

$$\dot{X}(t) = (M + P)X(t) \tag{10}$$

with $X = (x_1, \dots, x_n)^\top$ is not exponentially stable. Consider a solution $V : [-\tau, +\infty) \to [0, +\infty)^n$ of (3) such that $v_i(r) = 2$ for all $i \in \{1, \dots, n\}$ and $r \in [-\tau, 0]$ and the solution X of (10) with the initial condition $X(0) = (1, \dots, 1)^\top$. Then, $X(t) \geq 0$ for all $t \geq 0$, because $M + P$ is Metzler; see [11, Lemma 1]. We prove that

$$V(t) > X(t) \tag{11}$$

for all $t \geq 0$ by proceeding through contradiction.

Let us assume there is a $t_e > 0$ such that $V(t) > X(t)$ for all $t \in [0, t_e)$ and that there is $i \in \{1, \dots, n\}$ such that $v_i(t_e) = x_i(t_e)$. By integrating, we obtain the following:

$$\begin{aligned} V(t_e) &= e^{Mt_e}V(0) + \textstyle\int_0^{t_e} e^{M(t_e-\ell)}P\mathcal{S}(V_\ell)\mathrm{d}\ell \text{ and} \\ X(t_e) &= e^{Mt_e}X(0) + \textstyle\int_0^{t_e} e^{M(t_e-\ell)}PX(\ell)\mathrm{d}\ell. \end{aligned} \tag{12}$$

As in Remark 1, it follows that $V(t_e) > X(t_e)$ because M is Metzler. This yields a contradiction. Since $X(t)$ does not converge to 0 as $t \to +\infty$ and is nonnegative valued, it follows from (11) that $V(t)$ also does not exponentially converge to zero. This proves the necessity of Hurwitzness of $M + P$ for the convergence condition in Theorem 1.

2) Second Part. Sufficiency: We show that if (3) is such that $M + P$ is Hurwitz, then the convergence conclusion of Theorem 1 holds. To this end, we introduce the function $W(t) = e^{-Mt}V(t)$ for all $t \geq 0$. Then, (3) gives

$$\dot{W}(t) = e^{-Mt}P\mathcal{S}(V_t). \tag{13}$$

By integrating (13) on $[t - h, t]$ for any $t \geq h + \tau$ and $h > 0$, we obtain the following:

$$\begin{aligned} e^{-Mt}V(t) &= e^{M(h-t)}V(t-h) \\ &\quad + \textstyle\int_{t-h}^{t} e^{-M\ell}P\mathcal{S}(V_\ell)\mathrm{d}\ell \end{aligned} \tag{14}$$

which gives

$$V(t) = e^{Mh}V(t-h) + \textstyle\int_{t-h}^{t} e^{M(t-\ell)}P\mathcal{S}(V_\ell)\mathrm{d}\ell. \tag{15}$$

For all $\ell \leq t$, the matrix $e^{M(t-\ell)}P$ is nonnegative, because $P \geq 0$ and M is Metzler; see, e.g., [11, Lemma 1]. Hence, we have the following:

$$\begin{aligned} V(t) &\leq e^{Mh}V(t-h) + \textstyle\int_{t-h}^{t} e^{M(t-\ell)}P\mathrm{d}\ell\mathcal{S}_{h+\tau}(V_t) \\ &= e^{Mh}V(t-h) + M^{-1}\left(e^{Mh} - I\right)P\mathcal{S}_{h+\tau}(V_t) \end{aligned} \tag{16}$$

where $\mathcal{S}_{h+\tau}(V_t) = \sup_{\ell\in[t-h-\tau,t]} V(\ell)$. Thus

$$V(t) \leq \left(e^{Mh} + M^{-1}e^{Mh}P + R\right)\mathcal{S}_{h+\tau}(V_t) \tag{17}$$

with

$$R = -M^{-1}P. \tag{18}$$

By Lemma 1 in the Appendix, the matrix R is nonnegative and Schur stable. Hence, we can find a real value $\mu > 0$ such that $R + \mu\mathbb{H}$ is Schur stable and positive, where $\mathbb{H}$ is the constant $n \times n$ matrix each of whose entry is 1 (because of the continuity of eigenvalues of a matrix as a function of the entries of the matrix). Also, since M is Hurwitz, it follows that $\lim_{h\to+\infty} |e^{Mh}| = 0$. Hence, since R is Schur stable, there is a constant $h_\star > 0$ such that for all $h \geq h_\star$, the matrix $e^{Mh} + M^{-1}e^{Mh}P + R + \mu\mathbb{H}$ is Schur stable and positive.

Moreover, (17) is satisfied with $e^{Mh} + M^{-1}e^{Mh}P + R$ replaced by $e^{Mh} + M^{-1}e^{Mh}P + R + \mu\mathbb{H}$, since V is nonnegative valued. Then, by the nonnegativity and Schur property of $e^{Mh} + M^{-1}e^{Mh}P + R + \mu\mathbb{H}$, it follows from the proof of [1, Lemma 1] that $V(t)$ converges exponentially to zero as $t \to +\infty$.

III. Time-Varying Case

A. Studied Problem

Our main assumption throughout this section is given as follows.

Assumption 1: The matrix-valued functions $M : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $P : \mathbb{R} \to \mathbb{R}^{n\times n}$ are bounded piecewise continuous functions satisfying the following properties:

$M(t)$ is Metzler for all $t \in \mathbb{R}$, $P(t)$ is nonnegative for all $t \in \mathbb{R}$, and the system

$$\dot{X}(t) = M(t)X(t) \tag{19}$$

is uniformly globally exponentially stable on $\mathbb{R}^n$.

Let Φ be the state transition matrix of M. Note for later use that, for all $s_1 \le s_2$, we have $\Phi(s_2, s_1) \ge 0$ because M is Metzler (e.g., by [11, Lemma 1]). Also, there are constants $c_1 > 0$ and $c_2 > 0$ such that $|\Phi(t, s)r| \le c_1 e^{-c_2(t-s)}$ when $t \ge s \ge 0$ for all unit vectors r.

B. General Case

We introduce the following assumption.

Assumption 2: There are a constant $\alpha > 0$ and a Schur stable matrix $\overline{R} \ge 0$ such that

$$\Phi(t, s) + \textstyle\int_s^t \Phi(t, \ell)P(\ell)\mathrm{d}\ell \le \overline{R} \tag{20}$$

for all $s \ge 0$ and $t \ge s + \alpha$.

Since M and P are bounded, it follows from the exponential stability condition on (19) that we can satisfy Assumption 2 when $\alpha > 0$ is large enough and $|P|$ is small enough. Note for later use that since $\overline{R}$ is nonnegative and Schur stable, [7, Lemma 2.7, p. 79] implies that there are a vector $\mathcal{U} > 0$ and a constant $q \in (0, 1)$ such that

$$\mathcal{U}^\top \overline{R} \le q\mathcal{U}^\top. \tag{21}$$

We prove the following.

Theorem 2: Let Assumptions 1 and 2 hold. Consider a constant $\tau > 0$ and a vector-valued function $V : [0, +\infty) \to [0, +\infty)^n$ of class C^1 such that for all $t \ge 0$,

$$\dot{V}(t) \le M(t)V(t) + P(t)\mathcal{S}(V_t) \tag{22}$$

where $\mathcal{S}(V_t)$ is defined in (4). Then, $\lim_{t\to+\infty} V(t) = 0$.

Proof: First Part. First, let us establish that $V(t)$ is bounded. By variation of the parameters, we obtain

$$V(t) \le \Phi(t, s)V(s) + \textstyle\int_s^t \Phi(t, m)P(m)\mathcal{S}(V_m)\mathrm{d}m \tag{23}$$

when $t \ge s \ge 0$. It follows that

$$V(t) \le \left[\Phi(t, s)+\textstyle\int_s^t\Phi(t, m)P(m)\mathrm{d}m\right]\sup_{\ell\in[s-\tau,t]} V(\ell). \tag{24}$$

Assumption 2 ensures that for all $t \ge s + \alpha$

$$V(t) \le \overline{R} \sup_{\ell\in[s-\tau,t]} V(\ell). \tag{25}$$

Now, consider $t_\star \ge 0$ and $t \ge t_\star + \alpha$ where α is from Assumption 2. Then, from (25), it follows that for all $m \in [t_\star + \alpha, t]$, the inequalities

$$V(m) \le \overline{R} \sup_{\ell\in[t_\star-\tau,m]} V(\ell) \le \overline{R} \sup_{\ell\in[t_\star-\tau,t]} V(\ell) \tag{26}$$

are satisfied. It follows that

$$\sup_{m\in[t_\star+\alpha,t]} V(m) \le \overline{R} \sup_{\ell\in[t_\star-\tau,t]} V(\ell). \tag{27}$$

Therefore

$$\mathcal{U}^\top \sup_{m\in[t_\star+\alpha,t]} V(m) \le q\mathcal{U}^\top \sup_{\ell\in[t_\star-\tau,t]} V(\ell) \tag{28}$$

where $\mathcal{U}$ is the vector in (21). Using

$$\sup_{\ell\in[t_\star-\tau,t]} V(\ell) \le \sup_{\ell\in[t_\star-\tau,t_\star+\alpha]} V(\ell) + \sup_{\ell\in[t_\star+\alpha,t]} V(\ell) \tag{29}$$

it follows from (28) that

$$\begin{aligned}\mathcal{U}^\top \sup_{m\in[t_\star+\alpha,t]} V(m) \le{}& q\mathcal{U}^\top \sup_{\ell\in[t_\star-\tau,t_\star+\alpha]} V(\ell)\\ &+ q\mathcal{U}^\top \sup_{\ell\in[t_\star+\alpha,t]} V(\ell).\end{aligned} \tag{30}$$

Since $q \in (0, 1)$, this inequality is equivalent to

$$\mathcal{U}^\top \sup_{m\in[t_\star+\alpha,t]} V(m) \le \tfrac{q}{1-q}\mathcal{U}^\top \sup_{\ell\in[t_\star-\tau,t_\star+\alpha]} V(\ell). \tag{31}$$

It follows that

$$\mathcal{U}^\top V(t) \le \tfrac{q}{1-q}\mathcal{U}^\top \sup_{\ell\in[t_\star-\tau,t_\star+\alpha]} V(\ell). \tag{32}$$

Hence, since $\mathcal{U}$ is a positive vector, $V(t)$ is bounded.

Second Part. Let us prove that $\lim_{t\to+\infty} V(t) = 0$. Let us introduce the following functions:

$$\mathcal{N}_i^1(a, b) = \sup_{m\in[a+\alpha,b]} v_i(m) \text{ and}$$
$$\mathcal{N}_i^2(a, b) = \sup_{m\in[a-\tau,b]} v_i(m) \tag{33}$$

for $i = 1$ to n, having the domains $\mathcal{E}_1 = \{(a, b) \in [0, +\infty)^2 : b \ge a + \alpha\}$ and $\mathcal{E}_2 = \{(a, b) \in [0, +\infty)^2 : b \ge a \ge \tau\}$, respectively, where we continue using the notation from Section II-A. We have proved in the first part of the proof that the functions $\mathcal{N}_i^j$ are bounded. Moreover, they are continuous, nonincreasing in their first argument, and nondecreasing in their second argument.

Hence, there are bounded functions $\mathcal{I}_i^j(a)$ such that

$$\lim_{b\to+\infty} \mathcal{N}_i^j(a, b) = \mathcal{I}_i^j(a) \tag{34}$$

for $j \in \{1, 2\}$ and $i \in \{1, \ldots, n\}$. The functions $\mathcal{I}_i^j$ are nonincreasing and lower bounded by 0. It follows that there are constants $\mathcal{I}_{i,\infty}^j \ge 0$ such that

$$\lim_{a\to+\infty} \mathcal{I}_i^j(a) = \mathcal{I}_{i,\infty}^j \tag{35}$$

for all $j \in \{1, 2\}$ and $i \in \{1, \ldots, n\}$. Now, recall that for any $t_\star \ge 0$, the inequality (27) is satisfied for all $t \ge t_\star + \alpha$, which we rewrite as

$$\begin{pmatrix}\mathcal{N}_1^1(t_\star, t)\\ \vdots\\ \mathcal{N}_n^1(t_\star, t)\end{pmatrix} \le \overline{R}\begin{pmatrix}\mathcal{N}_1^2(t_\star, t)\\ \vdots\\ \mathcal{N}_n^2(t_\star, t)\end{pmatrix}. \tag{36}$$

It follows that

$$\begin{pmatrix}\mathcal{I}_1^1(t_\star)\\ \vdots\\ \mathcal{I}_n^1(t_\star)\end{pmatrix} \le \overline{R}\begin{pmatrix}\mathcal{I}_1^2(t_\star)\\ \vdots\\ \mathcal{I}_n^2(t_\star)\end{pmatrix}. \tag{37}$$

Since

$$\lim_{a\to+\infty} \sup_{m\in[a+\alpha,+\infty)} v_i(m) = \mathcal{I}_{i,\infty}^1 \tag{38}$$

and

$$\lim_{a\to+\infty} \sup_{m\in[a-\tau,+\infty)} v_i(m) = \mathcal{I}^2_{i,\infty} \tag{39}$$

we deduce that $\mathcal{I}^1_{i,\infty} = \mathcal{I}^2_{i,\infty}$ for all $i \in \{1,\ldots,n\}$. Thus

$$\mathcal{V}_\mathcal{I} \le \overline{R}\mathcal{V}_\mathcal{I} \tag{40}$$

with $\mathcal{V}_\mathcal{I} = [\mathcal{I}^1_{1,\infty},\ldots,\mathcal{I}^1_{n,\infty}]^\top$, by (37). Hence, $\mathcal{U}^\top\mathcal{V}_\mathcal{I} \le q\mathcal{U}^\top\mathcal{V}_\mathcal{I}$. Since $q\in(0,1)$, it follows that $\mathcal{I}^1_{i,\infty}=0$ for all $i\in\{1,\ldots,n\}$. Thus, $\lim_{a\to+\infty}\sup_{m\in[a+\alpha,+\infty)} v_i(m) = 0$ for each i, so $\lim_{t\to+\infty} V(t) = 0$. ■

C. Periodic Case

Consider the particular case where M and P are both periodic of some period $\omega>0$; many systems are periodic. However, one cannot simply apply the Floquet theory to reduce the periodic case to the constant coefficient case from Theorem 1, because the Floquet theory is usually nonconstructive and the time-varying changes of coordinates from the Floquet theory applied to positive systems do not necessarily yield a positive system. We propose a stability condition that can be more easily checked than Assumption 2 in this case. Let us introduce the following function:

$$\xi(t) = (I-\Phi(t+\omega,t))^{-1}\int_{t-\omega}^{t}\Phi(t,m)P(m)\mathrm{d}m \tag{41}$$

where the existence of the inverse follows from the exponential stability of (19), because if there were a nonzero vector $z\in\mathbb{R}^n$ and a $t\ge 0$ such that $\Phi(t+\omega,t)z=z$, then the periodicity and semigroup properties and the global asymptotic stability of (19) would give the contradiction $z=\Phi(t+\omega,t)^k z = \Phi(t+k\omega,t)z\to 0$ as $k\to+\infty$ with $k\in\mathbb{N}$. We also use the following condition.

Condition 1: There is a positive Schur stable matrix $\mathcal{B}$ such that

$$\xi(t)\le\mathcal{B} \tag{42}$$

for all $t\in[0,\omega]$.

We state and prove the following result.

Corollary 1: Let Assumption 1 hold, with M and P both periodic with some period $\omega>0$. Then, Condition 1 is satisfied if and only if Assumption 2 is satisfied.

Proof: First Part. Assume that Assumption 2 is satisfied. For notational convenience, we use the following functions:

$$\zeta(t,s) = \int_s^t \Phi(t,m)P(m)\mathrm{d}m \text{ and}$$
$$\Lambda(t,s) = \Phi(t,s)+\zeta(t,s). \tag{43}$$

Then, there are $h\in\mathbb{N}$ and a Schur stable matrix $\overline{R}\ge 0$ such that

$$\Phi(t,s)+\zeta(t,s)\le\overline{R} \tag{44}$$

for all $s\ge 0$ and $t\ge s+h\omega$.

Consequently, for all $t\ge\overline{h}\omega$ where $\overline{h}$ is any integer larger than h, the inequality $\Phi(t,t-\overline{h}\omega)+\zeta(t,t-\overline{h}\omega)\le\overline{R}$ holds. It follows from the semigroup property of Φ that

$$\varpi(t,\overline{h}) + \sum_{k=0}^{\overline{h}-1}\int_{t-(k+1)\omega}^{t-k\omega}\Phi(t,m)P(m)\mathrm{d}m \le \overline{R} \tag{45}$$

with $\varpi(t,\overline{h}) = \Phi(t,t-\omega)^{\overline{h}}$. This equality implies that

$$\mathcal{H}(\overline{h},t)\le\overline{R} \tag{46}$$

with $\mathcal{H}(\overline{h},t) = \sum_{k=0}^{\overline{h}-1}\int_{t-\omega}^{t}\Phi(t,t-k\omega)\Phi(t-k\omega,m-k\omega)P(m)\mathrm{d}m$, since $\varpi(t,\overline{h})\ge 0$ and again using the semigroup and periodicity properties. Since the matrix M is periodic of period $\omega>0$, (46) is equivalent to

$$\sum_{k=0}^{\overline{h}-1}\Phi(t,t-k\omega)\zeta(t,t-\omega)\le\overline{R}. \tag{47}$$

In terms of (41), (47) can be rewritten as follows:

$$\xi(t)\le\overline{R}+\kappa(t,\overline{h})\zeta(t,t-\omega) \tag{48}$$

where

$$\kappa(t,\overline{h}) = (I-\Phi(t+\omega,t))^{-1} - \sum_{k=0}^{\overline{h}-1}\Phi(t,t-k\omega). \tag{49}$$

For each $\epsilon>0$, there is a $\overline{\overline{h}}_\epsilon\in\mathbb{N}$ with $\overline{\overline{h}}_\epsilon\ge\overline{h}$ such that for all $h_\triangle\ge\overline{\overline{h}}_\epsilon$, we have $|\kappa(t,h_\triangle)\zeta(t,t-\omega)|\le\epsilon$ for all $t\ge 0$; this follows because of the geometric sum formula

$$\sum_{k=0}^{+\infty}\Phi(t+\omega,t)^k = (I-\Phi(t+\omega,t))^{-1} \tag{50}$$

and boundedness of M and P. Hence, Condition 1 holds.

Second Part. Let us assume that Condition 1 is satisfied. Let $\overline{\varepsilon}>0$ be a matrix such that $\mathcal{B}+\overline{\varepsilon}$ is Schur stable. Since (19) is uniformly exponentially stable, there is $\alpha_1>0$ such that for all $t\ge s+\alpha_1$, $|\Phi(t,s)|\le\frac{1}{2}\overline{\varepsilon}$ and

$$\int_s^{t-\lfloor\frac{t-s}{\omega}\rfloor\omega}\Phi(t,m)P(m)\mathrm{d}m\le\tfrac{1}{2}\overline{\varepsilon} \tag{51}$$

by picking $\alpha_1>\omega$ such that $\sup_{\ell\in[-1,1]}\Phi(t,s+\ell\omega)$ is small enough. Thus, if $t\ge s+\alpha_1$, then (43) gives the following:

$$\Lambda(t,s)\le\int_{t-j\omega}^{t}\Phi(t,m)P(m)\mathrm{d}m+\overline{\varepsilon} \tag{52}$$

with $j=\lfloor\frac{t-s}{\omega}\rfloor$. We deduce that

$$\Lambda(t,s)\le\sum_{k=0}^{j-1}\int_{t-(k+1)\omega}^{t-k\omega}\Phi(t,m)P(m)\mathrm{d}m+\overline{\varepsilon}. \tag{53}$$

Recalling the periodicity of M and P and using the semigroup property of Φ, it follows that $\Phi(t,l-k\omega)=\Phi(t,t-k\omega)\Phi(t-k\omega,l-k\omega)=\Phi(t,t-k\omega)\Phi(t,l)$ for all $l\in[t-\omega,t]$, and so also

$$\begin{aligned}\Lambda(t,s)&\le\sum_{k=0}^{j-1}\int_{t-\omega}^{t}\Phi(t,l-k\omega)P(l)\mathrm{d}l+\overline{\varepsilon}\\ &=\sum_{k=0}^{j-1}\Phi(t,t-k\omega)\zeta(t,t-\omega)+\overline{\varepsilon}\\ &\le\xi(t)+\overline{\varepsilon}\end{aligned} \tag{54}$$

where the last inequality can be deduced from the definition (41) of ξ, and (50), the fact that the partial sums in (54) form a nondecreasing sequence, the nonnegative valuedness of ζ, and the fact that the periodicity of M and the semigroup property of the state transition matrices give $\Phi(t+\omega,t)^k=\Phi(t,t-\omega)^k=\Phi(t,t-k\omega)$ for all integers $k\ge 0$. Hence, for all $t\ge s+\alpha_1$, we have $\Lambda(t,s)\le\xi(t)+\overline{\varepsilon}\le\mathcal{B}+\overline{\varepsilon}$. Hence, since $\mathcal{B}+\overline{\varepsilon}$ is Schur stable, Assumption 2 is satisfied with $\overline{R}=\mathcal{B}+\overline{\varepsilon}$. ■

IV. Illustrative Examples

A. First Illustration of Time-Invariant Case

Consider the n-dimensional dynamics

$$\begin{cases} \dot{Z}_1(t) = -c_1 Z_1(t) + Z_2(t-\tau_1) \\ \dot{Z}_2(t) = -c_2 Z_2(t) + Z_3(t-\tau_2) \\ \quad\vdots \\ \dot{Z}_{n-1}(t) = -c_{n-1} Z_{n-1}(t) + Z_n(t-\tau_{n-1}) \\ \dot{Z}_n(t) = -c_n Z_n(t) + W(t) \end{cases} \tag{55}$$

with the input W, which occurs in [3, Lemma 2] in the context of stabilization of linear strict-feedback systems with delayed integrators, where the constants c_i and τ_i are positive for $i = 1, 2, \ldots, n-1$ (but similar reasoning applies if the delays τ_i and τ_{ij} in this section are bounded continuous time-varying functions, provided the upper bound τ in the following analysis is taken to be a positive constant). We assume that W takes the form $W(t) = d_1 Z_1(t-\tau_{n1}) + \cdots + d_n Z_n(t-\tau_{nn})$ for constants $\tau_{ij} \geq 0$ and $d_i > 0$ for $i = 1, 2, \ldots, n$.

For such a function W, the system (55) is co-operative; see [11, Lemma 1]. Thus, when proving asymptotic stability properties for (55), we can restrict our attention to its nonnegative-valued solutions. Choosing $\tau > 0$ such that $\tau > \max\{\tau_1, \ldots, \tau_{n-1}, \tau_{n1}, \ldots, \tau_{nn}\}$, it follows that all C^1 solutions $Z : [-\tau, +\infty) \to [0, +\infty)^n$ of (55) are solutions of (6) in the special case where $M = \mathrm{diag}\{-c_1, \ldots, -c_n\}$ is a diagonal matrix having $-c_i$ as its ith main diagonal entry for $i = 1, \ldots, n$ and P is the $n \times n$ nonnegative matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ d_1 & d_2 & d_3 & \ldots & d_{n-1} & d_n \end{bmatrix}. \tag{56}$$

Then, we can find conditions on the c_i's and d_i's such that the corresponding matrix $M + P$ is Hurwitz, to ensure that all C^1 solutions $X : [-\tau, +\infty) \to [0, +\infty)^n$ of (55) exponentially converge to the origin as $t \to +\infty$.

For instance, in the special case where $n = 2$, it follows from the quadratic formula that $M + P$ is Hurwitz provided $c_1 + c_2 - d_2 > 0$ and $c_1(c_2 - d_2) > d_1$. For $n = 3$, the Hurwitzness condition on $M + P$ is that all roots of the characteristic polynomial $\chi_{M+P}(\lambda) = \lambda^3 + (c_1 + c_2 + c_3 - d_3)\lambda^2 + [c_2(c_3 - d_3) + c_1(c_3 + c_2 - d_3) - d_2]\lambda + c_1[c_2(c_3 - d_3) - d_2] - d_1$ of $M + P$ have negative real parts, which is equivalent to the requirements $(c_1 + c_2 + c_3 - d_3)[c_2(c_3 - d_3) + c_1(c_3 + c_2 - d_3) - d_2] > c_1[c_2(c_3 - d_3) - d_2] - d_1 > 0$ and $c_1 + c_2 + c_3 - d_3 > 0$ (by the Routh–Hurwitz criterion for the third-order polynomials). The preceding conditions can be checked even if there is uncertainty in the positive values c_i or in the nonnegative values d_i, under suitable conditions on known intervals containing these unknown parameter values. Moreover, we can allow uncertainty in the positive delays τ_i and τ_{ni} (including continuous and time-varying delays), if we know a bound $\tau > 0$ satisfying our conditions above. This illustrates how our work applies for delayed linear systems with uncertain coefficients and uncertain delays.

B. Second Illustration of Time-Invariant Case

Consider the following system:

$$\dot{X}(t) = AX(t) + \sum_{i=1}^{p} B_i X(t - \tau_i(t)) \tag{57}$$

for any integer $p \geq 1$ with X valued in $\mathbb{R}^n$ with constant matrices A and B_i for $i = 1, \ldots, p$ where A is Hurwitz (but not necessarily Metzler), and where $\overline{\tau} > 0$ will denote a known bound on the piecewise continuous delays $\tau_i : [0, +\infty) \to [0, \overline{\tau}]$ for all i. We provide novel conditions that are independent of the delays τ_i's and that ensure that one can build an interval observer whose existence implies that (57) is globally exponentially stable to the origin; see, for instance, Mazenc and Bernard [13] for the notion of interval observer. While more complicated than standard analysis for the linear time-invariant systems, our analysis is called for because of the mildness of the conditions on the delays and coefficient matrices, which puts this example outside the scope of existing results for the linear time-invariant systems. One key ingredient in our interval observer design will be the proof of [12, Th. 2] for Hurwitz matrices A, which constructs a C^1 function $Q : [0, +\infty) \to \mathbb{R}^{n\times n}$ with a bounded inverse and a constant Metzler matrix M such that $\dot{Q}(t)Q(t)^{-1} + Q(t)AQ(t)^{-1} = M$ for all $t \geq 0$.

To build the interval observer, first note that in terms of any Q that satisfies the requirements from the preceding paragraph, the new variable $Z(t) = Q(t)X(t)$ satisfies the following:

$$\dot{Z}(t) = MZ(t) + \sum_{i=1}^{p} L_i(t) Z(t - \tau_i(t)) \tag{58}$$

for all $t \geq \overline{\tau}$, where $L_i(t) = Q(t)B_iQ(t - \tau_i(t))^{-1}$ for all i and $t \geq \overline{\tau}$. Next, we introduce the following dynamic extension:

$$\begin{cases} \dot{\overline{Z}}(t) = M\overline{Z}(t) + \sum_{i=1}^{p} L_i(t)^+ \overline{Z}(t - \tau_i(t)) \\ \qquad - \sum_{i=1}^{p} L_i(t)^- \underline{Z}(t - \tau_i(t)) \\ \dot{\underline{Z}}(t) = M\underline{Z}(t) + \sum_{i=1}^{p} L_i(t)^+ \underline{Z}(t - \tau_i(t)) \\ \qquad - \sum_{i=1}^{p} L_i(t)^- \overline{Z}(t - \tau_i(t)) \end{cases} \tag{59}$$

where $C^+ = [\max\{c_{i,j}, 0\}]$ and $C^- = C^+ - C$ for all matrices $C = [c_{i,j}]$. The change of coordinates $Z_\ddagger(t) = -\underline{Z}(t)$ yields

$$\begin{cases} \dot{\overline{Z}}(t) = M\overline{Z}(t) + \sum_{i=1}^{p} L_i(t)^+ \overline{Z}(t - \tau_i(t)) \\ \qquad + \sum_{i=1}^{p} L_i(t)^- Z_\ddagger(t - \tau_i(t)) \\ \dot{Z}_\ddagger(t) = MZ_\ddagger(t) + \sum_{i=1}^{p} L_i(t)^+ Z_\ddagger(t - \tau_i(t)) \\ \qquad + \sum_{i=1}^{p} L_i(t)^- \overline{Z}(t - \tau_i(t)) \end{cases}. \tag{60}$$

Since M is Metzler, it follows that (60) is cooperative; this follows by a variant of the argument from the appendix in [14], which also explains why global exponential stability of (60) to the origin follows if all positive-valued solutions of (60) exponentially converge to the origin as $t \to +\infty$.

Thus, we focus on the positive solutions of (60). Let

$$\tilde{Z}(t) = \overline{Z}(t) + Z_\ddagger(t). \tag{61}$$

Then, we have the following:

$$\dot{\tilde{Z}}(t) = M\tilde{Z}(t) + \sum_{i=1}^{p} [L_i(t)^+ + L_i(t)^-]\tilde{Z}(t - \tau_i(t)). \tag{62}$$

Setting $\mathcal{S}_{\overline{\tau}}(\tilde{Z}_t) = \sup_{\ell \in [t-\overline{\tau}, t]} \tilde{Z}(\ell)$ gives the following:

$$\begin{aligned} \dot{\tilde{Z}}(t) &\leq M\tilde{Z}(t) + \sum_{i=1}^{p} [L_i(t)^+ + L_i(t)^-]\, \mathcal{S}_{\overline{\tau}}(\tilde{Z}_t) \\ &\leq M\tilde{Z}(t) + \overline{L}\mathcal{S}_{\overline{\tau}}(\tilde{Z}_t) \end{aligned} \tag{63}$$

where the matrix $\overline{L} \geq 0$ is such that

$$\sum_{i=1}^{p} [L_i(t)^+ + L_i(t)^-] \leq \overline{L} \tag{64}$$

for all $t \geq 0$. Hence, if $M + \overline{L}$ is Hurwitz, then we can apply Remark 1 to conclude that (62) is globally exponentially stable to the origin. Since $\overline{Z}$ and $Z_\ddagger$ are nonnegative valued, it follows from (61) that (60) is also globally exponentially stable to the origin, so the origin of (59) is also a globally exponentially stable equilibrium.

Since $L_i = L_i^+ - L_i^-$ for each i, the reasoning we used in our proof of cooperativity of (60) shows cooperativity of the dynamics for $(Z_+, Z_-) = (\overline{Z} - Z, Z - \underline{Z})$. Hence, $\overline{Z}(t) \geq Z(t) \geq \underline{Z}(t)$ for all $t \geq 0$ if we choose any initial functions for $\overline{Z}$ and $\underline{Z}$ such that $\overline{Z}(t) \geq Z(t) \geq \underline{Z}(t)$ for all $t \in [-\bar{\tau}, 0]$. Therefore, (59) provides an interval observer for (58) and all the solutions $Z(t)$ exponentially converge to the origin as $t \to +\infty$. The inequality $|X(t)| \leq |Q(t)^{-1}||Z(t)|$ for all $t \geq 0$ and the boundedness of $Q(t)^{-1}$ allow us to conclude that the X dynamics are globally exponentially stable to the origin.

Remark 3: In many cases, we can take Q to be constant, notably when all eigenvalues of A are real, by picking Q such that $QAQ^{-1} = M$ is the Jordan canonical form of A. Then, our sufficient condition for (57) to be globally exponentially stable to the origin is that $QAQ^{-1} + \overline{L}$ is Hurwitz. When A is Metzler, we can take $Q = I$ and then our sufficient condition is that $A + \sum_{i=1}^{p} [B_i^+ + B_i^-]$ is Hurwitz. See also [12, Sec. 4.3] for a Hurwitz matrix A that has a conjugate pair of complex (nonreal) eigenvalues and that calls for a nonconstant choice of the matrix Q.

C. Illustration of Corollary 1

We show that Corollary 1 makes it possible to prove the exponential stability in cases where (6) is satisfied and some coefficients of P take large values at some instants, and without any restriction on the delay bound, which we believe puts this example outside the scope of previous results. Given $p \in \mathbb{N}$, consider the system

$$\begin{cases} \dot{v}_1(t) = -3v_1(t) + (1 - \frac{1}{4}\cos(t))v_2(t) \\ \qquad + c_* \sin^{2p}(t) \sup\limits_{l \in [t-\tau, t]} v_2(l) \\ \dot{v}_2(t) = -2v_2(t) + \frac{9}{10}(1 - \sin(t)) \sup\limits_{l \in [t-\tau, t]} v_1(l) \end{cases} \tag{65}$$

where τ and c_* are positive constants, and v_1 and v_2 are nonnegative valued. Let us show that for any $c_* > 0$, the origin of (65) is globally exponentially stable when

$$c_* < 0.12\sqrt{1+p}. \tag{66}$$

With the notation of Section III, we can take $\omega = 2\pi$

$$M(t) = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \text{ and}$$

$$P(t) = \begin{bmatrix} 0 & \mathcal{H}(t) \\ \frac{9}{10}(1 - \sin(t)) & 0 \end{bmatrix} \tag{67}$$

where $\mathcal{H}(t) = 1 - \frac{1}{4}\cos(t) + c_* \sin^{2p}(t)$. Thus

$$\Phi(t, r) = \begin{bmatrix} e^{-3(t-r)} & 0 \\ 0 & e^{-2(t-r)} \end{bmatrix}. \tag{68}$$

Consequently

$$(I - \Phi(t + 2\pi, t))^{-1} = \begin{bmatrix} \frac{1}{1-e^{-6\pi}} & 0 \\ 0 & \frac{1}{1-e^{-4\pi}} \end{bmatrix}. \tag{69}$$

Also, since

$$\begin{aligned} &\Phi(t, \ell)P(\ell) \\ &= \begin{bmatrix} e^{-3(t-\ell)} & 0 \\ 0 & e^{-2(t-\ell)} \end{bmatrix} \begin{bmatrix} 0 & \mathcal{H}(\ell) \\ \frac{9}{10}(1 - \sin(\ell)) & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & e^{-3(t-\ell)}\mathcal{H}(\ell) \\ \frac{9}{10}e^{-2(t-\ell)}(1 - \sin(\ell)) & 0 \end{bmatrix} \end{aligned} \tag{70}$$

the choice $\mathcal{H}_*(t) = \int_{t-2\pi}^{t} e^{-3(t-\ell)}\mathcal{H}(\ell)\mathrm{d}\ell$ gives

$$\begin{aligned} &\int_{t-2\pi}^{t} \Phi(t, \ell)P(\ell)\mathrm{d}\ell \\ &= \begin{bmatrix} 0 & \mathcal{H}_*(t) \\ \frac{9}{10}\int_{t-2\pi}^{t} e^{-2(t-\ell)}(1 - \sin(\ell))\mathrm{d}\ell & 0 \end{bmatrix}. \end{aligned} \tag{71}$$

Then, the function

$$\xi(t) = (I - \Phi(t + 2\pi, t))^{-1} \int_{t-2\pi}^{t} \Phi(t, \ell)P(\ell)\mathrm{d}\ell \tag{72}$$

from (41) satisfies

$$\xi(t) = \begin{bmatrix} 0 & \theta_1(t) + \theta_2(t) \\ \theta_3(t) & 0 \end{bmatrix} \tag{73}$$

with the nonnegative-valued functions

$$\theta_1(t) = \frac{1}{1-e^{-6\pi}} \int_{t-2\pi}^{t} e^{-3(t-\ell)} \left(1 - \tfrac{1}{4}\cos(\ell)\right) \mathrm{d}\ell \tag{74}$$

$$\theta_2(t) = \frac{c_*}{1-e^{-6\pi}} \int_{t-2\pi}^{t} e^{-3(t-\ell)} \sin^{2p}(\ell)\mathrm{d}\ell \tag{75}$$

and

$$\theta_3(t) = \frac{9}{10(1-e^{-4\pi})} \int_{t-2\pi}^{t} e^{-2(t-\ell)}(1 - \sin(\ell))\mathrm{d}\ell. \tag{76}$$

Then, the simple mathematica calculations give the following:

$$\theta_1(t) = \tfrac{1}{3} - \tfrac{3}{40}\cos(t) - \tfrac{1}{40}\sin(t) \leq 0.413 \tag{77}$$

and

$$\begin{aligned} &\theta_3(t) \\ &= \frac{9}{10(1-e^{-4\pi})}(1 - e^{-4\pi})\left(\tfrac{1}{2} + \tfrac{1}{5}(\cos(t) - 2\sin(t))\right) \\ &\leq 0.853 \end{aligned} \tag{78}$$

for all $t \in \mathbb{R}$. We deduce that

$$\xi(t) \leq \begin{bmatrix} 0 & 0.413 + \theta_2(t) \\ 0.853 & 0 \end{bmatrix} \tag{79}$$

for all $t \geq 0$. Also, for each $p \in \mathbb{N}$ and $t \geq 0$, we have the following:

$$\begin{aligned} \theta_2(t) &\leq \frac{c_*}{1-e^{-6\pi}} \int_{t-2\pi}^{t} \sin^{2p}(\ell)\mathrm{d}\ell \\ &\leq \frac{4c_*}{1-e^{-6\pi}} \int_0^{\frac{\pi}{2}} \sin^{2p}(\ell)\mathrm{d}\ell. \end{aligned} \tag{80}$$

By the integration by parts formula $\int u(\ell)v'(\ell)\mathrm{d}\ell = u(\ell)v(\ell) - \int u'(\ell)v(\ell)\mathrm{d}\ell$ with $u = \sin^{2p-1}$ and $v = -\cos$ and the formula $\cos^2 = 1 - \sin^2$, we solve for the second integral in (80) to conclude that for all $p \in \mathbb{N}$, we have

$$\int_0^{\frac{\pi}{2}} \sin^{2p}(\ell)\mathrm{d}\ell = \left(1 - \tfrac{1}{2p}\right) \int_0^{\frac{\pi}{2}} \sin^{2(p-1)}(\ell)\mathrm{d}\ell. \tag{81}$$

Thus, since $\ln(1 - a) \leq -a$ for all $a \in (0, 1)$, we get the following:

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \sin^{2p}(\ell)\mathrm{d}\ell = \tfrac{\pi}{2} \prod_{k=1}^{p} \left(1 - \tfrac{1}{2k}\right) \\ &= \tfrac{\pi}{2} e^{\sum\limits_{k=1}^{p} \ln\left(1 - \frac{1}{2k}\right)} \leq \tfrac{\pi}{2} e^{-\frac{1}{2}\sum\limits_{k=1}^{p} \frac{1}{k}} \end{aligned} \tag{82}$$

by applying (81) recursively to reduce the power of sin in the integer and to 0. Since

$$\sum_{k=1}^{p} \frac{1}{k} \geq \int_1^{p+1} \frac{1}{s} \mathrm{d}s = \ln(1+p) \tag{83}$$

we obtain

$$\int_0^{\frac{\pi}{2}} \sin^{2p}(\ell) \mathrm{d}\ell \leq \frac{\pi}{2} e^{-\frac{\ln(1+p)}{2}} = \frac{\pi}{2} \frac{1}{\sqrt{1+p}}. \tag{84}$$

It follows from (80) that

$$\theta_2(t) \leq \frac{4c_*}{1-e^{-6\pi}} \frac{1}{\sqrt{1+p}} \frac{\pi}{2}. \tag{85}$$

Thus, $\xi(t) \leq \mathcal{G}$, where

$$\mathcal{G} = \begin{bmatrix} 0 & 0.413 + \frac{4c_*}{1-e^{-6\pi}} \frac{1}{\sqrt{1+p}} \frac{\pi}{2} \\ 0.853 & 0 \end{bmatrix}. \tag{86}$$

The matrix $\mathcal{G}$ is Schur stable if and only if the inequality

$$\left(0.413 + \frac{4c_*}{1-e^{-6\pi}} \frac{\pi/2}{\sqrt{1+p}}\right) 0.853 < 1 \tag{87}$$

is satisfied. Condition (87) holds if (66) is satisfied. Hence, Condition 1 is satisfied. Then, Corollary 1 implies that Assumption 2 is satisfied, so Theorem 2 applies.

V. Conclusion

We proved extensions of the stability analysis technique based on Halanay's inequality, which are suitable for the analysis of interconnected systems. Key features included our allowing time-varying delays and our novel use of positive systems and interval observers. This produced vector analogs of Halanay's inequality. Our results can be used to study time-varying systems with uncertain time-varying delays that were beyond the scope of the literature for the linear time-invariant systems. The ISS property with respect to additive disturbances can be proved. We hope to obtain extensions for PDEs and sampling, where instead of continuous time systems, we have continuous-discrete systems whose states are reset at the sample times.

Appendix

A. Schur Stable Matrix

We used this lemma in our proof of Theorem 1.

Lemma 1: Let the matrix M be Metzler and Hurwitz. Let $P \geq 0$ be a matrix such that $M + P$ is Hurwitz. Then, the matrix $-M^{-1}P$ is nonnegative and Schur stable.

Proof: By [21, Prop. 1], $-M^{-1} \geq 0$. Hence, $-M^{-1}P$ is nonnegative. Also, [21, Prop. 1] provides a vector $V > 0$ and a real number $c > 0$ such that $(M + P)V \leq -cPV$. Since $-M^{-1} \geq 0$, we deduce that $-M^{-1}(M + P)V \leq cM^{-1}PV$, which is equivalent to $-M^{-1}PV \leq \frac{1}{1+c}V$. Since $\frac{1}{1+c} \in (0, 1)$, [21, Prop. 2] allows us to conclude. ■

References

[1] S. Ahmed, F. Mazenc, and H. Ozbay, "Dynamic output feedback stabilization of switched linear systems with delay via a trajectory based approach," *Automatica*, vol. 93, pp. 92–97, 2018.

[2] C. Baker, "Development and application of Halanay-type theory: Evolutionary differential and difference equations with time lag," *J. Comput. Appl. Math.*, vol. 234, pp. 2663–2682, 2010.

[3] N. Bekiaris-Liberis and M. Krstic, "Stabilization of linear strict-feedback systems with delayed integrators," *Automatica*, vol. 46, pp. 1902–1910, 2010.

[4] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. Philadelphia, PA, USA: SIAM, 1994.

[5] S. Dashkovskiy, B. Ruffer, and F. Wirth, "An ISS small-gain theorem for general networks," *Math. Control, Signals, Syst.*, vol. 19, pp. 93–122, 2007.

[6] M. Di Ferdinando, P. Pepe, and E. Fridman, "Exponential input-to-state stability of globally Lipschitz time-delay systems under sampled-data noisy output feedback and actuation disturbances," *Int. J. Control*, New York, NY, USA: Taylor & Francis, 2020.

[7] W. M. Haddad, V. Chellaboina, and Q. Hui, *Nonnegative and Compartmental Dynamical Systems*. Princeton, NJ, USA: Princeton Univ. Press, 2010.

[8] A. Halanay, *Differential Equations: Stability, Oscillations, Time Lags*. New York, NY, USA: Academic, 1966.

[9] O. Mason and R. Shorten, "On linear copositive Lyapunov functions and the stability of switched positive linear systems," *IEEE Trans. Autom. Control*, vol. 52, no. 7, pp. 1346–1349, Jul. 2007.

[10] V. M. Matrosov, "Vector Lyapunov function method: Theory and application to complex industrial systems," *IFAC Proc. Vol.*, vol. 30, no. 6, pp. 49–62, 1997.

[11] F. Mazenc, V. Andrieu, and M. Malisoff, "Design of continuous-discrete observers for time-varying nonlinear systems," *Automatica*, vol. 57, no. 7, pp. 135–144, 2015.

[12] F. Mazenc and O. Bernard, "Interval observers for linear time-invariant systems with disturbances," *Automatica*, vol. 47, no. 1, pp. 140–147, 2011.

[13] F. Mazenc and O. Bernard, "ISS interval observers for nonlinear systems transformed into triangular systems," *Int. J. Robust Nonlinear Control*, vol. 24, no. 7, pp. 1241–1261, 2014.

[14] F. Mazenc and M. Malisoff, "Stability analysis for time-varying systems with delay using linear Lyapunov functions and a positive systems approach," *IEEE Trans. Autom. Control*, vol. 61, no. 3, pp. 771–776, Mar. 2016.

[15] F. Mazenc and M. Malisoff, "Extension of Razumikhin's theorem and Lyapunov-Krasovskii functional constructions for time-varying systems with delay," *Automatica*, vol. 78, pp. 1–13, 2017.

[16] F. Mazenc, M. Malisoff, and M. Krstic, "Stability analysis using new variant of Halanay's inequality," in *Proc. 24th Int. Symp. Math. Theory Netw. Syst.*, Aug. 2020, pp. 24–28.

[17] F. Mazenc, M. Malisoff, and M. Krstic, "Stability analysis for time-varying systems with asynchronous sampling using contractivity approach," *IEEE Control Syst. Lett.*, vol. 5, no. 1, pp. 49–54, 2021.

[18] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*. Philadelphia, PA, USA: SIAM, 2000.

[19] P. H. A. Ngoc, "A Perron-Frobenius theorem for a class of positive quasi-polynomial matrices," *Appl. Math. Lett.*, vol. 19, pp. 747–751, 2006.

[20] P. Pepe and E. Fridman, "On global exponential stability preservation under sampling for globally Lipschitz time-delay systems," *Automatica*, vol. 82, pp. 295–300, 2017.

[21] A. Rantzer, "Distributed control of positive systems," *Eur. J. Control*, vol. 24, pp. 72–80, 2015.

[22] W. Ren and J. Xiong, "Vector-Lyapunov-function-based input-to-state stability of stochastic impulsive switched time-delay systems," *IEEE Trans. Autom. Control*, vol. 64, no. 2, pp. 654–669, Feb. 2019.

[23] A. Selivanov and E. Fridman, "Distributed event-triggered control of transport-reaction system," *IFAC PapersOnLine*, vol. 48, no. 11, pp. 593–597, 2015.

[24] A. Selivanov and E. Fridman, "Distributed event-triggered control of diffusion semilinear PDEs automatica," *Automatica*, vol. 68, pp. 344–351, 2016.

[25] E. D. Sontag, *Mathematical Control Theory*. 2nd ed. New York, NY, USA: Springer-Verlag, 1998.

[26] D. Xu, X. Wang, Y. Hong, Z. Jiang, and S. Xu "Output feedback stabilization and estimation of the region of attraction for nonlinear systems: A vector control Lyapunov function perspective," *IEEE Trans. Autom. Control*, vol. 61, no. 12, pp. 4034–4040, Dec. 2016.

[27] B. Zhou and A. V. Egorov, "Time-varying Razumikhin and Krasovskii stability theorems for time-varying delay systems," *Automatica*, vol. 71, pp. 281–291, 2016.