New Versions of Halanay's Inequality With Multiple Gain Terms

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***Abstract*—Halanay's inequality in its standard form is a widely used tool for the analysis of systems with delays and uncertainties, but the assumptions needed to use it are sometimes too restrictive to cover applications of interest. This letter provides new generalizations of Halanay's inequality where, instead of the usual supremum in the gain term in standard versions of Halanay's inequality, we use a weighted sum of suprema over different intervals. This allows us to derive sufficient conditions for asymptotic and input-to-state stability. We apply our results to linear systems with switched delays and other examples that illustrate how our results are less restrictive than those of other contributions available in the literature.**

***Index Terms*—Stability, delays, time-varying.**

I. INTRODUCTION

DELAYED systems play an essential role in control theory and control applications, owing to time delayed information transmission (leading to sensor delays) and time delays in control actuation (which can be represented by input delays). Lyapunov-Krasovskii techniques are well suited for systems with known constant delays, but usually are not as easily applied to systems with time-varying or unknown delays. By contrast, Halanay's inequality technique (which was initiated in [3]) is a useful stability analysis tool for the stability analysis of systems with poorly known time-varying delays. This inequality and variants of it, which complement Razumikhin's theorem (which was used in contributions such as [4] and [16]), have been studied in several papers, such as [2], [5], [8], [9], [11], [12], and [15], to extend the domain of application of Halanay's stability analysis strategy.

In particular, the paper [13] presents notable results for time-varying Halanay inequalities of the type

$$\dot{v}(t) \leq -a(t)v(t) + b(t) \sup_{\ell \in [t-T,t]} v(\ell) \tag{1}$$

where $T \geq 0$ is a constant, v is nonnegative valued, and the term containing b is called the gain term. In [13], sufficient conditions for $\lim_{t\to+\infty} v(t) = 0$ to hold are given for cases where (1) holds and where $a(t)$ can take both positive and negative values and where $b(t)$ is larger than $a(t)$ on some arbitrarily large time intervals. This contrasts with the standard Halanay's inequality (e.g., [1, Lemma 4.2, p. 138]) having the form (1) where $a > 0$ and $b \in [0, a)$ are constants.

Motivated by the fact that inequalities of the type (1) can be used to study time-varying systems with time-varying delays, we revisit the main result of [13]. To obtain less restrictive conditions and establish input-to-state stability (or ISS) like inequalities, we consider generalized Halanay's inequalities with multiple gain terms of the type

$$\dot{v}(t) \leq -a(t)v(t) + \sum_{i=1}^{k} b_i(t) \sup_{\ell \in [t-T_{2,i}, t-T_{1,i}]} v(\ell) + \delta(t) \tag{2}$$

where a, the b_i's, and δ (which can represent an uncertainty) are nonnegative scalar valued piecewise continuous functions; see [14] for a presentation of the ISS notion. The b_i's are the coefficients in our multiple gain terms that are summed in (2). Later in Section II, we will describe persistence of excitation relations involving a and the b_i's that will ensure ISS conditions. The key idea which guides us consists of taking advantage of the knowledge of the constants $T_{1,i}$ and $T_{2,i} \geq T_{1,i}$, which can be derived from information on the delays of a system with poorly known time-varying delays. As in [13], our assumptions are satisfied by functions a which take both positive and negative values (which contrasts with our earlier results on generalized Halanay's inequalities from [8]–[11], which required a to be nonnegative valued) and the largest value of b_i can be arbitrarily large over arbitrarily large intervals, provided that the values $a(t)$ of the function a are positive and large on sufficiently long time intervals. Also, our Halanay inequality generalizations [8]–[11] did not use the comparison function approach that we use here, and they use only one gain term, instead of the multiple gain terms that we use here.

By contrast with the main result of [13], we establish ISS like inequalities. By the definition of ISS, this ensures that $\lim_{t+\infty} v(t) = 0$ when δ is the zero function. Our examples in Section III below show that the results we obtain are less restrictive than the latest existing results about Halanay's inequality, in several circumstances. For instance, see Section III for an example where our use of multiple gain terms is shown to be useful for obtaining less conservative results than [8]–[10]. By contrast with [7], the technique we propose does not use the strictification technique from [6].

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We use standard notation, which is simplified when no confusion would arise. The dimensions of our Euclidean spaces are arbitrary unless otherwise noted. The standard Euclidean norm and induced matrix norm are denoted by $|\cdot|$, $|\cdot|_\infty$ is the corresponding sup norm, and $|\cdot|_S$ is the supremum over a set S. We set $\mathbb{Z}_{\geq 0}=\{0, 1, \ldots\}$, $\mathbb{N}=\mathbb{Z}_{\geq 0}\backslash\{0\}$, $\mathbb{R}$ is the set of all real numbers, $[0, +\infty)$ is the set of all nonnegative real numbers, and I is the identity matrix.

II. Generalized Halanay's Inequalities

This section provides our main theorems, which we apply to systems with switched delays and other cases in Section III. Our first theorem covers cases of the form (2) with multiple suprema, and is based on a novel comparison approach. Our second theorem only allows one supremum term (i.e., $k = 1$ in (2)), but it provides very different sufficient conditions as compared with our first theorem.

A. Halanay's Result With Several Sup Terms

Consider any $k \in \mathbb{N}$ and constants $T_{1,i}$ and $T_{2,i}$ such that

$$\begin{aligned} &0 \leq T_{1,i} \leq T_{2,i} \text{ for all } i \in \{1, \ldots, k\} \\ &\text{and } T_{2,1} \leq \cdots \leq T_{2,k-1} \leq T_{2,k}. \end{aligned} \tag{3}$$

Consider a piecewise C^1 continuous function $v : [-T_{2,k}, +\infty) \to [0, +\infty)$, and locally bounded piecewise continuous functions $\delta : [0, +\infty) \to [0, +\infty)$, $a : \mathbb{R} \to \mathbb{R}$, and $b_i : [0, +\infty) \to [0, +\infty)$ for $i = 1, \ldots, k$ such that (2) holds for all $t \geq 0$, where δ represents uncertainty.

To analyze the behavior of the function v, we fix constants $\epsilon > 0$ and $t_0 \geq 0$, we define the functions $\mathcal{S}_i$ by

$$\mathcal{S}_i(t) = \max\left\{0, \sup_{\ell\in[t-T_{2,i},t-T_{1,i}]} \int_{\ell}^{t-T_{1,i}} a(m)\mathrm{d}m\right\} \tag{4}$$

for $i = 1, \ldots, k$, and we introduce the comparison system

$$\begin{cases} \dot{y}_\epsilon(t) = -a(t)y_\epsilon(t) + \delta(t) \\ \qquad + \sum_{i=1}^{k} [b_i(t) + \epsilon]e^{\mathcal{S}_i(t)}y_\epsilon(t - T_{1,i}) \\ \qquad \text{for all } t \geq t_0 \\ y_\epsilon(t) = \sup_{\ell\in[t_0-T_{2,k},t_0]} v(\ell) + \epsilon \\ \qquad \text{for all } t \in [t_0 - T_{2,k}, t_0]. \end{cases} \tag{5}$$

We prove the following result, whose conclusion ensures that v satisfies an ISS estimate for all $t \geq t_0$ when y_ϵ satisfies such an ISS estimate, and where we will provide sufficient conditions for y_ϵ to satisfy the required ISS estimates in our ISS corollary below, which also provides explicit ISS estimates for the function v with (ϵ, δ) viewed as the perturbation and which gives an ISS result with respect to δ as $\epsilon \to 0$.

Theorem 1: Consider the functions v and y_ϵ defined above. Then the inequality

$$v(t) \leq y_\epsilon(t) \tag{6}$$

holds for all $t \geq t_0 - T_{2,k}$.

Proof: The second equality in (5) and the positiveness of ϵ imply that $v(t) < y_\epsilon(t)$ for all $t \in [t_0 - T_{2,k}, t_0]$. Next, we proceed by contradiction. Bearing in mind that v and y_ϵ are continuous, let us suppose that there were a $t_\sharp > t_0$ such that $v(t_\sharp) = y_\epsilon(t_\sharp)$ and $v(t) < y_\epsilon(t)$ for all $t \in [t_0 - T_{2,k}, t_\sharp)$, i.e., $t_\sharp$ is the first time t when $v(t) = y_\epsilon(t)$. Then from (2) and (5), it follows that the inequality

$$\begin{aligned} \dot{v}(t_\sharp) - \dot{y}_\epsilon(t_\sharp) \leq& \sum_{i=1}^{k} b_i(t_\sharp) \sup_{\ell\in[t_\sharp-T_{2,i},t_\sharp-T_{1,i}]} v(\ell) \\ &- \sum_{i=1}^{k} [b_i(t_\sharp)+\epsilon]e^{\mathcal{S}_i(t_\sharp)}y_\epsilon(t_\sharp-T_{1,i}) \end{aligned} \tag{7}$$

is satisfied. By the second equality in the comparison system (5) and the positiveness of ϵ and the nonnegativity of the v values, we have $y_\epsilon(t) > 0$ for all $t \in [t_0 - T_{2,k}, t_0]$. Hence, since δ and the b_i's are nonnegative valued, we can collect terms on the right side of the first equality in (5) to find a continuous function $\mathcal{A}$ such that $\dot{y}_\epsilon(t) \geq \mathcal{A}(t)y_\epsilon(t)$ for all $t \geq t_0$ if each $T_{1,i}$ is zero, or such that $\dot{y}_\epsilon(t) \geq \mathcal{A}(t)y_\epsilon(t)$ on $[t_0, t_0 + \min_{i\in\mathcal{N}} T_{1,i}]$ where $\mathcal{N} \subseteq \{1, \ldots, k\}$ is the index set of all i values such that $T_{1,i} > 0$ if not all of the $T_{1,i}$'s are 0. In the first case, we can apply the method of variation of parameters to the inequality $\dot{y}_\epsilon(t) \geq \mathcal{A}(t)y_\epsilon(t)$ on any interval of the form $[t_0, T_*)$ to get $y_\epsilon(t) \geq e^{\int_{t_0}^{t} \mathcal{A}(\ell)\mathrm{d}\ell}y_\epsilon(t_0) > 0$ for all $t \geq t_0$. In the second case, we can apply variation of parameters to the same inequality in a similar way to get $y_\epsilon(t) > 0$ for all $t \in [t_0, t_0 + \min_{i\in\mathcal{N}} T_{1,i}]$. Repeating this process in a method of steps in the case where $\mathcal{N} \neq \emptyset$ gives $y_\epsilon(t) > 0$ for all $t \in [t_0 + \ell \min_{i\in\mathcal{N}} T_{1,i}, t_0 + (\ell + 1) \min_{i\in\mathcal{N}} T_{1,i}]$ for all $\ell \in \mathbb{Z}_0$. By combining both cases, it now follows that $y_\epsilon(t) > 0$ for all $t \geq t_0 - T_{2,k}$. Hence, for all $i \in \{1, \ldots, k\}$, we have $\epsilon e^{\mathcal{S}_i(t_\sharp)}y_\epsilon(t_\sharp - T_{1,i}) > 0$, so

$$\begin{aligned} \dot{v}(t_\sharp) - \dot{y}_\epsilon(t_\sharp) <& \sum_{i=1}^{k} b_i(t_\sharp)\Big[\sup_{\ell\in[t_\sharp-T_{2,i},t_\sharp-T_{1,i}]} v(\ell) \\ &- e^{\mathcal{S}_i(t_\sharp)}y_\epsilon(t_\sharp - T_{1,i})\Big]. \end{aligned} \tag{8}$$

Since v is continuous, it follows that for each $i \in \{1, \ldots, k\}$, there is an $s_{i\star} \in [t_\sharp - T_{2,i}, t_\sharp - T_{1,i}]$ such that $v(s_{i\star}) = \sup_{\ell\in[t_\sharp-T_{2,i},t_\sharp-T_{1,i}]} v(\ell)$. Then (8) implies that

$$\dot{v}(t_\sharp)-\dot{y}_\epsilon(t_\sharp) < \sum_{i=1}^{k}b_i(t_\sharp)\big[v(s_{i\star})-e^{\mathcal{S}_i(t_\sharp)}y_\epsilon(t_\sharp-T_{1,i})\big]. \tag{9}$$

Next consider any $i \in \{1, \ldots, k\}$, and three cases.

Case 1 ($s_{i\star} > t_0$): Recalling the nonnegative valuedness of y_ϵ and δ, it follows from (5) that we have

$$\dot{y}_\epsilon(t) \geq -a(t)y_\epsilon(t) \tag{10}$$

for all $t \geq t_0$. By integrating (10), we obtain

$$y_\epsilon(t_\sharp - T_{1,i}) \geq e^{-\int_{s_{i\star}}^{t_\sharp-T_{1,i}} a(m)\mathrm{d}m}y_\epsilon(s_{i\star}). \tag{11}$$

Since $s_{i\star} \in (t_0, t_\sharp]$, we deduce from the definition of $t_\sharp$ that $v(s_{i\star}) \leq y_\epsilon(s_{i\star})$, which we can combine with (11) to obtain

$$y_\epsilon(t_\sharp - T_{1,i}) \geq e^{-\int_{s_{i\star}}^{t_\sharp-T_{1,i}} a(m)\mathrm{d}m}v(s_{i\star}). \tag{12}$$

It follows that $y_\epsilon(t_\sharp - T_{1,i}) \geq e^{-\mathcal{S}_i(t_\sharp)}v(s_{i\star})$.

Case 2 ($s_{i\star} \leq t_0 \leq t_\sharp - T_{1,i}$): Integrating (10) gives

$$\begin{aligned} y_\epsilon(t_\sharp - T_{1,i}) &\geq e^{-\int_{t_0}^{t_\sharp-T_{1,i}} a(m)\mathrm{d}m}y_\epsilon(t_0) \\ &= e^{-\int_{t_0}^{t_\sharp-T_{1,i}} a(m)\mathrm{d}m}\left[\sup_{\ell\in[t_0-T_{2,k},t_0]} v(\ell) + \epsilon\right]. \end{aligned} \tag{13}$$

Since $t_0 - T_{2,i} \le t_\sharp - T_{2,i} \le s_{i\star} \le t_0$, it follows that $v(s_{i\star}) \le \sup_{\ell\in[t_0-T_{2,k},t_0]} v(\ell)$. Also, (13) gives

$$y_\epsilon(t_\sharp - T_{1,i}) \ge e^{-\int_{t_0}^{t_\sharp - T_{1,i}} a(m)\mathrm{d}m} v(s_{i\star}). \quad (14)$$

Since $t_\sharp - T_{2,k} \le s_{i*} \le t_0$, we can use (14) to conclude that

$$y_\epsilon(t_\sharp - T_{1,i}) \ge e^{-\mathcal{S}_i(t_\sharp)} v(s_{i\star}) \quad (15)$$

is satisfied.

Case 3 ($s_{i\star} \le t_\sharp - T_{1,i} \le t_0$): Recalling that our formula (4) for $\mathcal{S}_i$ ensures that $\mathcal{S}_i$ is nonnegative valued, we get

$$\begin{aligned} v(s_{i*}) - e^{\mathcal{S}_i(t_\sharp)} y_\epsilon(t_\sharp - T_{1,i}) &\le v(s_{i*}) - y_\epsilon(t_\sharp - T_{1,i}) \\ = v(s_{i*}) - \sup_{\ell\in[t_0-T_{2,k},t_0]} v(\ell) - \epsilon &\le -\epsilon. \end{aligned} \quad (16)$$

Hence, in all three cases, we have $e^{\mathcal{S}_i(t_\sharp)} y_\epsilon(t_\sharp - T_{1,i}) \ge v(s_{i\star})$ for $i = 1, \dots, k$. By combining the preceding inequalities with (9), it follows that $\dot{v}(t_\sharp) - \dot{y}_\epsilon(t_\sharp) < 0$, so there is $t_\triangle \in [t_0 - T_{2,k}, t_\sharp)$ such that $v(t_\triangle) - y_\epsilon(t_\triangle) > 0$, because $v(t_\sharp) - y_\epsilon(t_\sharp) = 0$. This contradicts the definition of $t_\sharp$. This concludes the proof. ■

Remark 1: Theorem 1 implies that if there is a constant $\epsilon > 0$ such that y_ϵ satisfies an ISS inequality, then v satisfies an ISS inequality too that is valid for all $t \ge t_0$. Hence, Theorem 1 provides a way to conclude ISS properties from its generalized Halanay's inequalitites; see Corollary 1.

Remark 2: Theorem 1 has the following two crucial advantages. First, when several poorly known delays are present in a studied system, it improves on the stability conditions for the results available in the literature by taking advantage of the information on the delays; see our illustrations below. Second, when the constants $T_{2,i}$ converge to $T_{1,i}$, then the stability conditions we obtain converge to those of the case where $T_{2,i} = T_{1,i}$ for all $i \in \{1, \dots, k\}$. This is in sharp contrast with the conditions of [13, Th. 1].

Remark 3: The preceding theorem is new, even in the special case where the $T_{1,i}$'s are all zero and when a and the b_i's all have the same period $P > 0$ and $\delta = 0$. In that special case, y_ϵ will exponentially converge to zero provided

$$\int_0^P g(\ell)\mathrm{d}\ell > 0, \text{ where } g(\ell) = a(\ell) - \sum_{i=1}^{k} [b_i(\ell) + \epsilon] e^{\mathcal{S}_i(\ell)}. \quad (17)$$

This follows by first noting that, in this case, g also has period P. Hence, for any $N \in \mathbb{N}$ such that $NP \ge t_0$, condition (17) gives $y_\epsilon((N+j)P) \le e^{-jI_*} y_\epsilon(NP)$ for all integers $j \ge 0$, where I_* is the integral in (17); this follows by applying variation of parameters to the first equation in (5) on the interval $[NP, (N+j)P]$. For $t \ge NP$, this gives

$$\begin{aligned} y_\epsilon(t) &= e^{-\int_{t_f P}^{t} g(\ell)\mathrm{d}\ell} y_\epsilon(t_f P) \\ &\le B_* e^{-I_*(t_f - N)} y_\epsilon(NP) \ \to\ 0 \text{ as } t \to +\infty, \end{aligned} \quad (18)$$

where $t_f = \mathrm{Floor}(t/P)$ and

$$B_* = \sup\left\{ e^{-\int_s^t g(\ell)\mathrm{d}\ell} : s \ge 0, 0 \le t - s \le P \right\}, \quad (19)$$

and Floor is the floor function, i.e., Floor(s) is the largest integer j such that $j \le s$. Similar reasoning shows that (17) implies that v satisfies an ISS estimate with disturbance (δ, ϵ). We illustrate the use of the criteria (17) in Section III.

B. ISS Corollary

We next present a consequence of Theorem 1, which provides an ISS estimate with respect to (δ, ϵ) that converges to an ISS estimate with respect to δ as $\epsilon \to 0^+$. The linear ISS Lyapunov-Krasovskii functional (24) in its proof motivated us to propose its persistence of excitation condition (20) (but see Remark 3 for alternative sufficient persistence of excitation conditions that ensure the required stability property for (5) under periodicity assumptions).

Corollary 1: Let the assumptions of Theorem 1 hold and $t_0 \ge 0$ be given. Assume that a and the b_i's are bounded, and that there is a constant $c > 0$ such that

$$a(t) - \sum_{i=1}^{k} [b_i(t + T_{1,i}) + \epsilon] e^{\mathcal{S}_i(t + T_{1,i})} \ge c \quad (20)$$

holds for all $t \ge t_0$. Then we can find constants $\zeta_i > 0$ for $i = 1, 2$ such that $v(t) \le \zeta_1 e^{-\zeta_2 t} |v|_{[t_0 - T_{2,k}, t_0]} + \zeta_2(\delta(t) + e^{-\zeta_2 t}\epsilon)$ for all $t \ge t_0$.

Proof: Consider the functional

$$\begin{aligned} V_{1,\epsilon}(t, y_{\epsilon,t}) &= y_\epsilon(t) \\ &+ \sum_{i=1}^{k} \int_{t - T_{1,i}}^{t} [b_i(\ell + T_{1,i}) + \epsilon] e^{\mathcal{S}_i(\ell + T_{1,i})} y_\epsilon(\ell)\mathrm{d}\ell. \end{aligned} \quad (21)$$

Its time derivative along all solutions of (5) satisfies

$$\dot{V}_{1,\epsilon}(t) = \left[-a(t) + \sum_{i=1}^{k} [b_i(t + T_{1,i}) + \epsilon] e^{\mathcal{S}_i(t + T_{1,i})} \right] y_\epsilon(t) + \delta(t) \quad (22)$$

for all $t \ge t_0$. Then, along solutions of (5), $\dot{V}_{1,\epsilon}(t) \le -c y_\epsilon(t) + \delta(t)$ for all $t \ge t_0$. Then let

$$V_{2,\epsilon}(t, y_{\epsilon,t}) = V_{1,\epsilon}(t, y_{\epsilon,t}) + \frac{c}{2\mu} \int_{t-\mu}^{t} \int_{m}^{t} y_\epsilon(s)\mathrm{d}s \quad (23)$$

with $\mu = \max_{i\in\{1,\dots,k\}}\{T_{1,i}\}$. Then, along solutions of (5),

$$\dot{V}_{2,\epsilon}(t, y_{\epsilon,t}) \le -\frac{c}{2} y_\epsilon(t) - \frac{c}{2\mu} \int_{t-\mu}^{t} y_\epsilon(s)\mathrm{d}s + \delta(t) \quad (24)$$

for all $t \ge t_0$. Since the functions a and b_i are bounded, there is a constant $\varsigma > 0$ such that

$$\dot{V}_{2,\epsilon}(t, y_{\epsilon,t}) \le -\varsigma V_{2,\epsilon}(t, y_{\epsilon,t}) + \delta(t) \quad (25)$$

for all $t \ge t_0$. For instance, since

$$V_{2,\epsilon}(t, y_{\epsilon,t}) \le y_\epsilon(t) + \bar{c} \int_{t-\mu}^{t} y_\epsilon(s)\mathrm{d}s \quad (26)$$

holds for all $t \ge t_0$ with the choices $\bar{c} = k(|b|_\infty + \epsilon)e^{|\mathcal{S}|_\infty} + \frac{c}{2}$ and $b = (b_1, \dots, b_k)$, we can choose $\varsigma = \frac{c}{2}\min\{1/\mu, 1\}/\max\{\bar{c}, 1\}$. Then we can apply the method of variation of parameters to (25) and recall the structure of $V_{2,\epsilon}$ to obtain the required ISS estimate for y_ϵ. In fact, Theorem 1 gives

$$\begin{aligned} v(t) &\le y_\epsilon(t) \le V_{2,\epsilon}(t, y_{\epsilon,t}) \\ &\le e^{-\varsigma t}(1 + \bar{c}\mu)|y_\epsilon|_{[t_0 - T_{2,k}, t_0]} + \frac{\delta(t)}{\varsigma} \\ &= e^{-\varsigma t}(1 + \bar{c}\mu)|v|_{[t_0 - T_{2,k}, t_0]} + \frac{\delta(t)}{\varsigma} + e^{-\varsigma t}(1 + \bar{c}\mu)\epsilon \end{aligned} \quad (27)$$

for all $t \ge t_0$, proving the corollary. ■

C. Halanay's Result With Only One Sup Term

Let $T \geq 0$. Consider the inequality

$$\dot{v}(t) \leq -a(t)v(t) + b(t) \sup_{\ell \in [t-T,t]} v(\ell) + \delta(t) \quad (28)$$

where $a : \mathbb{R} \to \mathbb{R}$, $b : [0,+\infty) \to [0,+\infty)$, and $\delta : [0,+\infty) \to [0,+\infty)$ are locally bounded and piecewise continuous, and where $v : [-T,+\infty) \to [0,+\infty)$ is of class C^1. We use any constants $\epsilon > 0$ and $t_0 \geq 0$, the function

$$\Gamma(t) = \max\left\{0, \sup_{\ell \in [t-T,t]} \int_{\ell}^{t} [a(r) - b(r)]\mathrm{d}r\right\}, \quad (29)$$

and the comparison system

$$\begin{cases} \dot{y}_\epsilon(t) = \left[-a(t) + (b(t) + \epsilon)e^{\Gamma(t)}\right] y_\epsilon(t) + \delta(t) \\ \qquad \text{for all } t \geq t_0 \\ y_\epsilon(t) = \sup_{\ell \in [t_0-T,t_0]} v(\ell) + \epsilon \text{ for all } t \in [t_0 - T, t_0]. \end{cases} \quad (30)$$

The system (30) differs from system (5), because (a) there is only one (instead of k) overshoot terms in the y_ϵ dynamics in (30), (b) the exponential term (4) in (5) having the integrand a has been replaced by a new exponential term (29) with the integrand $a - b$, and (c) the delays $T_{1,i}$ in (5) are not used in (29)-(30). See Section III-C for an example where (29)-(30) lead to less conservative conditions than Theorem 1.

We prove the following, which provides ISS properties for v when y_ϵ satisfies ISS, and where the required stability properties for y_ϵ can be checked using classical methods like in the preceding subsection, which can lead to explicit ISS like estimates for v as in Corollary 1.

Theorem 2: Consider the v and y_ϵ defined above. Then

$$v(t) \leq y_\epsilon(t) \quad (31)$$

holds for all $t \geq t_0 - T$.

Proof: First note that the second equality in (30) implies that $v(t) < y_\epsilon(t)$ for all $t \in [t_0 - T, t_0]$. Now, we proceed by contradiction. Bearing in mind that v and y_ϵ are continuous, suppose that there is a $t_\sharp > t_0$ such that $v(t_\sharp) = y_\epsilon(t_\sharp)$ and

$$v(t) < y_\epsilon(t) \quad (32)$$

for all $t \in [t_0 - T, t_\sharp)$ by choosing $t_\sharp$ to be the first time $t \geq t_0$ when the inequality (32) is violated as before. Then

$$\begin{aligned} \dot{v}(t_\sharp) - \dot{y}_\epsilon(t_\sharp) \leq\ & b(t_\sharp) \sup_{\ell \in [t_\sharp - T, t_\sharp]} v(\ell) \\ & - (b(t_\sharp) + \epsilon)e^{\Gamma(t_\sharp)} y_\epsilon(t_\sharp). \end{aligned} \quad (33)$$

Since $\epsilon e^{\Gamma(t_\sharp)} y_\epsilon(t_\sharp) > 0$, we have

$$\dot{v}(t_\sharp) - \dot{y}_\epsilon(t_\sharp) < b(t_\sharp)\left[\sup_{\ell \in [t_\sharp - T, t_\sharp]} v(\ell) - e^{\Gamma(t_\sharp)} y_\epsilon(t_\sharp)\right]. \quad (34)$$

Let $s_\star \in [t_\sharp - T, t_\sharp]$ be such that $v(s_\star) = \sup_{\ell \in [t_\sharp - T, t_\sharp]} v(\ell)$, which exists because v is continuous. Then

$$\dot{v}(t_\sharp) - \dot{y}_\epsilon(t_\sharp) < b(t_\sharp)\left[v(s_\star) - e^{\Gamma(t_\sharp)} y_\epsilon(t_\sharp)\right]. \quad (35)$$

Next note that since $y_\epsilon(t) \geq 0$ for all $t \geq t_0$, we have

$$\dot{y}_\epsilon(t) \geq [-a(t) + b(t)]y_\epsilon(t) \quad (36)$$

for all $t \geq t_0$. Next, let us distinguish between 2 cases.

Case 1 ($s_\star > t_0$): Then integrating (36) gives

$$y_\epsilon(t_\sharp) \geq e^{-\int_{s_\star}^{t_\sharp}[a(r)-b(r)]\mathrm{d}r} y_\epsilon(s_\star). \quad (37)$$

Since $s_\star \in [t_\sharp - T, t_\sharp]$, it follows from (29) that

$$\begin{aligned} y_\epsilon(t_\sharp) &\geq e^{-\int_{s_\star}^{t_\sharp}[a(r)-b(r)]\mathrm{d}r} v(s_\star) \\ &\geq e^{-\sup_{\ell \in [t_\sharp - T, t_\sharp]} \int_{\ell}^{t_\sharp}[a(r)-b(r)]\mathrm{d}r} v(s_\star) \\ &\geq e^{-\Gamma(t_\sharp)} v(s_\star). \end{aligned} \quad (38)$$

Case 2 ($s_\star \leq t_0$): Then integrating (36) gives

$$\begin{aligned} y_\epsilon(t_\sharp) &\geq e^{-\int_{t_0}^{t_\sharp}[a(r)-b(r)]\mathrm{d}r} y_\epsilon(t_0) \\ &\geq e^{-\int_{t_0}^{t_\sharp}[a(r)-b(r)]\mathrm{d}r} \sup_{\ell \in [t_0 - T, t_0]} v(\ell), \end{aligned} \quad (39)$$

by (30). Since $s_\star \in [t_\sharp - T, t_0]$ and $t_\sharp \geq t_0$, we deduce that

$$\begin{aligned} y_\epsilon(t_\sharp) &\geq e^{-\int_{t_0}^{t_\sharp}[a(r)-b(r)]\mathrm{d}r} v(s_\star) \\ &\geq e^{-\sup_{\ell \in [t_\sharp - T, t_\sharp]} \int_{\ell}^{t_\sharp}[a(r)-b(r)]\mathrm{d}r} v(s_\star) \\ &\geq e^{-\Gamma(t_\sharp)} v(s_\star). \end{aligned} \quad (40)$$

In both cases, we get $e^{\Gamma(t_\sharp)} y_\epsilon(t_\sharp) \geq v(s_\star)$, which we can combine with (35) to obtain $\dot{v}(t_\sharp) - \dot{y}_\epsilon(t_\sharp) < 0$. Hence, there is a $t_\triangle \in [t_0, t_\sharp)$ such that $v(t_\triangle) - y_\epsilon(t_\triangle) > 0$, because $v(t_\sharp) - y_\epsilon(t_\sharp) = 0$. This contradicts the definition of $t_\sharp$, so no such $t_\sharp$ can exist. This concludes the proof. ■

III. Illustrations

A. Systems With Switching Delays

Systems with switching delays commonly arise when controls need to switch between different sensors or different actuators that have different latencies, and so are of considerable research interest in the control theory community. Therefore, we first illustrate Theorem 1 using the class of systems with switching delays from [8] and [10], using our less restrictive new generalized Halanay's conditions (2).

Let $\underline{T} > 0$ and $\bar{T} \geq \underline{T}$ be any constants, and consider any sequence t_i satisfying $\underline{T} \leq t_{i+1} - t_i \leq \bar{T}$ for all $i \geq 0$. As in [8, Sec. 3.4], let τ_l and τ_s be any constants such that

$$\underline{T} > 5(\tau_l + \tau_s) \quad (41)$$

and $\tau_l > \tau_s \geq 0$, and we consider the system

$$\dot{x}(t) = Mx(t) + Nx(t - \tau(t)) \quad (42)$$

with x valued in $\mathbb{R}^n$, where τ is a time-varying piecewise continuous unknown delay such that

$$\begin{gathered} 0 \leq \tau(t) \leq \tau_s \text{ if } t \notin E, \text{ and } 0 \leq \tau(t) \leq \tau_l \text{ if } t \in E, \\ \text{where } E = \cup_{i \in \mathbb{N}}[t_i, t_i + T) \text{ and } T = \tau_s + \tau_l, \end{gathered} \quad (43)$$

and $M \in \mathbb{R}^{n \times n}$ and $N \in \mathbb{R}^{n \times n}$ are constant (which includes cases where τ does not switch, e.g., with $T = t_{i+1} - t_i$ for all i). Following [8, Sec. 3.4] and [10, Sec. 4.2], we also assume the following, where $M_1 \geq M_2$ for square matrices means that $M_1 - M_2$ is nonnegative definite.

Assumption 1: There are a symmetric positive definite matrix $Q \in \mathbb{R}^{n \times n}$ and a constant $q > 0$ such that

$$Q(M + N) + (M + N)^\top Q \leq -qQ \quad (44)$$

and $I \leq Q$ are satisfied.

In terms of the notation

$$L = \frac{2|N^\top Q N|(|M| + |N|)^2}{q}, \quad (45)$$

we also use the following lemma, whose proof consists of the first part of the proof of [10, Proposition 1].

Lemma 1: With the preceding notation and under Assumption 1, the time derivative of the function $U(x) = x^\top Qx$ along all solutions of (42) is such that

$$\dot{U}(t) \leq -\frac{q}{2}U(x(t)) + L\tau_s^2 \sup_{m\in[t-\tau_l-\tau_s,t]} U(x(m)) \tag{46}$$

for all $t \in [0, +\infty)\backslash E$ and

$$\dot{U}(t) \leq -\frac{q}{2}U(x(t)) + \frac{8|N^\top QN|}{q} \sup_{l\in[t-\tau_l,t]} U(x(l)) \tag{47}$$

for all $t \in E$.

In terms of the constant a_* and the function b_* in

$$a_* = e^{-q(\tau_s+\tau_l)/2} \text{ and } b_*(\ell) = \frac{2(1-a_*)\ell}{q} e^{2(R-q/2)(\tau_l+\tau_s)} \tag{48}$$

where $R = 8|N^\top QN|/q$, the main result in [8] for (42) is:

Proposition 1: With the above notation, let Assumption 1 hold, and assume that

$$L\tau_s^2 \leq \frac{q}{2} < \frac{8|N^\top QN|}{q}$$
$$\text{and } \left(a_* + b_*(L\tau_s^2)\right)\left(a_* + b_*\left(\frac{8|N^\top QN|}{q}\right)\right) < 1. \tag{49}$$

Then the origin of (42) is a globally exponentially stable equilibrium point on $\mathbb{R}^n$.

On the other hand, we can apply Theorem 1 from Section II-A above in the preceding case, by choosing

$$v(t) = U(x(t)),\ a(t) = \frac{q}{2},\ k = 2,$$
$$b_1(t) = \frac{8}{q}|N^\top QN|\chi_{S_a}(t),\ b_2(t) = L\tau_s^2\chi_{S_b}(t),$$
$$T_{1,1} = T_{1,2} = 0,\ T_{2,1} = \tau_l, \text{ and } T_{2,2} = \tau_l + \tau_s \tag{50}$$

under suitable conditions on the parameters, where χ_S is the indicator (or characteristic) function for each set $S \subseteq \mathbb{R}$, meaning $\chi_S(\ell) = 1$ if $\ell \in S$ and $\chi_S(\ell) = 0$ if $\ell \in \mathbb{R}\backslash S$, and where $S_a = E$ and $S_b = \mathbb{R}\backslash E$. For example, if we choose $\tau_s = 0$, $\tau_l = 0.01$, $t_i = i$ for all $i \geq 0$, the matrices

$$M = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix},\ Q = \begin{bmatrix} 2.68767 & 0.789041 \\ 0.789041 & 1.70137 \end{bmatrix}, \tag{51}$$

and $N = -2I$, and $q = 1.5$, then the assumptions of Proposition 1 would not hold, but the assumptions of Theorem 1 would hold with $\epsilon = 0.001$, where Q and q can be found by solving the Lyapunov equation $Q_a(M+N) + (M+N)^\top Q_a = -I$ for Q_a, and then scaling Q_a to obtain a Q that satisfies the requirements of Assumption 1 for a small enough constant $q > 0$. The fact that the requirements from Theorem 1 hold for any $t_0 \geq 0$ in this case follows from Remark 3 above with $P = 1$.

In Fig. 1, we plot the solutions of (42) with the preceding values and $\tau(t) = 0.01$ for all $t \in E$ for three sets of constant initial functions (i.e., initial states), using Mathematica. They show convergence to the desired equilibrium. This illustrates how Theorem 1 can provide less restrictive conditions than the conditions in [8] for systems with switched delays.

B. Example With Comparison With [13]

Consider the special case

$$\dot{v}(t) \leq -a(t)v(t) + b_1(t) \sup_{\ell\in[t-T_{2,1},t-T_{1,1}]} v(\ell) + b_2(t) \sup_{\ell\in[t-T_{2,2},t-T_{1,2}]} v(\ell) \tag{52}$$

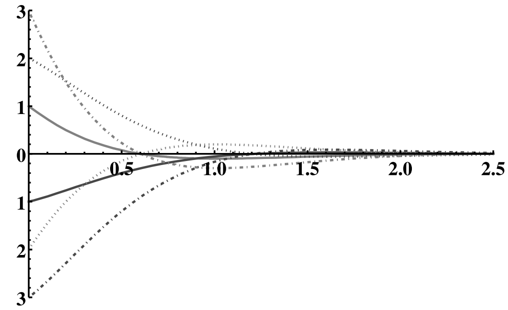

Fig. 1. Simulations of (42) Showing $x_1(t)$ (Red) and $x_2(t)$ (Blue) for Initial States (1, 1) (Solid), (−2, 2) (Dotted) and (3, −3) (Dashed-Dotted).

of (2) with $k = 2$ and $\delta = 0$, and with the choices

$$T_{2,1} = \frac{\pi}{2}, T_{1,1} = \frac{\pi}{2} - \frac{\pi}{100}, a(t) = \frac{1}{4} + 8\sin^2(t),$$
$$b_1(t) = 4e^{-\frac{\pi}{100}}\sin^2\left(t - \frac{49\pi}{100}\right),\ T_{2,2} = 2\pi, \text{ and}$$
$$b_2(t) = 4e^{-\frac{\pi}{100}}\sin^2\left(t - \frac{199\pi}{100}\right), \text{ and } T_{1,2} = 2\pi - \frac{\pi}{100}. \tag{53}$$

Then the sufficient condition (20) for $\lim_{t\to+\infty} v(t) = 0$ to hold is that there is a constant $c > 0$ such that

$$\frac{1}{4} + 8\sin^2(t) - [b_1(t+T_{1,1}) + \epsilon]e^{\mathcal{S}_1(t+T_{1,1})} - [b_2(t+T_{1,2}) + \epsilon]e^{\mathcal{S}_2(t+T_{1,2})} \geq c \tag{54}$$

for all $t \geq 0$. Let us check that this inequality is satisfied for sufficiently small positive values ϵ and c.

To this end, notice that using the notation from Theorem 1, it follows that for all $t \geq 0$, we have

$$\mathcal{S}_1(t+T_{1,1}) = \int_{t-T_{2,1}+T_{1,1}}^{t} a(m)\mathrm{d}m \leq 0.26 \text{ and}$$
$$\mathcal{S}_2(t+T_{1,2}) = \int_{t-T_{2,2}+T_{1,2}}^{t} a(m)\mathrm{d}m \leq 0.26 \tag{55}$$

where the equalities in (55) follow because a is nonnegative valued. Hence, the left side of (54) is bounded below by

$$\frac{1}{4} + 8\sin^2(t) - \left[b_1\left(t + \frac{49\pi}{100}\right) + b_2\left(t + \frac{199\pi}{100}\right) + 2\epsilon\right]e^{0.26} \tag{56}$$

for all $t \geq 0$. Since

$$b_1\left(t + \frac{49\pi}{100}\right) = b_2\left(t + \frac{199\pi}{100}\right) = 4e^{-\frac{\pi}{100}}\sin^2(t) \tag{57}$$

it follows that when $\epsilon > 0$ is small enough, we have

$$\frac{1}{4} + 8\sin^2(t) - [b_1(t+T_{1,1}) + \epsilon]e^{\mathcal{S}_1(t+T_{1,1})} - [b_2(t+T_{1,2}) + \epsilon]e^{\mathcal{S}_2(t+T_{1,2})} \geq \frac{1}{8} \tag{58}$$

which ensures that $\lim_{t\to+\infty} v(t) = 0$, by Corollary 1.

On the other hand, [13, Th. 1] would not apply to the preceding example. To see why, first observe that (52) implies that

$$\dot{v}(t) \leq -a(t)v(t) + [b_1(t) + b_2(t)] \sup_{\ell\in[t-T_{2,2},t]} v(\ell) \tag{59}$$

for all $t \geq 0$. Then, with the choice $\mathcal{H}(\ell) = \sin^2(\ell - 0.49\pi) + \sin^2(\ell - 1.99\pi)$, the function

$$G(t) = \int_0^t \left[-a(\ell) + (b_1(\ell) + b_2(\ell))e^{\int_{\ell-T_{2,2}}^{\ell} a(s)\mathrm{d}s}\right]\mathrm{d}\ell \tag{60}$$

satisfies

$$
\begin{aligned}
G(t) &= -\frac{t}{4} - 8\int_0^t \sin^2(\ell)\mathrm{d}\ell \\
&\quad + 4e^{-\frac{\pi}{100}}\int_0^t \mathcal{H}(\ell)e^{\frac{17\pi}{2}}\mathrm{d}\ell \\
&= -\frac{t}{4} - 8\int_0^t \sin^2(\ell)\mathrm{d}\ell \\
&\quad + 4e^{\frac{849\pi}{100}}\int_{-\frac{49\pi}{100}}^{t-\frac{49\pi}{100}} \sin^2(\ell)\mathrm{d}\ell \\
&\quad + 4e^{\frac{849\pi}{100}}\int_{-\frac{199\pi}{100}}^{t-\frac{199\pi}{100}} \sin^2(\ell)\mathrm{d}\ell \\
&\geq -\frac{17t}{4} + 2\sin(2t) + 4(t-2)e^{\frac{849\pi}{100}}, \qquad (61)
\end{aligned}
$$

which follows by using $\sin^2(\ell) = \frac{1}{2}(1-\cos(2\ell))$ to evaluate and then bound the integrals in (61). Since $\lim_{t\to+\infty} G(t) = +\infty$, it follows that [13, Th. 1] does not allow us to prove that $\lim_{t\to+\infty} v(t) = 0$. By covering the preceding example which is not covered by [13, Th. 1], it follows that Theorem 1 is less restrictive than [13, Th. 1].

C. Illustration of Theorem 2

Using the notation from Theorem 2, and considering any constant $b_0 \in (0,1)$, consider the special case where $a(t) = \sin^2(t)$, $b(t) = b_0\sin^2(t)$, and $T = 2\pi$. Then

$$\int_\ell^t [a(r) - b(r)]\mathrm{d}r = \int_\ell^t [1-b_0]\sin^2(r)\mathrm{d}r \qquad (62)$$

for all $t \geq 0$. It follows that

$$
\begin{aligned}
&\sup_{\ell\in[t-2\pi,t]}\int_\ell^t [a(r)-b(r)]\mathrm{d}r \\
&= \int_{t-2\pi}^t [1-b_0]\sin^2(r)\mathrm{d}r = \pi[1-b_0]. \qquad (63)
\end{aligned}
$$

To apply Theorem 2, we use the comparison system

$$
\begin{cases}
\dot{y}_\epsilon(t) = \left[-\sin^2(t) + (b_0\sin^2(t)+\epsilon)e^{\pi[1-b_0]}\right]y_\epsilon(t) \\
\qquad + \delta(t) \text{ for all } t \geq t_0 \\
y_\epsilon(t) = \sup_{\ell\in[t_0-T,t_0]} v(\ell) + \epsilon \text{ if } t \in [t_0-T, t_0].
\end{cases} \qquad (64)
$$

Bearing Theorem 2 and Remark 3 in mind, we obtain the following sufficient condition for stability:

$$\int_{t-2\pi}^t \left[-\sin^2(m) + (b_0\sin^2(m)+\epsilon)e^{\pi(1-b_0)}\right]\mathrm{d}m < 0 \qquad (65)$$

for all $t \geq 0$. Condition (65) holds if and only if $b_0e^{\pi(1-b_0)} + 2\epsilon e^{\pi(1-b_0)} < 1$. By choosing $\epsilon > 0$ small enough, we obtain the condition

$$b_0e^{-\pi b_0} < e^{-\pi}. \qquad (66)$$

For a comparison, we next apply Theorem 1, with $k = 1$, $T_{1,1} = 0$, and $T_{2,1} = 2\pi$, using the comparison system

$$
\begin{cases}
\dot{y}_\epsilon(t) = -\sin^2(t)y_\epsilon(t) + [b_0\sin^2(t)+\epsilon]e^{\pi}y_\epsilon(t) \\
\qquad + \delta(t) \text{ if } t \geq t_0 \\
y_\epsilon(t) = \sup_{\ell\in[t_0-T_{2,k},t_0]} v(\ell) + \epsilon \text{ if } t \in [t_0-T, t_0].
\end{cases} \qquad (67)
$$

Reasoning as in Remark 3, we obtain the stability condition

$$\int_0^{2\pi}\left[-\sin^2(s) + (b_0\sin^2(s)+\epsilon)e^{\pi}\right]\mathrm{d}s < 0 \qquad (68)$$

which is equivalent to $-1 + b_0e^{\pi} < -2\epsilon e^{\pi}$. Since $\epsilon > 0$ is arbitrarily small, we obtain the stability condition

$$b_0 < e^{-\pi}. \qquad (69)$$

Note that $e^{-\pi}e^{-\pi e^{-\pi}} < e^{-\pi}$, so if (69) holds, then so does (66) (because $xe^{-\pi x}$ increases over $x \in [0, e^{-\pi}]$), and (66) holds with $b_0 = e^{-\pi}$. Hence, (66) is less restrictive than (69). This example shows how Theorem 2 is less restrictive than Theorem 1, because it can cover the case $b_0 = e^{-\pi}$ that was not covered by Theorem 1.

IV. Conclusion

We proposed new stability analysis results for functions that satisfy generalized time-varying inequalities of Halanay's type. We illustrated how our results can provide less restrictive conditions than ones in the literature. Since earlier generalizations of Halanay's inequality have been shown to be effective for solving observer design problems that were beyond the scope of the observers literature (e.g., in [10]), we aim to apply our work to observer designs for continuous-discrete, event-triggered, and switched systems.

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