New Finite-Time and Fast Converging Observers With a Single Delay

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***Abstract*—We provide new reduced order observer designs for a key class of nonlinear dynamics. When continuous output measurements are available, we prove that our observers converge in a fixed finite time in the absence of perturbations, and we prove a robustness result under uncertainties in the output measurements and in the dynamics, which bounds the observation error in terms of bounds on the uncertainties. The observers contain a dynamic extension with only one pointwise delay, and they use the observability Gramian to eliminate an invertibility condition that was present in earlier finite time observer designs. We also provide analogs for cases where the measurements are only available at discrete times, where we prove exponential input-to-state stability. We illustrate the advantages of our new observers using a DC motor dynamics.**

***Index Terms*—Observer, nonlinear, robust.**

I. Introduction

FINITE and fixed time observers present an obvious advantage by providing estimates of the state variables of systems in finite time [1]. Fixed time observers are special cases of finite time observers where the finite convergence time is independent of the initial state. Several types of fixed time observers are available. Some use discontinuous dynamic extensions [2], time-varying high gains [3], delays, or homogeneity conditions [4].

In earlier works, e.g., [5], [6], and [7], fixed time observers are designed using dynamic extensions and a delay τ. The designs rely on the invertibility of a matrix which can be problematic because it is not invertible for all τ's and because, when it exists, the inverse can contain big terms when the delays are close to values where it is not invertible. We refer to such delays as artificial delays, because although they are not present in the given dynamics, they occur in the observers. The work [8] provides an exact calculation of state variables using a formula with several delays and the inversion of a matrix, which can also be problematic because it may be noninvertible for some delay values. Moreover, the observers in [5], [6], [7], and [8] are not reduced order.

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To overcome these shortcomings, we revisit the problem of estimating the state variables of a system in finite time using an artificial delay. For a family of unperturbed systems that are affine in the unmeasured state, we propose a new family of observers that converge in fixed time when continuous output measurements are available. The observers only estimate unmeasured variables, and so are reduced order. A key aspect of the observer design we propose is that it relies on the introduction of only one pointwise delay, which can be arbitrarily chosen. The delay is the fixed convergence time. We also establish a robustness result for the observers with respect to additive disturbances on the output measurement and dynamics. We then provide an analog for cases where the measurements are only available at discrete instants. In this case, the exponential convergence rate is proportional to the logarithm of the size of the largest sampling interval.

We use standard notation, where the dimensions of our Euclidean spaces are arbitrary, unless we indicate otherwise. The standard Euclidean 2-norm, and its induced matrix norm, are denoted by $|\cdot|$, $|\cdot|_\infty$ is the $\mathcal{L}_\infty$ sup norm, $|\cdot|_S$ is the essential supremum over sets S, and I is the identity matrix.

II. Studied System

We consider the class of continuous-time systems

$$\begin{cases} \dot{\chi}(t) = M\chi(t) + \Psi(N\chi(t), t) + \delta_1(t) \\ Y(t) = N\chi(t) + \delta_2(t) \end{cases} \tag{1}$$

where χ is valued in $\mathbb{R}^n$, the output Y is valued in $\mathbb{R}^q$, the time dependence in Ψ can represent the effects of a control, and the locally essentially bounded measurable functions δ_1 and δ_2 represent disturbances (but see Remark 3 for a method to use the uncertainties δ_i to incorporate the effects of nonlinearities in more general systems or nonlinearities in the measurements). The structure (1) and the main assumptions below are motivated by the facts that they hold for permanent magnet DC motors, pendulums, and dynamics for elastic membranes; see [9] and [10]. We assume that (1) is forward complete, and that Ψ is locally Lipschitz. We also assume that the pair (M, N) is observable and that N has full rank.

From [11, pp. 304–306], we can deduce that there is a linear change of coordinates which yields the system

$$\begin{cases} \dot{\xi}_1(t) = A_1\xi_1(t) + F_1(Y(t) - \delta_2(t), t) + \epsilon_1(t) \\ \dot{\xi}_2(t) = A_2\xi_1(t) - k\xi_2(t) + F_2(Y(t) - \delta_2(t), t) \\ \qquad + \epsilon_2(t) \\ Y(t) = \xi_2(t) + \delta_2(t) \end{cases} \tag{2}$$

which is affine in the unmeasured variable ξ_1, where ξ_1 is valued in $\mathbb{R}^{n-q}$, ξ_2 is valued in $\mathbb{R}^q$, $A_1 \in \mathbb{R}^{(n-q)\times(n-q)}$, $A_2 \in \mathbb{R}^{q\times(n-q)}$, the pair (A_1, A_2) is observable, and $k > 0$ is a constant such that $A_1 + kI$ is invertible. Then F_1 and F_2 are locally Lipschitz, and the measurable locally essentially bounded functions ϵ_i represent disturbances. Although the $-k\xi_2(t)$ term can be incorporated into the function F_2 in (2), we keep it separate to facilitate the analysis that follows, and we write ξ_2(t) as $Y(t)-\delta_2(t)$ in the F_i's in (2) to facilitate our study of the key special case where δ_2 is the zero function. Changing the parameter k can be done by changing F_2. One can always choose it such that A_1+kI is invertible, by taking k larger than the spectral radius of A_1.

III. Continuous Measurement Cases

A. Assumptions and Statement of Theorem

We construct an fixed time observer for (2), assuming:

Assumption 1: Either (i) there are two constants $K_1 \geq 0$ and $K_2 \geq 0$ such that

$$|F_i(a,t)-F_i(b,t)| \leq K_i|a-b| \text{ for } i=1,2 \tag{3}$$

for all $t \geq 0$ and a and b in $\mathbb{R}^q$ or (ii) $\delta_2(t)=0$ for all $t \geq 0$.

Let us introduce any positive constant τ and the function $\lambda : \mathbb{R} \to \mathbb{R}^{q\times(n-q)}$ defined by

$$\lambda(r) = A_2(A_1+kI)^{-1}\Big[I-e^{(A_1+kI)r}\Big] \tag{4}$$

which is well-defined because we choose $k>0$ such that the matrix A_1+kI is invertible. We also use the matrix

$$\mathcal{S} = \int_{-\tau}^{0} \lambda(m)^\top\lambda(m)\mathrm{d}m \in \mathbb{R}^{(n-q)\times(n-q)}. \tag{5}$$

In the appendix below, we prove that since the pair (A_1, A_2) is observable, $\mathcal{S}$ is invertible. Then we define the matrices

$$\begin{aligned}
\mathcal{N} &= \int_{-\tau}^{0}\lambda(m)^\top\mathrm{d}m \in \mathbb{R}^{(n-q)\times q},\\
\mathcal{R} &= \mathcal{S}^{-1}\mathcal{N} \in \mathbb{R}^{(n-q)\times q}, \text{ and}\\
\mathcal{H} &= ((A_1+kI)^{-1})^\top A_2^\top \in \mathbb{R}^{(n-q)\times q}
\end{aligned} \tag{6}$$

and we introduce the dynamic extension

$$\begin{cases}
\dot{\hat{\xi}}_1(t) = A_1\hat{\xi}_1(t) + F_1(Y(t),t)\\
\dot{\hat{\xi}}_2(t) = A_2\hat{\xi}_1(t) - k\hat{\xi}_2(t) + F_2(Y(t),t)\\
\dot{\psi}_1(t) = -k\psi_1(t) + \mathcal{H}[Y(t)-\hat{\xi}_2(t)]\\
\dot{\psi}_2(t) = -(A_1^\top+2kI)\psi_2(t) + \mathcal{H}[Y(t)-\hat{\xi}_2(t)]
\end{cases} \tag{7}$$

where $\hat{\xi}_1$ is valued in $\mathbb{R}^{n-q}$, $\hat{\xi}_2$ is valued in $\mathbb{R}^q$, and ψ_1 and ψ_2 are valued in $\mathbb{R}^{n-q}$. Finally, in terms of the functions

$$\begin{aligned}
\Delta_*(p) &= e^{A_1p} - e^{-kpI} \text{ and}\\
\Delta_{**}(p,q) &= \int_p^q e^{A_1(p-\ell)}\epsilon_1(\ell)\mathrm{d}\ell
\end{aligned} \tag{8}$$

where k is from (2), we let $\epsilon_\ddagger$ be the $\mathbb{R}^{n-q}$-valued function

$$\begin{aligned}
\epsilon_\ddagger(t) &= \mathcal{S}^{-1}\int_{t-\tau}^{t}\lambda(s-t)^\top\mathcal{H}^\top\Delta_*(t-s)\Delta_{**}(s,t)\mathrm{d}s\\
&- \mathcal{S}^{-1}\int_{t-\tau}^{t}\lambda(s-t)^\top\Big[-\int_s^t e^{k(m-t)}A_2\Delta_{**}(m,s)\mathrm{d}m\\
&+ \int_s^t e^{k(\ell-t)}\epsilon_2(\ell)\mathrm{d}\ell\Big]\mathrm{d}s.
\end{aligned} \tag{9}$$

In terms of the preceding notation and the functions

$$\epsilon_i^\sharp(m) = K_i|\delta_2(m)| + |\epsilon_i(m)| \tag{10}$$

for $i=1,2$ and the constants

$$\begin{aligned}
\overline{S} &= |\mathcal{S}^{-1}| \text{ and } c_\Delta(\tau) = \tau\overline{S}|A_2|e^{|A_1|\tau}\\
&+ \overline{S}\Big|A_2(A_1+kI)^{-1}\Big|\Big[e^{|A_1|\tau}+1\Big]e^{|A_1|\tau},
\end{aligned} \tag{11}$$

our first theorem is then as follows:

Theorem 1: Let (2) satisfy Assumption 1. Then, with the preceding notation, when δ_2 is the zero function, we have

$$\xi_1(t) = \xi_e(t) + \epsilon_\ddagger(t) \text{ for all } t \geq \tau, \text{ where} \tag{12}$$

$$\begin{aligned}
\xi_e(t) &= \hat{\xi}_1(t) + \mathcal{R}(\xi_2(t)-\hat{\xi}_2(t))\\
&+ \mathcal{S}^{-1}\Big[e^{-k\tau}\psi_1(t-\tau) - \psi_1(t)\Big]\\
&+ \mathcal{S}^{-1}\Big[\psi_2(t) - e^{-(A_1^\top+2kI)\tau}\psi_2(t-\tau)\Big].
\end{aligned} \tag{13}$$

Also, if F_1 and F_2 satisfy (3) and $\delta_2 \neq 0$, then

$$\xi_1(t) = \xi_e(t) + \epsilon_\bigstar(t) \tag{14}$$

holds for all $t \geq \tau$ where $\epsilon_\bigstar$ is a function such that

$$\begin{aligned}
|\epsilon_\bigstar(t)| &\leq c_\Delta(\tau)\int_{t-\tau}^{t}|\lambda(s-t)|\int_s^t \epsilon_1^\sharp(m)\mathrm{d}m\mathrm{d}s\\
&+ \overline{S}\int_{t-\tau}^{t}|\lambda(s-t)|\int_s^t\epsilon_2^\sharp(m)\mathrm{d}m\mathrm{d}s\\
&+ \overline{S}|\mathcal{H}|\Big\{1+e^{|A_1^\top+2kI|\tau}\Big\}\int_{t-\tau}^{t}|\delta_2(s)|\mathrm{d}s
\end{aligned} \tag{15}$$

for all $t \geq \tau$, and the $\epsilon_i^\sharp$'s are from (10).

Remark 1: A key feature of the observer (13) is that it incorporates only one delay τ, which can be any positive value because for any $\tau > 0$, $\mathcal{S}$ is invertible; see the Appendix.

Remark 2: Since

$$|\lambda(r)| \leq |A_2(A_1+kI)^{-1}|\Big[1+e^{|A_1+kI||r|}\Big] \tag{16}$$

for all $r \in \mathbb{R}$, there are constants $c_\natural \geq 0$ and $c_\diamond \geq 0$ such that $|\epsilon_\ddagger(t)| \leq c_\natural|(\epsilon_1,\epsilon_2)|_{[t-\tau,t]}$ and $|\epsilon_\bigstar(t)| \leq c_\diamond|(\epsilon_1,\epsilon_2,\delta_2)|_{[t-\tau,t]}$ for all $t \geq \tau$, namely, $c_\natural = \overline{S}\tau^2\beta_*(|\mathcal{H}||\Delta_*|_{[0,\tau]}e^{|A_1|\tau} + \tau|A_2|e^{|A_1|\tau}+1)$ and $c_\diamond = \tau^2\beta_*(c_\Delta(\tau)(K_1+1)+\overline{S}(K_2+1)) + \overline{S}|\mathcal{H}|\beta_{**}\tau$, where β_* is the right side of (16) and β_{**} is the quantity in curly braces in (15). We illustrate the effects of these error terms in Section V.

Remark 3: In terms of the δ_i's from (1), the arguments from [11, pp. 304–306] imply that the ϵ_i's in (2) are $\epsilon_1 = P\delta_1$ and $\epsilon_2 = N\delta_1$, where the matrix P is chosen such that $[P^\top, N^\top]^\top$ is invertible, and then $\xi = [P^\top, N^\top]^\top\chi$. The δ_i's can be used to represent the effects of unmodeled nonlinearities in the dynamics or the measurements (by letting the δ_i's be the remainder terms in the Taylor approximations). This allows us to incorporate the effects of the nonlinearities in the observer error terms $\epsilon_\ddagger$ and ϵ_* from Theorem 1.

B. Proof of Theorem 1

We introduce the variables $y = Y - \hat{\xi}_2$ and

$$\begin{aligned}
\Delta_i(t) &= F_i(Y(t)-\delta_2(t),t) - F_i(Y(t),t) \text{ and}\\
x_i(t) &= \xi_i(t) - \hat{\xi}_i(t) \text{ for } i=1,2.
\end{aligned} \tag{17}$$

Then simple calculations based on (2) and (7) give

$$\begin{cases}
\dot{x}_1(t) = A_1x_1(t) + \Delta_1(t) + \epsilon_1(t)\\
\dot{x}_2(t) = A_2x_1(t) - kx_2(t) + \Delta_2(t) + \epsilon_2(t)\\
y(t) = x_2(t) + \delta_2(t).
\end{cases} \tag{18}$$

Here and in the sequel, all equalities and inequalities hold for all $t \geq 0$, unless otherwise indicated.

By applying variation of parameters to (18), we obtain

$$\begin{aligned}
x_1(t) &= e^{A_1(t-s)}x_1(s)\\
&+ \int_s^t e^{A_1(t-m)}[\Delta_1(m)+\epsilon_1(m)]\mathrm{d}m
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
x_2(t) - e^{-k(t-s)}x_2(s) &= \rho_1(t,s)\\
&+A_2(A_1+kI)^{-1}\Big[e^{A_1(t-s)} - e^{-k(t-s)I}\Big]x_1(s)
\end{aligned} \tag{20}$$

for all $s \geq 0$ and $t \geq s$, where

$$\rho_1(t,s) = \int_s^t e^{k(\ell-t)}[\Delta_2(\ell)+\epsilon_2(\ell)]\mathrm{d}\ell$$

$$+\int_s^t e^{k(m-t)}A_2\int_s^m e^{A_1(m-\ell)}[\Delta_1(\ell)+\epsilon_1(\ell)]\mathrm{d}\ell\mathrm{d}m, \tag{21}$$

and where we used the fact that

$$\begin{aligned}&\int_s^t e^{k(m-t)}A_2e^{A_1(m-s)}x_1(s)\mathrm{d}m\\&=e^{k(s-t)}A_2\int_s^t e^{(A_1+kI)(m-s)}\mathrm{d}mx_1(s)\\&=e^{k(s-t)}A_2(A_1+kI)^{-1}\Big(e^{(A_1+kI)(t-s)}-I\Big)x_1(s),\end{aligned} \tag{22}$$

where the $e^{k(s-t)}$ in (22) occurs because of the relation $e^{k(m-t)}A_2e^{A_1(m-s)} = e^{k(m-t)}A_2e^{(A_1+kI)(m-s)}e^{k(s-m)} = e^{k(s-t)}A_2e^{(A_1+kI)(m-s)}$.

According to (19), we have

$$x_1(s)=e^{A_1(s-t)}x_1(t)-\int_s^t e^{A_1(s-m)}[\Delta_1(m)+\epsilon_1(m)]\mathrm{d}m, \tag{23}$$

and our formulas (4) and (8) give $\lambda(s-t)=\mathcal{H}^\top\Delta_*(t-s)e^{A_1(s-t)}$. Hence, we can substitute (23) into (20) to obtain

$$\lambda(s-t)x_1(t)=x_2(t)-e^{-k(t-s)}x_2(s)+\rho_2(t,s) \tag{24}$$

where

$$\begin{aligned}\rho_2(t,s)=&-\rho_1(t,s)\\&+\mathcal{H}^\top\Delta_*(t-s)\int_s^t e^{A_1(s-m)}[\Delta_1(m)+\epsilon_1(m)]\mathrm{d}m.\end{aligned} \tag{25}$$

By left multiplying both sides of (24) by $\lambda(s-t)^\top$, we obtain

$$\begin{aligned}&\lambda(s-t)^\top\lambda(s-t)x_1(t)=\lambda(s-t)^\top\rho_2(t,s)\\&+\lambda(s-t)^\top x_2(t)-e^{-k(t-s)}\lambda(s-t)^\top x_2(s).\end{aligned} \tag{26}$$

By integrating (26) with respect to s over $[t-\tau,t]$, we obtain

$$\begin{aligned}&\int_{t-\tau}^t\lambda(s-t)^\top\lambda(s-t)\mathrm{d}sx_1(t)\\&=\int_{t-\tau}^t\lambda(s-t)^\top\mathrm{d}sx_2(t)\\&\quad-\int_{t-\tau}^t e^{-k(t-s)}\lambda(s-t)^\top x_2(s)\mathrm{d}s\\&\quad+\int_{t-\tau}^t\lambda(s-t)^\top\rho_2(t,s)\mathrm{d}s \text{ for all } t\geq\tau.\end{aligned} \tag{27}$$

Hence, our choices in (5)-(6), and the invertibility of $\mathcal{S}$, give

$$\begin{aligned}x_1(t)=&\mathcal{R}x_2(t)+\mathcal{S}^{-1}\int_{t-\tau}^t\lambda(s-t)^\top\rho_2(t,s)\mathrm{d}s\\&-\mathcal{S}^{-1}\int_{t-\tau}^t e^{-k(t-s)}\lambda(s-t)^\top x_2(s)\mathrm{d}s.\end{aligned} \tag{28}$$

Using the formula for λ from (4), we obtain

$$\begin{aligned}x_1(t)=&\mathcal{R}x_2(t)+\mathcal{S}^{-1}\int_{t-\tau}^t\lambda(s-t)^\top\rho_2(t,s)\mathrm{d}s\\&-\mathcal{S}^{-1}\int_{t-\tau}^t e^{-k(t-s)}\Big[I-e^{(A_1^\top+kI)(s-t)}\Big]\mathcal{H}x_2(s)\mathrm{d}s\end{aligned} \tag{29}$$

with $\mathcal{H}$ defined in (6). This equality can be rewritten as

$$\begin{aligned}x_1(t)=&\mathcal{R}x_2(t)+\mathcal{S}^{-1}\int_{t-\tau}^t\lambda(s-t)^\top\rho_2(t,s)\mathrm{d}s\\&-\mathcal{S}^{-1}\int_{t-\tau}^t e^{-k(t-s)}\mathcal{H}x_2(s)\mathrm{d}s\\&+\mathcal{S}^{-1}\int_{t-\tau}^t e^{-(A_1^\top+2kI)(t-s)}\mathcal{H}x_2(s)\mathrm{d}s.\end{aligned} \tag{30}$$

Since $\xi_2(s)=Y(s)-\delta_2(s)$, we have $Y-\hat{\xi}_2=x_2+\delta_2$, so we deduce from (7) and (30) that

$$\begin{aligned}x_1(t)=&\mathcal{R}x_2(t)-\mathcal{S}^{-1}\Big[\psi_1(t)-e^{-k\tau}\psi_1(t-\tau)\Big]\\&+\mathcal{S}^{-1}\Big[\psi_2(t)-e^{-(A_1^\top+2kI)\tau}\psi_2(t-\tau)\Big]+\epsilon_\bigstar(t),\end{aligned} \tag{31}$$

where

$$\begin{aligned}\epsilon_\bigstar(t)=&\mathcal{S}^{-1}[\int_{t-\tau}^t\lambda(s-t)^\top\rho_2(t,s)\mathrm{d}s\\&+\int_{t-\tau}^t e^{-k(t-s)}\mathcal{H}\delta_2(s)\mathrm{d}s\\&.-\int_{t-\tau}^t e^{-(A_1^\top+2kI)(t-s)}\mathcal{H}\delta_2(s)\mathrm{d}s].\end{aligned} \tag{32}$$

Hence, (13) and (17) give $\xi_1=\xi_e+\epsilon_\bigstar$ for all $t\geq\tau$.

Recalling the formula for ρ_2 in (25) and (17) and our Lipshitz condition (3) on the F_i's, it follows that (32) satisfies

$$\begin{aligned}|\epsilon_\bigstar(t)|\leq&\overline{S}\int_{t-\tau}^t|\lambda(s-t)|\rho_1(t,s)|\mathrm{d}s\\&+\overline{S}\int_{t-\tau}^t|\lambda(s-t)||\mathcal{H}^\top|J_1(t,s)\mathrm{d}s\\&+\overline{S}|\mathcal{H}|\int_{t-\tau}^t e^{-k(t-s)}|\delta_2(s)|\mathrm{d}s\\&+\overline{S}|\mathcal{H}|\int_{t-\tau}^t e^{|A_1^\top+2kI|(t-s)}|\delta_2(s)|\mathrm{d}s\\\leq&\overline{S}\int_{t-\tau}^t|\lambda(s-t)|\rho_1(t,s)|\mathrm{d}s\\&+\overline{S}\int_{t-\tau}^t|\lambda(s-t)||\mathcal{H}^\top|J_2(t,s)\mathrm{d}s\\&+\overline{S}|\mathcal{H}|\int_{t-\tau}^t J_3(t-s)|\delta_2(s)|\mathrm{d}s, \text{ where}\end{aligned} \tag{33}$$

$$J_1(t,s)=|\Delta_*(t-s)|\int_s^t e^{|A_1|(m-s)}[|\Delta_1(m)|+|\epsilon_1(m)|]\mathrm{d}m \tag{34}$$

and the function Δ_* was defined in (8), and where

$$J_2(t,s)=\Big[e^{|A_1|\tau}+1\Big]\int_s^t e^{|A_1|(m-s)}[K_1|\delta_2(m)|+|\epsilon_1(m)|]\mathrm{d}m \tag{35}$$

and $J_3(r)=e^{-kr}+e^{|A_1^\top+2kI|r}$, and where $\overline{S}$ is defined in (11). Also, when $\delta_2=0$, we can use (17) to get $\Delta_1=\Delta_2=0$, so our formula (9) for $\epsilon_\ddagger$ and (21) and (25) give $\epsilon_\bigstar=\epsilon_\ddagger$ when $\delta_2=0$. Hence, since the right side of (15) is an upper bound for the right side of (33), this allows us to conclude that (15) holds for all $t\geq\tau$.

IV. Discrete Measurements Cases

A. Assumptions and Statement of Theorem

While the observer from Section III enjoys fixed time convergence and robustness properties, it requires continuous measurements of the output, which might not always be available in practice. Therefore, we next consider cases where the variables are only measured at discrete instants.

Let t_j be a sequence such that $t_0=0$ and such that there are two constants $\underline{T}>0$ and $\overline{T}>\underline{T}$ such that

$$\underline{T}\leq t_{j+1}-t_j\leq\overline{T} \text{ for all } j\geq 0. \tag{36}$$

We continue the notation from Section III except we consider

$$\begin{cases} \dot{\xi}_1(t) = A_1\xi_1(t) + F_1(\xi_2(t), t) + \epsilon_1(t) \\ \dot{\xi}_2(t) = A_2\xi_1(t) - k\xi_2(t) + F_2(\xi_2(t), t) + \epsilon_2(t) \\ Y(t_j) = \xi_2(t_j) + \delta_2(t_j) \text{ for all } j \geq 0, \end{cases} \tag{37}$$

under the assumption that F_1 and F_2 satisfy (3) and where k is selected as in Section II. We also use these constants:

$$\begin{aligned} \varsigma_1 &= \frac{\bar{S}|A_2|e^{|A_1|\tau}}{k^2}\int_{-\tau}^{0}|\lambda(s)|(e^{ks} - sk - 1)\mathrm{d}s \\ \varsigma_2 &= \frac{\bar{S}}{k}\int_{-\tau}^{0}|\lambda(s)|(1 - e^{ks})\mathrm{d}s \\ \varsigma_3 &= \bar{S}\tau e^{|A_1|\tau}|\mathcal{H}^\top|\int_{-\tau}^{0}|\lambda(s)|(e^{|A_1|\tau} + e^{-ks})\mathrm{d}s \\ \varsigma_4 &= \bar{S}|\mathcal{H}|\left(\frac{1}{k}(1 - e^{-k\tau}) + \tau e^{|A_1^\top + 2kI|\tau}\right) \end{aligned} \tag{38}$$

where $\mathcal{H}$ is from (6) as before, and τ satisfies the requirements from Section II. We introduce the dynamic extension

$$\begin{cases} \dot{\hat{\xi}}_1(t) = A_1\hat{\xi}_1(t) + F_1(\omega(t), t) \\ \dot{\hat{\xi}}_2(t) = A_2\hat{\xi}_1(t) - k\hat{\xi}_2(t) + F_2(\omega(t), t) \\ \dot{\psi}_1(t) = -k\psi_1(t) + \mathcal{H}[\omega(t) - \hat{\xi}_2(t)] \\ \dot{\psi}_2(t) = -(A_1^\top + 2kI)\psi_2(t) + \mathcal{H}[\omega(t) - \hat{\xi}_2(t)] \\ \dot{\omega}(t) = A_2\xi_e(t) - k\omega(t) + F_2(\omega(t), t) \\ \qquad \text{for all } t \in [t_j, t_{j+1}) \text{ and } j \geq 0 \\ \omega(t_j) = Y(t_j) \text{ for all } j \geq 0 \end{cases} \tag{39}$$

where ξ_e is defined as in (13). We prove:

Theorem 2: Let the constant $\overline{T}$ in (36) be such that

$$\overline{T}\mu < 1, \text{ where} \tag{40}$$

$$\mu = |A_2|q_1 + k + K_2 \text{ and } q_1 = (\varsigma_1 + \varsigma_3)K_1 + \varsigma_2K_2 + \varsigma_4 \tag{41}$$

using the constants (38). Then we can find positive constants a_1 and a_2 such that all solutions of (37) and (39) satisfy

$$\begin{aligned} |\xi_1(t) - \xi_e(t)| \ &\leq \ a_1|\xi_2 - \omega|_{[r-2\tau-\overline{T},r]}e^{\frac{\ln(\overline{T}\mu)}{\tau+\overline{T}}(t-r)} \\ &+ a_2 \sup_{\ell\in[r-\overline{T}-2\tau,t]}[|\epsilon_1(\ell)| + |\epsilon_2(\ell)| + |\delta_2(\ell)|] \end{aligned} \tag{42}$$

for all $r \geq 2\tau + \overline{T}$ and $t \geq r + \tau$.

Remark 4: The inequality (42) is of ISS type because

$$\frac{\ln(\overline{T}\mu)}{\tau + \overline{T}} < 0, \tag{43}$$

by (40). Moreover, since μ is independent of $\overline{T}$, the left side of (43) converges to $-\infty$ as $\overline{T} \to 0^+$. Therefore, we can have arbitrarily large rates of convergence of the estimation error $|\xi_1(t) - \xi_e(t)|$ to 0 when the ϵ_i's are zero, by choosing the sample times t_i such that $\overline{T}$ is small enough. In practice, this faster sampling can often be achieved by upgrading to a faster digital signal processor (or DSP) in a lab. Our proof of Theorem 2 can be used to easily get formulas for the a_i's in (42). For example, our proof shows that we can choose $a_1 = q_1e^{-R_*\tau}$ and $a_2 = q_2+q_1\overline{T}^\sharp/(1-\overline{T}\mu)$, where R_* is the left side of (43), $\overline{T}^\sharp = \overline{T}(|A_2|q_2 + 1) + 1$ and $q_2 = \max\{\varsigma_1 + \varsigma_3, \varsigma_2\}$. Also, since the F_i's are known, we can readily compute the Lipschitz constants K_i that are needed to compute μ.

B. Proof of Theorem 2

Our proof will use the variables

$$\tilde{\omega}(t) = \xi_2(t) - \omega(t), \text{ and } x_i(t) = \xi_i(t) - \hat{\xi}_i(t)$$

$$\text{and } \kappa_i(t) = F_i(\xi_2(t), t) - F_i(\omega(t), t) \text{ for } i = 1, 2. \tag{44}$$

Then $\omega - \hat{\xi}_2 = \xi_2 - \tilde{\omega} - \hat{\xi}_2 = x_2 - \tilde{\omega}$, so we obtain

$$\begin{cases} \dot{x}_1(t) = A_1x_1(t) + \kappa_1(t) + \epsilon_1(t) \\ \dot{x}_2(t) = A_2x_1(t) - kx_2(t) + \kappa_2(t) + \epsilon_2(t) \\ \dot{\psi}_1(t) = -k\psi_1(t) + \mathcal{H}x_2(t) - \mathcal{H}\tilde{\omega}(t) \\ \dot{\psi}_2(t) = -(A_1^\top + 2kI)\psi_2(t) + \mathcal{H}x_2(t) - \mathcal{H}\tilde{\omega}(t) \\ \dot{\tilde{\omega}}(t) = A_2[\xi_1(t) - \xi_e(t)] - k\tilde{\omega}(t) + \kappa_2(t) + \epsilon_2(t) \\ \qquad \text{for all } t \in [t_j, t_{j+1}) \text{ and } j \geq 0 \\ \tilde{\omega}(t_j) = -\delta_2(t_j) \text{ for all } j \geq 0. \end{cases} \tag{45}$$

We also use the $\mathbb{R}^{n-q}$-valued variables

$$\begin{aligned} \gamma_1(t) = &-\mathcal{S}^{-1}\int_{t-\tau}^{t}\lambda(s-t)^\top\mathcal{H}^\top\Delta_*(t-s)J_a(t,s)\mathrm{d}s \\ &- \mathcal{S}^{-1}\int_{t-\tau}^{t}\lambda(s-t)^\top\int_{s}^{t}e^{k(m-t)}A_2J_a(s,m)\mathrm{d}m\mathrm{d}s \\ &- \mathcal{S}^{-1}\int_{t-\tau}^{t}\lambda(s-t)^\top\int_{s}^{t}e^{k(\ell-t)}[\kappa_2(\ell) + \epsilon_2(\ell)]\mathrm{d}\ell\mathrm{d}s, \end{aligned} \tag{46}$$

$$\begin{aligned} \gamma_2(t) = &-\mathcal{S}^{-1}\int_{t-\tau}^{t}e^{-k(t-s)}\mathcal{H}\tilde{\omega}(s)\mathrm{d}s \\ &+ \mathcal{S}^{-1}\int_{t-\tau}^{t}e^{(A_1^\top+2kI)(-t+s)}\mathcal{H}\tilde{\omega}(s)\mathrm{d}s, \end{aligned} \tag{47}$$

and

$$\begin{aligned} x_a(t) = \mathcal{R}x_2(t) &- \mathcal{S}^{-1}\Big[\psi_1(t) - e^{-k\tau}\psi_1(t-\tau)\Big] \\ &+ \mathcal{S}^{-1}\Big[\psi_2(t) - e^{-(A_1^\top+2kI)\tau}\psi_2(t-\tau)\Big], \end{aligned} \tag{48}$$

$$\text{where } J_a(s,m) = \int_{s}^{m}e^{A_1(m-\ell)}[\kappa_1(\ell) + \epsilon_1(\ell)]\mathrm{d}\ell. \tag{49}$$

The rest of the proof of the theorem consists of two steps. In the first step, we prove that the preceding variables satisfy

$$\xi_1(t) = \xi_e(t) + \gamma_1(t) + \gamma_2(t) \text{ for all } t \geq \tau. \tag{50}$$

Then, we bound $\gamma_1(t) + \gamma_2(t)$ by the right side of (42).

First Step. Since the (x_1, x_2)-dynamics of (45) agree with the first two equations of (18) except with the Δ_i's replaced by the κ_i's, the same reasoning that led to (30) gives

$$\begin{aligned} x_1(t) = \mathcal{R}x_2(t) &- \mathcal{S}^{-1}\int_{t-\tau}^{t}e^{-k(t-s)}\mathcal{H}x_2(s)\mathrm{d}s \\ &+ \mathcal{S}^{-1}\int_{t-\tau}^{t}e^{-(A_1^\top+2kI)(t-s)}\mathcal{H}x_2(s)\mathrm{d}s + \gamma_1(t). \end{aligned} \tag{51}$$

Also, by applying the method of variation of parameters separately to the ψ_1 and ψ_2 dynamics in (45), we obtain

$$\begin{aligned} &\int_{t-\tau}^{t}e^{-k(t-s)}\mathcal{H}x_2(s)\mathrm{d}s \\ &= \psi_1(t) - e^{-k\tau}\psi_1(t-\tau) + \int_{t-\tau}^{t}e^{-k(t-s)}\mathcal{H}\tilde{\omega}(s)\mathrm{d}s \end{aligned} \tag{52}$$

and

$$\begin{aligned} &\int_{t-\tau}^{t}e^{-(A_1^\top+2kI)(t-s)}\mathcal{H}x_2(s)\mathrm{d}s \\ &= \psi_2(t) - e^{-(A_1^\top+2kI)\tau}\psi_2(t-\tau) \\ &+ \int_{t-\tau}^{t}e^{(A_1^\top+2kI)(-t+s)}\mathcal{H}\tilde{\omega}(s)\mathrm{d}s \end{aligned} \tag{53}$$

for all $t \geq \tau$. By combining (51)-(53), we obtain

$$x_1(t) = \mathcal{R}x_2(t) - \mathcal{S}^{-1}\Big[\psi_1(t) - e^{-k\tau}\psi_1(t-\tau)$$

$$+\int_{t-\tau}^{t} e^{-k(t-s)}\mathcal{H}\tilde{\omega}(s)\mathrm{d}s\Big] + \gamma_1(t)$$
$$+\ \mathcal{S}^{-1}\Big[\psi_2(t) - e^{-(A_1^\top+2kI)\tau}\psi_2(t-\tau)$$
$$+\int_{t-\tau}^{t} e^{(A_1^\top+2kI)(-t+s)}\mathcal{H}\tilde{\omega}(s)\mathrm{d}s\Big] \quad (54)$$

for all $t \geq \tau$ Hence, our choices (46)-(48), and our choice of x_1 in (44) and our formula (13) for ξ_e, give $x_1 = x_a + \gamma_1 + \gamma_2$ and $x_1 - x_a = \xi_1 - \xi_e$, which we can combine to obtain (50).

Second Step. From (50), it follows that

$$\begin{aligned}
&|\xi_1(t) - \xi_e(t)|\\
&\leq \overline{S}\int_{t-\tau}^{t} |\lambda(s-t)|\int_{s}^{t} |A_2|e^{k(m-t)}J_4(s,m)\mathrm{d}m\mathrm{d}s\\
&+\overline{S}\int_{t-\tau}^{t} |\lambda(s-t)|\int_{s}^{t} e^{k(\ell-t)}[|\kappa_2(\ell)| + |\epsilon_2(\ell)|]\mathrm{d}\ell\mathrm{d}s\\
&+\overline{S}\int_{t-\tau}^{t} |\lambda(s-t)||\mathcal{H}^\top|\Big[e^{|A_1|\tau} + e^{k(s-t)}\Big]J_4(s,t)\mathrm{d}s\\
&+\overline{S}\int_{t-\tau}^{t} \Big(e^{k(s-t)} + e^{|A_1^\top+2kI|\tau}\Big)|\mathcal{H}||\tilde{\omega}(s)|\mathrm{d}s \qquad (55)
\end{aligned}$$

for all $t \geq \tau$, where

$$J_4(s,m) = \int_{s}^{m} e^{|A_1|\tau}[|\kappa_1(\ell)| + |\epsilon_1(\ell)|]\mathrm{d}\ell. \qquad (56)$$

Moreover, our choices of the κ_i's in (44) and (3) give $|\kappa_2(\ell)| \leq K_2|\tilde{w}(\ell)|$ when $s \leq \ell \leq t$, and so also

$$J_4(s,m) \leq (m-s)e^{|A_1|\tau}(K_1|\tilde{w}|_{[s,m]} + |\epsilon_1|_{[s,m]}) \qquad (57)$$

when $s \leq m \leq t$. Therefore, by upper bounding the right side of (55) and then collecting coefficients of $|\tilde{w}|_{[t-\tau,t]}$ and $|\epsilon_i|_{[t-\tau,t]}$ for $i = 1, 2$ in the result, it follows from (38) that

$$\begin{aligned}
&|\xi_1(t) - \xi_e(t)|\\
&\leq q_1|\tilde{w}|_{[t-\tau,t]} + q_2(|\epsilon_1|_{[t-\tau,t]} + |\epsilon_2|_{[t-\tau,t]}) \text{ for all } t \geq \tau,
\end{aligned} \qquad (58)$$

where q_1 is from (41) and $q_2 = \max\{\varsigma_1 + \varsigma_3, \varsigma_2\}$.

By combining (45) and (58), and recalling (3), we get

$$\begin{aligned}
|\dot{\tilde{\omega}}(t)| &\leq |A_2||\xi_1(t) - \xi_e(t)| + k|\tilde{\omega}(t)| + |\epsilon_2(t)|\\
&\quad + |F_2(\xi_2(t),t) - F_2(\omega(t),t)|\\
&\leq (|A_2|q_1 + k + K_2)|\tilde{w}|_{[t-\tau,t]} + \epsilon_\pounds(t) \qquad (59)
\end{aligned}$$

for all $t \in [t_j, t_{j+1})$ and all $j \geq 0$ when $t \geq \tau$, where $\epsilon_\pounds(t) = |A_2|q_2(|\epsilon_1|_{[t-\tau,t]} + |\epsilon_2|_{[t-\tau,t]}) + |\epsilon_2(t)|$. Since

$$\tilde{\omega}(t) = \tilde{\omega}(t_j) + \int_{t_j}^{t} \dot{\tilde{\omega}}(\ell)\mathrm{d}\ell \qquad (60)$$

for all $t \in [t_j, t_{j+1})$, we deduce that, for all $t \geq \overline{T} + \tau$,

$$\begin{aligned}
|\tilde{\omega}(t)| &\leq |\tilde{\omega}(t_j)| + \overline{T}(|A_2|q_1 + k + K_2)|\tilde{w}|_{\mathcal{S}_t}\\
&\quad + \overline{T}|\epsilon_\pounds|_{[t-\overline{T},t]}\\
&\leq \overline{T}\mu|\tilde{\omega}|_{\mathcal{S}_t} + \overline{T}^\sharp\left(\sum_{j=1}^{2} |\epsilon_i|_{\mathcal{S}_t} + |\delta_2|_{\mathcal{S}_t}\right) \qquad (61)
\end{aligned}$$

when $t \in [t_j, t_{j+1})$, $j \geq 0$, $t \geq r$, and $r \geq \overline{T} + \tau$, where the suprema are over $\mathcal{S}_t = [t-\overline{T}-\tau, t]$, $\overline{T}^\sharp = \overline{T}(|A_2|q_2+1)+1$, the last inequality in (61) used the bound $|\tilde{\omega}(t_j)| = |\delta_2(t_j)|$, and μ is from (41). Using (40), it follows from applying [12, Lemma 1] to $w_0(t) = |\tilde{\omega}(t+r)|$ that, for all $t \geq r$,

$$|\tilde{\omega}(t)| \leq |\tilde{\omega}|_{[r-\tau-\overline{T},r]}e^{\frac{\ln(\overline{T}\mu)}{\tau+\overline{T}}(t-r)} + \mathcal{T}_\epsilon(t,r), \quad (62)$$

$$\text{where } \mathcal{T}_\epsilon(t,r) = \frac{\overline{T}^\sharp}{1-\overline{T}\mu}\left(\sum_{j=1}^{2} |\epsilon_i|_{\mathcal{S}_t} + |\delta_2|_{\mathcal{S}_t}\right). \qquad (63)$$

The theorem now follows by using (62) to upper bound the first right side term in (58).

V. Illustrations

Consider this model for a single-link direct-drive manipulator actuated by a permanent magnet DC brush motor [13]:

$$\begin{aligned}
&M\ddot{q} + B\dot{q} + N\sin(q) = \mathcal{I} \ \text{ and}\\
&L\dot{\mathcal{I}} = V_e - R\mathcal{I} - K_B\dot{q},\\
\text{where}\quad &M = \frac{J}{K_\tau} + \frac{mL_0^2}{3K_\tau} + \frac{M_0L_0^2}{K_\tau} + \frac{2M_0R_0^2}{5K_\tau},\\
&N = \frac{mL_0G}{2K_\tau} + \frac{M_0L_0G}{K_\tau}, \text{ and } B = \frac{B_0}{K_\tau} \qquad (64)
\end{aligned}$$

where the physical meanings of the positive constants m, J, L_0, M_0, B_0, R_0, G, K_τ, R, L, K_B, and V_e is explained in [13], $q(t)$ is the position of the load (which is the angular motor position), and $\mathcal{I}(t)$ is the motor armature current. We assume that perturbed measurements of q are available.

The model (64) has been studied extensively. For instance, see [10] for continuous-discrete observers for (64), and [14] for full order observers with sampling and input delays. However, we believe that the problem we will solve of building reduced order observers for (64) with arbitrarily small fixed convergence times τ and a single delay was open.

By also allowing additive uncertainties in the model (64) and in the measurements, we obtain the dynamics

$$\begin{cases}
\dot{\chi}_1(t) = \chi_2(t) + \delta_{1,1}(t)\\
\dot{\chi}_2(t) = b_1\chi_3(t) - a_1\sin(\chi_1(t)) - a_2\chi_2(t)\\
\qquad\quad + \delta_{1,2}(t)\\
\dot{\chi}_3(t) = b_0u(t) - a_3\chi_2(t) - a_4\chi_3(t) + \delta_{1,3}(t)\\
Y(t) = \chi_1(t) + \delta_2(t)
\end{cases} \qquad (65)$$

where $\chi_1 = q$, $\chi_2 = \dot{q}$, $\chi_3 = \mathcal{I}$, $a_1 = N/M$, $a_2 = B/M$, $a_3 = K_B/L$, $a_4 = R/L$, $b_0 = 1/L$, and $b_1 = 1/M$, and $u = V_e$ is the control. As in [6], we choose $b_0 = 40$, $b_1 = 15$, $a_1 = 35$, $a_2 = 1$, $a_3 = 36.4$ and $a_4 = 200$.

Adopting the notation $\xi_2 = \chi_1$, $\xi_{1,1} = \chi_2$, $\xi_{1,2} = \chi_3$, $\varepsilon_{1,1}(t) = \delta_{1,2}(t)$, $\varepsilon_{1,2}(t) = \delta_{1,3}(t)$, $\varepsilon_2(t) = \delta_{1,1}(t)$, and $\varepsilon_3(t) = \delta_2(t)$, we can rewrite the system (65) as

$$\begin{cases}
\dot{\xi}_{1,1}(t) = -a_2\xi_{1,1}(t) + b_1\xi_{1,2}(t)\\
\qquad\quad - a_1\sin(Y(t) - \delta_2(t)) + \varepsilon_{1,1}(t)\\
\dot{\xi}_{1,2}(t) = -a_3\xi_{1,1}(t) - a_4\xi_{1,2}(t) + b_0u(t)\\
\qquad\quad + \varepsilon_{1,2}(t)\\
\dot{\xi}_2(t) = \xi_{1,1}(t) - k\xi_2(t) + k[Y(t) - \delta_2(t)] + \varepsilon_2(t)\\
Y(t) = \xi_2(t) + \delta_2(t)
\end{cases} \qquad (66)$$

for a constant $k > 0$ that will be specified.

Then the notation of Sections II–III produces the choices

$$A_1 = \begin{bmatrix} -a_2 & b_1\\ -a_3 & -a_4 \end{bmatrix}, \ A_2 = [1 \quad 0], \qquad (67)$$

$F_1(s,t) = (-a_1\sin(s), b_0u(t))$, $F_2(s,t) = ks$, $K_1 = a_1$ and $K_2 = k$. With the preceding parameter choices, $A_1 + kI$ is invertible when $k^2 - 201k + 746 \neq 0$. Thus we can take any $k > 0$ that is not a root of $k^2 - 201k + 746$. The preceding choices can then be used to write the dynamic extensions from our theorems, and then Theorem 1 provides the exact value of ξ_1. The preceding observer contrasts significantly with the fixed time observer for (65) that was presented in [6, Sec. 5.2], whose fixed convergence time τ is required to be such that

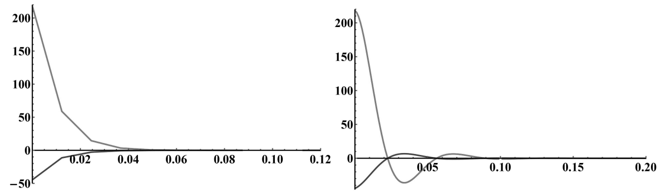

Fig. 1. Observer Error Components for (66) from Theorem 1 (Left) and Theorem 2 (Right) with $k = 100$ using Parameter Values from [6].

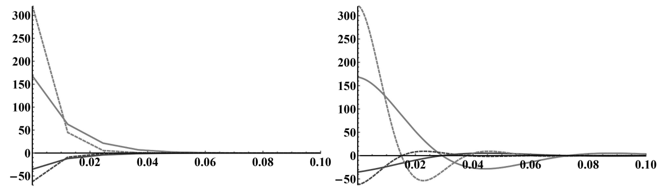

Fig. 2. Observer Error Components for (66) from Theorem 1 (Left) and Theorem 2 (Right) with $k = 75$ (Solid) and $k = 150$ (Dashed) using Parameter Values from [6].

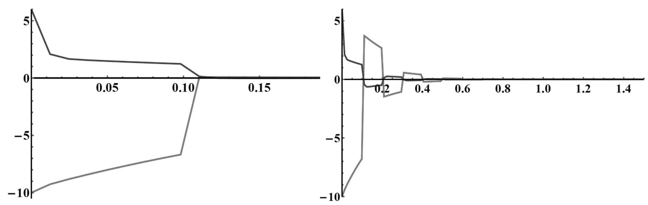

Fig. 3. Observer Error Components for (66) using [10, Th. 1] (Left) and [10, Th. 2] (Right) using Parameter Values from [6].

$e^{-H\tau} - e^{-\tau A}$ is invertible where $H = A + L_A C$ for a suitable matrix L_A, and where C is from the representation $y = Cx$ of the output in terms of the state x. Moreover, [6, Sec. 5.2] produces large coefficients in the final estimation error under discrete time measurements for small $\tau > 0$ values. Hence, we believe that the observer designs from this work offer potential advantages over previously available observers.

We simulated the observers from Theorems 1–2 from Sections III–IV for the preceding motor model using the program Mathematica, and we report the results in Fig. 1. Fig. 2 shows the effects of changing the observer parameter k (e.g., from (4)), with all other parameters kept the same as the simulations in Fig. 1. In both figures, we used the same motor model parameter values, initial states for (66), parameter $\tau = 0.1$, and uncertainties that were used in the observers for the motor dynamics in [6, Sec. 5.2], which used observers from [6, Ths. 2 and 4] (which were not reduced order). We chose $\bar{T} = \underline{T} = 0.1$. We plotted the first component of the observer error (for estimating $\xi_{1,1}$) in red, and the second observer error component (for $\xi_{1,2}$) in blue.

In Fig. 3 below, we also show Mathematica simulations using the observers from both theorems from [10], using the preceding parameter values, and the matrix $L = [-1, 3]^\top$ and parameter values $\nu = 0.1$ and $g = 20$ from [10]. Since Fig. 1–Fig. 2 show improved performance in terms of settling terms and observer errors after time $\tau = 0.1$ (e.g., $\epsilon_\ddagger$ from Theorem 1) compared with [6] and [10] (while reducing the order of the observer compared with [6]), they help illustrate this work; see, e.g., [6, Fig. 6], where the observation errors provided by the upper or lower bound from [6, Th. 4] were greater than 9 after 1.5 seconds when nonzero uncertainties were present, whereas our observation errors from using Theorem 2 from Section IV above stayed below 1 after 0.15 seconds.

Appendix: Invertibility of the Matrix $\mathcal{S}$

Let us prove that the matrix (5) is invertible, which was needed for the observer designs from our theorems. Let $V \in \mathbb{R}^{n-q}$ be a vector such that $\mathcal{S}V = 0$. Then

$$\int_{-\tau}^{0} V^\top \lambda(m)^\top \lambda(m) V \mathrm{d}m = 0. \tag{A.1}$$

As an immediate consequence, we get $\lambda(m)V = 0$ for all $m \in [-\tau, 0]$. It follows that for all integers $j > 0$, $\lambda^{(j)}(0)V = 0$. Also, simple calculations give $\lambda^{(j)}(0) = -A_2(A_1+kI)^{j-1}$. Thus $A_2(A_1 + kI)^l V = 0$ for all integers $l \geq 0$. Using the fact that these equalities are equivalent to

$$A_2 \sum_{j=0}^{l} C_l^j A_1^j V = 0 \tag{A.2}$$

for suitable nonzero integers C_l^j, we deduce that $A_2 A_1^l V = 0$ for all integers $l \geq 0$, by induction on l. Since (A_1, A_2) is observable, it follows that $V = 0$. This allows us to conclude.

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