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New Fixed Time and Fast Converging Reduced Order Observers

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Abstract— **For nonlinear continuous-time systems with continuous measurements of the output, we provide new reduced order observers that converge in finite time. The convergence time is independent of the initial state. For cases where the measurements are discrete, we provide asymptotically converging observers, whose rate of convergence is proportional to the negative of the logarithm of the size of the sampling interval. Our observers are based on the observability Gramian.**

I. Introduction

As explained in [12], finite time observers offer considerable promise for an ever-growing range of practical applications, because of their ability to compute exact values of states in a finite time, and many contributions are devoted to the design of observers of this type. Some observers use delays, dynamic extensions, homogenous functions, sliding mode, or unbounded gains; see for instance, [2], [3], [5], [8], [15], and [18]. The works [12] and [16] are different because they use observers with impulses and no delay.

In this paper, we continue our work on the design of fixed time observers. Fixed time convergence means that the convergence time is independent of the initial state. This differs from semi-global works such as [19] whose finite convergence time depends on the initial state. As in [12], we provide new reduced order observers for continuous-time nonlinear systems, first when there are continuous output measurements and next in the case where there are only discrete output measurements. When continuous measurements are available, we provide observers that converge in finite time. When only discrete measurements are available, we provide observers that do not converge in finite time, but which do converge asymptotically with a rate of convergence that is proportional to the negative of the logarithm of the size of the sampling interval. This ensures arbitrarily fast convergence, by picking the sampling interval small enough.

The fundamental difference between [12] and the present paper is that the observer we introduce is not based on the one proposed in [16]. Instead, the novel observer design that we propose here uses discrete variables and the observability Gramian and the solutions of the observer are continuous. It is also very different from those of [1], [4], [10], [13], and [17], which include observers with delays. By not using the delays that occur in earlier observer designs, we obtain simpler reduced order controllers that still enjoy the required fixed time or arbitrarily fast convergence of the observation error to zero. Our convergence proof for our second observer uses the trajectory based approach that was introduced in [11] and developed in several papers such as [14]. See, e.g., [7], for more motivation for observer design.

Our observer result under continuous measurements is stated and proven in Section II, and Section III provides our analog where only discrete measurements are available. We illustrate our approach in Section IV. We summarize our results and our suggestions for further research in Section V.

Notation. We use standard notation, which we simplify when no confusion would arise. The dimensions of our Euclidean spaces are arbitrary, unless we indicate otherwise. The standard Euclidean 2-norm, and its induced matrix norm, are denoted by $|\cdot|$, $|\cdot|_S$ denotes the essential supremum over any set S, $\mathbb{N} = \{1, 2, \ldots\}$, and $\mathbb{Z}_{\geq 0} = \{0\} \cup \mathbb{N}$. We let I denote the identity matrix of any dimension.

II. Observers for Continuous Measurements

A. Statement of Result

We consider the system

$$\left\{ \begin{array}{rcl} \dot{\xi}(t) & = & \mathcal{A}\xi(t) + \mathcal{F}(\mathcal{C}\xi(t), u(t)) + \kappa(t) \\ Y(t) & = & \mathcal{C}\xi(t) + \epsilon(t) \end{array} \right. \tag{1}$$

where ξ is valued in $\mathbb{R}^n$, the piecewise continuous locally bounded function u is valued in $\mathbb{R}^p$, the output Y is valued in $\mathbb{R}^q$, $\mathcal{C}$ has full rank, κ and ϵ are locally bounded and piecewise continuous and represent disturbances, and $\mathcal{F}$ is a locally Lipschitz function such that (1) is forward complete. Assume:

Assumption 1: The pair $(\mathcal{A}, \mathcal{C})$ is observable. □

Then (e.g. from [9, pp. 304-306]) with an appropriate decomposition of the state vector ξ, we obtain

$$\left\{ \begin{array}{rcl} \dot{\xi}_1(t) & = & A_1\xi_1(t) + F_1(\xi_2(t), u(t)) + \kappa_1(t) \\ \dot{\xi}_2(t) & = & A_2\xi_1(t) + F_2(\xi_2(t), u(t)) + \kappa_2(t) \\ Y(t) & = & \xi_2(t) + \epsilon(t) \end{array} \right. \tag{2}$$

where F_1 and F_2 are locally Lipschitz functions, and where the pair (A_1, A_2) is observable. By a change of coordinates, we can assume that $A_1 \in \mathbb{R}^{(n-q)\times(n-q)}$ is an invertible matrix. (If A_1 were not invertible, then we could replace it by a new one $A_1 + LA_2$ where L is such that $A_1 + LA_2$ is Hurwitz, by applying the change of coordinates $\xi_3(t) = \xi_1(t) + L\xi_2(t)$, using the fact that if (A_1, A_2) is observable then so is $(A_1 + LA_2, A_2)$, by using the new coordinates (ξ_3, ξ_2).) Observability of (A_1, A_2) and the invertibility of A_1 imply that for any constant $\nu > 0$ and with the choice

$$H = A_2 A_1^{-1} \in \mathbb{R}^{q\times(n-q)}, \tag{3}$$

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the $(n-q)\times(n-q)$ inverse matrix

$$W=\left(\int_0^\nu\left(e^{A_1^\top\ell}-I\right)H^\top H\left(e^{A_1\ell}-I\right)\mathrm{d}\ell\right)^{-1} \quad (4)$$

exists. This follows because if there were a nonzero vector V such that $H\left(e^{A_1\ell}-I\right)V=0$ for all $\ell\in[0,\nu]$ then all of the derivatives of $H\left(e^{A_1\ell}-I\right)V$ with respect to ℓ are zero, which implies that $HA_1V=0$, $HA_1^2V=0$, ..., $HA_1^nV=0$, which yields a contradiction with the fact that (A_1,A_2) is observable. We fix a constant $\nu>0$ in what follows.

Let $\alpha:\mathbb{R}\to\mathbb{R}^{(n-q)\times q}$ be defined by

$$\alpha(\ell)=W\left(e^{A_1^\top\ell}-I\right)H^\top. \quad (5)$$

Let $t_i=i\nu$ for all $i\in\mathbb{Z}_{\geq 0}$ and σ be the function defined by $\sigma(t)=t_i$ when $t\in[t_i,t_{i+1})$. We define $\zeta:\mathbb{R}\to\mathbb{R}^{n-q}$ by

$$\begin{aligned}&\zeta(t)=\int_{\sigma(t)}^t e^{A_1(t-s)}\kappa_1^\sharp(s)\mathrm{d}s\\&-e^{A_1(t-\sigma(t))}\int_{\sigma(t)}^t\alpha(s-\sigma(t))\left[\int_{\sigma(t)}^s\kappa_2^\sharp(\ell)\mathrm{d}\ell\right.\\&\left.+A_2\int_{\sigma(t)}^s\int_{\sigma(t)}^m e^{A_1(m-\ell)}\kappa_1^\sharp(\ell)\mathrm{d}\ell\mathrm{d}m\right]\mathrm{d}s\\&-e^{A_1(t-\sigma(t))}\int_{\sigma(t)}^t\alpha(s-\sigma(t))(\epsilon(s)-\epsilon(\sigma(t)))\mathrm{d}s\end{aligned} \quad (6)$$

where for $i=1$ and 2,

$$\begin{aligned}&\kappa_i^\sharp=\kappa_i+\Delta_i\ \text{ and}\\&\Delta_i(t)=F_i(\xi_2(t),u(t))-F_i(Y(t),u(t)),\end{aligned} \quad (7)$$

and the κ_i's are from (2). We propose a candidate observer:

$$\left\{\begin{array}{rcl}\dot{\xi}_{\star,1}(t)&=&A_1\xi_{\star,1}(t)+F_1(Y(t),u(t))\\\dot{\xi}_{\star,2}(t)&=&A_2\xi_{\star,1}(t)+F_2(Y(t),u(t))\\\dot{\hat{x}}_1(t)&=&A_1\hat{x}_1(t)+e^{A_1(t-t_i)}\alpha(t-t_i)\left[Y(t)\right.\\&&-\xi_{\star,2}(t)-Y(t_i)+\xi_{\star,2}(t_i)\\&&\left.-H\left(e^{A_1(t-t_i)}-I\right)\hat{x}_1(t_i)\right],\\&&\text{for all } t\in[t_i,t_{i+1}) \text{ and } i\in\mathbb{Z}_{\geq 0},\end{array}\right. \quad (8)$$

whose first two equations are solved for all $t\geq 0$, and whose last equation is solved successively on the intervals $[t_i,t_{i+1})$ for all $i\in\mathbb{Z}_{\geq 0}$ with the initial state $\hat{x}_1(t_0)=0$ at time $t_0=0$ and the initial states for $\hat{x}_1$ at the times t_i for $i\geq 1$ given by the left limits $\hat{x}_1(t_i^-)$ for $i\geq 1$ (which we write more concisely as $\hat{x}_1(t_i)$ in our observer formula).

We are ready to state and prove the following result:

Theorem 1: Let Assumption 1 hold. Then for all initial states of (2), the solutions of (2) and (8) are such that

$$\xi_1(t)=\hat{x}_1(t)+\xi_{\star,1}(t)+\zeta(t) \quad (9)$$

for all $t\geq\nu$, where ζ is defined by (6). □

Remark 1: In order to describe precisely how $\hat{x}_1(t)+\xi_{\star,1}(t)$ estimates $\xi_1(t)$, let us define the constant

$$\bar{\alpha}=|\alpha|_{[0,\nu]} \quad (10)$$

and observe that

$$\begin{array}{rcl}|\zeta(t)|&\leq&\nu e^{|A_1|\nu}\overline{\alpha}\left[\int_{\sigma(t)}^t|\kappa_2^\sharp(\ell)|\mathrm{d}\ell\right.\\&&\left.+\nu|A_2|e^{\nu|A_1|}\int_{\sigma(t)}^t|\kappa_1^\sharp(\ell)|\mathrm{d}\ell\right]\\&&+e^{\nu|A_1|}\int_{\sigma(t)}^t|\kappa_1^\sharp(\ell)|\mathrm{d}\ell\\&&+e^{\nu|A_1|}\bar{\alpha}\int_{\sigma(t)}^t|\epsilon(s)-\epsilon(\sigma(t))|\mathrm{d}s\end{array} \quad (11)$$

for all $t\geq 0$. Consequently, if there are constants $K_i\geq 0$ that satisfy the requirements from Assumption 2 below, then

$$\begin{aligned}&|\zeta(t)|\leq\overline{\zeta}_1\int_{\sigma(t)}^t(|\kappa_1(\ell)|+|\kappa_2(\ell)|)\mathrm{d}\ell+\overline{\zeta}_2\int_{\sigma(t)}^t|\epsilon(\ell)|\mathrm{d}\ell\\&+e^{\nu|A_1|}\bar{\alpha}\int_{\sigma(t)}^t|\epsilon(s)-\epsilon(\sigma(t))|\mathrm{d}s\end{aligned} \quad (12)$$

for all $t\geq 0$, where

$$\overline{\zeta}_1=\max\left\{\nu^2\overline{\alpha}|A_2|e^{2\nu|A_1|}+e^{\nu|A_1|},\nu e^{|A_1|\nu}\overline{\alpha}\right\}\text{ and} \quad (13)$$

$$\overline{\zeta}_2=\max\left\{\left(\nu^2\overline{\alpha}|A_2|e^{2\nu|A_1|}+e^{\nu|A_1|}\right)K_1,\nu e^{|A_1|\nu}\overline{\alpha}K_2\right\}. \quad (14)$$

Thus,

$$\begin{aligned}&|\xi_1(t)-\hat{x}_1(t)-\xi_{\star,1}(t)|\leq\nu\overline{\zeta}_2\sup_{\ell\in[\sigma(t),t]}|\epsilon(\ell)|\\&+\nu\overline{\zeta}_1\sup_{\ell\in[\sigma(t),t]}(|\kappa_1(\ell)|+|\kappa_2(\ell)|)\\&+\nu e^{\nu|A_1|}\bar{\alpha}\sup_{\ell\in[\sigma(t),t]}|\epsilon(\ell)-\epsilon(\sigma(t))|\end{aligned} \quad (15)$$

for all $t\geq\nu$.

B. Proof of Theorem 1

We introduce the variables

$$\begin{aligned}&x_1(t)=\xi_1(t)-\xi_{\star,1}(t),\ x_2(t)=\xi_2(t)-\xi_{\star,2}(t),\\&\tilde{x}_1(t)=x_1(t)-\hat{x}_1(t),\ H^\sharp(t,s)=H\left(e^{A_1(t-s)}-I\right),\\&\text{and }\epsilon^\sharp(t,s)=-e^{A_1(t-s)}\alpha(t-s)[\epsilon(t)-\epsilon(s)].\end{aligned} \quad (16)$$

We observe that

$$\begin{array}{rcl}\dot{\hat{x}}_1(t)&=&A_1\hat{x}_1(t)+e^{A_1(t-t_i)}\alpha(t-t_i)\left[y(t)-y(t_i)\right.\\&&\left.-H\left(e^{A_1(t-t_i)}-I\right)\hat{x}_1(t_i)\right]\end{array} \quad (17)$$

with $y(t)=x_2(t)+\epsilon(t)$ and

$$\left\{\begin{array}{rcl}\dot{x}_1(t)&=&A_1x_1(t)+\kappa_1^\sharp(t)\\\dot{x}_2(t)&=&A_2x_1(t)+\kappa_2^\sharp(t)\\\dot{\tilde{x}}_1(t)&=&A_1\tilde{x}_1(t)+\epsilon^\sharp(t,t_i)\\&&-e^{A_1(t-t_i)}\alpha(t-t_i)\left[x_2(t)-x_2(t_i)\right.\\&&\left.-H\left(e^{A_1(t-t_i)}-I\right)\hat{x}_1(t_i)\right]+\kappa_1^\sharp(t).\end{array}\right. \quad (18)$$

By applying variation of parameters to the x_1-subsystem in (18), then integrating the result for each $i\geq 0$, we obtain

$$\begin{aligned}&x_2(t)=x_2(t_i)+H\left(e^{A_1(t-t_i)}-I\right)x_1(t_i)\\&+\int_{t_i}^t\kappa_2^\sharp(\ell)\mathrm{d}\ell+A_2\int_{t_i}^t\int_{t_i}^m e^{A_1(m-\ell)}\kappa_1^\sharp(\ell)\mathrm{d}\ell\mathrm{d}m\end{aligned} \quad (19)$$

for all $t\in[t_i,t_{i+1})$. By combining (18)-(19), we get

$$\begin{array}{rcl}\dot{\tilde{x}}_1(t)&=&A_1\tilde{x}_1(t)-e^{A_1(t-t_i)}\alpha(t-t_i)\left[H^\sharp(t,t_i)x_1(t_i)\right.\\&&\left.-H^\sharp(t,t_i)\hat{x}_1(t_i)\right]+\epsilon^\sharp(t,t_i)\\&&-e^{A_1(t-t_i)}\alpha(t-t_i)\left[\int_{t_i}^t\kappa_2^\sharp(\ell)\mathrm{d}\ell\right.\\&&\left.+A_2\int_{t_i}^t\int_{t_i}^m e^{A_1(m-\ell)}\kappa_1^\sharp(\ell)\mathrm{d}\ell\mathrm{d}m\right]+\kappa_1^\sharp(t)\\&=&A_1\tilde{x}_1(t)-e^{A_1(t-t_i)}\alpha(t-t_i)H^\sharp(t,t_i)\tilde{x}_1(t_i)\\&&+\epsilon^\sharp(t,t_i)-e^{A_1(t-t_i)}\alpha(t-t_i)\left[\int_{t_i}^t\kappa_2^\sharp(\ell)\mathrm{d}\ell\right.\\&&\left.+A_2\int_{t_i}^t\int_{t_i}^m e^{A_1(m-\ell)}\kappa_1^\sharp(\ell)\mathrm{d}\ell\mathrm{d}m\right]+\kappa_1^\sharp(t).\end{array} \quad (20)$$

By integrating this equation over $[t_i,t)$, we obtain

$$\begin{aligned}&\tilde{x}_1(t)=e^{A_1(t-t_i)}\tilde{x}_1(t_i)+\zeta(t)\\&-\int_{t_i}^t e^{A_1(t-m)}e^{A_1(m-t_i)}\alpha(m-t_i)H^\sharp(m,t_i)\mathrm{d}m\tilde{x}_1(t_i)\end{aligned} \quad (21)$$

with ζ defined in (6). Thus

$$\begin{aligned}&\tilde{x}_1(t) = \zeta(t)\\&+e^{A_1(t-t_i)}\left[I - \int_{t_i}^{t}\alpha(m-t_i)H^\sharp(m,t_i)\mathrm{d}m\right]\tilde{x}_1(t_i)\end{aligned} \tag{22}$$

and so also

$$\begin{aligned}&\tilde{x}_1(t) = \zeta(t)\\&+e^{A_1(t-t_i)}\left[I-\int_{t_i}^{t}W^\sharp(m,t_i)H^\top H^\sharp(m,t_i)\mathrm{d}m\right]\tilde{x}_1(t_i)\end{aligned} \tag{23}$$

for all $t \in [t_i, t_{i+1})$ and $i \geq 0$, where

$$W^\sharp(m,t_i) = W(e^{A_1^\top(m-t_i)} - I), \tag{24}$$

and where the last equality in (23) is a consequence of the definition in (5) of α. Thus in particular, since

$$\zeta(t_{i+1}) = 0, \tag{25}$$

we can specialize (23) to the case where $t = t_{i+1}$ to get

$$\begin{aligned}&\tilde{x}_1(t_{i+1}) =\\&= e^{A_1\nu}\left[I - W\int_0^\nu (H^\sharp(m,0))^\top H^\sharp(m,0)\mathrm{d}m\right]\tilde{x}_1(t_i)\end{aligned} \tag{26}$$

for all $i \in \mathbb{Z}_{\geq 0}$. The definition (4) of W ensures that

$$\tilde{x}_1(t_{i+1}) = 0. \tag{27}$$

From this equality and (23), we deduce that, for all $t \geq \nu$,

$$\tilde{x}_1(t) = \zeta(t). \tag{28}$$

Since $\tilde{x}_1 = \xi_1 - \xi_{\star,1} - \hat{x}_1$, this allows us to conclude.

III. Observers for Discrete Measurements

A. Statement of Result

In this part, we consider the case where instead of having continuous measurements of the output, the measurements are only available at discrete instants. For simplicity, we assume that the additive uncertainty ϵ on the measurements is zero, but this section can be generalized to cases where this uncertainty is nonzero. We introduce the sequence $s_k = k\varsigma$ for all $k \in \mathbb{Z}_{\geq 0}$ with a constant $\varsigma > 0$. Consider the system

$$\left\{\begin{array}{rcl}\dot{\xi}_1(t) &=& A_1\xi_1(t) + F_1(\xi_2(t),u(t)) + \kappa_1(t)\\ \dot{\xi}_2(t) &=& A_2\xi_1(t) + F_2(\xi_2(t),u(t)) + \kappa_2(t)\\ Y(t) &=& \xi_2(s_k) \text{ for all } t \in [s_k, s_{k+1})\\ && \text{and } k \in \mathbb{Z}_{\geq 0}\end{array}\right. \tag{29}$$

with ξ_1 valued in $\mathbb{R}^{n-q}$, ξ_2 valued in $\mathbb{R}^q$, and u, κ_1, and κ_2 being piecewise continuous and locally bounded. As in Section II, we let (A_1, A_2) be observable and A_1 be invertible. We assume:

Assumption 2: The functions F_1 and F_2 are locally Lipschitz and there are two constants $K_1 \geq 0$ and $K_2 \geq 0$ such that

$$\begin{array}{l}|F_1(a,u) - F_1(b,u)| \leq K_1|a-b| \text{ and}\\ |F_2(a,u) - F_2(b,u)| \leq K_2|a-b|\end{array} \tag{30}$$

hold for all $a \in \mathbb{R}^q, b \in \mathbb{R}^q$ and $u \in \mathbb{R}^p$. □

Assumption 2 (which is also used in [14]) ensures that (29) is forward complete. However, the main reason why we impose Assumption 2 is that it will be needed in the proof of Theorem 2. We introduce the candidate observer:

$$\left\{\begin{array}{rcl}\dot{\xi}_{\star,1}(t) &=& A_1\xi_{\star,1}(t) + F_1(\omega(t),u(t))\\ \dot{\xi}_{\star,2}(t) &=& A_2\xi_{\star,1}(t) + F_2(\omega(t),u(t))\\ \dot{\hat{x}}_1(t) &=& A_1\hat{x}_1(t)\\ && +e^{A_1(t-t_i)}\alpha(t-t_i)\left[\omega(t) - \xi_{\star,2}(t)\right.\\ && -\omega(t_i) + \xi_{\star,2}(t_i)\\ && \left.-H\left(e^{A_1(t-t_i)} - I\right)\hat{x}_1(t_i)\right]\\ && \text{for all } t \in [t_i, t_{i+1}) \text{ and } i \in \mathbb{Z}_{\geq 0}\\ \dot{\omega}(t) &=& A_2[\hat{x}_1(t) + \xi_{\star,1}(t)] + F_2(\omega(t),u(t))\\ && \text{for all } t \in [s_k, s_{k+1}) \text{ and } k \in \mathbb{Z}_{\geq 0}\\ \omega(s_k) &=& \xi_2(s_k) \text{ for all } k \in \mathbb{Z}_{\geq 0}\end{array}\right. \tag{31}$$

with α defined in (5) and $H = A_2A_1^{-1}$ as before, and where the solutions of (5) are defined analogously to those of (8) with $\omega(0) = 0$. This observer is inspired by the one used in [6]. We also use the constant $\overline{\alpha}$ from (10), and W from (4). Let us introduce the function $\beta : \mathbb{R} \to \mathbb{R}^{(n-q)\times(n-q)}$ defined by

$$\beta(\ell) = e^{A_1\ell}\left[I - W\int_0^\ell (H^\sharp(m,0))^\top H^\sharp(m,0)\mathrm{d}m\right] \tag{32}$$

with $H^\sharp(t,s) = H\left(e^{A_1(t-s)} - I\right)$ as before, and the constant

$$\overline{\beta} = |\beta|_{[0,\nu]}. \tag{33}$$

In terms of our sample rates ν and ς for the t_i's and s_k respectively, the constant $\overline{\alpha}$ from (10), and the constants

$$\overline{c}_1 = |A_2|\max\{\overline{\beta},1\}e^{|A_1|\nu}\overline{c}_5(\varsigma+2\nu) + K_2, \tag{34}$$

$$\overline{c}_2 = 2\max\{\overline{\beta},1\}\overline{c}_5 e^{|A_1|\nu + \frac{\ln(\varsigma\overline{c}_1)}{3\nu+\varsigma}(-6\nu-2\varsigma)}, \tag{35}$$

$$\overline{c}_3 = 2\max\{\overline{\beta},1\}e^{|A_1|\nu}\overline{c}_5\varsigma\frac{\nu|A_2|\overline{c}_4+1}{1-\varsigma\overline{c}_1} + \overline{c}_4, \tag{36}$$

$$\overline{c}_4 = e^{|A_1|\nu}\max\left\{\nu\overline{\alpha}, \nu\overline{\alpha}|H|(e^{|A_1|\nu}-1)+1\right\}, \tag{37}$$

and

$$\begin{aligned}\overline{c}_5 = {}& e^{|A_1|\nu}\overline{\alpha}\nu\left[|H|K_1\left(e^{|A_1|\nu}-1\right)+K_2\right]\\ &+K_1 + 2e^{|A_1|\nu}\overline{\alpha},\end{aligned} \tag{38}$$

our main result of this section is:

Theorem 2: Let (29) satisfy Assumption 2 and let

$$\varsigma\overline{c}_1 < 1 \tag{39}$$

hold. Then all solutions of (29) and (31) are such that

$$\begin{aligned}&|\xi_1(t) - \hat{x}_1(t) - \xi_{\star,1}(t)|\\ &\leq \nu\overline{c}_2 \sup_{m\in[0,3\nu+\varsigma]}|\omega(m) - \xi_2(m)|e^{\frac{\ln(\varsigma\overline{c}_1)}{3\nu+\varsigma}t}\\ &\quad+\nu\overline{c}_3\sup_{m\in[0,t]}(|\kappa_1(m)| + |\kappa_2(m)|)\end{aligned} \tag{40}$$

holds for all $t \geq \varsigma + 6\nu$.

Remark 2: Condition (40) gives an exponential convergence rate of $-\ln(\varsigma\overline{c}_1)/(3\nu+\varsigma)$, which converges to $+\infty$ as the sample rate ς for the sequence $\{s_k\}$ converges to 0. Hence, we can ensure arbitrarily fast convergence. Condition (39) imposes a constraint on the size of the sampling interval length ς, and ν can be chosen by the designer of the observer.

B. Proof of Theorem 2

The proof has three parts. In the first part, we perform changes of variables that produce an error dynamics associated with the observation error. In the second step, we perform a stability analysis for the error variables. In the final step, we use the trajectory based approach from [11] and the contractivity condition from (39) to obtain the final error estimation from our theorem.

First Step. We use the new error variables

$$\begin{array}{l} x_1(t) = \xi_1(t) - \xi_{\star,1}(t),\ x_2(t) = \xi_2(t) - \xi_{\star,2}(t), \\ \tilde{x}_1(t) = x_1(t) - \hat{x}_1(t),\ \text{and}\ r(t) = \omega(t) - \xi_2(t). \end{array} \tag{41}$$

and the function $H^\sharp(t,s) = H(e^{A_i(t-s)} - I)$ as before. Simple calculations give

$$\left\{ \begin{array}{rcl} \dot{x}_1(t) &=& A_1 x_1(t) + F_1(\xi_2(t), u(t)) \\ && - F_1(\omega(t), u(t)) + \kappa_1(t) \\ \dot{x}_2(t) &=& A_2 x_1(t) + F_2(\xi_2(t), u(t)) \\ && - F_2(\omega(t), u(t)) + \kappa_2(t) \\ \dot{r}(t) &=& A_2\hat{x}_1(t) + A_2\xi_{\star,1}(t) - A_2\xi_1(t) - \kappa_2(t) \\ && + F_2(\omega(t), u(t)) - F_2(\xi_2(t), u(t)) \\ && \text{for all } t \in [s_k, s_{k+1}) \text{ and } k \in \mathbb{Z}_{\geq 0} \\ r(s_k) &=& 0 \text{ for all } k \in \mathbb{Z}_{\geq 0}. \end{array} \right. \tag{42}$$

Also, $x_2 + r = \omega - \xi_{\star,2}$. Hence

$$\left\{ \begin{array}{rcl} \dot{\hat{x}}_1(t) &=& A_1\hat{x}_1(t) \\ && + e^{A_1(t-t_i)}\alpha(t-t_i)\left[r(t) - r(t_i)\right. \\ && + x_2(t) - x_2(t_i) \\ && \left. - H\left(e^{A_1(t-t_i)} - I\right)\hat{x}_1(t_i)\right] \\ && \text{for all } t \in [t_i, t_{i+1}) \text{ and } i \in \mathbb{Z}_{\geq 0} \\ \dot{r}(t) &=& A_2(\hat{x}_1(t) - x_1(t)) + F_2(\omega(t), u(t)) \\ && - F_2(\xi_2(t), u(t)) - \kappa_2(t) \\ && \text{for all } t \in [s_k, s_{k+1}) \text{ and } k \in \mathbb{Z}_{\geq 0} \\ r(s_k) &=& 0 \text{ for all } k \in \mathbb{Z}_{\geq 0}. \end{array} \right. \tag{43}$$

We deduce that

$$\begin{array}{rcl} \dot{\tilde{x}}_1(t) &=& A_1\tilde{x}_1(t) - e^{A_1(t-t_i)}\alpha(t-t_i)\left[x_2(t)\right. \\ && \left. - x_2(t_i) - H\left(e^{A_1(t-t_i)} - I\right)\hat{x}_1(t_i)\right] \\ && - e^{A_1(t-t_i)}\alpha(t-t_i)[r(t) - r(t_i)] + \kappa_1(t) \\ && + F_1(\xi_2(t), u(t)) - F_1(\omega(t), u(t)). \end{array} \tag{44}$$

By applying variation of parameters to the x_1-subsystem of (42), we obtain

$$\begin{array}{rcl} x_2(t) &=& x_2(t_i) + H\left(e^{A_1(t-t_i)} - I\right)x_1(t_i) + \psi_1(t) \\ && + \int_{t_i}^t \left[\kappa_2(\ell) + H\left(e^{A_1(t-\ell)} - I\right)\kappa_1(\ell)\right]\mathrm{d}\ell \end{array} \tag{45}$$

for all $t \in [t_i, t_{i+1})$ and $i \geq 0$, where

$$\begin{array}{rcl} \psi_1(t) &=& H\int_{t_i}^t \left(e^{A_1(t-\ell)} - I\right)\left[F_1(\xi_2(\ell), u(\ell))\right. \\ && \left. - F_1(\omega(\ell), u(\ell))\right]\mathrm{d}\ell + \int_{t_i}^t \left[F_2(\xi_2(\ell), u(\ell))\right. \\ && \left. - F_2(\omega(\ell), u(\ell))\right]\mathrm{d}\ell. \end{array} \tag{46}$$

By using (45) to obtain a formula for $x_2(t) - x_2(t_i)$ and then replacing the $x_2(t) - x_2(t_i)$ in (44) by this formula, and then collecting terms, we immediately obtain

$$\begin{array}{rcl} \dot{\tilde{x}}_1(t) &=& A_1\tilde{x}_1(t) - e^{A_1(t-t_i)}\alpha(t-t_i)H^\sharp(t,t_i)x_1(t_i) \\ && - e^{A_1(t-t_i)}\alpha(t-t_i)\left[\psi_1(t) + \int_{t_i}^t [\kappa_2(\ell)\right. \\ && \left. + H^\sharp(t,\ell)\kappa_1(\ell)]\,\mathrm{d}\ell - H^\sharp(t,t_i)\hat{x}_1(t_i)\right] \\ && - e^{A_1(t-t_i)}\alpha(t-t_i)[r(t) - r(t_i)] \\ && + F_1(\xi_2(t), u(t)) - F_1(\omega(t), u(t)) + \kappa_1(t) \\ &=& A_1\tilde{x}_1(t) - e^{A_1(t-t_i)}\alpha(t-t_i)H^\sharp(t,t_i)\tilde{x}_1(t_i) \\ && + \psi_2(t) + \kappa_1(t) \\ && - e^{A_1(t-t_i)}\alpha(t-t_i)\int_{t_i}^t \kappa_3(t,\ell)\mathrm{d}\ell \end{array} \tag{47}$$

where $\kappa_3(t,\ell) = \kappa_2(\ell) + H^\sharp(t,\ell)\kappa_1(\ell)$ and

$$\begin{array}{rcl} \psi_2(t) &=& -e^{A_1(t-\sigma(t))}\alpha(t-\sigma(t))\psi_1(t) \\ && - e^{A_1(t-\sigma(t))}\alpha(t-\sigma(t))[r(t) - r(\sigma(t))] \\ && - F_1(\omega(t), u(t)) + F_1(\xi_2(t), u(t)). \end{array} \tag{48}$$

By applying the variation of parameter to the last equality in (47) over $[t_i, t)$ with $t \in [t_i, t_{i+1}]$, we obtain

$$\begin{array}{rcl} \tilde{x}_1(t) &=& e^{A_1(t-t_i)}\tilde{x}_1(t_i) \\ && - \int_{t_i}^t e^{A_1(t-\ell)}e^{A_1(\ell-t_i)}\alpha(\ell-t_i)H^\sharp(\ell,t_i)\tilde{x}_1(t_i)\mathrm{d}\ell \\ && + \int_{t_i}^t e^{A_1(t-\ell)}\psi_2(\ell)\mathrm{d}\ell + \kappa_\star(t) \\ &=& e^{A_1(t-t_i)}\left[I - \int_{t_i}^t \alpha(\ell-t_i)H^\sharp(\ell,t_i)d\ell\right]\tilde{x}_1(t_i) \\ && + \int_{t_i}^t e^{A_1(t-\ell)}\psi_2(\ell)\mathrm{d}\ell + \kappa_\star(t), \end{array}$$

where

$$\begin{array}{l} \kappa_\star(t) = \int_{\sigma(t)}^t e^{A_1(t-\ell)}\kappa_1(\ell)\mathrm{d}\ell \\ - e^{A_1(t-\sigma(t))}\int_{\sigma(t)}^t \alpha(m-\sigma(t))\int_{\sigma(t)}^m \kappa_3(t,\ell)\mathrm{d}\ell\mathrm{d}m. \end{array} \tag{49}$$

From the definition of α in (5), we get

$$\begin{array}{l} \tilde{x}_1(t) = \int_{t_i}^t e^{A_1(t-\ell)}\psi_2(\ell)\mathrm{d}\ell + \kappa_\star(t) \\ + e^{A_1(t-t_i)}\left[I - \int_{t_i}^t W(H^\sharp(\ell,t_i))^\top H^\sharp(\ell,t_i)\mathrm{d}\ell\right]\tilde{x}_1(t_i) \end{array} \tag{50}$$

for all $t \in [t_i, t_{i+1})$ and all $i \geq 0$.

Thus, we deduce from the definition (4) of W that

$$\begin{array}{rcl} \tilde{x}_1(t_{i+1}) &=& \int_{t_i}^{t_{i+1}} e^{A_1(t_{i+1}-\ell)}\psi_2(\ell)\mathrm{d}\ell + \kappa_\star(t_{i+1}) \\ &=& \int_{t_i}^{t_{i+1}} e^{A_1(t_{i+1}-\ell)}\psi_2(\ell)\mathrm{d}\ell \end{array} \tag{51}$$

for all $i \geq 0$. Combining (51) and (50), we deduce that

$$\begin{array}{l} \tilde{x}_1(t) = \int_{t_i}^t e^{A_1(t-\ell)}\psi_2(\ell)\mathrm{d}\ell + \kappa_\star(t) \\ + e^{A_1(t-t_i)}\left[I - \tilde{W}(t-t_i)\right]\int_{t_{i-1}}^{t_i} e^{A_1(t_i-\ell)}\psi_2(\ell)\mathrm{d}\ell \end{array} \tag{52}$$

for all $t \geq \nu$, where

$$\tilde{W}(s) = W\int_0^s (H^\sharp(\ell,0))^\top H^\sharp(\ell,0)\mathrm{d}\ell \tag{53}$$

This gives

$$\left\{ \begin{array}{rcl} \tilde{x}_1(t) &=& \beta(t-\sigma(t))\int_{\sigma(t)-\nu}^{\sigma(t)} e^{A_1(\sigma(t)-\ell)}\psi_2(\ell)\mathrm{d}\ell \\ && + \int_{\sigma(t)}^t e^{A_1(t-\ell)}\psi_2(\ell)\mathrm{d}\ell + \kappa_\star(t) \\ \dot{r}(t) &=& -A_2\tilde{x}_1(t) + F_2(\omega(t), u(t)) \\ && - F_2(\xi_2(t), u(t)) - \kappa_2(t) \\ && \text{for all } t \in [s_k, s_{k+1}) \text{ and } k \in \mathbb{Z}_{\geq 0} \\ r(s_k) &=& 0 \text{ for all } k \in \mathbb{Z}_{\geq 0} \end{array} \right. \tag{54}$$

with β defined in (32) for all $t \geq \nu$.

Second Step. Now, we perform a stability analysis of the system (54). Integrating the second equality in (54) over $[s_k, t)$ with $t \in [s_k, s_{k+1})$, we obtain

$$\begin{aligned} r(t) \;=\; & -A_2 \int_{s_k}^t \tilde{x}_1(m)\mathrm{d}m - \int_{s_k}^t \kappa_2(m)\mathrm{d}m \\ & + \int_{s_k}^t [F_2(\omega(m), u(m)) \\ & - F_2(\xi_2(m), u(m))]\mathrm{d}m \end{aligned} \tag{55}$$

for all $k \in \mathbb{Z}_{\geq 0}$. Consequently,

$$\begin{aligned} |r(t)| \;\leq\; & |A_2| \int_{s_k}^t |\tilde{x}_1(m)|\mathrm{d}m + K_2 \int_{s_k}^t |r(m)|\mathrm{d}m \\ & \int_{s_k}^t |\kappa_2(m)|\mathrm{d}m \end{aligned} \tag{56}$$

for all $t \in [s_k, s_{k+1})$. On the other hand, (54) gives

$$\begin{aligned} |\tilde{x}_1(t)| \;\leq\; & \overline{\beta} \int_{\sigma(t)-\nu}^{\sigma(t)} e^{|A_1|\nu} |\psi_2(\ell)|\mathrm{d}\ell \\ & + \int_{\sigma(t)}^t e^{|A_1|\nu} |\psi_2(\ell)|\mathrm{d}\ell + |\kappa_\star(t)| \\ \leq\; & \max\left\{\overline{\beta}, 1\right\} e^{|A_1|\nu} \int_{\sigma(t)-\nu}^t |\psi_2(\ell)|\mathrm{d}\ell + |\kappa_\star(t)|. \end{aligned} \tag{57}$$

for all $t \geq 0$. The last inequality and (56) give

$$\begin{aligned} & |r(t)| \leq \\ & |A_2| \max\{\overline{\beta}, 1\} e^{|A_1|\nu} \int_{s_k}^t \int_{\sigma(m)-\nu}^m |\psi_2(\ell)|\mathrm{d}\ell\mathrm{d}m \\ & + |A_2| \int_{s_k}^t |\kappa_\star(m)|\mathrm{d}m + \int_{s_k}^t (K_2|r(m)| + |\kappa_2(m)|)\mathrm{d}m. \end{aligned} \tag{58}$$

As an immediate consequence,

$$\begin{aligned} & |r(t)| \leq \varsigma |A_2| \max\{\overline{\beta}, 1\} e^{|A_1|\nu} \int_{t-\varsigma-2\nu}^t |\psi_2(\ell)|\mathrm{d}\ell \\ & + \int_{t-\varsigma}^t (K_2|r(m)| + |A_2||\kappa_\star(m)| + |\kappa_2(m)|)\mathrm{d}m \end{aligned} \tag{59}$$

for all $t \geq \varsigma + 2\nu$. Also, Assumption 2 implies that

$$\begin{aligned} |\psi_1(t)| \;\leq\; & |H| K_1 \int_{\sigma(t)}^t \left|e^{A_1(t-\ell)} - I\right| |r(\ell)|\mathrm{d}\ell \\ & + K_2 \int_{\sigma(t)}^t |r(\ell)|\mathrm{d}\ell \text{ and} \\ |\psi_2(t)| \;\leq\; & e^{|A_1|\nu}\overline{\alpha}|\psi_1(t)| + e^{|A_1|\nu}\overline{\alpha}|r(t) - r(t_i)| \\ & + K_1|r(t)|. \end{aligned} \tag{60}$$

Thus, with the choice

$$K^\sharp(s) = K_1|H| \left|e^{A_1 s} - I\right| + K_2, \tag{61}$$

we get

$$\begin{aligned} |\psi_2(t)| \;\leq\; & e^{|A_1|\nu}\overline{\alpha}|H|K_1 \int_{\sigma(t)}^t \left|e^{A_1(t-\ell)} - I\right| |r(\ell)|\mathrm{d}\ell \\ & + e^{|A_1|\nu}\overline{\alpha}K_2 \int_{\sigma(t)}^t |r(\ell)|\mathrm{d}\ell \\ & + e^{|A_1|\nu}\overline{\alpha}|r(t) - r(\sigma(t))| + K_1|r(t)| \\ \leq\; & e^{|A_1|\nu}\overline{\alpha} \int_{\sigma(t)}^t K^\sharp(t-\ell)|r(\ell)|\mathrm{d}\ell \\ & + e^{|A_1|\nu}\overline{\alpha}|r(\sigma(t))| + \left(K_1 + e^{|A_1|\nu}\overline{\alpha}\right)|r(t)| \\ \leq\; & \overline{c}_5 |r|_{[t-\nu,t]} \end{aligned} \tag{62}$$

for all $t \geq \nu$ with $\overline{c}_5$ defined in (38). Combining (62) with (59), we obtain

$$\begin{aligned} & |r(t)| \leq \int_{t-\varsigma}^t (|A_2||\kappa_\star(m)| + |\kappa_2(m)|)\mathrm{d}m \\ & + \varsigma|A_2| \max\{\overline{\beta}, 1\} e^{|A_1|\nu} \int_{t-\varsigma-2\nu}^t \overline{c}_5 |r|_{[\ell-\nu,\ell]}\mathrm{d}\ell \\ & + K_2 \int_{t-\varsigma}^t |r(m)|\mathrm{d}m \end{aligned} \tag{63}$$

for all $t \geq \varsigma + 3\nu$. Hence, with $\overline{c}_1$ as defined in (39), we have

$$\begin{aligned} |r(t)| \;\leq\; & \varsigma|A_2| \max\{\overline{\beta}, 1\} e^{|A_1|\nu} \overline{c}_5(\varsigma + 2\nu)|r|_{[t-\varsigma-3\nu,t]} \\ & + K_2\varsigma|r|_{[t-\varsigma-3\nu,t]} \\ & + \int_{t-\varsigma}^t (|A_2||\kappa_\star(m)| + |\kappa_2(m)|)\mathrm{d}m \\ \leq\; & \varsigma\left[|A_2| \max\{\overline{\beta}, 1\} e^{|A_1|\nu} \overline{c}_5(\varsigma + 2\nu)\right. \\ & \left.+K_2\right] |r|_{[t-\varsigma-3\nu,t]} \\ & + \int_{t-\varsigma}^t (|A_2||\kappa_\star(m)| + |\kappa_2(m)|)\mathrm{d}m \\ \leq\; & \varsigma\overline{c}_1 |r|_{[t-\varsigma-3\nu,t]} \\ & + \int_{t-\varsigma}^t (|A_2||\kappa_\star(m)| + |\kappa_2(m)|)\mathrm{d}m. \end{aligned} \tag{64}$$

From the definition of $\kappa_\star$ in (49), we deduce that

$$\begin{aligned} |\kappa_\star(t)| \;\leq\; & \overline{\alpha} e^{|A_1|\nu} \int_{\sigma(t)}^t \left[\int_{\sigma(t)}^s |\kappa_2(\ell)|\mathrm{d}\ell\right. \\ & \left.+ |H| \int_{\sigma(t)}^s \left|e^{A_1(t-\ell)} - I\right| |\kappa_1(\ell)|\mathrm{d}\ell\right] \mathrm{d}s \\ & + e^{|A_1|\nu} \int_{\sigma(t)}^t |\kappa_1(\ell)|\mathrm{d}\ell \\ \leq\; & \nu\overline{\alpha} e^{|A_1|\nu} \left[\int_{\sigma(t)}^t |\kappa_2(\ell)|\mathrm{d}\ell\right. \\ & \left.+ |H| \int_{\sigma(t)}^s \left(e^{|A_1|(t-\ell)} - 1\right) |\kappa_1(\ell)|\mathrm{d}\ell\right] \\ & + e^{|A_1|\nu} \int_{\sigma(t)}^t |\kappa_1(\ell)|\mathrm{d}\ell \\ \leq\; & \nu\overline{\alpha} e^{|A_1|\nu} \int_{\sigma(t)}^t (|\kappa_2(\ell)| \\ & + |H| \left(e^{|A_1|\nu} - 1\right) |\kappa_1(\ell)|) \mathrm{d}\ell \\ & + e^{|A_1|\nu} \int_{\sigma(t)}^t |\kappa_1(\ell)|\mathrm{d}\ell \\ =\; & \int_{\sigma(t)}^t \left[\nu\overline{\alpha} e^{|A_1|\nu} |\kappa_2(\ell)| + e^{|A_1|\nu} \nu^\sharp |\kappa_1(\ell)|\right] \mathrm{d}\ell \\ \leq\; & \overline{c}_4 \int_{\sigma(t)}^t \left[\kappa_1(\ell)| + |\kappa_2(\ell)|\right] \mathrm{d}\ell \end{aligned} \tag{65}$$

where $\nu^\sharp = \nu\overline{\alpha}|H|e^{|A_1|\nu} - \nu\overline{\alpha}|H| + 1$ and $\overline{c}_4$ is from (37). Thus, with the choice $\tilde{\kappa}(\ell) = |\kappa_1(\ell)| + |\kappa_2(\ell)|$, we have

$$\begin{aligned} |r(t)| \;\leq\; & \varsigma\overline{c}_1 |r|_{[t-\varsigma-3\nu,t]} \\ & + \int_{t-\varsigma}^t \left(|A_2|\overline{c}_4 \int_{\sigma(m)}^m \tilde{\kappa}(\ell)\mathrm{d}\ell + |\kappa_2(m)|\right) \mathrm{d}m \\ \leq\; & \varsigma\overline{c}_1 |r|_{[t-\varsigma-3\nu,t]} + \varsigma\nu|A_2|\overline{c}_4 |\tilde{\kappa}|_{[t-\nu-\varsigma,t]} \\ & + \varsigma|\kappa_2|_{[t-\nu-\varsigma,t]} \\ =\; & \varsigma\overline{c}_1 |r|_{[t-\varsigma-3\nu,t]} + \varsigma(\nu|A_2|\overline{c}_4 + 1)|\tilde{\kappa}|_{[t-\nu-\varsigma,t]} \end{aligned} \tag{66}$$

for all $t \geq \varsigma + 3\nu$.

Third Step. We now apply the trajectory based contractivity method from [14, Lemma 1] to the function $w(t) = |r(t + \varsigma + 3\nu)|$. This gives

$$\begin{aligned} |r(t)| \;\leq\; & \sup_{m \in [0, \varsigma+3\nu]} |r(m)| e^{\frac{\ln(\varsigma\overline{c}_1)}{3\nu+\varsigma}(t-3\nu-2\varsigma)} \\ & + \varsigma \frac{\nu|A_2|\overline{c}_4 + 1}{1 - \varsigma\overline{c}_1} \sup_{m \in [0,t]} \tilde{\kappa}(m) \end{aligned} \tag{67}$$

for all $t \geq \varsigma + 3\nu$. We deduce from (54) and (62) that

$$\begin{aligned} |\tilde{x}_1(t)| \;\leq\; & \overline{\beta} \int_{\sigma(t)-\nu}^{\sigma(t)} e^{|A_1|\nu} |\psi_2(\ell)|\mathrm{d}\ell \\ & + \int_{\sigma(t)}^t e^{|A_1|\nu} |\psi_2(\ell)|\mathrm{d}\ell + |\kappa_\star(t)| \\ \leq\; & \max\left\{\overline{\beta}, 1\right\} e^{|A_1|\nu} \int_{t-2\nu}^t |\psi_2(\ell)|\mathrm{d}\ell + |\kappa_\star(t)| \\ \leq\; & \max\left\{\overline{\beta}, 1\right\} e^{|A_1|\nu} \int_{t-2\nu}^t \overline{c}_5 |r|_{[\ell-\nu,\ell]}\mathrm{d}\ell \\ & + |\kappa_\star(t)| \\ \leq\; & \overline{c}_6 \nu |r|_{[t-3\nu,t]} + |\kappa_\star(t)| \end{aligned} \tag{68}$$

for all $t \geq \varsigma + 3\nu$, where $\overline{c}_6 = 2\max\{\overline{\beta}, 1\} e^{|A_1|\nu} \overline{c}_5$. Combining this inequality with (67) and (65), we obtain

$$\begin{array}{l} |\tilde{x}_1(t)| \leq \overline{c}_4 \int_{\sigma(t)}^{t} [|\kappa_1(\ell)| + |\kappa_2(\ell)|] \, \mathrm{d}\ell \\ +\overline{c}_6 \nu \sup_{\ell \in [t-3\nu, t]} \left[\sup_{m \in [0, 2\varsigma+3\nu]} |r(m)| e^{\frac{\ln(\varsigma \overline{c}_1)}{3\nu+2\varsigma}(\ell - 3\nu - 2\varsigma)} \right. \\ \left. +\varsigma^{\frac{\nu|A_2|\overline{c}_4+1}{1-\varsigma\overline{c}_1}} \sup_{m \in [0,t]} (|\kappa_1(m)| + |\kappa_2(m)|) \right] \end{array} \tag{69}$$

for all $t \geq \varsigma + 6\nu$. It follows that

$$\begin{array}{l} |\tilde{x}_1(t)| \leq \\ 2\max\{\overline{\beta}, 1\} e^{|A_1|\nu} \overline{c}_5 \nu |r|_{[0, \varsigma+3\nu]} e^{\frac{\ln(\varsigma \overline{c}_1)}{3\nu+\varsigma}(t - 6\nu - 2\varsigma)} \\ +\nu \left[2\max\{\overline{\beta}, 1\} e^{|A_1|\nu} \overline{c}_5 \varsigma^{\frac{\nu|A_2|\overline{c}_4+1}{1-\varsigma\overline{c}_1}} + \overline{c}_4 \right] |\tilde{\kappa}|_{[0,t]}. \end{array} \tag{70}$$

This allows us to conclude the proof.

IV. Illustrations

As in [12], we study a pendulum model

$$\left\{ \begin{array}{l} \dot{a}_1(t) = a_2(t), \;\; \dot{a}_2(t) = -\sin(a_1(t)) \\ z(t) = a_1(s_j) \;\text{ if }\; t \in [s_j, s_{j+1}) \end{array} \right. \tag{71}$$

with the a_i's valued in $\mathbb{R}$.

The change of coordinates given by $\xi_1(t) = a_1(t) + a_2(t)$ and $\xi_2(t) = a_1(t)$ transforms the system (71) into

$$\left\{ \begin{array}{rcl} \dot{\xi}_1(t) & = & \xi_1(t) - \xi_2(t) - \sin(\xi_2(t)) \\ \dot{\xi}_2(t) & = & \xi_1(t) - \xi_2(t) \\ y(t) & = & \xi_2(s_j) \;\text{ if }\; t \in [s_j, s_{j+1}) \end{array} \right. \tag{72}$$

which is covered by Theorem 2.

Then, with the notation of the previous section, we take $A_1 = A_2 = 1$, $F_1(\xi_2, u) = -\xi_2 - \sin(\xi_2)$ and $F_2(\xi_2, u) = -\xi_2$, $K_1 = 2$, and $K_2 = 1$. Simple calculations show that in this case, the function α that we defined in (5) is

$$\alpha(m) = \frac{e^m - 1}{\frac{e^{2\nu}-1}{2} - 2(e^\nu - 1) + \nu} \tag{73}$$

so we get the observer

$$\left\{ \begin{array}{rcl} \dot{\xi}_{\star,1}(t) & = & \xi_{\star,1}(t) - \omega(t) - \sin(\omega(t)) \\ \dot{\xi}_{\star,2}(t) & = & \xi_{\star,1}(t) - \omega(t) \\ \dot{\hat{x}}_1(t) & = & \hat{x}_1(t) \\ & & + \frac{2e^{t-t_i}\left(e^{t-t_i}-1\right)}{e^{2\nu}-1-4(e^\nu-1)+2\nu} [\omega(t) - \omega(t_i) \\ & & -\xi_{\star,2}(t) + \xi_{\star,2}(t_i) \\ & & - \left(e^{(t-t_i)} - I\right) \hat{x}_1(t_i)] \\ & & \text{for all } t \in [t_i, t_{i+1}) \text{ and } i \in \mathbb{Z}_{\geq 0} \\ \dot{\omega}(t) & = & \hat{x}_1(t) + \xi_{\star,1}(t) - \omega(t) \\ & & \text{for all } t \in [s_k, s_{k+1}) \text{ and } k \in \mathbb{Z}_{\geq 0} \\ \omega(s_k) & = & \xi_2(s_k) \text{ for all } k \in \mathbb{Z}_{\geq 0} \end{array} \right. \tag{74}$$

which differs from the observers in [12] which have delays.

We can also apply Theorem 1 to design observers for the single link robotic manipulator dynamics in [12], to achieve a similar advantage of having observers that are free of delay terms while providing a fixed convergence time (which is a stronger result than the finite time convergence result in [19]), but without requiring the use of delayed output measurements that were required in [12].

V. Conclusion

We provided a new class of fixed time observers, when continuous measurements are available. By not requiring delayed output values in the observer, and providing a reduced order observer structure, our methods may offer computational advantages as compared with earlier methods. Our methods are based on the observability Gramian. We adapted this observer to cases where only discrete time measurements are available, where one instead obtains arbitrarily fast convergence. Extensions including new robustness results can be proved and time-varying version are expected.

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