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The Kaczmarz Algorithm in Banach Spaces

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Abstract

For a Banach space X and its dual X^*, a sequence $\{(\phi_n, \psi_n)\} \subset X^* \times X$ is effective if the Kaczmarz algorithm provides a reconstruction for every vector in X. We give necessary and sufficient conditions for a sequence to be effective. Starting with the mixed Gram matrix, we derive necessary matrix Eq.s for an effective sequence. When certain boundedness conditions are met, we show that these matrix Eq.s are also sufficient. We also give necessary conditions for related sequences to form a resolution of the identity. Finally, we provide examples of effective sequences in infinite dimensional Banach spaces.

Keywords Kaczmarz algorithm · Effective sequence · Gram matrix · Banach space

Mathematics Subject Classification Primary 41A65 · 65D15; Secondary 42C15 · 65F10

1 Introduction

In 1937, Stefan Kaczmarz introduced an iterative process for solving linear systems which is now known as the *Kaczmarz algorithm* [7]. The algorithm can be extended to infinite dimensional spaces [5, 9] as follows. Given a sequence of unit vectors $\{e_n\}_{n=0}^{\infty}$

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in a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$, we define a sequence of approximations $\{x_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} x_0 &= \langle x, e_0 \rangle e_0, \\ x_n &= x_{n-1} + \langle x - x_{n-1}, e_n \rangle e_n, \quad n \geq 1. \end{aligned} \tag{1.1}$$

The sequence $\{e_n\}_{n=0}^{\infty}$ is *effective* if $\|x_n - x\| \to 0$ for every $x \in \mathcal{H}$. Kaczmarz showed in [8] that any periodic, linearly dense sequence of unit vectors $\{e_n\}_{n=0}^{\infty}$ in a finite-dimensional Hilbert space is effective. It is observed in [9] that since the sequence of approximations $\{x_n\}$ is bounded for any input vector x, to prove a sequence is effective it is sufficient to prove that $\|x_n - x\| \to 0$ for every x in a dense subspace of $\mathcal{H}$.

We consider the extension of the Kaczmarz algorithm to a Banach space X (with dual X^*). Introduced in [1], for a sequence $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset X^* \times X$ with $\phi_n(\psi_n) = 1$ and $x \in X$, we define the sequence of approximations as

$$\begin{aligned} x_0 &= \phi_0(x)\psi_0, \\ x_n &= x_{n-1} + \phi_n(x - x_{n-1})\psi_n, \quad n \geq 1. \end{aligned} \tag{1.2}$$

Definition 1.1 The sequence $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset X^* \times X$, with $\phi_n(\psi_n) = 1$, is *effective* in X if for any input vector $x \in X$, the sequence of approximations $\{x_n\}_{n=0}^{\infty}$ as in Eq. (1.2) satisfies $x_n \to x$ in the strong topology on X. The sequence is *weakly effective* in X if $x_n \to x$ in the weak topology on X. We refer to $\{\phi_n\}_{n=0}^{\infty}$ as the analysis sequence and the $\{\psi_n\}_{n=0}^{\infty}$ as the synthesis sequence.

For an input vector $x \in X$, we let $\mathcal{A}_n : X \to X$ be given by $\mathcal{A}_n(x) = x_n$ as in Eq. (1.2). An application of the Banach-Steinhaus theorem yields the following theorem.

Theorem 1.2 *Let* $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset X^* \times X$, *with* $\phi_n(\psi_n) = 1$. *The sequence is effective (weakly effective) in X if and only if the following two conditions are met:*

(1) *the sequence* $\{\mathcal{A}_n(x)\}_{n=0}^{\infty}$ *is pointwise bounded for every* $x \in X$*;*
(2) $\mathcal{A}_n(x) \to x$ *in the strong (weak) topology for every x in a dense subspace of X.*

Our main results (Theorems 2.5 and 2.9 and Corollaries 2.6 and 2.10) focus on necessary conditions for the sequence $\{(\phi_n, \psi_n)\}_{n=0}^{\infty}$ to be effective or weakly effective.

These conditions will be in terms of matrix Eq.s similar to the criteria given in [5] for an effective sequence in a Hilbert space. We will also prove matrix Eq.s for duality conditions between two other related sequences. As we shall see, (Theorems 2.5, 2.9, and 2.11), certain duality and matrix conditions that are equivalent in the case of [5] are distinct in our setting.

1.1 Prior Results

In [9], Kwapień and Mycielski made progress toward characterizing effective sequences in an infinite-dimensional Hilbert space $\mathcal{H}$ by utilizing the auxiliary sequence $\{h_n\}_{n=0}^{\infty}$ defined recursively as

$$\begin{aligned} h_0 &= e_0, \\ h_n &= e_n - \sum_{k=0}^{n-1} \langle e_n, e_k \rangle h_k, \quad n \geq 1. \end{aligned} \tag{1.3}$$

They showed the following characterization for effective sequences (this precise statement does not appear in [9] but is an immediate consequence of results therein).

Theorem A *Let $\{e_n\}_{n=0}^{\infty}$ be a sequence of unit vectors in the Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(1) *$\{e_n\}_{n=0}^{\infty}$ is an effective sequence;*
(2) *$x = \sum_{n=0}^{\infty} \langle x, h_n \rangle e_n$ for every $x \in \mathcal{H}$, with convergence in the norm;*
(3) *$x = \sum_{n=0}^{\infty} \langle x, h_n \rangle h_n$ for every $x \in \mathcal{H}$, with convergence in the norm.*

Condition (3) is equivalent to $\{h_n\}_{n=0}^{\infty}$ being a Parseval frame [3, 6], namely that, for every $x \in \mathcal{H}$,

$$\|x\|^2 = \sum_{n=0}^{\infty} |\langle x, h_n \rangle|^2.$$

The properties involved in Theorem A can be more concisely described using the following definition.

Definition 1.3 The sequence $\{(\eta_n, \rho_n)\}_{n=0}^{\infty} \subset X^* \times X$ forms a resolution (weak resolution) of the identity if for every $x \in X$,

$$x = \sum_{n=0}^{\infty} \eta_n(x) \rho_n \tag{1.4}$$

with convergence in the strong (weak) topology.

Note that we do not require unconditional convergence of the series in Eq. (1.4). Theorem A then says that the sequence $\{e_n\}_{n=0}^{\infty} \subset \mathcal{H}$ is effective if and only if $\{(h_n, e_n)\}_{n=0}^{\infty}$ forms a resolution of the identity if and only if $\{(h_n, h_n)\}_{n=0}^{\infty}$ forms a resolution of the identity.

In [5], Haller and Szwarc approached the characterization of an effective sequence in $\mathcal{H}$ from a different perspective, using the matrix of inner products

$$I + N = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ \langle e_1, e_0 \rangle & 1 & 0 & \cdots \\ \langle e_2, e_0 \rangle & \langle e_2, e_1 \rangle & 1 & \cdots \\ \langle e_3, e_0 \rangle & \langle e_3, e_1 \rangle & \langle e_3, e_2 \rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.5}$$

The main result of [5] is the following.

Theorem B *Let $\{e_n\}_{n=0}^{\infty}$ be a linearly dense sequence of unit vectors in the Hilbert space $\mathcal{H}$. Let N be the matrix defined in Eq. (1.5), and let V be the matrix such that $(I+N)(I+V)=I$. Then the following are equivalent:*

(1) *$\{e_n\}_{n=0}^{\infty}$ is an effective sequence;*
(2) *The matrix V is a partial isometry on $\ell^2(\mathbb{N})$.*

2 Matrix Conditions and Effectivity

A characterization of when a sequence $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset X^* \times X$ is effective would be desirable, particularly a characterization similar to those in Theorems A and B. In this section, we will determine necessary conditions for a sequence to satisfy condition (2) of Theorem 1.2. These conditions are matrix Eq.s inspired by the proof of Theorem B, although there are significant differences. Certain matrix Eq.s that are equivalent in [5]—where the analysis and synthesis sequences are identical—are not equivalent in our setting. Consequently, when the algorithm uses different analysis and synthesis sequences, the analogues of condition (3) of Theorem A and condition (2) of Theorem B are no longer equivalent to the effectivity of $\{(\phi_n, \psi_n)\}_{n=0}^{\infty}$ in X.

2.1 Notation

For the remainder of this paper, we shall work in a Banach space, X, with dual space denoted by X^*. We will consistently assume both the analysis and synthesis sequences are linearly dense sequence in X^* and X, respectively. For the synthesis sequence $\{\psi_n\}_{n=0}^{\infty} \subset X$, we let Ψ denote the subspace of finite linear combinations of the vectors in the sequence; we similarly use Φ to denote the subspace of finite linear combinations of the vectors in the sequence $\{\phi_n\}_{n=0}^{\infty} \subset X^*$. All of the presented results also hold in a Hilbert space, substituting inner products in the natural way.

For a matrix A whose entries are indexed by $\mathbb{N}_0$, we denote the entries of A by both $A = (a_{ij})$ and $\langle A\delta_j, \delta_i\rangle$. We will encounter matrices that need not be bounded operators on $\ell^2(\mathbb{N}_0)$, and, as such, products need not be well-defined. If A and B are two such matrices, when we write $\langle AB\delta_j, \delta_i\rangle = c$, we mean that the series $\sum_{k=0}^{\infty} a_{ik}b_{kj}$ converges to c. No assumption on the mode of convergence is made. We use the inner-product notation because we will consider certain matrices acting on finite sequences. For the matrix A, we use A^* to denote the matrix such that $\langle A^*\delta_j, \delta_i\rangle = \overline{\langle A\delta_i, \delta_j\rangle}$. To aid our investigations, we define a lower triangular matrix reminiscent of a mixed Gramian (similar to Eq. (1.5)). Let

$$I + M = \begin{pmatrix} 1 & 0 & 0 & 0 \cdots \\ \overline{\phi_1(\psi_0)} & 1 & 0 & 0 \cdots \\ \overline{\phi_2(\psi_0)} & \overline{\phi_2(\psi_1)} & 1 & 0 \cdots \\ \overline{\phi_3(\psi_0)} & \overline{\phi_3(\psi_1)} & \overline{\phi_3(\psi_2)} & 1 \cdots \\ \vdots & \vdots & \vdots & \vdots \ddots \end{pmatrix} \tag{2.1}$$

and let U be the lower triangular matrix such that

$$(I+U)(I+M)=I. \tag{2.2}$$

We will denote the entries of $I+U$ by $\left(c_{jk}\right)_{j,k\in\mathbb{N}_0}$. Similarly, we define

$$I+\widetilde{M}=\begin{pmatrix} 1 & 0 & 0 & 0\cdots \\ \phi_0(\psi_1) & 1 & 0 & 0\cdots \\ \phi_0(\psi_2) & \phi_1(\psi_2) & 1 & 0\cdots \\ \phi_0(\psi_3) & \phi_1(\psi_3) & \phi_2(\psi_3) & 1\cdots \\ \vdots & \vdots & \vdots & \vdots\ddots \end{pmatrix} \tag{2.3}$$

and let $\widetilde{U}$ be the lower triangular matrix such that

$$(I+\widetilde{U})(I+\widetilde{M})=I. \tag{2.4}$$

We will denote the entries of $I+\widetilde{U}$ by $\left(\tilde{c}_{jk}\right)_{j,k\in\mathbb{N}_0}$. Note that $I+\widetilde{M}+M^*=\left(\phi_k(\psi_j)\right)_{j,k\in\mathbb{N}_0}$.

For the sequence $\{e_n\}_{n=0}^{\infty}\subset\mathcal{H}$, Haller and Szwarc [5] prove Theorem B. An intermediate step of the proof is to prove that the sequence $\{e_n\}_{n=0}^{\infty}$ is effective if and only if

$$\langle (NV^*VN^*-NN^*)\delta_j,\delta_i\rangle=0 \tag{2.5}$$

for all $i,j\in\mathbb{N}_0$. Moreover, although they do not explicitly state this, it is possible to glean from their work that $\{e_n\}_{n=0}^{\infty}$ is effective if and only if

$$\langle N^*(VN^*+N^*+I)\delta_j,\delta_i\rangle=0 \tag{2.6}$$

for all $i,j\in\mathbb{N}_0$. In the case of distinct analysis and synthesis sequences, we derive matrix conditions analogous to Eqs. (2.5) and (2.6) (see Theorems 2.5, 2.9, and 2.11). As we will see in Example 4.2, these analogues to Eqs. (2.5) and (2.6) are actually not equivalent.
We next define the analysis auxiliary sequence $\{g_n\}_{n=0}^{\infty}\subset X^*$ for $\{(\phi_n,\psi_n)\}_{n=0}^{\infty}$.

Definition 2.1 The analysis auxiliary sequence for $\{(\phi_n,\psi_n)\}_{n=0}^{\infty}\subset X^*\times X$ is given by

$$\begin{aligned} g_0&=\phi_0\\ g_n&=\phi_n-\sum_{k=0}^{n-1}\phi_n(\psi_k)g_k \text{ for } n\geq 1. \end{aligned} \tag{2.7}$$

Using induction, one can show that $\mathcal{A}_n(x) = \sum_{k=0}^{n} g_k(x)\psi_k$ for any $x \in X$. From this, we obtain an immediate lemma.

Lemma 2.2 *Suppose that* $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset X^* \times X$ *has the analysis auxiliary sequence* $\{g_n\}_{n=0}^{\infty}$ *and that* $\phi_n(\psi_n) = 1$ *for all* n*. The sequence* $\{(\phi_n, \psi_n)\}_{n=0}^{\infty}$ *is effective (weakly effective) in* X *if and only if*

$$x = \sum_{n=0}^{\infty} g_n(x)\psi_n \tag{2.8}$$

for every $x \in X$*, with convergence in the strong (weak) topology.*

We sometimes refer to the resolution of the identity in Eq. (2.8) by saying that $\{g_n\}_{n=0}^{\infty}$ and $\{\psi_n\}_{n=0}^{\infty}$ are dual (as in the frame theory context), but there is no assumption here that either sequence is a frame.

We will have use for a second auxiliary sequence. It will provide symmetry in our development of the matrix characterizations as well as play a crucial role in whether a sequence $\{(\phi_n, \psi_n)\}_{n=0}^{\infty}$ can be effective in both X and X^*.

Definition 2.3 The synthesis auxiliary sequence for $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset X^* \times X$ is given by

$$\begin{aligned} \tilde{g}_0 &= \psi_0 \\ \tilde{g}_n &= \psi_n - \sum_{k=0}^{n-1} \phi_k(\psi_n)\tilde{g}_k. \end{aligned} \tag{2.9}$$

In the case of Haller and Szwarc [5], where the analysis and synthesis sequences are identical, the two auxiliary sequences are also identical. When a sequence is effective, this auxiliary sequence, $\{h_n\}_{n=0}^{\infty}$, provides a resolution of the identity with $\{e_n\}_{n=0}^{\infty}$, and is also a Parseval frame (i.e., forms a resolution of the identity with itself). When the analysis and synthesis sequences are distinct, there are two separate matrix conditions: one associated with the duality of $\{g_n\}_{n=0}^{\infty}$ and $\{\psi_n\}_{n=0}^{\infty}$ and one associated with the duality of $\{g_n\}_{n=0}^{\infty}$ and $\{\tilde{g}_n\}_{n=0}^{\infty}$. These are represented in Theorems 2.5 and 2.11, respectively.

There are several helpful interactions between the auxiliary sequences and the matrices in Eqs. (2.1), (2.2), (2.3), and (2.4). If $u \in X^*$, define $\overline{u}(x) = \overline{u(x)}$ for all $x \in X$. It follows that

$$(I + M) \cdot \begin{pmatrix} \overline{g}_0 \\ \overline{g}_1 \\ \overline{g}_2 \\ \overline{g}_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \cdots \\ \overline{\phi_1(\psi_0)} & 1 & 0 & 0 \cdots \\ \overline{\phi_2(\psi_0)} & \overline{\phi_2(\psi_1)} & 1 & 0 \cdots \\ \overline{\phi_3(\psi_0)} & \overline{\phi_3(\psi_1)} & \overline{\phi_3(\psi_2)} & 1 \cdots \\ \vdots & \vdots & \vdots & \vdots \ddots \end{pmatrix} \cdot \begin{pmatrix} \overline{g}_0 \\ \overline{g}_1 \\ \overline{g}_2 \\ \overline{g}_3 \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \overline{g}_0 \\ \sum_{k=0}^{1} \overline{\phi_1(\psi_k)}\overline{g_k} \\ \sum_{k=0}^{2} \overline{\phi_2(\psi_k)}\overline{g_k} \\ \sum_{k=0}^{3} \overline{\phi_3(\psi_k)}\overline{g_k} \\ \vdots \end{pmatrix} = \begin{pmatrix} \overline{\phi}_0 \\ \overline{\phi}_1 \\ \overline{\phi}_2 \\ \overline{\phi}_3 \\ \vdots \end{pmatrix}.$$

Because $(I + M)(I + U) = I$, we also infer that

$$\overline{g}_n = \sum_{k=0}^{n} c_{nk}\overline{\phi}_k. \tag{2.10}$$

Similarly,

$$(I + \widetilde{M}) \cdot \begin{pmatrix} \tilde{g}_0 \\ \tilde{g}_1 \\ \tilde{g}_2 \\ \tilde{g}_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \cdots \\ \phi_0(\psi_1) & 1 & 0 & 0 \cdots \\ \phi_0(\psi_2) & \phi_1(\psi_2) & 1 & 0 \cdots \\ \phi_0(\psi_3) & \phi_1(\psi_3) & \phi_2(\psi_3) & 1 \cdots \\ \vdots & \vdots & \vdots & \vdots \ddots \end{pmatrix} \cdot \begin{pmatrix} \tilde{g}_0 \\ \tilde{g}_1 \\ \tilde{g}_2 \\ \tilde{g}_3 \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{g}_0 \\ \sum_{k=0}^{1} \phi_k(\psi_1)\tilde{g}_k \\ \sum_{k=0}^{2} \phi_k(\psi_2)\tilde{g}_k \\ \sum_{k=0}^{3} \phi_k(\psi_3)\tilde{g}_k \\ \vdots \end{pmatrix} = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \end{pmatrix}$$

and

$$\tilde{g}_n = \sum_{k=0}^{n} \tilde{c}_{nk}\psi_k. \tag{2.11}$$

2.2 Main Results

The following lemma will prove useful in the proof of Theorem 2.5.

Lemma 2.4 *Suppose* $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset X^* \times X$ *has analysis auxiliary sequence* $\{g_n\}_{n=0}^{\infty}$ *and that* $\phi_n(\psi_n) = 1$ *for all* n*. Let* M*,* $\widetilde{M}$*,* U*, and* $\widetilde{U}$ *be as defined in Eqs.* (2.1) *and* (2.3)*. Then, for all* $j, n \in \mathbb{N}_0$*,*

$$\langle (U\widetilde{M}^* + \widetilde{M}^* + I)\delta_j, \delta_n \rangle = \overline{g_n(\psi_j)}. \tag{2.12}$$

Proof Note that the multiplication $U\widetilde{M}^*$ is defined, since U is lower triangular. We see that the (n, j) entry of $U\widetilde{M}^*$ is

$$\begin{cases} 0, \text{ if } j \geq 0, n = 0 \\ \sum_{k=0}^{j-1} c_{nk}\overline{\phi_k(\psi_j)}, \text{ if } n \geq j \geq 1 \\ \sum_{k=0}^{n-1} c_{nk}\overline{\phi_k(\psi_j)}, \text{ if } n < j \end{cases}. \tag{2.13}$$

Recalling that $\overline{\phi_j(\psi_j)} = c_{jj} = 1$, we have that the (n, j) entry of $U\widetilde{M}^* + I + \widetilde{M}^*$ is given by

$$\begin{cases} \overline{\phi_n(\psi_j)}, \text{ if } j \geq 0, n = 0 \\ \sum_{k=0}^{j-1} c_{nk}\overline{\phi_k(\psi_j)}, \text{ if } n > j \geq 1 \\ \overline{\phi_n(\psi_j)} + \sum_{k=0}^{n-1} c_{nk}\overline{\phi_k(\psi_j)}, \text{ if } n \leq j \end{cases} = \begin{cases} \sum_{k=0}^{j-1} c_{nk}\overline{\phi_k(\psi_j)}, \text{ if } n > j \\ \sum_{k=0}^{n} c_{nk}\overline{\phi_k(\psi_j)}, \text{ if } n \leq j \end{cases} \tag{2.14}$$

We exploit the relationship between $(I + M)$ and $(I + U)$ to more cleanly write the entries of $U\widetilde{M}^* + \widetilde{M}^* + I$. Specifically, because $(I + U)(I + M) = I$, we know that if $n > j$, then

$$\sum_{k=j}^{n} c_{nk}\overline{\phi_k(\psi_j)} = 0. \tag{2.15}$$

We infer from (2.14) that the (n, j) entry of $U\widetilde{M}^* + \widetilde{M}^* + I$ is given by

$$\sum_{k=0}^{n} c_{nk}\overline{\phi_k(\psi_j)}.$$

Because $\overline{g_n} = \sum_{k=0}^{n} c_{nk}\overline{\phi_k}$, by Eq. (2.10), we see that

$$\overline{g_n}(\psi_j) = \left(\sum_{k=0}^{n} c_{nk}\overline{\phi_k}\right)\psi_j = \sum_{k=0}^{n} c_{nk}\overline{\phi_k(\psi_j)} = \langle (U\widetilde{M}^* + \widetilde{M}^* + I)\delta_j, \delta_n\rangle.$$

□

Theorem 2.5 *Suppose* $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset X^* \times X$ *has analysis auxiliary sequence* $\{g_n\}_{n=0}^{\infty}$ *and that* $\phi_n(\psi_n) = 1$ *for all* n. *Let* M, $\widetilde{M}$, U, *and* $\widetilde{U}$ *be as defined in Eqs.* (2.1) *and* (2.3). *Then the following are equivalent:*

(1) *For all* $i, j \in \mathbb{N}_0$,

$$\langle \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)\delta_j, \delta_i\rangle = 0; \tag{2.16}$$

(2) *For every* $\phi \in \Phi$ *and* $\psi \in \Psi$,

$$\lim_{n\to\infty} \phi(\mathcal{A}_n(\psi)) = \phi(\psi);$$

(3) *For every* $\phi \in \Phi$ *and* $\psi \in \Psi$,

$$\lim_{n\to\infty} \phi\left(\sum_{j=0}^{n} g_j(\psi)\psi_j\right) = \phi(\psi).$$

Proof Note that conditions (2) and (3) are equivalent by virtue of the fact that $\mathcal{A}_n(\psi) = \sum_{j=0}^{n} g_j(\psi)\psi_j$ (Lemma 2.2). It suffices to show that condition (1) and condition (3) are equivalent on the sequences $\{\phi_n\}_{n=0}^{\infty}$ and $\{\psi_n\}_{n=0}^{\infty}$.

Fix $i, j \in \mathbb{N}_0$. Notice that

$$\sum_{k=0}^{\infty} \phi_i(\psi_k)g_k(\psi_j) = \sum_{k=0}^{i} \phi_i(\psi_k)g_k(\psi_j) + \sum_{k=i+1}^{\infty} \phi_i(\psi_k)g_k(\psi_j). \tag{2.17}$$

Because $\phi_i = \sum_{k=0}^{i} \phi_i(\psi_k)g_k$, by Eq. (2.7), we see that

$$\sum_{k=0}^{i} \phi_i(\psi_k)g_k(\psi_j) = \left(\sum_{k=0}^{i} \phi_i(\psi_k)g_k\right)\psi_j = \phi_i(\psi_j)$$

and from the above derivations, we have

$$\sum_{k=0}^{\infty} \phi_i(\psi_k)g_k(\psi_j) = \phi_i(\psi_j) + \sum_{k=i+1}^{\infty} \phi_i(\psi_k)g_k(\psi_j). \tag{2.18}$$

Assume that condition (1) holds. Using Lemma 2.4, we see that the (i, j) entry of $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)$ is given by $\sum_{k=i+1}^{\infty} \overline{\phi_i(\psi_k)g_k(\psi_j)}$, which converges by the assumption of Eq. (2.16). By Eq. (2.18), we then obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \phi_i(\psi_k)g_k(\psi_j) &= \phi_i(\psi_j) + \overline{\langle \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)\delta_j, \delta_i\rangle} \\ &= \phi_i(\psi_j).\end{aligned}$$

Conversely, suppose condition (3) holds. Then

$$\sum_{k=0}^{\infty} \phi_i(\psi_k)g_k(\psi_j) = \phi_i(\psi_j)$$

for all $i, j \in \mathbb{N}_0$ and by Eq. (2.18), we know that

$$\sum_{k=i+1}^{\infty} \phi_i(\psi_k) g_k(\psi_j) = 0 = \langle \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)\delta_j, \delta_i \rangle.$$

□

A standard argument using the Banach-Steinhaus theorem yields the following:

Corollary 2.6 *Suppose $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset X^* \times X$ with $\phi_n(\psi_n) = 1$ for all n. Suppose that for every $x \in X$, the sequence $\{\mathcal{A}_n(x)\}_{n=0}^{\infty}$ is bounded. Then the sequence $\{(\phi_n, \psi_n)\}_{n=0}^{\infty}$ is weakly effective if and only if Eq. (2.16) is satisfied.*

For a Hilbert space $\mathcal{H}$, we may reverse the roles of the analysis and synthesis sequences, although this may impact effectivity (see Example 4.2). Conditions which guarantee effectivity for both roles are given in [1].

For a Banach space X, the roles may not be directly reversed; however, we can still define a Kaczmarz algorithm to perform reconstruction in X^*. We retain the assumptions that $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset X^* \times X$ and $\phi_n(\psi_n) = 1$. For $y \in X^*$, given the data $\{y(\psi_n)\}_{n=0}^{\infty}$, we define the sequence of approximations

$$\begin{aligned} y_0 &= y(\psi_0)\phi_0, \\ y_n &= y_{n-1} + [y - y_{n-1}](\psi_n)\phi_n, \quad n \geq 1. \end{aligned} \tag{2.19}$$

Moreover, we define the sequence of bounded linear operators $\mathcal{B}_n : X^* \to X^*$ by $\mathcal{B}_n(y) = y_n$.
By induction, we have that

$$\mathcal{B}_n(y) = \sum_{j=0}^{n} y(\tilde{g}_j)\phi_j,$$

where $\{\tilde{g}_n\}_{n=0}^{\infty}$ is as given in Eq. (2.9).

Definition 2.7 We will say that the sequence $\{(\phi_n, \psi_n)\}_{n=0}^{\infty}$ is *effective (weak-$*$ effective) in X^** if for every y,

$$\lim_{n\to\infty} y_n = y,$$

with convergence in the strong (weak-$*$) topology.

With this terminology, we produce results analogous to Lemma 2.4 and Theorem 2.5.

Lemma 2.8 *Suppose $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset X^* \times X$ has synthesis auxiliary sequence $\{\tilde{g}_n\}_{n=0}^{\infty}$ and that $\phi_n(\psi_n) = 1$ for all n. Let M, $\widetilde{M}$, U, and $\widetilde{U}$ be as defined in Eqs. (2.1) and (2.3). Then, for all $j, n \in \mathbb{N}_0$,*

$$\langle (\widetilde{U}M^* + M^* + I)\delta_j, \delta_n \rangle = \phi_j(\tilde{g}_n). \tag{2.20}$$

Proof Similar to the proof of Lemma 2.4. □

Theorem 2.9 *Suppose* $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset X^* \times X$ *has synthesis auxiliary sequence* $\{\tilde{g}_n\}_{n=0}^{\infty}$ *and that* $\phi_n(\psi_n) = 1$ *for all* n. *Let* M, $\widetilde{M}$, U, *and* $\widetilde{U}$ *be as defined in Eqs.* (2.1) *and* (2.3). *Then the following are equivalent:*

(1) *For all* $i, j \in \mathbb{N}_0$,

$$\langle M^*(\widetilde{U} M^* + M^* + I)\delta_j, \delta_i \rangle = 0; \tag{2.21}$$

(2) *For every* $\phi \in \Phi$ *and* $\psi \in \Psi$,

$$\lim_{n\to\infty} [\mathcal{B}_n(\phi)]\,(\psi) = \phi(\psi);$$

(3) *For every* $\phi \in \Phi$ *and* $\psi \in \Psi$,

$$\lim_{n\to\infty} \left[\sum_{j=0}^{n} \phi(\tilde{g}_j)\phi_j \right] (\psi) = \phi(\psi).$$

Proof Similar to the proof of Theorem 2.5. □

Corollary 2.10 *Suppose* $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset X^* \times X$ *with* $\phi_n(\psi_n) = 1$ *for all* n. *Suppose that for every* $y \in X^*$, *the sequence* $\{\mathcal{B}_n(y)\}_{n=0}^{\infty}$ *is bounded. Then the sequence* $\{(\phi_n, \psi_n)\}_{n=0}^{\infty}$ *is weak-$*$ effective in* X^* *if and only if Eq.* (2.21) *is satisfied.*

In Theorem A, we see that the sequence $\{e_n\}_{n=0}^{\infty}$ is effective in $\mathcal{H}$ if and only if either $\{(h_n, e_n)\}_{n=0}^{\infty}$ or $\{(h_n, h_n)\}_{n=0}^{\infty}$ forms a resolution of the identity. By definition, the sequence $\{(\phi_n, \psi_n)\}_{n=0}^{\infty}$ is effective precisely when the sequence $\{(g_n, \psi_n)\}_{n=0}^{\infty}$ forms a resolution of the identity. Analogously, $\{(\phi_n, \psi_n)\}_{n=0}^{\infty}$ is effective in X^* if and only if $\{(\phi_n, \tilde{g}_n)\}_{n=0}^{\infty}$ forms a resolution of the identity in X^* (at least in the case when X is reflexive). It is thus natural to consider under what conditions the sequence of auxiliaries $\{(\tilde{g}_n, g_n)\}_{n=0}^{\infty}$ forms a resolution of the identity (the analogue to condition (3) of Theorem A). The matrix condition associated with this duality will connect to condition (2) of Theorem B. Specifically, Haller and Szwarc show that their matrix condition for $NV^*VN^* - NN^*$ is equivalent to the operator V^*V being an orthogonal projection [5]. In our case, we will show that the analogous matrix condition for $\widetilde{M}U^*\widetilde{U}M^* - \widetilde{M}M^*$ holds precisely when a particular component of the operator $U^*\widetilde{U}$ is a projection with restricted domain.

We first introduce a collection of subspaces inside of $\ell^2(\mathbb{N}_0)$. Let $\mathcal{F}(\mathbb{N}_0) \subset \ell^2(\mathbb{N}_0)$ denote the set of all finite linear combinations of the collection $\{\delta_n\}_{n=0}^{\infty}$ and define

$$\begin{aligned} \mathcal{H}_0 &= \overline{M^*(\mathcal{F}(\mathbb{N}_0))} \subset \ell^2(\mathbb{N}_0) \\ \tilde{\mathcal{H}}_0 &= \overline{\widetilde{M}^*(\mathcal{F}(\mathbb{N}_0))} \subset \ell^2(\mathbb{N}_0). \end{aligned} \tag{2.22}$$

If U and $\widetilde{U}$ are both bounded operators on $\ell^2(\mathbb{N}_0)$, then we may represent $U^*\widetilde{U}$ as

$$U^*\widetilde{U} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.23}$$

where

$$A : \mathcal{H}_0 \to \tilde{\mathcal{H}}_0, \quad B : \mathcal{H}_0^\perp \to \tilde{\mathcal{H}}_0, \quad C : \mathcal{H}_0 \to \tilde{\mathcal{H}}_0^\perp, \quad D : \mathcal{H}_0^\perp \to \tilde{\mathcal{H}}_0^\perp. \tag{2.24}$$

Theorem 2.11 *Suppose $\{(\phi_n, \psi_n)\}_{n=0}^\infty \subset X^* \times X$ with $\phi_n(\psi_n) = 1$ for all n. Let M, $\widetilde{M}$, U, and $\widetilde{U}$ be as defined in Eqs. (2.1) and (2.3), and let $\{g_n\}_{n=0}^\infty$ and $\{\tilde{g}_n\}_{n=0}^\infty$ be the analysis and synthesis auxiliary sequences for $\{(\phi_n, \psi_n)\}_{n=0}^\infty$. Suppose that U and $\widetilde{U}$ are bounded operators on $\ell^2(\mathbb{N}_0)$. Then the following are equivalent:*

(1) *For all $i, j \in \mathbb{N}_0$,*

$$\langle (\widetilde{M}U^*\widetilde{U}M^* - \widetilde{M}M^*)\delta_j, \delta_i\rangle = 0; \tag{2.25}$$

(2) *For every $\phi \in \Phi$ and $\psi \in \Psi$,*

$$\lim_{n\to\infty} \sum_{j=0}^{n} \phi(\tilde{g}_n) g_n(\psi) = \phi(\psi); \tag{2.26}$$

(3) $A = P_{\tilde{\mathcal{H}}_0}\Big|_{\mathcal{H}_0}$ *(the projection onto $\tilde{\mathcal{H}}_0$ restricted to $\mathcal{H}_0$).*

Proof It suffices to consider condition (2) on the collections $\{\phi_n\}_{n=0}^\infty$ and $\{\psi_n\}_{n=0}^\infty$. Because M^* and $\widetilde{M}^*$ are upper triangular, their columns are elements in $\ell^2(\mathbb{N}_0)$. As $\widetilde{U}$ and U are assumed to be bounded operators on $\ell^2(\mathbb{N}_0)$, we also know that $(\widetilde{U}M^* + M^* + I)\delta_j \in \ell^2(\mathbb{N}_0)$ and $(U\widetilde{M}^* + \widetilde{M}^* + I)\delta_i \in \ell^2(\mathbb{N}_0)$ for all $i, j \in \mathbb{N}_0$. We may then use Lemma 2.4 and Lemma 2.8 to derive

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_j(\tilde{g}_n) g_n(\psi_i) &= \sum_{n=0}^{\infty} \langle (\widetilde{U}M^* + M^* + I)\delta_j, \delta_n\rangle \langle \delta_n, (U\widetilde{M}^* + \widetilde{M}^* + I)\delta_i\rangle \\ &= \langle (U\widetilde{M}^* + \widetilde{M}^* + I)^*(\widetilde{U}M^* + M^* + I)\delta_j, \delta_i\rangle. \end{aligned} \tag{2.27}$$

We define the truncations M_n, $\widetilde{M}_n$, U_n, and $\widetilde{U}_n$ (which are all bounded operators on $\ell_2(\mathbb{N}_0)$ and also strictly lower triangular) as follows:

$$M_n = \begin{pmatrix} 0 & & & & \\ \frac{0}{\phi_1(\psi_0)} & 0 & & & \\ \vdots & \ddots & 0 & & \\ \frac{0}{\phi_n(\psi_0)} & \cdots & \frac{0}{\phi_n(\psi_{n-1})} & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad U_n = \begin{pmatrix} 0 & & & & \\ c_{10} & 0 & & & \\ \vdots & \ddots & 0 & & \\ c_{n0} & \cdots & c_{n,n-1} & 0 & \cdots \\ 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Define $\widetilde{M}_n$ and $\widetilde{U}_n$ similarly. Because $(I+U_n)(I+M_n) = I$ and $(I+\widetilde{U}_n)(I+\widetilde{M}_n) = I$, we have

$$\begin{aligned} U_n M_n &= -U_n - M_n = M_n U_n \\ \widetilde{U}_n \widetilde{M}_n &= -\widetilde{U}_n - \widetilde{M}_n = \widetilde{M}_n \widetilde{U}_n \end{aligned}$$

and can calculate

$$(U_n \widetilde{M}_n^* + \widetilde{M}_n^* + I)^*(\widetilde{U}_n M_n^* + M_n^* + I) = \widetilde{M}_n U_n^* \widetilde{U}_n M_n^* - \widetilde{M}_n M_n^* + \widetilde{M}_n + M_n^* + I.$$

From this we conclude that

$$\begin{aligned} &\langle (U_n \widetilde{M}_n^* + \widetilde{M}_n^* + I)^*(\widetilde{U}_n M_n^* + M_n^* + I)\delta_j, \delta_i \rangle \\ &\quad = \langle \widetilde{U}_n M_n^* \delta_j, U_n \widetilde{M}_n^* \delta_i \rangle - \langle \widetilde{M}_n M_n^* \delta_j, \delta_i \rangle + \langle (\widetilde{M}_n + M_n^* + I)\delta_j, \delta_i \rangle. \end{aligned} \tag{2.28}$$

At this point we wish to take limits as $n \to \infty$. Since U and $\widetilde{U}$ are bounded operators, we have $U_n \to U$, $\widetilde{U}_n \to \widetilde{U}$, $U_n^* \to U^*$, and $\widetilde{U}_n^* \to \widetilde{U}^*$ in the strong operator topology on $\ell^2(\mathbb{N}_0)$. As M^* and $\widetilde{M}^*$ are upper triangular, we know that for $n > j, i$ we have $M_n^* \delta_j = M^* \delta_j$ and $\widetilde{M}_n^* \delta_i = \widetilde{M}^* \delta_i$, from which we conclude that

$$\begin{aligned} \lim_{n\to\infty} \langle M_n^* \delta_j, \delta_i \rangle &= \langle M^* \delta_j, \delta_i \rangle \\ \lim_{n\to\infty} \langle \widetilde{M}_n \delta_j, \delta_i \rangle &= \lim_{n\to\infty} \langle \delta_j, \widetilde{M}_n^* \delta_i \rangle = \langle \delta_j, \widetilde{M}^* \delta_i \rangle = \langle \widetilde{M} \delta_j, \delta_i \rangle \\ \lim_{n\to\infty} \langle \widetilde{M}_n M_n^* \delta_j, \delta_i \rangle &= \lim_{n\to\infty} \langle M_n^* \delta_j, \widetilde{M}_n^* \delta_i \rangle = \langle M^* \delta_j, \widetilde{M}^* \delta_i \rangle = \langle \widetilde{M} M^* \delta_j, \delta_i \rangle. \end{aligned}$$

By virtue of the upper triangular structure of M and $\widetilde{M}$, we know that $M^* \delta_j$ and $\widetilde{M}^* \delta_i$ are both sequences in $\ell^2(\mathbb{N}_0)$ and thus

$$\begin{aligned} \lim_{n\to\infty} \langle \widetilde{M}_n U_n^* M_n^* \delta_j, \delta_i \rangle &= \langle \widetilde{M} U^* M^* \delta_j, \delta_i \rangle \\ \lim_{n\to\infty} \langle \widetilde{M}_n U_n^* \widetilde{U}_n M_n^* \delta_j, \delta_i \rangle &= \langle \widetilde{M} U^* \widetilde{U} M^* \delta_j, \delta_i \rangle. \end{aligned}$$

For any $i, j \in \mathbb{N}_0$, we may now take the limit of both sides of Eq. (2.28) as $n \to \infty$ to obtain

$$\begin{aligned} &\langle (U \widetilde{M}^* + \widetilde{M}^* + I)^*(\widetilde{U} M^* + M^* + I)\delta_j, \delta_i \rangle \\ &\quad = \langle \widetilde{U} M^* \delta_j, U \widetilde{M}^* \delta_i \rangle - \langle M^* \delta_j, \widetilde{M}^* \delta_i \rangle + \langle (M^* + \widetilde{M} + I)\delta_j, \delta_i \rangle. \end{aligned} \tag{2.29}$$

Note that the (i, j) entry of $M^* + \widetilde{M} + I$ is simply $\phi_j(\psi_i)$. The equation then becomes:

$$\begin{aligned} &\langle (U \widetilde{M}^* + \widetilde{M}^* + I)^*(\widetilde{U} M^* + M^* + I)\delta_j, \delta_i \rangle \\ &\quad = \langle \widetilde{U} M^* \delta_j, U \widetilde{M}^* \delta_i \rangle - \langle M^* \delta_j, \widetilde{M}^* \delta_i \rangle + \phi_j(\psi_i). \end{aligned} \tag{2.30}$$

Assume that condition (2) holds. By Eq. (2.27), we know that

$$\phi_j(\psi_i) = \sum_{n=0}^{\infty} \phi_j(\tilde{g}_n) g_n(\psi_i) = \langle (U\widetilde{M}^* + \widetilde{M}^* + I)^*(\widetilde{U}M^* + M^* + I)\delta_j, \delta_i\rangle.$$

Combining this with Eq. (2.30), we obtain

$$\begin{aligned} &\langle \widetilde{U}M^*\delta_j, U\widetilde{M}^*\delta_i\rangle - \langle M^*\delta_j, \widetilde{M}^*\delta_i\rangle = 0 \text{ for all } i, j \in \mathbb{N}_0 \\ &\Rightarrow \langle (\widetilde{M}U^*\widetilde{U}M^* - \widetilde{M}M^*)\delta_j, \delta_i\rangle = 0 \text{ for all } i, j \in \mathbb{N}_0. \end{aligned}$$

Conversely, suppose condition (1) holds. By Eq. (2.30),

$$\phi_j(\psi_i) = \langle (U\widetilde{M}^* + \widetilde{M}^* + I)^*(\widetilde{U}M^* + M^* + I)\delta_j, \delta_i\rangle \text{ for all } i, j \in \mathbb{N}_0$$

and by Eq. (2.27), we see that

$$\phi_j(\psi_i) = \sum_{n=0}^{\infty} \phi_j(\tilde{g}_n) g_n(\psi_i),$$

as desired.

We will now show that condition (1) is equivalent to condition (3). Assume that condition (1) holds. We begin by proving that

$$\langle U^*\widetilde{U}x, y\rangle = \langle x, y\rangle \text{ if } x \in \mathcal{H}_0 \text{ and } y \in \tilde{\mathcal{H}}_0. \tag{2.31}$$

Let $x \in \mathcal{H}_0$ and suppose that $x = M^*w$ where $w = \sum_{i=0}^{n} \alpha_i \delta_i$ for some $\{\alpha_k\}_{k=0}^{\infty} \subset \mathbb{C}$ and $n \in \mathbb{N}_0$. Similarly, let $y \in \tilde{\mathcal{H}}_0$ and suppose that $y = \widetilde{M}^*z$, where and $z = \sum_{k=0}^{m} \beta_k \delta_k$ for some $\{\beta_k\}_{k=0}^{\infty} \subset \mathbb{C}$ and $m \in \mathbb{N}_0$. We derive

$$\begin{aligned} \langle U^*\widetilde{U}x, y\rangle &= \langle U^*\widetilde{U}M^*w, \widetilde{M}^*z\rangle \\ &= \sum_{i=0}^{n} \alpha_i \sum_{k=0}^{m} \overline{\beta}_k \langle U^*\widetilde{U}M^*\delta_i, \widetilde{M}^*\delta_k\rangle \\ &= \sum_{i=0}^{n} \alpha_i \sum_{k=0}^{m} \overline{\beta}_k \langle M^*\delta_i, \widetilde{M}^*\delta_k\rangle \quad \text{by (1)} \\ &= \langle M^*w, \widetilde{M}^*z\rangle \\ &= \langle x, y\rangle. \end{aligned}$$

Write $x = \lim_{n\to\infty} M^*w_n$ and $y = \lim_{k\to\infty} \widetilde{M}^*z_k$, where $w_n, z_k \in \mathcal{F}(\mathbb{N}_0)$ for all n, k. As U and $\widetilde{U}$ are bounded operators on $\ell^2(\mathbb{N}_0)$, by the work above we have that

$$\langle U^*\widetilde{U}x, y\rangle = \langle U^*\widetilde{U} \lim_{n\to\infty} M^*w_n, \lim_{k\to\infty} \widetilde{M}^*z_k\rangle$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \lim_{k\to\infty} \langle U^* \widetilde{U} M^* w_n, \widetilde{M}^* z_k \rangle \\
&= \lim_{n\to\infty} \lim_{k\to\infty} \langle M^* w_n, \widetilde{M}^* z_k \rangle \\
&= \langle \lim_{n\to\infty} M^* w_n, \lim_{k\to\infty} \widetilde{M}^* z_k \rangle \\
&= \langle x, y \rangle.
\end{aligned}
$$

We conclude that Eq. (2.31) holds, noting that there are no continuity requirements on $\widetilde{M}^*$ or M^* to achieve the above derivations.
Let $x = x_1 + x_2$ and $y = y_1 + y_2 \in \ell^2(\mathbb{N}_0)$ where $x_1 \in \mathcal{H}_0$, $x_2 \in \mathcal{H}_0^\perp$, $y_1 \in \tilde{\mathcal{H}}_0$, and $y_2 \in \tilde{\mathcal{H}}_0^\perp$. Using A, B, C, and D as defined in (2.24), we may then write

$$
\left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle_{\tilde{\mathcal{H}}_0 \oplus \tilde{\mathcal{H}}_0^\perp} = \langle Ax_1 + Bx_2, y_1 \rangle_{\tilde{\mathcal{H}}_0} + \langle Cx_1 + Dx_2, y_2 \rangle_{\tilde{\mathcal{H}}_0^\perp}. \tag{2.32}
$$

Suppose that $x \in \mathcal{H}_0$ and $y \in \tilde{\mathcal{H}}_0$ so that $x_2 = y_2 = 0$. Furthermore, write $x_1 = z_1 + z_2$, where $z_1 \in \tilde{\mathcal{H}}_0$ and $z_2 \in \tilde{\mathcal{H}}_0^\perp$. From Eq. (2.32) we see that

$$
\langle U^* \widetilde{U} x_1, y_1 \rangle = \langle Ax_1, y_1 \rangle. \tag{2.33}
$$

Now calculate

$$
\langle x_1, y_1 \rangle = \langle z_1, y_1 \rangle + \langle z_2, y_1 \rangle = \langle z_1, y_1 \rangle. \tag{2.34}
$$

By Eq. (2.31), $\langle x_1, y_1 \rangle = \langle U^* \widetilde{U} x_1, y_1 \rangle$, so we infer from Eqs. (2.33) and (2.34) that

$$
\begin{aligned}
&\langle Ax_1, y_1 \rangle = \langle z_1, y_1 \rangle \\
&\Rightarrow \langle Ax_1 - z_1, y_1 \rangle = 0
\end{aligned}
$$

for all $x_1 \in \mathcal{H}_0$ and $y_1 \in \tilde{\mathcal{H}}_0$. As $z_1 = P_{\tilde{\mathcal{H}}_0} x_1$ and $x_1 \in \mathcal{H}_0$, we conclude that $A = P_{\tilde{\mathcal{H}}_0}\big|_{\mathcal{H}_0}$.
Conversely, assume that $A = P_{\tilde{\mathcal{H}}_0}\big|_{\mathcal{H}_0}$. Let $x = x_1 + x_2$ and $y = y_1 + y_2 \in \ell^2(\mathbb{N}_0)$ where $x_1 \in \mathcal{H}_0$, $x_2 \in \mathcal{H}_0^\perp$, $y_1 \in \tilde{\mathcal{H}}_0$, and $y_2 \in \tilde{\mathcal{H}}_0^\perp$. Suppose that $x \in \mathcal{H}_0$ and $y \in \tilde{\mathcal{H}}_0$, so that $x_2 = y_2 = 0$. By Eq. (2.32) we derive

$$
\begin{aligned}
\langle U^* \widetilde{U} x, y \rangle &= \langle Ax_1, y_1 \rangle \\
&= \langle P_{\tilde{\mathcal{H}}_0} x_1, y_1 \rangle \\
&= \langle x_1, P_{\tilde{\mathcal{H}}_0} y_1 \rangle \\
&= \langle x_1, y_1 \rangle \\
&= \langle x, y \rangle
\end{aligned}
$$

and we conclude that for all $x \in \mathcal{H}_0$ and $y \in \tilde{\mathcal{H}}_0$,

$$\langle U^* \widetilde{U} x, y \rangle = \langle x, y \rangle \text{ if } x \in \mathcal{H}_0.$$

As $M^* \delta_j \in \mathcal{H}_0$ and $\widetilde{M}^* \delta_i \in \tilde{\mathcal{H}}_0$ for any $i, j \in \mathbb{N}_0$, it follows that

$$\begin{aligned} \left\langle U^* \widetilde{U} M^* \delta_j, \widetilde{M}^* \delta_i \right\rangle &= \left\langle M^* \delta_j, \widetilde{M}^* \delta_i \right\rangle \\ \Rightarrow \left\langle \widetilde{M} U^* \widetilde{U} M^* \delta_j, \delta_i \right\rangle - \left\langle \widetilde{M} M^* \delta_j, \delta_i \right\rangle &= 0 \\ \Rightarrow \left\langle \left(\widetilde{M} U^* \widetilde{U} M^* - \widetilde{M} M^* \right) \delta_j, \delta_i \right\rangle &= 0 \end{aligned}$$

for any $i, j \in \mathbb{N}_0$, and the proof is finished. □

3 A Characterization of Weakly Effective Sequences

In the following theorem, we show that the assumption of a weak bound on $\{\mathcal{A}_n\}_{n=0}^{\infty}$ relative to $\Phi \times \Psi$ combined with the matrix condition in Eq. (2.16) is enough to characterize weak convergence of the algorithm on the entire space in question.

Theorem 3.1 *Suppose $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset X^* \times X$ with $\phi_n(\psi_n) = 1$ for all n. Let M, $\widetilde{M}$, U, and $\widetilde{U}$ be as defined in Eqs. (2.1) and (2.3). The sequence $\{(\phi_n, \psi_n)\}_{n=0}^{\infty}$ is weakly effective in X if and only if Eq. (2.16) holds and there exists some $C > 0$ such that*

$$|\phi(\mathcal{A}_n(\psi))| \leq C \|\psi\| \|\phi\| \textit{ for all } (\phi, \psi) \in \Phi \times \Psi, n \in \mathbb{N}_0. \tag{3.1}$$

Proof The necessity of Eq. (2.16) follows from Theorem 2.5, whereas the necessity of Eq. (3.1) follows from the Banach-Steinhaus theorem.
We now turn to the sufficiency. If Eq. (3.1) holds, then a standard continuity argument demonstrates that

$$|y(\mathcal{A}_n(x))| \leq C \|x\| \|y\| \tag{3.2}$$

for all $x \in X$ and $y \in X^*$.

We will next show that $y(\mathcal{A}_n(\psi)) \to y(\psi)$ for $y \in X^*$ and $\psi \in \Psi$. Choose $\{\phi_k\}_{k=0}^{\infty} \subset \Phi$ such that $\{\phi_k\}_{k=0}^{\infty}$ converges to y.

Let $\varepsilon > 0$. Using Inequality (3.2), we derive

$$\begin{aligned} |y(\mathcal{A}_n(\psi)) - y(\psi)| &\leq |(y - \phi_k)(\mathcal{A}_n(\psi))| + |\phi_k(\mathcal{A}_n(\psi)) - \phi_k(\psi)| + |(\phi_k - y)(\psi)| \\ &\leq C \|\psi\| \|y - \phi_k\| + |\phi_k(\mathcal{A}_n(\psi)) - \phi_k(\psi)| + \|\phi_k - y\| \|\psi\|. \end{aligned}$$

Choose k large enough that $\|y - \phi_k\| \leq \min \left\{ \frac{\varepsilon}{3C\|\psi\|}, \frac{\varepsilon}{3\|\psi\|} \right\}$. We then obtain

$$|y(\mathcal{A}_n(\psi)) - y(\psi)| \leq \frac{\varepsilon}{3} + |\phi_k(\mathcal{A}_n(\psi)) - \phi_k(\psi)| + \frac{\varepsilon}{3}.$$

Because $\psi \in \Psi$ and $\phi_k \in \Phi$, by Theorem 2.5, there is some $N \in \mathbb{N}_0$ such that $|\phi_k(\mathcal{A}_n(\psi)) - \phi_k(\psi)| < \frac{\varepsilon}{3}$ for all $n > N$. For such an n,

$$|y(\mathcal{A}_n(\psi)) - y(\psi)| < \varepsilon. \tag{3.3}$$

Now, choose $x \in X$, $y \in X^*$, and $\{\psi_k\}_{k=0}^{\infty} \subset \Psi$ such that $\{\psi_k\}_{k=0}^{\infty}$ converges to x. Again using Inequality (3.2), we obtain

$$\begin{aligned}|y(\mathcal{A}_n(x)) - y(x)| &\leq |y(\mathcal{A}_n(x - \psi_k))| + |y(\mathcal{A}_n(\psi_k)) - y(\psi_k)| + |y(\psi_k - x)| \\ &\leq C\|x - \psi_k\|\|y\| + |y(\mathcal{A}_n(\psi_k)) - y(\psi_k)| + \|y\|\|\psi_k - x\|.\end{aligned}$$

Choose k large enough that $\|x - \psi_k\| \leq \min\left\{\frac{\varepsilon}{3C\|y\|}, \frac{\varepsilon}{3\|y\|}\right\}$. The inequality then becomes

$$|y(\mathcal{A}_n(x)) - y(x)| \leq \frac{\varepsilon}{3} + |y(\mathcal{A}_n(\psi_k)) - y(\psi_k)| + \frac{\varepsilon}{3}.$$

As $y \in X^*$ and $\psi_k \in \Psi$, by Inequality (3.3), there is some $N \in N_0$ such that $|y(\mathcal{A}_n(\psi_k)) - y(\psi_k)| < \frac{\varepsilon}{3}$ for all $n > N$. Choosing such an n, we see that

$$|y(\mathcal{A}_n(x)) - y(x)| < \varepsilon.$$

□

It would be desirable for Inequality (3.1) to be characterized by a matrix condition, but, as discussed in Remark 4.8, this is unlikely to be possible.

4 Examples

The following examples demonstrate the lack of equivalence between the three matrix conditions in Eqs. (2.16), (2.21), and (2.25), and also establish the insufficiency of matrix equations to completely encode information about condition (1) of Theorem 1.2.

4.1 Finite Dimensions

Suppose $x \in X$ and define $P_n : X \to X$ by $P_n x = x - \phi_n(x)\psi_n$. Manipulating Eq. (1.2), we obtain

$$x - x_n = P_n P_{n-1} \cdots P_0 x. \tag{4.1}$$

Recall that $\{(\phi_n, \psi_n)\}_{n=0}^{\infty}$ is effective in X when $\|x - x_n\| \to 0$ for all $x \in X$. For a bounded operator B on X, we use $\rho(B)$ to denote the spectral radius of B. A sequence $\{a_n\}_{n=0}^{\infty}$ is *k-periodic* if $a_{n+k} = a_n$ for $n \in \mathbb{N}_0$.

Theorem 4.1 *Let X be a finite-dimensional Banach space and let $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset X^* \times X$ be a k-periodic sequence. Then $\{(\phi_n, \psi_n)\}_{n=0}^{\infty}$ is effective if and only if $\rho(P_{k-1}P_{k-2} \cdots P_0) < 1$.*

Proof It is well known that $(P_{k-1}P_{k-2} \cdots P_0)^m \to 0$ in the strong operator topology if and only if $\rho(P_{k-1}P_{k-2} \cdots P_0) < 1$ [10]. Therefore, $\|x - x_{mk}\| \to 0$ as $m \to \infty$ if and only if $\rho(P_{k-1}P_{k-2} \cdots P_0) < 1$. □

Theorem 4.1 provides an efficient method of testing whether or not periodic sequences in finite dimensions are effective.

Example 4.2 Let $\{\phi_n\}_{n=0}^{\infty}, \{\psi_n\}_{n=0}^{\infty} \subset \mathbb{R}^2$, $\phi_n = \phi_{n+3}$, and $\psi_n = \psi_{n+3}$ for all n, where

$$\begin{bmatrix}\phi_0 & \phi_1 & \phi_2 & \psi_0 & \psi_1 & \psi_2\end{bmatrix} = \begin{bmatrix} 1 & 1 & .5 & 1 & 1 & 1.5 \\ -1 & 1 & -.5 & 0 & 0 & -.5 \end{bmatrix}.$$

One can easily calculate $\rho(P_2P_1P_0) = \frac{1}{2}$ and $\rho(Q_2Q_1Q_0) = 2$ for $P_n x = x - \langle x, \phi_n\rangle \psi_n$ and $Q_n x = x - \langle x, \psi_n\rangle \phi_n$. Using Theorem 4.1, we infer that $\{(\phi_n, \psi_n)\}_{n=0}^{\infty}$ is effective in $\mathbb{R}^2$, but $\{(\psi_n, \phi_n)\}_{n=0}^{\infty}$ is not.

Applying Theorems 2.5 and 2.9, we infer that for M, $\widetilde{M}$, U, and $\widetilde{U}$ as defined Equations (2.1) and (2.3),

$$\langle \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)\delta_j, \delta_i\rangle = 0 \text{ for all } i, j \in \mathbb{N}_0 \tag{4.2}$$

$$\langle M^*(\widetilde{U}M^* + M^* + I)\delta_j, \delta_i\rangle \neq 0 \text{ for some } i, j \in \mathbb{N}_0. \tag{4.3}$$

In the Appendix, we confirm this algebraically by explicitly calculating the matrices in Eqs. (4.2) and (4.3). More interestingly, we also calculate the matrix $\widetilde{M}U^*\widetilde{U}M^* - \widetilde{M}M^*$ from Theorem 2.11, discovering that

$$\langle (\widetilde{M}U^*\widetilde{U}M^* - \widetilde{M}M^*)\delta_j, \delta_i\rangle = 0 \text{ for all } i, j \in \mathbb{N}_0. \tag{4.4}$$

Equations (4.2), (4.3), and (4.4) together confirm that the matrix conditions in the conclusions of Theorems 2.5, 2.9, and 2.11 are not equivalent to each other in general. This is in contrast to the case of $\{e_n\}_{n=0}^{\infty} \subset \mathcal{H}$ in [5] (or that of $\{(\phi_n, \psi_n)\}_{n=0}^{\infty} \subset \mathcal{H} \times \mathcal{H}$ subject to relatively strong additional hypotheses [1]).

4.2 Infinite Dimensions

We aim to show that for certain singular measures μ on $[0, 1] = \mathbb{T}$, the sequence $\{(e^{2\pi i n x}, e^{2\pi i n x})\}_{n=0}^{\infty} \subset L^q(\mu) \times L^p(\mu)$ is an effective sequence. To do so, we require some background material from the spectral theory of the backward shift operator [4, 11]. For an analytic function f on $\mathbb{D}$, we define the backward shift to be

$$[S^* f](z) = \frac{f(z) - f(0)}{z}.$$

The Herglotz Representation yields a one-to-one correspondence between the inner functions θ on $\mathbb{D}$ and singular measures μ on $\mathbb{T}$ as follows:

$$\frac{1+\theta(z)}{1-\theta(z)} = \int_{\mathbb{T}} \frac{w+z}{w-z} d\mu(w) + iIm\frac{1+\theta(0)}{1-\theta(0)} \tag{4.5}$$

For the remainder of this section, the measure μ and inner function θ will be related in this way.
We define $H(\mathbb{D})$ to be the set of holomorphic functions on $\mathbb{D}$, and H^p, $1 \leq p < \infty$, to be the set of functions $f \in H(\mathbb{D})$ that satisfies

$$\|f\|_p = \sup_{0<r<1} \left(\int_{\mathbb{T}} |f(r\xi)|^p d\xi \right)^{1/p} < +\infty.$$

By virtue of Fatou's theorem, H^p may be identified with a subspace of $L^p(\mathbb{T})$. We then define

$$H_0^p = \{f \in H^p | f(0) = 0\}; \qquad H_-^p = \{f \in L^p(\mathbb{T}) | \overline{f} \in H_0^p\},$$

and for the inner function θ,

$$\theta^*(H^p) = \{f \in H^p | \overline{\theta} f \in H_-^p\}.$$

The model space $\theta^*(H^p)$ has a reproducing kernel given by

$$k_\lambda(z) = \frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\overline{\lambda}z}. \tag{4.6}$$

The Clark transform [4] $U = U_1 : \theta^*(H^p) \to L^p(\mu)$ is defined on the kernel functions by:

$$[U_1 k_\lambda](z) = \frac{1-\overline{\theta(\lambda)}}{1-\overline{\lambda}z}.$$

The operator U_1 can be extended to all of $\theta^*(H^p)$ only under certain conditions [2].

The Normalized Cauchy Transform (NCT) $V_\mu : L^p(\mu) \to H(\mathbb{D})$ is defined by

$$[V_\mu f](z) = (1-\theta(z)) \int_{\mathbb{T}} \frac{f(w)}{1-\overline{w}z} d\mu(w). \tag{4.7}$$

Lemma 4.3 *Suppose μ is a measure on $\mathbb{T}$ and $1 < p < \infty$. For the sequence $\{(e^{2\pi inx}, e^{2\pi inx})\}_{n=0}^\infty \subset L^q(\mu) \times L^p(\mu)$ with analysis auxiliary sequence $\{g_n\}_{n=0}^\infty \subset L^q(\mu)$, and $f \in L^p(\mu)$,*

$$[V_\mu f](z) = \sum_{n=0}^{\infty} \langle f, g_n \rangle z^n. \tag{4.8}$$

Proof A simple calculation shows that

$$\int_{\mathbb{T}} \frac{f(w)}{1-\overline{w}z} d\mu(w) = \sum_{n=0}^{\infty} \langle f, e^{2\pi i n x} \rangle z^n. \tag{4.9}$$

The product of $(1-\theta(z))$ and Eq. (4.9) is given by

$$\sum_{n=0}^{\infty} \sum_{j=0}^{n} \alpha_{n-j} \langle f, e^{2\pi i n x} \rangle z^n = \sum_{n=0}^{\infty} \langle f, g_n \rangle z^n.$$

□

Theorem 4.4 *Let μ be a singular measure and $1 < p < \infty$. Suppose that $U_1(\theta^*(H^p)) = L^p(\mu)$. Then, for every $F \in \theta^*(H^p)$ with Fourier series $F(z) = \sum_{n=0}^{\infty} a_n z^n$, the sequence of partial sums $\sum_{n=0}^{N} a_n z^n$ converges to $U_1 F$ in the $L^p(\mu)$-norm.*

We follow the argument given in Poltoratskii [12, Theorem 1.1]. Note that by assumption, U_1 is bounded (and bounded below) from $\theta^*(H^p)$ to $L^p(\mu)$.

Proof As argued in Poltoratskii, the partial sum $\sum_{n=0}^{N} a_n z^n$ coincides a.e. μ with $U_1 F - z^{N+1} U_1 S^{*N} F$. Then, since U_1 is bounded, we have that

$$\begin{aligned} \left\| \sum_{n=0}^{N} a_n z^n - U_1 F \right\|_{L^p(\mu)} &= \| z^{N+1} U_1 S^{*N} F \|_{L^p(\mu)} = \| U_1 S^{*N} F \|_{L^p(\mu)} \\ &\leq \| U_1 \|_p \| S^{*N} F \|_{L^p(\mu)} \to 0. \end{aligned}$$

□

Theorem 4.5 *Let μ be a singular measure and $1 < p < \infty$. Suppose that $U_1(\theta^*(H^p)) = L^p(\mu)$. Then $V_\mu = U_1^{-1}$.*

Proof We first note that it is well-known that the conclusion holds under the hypothesis that $1 < p \leq 2$ [2]. Our proof requires several steps.

We consider first the kernel functions $k_\lambda(z)$. Using Eqs. (4.6), (4.7), and (4.5), we calculate $V_\mu U_1 k_\lambda$:

$$\begin{aligned} [V_\mu U_1 k_\lambda](z) &= [1-\theta(z)] \int_{\mathbb{T}} \left(\frac{1-\overline{\theta(\lambda)}}{1-\overline{\lambda} w} \right) \left(\frac{1}{1-\overline{w}z} \right) d\mu(w) \\ &= [1-\theta(z)][1-\overline{\theta(\lambda)}] \int_{\mathbb{T}} \frac{1}{2(1-\overline{\lambda}z)} \left(\frac{\overline{w}+\overline{\lambda}}{\overline{w}-\overline{\lambda}} + \frac{w+z}{w-z} \right) d\mu(w) \\ &= [1-\theta(z)][1-\overline{\theta(\lambda)}] \frac{1}{2(1-\overline{\lambda}z)} \left(\frac{1+\overline{\theta(\lambda)}}{1-\overline{\theta(\lambda)}} + \frac{1+\theta(z)}{1-\theta(z)} \right) \end{aligned}$$

$$= \frac{1 - \overline{\theta(\lambda)}\theta(z)}{1 - \overline{\lambda} z}.$$

Since the linear hull of $\{k_\lambda\} \subset L^p(\mu)$ is dense, we have that $V_\mu U_1 = I$ on a dense subspace. Because $U_1 : \theta^*(H^p) \to L^p(\mu)$ is continuous, we only need to demonstrate that $V_\mu : L^p(\mu) \to \theta^*(H^p)$ is continuous to complete the proof. We take a slightly different approach here. Let $F \in \theta^*(H^p)$ have Fourier series given by $F(z) = \sum a_n z^n$. We define a functional $\eta_n(F) = a_n$. We have that this functional is continuous. We also define the functional $\gamma_n(F) = \langle U_1 F, g_n \rangle$; this functional is continuous as well. Note that $\eta_n(V_\mu f) = \langle f, g_n \rangle$ by Lemma 4.3. Moreover, $\eta_n = \gamma_n$ on the linear hull of $\{k_\lambda\}$, so they agree everywhere. It now follows that for $F \in \theta^*(H^p)$,

$$[V_\mu U_1 F](z) = \sum \langle U_1 F, g_n \rangle z^n = \sum \eta_n(F) z^n = \sum a_n z^n = F(z).$$

□

Theorem 4.6 *Suppose μ is a singular measure and $1 < p < \infty$. Suppose that $U_1(\theta^*(H^p)) = L^p(\mu)$. Then $\{(e^{2\pi i n x}, e^{2\pi i n x})\}_{n=0}^{\infty} \subset L^q(\mu) \times L^p(\mu)$ is effective in $L^p(\mu)$.*

Proof By Theorem 4.5, we have that for any $f \in L^p(\mu)$, $V_\mu f \in \theta^*(H^p)$. By Theorem 4.4, the Fourier series for $V_\mu f$ converges in $L^p(\mu)$. But, by Lemma 4.3, the sequence of partial sums of the Fourier series for $V_\mu f$ is precisely the sequence of approximations given by the Kaczmarz algorithm in Eq. (1.2). Since this Fourier series converges, we have for every $f \in L^p(\mu)$ that $\lim_{n\to\infty} \mathcal{A}_n(f) = f$. □

Corollary 4.7 *Suppose μ is a singular measure and the corresponding inner function satisfies the "one-component condition." Then $\{(e^{2\pi i n x}, e^{2\pi i n x})\} \subset L^q(\mu) \times L^p(\mu)$ is effective in $L^p(\mu)$.*

Proof If the inner function satisfies the one-component condition, then $U_1(\theta^*(H^p)) = L^p(\mu)$ for all $1 < p < \infty$ [2]. □

Remark 4.8 The exponentials in $L^2(\mu)$ (and $L^p(\mu)$) demonstrate that the matrix conditions in Theorems 2.5 and 2.11 are not sufficient to characterize effectivity. There are several aspects to this.

For a fixed measure μ, the matrix N for $\{e^{2\pi i n x}\} \subset L^2(\mu)$ as defined in Eq. (1.5) is identical to the matrix M for $\{(e^{2\pi i n x}, e^{2\pi i n x})\} \subset L^q(\mu) \times L^p(\mu)$ (and the matrix $\widetilde{M}$ also). Therefore, the matrix V as in Theorem B is equal to the matrix U (and $\widetilde{U}$). Therefore, if μ is singular, then the matrix conditions in Eqs. (2.5) and (2.6) are satisfied [5] but $\{(e^{2\pi i n x}, e^{2\pi i n x})\} \subset L^q(\mu) \times L^p(\mu)$ need not be effective in $L^p(\mu)$. Indeed, there are singular measures for which U_1 is not bounded for $p > 2$ [2], and for those measures μ, the pointwise boundedness condition of (1) in Theorem 1.2 is not satisfied.

As just observed, there are singular measures μ for which the matrix conditions are satisfied but the pointwise boundedness condition is not satisfied. However, there exist other singular measures μ for which both the matrix condition and the pointwise boundedness condition hold. We conclude from these examples that matrices alone cannot characterize effectivity. In particular, a matrix condition to guarantee the boundedness condition in Eq. (3.1) should not be expected.

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Data Availability Statement Code is available from the following repository: https://www.bitbucket.org/esweber/kaczmarz-in-banach-spaces/.

Appendix

This appendix contains the calculations for Eqs. (4.2), (4.3), and (4.4) from Example 4.2. SAGE code is available from the repository specified in the Data Availability Statement.

Recall that $\{\phi_n\}_{n=0}^{\infty}, \{\psi_n\}_{n=0}^{\infty} \subset \mathbb{R}^2$, $\phi_n = \phi_{n+3}$, and $\psi_n = \psi_{n+3}$ for all n, where

$$\begin{bmatrix}\phi_0 \ \phi_1 \ \phi_2 \ \psi_0 \ \psi_1 \ \psi_2\end{bmatrix} = \begin{bmatrix} 1 & 1 & .5 & 1 & 1 & 1.5 \\ -1 & 1 & -.5 & 0 & 0 & -.5 \end{bmatrix}.$$

Let M, $\widetilde{M}$, U, and $\widetilde{U}$ be as defined in Eqs. (2.1) and (2.3). We exploit the block structure engendered by the periodicity of $\{\phi_n\}_{n=0}^{\infty}$ and $\{\psi_n\}_{n=0}^{\infty}$ to explicitly calculate the involved matrices.

Claim 4.9 *If*

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ .5 & .5 & 1 \end{pmatrix},\ D = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ .5 & .5 & 1 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & -2 \\ 0 & -.5 & 1 \\ 0 & .25 & -.5 \end{pmatrix},\ C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -.25 & .5 \\ 0 & .125 & -.25 \end{pmatrix},$$
$$R = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix},\ S = \begin{pmatrix} 1 & 1 & .5 \\ 1 & 1 & .5 \\ 2 & 1 & 1 \end{pmatrix},\ W = \begin{pmatrix} .5 & -.5 & -.5 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix},\ T = \begin{pmatrix} .5 & -.5 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix},$$

then

$$I + M = \begin{pmatrix} F & 0 & 0 & 0 & \cdots \\ D & F & 0 & 0 & \cdots \\ D & D & F & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},\quad I + U = \begin{pmatrix} F^{-1} & 0 & 0 & 0 & 0 & \cdots \\ B & F^{-1} & 0 & 0 & 0 & \cdots \\ C & B & F^{-1} & 0 & 0 & \cdots \\ \frac{1}{2}C & C & B & F^{-1} & 0 & \cdots \\ \frac{1}{4}C & \frac{1}{2}C & C & B & F^{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.10}$$

$$I+\widetilde{M}=\begin{pmatrix} R & 0 & 0 & 0 & \cdots \\ S & R & 0 & 0 & \cdots \\ S & S & R & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad I+\widetilde{U}=\begin{pmatrix} R^{-1} & 0 & 0 & 0 & 0 & \cdots \\ W & R^{-1} & 0 & 0 & 0 & \cdots \\ T & W & R^{-1} & 0 & 0 & \cdots \\ T & T & W & R^{-1} & 0 & \cdots \\ T & T & T & W & R^{-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{4.11}$$

Given a matrix A, we will use $A_{(n,j)}$ to denote the three-by-three block matrix of A in the (n, j) position. We will use this notation for the remainder of the appendix. It is straightforward to check that $(I + M)(I + U) = I$ and $(I + \widetilde{M})(I + \widetilde{U}) = I$. First, note that $(I + M)(I + U)_{(n,j)}$ is equal to zero for $n < j$ and equal to I_3 for $n = j$. We calculate the following, using induction for the last equation.

$$\begin{aligned}
(I+M)(I+U)_{(n+1,n)} &= DF^{-1} + FB = 0 \\
(I+M)(I+U)_{(n+2,n)} &= DF^{-1} + DB + FC = 0 \\
(I+M)(I+U)_{(n+3,n)} &= DC - \frac{1}{2}FC = 0 \\
(I+M)(I+U)_{(n+k,n)} &= \frac{1}{2^{k-3}}(DC - \frac{1}{2}FC) = 0 \text{ for } k > 3.
\end{aligned}$$

We now show that $(I+\widetilde{M})(I+\widetilde{U}) = I$. Again, it is clear that $(I+\widetilde{M})(I+\widetilde{U})_{(n,j)}$ is equal to zero if $n < j$ and that $(I+\widetilde{M})(I+\widetilde{U})_{(n,j)} = I_3$ if $n = j$. Using calculation and induction,

$$\begin{aligned}
(I+\widetilde{M})(I+\widetilde{U})_{(n+1,n)} &= SR^{-1} + RW = 0 \\
(I+\widetilde{M})(I+\widetilde{U})_{(n+2,n)} &= SR^{-1} + SW + RT = 0 \\
(I+\widetilde{M})(I+\widetilde{U})_{(n+k,n)} &= ST = 0 \text{ for } k \geq 3
\end{aligned}$$

and conclude that $(I + \widetilde{M})(I + \widetilde{U}) = I$.

Proposition 4.10 $\langle \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)\delta_n, \delta_j\rangle = 0$ *for all* n, j.

Proof We first calculate the matrix $U\widetilde{M}^* + \widetilde{M}^* + I$.

$$\begin{aligned}
&(I+\widetilde{M}^*) + U\widetilde{M}^* \\
&= \begin{pmatrix} R^* & S^* & S^* & S^* & S^* & \cdots \\ 0 & R^* & S^* & S^* & S^* & \cdots \\ 0 & 0 & R^* & S^* & S^* & \cdots \\ 0 & 0 & 0 & R^* & S^* & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
\end{aligned}$$

$$
+\begin{pmatrix}
F^{-1}-I_3 & 0 & 0 & 0 & 0 & \cdots \\
B & F^{-1}-I_3 & 0 & 0 & 0 & \cdots \\
C & B & F^{-1}-I_3 & 0 & 0 & \cdots \\
\frac{1}{2}C & C & B & F^{-1}-I_3 & 0 & \cdots \\
\frac{1}{4}C & \frac{1}{2}C & C & B & F^{-1}-I_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

$$
\cdot\begin{pmatrix}
R^*-I_3 & S^* & S^* & S^* & S^* & \cdots \\
0 & R^*-I_3 & S^* & S^* & S^* & \cdots \\
0 & 0 & R^*-I_3 & S^* & S^* & \cdots \\
0 & 0 & 0 & R^*-I_3 & S^* & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

$$
=\begin{pmatrix}
R^*+(F^{-1}-I_3)(R^*-I_3) & S^*+(F^{-1}-I_3)S^* \\
B(R^*-I_3) & R^*+BS^*+(F^{-1}-I_3)(R^*-I_3) \\
C(R^*-I_3) & CS^*+B(R^*-I_3) \\
\frac{1}{2}C(R^*-I_3) & \frac{1}{2}CS^*+C(R^*-I_3) \\
\vdots & \vdots
\end{pmatrix.}
$$

$$
\begin{matrix}
S^*+(F^{-1}-I_3)S^* & S^*+(F^{-1}-I_3)S^* \\
S^*+BS^*+(F^{-1}-I_3)S^* & S^*+BS^*+(F^{-1}-I_3)S^* \\
R^*+CS^*+BS^*+(F^{-1}-I_3)(R^*-I_3) & S^*+CS^*+BS^*+(F^{-1}-I_3)S^* \\
\frac{1}{2}CS^*+CS^*+B(R^*-I_3) & R^*+\frac{1}{2}CS^*+CS^*+BS^*+(F^{-1}-I_3)(R^*-I_3) \\
\vdots & \vdots
\end{matrix}
$$

$$
\left.\begin{matrix}
S^*+(F^{-1}-I_3)S^* & \cdots \\
S^*+BS^*+(F^{-1}-I_3)S^* & \cdots \\
S^*+CS^*+BS^*+(F^{-1}-I_3)S^* & \cdots \\
S^*+\frac{1}{2}CS^*+CS^*+BS^*+(F^{-1}-I_3)S^* & \cdots \\
\vdots & \ddots
\end{matrix}\right).
$$

□

To show that every entry of $\widetilde{M}^*(U\widetilde{M}^*+\widetilde{M}^*+I)$ is equal to zero, we present a series of lemmas exploiting its block structure. Specifically, we use induction across select diagonals, rows, and columns.

Lemma 4.11 (Diagonal) $\widetilde{M}^*(U\widetilde{M}^*+\widetilde{M}^*+I)_{(i,i)}=0$ *for all* $i\in\mathbb{N}$.

Proof Using block matrix multiplication combined with the previous derivations, we compute the 3×3 principal submatrix of $\widetilde{M}^*(U\widetilde{M}^*+\widetilde{M}^*+I)$:

$$
\begin{aligned}
\widetilde{M}^*(U\widetilde{M}^*+\widetilde{M}^*+I)_{(0,0)} &= (R^*-I_3)\Big(R^*+(F^{-1}-I_3)(R^*-I_3)\Big)+S^*B(R^*-I_3) \\
&\quad +S^*C(R^*-I_3)+\frac{1}{2}S^*C(R^*-I_3)+\frac{1}{4}S^*C(R^*-I_3)+\cdots \\
&= (R^*-I_3)\Big(R^*+(F^{-1}-I_3)(R^*-I_3)\Big)+S^*B(R^*-I_3) \\
&\quad +2S^*C(R^*-I_3) \\
&= 0.
\end{aligned}
$$

We also calculate

$$
\begin{aligned}
&\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(1,1)} \\
&\quad = (R^* - I_3)\left(R^* + BS^* + (F^{-1} - I_3)(R^* - I_3)\right) \\
&\qquad + S^*\left(CS^* + B(R^* - I_3)\right) \\
&\qquad + S^*\left(\frac{1}{2}CS^* + C(R^* - I_3)\right) + S^*\left(\frac{1}{4}CS^* + \frac{1}{2}C(R^* - I_3)\right) \\
&\qquad + S^*\left(\frac{1}{8}CS^* + \frac{1}{4}C(R^* - I_3)\right) + S^*\left(\frac{1}{16}CS^* + \frac{1}{8}C(R^* - I_3)\right) \\
&\qquad + \cdots \\
&\quad = (R^* - I_3)\left(R^* + BS^* + (F^{-1} - I_3)(R^* - I_3)\right) + S^*B(R^* - I_3) \\
&\qquad + S^*CS^* + \frac{1}{2}S^*CS^* + \frac{1}{4}S^*CS^* + \frac{1}{8}S^*CS^* + \cdots \\
&\qquad + S^*C(R^* - I_3) + \frac{1}{2}S^*C(R^* - I_3) + \frac{1}{4}S^*C(R^* - I_3) + \cdots \\
&\quad = (R^* - I_3)\left(R^* + BS^* + (F^{-1} - I_3)(R^* - I_3)\right) + S^*B(R^* - I_3) \\
&\qquad + 2S^*CS^* + 2S^*C(R^* - I_3) \\
&\quad = 0.
\end{aligned}
$$

Notice that during both of these calculations we encountered geometric series when looking at the coefficients of the S^*CS^* and $S^*C(R^* - I_3)$ terms. This is a result of the structure of the matrix $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)$. Eventually, for large enough n, every (n, i) block in the matrix $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)$ is equal to $\frac{1}{2}\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(n-1,i)}$. This provides us with the iterative structure necessary for the blocks in question to be well defined. Furthermore, it allows us to make successful induction arguments concerning the entries of $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)$.

Specifically, we know that for $n \geq 2$,

$$
\begin{aligned}
\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(n,n)} &= \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(n-1,n-1)} \\
&\quad + \frac{1}{2^{n-1}}(R^* - I_3)CS^* + \frac{1}{2^{n-1}}S^*CS^* \\
&= \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(n-1,n-1)} \\
&\quad + \frac{1}{2^{n-1}}\left((R^* - I_3)CS^* + S^*CS^*\right). \qquad (4.12)
\end{aligned}
$$

One can confirm that $(R^* - I_3)CS^* + S^*CS^* = 0$. We have already shown that $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(n,n)} = 0$ for $n = 0$ and $n = 1$. Suppose that $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* +$

$I)_{(k,k)} = 0$ for some $k \geq 1$. By (4.12),

$$\begin{aligned}\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(k+1,k+1)} &= \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(k,k)} \\ &\quad + \frac{1}{2^k}\left((R^* - I_3)CS^* + S^*CS^*\right) \\ &= 0 + \frac{1}{2^k}\cdot 0 \\ &= 0.\end{aligned}$$

By induction, the lemma is proven. □

Lemma 4.12 *(Subdiagonal)* $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(i+1,i)} = 0$ *for all* $i \in \mathbb{N}$.

Proof We calculate

$$\begin{aligned}\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(1,0)} &= (R^* - I_3)B(R^* - I_3) \\ &\quad + S^*C(R^* - I_3) + \frac{1}{2}S^*C(R^* - I_3) \\ &\quad + \frac{1}{4}S^*C(R^* - I_3) + \cdots \\ &= (R^* - I_3)B(R^* - I_3) + 2S^*C(R^* - I_3) \\ &= 0.\end{aligned}$$

Next, we derive

$$\begin{aligned}\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(2,1)} &= (R^* - I_3)\left(CS^* + B(R^* - I_3)\right) \\ &\quad + S^*\left(\frac{1}{2}CS^* + C(R^* - I_3)\right) \\ &\quad + S^*\left(\frac{1}{4}CS^* + \frac{1}{2}C(R^* - I_3)\right) \\ &\quad + S^*\left(\frac{1}{8}CS^* + \frac{1}{4}C(R^* - I_3)\right) \\ &\quad + \cdots \\ &= (R^* - I_3)\left(CS^* + B(R^* - I_3)\right) \\ &\quad + \frac{1}{2}S^*CS^* + \frac{1}{4}S^*CS^* + \frac{1}{8}S^*CS^* + \cdots \\ &\quad + S^*C(R^* - I_3) + \frac{1}{2}S^*C(R^* - I_3) \\ &\quad + \frac{1}{4}S^*C(R^* - I_3) + \cdots \\ &= (R^* - I_3)\left(CS^* + B(R^* - I_3)\right) \\ &\quad + S^*CS^* + 2S^*C(R^* - I_3) \\ &= 0.\end{aligned}$$

We see that $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(n+1,n)} = 0$ holds for $n = 0, 1$. Recall that $(R^* - I_3)CS^* + S^*CS^* = 0$ and note that, for $n \geq 2$,

$$\begin{aligned}
\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(n+1,n)} &= \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(n,n-1)} \\
&\quad + \frac{1}{2^{n-1}}(R^* - I_3)CS^* + \frac{1}{2^n}S^*CS^* \\
&= \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(n,n-1)} \\
&\quad + \frac{1}{2^{n-1}}\left((R^* - I_3)CS^* + S^*CS^*\right). \qquad (4.13)
\end{aligned}$$

Assume that $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(k+1,k)} = 0$ for some $k \geq 1$. By (4.13),

$$\begin{aligned}
\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(k+2,k+1)} &= \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(k+1,k)} \\
&\quad + \frac{1}{2^k}\left((R^* - I_3)CS^* + S^*CS^*\right) \\
&= 0 + \frac{1}{2^k} \cdot 0 \\
&= 0
\end{aligned}$$

and the result follows by induction. □

Having shown the subdiagonal is equal to zero, we will now show that the entire lower triangular portion of the matrix is equal to zero.

Lemma 4.13 *(Lower Triangle)* $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(i,n)} = 0$*,* $n < i$.

Proof We first verify that

$$B(R^* - I_3) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & -.5 & 1 \\ 0 & .25 & -.5 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -.25 & .5 \\ 0 & .125 & -.2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 2C(R^* - I_3).$$

We then see that, for $i > n + 1$, $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(i,n)} = \frac{1}{2}\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(i-1,n)}$. By Lemma 4.12, the result holds. □

We will now use a series of three lemmas to show that every upper triangular entry of the matrix $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)$ is equal to zero.

Lemma 4.14 *(0th Row)* $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(0,n)} = 0$*,* $n > 0$.

Proof We first calculate

$$\begin{aligned}
\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(0,1)} &= (R^* - I_3)\left(S^* + (F^{-1} - I_3)S^*\right) \\
&\quad + S^*\left(R^* + BS^* + (F^{-1} - I_3)(R^* - I_3)\right) \\
&\quad + S^*\left(CS^* + B(R^* - I_3)\right) + S^*\left(\frac{1}{2}CS^* + C(R^* - I_3)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ S^*\left(\frac{1}{4}CS^* + \frac{1}{2}C(R^* - I_3)\right) + S^*\left(\frac{1}{8}CS^* + \frac{1}{4}C(R^* - I_3)\right)\\
&+ \cdots\\
&= (R^* - I_3)\left(S^* + (F^{-1} - I_3)S^*\right)\\
&+ S^*\left(R^* + BS^* + (F^{-1} - I_3)(R^* - I_3)\right) + S^*B(R^* - I_3)\\
&+ S^*CS^* + \frac{1}{2}S^*CS^* + \frac{1}{2}S^*CS^* + \cdots\\
&+ S^*C(R^* - I_3) + \frac{1}{2}S^*C(R^* - I_3) + \frac{1}{4}S^*C(R^* - I_3) + \cdots\\
&= (R^* - I_3)\left(S^* + (F^{-1} - I_3)S^*\right)\\
&+ S^*\left(R^* + BS^* + (F^{-1} - I_3)(R^* - I_3)\right) + S^*B(R^* - I_3)\\
&+ 2S^*CS^* + 2S^*C(R^* - I_3)\\
&= 0.
\end{aligned}
$$

We see that $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(0,n)} = 0$ holds for $n = 1$ and note that for $n \geq 1$,

$$
\begin{aligned}
\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(0,n+1)} &= \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(0,n)}\\
&\quad + S^*(F^{-1} - I_3)S^* + S^*S^* + S^*BS^* + 2S^*CS^*.
\end{aligned} \tag{4.14}
$$

Assume that $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(0,k)} = 0$ for some $k \geq 1$. We calculate $S^*(F^{-1} - I_3)S^* + S^*S^* + S^*BS^* + 2S^*CS^* = 0$. By (4.14),

$$
\begin{aligned}
\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(0,k+1)} &= \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(0,k)}\\
&\quad + S^*(F^{-1} - I_3)S^* + S^*S^* + S^*BS^* + 2S^*CS^*\\
&= 0
\end{aligned}
$$

and the result follows by induction. □

Lemma 4.15 *(1st Row)* $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(1,n)} = 0$, $n > 1$.

Proof By Lemma 4.14, we know that $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(0,n)} = 0$ for all $n > 0$. Note that for all $n > 1$,

$$
\begin{aligned}
\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(1,n)} &= \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(0,n-1)}\\
&\quad + (R^* - I_3)BS^* + 2S^*CS^*.
\end{aligned}
$$

We calculate $(R^* - I_3)BS^* + 2S^*CS^* = 0$ and conclude inductively that $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(1,n)} = 0$ for $n > 1$. □

Lemma 4.16 *(Diagonals in Upper Triangle)* $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(n,n+k)} = 0$, $n \geq 2$, $k \in \mathbb{N}_0$.

Proof We first note that for $n \geq 2$ and $k \in \mathbb{N}_0$,

$$\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(n,n+k)} = \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(n-1,n+k-1)} + \frac{1}{2^{n-2}}\left((R^* - I_3)CS^* + S^*CS^*\right). \tag{4.15}$$

By Lemma 4.15, we know that $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(n,n+k)} = 0$ holds for $n = 1$. Suppose $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(i,i+k)} = 0$ for some $i \geq 1$. Recall that $(R^* - I_3)CS^* + S^*CS^* = 0$. Thus, by (4.15),

$$\begin{aligned}\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(i+1,i+1+k)} &= \widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)_{(i,i+k)} \\ &\quad + \frac{1}{2^{n-2}}\left((R^* - I_3)CS^* + S^*CS^*\right) \\ &= 0.\end{aligned}$$

The result holds by induction. □

We have now shown that every entry of $\widetilde{M}^*(U\widetilde{M}^* + \widetilde{M}^* + I)$ is equal to zero. □

Proposition 4.17 *There exists some $i, j \in \mathbb{N}_0$ such that $\langle M^*(\widetilde{U}M^* + M^* + I)\delta_i, \delta_j\rangle \neq 0$.*

Proof We first calculate

$$\begin{aligned}&(I + M^*) + \widetilde{U}M^* \\ &= \begin{pmatrix} F^* & D^* & D^* & D^* & \cdots \\ 0 & F^* & D^* & D^* & \cdots \\ 0 & 0 & F^* & D^* & \cdots \\ 0 & 0 & 0 & F^* & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\ &\quad + \begin{pmatrix} R^{-1} - I_3 & 0 & 0 & 0 & \cdots \\ W & R^{-1} - I_3 & 0 & 0 & \cdots \\ T & W & R^{-1} - I_3 & 0 & \cdots \\ T & T & W & R^{-1} - I_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} F^* - I_3 & D^* & D^* & D^* & \cdots \\ 0 & F^* - I_3 & D^* & D^* & \cdots \\ 0 & 0 & F^* - I_3 & D^* & \cdots \\ 0 & 0 & 0 & F^* - I_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} F^* + (R^{-1} - I_3)(F^* - I_3) & D^* + (R^{-1} - I_3)D^* \\ W(F^* - I_3) & F^* + WD^* + (R^{-1} - I_3)(F^* - I_3) \\ T(F^* - I_3) & TD^* + W(F^* - I_3) \\ T(F^* - I_3) & TD^* + T(F^* - I_3) \\ \vdots & \vdots \end{pmatrix.$$

$$\begin{matrix} D^* + (R^{-1} - I_3)D^* \\ D^* + WD^* + (R^{-1} - I_3)D^* \\ F^* + TD^* + WD^* + (R^{-1} - I_3)(F^* - I_3) \\ TD^* + WD^* + W(F^* - I_3) \\ \vdots \end{matrix}$$

$$\left. \begin{matrix} D^* + (R^{-1} - I_3)D^* & \cdots \\ D^* + WD^* + (R^{-1} - I_3)D^* & \cdots \\ D^* + TD^* + WD^* + (R^{-1} - I_3)D^* & \cdots \\ \vdots & \ddots \end{matrix} \right).$$

Consider the (0, 0) block entry of $M^*(\widetilde{U}M^* + M^* + I)$ given by

$$(F^* - I_3)\left(F^* + (R^{-1} - I_3)(F^* - I_3)\right) + D^*W(F^* - I_3) + D^*T(F^* - I_3) + \cdots \tag{4.16}$$

where the term $D^*T(F^* - I_3)$ continues infinitely. We compute $D^*W(F^* - I_3) = D^*T(F^* - I_3) = 0$ and conclude that

$$(F^* - I_3)\left(F^* + (R^{-1} - I_3)(F^* - I_3)\right) = \begin{pmatrix} 0 & -.5 & 0 \\ 0 & -.5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0. \tag{4.17}$$

□

Proposition 4.18 $\langle (MU^*\widetilde{U}\widetilde{M}^* - M\widetilde{M}^*)\delta_i, \delta_j \rangle = 0$ *for all* $i, j \in \mathbb{N}_0$.

Proof It is clear that $U^*\widetilde{U} = I$ implies $\langle (MU^*\widetilde{U}\widetilde{M}^* - M\widetilde{M}^*)\delta_i, \delta_j \rangle = 0$ for all $i, j \in \mathbb{N}_0$. We will show $U^*\widetilde{U} = I$ using block matrix multiplication and exploiting the diagonal structure of U^* and $\widetilde{U}$.

First consider

$$U^*\widetilde{U} = \begin{pmatrix} F^{-*} - I_3 & B^* & C^* & \frac{1}{2}C^* & \frac{1}{4}C^* & \cdots \\ 0 & F^{-*} - I_3 & B^* & C^* & \frac{1}{2}C^* & \cdots \\ 0 & 0 & F^{-*} - I_3 & B^* & C^* & \cdots \\ 0 & 0 & 0 & F^{-*} - I_3 & B^* & \cdots \\ 0 & 0 & 0 & 0 & F^{-*} - I_3 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\cdot \begin{pmatrix} R^{-1}-I_3 & 0 & 0 & 0 & 0 & \cdots \\ W & R^{-1}-I_3 & 0 & 0 & 0 & \cdots \\ T & W & R^{-1}-I_3 & 0 & 0 & \cdots \\ T & T & W & R^{-1}-I_3 & 0 & \cdots \\ T & T & T & W & R^{-1}-I_3 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that $U^*\widetilde{U}$ will have diagonal bands of identical block matrices. Specifically, $U^*\widetilde{U}_{(n,k)}$ will be equal to $U^*\widetilde{U}_{(n+j,k+j)}$ for any $n, k, j \in \mathbb{N}_0$. It suffices then, to show that $U^*\widetilde{U}_{(0,k)} = U^*\widetilde{U}_{(n,0)} = 0$ and $U^*\widetilde{U}_{(0,0)} = I_3$ for $n, k \in \mathbb{N}_0$.

We begin by calculating

$$\begin{aligned} U^*\widetilde{U}_{(0,0)} &= (F^{-*} - I_3)(R^{-1} - I_3) + B^*W + C^*T + \frac{1}{2}C^*T + \frac{1}{4}C^*T + \cdots \\ &= (F^{-*} - I_3)(R^{-1} - I_3) + B^*W + 2C^*T \\ &= I_3. \end{aligned}$$

We next consider the 0th block column of $U^*\widetilde{U}$. First notice that $U^*\widetilde{U}_{(2,0)} = U^*\widetilde{U}_{(n,0)}$ for all $n \geq 2$. It is sufficient to calculate

$$\begin{aligned} U^*\widetilde{U}_{(1,0)} &= (F^{-*} - I_3)W + B^*T + C^*T + \frac{1}{2}C^*T + \frac{1}{4}C^*T + \cdots \\ &= (F^{-*} - I_3)W + B^*T + 2C^*T \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} U^*\widetilde{U}_{(2,0)} &= (F^{-*} - I_3)T + B^*T + C^*T + \frac{1}{2}C^*T + \frac{1}{4}C^*T + \cdots \\ &= (F^{-*} - I_3)T + B^*T + 2C^*T \\ &= 0. \end{aligned}$$

Finally, we consider the 0th row of $U^*\widetilde{U}$. Notice that

$$U^*\widetilde{U}_{(0,k)} = \frac{1}{2^{k-1}}\Big(2C^*(R^{-1} - I_3) + C^*W + C^*T\Big) \text{ for } k \geq 2. \tag{4.18}$$

We confirm that

$$\begin{aligned} C^*W &= C^*T \\ C^*(R^{-1} - I_3) &= -C^*T \end{aligned}$$

and conclude from (4.18) that $U^*\widetilde{U}_{(0,k)} = 0$ for ≥ 2.

It remains only to note that $B^*(R^{-1} - I_3) = -2C^*T$ and show

$$\begin{aligned} U^*\widetilde{U}_{(0,1)} &= B^*(R^{-1} - I_3) + C^*W + \frac{1}{2}C^*T + \frac{1}{4}C^*T + \cdots \\ &= B^*(R^{-1} - I_3) + C^*W + C^*T \\ &= 0. \end{aligned}$$

We see that $U^*\widetilde{U} = I$ and thus $\langle (MU^*\widetilde{U}\widetilde{M}^* - M\widetilde{M}^*)\delta_i, \delta_j \rangle = 0$ for all $i, j \in \mathbb{N}_0$. $\square$

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